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Thesis submitted for the degree of Magister Philosophiae

ASYMPTOTIC BEHAVIOUR OF DIRICHLET PROBLEMS IN RIEMANNIAN MANIFOLDS

candidate:
Lino Notarantonio

Supervisor:
Prof. Gianni Dal Maso

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INTRODUCTION

The aim of this thesis is the study of the asymptotic behaviour of Dirichlet problems in a Riemannain manifold.

More precisely, we are in a Riemannian manifold *(M,g)*, g is the metric tensor; let $(E_h)_h$ be a sequence of closed sets in M assuming (for simplicity) that they have smooth boundary. We study the Dirichlet problem

$$(0.1)\qquad \begin{aligned} -\Delta_g u_h + \lambda u_h = f \qquad & \text{in } M \setminus E_h \\ u_h = 0 \qquad & \text{on} \partial(M \setminus E_h)\,, \end{aligned}$$

and it is known that it has a solution u_h for every $\lambda > 0$.

Now we let h tends to $+\infty$ and we find what happens to the solutions. The limit problem could have an "extra" term μu, where μ is a Borel measure satisfying certain suitable conditions that will be discussed later; so we may have

$$(0.2)\qquad \begin{aligned} -\Delta_g u + \mu u + \lambda u = f \qquad & \text{in } M \\ u = 0 \qquad & \text{on } \partial M\,, \end{aligned}$$

that has to be interpreted in an appropriate way; it was proposed the following definition: We say that u is a *weak solution* to the problem (0.2), if

$$\int_M \langle \nabla u \,, \nabla z \rangle_g \mathrm{d}V_g + \int_M uz \mathrm{d}\mu + \lambda \int_M uz \,\mathrm{d}V_g = \int_M fz \mathrm{d}V_g$$

for every $z \in H_0^1(M) \cap L^2(M, \mu)$.

Associates to (0,1) and (0.2), we have two functionals, respectively:

$$(0.3)\qquad F_h(v) = \int_M [| \nabla v |^2 + \lambda v^2] \mathrm{d}V_g + \int_M v^2 \mathrm{d}\mu - \int_M fv \mathrm{d}V_g.$$

and the μ–energy functional

$$(0.4)\qquad F(u) = \int_M [| \nabla u |^2 + \lambda u^2] \mathrm{d}V_g + \int_M u^2 \mathrm{d}\mu - \int_M fu \mathrm{d}V_g.$$

Since it is possible to prove, by variational method, that the solutions u_h and u of (0.1) and (0.2) are the minimum point of F_h and F respectively, we will focus on the functional of the μ–energy for which we prove some results in Γ– convergence and we shall go back to (0.2) only in the fourth chapter.

We study the Γ–limit of F_h and give an integral representation for the Γ–limit (Theorem 2.2), adapting a Daniell's type argument to our scope; more in general we can study the Γ–limit of functionals of the μ–energy of type (0.4) and Theorem 2.2 gives in turn, a compactness result for a suitable class of measures. The Daniell's type argument was used in a paper of Buttazzo-Dal Maso-Mosco [1] for a similar purpose. Then we prove a result (Theorem 3.19) which concerns with the continuity property of the restriction operator μ^E of a measure μ belonging to the class of all Borel measures which are absolutely continuous w.r.t. the capacity on M. (Definition 1.19). This class is indicated by $\mathcal{M}_0(M)$. In the case of R^N, such a continuity property was proved in Dal Maso [2]. Finally, using an abstract result about the Γ convergence in Attouch [1] (Theorem 3.26), we can give first a result on the strongly convergence of the resolvent operator associated to our functionals, and consequently the convergence of the eigenvalues.

The same problem (0.1), under more restrictive conditions, was treated in Chavel [1], using Brownian motion methods.

When we have an involutive map on the Riemannian manifold, the problems (0.1) and (0.2) were investigated by Dal Maso-Gulliver-Mosco [1] using the methods of Γ convergence.

In the case that $M = \Omega \subset I\!R^n$, problems similar to (0.1) were studied by Rauch [1], Rauch and Taylor [1], using scattering methods; a compactness result was given by Dal Maso-Mosco [1] using methods of the Γ–convergence. See also the Bibliography for more references.

The first chapter is a collection of preliminaries, in which we present, in the first section, some classical facts of the Riemannian geometry; in the second we define the capacity of a Borel subset of M and in the third some definitions about the quasi–topology.

The second chapter is devoted to an abstract method, which allows us to construct the measure μ that will give the required representation of the limit functional. This method is a specialization to the bilinear form of the construction of the Daniell's integral; the abstract part of the theory is clearly spotted for possible future use.

In the third chapter we introduce the notion of γ–convergence for measures belonging to the class $\mathcal{M}_0(M)$, defiined in term of the Γ–convergence of the μ–energy functionals. The main result is the equivalence between the γ–convergence and the continuity of the restriction operator under the γ–convergence. in this chapter we make an extensive use of a series of general results in Dal Maso [2].

The main result in the fourth chapter is the convergence of the resolvent operators if and only if the corrispective functionals Γ–converge. From this equivalence, by a general result in Functional Analysis, we get the convergence of the spectrum.

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CHAPTER 1

Let M be a differentiable manifold. We say that M is a *Riemannian manifold* if there exists a $(0,2)$ tensor g, i.e. a section of $T^*(X) \otimes T^*(X)$, *symmetric* and *positively definite.* Such a tensor g is called *metric tensor.* With this tensor we can define a scalar product on T_pM, the tangent space at $p \in M$.

In fact, let $(\frac{\partial}{\partial x^i})_{1 \leq i \leq n}$ be the coordinate vector fields in a local chart (U, ζ) around $p \in M$ and let $u, v \in T_p(M)$

$$u = u^i \frac{\partial}{\partial x^i}|_p \qquad v = v^j \frac{\partial}{\partial x^j}|_p$$

then

$$g(u,v)|_p = g_{ij}|_p u^i v^j,$$

where

$$g_{ij}|_p = g(\frac{\partial}{\partial x^i}|_p, \frac{\partial}{\partial x^j}|_p).$$

Here and after we use the Einstein convention over repeated indeces and we shall suppress the index p in order to avoid heavy notations, if no confusion will arise.

Let $f : M \longrightarrow I\!R$ be a smooth function; we now recall that the gradient of f is the unique element of the tangent space T_pM, for all $p \in M$ such that $\langle \nabla f_p, v \rangle_g = df_p(v)$, for every vector field v. Let $(\frac{\partial}{\partial x^i})_{1 \leq i \leq n}$ be the coordinate vector field in a local chart (U, ζ) around a point $p \in M$; the gradient of f reads then

$$\nabla f = g_{ij} \frac{\partial f}{\partial x^j} \frac{\partial}{\partial x^i}.$$

We can define another differential operator, the *divergence* of a smooth vector field Y, as

$$\mathrm{div} Y = \mathrm{tr} DY$$

where D is the Levi–Civita connection.

In local coordinates $(\frac{\partial}{\partial x^i})_{1 \leq i \leq n}$ we have

$$(\nabla_i Y)^j = \frac{\partial Y^j}{\partial x^i} + \Gamma^j_{ik} Y^k$$

where ∇_i is the covariant derivation along the direction $\frac{\partial}{\partial x^i}$, Γ^j_{ik} is the Christoffel symbol, defined as

$$\Gamma^j_{ik} = \frac{1}{2} g^{jl} (\frac{\partial g^{kl}}{\partial x^i} + \frac{\partial g^{li}}{\partial x^k} - \frac{\partial g^{ik}}{\partial x^l});$$

so

$$\mathrm{div} Y = \mathrm{tr} DY = (\nabla_i Y)^i = \frac{\partial Y^i}{\partial x^i} + \Gamma^i_{ik} Y^k.$$

Taking into account the formula

$$\Gamma^i_{ik} = \frac{\partial}{\partial x^k} \log \sqrt{g},$$

where $|g| = \det g_{ij}$, we have

(1.1). $$\mathrm{div} Y = \frac{\partial Y^i}{\partial x^i} + \frac{\partial \log \sqrt{g}}{\partial x^i} Y^i$$

REMARK 1.1

In the Euclidean case (or more generally, in the flat case) div $Y = \frac{\partial Y^i}{\partial x^i}$, since in this case the Christoffel symbols vanish and so the second member of the right hand side of the formula above reduces to zero.

DEFINITION 1.2

Let M be a differential manifold; we say that M is *compact* if for every open covering there exists a *finite* sub–covering.

MANIFOLD WITH BOUNDARY

A smooth manifold with boundary is a Hausdorff topological space equipped with coordinate charts $(U_i, \varphi_i)_{i \in I}$ such that:

i) the U_i gives an open covering of M;

ii) φ_i is an homeomorphism of U_i onto an open subset of $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x^1 \geq 0\}$;

iii) for every pair of indeces (i,j) such that $U_i \cap U_j \neq \emptyset$, the coordinate changes

$$\varphi_i \circ \varphi_j^{-1} : \varphi_j(U_j) \longrightarrow \varphi_i(U_i)$$

is a smooth diffeomorphism.

For the sake of precision we say that a function $F : V \longrightarrow W$ for $V, W \in \mathbb{R}^n_+$ is a diffeomorphism if

a) F induces a diffeomorphism between $V \setminus \partial V$ onto $W \setminus \partial W$, where $\partial V = V \cap \partial I\!R^n_+$.

b) Any partial derivatives $D^{(k)}F$ of F extends to a continuous function on ∂V, the boundary of V.

c) The boundary value of any partial derivative is smooth on ∂V and

$$D^{(k)}(F|_{\partial V}) = (D^{(k)}F)|_{\partial V}, \qquad 2 \leq k \leq n.$$

d) $F|_{\partial V} : \partial V \longrightarrow \partial W$ is a smooth diffeomorphism of ∂V onto ∂W.

Therefore if $\varphi_i(p) \in \partial I\!R^n_+$ for $p \in M$ and $p \in U_i \cap U_j$, then $\varphi_j(p) \in \partial I\!R^n_+$. The *boundary* of M, denoted by ∂M, is the set of points which have the above property. A manifold with boundary will be denoted by $\overline{M}$. $\overline{M} \setminus \partial M$ is an n–dimensional manifold (without boundary) and it is called the *interior* of $\overline{M}$ and it will be denoted by M.

REMARK 1.3

A *closed* manifold is a compact manifold without boundary, such as S^2.

We need to introduce $i_X\omega$, where X is a vector field and ω is a form in order to give a meaning to the n–1 volume form of ∂M.

DEFINITION 1.4

Let X be a vector field and S be a $(0,p)$ tensor (a p–form); $i_X S$ is the $(0,p-1)$ tensor defined as

$$(i_X S)(x_1, \ldots, x_{p-1}) = S(X, x_1, \ldots, x_{p-1}).$$

This is the rigorous definition of the contraction operation on the indeces of a tensor.

With the Definition 1.4 we can define the n–1 volume form of the Riemannian sub-manifold $(\partial M, g_{|\partial M})$ as

$$\omega_{n-1} = i_\nu \omega_n$$

where ω_n is the n–volume form of $\overline{M} \setminus \partial M$; ν is the normal vector field pointing inside (that is ν_p for $p \in M$ is the unit normal vector to $T_p\partial M$). This notion will be used below, when we define the integral over the boundary of $\overline{M}$.

INTEGRATION ON M

DEFINITION 1.5

Let M be a Riemannian manifold and (U,ζ) be a local chart, with $(x^i)_{i=1}^n$ the associated coordinate system. Define

$$(1.2)\qquad \int_M f\mathrm{d}V_g := \int_{\zeta(U)} (\sqrt{\det g} f)\circ\zeta^{-1}\mathrm{d}x,$$

for the continuous function f on M, with $\mathrm{supp}(f)\subset U$. If (U',ζ') is another chart and if $\mathrm{supp}(f)\subset U'\bigcap U$, then by a change of coordinates we have

$$\int_U f\mathrm{d}V_g = \int_{\zeta(U)} f\sqrt{|g|}\circ\omega^{-1}\mathrm{d}x = \int_{\zeta'(U')} f\sqrt{|g|}\circ\omega^{-1}\mathrm{d}x,$$

since, denoting by $A_i^\alpha = \frac{\partial y^\alpha}{\partial x^i}$ and $B_j^\gamma = \frac{\partial x^\gamma}{\partial y^j}$, $AB = Id.$ and $\sqrt{\det g} = |B|^2\sqrt{\det g'}$, where g' is the metric tensor of M expressed in the chart (U',ζ').

If the support of f is not contained in a single chart, by a partition of the unity $(\alpha_i)_i$ subordinate to the atlas $(U_i,\zeta_i)_i$, we define

$$\Sigma_i\int_M \alpha_i f\mathrm{d}V_g = \Sigma_i\int_M \alpha_i f\mathrm{d}V_g.$$

If $(\gamma_i)_i$ is a partition of the unity subordinate to another atlas $(U'_j,\zeta'_j)_j$, we have

$$\Sigma_i\int_M \alpha_i f\mathrm{d}V_g = \Sigma_{i,j}\int_M \alpha_i\gamma_j f\mathrm{d}V_g = \Sigma_j\int_M \gamma_j f\mathrm{d}V_g.$$

In this way $f\mapsto\int_M f\,dV_g$ defines a positive, continuous and linear functional on $C_c(M)$; thus dV_g defines a positive Radon measure on M. We refer to $\mathrm{d}V_g$ as the volume measure (or Lebesgue measure) on M.

REMARK 1.6

Let (U,ζ) be a local chart and let $\Omega=\zeta(U)$. For every $f: U\longrightarrow I\!R$ we still indicate, with an abuse of notation, by f the function $f\circ\zeta^{-1}$. With this notation, we have

$$\int_U f\mathrm{d}V_g = \int_\Omega f(x)b(x)\mathrm{d}x,$$

where $b(x)=\sqrt{|g|}\circ\zeta^{-1}$.

DEFINITION 1.7

We define

$$L^2(M) = \left\{ f : \int_M f^2 dV_g < +\infty, \text{f Borel function on} M \right\}.$$

PROPOSITION 1.8 (Green's formula).

Let M be a Riemannian manifold and let $f : M \longrightarrow I\!R$ a smooth function (say C^1); for every $\mathbf{v} \in \Gamma(TM)$, i.e. a section of the tangent bundle, the following equality holds

$$\text{(1.3)} \qquad \int_M \text{div}(f\mathbf{v}) = \int_M \langle \nabla f, \mathbf{v} \rangle_g + \int_M f \text{div}\mathbf{v}.$$

PROOF.

In local coordinates, it follows that

$$(\nabla_i(f\mathbf{v}))^j = v^j \frac{\partial f}{\partial x^i} + f \frac{\partial v^j}{\partial x^i} + f\Gamma^j_{ik};$$

so

$$\text{div}(f\mathbf{v}) = v^i \frac{\partial f}{\partial x^i} + f \frac{\partial v^i}{\partial x^i} + f\Gamma^i_{ik}.$$

Since

$$g^{ij} g_{jk} = \delta^i_k \qquad \text{(Kroneker's symbol)},$$

we have

$$\frac{\partial f}{\partial x^i} v^i = g_{ik} g^{ki} \frac{\partial f}{\partial x^i} v^i = \langle \nabla f, \mathbf{v} \rangle_g.$$

The formula (1.2) is the first step in order to prove the Green's formula:

$$(GF) \qquad \int_M f \Delta h dV_g + \int_M \langle \nabla f, \nabla h \rangle_g dV_g = \int_{\partial M} \langle f \nabla h, \nu \rangle_g dV_{g|_{\partial M}}$$

The second step consists in proving the following equality:

$$\text{(1.4)} \qquad \int_M \text{div}(f \nabla h) dV_g = \int_{\partial M} \langle f \nabla h, \nu \rangle_g dV_{g|_{\partial M}}$$

To get (1.4) we need the Stokes' theorem and the following result.

PROPOSITION 1.9

(1.5) $$(\mathrm{div}X)v_g = \mathrm{d}(i_X(v_g)).$$

STOKES' THEOREM

For an oriented manifold M and for every n–1 form α, then

$$\int_M \mathrm{d}\alpha = \int_{\partial M} \alpha$$

holds true. If M is a manifold without boundary, then the right hand side is zero.

For a proof of this two results, we refer to [see Gallot-Hulin-Lafontaine, Lemma 4.8, pag 157].

From the formula (1.5), we get the following one

(1.6). $$\int_M \mathrm{div}X v_g = \int_{\partial M} \langle X, \nu\rangle_{g|\partial M} v_{|g\partial M}$$

In fact, using the Stokes' theorem we have

$$\int_M \mathrm{div}X v_g = \int_M \mathrm{d}(i_X v_g) = \int_{\partial M} i_X v_g$$

and if $(\nu, e_2, \ldots, e_n)$ is an orthonormal base in $T_m(M)$, then

$$(i_X v_g)(e_2, \ldots, e_n) = v_g(X, e_2, \ldots, e_n) = \langle X, \nu\rangle_g v_g(\nu, e_2, \ldots, e_n) = \langle X, \nu\rangle_g i_\nu v_g,$$

and the formula (1.4) is proven; (1.3) and (1.4) together give (GF).

PROPOSITION 1.10

Let be $u \in C^1(M)$; then the following formula holds true:

(1.7) $$\int_M \langle \nabla u, \varphi\rangle_g \mathrm{d}V_g = -\int_M (u\mathrm{div}\varphi)\mathrm{d}V_g,$$

$\forall \varphi \in \Gamma(TM)$ with compact support.

PROOF. The formula (1.7) follows from (1.3) and (1.6).

The formula (1.7) is the Riemannian analogous of the integration by part formula in (the flat case of) $\mathbb{R}^N$.

We say that a vector field is *measurable* if $X \circ \zeta^{-1} : \Omega \longrightarrow \mathbb{R}$ is a measurable function for every local chart (U, ζ), where $\Omega = \zeta(U)$.

Define now

$$\mathcal{L}^2(M) = \left\{ X : \int_M | X |_g^2 \, dV_g < +\infty, X \text{Borel function on M} \right\}.$$

Given any continuous vector field X and Y, we define the inner product on $\mathcal{L}^2(M)$ by

$$(X, Y)_{\mathcal{L}^2(M)} = \int_M \langle X, Y \rangle_g dV_g$$

inducing the norm

$$| X |_{\mathcal{L}^2(M)}^2 = \int_M | X |_g^2 \, dV_g.$$

With this notation, formula (1.7) reads

$$(\nabla u, \varphi)_{\mathcal{L}^2(M)} = -(u, \mathrm{div}\varphi)_{L^2(M)},$$

where the subscripts indicate in which space the scalar product is performed. However, since it is clear what are the spaces involved, we shall suppress these subscripts.

Formula (1.6) can be generalized as follows: we say that $f \in L^2(M)$ has a *weak derivative* $Y \in \mathcal{L}^2(M)$ if

$$(Y, \varphi)_{\mathcal{L}^2(M)} = -(f, \mathrm{div}\varphi)_{L^2(M)},$$

for every vector field φ with compact support.

<u>DEFINITION 1.11</u>

The Sobolev space $H^1(M)$ consists of those measurable functions $f \in L^2(M)$ having weak derivative in $\mathcal{L}^2(M)$. Since there exists at most one weak derivative Y that belongs to

$\mathcal{L}^2(M)$, we write in analogy with the smooth case $Y = \nabla f$. We remark that $H^1(M)$ inherits the scalar product

$$(f,h)_{H^1(M)} = (f,h) + (\nabla f, \nabla h),$$

and so $H^1(M)$ is a Hilbert space.

<u>DEFINITION 1.12</u>

With $H^1_0(M)$, we shall indicate the closure of $C_c^\infty(M)$ with respect to the norm induced by the scalar product in $H^1(M)$.

<u>REMARK 1.13</u>

If $M = \Omega \subset\subset \mathbb{R}^n$, it is a classical result that formula [5] defines a distribution a continuous and linear functional on $\mathcal{D}(\Omega)$ and, therefore, we can give a meani $\frac{\partial u}{\partial x^i}, i = 1, \dots, n$ for $u \in H^1(M)$ at least in the distribution sense.

<u>REMARK 1.14</u>

In the Euclidean case, we have for an open bounded set $\Omega \subset \mathbb{R}^n$

$$H^1_0(\Omega) \subset H^1(\Omega)$$

and the inclusion is strict; in the Riemannian case we may have

$$H^1_0(\Omega) = H^1(\Omega),$$

as the following result shows (see [Aubin]).

<u>THEOREM 1.15</u>

For a complete Riemannian manifold (without boundary)

$$H^1_0(\Omega) = H^1(\Omega).$$

<u>DEFINITION 1.16</u>

In a Riemannian manifold we can define in a natural way

$$F_M(u) = \int_M [|\nabla u|_g^2 + u^2] \tag{1.8}$$

in local coordinates, the functional in (1.8) is read as

$$F_M(u) = \int_M \left(g^{ij} D_i u D_j u + u^2\right) \mathrm{d}V_g.$$

We define the notion of *capacity*, as the infimum of (1.8) under constraints, i.e. let A a subset of M,

$$\mathrm{cap} A = \inf \left\{ F_M(u) : u \in H^1(M), \ u \geq 1 \quad \text{in an open neighbourhood of } A, \right\}$$

REMARK 1.17

The notion of capacity is *intrinsic*, that is, it does not depend on the choice of the coordinates.

Since $H^1(\mathbb{R}^N)$ functions are defined up to a set of capacity zero, by the uniform ellipticity of the metric tensor g this property holds true for the functions of $H^1(M)$.

REMARK 1.18

By $\mathcal{B}(M)$ we mean the class of the Borel subsets of M.

DEFINITION 1.19

We define $\mathcal{M}_0(M)$ as the family of all Borel measures that vanish on all set of capacity zero.

EXAMPLE 1.20

Let A and B two subsets of Ω, $B \subset \mathcal{B}(M)$, where Ω is a coordinate chart of M. where Ω is a coordinate chart of M. Define

$$\infty_A(B) = \begin{cases} 0, & \text{if } \mathrm{cap}(A \cap B) = 0; \\ +\infty, & \text{if } \mathrm{cap}(A \cap B) > 0. \end{cases}$$

Then $\infty_A(\cdot)$ belongs to $\mathcal{M}_0(M)$.

EXAMPLE 1.21

Let A a Borel subset of Ω, where Ω is as above; let V_g be the Lebesgue measure on M (also called the Riemannian volume of M), that is

$$V_g(A) = \int_M 1_A \, dV_g,$$

then V_g belongs to $\mathcal{M}_0(M)$.

DEFINITION 1.22

Let $\mu \in \mathcal{M}_0(M)$ be given and let

$$f : M \longrightarrow \mathbb{R}$$

be a function which is measurable with respect to the σ–algebra $\mathcal{B}(M)$ of the Borel sets of M; we define

$$\int_M f d\mu$$

as an (abstract) Lebesgue integral on the measure space $(M, \mathcal{B}(M), \mu)$.

The notion of *capacity* of a subset E of a Riemannian manifold M is given in an invariant way, see Definition above, in particular we can say that a property *P(x)* holds *quasi everywhere* if this property holds for all x in M except for a set Z, with $\text{cap}(Z) = 0$.

DEFINITION 1.23

We say that a set A, contained in M, is *quasi open* (resp. *quasi closed,quasi compact*) if for every $\varepsilon > 0$, there exists an open set (resp. closed, compact) U, such that

$$\text{cap}(U \Delta A) < \varepsilon,$$

where Δ is the symmetric difference between two sets. A set A is quasi open if and only if A^c is quasi closed, where A^c is the complementation w.r.t. M; moreover countable union (or finite intersection) is still quasi open.

DEFINITION 1.24

A function

$$f : M \longrightarrow \mathbb{R}$$

is said to be *quasi continuous* in M if, for every $\varepsilon > 0$ there exists a set E in M, with $\text{cap}(M \setminus E) < \varepsilon$, such that the restriction

$$f_{|_E} : E \longrightarrow \mathbb{R}$$

is continuous.

From the Remark 1.17, we have seen that every function in $H^1(M)$ is defined up to a set of capacity zero; actually a more striking property holds: every function $u$has a quasi continuous representative $\widetilde{u}$ in $H^1(M)$. This fact permits us to say that the following functional

$$\Phi_M(u) = \int_M u^2 d\mu, \qquad \forall u \in H^1(M)$$

is well defined. In the Riemannian case, we point out that this functional is *non–local*, that is, no chart is needed for its definition, as it has been done for the Lebesgue integral.

DEFINITION 1.25

Let $\mathcal{E}$ be a family of subsets in M; we say that $\mathcal{E}$ is *dense* in $\mathcal{P}(M)$ if for every pair (K, V), K compact, V open, $K \subset V$, there exists $E \in \mathcal{E}$, such that

$$K \subset E \subset V.$$

We say that $\mathcal{E}$ is *rich* in $\mathcal{P}(M)$ if for every chain $(E_t)_{t \in T}$ in $\mathcal{P}(M)$, the set

$$\{t \in T : E_t \notin \mathcal{E}\}$$

is at most countable.

By *chain* we mean a family of subsets of M such that T is a non-empty open interval of $I\!R$, $\overline{E}_t$ is compact for every $t \in T$ and $\overline{E}_s \subset \mathrm{int}(E_t)$ for every $s < t$, $t, s \in T$

PROPOSITION 1.26

Every rich family is dense.

PROOF

For the proof, we refer to [Dal Maso,Proposition 4.8]. This proof depends essentially on the Urysohn's Lemma, which holds true in any normal topological space, such as a differentiable manifold.

LEMMA 1.27

Let $\alpha : \mathcal{P}(M) \longrightarrow \overline{I\!R}$ be an increasing function, such that $\alpha(E_1) = \alpha(E_2)$ whenever $\mathrm{cap}(E_1 \Delta E_2) = 0$. Let $\mathcal{E}(\alpha)$ be the family of all subsets of M such that $\overline{E}$ is compact in M and $\alpha(\mathrm{int}E) = \alpha(\overline{E})$. Then $\mathcal{E}(\alpha)$ is rich in $\mathcal{P}(M)$.

PROOF

For the proof we address to [Dal Maso, Lemma 4.10].

CHAPTER 2

DEFINITION OF Γ–CONVERGENCE

We shall use the notion of Γ–convergence in the sequel; to this purpose we give below the definition and a result of Γ–convergence that we will use hereafter.

DEFINITION 2.1

Let be X a metric space, $(F_h)_h$ a sequence of functionals, such that

$$F_h : X \longrightarrow \overline{\mathbb{R}},$$

and let

$$F : X \longrightarrow \overline{\mathbb{R}}.$$

We say that the sequence $(F_h)_h$ Γ–converge to F in X if and only if the following conditions (a) and (b) hold true:

(a) for every sequence $(u_h)_h$ in X converging to some $u \in X$ as $h \to +\infty$, we have

$$F(u) \leq \liminf_{h \to +\infty} F_h(u_h);$$

(b) for every $u \in X$ there exists a sequence $(u_h)_h$ such that

$$F(u) \geq \limsup_{h \to +\infty} F_h(u_h).$$

The following compactness theorem holds (see [DG-F], Prop. 3.1)

THEOREM 2.1

Assume that X is a separable metric space. For every sequence $(F_h)_h$ of functionals there exists a subsequence $(F_{h_k})_k$ which Γ–converges in X to a lower–semicontinuous functional as $k \to +\infty$.

Let $(\mu_h)_h$ be a sequence of measures belonging to $\mathcal{M}_0(M)$ and for every $h \in \mathbf{N}$, let us consider the functional $F_h : L^2(M) \longrightarrow [0, +\infty]$, defined by

$$F_h(u) = \begin{cases} \Psi(u) + \Phi_h(u), & u \in H_0^1(M), \\ +\infty & \text{otherwise in } L^2(M), \end{cases}$$

where

$$\Phi_h(u) = \int_M \widetilde{u}^2 d\mu_h$$

and

$$\Psi(u) = \int_M [| \nabla u |^2 + u^2] dV_g.$$

By the previous theorem, a subsequence of $(F_h)_h$ Γ–converges in $L^2(M)$ to a functional $F : L^2(M) \to [0, +\infty]$. The following theorem provides an integral representation of the limit functional F.

<u>THEOREM 2.2</u>

Suppose that

$$F_h \overset{\Gamma(L^2(M))}{\longrightarrow} F \qquad \text{as} \quad h \to +\infty;$$

then there exists a measure $\mu \in \mathcal{M}_0(M)$, such that

$$F(u) = \int_M [| \nabla u |^2 + u^2] \mathrm{d}V_g + \int_M \widetilde{u}^2 \mathrm{d}\mu, \quad \forall u \in H_0^1(M),$$

while $F(u) = +\infty$ if $u \notin H_0^1(M)$.

<u>REMARK 2.3</u>

To prove the theorem we define the functional $\Phi : H_0^1(M) \to [0, +\infty]$ by

$$\Phi(u) = F(u) - \Psi(u), \quad \text{if } u \in H_0^1(M); \tag{2.1}$$

we have to show that

$$\Phi(u) = \int_M \widetilde{u}^2 \mathrm{d}\mu \qquad \forall u \in H_0^1(M)$$

for a suitable measure $\mu \in \mathcal{M}_0(M)$. This will be performed through various steps.

REMARK 2.4

Before giving the properties of the limit functional Φ, we want to precise what we mean for a extended valued quadratic functional $F : X \longrightarrow [0, +\infty]$, acting on any real vector space X. We say that F is a *quadratic functional* if it satisfies the following conditions

$$F(0) = 0, \quad F(u) \geq 0, \quad F(tu) = t^2 F(u), \; \forall t \in I\!R$$

$$F(u+v) + F(u-v) = 2[F(u) + F(v)].$$

Since the functional F admits the value $+\infty$, we shall follow hereafter the following (usual) convention: $0 \cdot +\infty = 0$, $+\infty + t = +\infty$ for every $t \in I\!R$.

THEOREM 2.5

Let $u, v \in H_0^1(M)$ and let Φ be the functional defined by (2.1). Then

(i) If $0 \leq u \leq v$ a.e. on M, then $\Phi(u) \leq \Phi(v)$;

(ii) $\Phi(|u|) \leq \Phi(u)$;

(iii) $\Phi(u+v) \leq \Phi(u) + \Phi(v)$, if $u \wedge v = 0$ a.e. on M;

(iv) $\Phi(u) = \lim_h \Phi(u_h)$, for every increasing sequence $(u_h)_h$ such that $\widetilde{u}_h \to \widetilde{u}$ q.e. on M;

(v) $\Phi(\cdot)$ is a real extended quadratic functional.

PROOF.

(i) Let $(u_h)_h, (v_h)_h$ be two sequences in $H_0^1(M)$ converging to u and v respectively, such that

$$\Psi(u) + \Phi(u) = \lim_{h \to +\infty} [\Psi(u_h) + \Phi_h(u_h)],$$

and

$$\Psi(v) + \Phi(v) = \lim_{h \to +\infty} [\Psi(v_h) + \Phi_h(v_h)].$$

Since $u \geq 0$ and $v \geq 0$, it is not restrictive to take $v_h, u_h \geq 0$; from the relations below, that can be proven very easily,

$$\Psi(u_h \wedge v_h) + \Psi(u_h \vee v_h) = \Psi(u_h) + \Psi(v_h)$$

$$\Phi_h(u_h \wedge v_h) \leq \Phi_h(v_h),$$

we have that $u_h \wedge v_h$ tends to u, while $u_h \vee v_h$ tends to v; by Γ–convergence we have

$$\Psi(u) + \Phi(u) \leq \liminf_h \left[Psi(u_h \wedge v_h) + \Phi(u_h \wedge v_h)\right],$$

and by the lower semicontinuity of Ψ

$$\Psi(v) \leq \liminf_h \left[\Psi(u_h \vee v_h)\right];$$

this two relations together give

$$\begin{aligned}\Psi(u) + \Psi(v) + \Phi(u) &\leq \liminf_h \left[\Psi(u_h \wedge v_h) + \Phi_h(u_h \wedge v_h)\right] + \\ &+ \liminf_h \Psi(u_h \vee v_h) \leq \\ &\leq \liminf_h [\Psi(u_h) + \Phi_h(v_h) + \Psi(v_h)] = \\ &= \Psi(u) + \Phi(v) + \Psi(v).\end{aligned}$$

(ii) To this aim, let u be a function in $H_0^1(M)$ and let $(u_h)_h$ be a sequence in $H_0^1(M)$ converging to u in $L^2(M)$, such that

$$\Psi(u) + \Phi(u) = \lim_h [\Psi(u_h) + \Phi_h(u_h)];$$

since $|u_h| \to |u|$ in $L^2(M)$ and $\Phi_h(|u|) = \Phi_h(u)$,

$$\begin{aligned}\Psi(|u|) + \Phi(|u|) &\leq \liminf_h [\Psi(|u_h|) + \Phi_h(|u_h|) \leq \\ &\leq \liminf_h [\Psi(u_h) + \Phi_h(u_h)] = \quad . \\ &= \Psi(u) + \Phi(u) < +\infty\end{aligned}$$

This yields that $\Phi(|u|) \leq \Phi(u)$, since $\Psi(|u|) = \Psi(u)$ for every $u \in H_0^1(M)$.

(iii) By definition of Γ–convergence, there exist two sequence $(u_h)_h$ and $(v_h)_h$ of non–negative functions converging in $L^2(M)$ respectively to u and v, such that

$$\Psi(u) + \Phi(u) = \lim_h [\Psi(u_h) + \Phi_h(u_h)]$$

$$\Psi(v) + \Phi(v) = \lim_h [\Psi(v_h) + \Phi_h(v_h)].$$

Since $u_h \vee v_h$ converges in $L^2(M)$ to $u \vee v = u + v$ as $h \to +\infty$, we have

$$\Psi(u+v) + \Phi(u+v) \leq \liminf_h [\Psi(u_h \vee v_h) + \Phi_h(u_h \vee v_h)] \leq$$
$$\leq \liminf_h [\Psi(u_h) + \Psi(v_h) + \Phi_h(u_h) + \Phi_h(v_h)] =$$
$$= \Psi(u) + \Psi(v) + \Phi(u) + \Phi(v).$$

Since $\Psi(u+v) = \Psi(u) + \Psi(v)$ for $u \wedge v = 0$, we have proved that $\Phi(u+v) \leq \Phi(u) + \Phi(v)$.

(iv) By lemma 1.6 [Dal Maso] there exists a sequence $(v_h)_h$ in $H_0^1(M)$ such that $0 \leq v_h \leq u_h$ a.e. and v_h converges to u strongly in $H_0^1(M)$. Since the functional Φ, by definition, is the difference between the limit functional F, which is lower semicontinuous, and the functional Ψ, which is continuous w.r.t. the weak topology of $H_0^1(M)$, we get that Φ is lower semicontinuous on the weak topology of $H_0^1(M)$, hence we have, using also item *(i)*,

$$\Phi(u) \leq \liminf_h \Phi(v_h) \leq \liminf_h \Phi(u_h).$$

On the other hand, for every h we have by *(i)* $\Phi(u_h) \leq \Phi(u)$, hence $\limsup_h \Phi(u_h) \leq \Phi(u)$, and the conclusion follows.

(v) Proposition V in [Sbordone] says that for every $u, v \in H_0^1(M)$, $t \in \mathbb{R}$, the limit functional F satisfies the conditions of Remark 2.4, that is the limit functional is quadratic, hence we get that the functional Φ itself satisfies the same conditions above, since the functional Ψ is quadratic. The proof of Theorem 2.4 is now complete.

In the remainder of this Chapter, we give an integral representation of the limit functional Φ, occurring in the Theorem 2.2, by means of a measure μ belonging to the class $\mathcal{M}_0(M)$. The methods we use do not depend mostly on the "concrete" spaces which enter in Theorem 2.2, rather on the structure of such spaces [see Butt-DM-M]. So we suppose that a *real valued* quadratic functional G is given on a Riesz space $\mathcal{L}$ and we assume that G satisfies the properties $(i), \ldots, (v)$ of the Definition below. Beside this functional, we consider its associated bilinear form β; by means of a Daniell's type extension result adapted to our situation, the bilinear form β is extended to $\widehat{\mathcal{L}} \times \widehat{\mathcal{L}}$, where $\widehat{\mathcal{L}}$ is the monotone class generated by $\mathcal{L}$, and the measure μ is characterized by this extension. At this point, we turn to our "concrete" functional Φ and we give the required representation.

DEFINITION 2.6

Let $\mathcal{L}$ be a (real) vector space of (real) functions defined on an arbitrary set Ω We say that $\mathcal{L}$ is a *Riesz space* whenever $|f| \in \mathcal{L}$,for all $f \in \mathcal{L}$.

REMARK 2.7

Since $f^+ = |f| - f$, we have that $\mathcal{L}$ is a Riesz space if and only if it contains f^+ (or f^-) for any $f \in \mathcal{L}$. This implies that a Riesz space is closed under the $\vee$ and $\wedge$ operations.

DEFINITION 2.8

Let $G : \mathcal{L} \longrightarrow [0, +\infty[$ be a *quadratic* functional, according to Remark 2.4, which satisfies the following properties:

(i) If $0 \leq u \leq v$, $u, v \in \mathcal{L}$, then $G(u) \leq G(v)$;

(ii) $G(|u|) \leq G(u)$;

(iii) $G(u + v) \leq G(u) + G(v)$, if $u, v \in \mathcal{L}$ such that $u \wedge v = 0$;

(iv) $G(u) = \lim_h G(u_h)$, for every increasing sequence $(u_h)_h$ converging to u.

(v) $\Phi(\cdot)$ is a real quadratic functional with *finite values.*

DEFINITION 2.9

A monotone class $\mathcal{S}$ on a set Ω is a class of real valued functions defined on Ω such that:

(i) if $(u_h)_h$ is an increasing sequence in $\mathcal{S}$ having a majorant in $\mathcal{S}$, then $u = \sup_h u_h \in \mathcal{S}$.

(ii) if $(u_h)_h$ is a decreasing sequence having a minorant in $\mathcal{S}$, then $u = \inf_h u_h \in \mathcal{S}$.

Let $\mathcal{L}$ be a Riesz space; the monotone class generated by $\mathcal{L}$ (i.e. the smallest monotone class generated by $\mathcal{L}$) is still a Riesz space which will be denoted by $\widehat{\mathcal{L}}$.

Let us define for $f, g \in \mathcal{L}$

$$\beta(f, g) = \frac{1}{2}[G(f + g) - G(f) - G(g)].$$

It is possible to prove that β is a bilinear form: the functional G is homogeneus of degree two, that satisfies a "parallelogram identity", so the proof that β is bilinear is similar to the proof that any norm that satisfies the parallelogram identity comes from a scalar product [see Yosida, Chapter I,pag. 39]

Since $\beta(f,f) = G(f)$, β is the bilinear form associated to the quadratic functional G. Observe that β is *symmetric*.

DEFINITION 2.10

Let $\mathcal{L}$ be a Riesz space. We say that a bilinear form β defined on $\mathcal{L} \times \mathcal{L}$ is :

-*positive* if

$$\beta(u,v) \geq 0, \qquad \text{for every} \qquad u,v \geq 0,\ u,v \in \mathcal{L};$$

-*local* if, given $u,v \in \mathcal{L}$ with $|u| \wedge |v| = 0$, we have $\beta(u,v) = 0$;

-*continuous on monotone sequences* if, given $u,v \in \mathcal{L}$, we have

$$\beta(u,v) = \lim_{h \to +\infty} \beta(u,v_h),$$

where $(v_h)_h$ is an increasing sequence in $\mathcal{L}$ with $v = \sup_{h \in \mathbf{N}} v_h$.

The following three propositions show that the functional G enjoys of the three properties listed in Definition 2.9.

PROPOSITION 2.11

Let us suppose that $u,v \geq 0$; then $\beta(u,v) \geq 0$.

Proof. Since $\beta(\cdot,\cdot)$ is the bilinear form associated to G, we have

$$\beta(u,v) = \lim_{t \to 0+} \frac{G(u+tv) - G(u)}{t},$$

since $u + tv \geq u$; then the proof follows from the property *(i)* of Definition 2.9.

PROPOSITION 2.12

If $f,g \in \mathcal{L}$, $|f| \wedge |g| = 0$, then $\beta(f,g) = 0$.

Proof. It is not restrictive to assume that both $f,g \geq 0$; from the positiveness of β, we have the inequality $\beta(f,g) \geq 0$. To prove the opposite inequality, by definition of β, we have to prove that

$$G(u+v) \leq G(u) + G(v);$$

this follows from the property *(iii)* of Definition 2.9.

PROPOSITION 2.13

Let $f, g \in \mathcal{L}$, $f \geq 0$ and let $(g_h)_h$ be an increasing sequence in $\mathcal{L}$ such that $g = \sup_{h \in \mathbb{N}} g_h$. Then

$$\beta(f,g) = \lim_{h \to +\infty} \beta(f, g_h).$$

Proof. It is not restrictive to assume $g \geq 0$ and $g_h \geq 0$. By the Schwarz's inequality

$$| \beta(f, g_h - g) |^2 \leq \beta(f,f)\beta(g_h - g, g_h - g) = G(f)G(g_h - g).$$

From the property *(v)* of Definition 2.9, we have

$$G(g_h - g) = 2G(g) + 2G(g_h) - G(g_h + g);$$

taking into account Theorem 1-(iv), if we pass to the limit

$$\lim_{h \to +\infty} G(g_h - g) = 4G(g) - G(2g) = 0$$

and the conclusion follows.

DEFINITION 2.14 (Condition (D_1))

Suppose we are given a functional

$$I : \mathcal{L} \longrightarrow \mathbb{R}$$

satisfying the following properties:

(1) I is linear, i.e. $I(af + bg) = aI(f) + bI(g)$, $\forall a, b \in \mathbb{R}$ and $\forall f, g \in \mathcal{L}$;

(2) I is increasing, i.e. $I(f) \geq 0$, $\forall\ 0 \leq f \in \mathcal{L}$;

(3) I is continuous on monotone sequences, i.e.

if $0 \leq f_n \in \mathcal{L}, f_n(x) \searrow 0^+$ as $n \to +\infty$,then $\lim_n I(f_n) = 0$.

We refer to the set of conditions in the Definition 2.14 above as (D_1) conditions.

REMARK 2.15

A functional satisfying the (D_1) conditions is called the *Daniell integral*; the set of conditions (D_1) are known as the Daniell's conditions.

We say that the bilinear form β associated to G satisfies the Daniell's conditions (D_2) below if for every fixed $u, v \in \mathcal{L}$, our bilinear form $\beta(u, \cdot)$ and $\beta(\cdot, v)$ satisfies conditions (D_1), that is

Conditions (D_2)

(1) $\beta(\cdot, v)$ and $\beta(u, \cdot)$ are linear, i.e.

$$\beta(\cdot, av_1 + bv_2) = a\beta(\cdot, v_1) + b\beta(\cdot, v_2)$$

and

$$\beta(au_1 + bu_2, \cdot) = a\beta(u_1, \cdot) + b\beta(u_2, \cdot),$$

$\forall a, b \in I\!R$ and $\forall u, v \in \mathcal{L}$;

(2) $\beta(\cdot, v)$ and $\beta(u, \cdot)$ are increasing, i.e.

$$\beta(\cdot, v) \geq 0, \ \forall \ 0 \leq v \in \mathcal{L}$$

and

$$\beta(u, \cdot) \geq 0, \ \forall \ 0 \leq u \in \mathcal{L};$$

(3) $\beta(\cdot, v)$ and $\beta(u, \cdot)$ are continuous on monotone sequences, i.e.

if $0 \leq u_n, v_n \in \mathcal{L}, u_n(x) \searrow 0$ and $v_n \searrow 0$ as $n \to +\infty$, then $\lim_n \beta(\cdot, v_n) = 0$ and $\lim_n \beta(u_n, \cdot) = 0$.

The following result is a classical one [see e.g. C–W–S, Chapter III].

<u>THEOREM 2.16</u>

Let $\mathcal{L}$ be a Riesz space and let $\widehat{\mathcal{L}}$ be the monotone class generated by $\mathcal{L}$. Let I_0 be a linear form satisfying the Daniell's condition (D_1) above; then there exists an unique positive linear form

$$I : \widehat{\mathcal{L}} \longrightarrow I\!R$$

still satisfying the conditions (D_1), such that $I = I_0$ on $\mathcal{L}$.

It is possible to prove a similar extension result for bilinear form

<u>PROPOSITION 2.17</u>

Suppose that β is a bilinear form satisfying the conditions (D_2) above; then there exists an unique bilinear form

$$\widehat{\beta} : \widehat{\mathcal{L}} \times \widehat{\mathcal{L}} \longrightarrow I\!R$$

which still satisfies (D_2) and extends β.

PROOF. For every $v \in \mathcal{L}^+$, $\beta(\cdot, v)$ may be extended to a form $\widetilde{\beta}(\cdot, v)$, defined on $\widehat{\mathcal{L}}$, which still satisfies (D_1). Set for every $v \in \mathcal{L}$, $v = v^+ - v^-$,

$$\widetilde{\beta}(\cdot, v) = \widetilde{\beta}(\cdot, v^+) - \widetilde{\beta}(\cdot, v^-).$$

Hence we have

$$\widetilde{\beta} : \widehat{\mathcal{L}} \times \mathcal{L} \longrightarrow I\!R.$$

For every $u \in \widehat{\mathcal{L}}$, $u \geq 0$, $\widetilde{\beta}(u, \cdot)$ still continues to satisfy (1) and (2) of (D). In order to prove condition (3), we first remark that, as a consequence of the definition of the monotone class, each element of $\widehat{\mathcal{L}}$ is between two elements of $\mathcal{L}$; with this remark, it is easy to realize that the continuity of $\widetilde{\beta}(u, \cdot)$ holds true.

For every $u \in \widehat{\mathcal{L}}^+$ the form $\widetilde{\beta}(u, \cdot)$ satisfies (D_1); again by Theorem 2.16 there exists an unique form

$$\widehat{\beta} : \widehat{\mathcal{L}}^+ \times \widehat{\mathcal{L}} \longrightarrow I\!R$$

that extends $\widetilde{\beta}$ and such that $\widehat{\beta}(\cdot, v)$ satisfies *(D)* (see above). For every $u \in \widehat{\mathcal{L}}$ we define

$$\widehat{\beta}(u, \cdot) = \widehat{\beta}(u^+, \cdot) - \widehat{\beta}(u^-, \cdot).$$

As before, it is possible to prove that $\widehat{\beta}$ still satifies the Daniell's conditions (D_1). Let us prove the uniqueness of the bilinear form $\widehat{\beta}$. Suppose that there exist two extensions of bilinear form β, let us call them β_1 and β_2, both satisfying the Daniell's conditions (D_2); let us consider for a fixed $u \in \mathcal{L}$,

$$A = \left\{ v \in \widehat{\mathcal{L}} : \beta_1(u, v) = \beta_2(u, v) \right\}.$$

Since $\beta_1|_{\mathcal{L}} = \beta_2|_{\mathcal{L}} = \beta$, A contains $\mathcal{L}$; moreover A is a monotone class, since β_1 and β_2 both satisfy conditions (D). So A is a monotone class containing $\mathcal{L}$, hence $A = \widehat{\mathcal{L}}$, by definition of monotone class. This shows that $\beta_1(u, v) = \beta_2(u, v)$ for every $u \in \mathcal{L}$ and for every $v \in \widehat{\mathcal{L}}$.

Now fix $v \in \mathcal{L}$ and consider

$$B = \left\{ u \in \widehat{\mathcal{L}} : \beta_1(u, v) = \beta_2(u, v) \right\};$$

by an analogous argument, it may be proved that B coincides with $\widehat{\mathcal{L}}$ and $\beta_1(u,v) = \beta_2(u,v)$ for every $u \in \widehat{\mathcal{L}}$ and for every $v \in \mathcal{L}$. Since every element of $\widehat{\mathcal{L}}$ is between two elements of $\mathcal{L}$, we can conclude the proof.

REMARK 2.18

We want to stress that the above extension $\widehat{\beta}$ of β, is *local* and *symmetric*, if such is β.

THEOREM 2.19

Let $\widehat{\mathcal{L}}$ be the monotone class generated by $\mathcal{L}$ and let $\beta : \mathcal{L} \times \mathcal{L} \longrightarrow I\!R$ which is *local*, *positive*, *symmetric* and *continuous on monotone sequences*; let us suppose that $\widehat{\mathcal{L}}$ satisfies the *Stone condition*:

$$(S) \qquad f \wedge 1 \in \widehat{\mathcal{L}}, \text{ if } f \in \widehat{\mathcal{L}}.$$

Denote by $\mathcal{E}$ the class

$$\mathcal{E} = \{E \subset \Omega : 1_E \in \widehat{\mathcal{L}}\}$$

and by μ the set function

$$\mu : \mathcal{E} \longrightarrow I\!R^+$$

defined by

$$\mu(E) = B(1_E, 1_E).$$

Then $\mathcal{E}$ is a δ–ring, μ is a measure on $\mathcal{E}$, $\widehat{\mathcal{L}}$ is a subset of $L^2(\Omega, \mathcal{E}, \mu)$ and

$$(2.2) \qquad \beta(f,g) = \int_\Omega fg\,\mathrm{d}\mu,$$

for every $f, g \in \widehat{\mathcal{L}}$.

PROOF. The proof of the theorem is achieved through six steps.

Step 1.

$u \in \widehat{\mathcal{L}},\, t > 0 \Rightarrow \{u > t\} \subset \mathcal{E}$. In fact for every $t > 0$ we have

$$(u-t)^+ = u - u \wedge t \in \widehat{\mathcal{L}}$$

by the Stone condition. Then

$$1_{\{u>t\}} = \sup_h (h(u-t)^+ \wedge 1) \in \widehat{\mathcal{L}}$$

as supremum of an increasing sequence of $\widehat{\mathcal{L}}$ majorized by $\frac{u}{t}$.

Step 2.

$E \in \mathcal{E}$, $u \in \widehat{\mathcal{L}} \Rightarrow u1_E \in \widehat{\mathcal{L}}$. It is not restrictive to assume $u \geq 0$; in this case,

$$u1_E = \lim_h [u \wedge h1_E];$$

this is a monotone sequence whose limit is in the monotone class generated by $\mathcal{L}$.

Step 3.

$E, F \in \mathcal{E}$, $E \subset F$ implies

$$\beta(1_E, 1_F) = \beta(1_E, 1_E) = \mu(E) \tag{2.3.}$$

This is a consequence of the local property of B, see Proposition 2.12. Moreover this property implies also

$$\beta(1_E, 1_{F \setminus E}) = 0, \ \forall E, F \in \mathcal{E}.$$

Step 4.

μ is a measure on the δ–ring $\mathcal{E}$. $\mathcal{E}$ is a δ–ring because $\widehat{\mathcal{L}}$ is a monotone Riesz space. The fact that μ is a measure, follows from conditions (D_2) and from the local property of β: in fact, taking into account the local property of B, we have that

- the *finite additivity* of μ comes from the linearity of B;
- the *countable additivity* follows from the continuity of B along monotone sequences.

Step 5.

Let $u \in \widehat{\mathcal{L}}$, $E \in \mathcal{E}$; then

$$B(u, 1_E) = \int_\Omega u1_E d\mu = \int_E u d\mu.$$

Assume, for a moment, that u is a positive step function

$$u = \sum_{i=1}^n a_i 1_{A_i};$$

then, by the local property of B, we can also suppose that $A_i \subset E$. By the linearity,

$$B(u, 1_E) = \sum_{i=1}^n a_i B(1_{A_i}, 1_E) = \sum_{i=1}^n a_i \mu(A_i) = \sum_{i=1}^n a_i \mu(A_i \cap E).$$

If $u \in \widehat{\mathcal{L}}$, $u \geq 0$, there exists a sequence of step function $0 \leq \varphi_n \nearrow u$; so the conclusion follows from the continuity of B along monotone sequences.

Step 6.

If $u, v \in \widehat{\mathcal{L}}$, then

$$B(u,v) = \int_\Omega uv d\mu.$$

We may assume that both $u, v \geq 0$. The equality above is true for every v which is a step function. The proof is then achieved by a monotone argument, as above.

Now we take up again to our concrete functional Φ defined by (2.1). In order to apply the abstract part of this Chapter, we introduce H_Φ as the class of all quasi–continuous Borel functions $f : M \to I\!R$ for which there exists a function $u \in H_0^1(M)$ such that $\widetilde{u} = f$ q.e. on M and $\Phi(u) < +\infty$; with $\widetilde{u}$ we mean the quasi–continuous representative of $u \in H_0^1(M)$. Since for every $u \in H_0^1(M)$ the quasi–continuous representative is unique, up to a set of capacity zero, we can define Φ on H_Φ by setting $\Phi(f) = \Phi(u)$. With this definition, Φ is a *finite real valued* quadratic functional on H_Φ.

REMARK 2.20

We want to stress that $\Phi(u) = \Phi(v)$ if $u = v$ up to a set of capacity zero.

REMARK 2.21

From Theorem 2.5-(v), we have that H_Φ is a vector space of real functions defined on M.

REMARK 2.22

From Theorem 2.5-(ii), we find that H_Φ is a Riesz space.

We define the bilinear form

$$B(u,v) = \frac{1}{2}[\Phi(u+v) - \Phi(u) - \Phi(v)] \tag{2.3}$$

which results to be *local*, *positive* and *continuous on monotone sequences*, as it has been proved in the Propositions 2.11, 2.12, 2.13 above. Now we apply Proposition 2.17, which assures the existence of the extension, $\widehat{B}$, enjoying of the same properties of B on the monotone class, $\widehat{H_\Phi}$, generated by H_Φ. By means of $\widehat{B}$, using Theorem 2.19, the measure μ, which gives the required representation of the limit functional Φ, is given by the formula (2.4) below. We sum up all these in the following

PROPOSITION 2.23

Let $\Phi : H_0^1(M) \longrightarrow [0,+\infty]$ be a functional satisfying conditions $(i),\ldots,(v)$ of Theorem 2.5. Then there exists a measure $\mu \in \mathcal{M}_0$ such that

$$\Phi(u) = \int_M \widetilde{u}^2 \mathrm{d}\mu \tag{2.4}$$

for every $u \in H_0^1(M)$.

PROOF. Let $\mathcal{E}_\sigma$ be the σ–ring generated by $\mathcal{E}_\Phi$. The measure μ of theorem 3 can be extended to a measure defined on the Borel σ–field on M, which we still denote by μ, such that $\mu(A) = +\infty$ whenever A is not in $\mathcal{E}_\sigma$; from Remark 2.24 below, it follows that μ is in $\mathcal{M}_0$.

From (1) we have that

$$\Phi(f) = \int_M f^2 \mathrm{d}\mu \tag{2.5}$$

for every $f \in H_\Phi$. To complete the proof we have to extend Φ to all functions in $H_0^1(M)$. Let u be a function in $H_0^1(M)$, being supposed fixed. If $\Phi(u) < +\infty$, then there exists $f \in H_\Phi$ and so (2.4) follows from (2.5). Now suppose that $\Phi(u) = +\infty$ and, in order to get a contradiction, let us suppose that

$$\int_M \widetilde{u}^2 \mathrm{d}\mu < +\infty.$$

For every $\varepsilon > 0$

$$\mu(\{|\widetilde{u}| > \varepsilon\}) \leq \varepsilon^{-2} \int_M \widetilde{u}^2 \mathrm{d}\mu < +\infty,$$

so $A = \{|\widehat{u}| > \varepsilon\} \in \mathcal{E}_\sigma$; hence let $(g_h)_h$ be an increasing sequence in H_Φ, such that $g_h \to +\infty$ on A and let

$$f_h = h \wedge (|\widetilde{u}| - \varepsilon)^+;$$

then $f_h \in H_\Phi$ and $\lim_h f_h = (|\widehat{u}| - \varepsilon)^+$, so that

$$\Phi(|\widetilde{u}| - \varepsilon = \lim_h \Phi(f_h) = \lim_h \int_M f_h^2 \mathrm{d}\mu = \int_M |\ (|\widetilde{u}| - \varepsilon)^+\ |^2\ \mathrm{d}\mu.$$

Since $(|\widetilde{u}| - \varepsilon)^+$, as ε goes to zero, converges to $|\widehat{u}|$, by Theorem 1-(iv) we have

$$\Phi(|\widetilde{u}|) = \int_M |\widetilde{u}|^2 \mathrm{d}\mu < +\infty.$$

So by theorem 1-(i) and (v)

$$\Phi(u) \leq 2[\Phi(u^+) + \Phi(u^-)] \leq 4\Phi(|u|) < +\infty$$

and we get the contradiction.

REMARK 2.24

The measure μ is absolutely continuous with respect to capacity, that is $\mu(E) = 0$ if E is a Borel set of capacity zero. In fact, by definition, $E \in \mathcal{E}$ if and only if $1_E \in \widehat{H_\Phi}$; if E has capacity zero, then 1_E coincides q. e. with the function identically zero. Hence

$$\mu(E) = B(1_E, 1_E) = 0.$$

REMARK 2.25

We wish to underline that a δ–ring is not assumed to be closed under the complement operation, hence in general A^c does not belong to $\mathcal{E}$ if $A \in \mathcal{E}$, as the following example shows.

EXAMPLE 2.26

Let us consider Ω a bounded open set of $\mathbb{R}^N$ and let C be a closed subset of Ω; let

$$\mu = \infty_C$$

[cfr. DM–M]. In this situation H_Φ consists of $H_0^1(\Omega)$ functions that are zero on C, up to a set of capacity zero and this property still continues to hold for functions in $\widehat{H}_\Phi$. By definition, $E \in \mathcal{E}$ if and only if $1_E \in \widehat{H}_\Phi$; so the property to be zero on C is true if $E \subset \Omega \setminus C$, then there is no hope to have $E^c \in \mathcal{E}$.

PROOF OF THE THEOREM 2.5

Now the proof is straightforward; it follows from theorem 2.5 and proposition 2.23.

CHAPTER 3

With the notations of Chapter 2, we say that the sequence $(\mu_h)_h$ of measures in $\mathcal{M}_0$ γ–converge to $\mu \in \mathcal{M}_0$ if the sequence of the corresponding functionals $(F_{\mu_h})_h$ Γ– converges in $L^2(M)$ to the functional F_μ.

DEFINITION 3.1

For every open submanifold D in M and for every $\nu \in \mathcal{M}_0$, we shall denote by F_ν^D the following functional

$$F_\nu^D(u) = \begin{cases} \int_D |\nabla u|_g^2 dV_g + \int_D u^2 d\nu, & \text{if } u \in H_0^1(D); \\ +\infty, & \text{otherwise in } L^2(D). \end{cases}$$

We have the following result

PROPOSITION 3.2

Let $(\mu_h)_h$be a sequence in $\mathcal{M}_0$ and let $\mu \in \mathcal{M}_0$. The following are equivalent:

(a) $(\mu_h)_h$ γ–converges to μ.

(b) $(F_{\mu_h}^D)_h$ Γ–converges to F_μ^D in $L^2(D)$, for every open submanifold D in M.

REMARK 3.3

For every open submanifold D in M, let us define

$$F_+^D(u) = \inf\left\{\limsup_h F_{\mu_h}^D(u_h) : u_h \to u \text{ in } L^2(D)\right\} \tag{3.1}$$

$$F_-^D(u) = \inf\left\{\liminf_h F_{\mu_h}^D(u_h) : u_h \to u \text{ in } L^2(D)\right\}; \tag{3.2}$$

If $D = M$ we denote the corresponding functionals by F_+ and F_-. By definition we have $F_+^D(u) \geq F_-^D(u)$ for every $u \in L^2(D)$. By a diagonal argument, it is easy to see that the infima in (3.1) and (3.2) are achieved by suitable sequences; moreover F_+^D and F_-^D are lower semicontinuous on $L^2(D)$ (see [De Giorgi-Franzoni]).

It is easy to realize that F_μ^D is the Γ–limit in $L^2(D)$ of $F_{\mu_h}^D$ if and only if $F_\mu^D = F_+^D = F_-^D$ on $L^2(D)$; since these functionals are equal to $+\infty$ outside $H_0^1(M)$ [see Chapter 2], the Γ– convergence of $F_{\mu_h}^D$ to F_μ^D is equivalent to the inequalities

$$F_+^D \leq F_\mu^D \leq F_-^D \qquad \text{on} \quad H_0^1(D).$$

REMARK 3.4

Let D be an open submanifold of M. If $u \in H_0^1(D)$, we can extend it to the whole manifold M by putting $u = 0$ outside D; so we get $u \in H_0^1(M)$ and this extension is still denoted by u.

Proof of the proposition.

$(b) \Rightarrow (a)$ is trivial, since we can always take $D = M$.

$(a) \Rightarrow (b)$ Let us assume (a), which is equivalent to suppose $F_+ = F_\mu = F_-$ on $L^2(M)$. Let us prove that

$$F_\mu^D \leq F_-^D \qquad \text{on} \quad H_0^1(D).$$

Let $u \in H_0^1(D)$ with $F_-^D(u) < +\infty$. By (3.2) there exists a sequence $(u_h)_h$ converging to u in $L^2(D)$ such that

$$\liminf_h F_{\mu_h}^D(u_h) = F_-^D(u);$$

since $F_-^D(u)$ is finite, we may assume that, up to a subsequence, $u_h \in H_0^1(D)$; hence $u_h \in H_0^1(M)$ and $F_{\mu_h}^D(u_h) = F_{\mu_h}(u_h)$, so that

$$F_-^D(u) = \liminf_h F_{\mu_h}^D(u_h) = \liminf_h F_{\mu_h}(u_h) \geq F_-(u) = F_\mu(u) = F_\mu^D(u).$$

It remains to prove that

$$F_+^D \leq F_\mu^D.$$

Let $u \in H_0^1(D)$ and such that $F_\mu^D(u) < +\infty$; it is not restrictive to assume that $u \in C_c^\infty(M)$. In fact, since the functional F_μ^D is continuous w.r.t. the strong topology of $H_0^1(D)$, while F_+^D is lower semi-continuous w.r.t. the strong topology of $L^2(D)$, we obtain the desired result by the density of $C_c^\infty(D)$ in $H_0^1(D)$.

Now extend the function u by putting $u = 0$ outside D, so that $u \in H_0^1(M)$ and $F_+(u) = F_\mu(u) = F_-(u)$. By (3.1) there exists a sequence $(u_h)_h$ converging to u in $L^2(M)$ such that

$$F_+(u) = F_\mu^D(u) = \limsup_h F_{\mu_h}(u_h) < +\infty.$$

This yields that $u_h \in H_0^1(M)$ for h large enough and

$$\limsup_h \int_M |\nabla u_h|^2 \mathrm{d}V_g \leq \limsup_h F_{\mu_h}(u_h) < +\infty$$

so that $(u_h)_h$ converges weakly to u in $H_0^1(M)$. Let ζ be a smooth function, compactely supported in D such that $\zeta = 1$ on $supp(u)$; then ζu_h belongs to $H_0^1(D)$ and ζu_h converges strongly in $L^2(D)$ to ζu; hence

$$F_+^D(u) \le \limsup_h F_{\mu_h}^D(\zeta u_h) =$$

$$= \limsup_h \{\int_D \left[|\nabla u_h|_g^2 \zeta^2 + 2\zeta u_h \langle \nabla u_h, \nabla \zeta \rangle_g + u_h^2 |\nabla \zeta|_g^2\right] dV_g + \int_D \zeta^2 u_h^2 d\mu_h\} \le$$

$$\le \limsup_h F_{\mu_h}(u_h) + 2\int_D \left[\zeta u \langle \nabla u, \nabla \zeta \rangle_g + u^2 |\nabla \zeta|_g^2\right] dV_g = F_\mu(u),$$

and so the proof of the proposition is complete.

THE μ–CAPACITY

With the notations of Chapter 2, let E be a Borel set in M and let $\mu \in \mathcal{M}_0$. The μ–capacity of E in M is defined by

$$\text{(3.3)} \qquad \text{cap}_\mu(E) = \inf \left\{ \Psi(u) + \int_E (u-1)^2 d\mu, \; u \in H_0^1(M) \right\}.$$

The infimum in (3.3) is attained by the lower semi-continuity of the functional in the weak topology of $H_0^1(M)$ and if $\text{cap}_\mu(E)$ is finite, then the minimum point is unique by the strict convexity.

REMARK 3.5

It is possible to extend the notion of μ– capacity to every subset of M; in order to do this , first of all we have to extend the measure from $\mathcal{B}(M)$ to $\mathcal{P}(M)$ by

$$\mu(E) = \inf \{\mu(B) : B \in \mathcal{B}(M), \; E \subset B\};$$

then define

$$\text{cap}_\mu(E) = \inf \left\{ \Psi(u) + \int_M (u-1)^2 d\mu^E, \; u \in H_0^1(M) \right\}$$

where μ^E is the restriction of the measure μ to E, that is

$$\mu^E(A) = \mu(A \cap E), \; \forall A \in \mathcal{B}(M).$$

REMARK 3.6

Note that

$$\int_M f \mathrm{d}\mu^E = \inf \left\{ \int_B f \mathrm{d}\mu, \ B \in \mathcal{B}(M), \ E \subset B \right\}$$

f is any Borel, positive valued function.

PROPERTIES OF THE μ–CAPACITY

In the remainder of this Chapter we face the problem of the continuity of the restriction operator μ^E [see Remark 3.5] w.r.t. the γ–convergence. This kind of result was firstly tackled in the "flat" case in [Dal Maso: γ–convergence and μ–capacity]. Since the methods used in [Dal Maso] apply to our case, except for Propositions 3.12 and 3.13 below, we will address to the above quoted paper for the proofs, when it is required. Our scheme is now the following: first of all we give a result about the asymptotic behaviour of the cap_{μ_h} when the sequence $(\mu_h)_h$ γ–converges to a measure $\mu \in \mathcal{M}_0(M)$. We use such a result in order to give a characterization of the γ–convergence; this characterization is then used to get an equivalence between the γ–convergence of the sequence $(\mu_h)_h$ and the γ– convergence of the restriction operators $(\mu_h^E)_h$, $E \in \mathcal{B}(M)$.

PROPOSITION 3.7

For every $\mu \in \mathcal{M}_0(M)$, the set function

$$\mathrm{cap}_\mu : \mathcal{P}(M) \longrightarrow [0, +\infty]$$

satisfies the following properties:

(a) $\mathrm{cap}_\mu(\emptyset) = 0$

(b) if $E_1 \subset E_2$, then

$$\mathrm{cap}_\mu(E_1) \leq \mathrm{cap}_\mu(E_2)$$

(c) if $(E_h)_h$ is an increasing sequence in M and

$$E = \bigcup_h E_h,$$

then

$$\mathrm{cap}_\mu(E) = \sup_h \mathrm{cap}_\mu(E_h)$$

(d) if $(E_h)_h$ is a sequence in M and $E = \bigcup_h E_h$, then

$$\mathrm{cap}_\mu(E) \leq \sum_h \mathrm{cap}_\mu(E_h)$$

(e)

$$\mathrm{cap}_\mu(E_1 \cup E_2) + \mathrm{cap}_\mu(E_1 \cap E_2) \leq \mathrm{cap}_\mu(E_1) + \mathrm{cap}_\mu(E_2)$$

(f) $\mathrm{cap}_\mu(E) \leq \mathrm{cap}(E)$, for every $E \subset M$

(g) $\mathrm{cap}_\mu(E) \leq \mu(E)$, for every $E \subset M$

(h)

$$\mathrm{cap}_\mu(E) = \inf\left\{\mathrm{cap}_\mu(B) : B \in \mathcal{B}(M),\ E \subset B\right\}$$

(i)

$$\mathrm{cap}_\mu(A) = \sup\left\{\mathrm{cap}_\mu(K) :\ K \text{ compact},\ K \subset A\right\}$$

for every quasi open set A in M

(j)

$$\mathrm{cap}_\mu(A) = \inf\left\{\mathrm{cap}_\mu(U) :\ U \text{ open},\ A \subset U\right\}$$

for every quasi open set A in M

For the proof of this result, we refer to [Dal Maso, Theorem 2.9].

REMARK 3.8

The proposition above tells us that cap_μ is *increasing* (item *(b)*) and *continuous on increasing sequence* (item *(c)*); cap_μ,however, is not a Choquet capacity, since, in general, the property

$$\mathrm{cap}_\mu(K) = \inf \mathrm{cap}_\mu(K_h)$$

for every decreasing sequence $(K_h)_h$ of compact sets in M, with $K = \bigcap_h K_h$, does not holds true [see DM-M,, J. Appl. Math. Opt.].

We now introduce a class of measures, denoted by $\mathcal{M}_0^*(M)$, whose interest lies in the fact that they lead to μ–capacities that are Choquet capacities.

DEFINITION 3.9

With $\mathcal{M}_0^*(M)$ we denote the class of measures in $\mu \in \mathcal{M}_0(M)$ such that

$$\mu(E) = \inf\{\ \mu(A) : E \subset A,\ A \text{ quasi open }\} \tag{3.3}$$

for every $E \subset M$. By Remark 3.5, it is sufficient to verify (3.3) for E Borel set.

REMARK 3.10

For every $\mu \in \mathcal{M}_0(M)$ define

$$\mu^*(E) = \inf\{\mu(A) : A \quad \text{quasi open},\ E \subset A\}.$$

It has been proved [see DM,theorem 3.5] that μ^* is equivalent to μ, according the equivalent relation given in Chapter 1, and $\text{cap}_\mu(A) = \text{cap}_{\mu^*}(A)$ for every A quasi open subset of M.

PROPOSITION 3.11

Let $\mu \in \mathcal{M}_0^*(M)$. Then cap_μ is a Choquet capacity,

$$\text{cap}_\mu(E) = \sup\left\{\text{cap}_\mu(K) :\ K\text{compact},\ K \subset E\right\}$$

for every E in M and

$$\text{cap}_\mu(E) = \inf\left\{\text{cap}_\mu(U) :\ U\text{open},\ E \subset U\right\}$$

for every E in M

We refer to [Dal Maso] for the proof of this result; we want to underline that a consequence of these results on the cap_μ is the possibility to reconstruct a measure $\mu \in \mathcal{M}_0^*(M)$ from the corresponding μ- -capacity (see theorem 4.5 in the above quoted work)

In order to tackle the problem of the continuity of the restriction operator μ^E, we need the following two results.

PROPOSITION 3.12

Let $(\mu_h)_h$ be a sequence in $\mathcal{M}_0(M)$ which converges to $\mu \in \mathcal{M}_0(M)$. Let A be an open set in M and N an open submanifold of M such that

$$A \subset N \subset M.$$

Then

$$\Psi(u) + \int_A u^2 \mathrm{d}\mu \leq \liminf_h \left[\Psi(u_h) + \int_A u_h^2 \mathrm{d}\mu_h\right], \tag{3.4}$$

for every $u \in H^1(N)$ and for every $u_h \in H^1(N)$ converging to u weakly in $L^2(N)$.

PROOF

First of all we remark that it is not restrictive to assume that the liminf in the right hand side of (3.4) is a finite limit; hence u_h is a bounded sequence in $H^1(N)$ converging to u weakly in $H^1(N)$. Now let us prove first the case of A contained in a single chart (U, ω) with coordinate system $(x_1, \ldots, x_N)$. Let K be a compact set, $K \subset A$, and consider $\tau \in C_c^\infty(A)$, $0 \leq \tau \leq 1$, $\tau = 1$ on K. Since $v_h \in H_0^1(N)$, $\tau v_h \in H^1(N)$, $supp(\tau v_h)$ is in A and τv_h converges to τv strongly in $L^2(N)$, so by condition (a) of Γ–convergence [Definition 2.1] we have

$$\Psi_M(\tau v) + \int_M \tau u^2 \mathrm{d}\mu \leq \liminf_h \left[\Psi_M(\tau v_h) + \int_M (\tau v_h \mathrm{d}\mu_h\right],$$

that in local coordinates it reads

$$\int_\Omega g_{ij} D_j(\tau v) D_i(\tau v) b \mathrm{d}x + \int_M (\tau v)^2 \mathrm{d}\mu \leq$$

$$\leq \liminf_h \left[\int_\Omega g_{ij} D_j(\tau v_h) D_i(\tau v_h) b \mathrm{d}x + \int_M (\tau v_h)^2 \mathrm{d}\mu\right]$$

where $b(x) = \sqrt{|g(x)|}$ is a function in $L^\infty(\Omega)$, $\Omega = \omega(U)$ is a bounded open set in $\mathbb{R}^N$ and D_i are the distributional derivatives (in $\mathbb{R}^N$). We have, expliciting the calculations,

$$\int_\Omega [g_{ij} D_j \tau D_i \tau]\, u^2 b \mathrm{d}x + \int_\Omega [g_{ij} D_j \tau D_i u]\, u \tau b \mathrm{d}x +$$

$$+ \int_\Omega [g_{ij} D_j u D_i u]\, \tau^2 b \mathrm{d}x + \int_M \tau^2 u^2 \mathrm{d}\mu \leq$$

$$\leq \liminf_h \left[\int_\Omega (g_{ij} D_j u_h D_i u_h) \tau^2 b \mathrm{d}x + \int_\Omega (g_{ij} D_j \tau D_i u_h) u_h \tau b \mathrm{d}x + \right.$$

$$\left. + \int_\Omega (g_{ij} D_j \tau D_i \tau) u_h^2 b \mathrm{d}x + \int_M (u_h \tau)^2 \mathrm{d}\mu\right].$$

Since u_h converges weakly to u in $H^1(M)$ the first and the second term on the left hand side tends to the third and to the second on the right hand side, hence

$$
\begin{aligned}
(3.5) \qquad & \int_\Omega [g_{ij}D_j u D_i u]\,\tau^2 b \mathrm{d}x + \int_M u^2\tau^2 \mathrm{d}\mu \leq \\
& \leq \liminf_h \left[\int_\Omega [g_{ij}D_j u_h D_i u_h]\, b\mathrm{d}x + \int_M \tau^2 u_h^2 \mathrm{d}\mu \right]
\end{aligned}
$$

By the lower semicontinuity of the functional Ψ w.r.t. the weak topology of $H^1(M)$, we have also

$$
\begin{aligned}
(3.6) \qquad & \int_N (g_{ij}D_j u D_i u)(1-\tau^2) b\mathrm{d}x \leq \\
& \leq \liminf_h \int_N (g_{ij}D_j u D_i u)(1-\tau^2) b\mathrm{d}x;
\end{aligned}
$$

adding (3.5) and (3.6), we obtain

$$
\int_N |\nabla u|^2 \mathrm{d}V_g + \int_K u^2 \mathrm{d}\mu \leq
$$

$$
\leq \liminf_h \left[\int_N |\nabla u_h|^2 \mathrm{d}V_g + \int_A u_h^2 \mathrm{d}\mu_h \right]
$$

and taking $K \nearrow A$, we have the desidered result.

If we have not A contained in a single local chart, then we may consider

$$
A \cap U_i
$$

where $\mathcal{U} = (U_i)_i$ is the family of open set given in Lemma A.1 in Appendix. We apply the above argument to $A \cap U_i$ and we get the assertion of the proposition, since

$$
\int_N |\nabla u|^2 \mathrm{d}V_g + \int_A u^2 \mathrm{d}\mu =
$$

$$
= \sum_i \left[\int_{U_i} |\nabla u|^2 \mathrm{d}V_g + \int_{A\cap U_i} u^2 \mathrm{d}\mu \right],
$$

and

$$
\sum_i \left[\liminf_h \int_{U_i} |\nabla u_h|^2 \mathrm{d}V_g + \int_{A\cap U_i} u_h^2 \mathrm{d}\mu_h \right] \leq
$$

$$\leq \liminf_h \left[\sum_i \int_{U_i} |\nabla u_h|^2 \mathrm{d}V_g + \int_{A\cap U_i} u_h^2 \mathrm{d}\mu_h \right] =$$

$$= \liminf_h \left[\int_N |\nabla u|^2 \mathrm{d}V_g + \int_A u_h^2 \mathrm{d}\mu_h \right].$$

PROPOSITION 3.13

Let $(\mu_h)_h$ be a sequence in $\mathcal{M}_0(M)$ which γ–converges to $\mu \in \mathcal{M}_)(M)$. Let N be an open submanifold of M, K a compact set and A an open set such that

$$K \subset A \subset N.$$

Then for every $u \in H^1(N)$ there exists a sequence $(u_h)_h$ in $H^1(N)$ such that u_h converges strongly in $L^2(N)$ to u and

$$\Psi_N(u) + \int_A u^2 \mathrm{d}\mu \geq \limsup_h \left[\Psi_N(u_h) + \int_K u_h^2 \mathrm{d}\mu_h \right]. \tag{3.7}$$

PROOF

We may assume that $u \in L^2(N, \mu)$; by a diagonal argument it is enough to show that for every $\varepsilon > 0$ there exists a sequence $(u_h)_h$ in $H^1(N)$ such that u_h converges to u strongly in $L^2(N)$ and

$$\Psi_N(u) + \int_A u^2 \mathrm{d}\mu + \varepsilon \geq \limsup_h \left[\Psi_N(u_h) + \int_K u_h^2 \mathrm{d}\mu_h \right].$$

Let ε be given and let W be an open set of N such that

$$K \subset W \subset \overline{W} \subset A$$

and $\Psi_{\overline{W}\setminus K} < \varepsilon$. We suppose, at first, that A is contained in a local chart (U, ω). Let $\zeta \in C_c^\infty(A)$, $\zeta = 1$ on $\overline{W}$ and $0 \leq \zeta \leq 1$ on A; define $v = u\zeta$ that belongs to $H_0^1(N) \cap L^2(N, \mu)$. By condition (b) of the definition 2.1, there exists a sequence $(v_h)_h$ in $H_0^1(N)$ converging to u strongly in $L^2(N)$ and such that

$$\Psi_M(v) + \int_M v^2 \mathrm{d}\mu \geq \limsup_h \left[\Psi_M(v_h) + \int_M v_h^2 \mathrm{d}\mu_h \right].$$

Set $E = M \setminus \overline{W}$; then we have

$$\Psi_{\overline{W}}(u) + \int_{\overline{W}} u^2 \mathrm{d}\mu + \Psi_E(v) + \int_E v^2 \mathrm{d}\mu \geq$$

$$\geq \limsup_h \left[\Psi_{\overline{W}}(v_h) + \int_{\overline{W}} v_h^2 \mathrm{d}\mu_h\right] + \liminf_h \left[\Psi_E(v_h) + \int_E v_h^2 \mathrm{d}\mu_h\right].$$

By the Proposition 3.9 we have that

$$\Psi_E(v) + \int_E v^2 \mathrm{d}\mu \leq \liminf_h \left[\Psi_E(v_h) + \int_E v_h^2 \mathrm{d}\mu_h\right]$$

and, by definition, $v \in L^2(E, \mu)$, so

$$\Psi_{\overline{W}}(u) + \int_{\overline{W}} u^2 \mathrm{d}\mu \geq \limsup_h \left[\Psi_{\overline{W}}(v_h) + \int_{\overline{W}} v_h^2 \mathrm{d}\mu_h\right]. \tag{3.8}$$

Let $\xi \in C_c^\infty(W)$, $\xi = 1$ on a neighbourhood of K, $0 \leq \xi \leq 1$. Define

$$u_h = \xi v_h + (1 - \xi) u;$$

then

$$u_h = v_h \qquad \text{in a neighbourhood of} \quad K$$

and converges strongly in $L^2(N)$ to u. We have, for every $\varepsilon \in (0,1)$

$$\begin{gathered} g_{ij} D_j u_h D_i u_h \leq \left[\frac{\xi}{1-\varepsilon}\right] [g_{ij} D_j v_h D_i v_h] + \\ \left[\frac{1-\xi}{1-\varepsilon}\right] [g_{ij} D_j u D_i u] + \left[\frac{v_h - u}{\varepsilon}\right] [g_{ij} D_j \xi D_i \xi]. \end{gathered} \tag{3.9}$$

Since v_h converges to u strongly in $L^2(W)$, from the (3.8) and (3.9) we get

$$\limsup_h \left[\Psi_N(u_h) + \int_K u_h^2 \mathrm{d}\mu_h\right] \leq$$

$$\leq \frac{1}{1-\varepsilon} \limsup_h \left[\Psi_{\overline{W}}(v_h) + \int_{\overline{W}} v_h^2 \mathrm{d}\mu_h\right] + \frac{1}{1-\varepsilon} \Psi_{\overline{W} \setminus K}(u) \leq$$

$$\leq \frac{1}{1-\varepsilon} \left[\Psi_{\overline{W}}(u) + \int_{\overline{W}} u^2 \mathrm{d}\mu + \Psi_{\overline{W} \setminus K}(u)\right] \leq$$

$$\leq \frac{1}{1-\varepsilon} \left[\Psi_N(u) + \int_W u^2 \mathrm{d}\mu + \varepsilon\right].$$

The proof is then achieved in the case that A is contained in a single chart. If this does not happen, we may consider, for a given ε strictly positive,

$$A \cap U_i,$$

where $\mathcal{U} = (U_i)_{i \in I}$ is the family of open sets given in Lemma A.2 in Appendix, I is a finite set of indeces. The measure σ in Lemma A.2 is

$$\sigma(E) = \int_E u^2 \mathrm{d}\mu$$

We apply to $A \cap U_i$ the above arguments and after a summation , we get the proof of the proposition. In fact,

$$\sum_i \left[\Psi_{N \cap U_i}(u) + \int_{A \cap U_i} u^2 \mathrm{d}\mu \right] =$$

$$= \sum_i \limsup_h \left[\Psi_{N \cap U_i}(u_h) + \int_{K \cap U_i} u_h^2 \mathrm{d}\mu_h \right] \geq$$

$$\geq \limsup_h \left[\sum_i \left(\Psi_{N \cap U_i}(u_h) + \int_{K \cap U_i} u_h^2 \mathrm{d}\mu_h \right) \right] \geq$$

$$\geq \limsup_h \left[\Psi_N(u_h) + \int_K u_h^2 \mathrm{d}\mu_h \right],$$

while

$$\sum_i \left[\Psi_{N \cap U_i}(u) + \int_{A \cap U_i} u^2 \mathrm{d}\mu \right] = \Psi_N(u) + \int_A u^2 \mathrm{d}\mu + \varepsilon$$

by property (2) in Lemma A.2.

The Propositions 3.10 and 3.11 allow us to prove the following result.

PROPOSITION 3.14

Let $(\mu_h)_h$ be a sequence in $\mathcal{M}_0(M)$ which γ–converges to $\mu \in \mathcal{M}_0(M)$. Then

$$\text{(3.10)} \qquad \operatorname{cap}_\mu(A) \leq \liminf_h \operatorname{cap}_{\mu_h}(A),$$

$$\text{(3.11)} \qquad \operatorname{cap}_\mu(A) \geq \limsup_h \operatorname{cap}_{\mu_h}(K),$$

for every K compact set and for every A open set in M such that $K \subset A$.

PROOF

We suppose that in the right hand side the liminf is a limit and it is finite; this implies that there exists a sequence $(v_h)_h$ bounded in $H_0^1(M)$ such that

$$\mathrm{cap}_{\mu_h}(A) = \left[\Psi_M(v_h) + \int_A (v_h - 1)^2 \mathrm{d}\mu_h\right].$$

By passing to a subsequence, we may assume that v_h converges to v weakly in $H_0^1(M)$. Therefore the inequalities

$$\mathrm{cap}_\mu(A) \leq \left[\Psi_M(v) + \int_A (v-1)^2 \mathrm{d}\mu\right] \leq$$

$$\leq \liminf_h \left[\Psi_M(v_h) + \int_A (v_h - 1)^2 \mathrm{d}\mu_h\right] = \mathrm{cap}_{\mu_h}(A)$$

follows from Proposition 3.9 with $u_h = v_h - \zeta$, where $\zeta \in C_c^\infty(M)$, $\zeta = 1$ on U.

Let us prove (3.11). Let w be a function contained in $H_0^1(M)$ such that

$$\mathrm{cap}_\mu(A) = \left[\Psi_M(w) + \int_A (w-1)^2 \mathrm{d}\mu\right].$$

By Proposition 3.11, with N=M and $u = w - \zeta$, where ζ is as above, we know that there exists a sequence $(w_h)_h$ in $H_0^1(M)$ such that it converges to w and

$$\mathrm{cap}_\mu(A) = \left[\Psi_M(w) + \int_A (w-1)^2 \mathrm{d}\mu\right] \leq$$

$$\leq \mathrm{cap}_{\mu_h}(A) = \limsup_h \left[\Psi_M(w_h) + \int_A (w_h - 1)^2 \mathrm{d}\mu_h\right] \leq$$

$$\leq \limsup_h \mathrm{cap}_\mu(K),$$

and the (3.11) follows.

First of all, the results given in Proposition 3.14 may be improved in

PROPOSITION 3.15

Let $(\mu_h)_h$ be a sequence in $\mathcal{M}_0(M)$ which γ–converges to μ in $\mathcal{M}_0(M)$. Then

$$\mathrm{cap}_\mu(A) = \sup\left\{\liminf_h \mathrm{cap}_{\mu_h}(K) : K \subset A,\ K \quad \text{compact}\right\}$$

for every quasi open A in M.

PROOF

The proof is in ([DM] theorem 5.10).

PROPOSITION 3.16

Let $(\mu_h)_h$ be a sequence in $\mathcal{M}_0(M)$ which γ–converges to μ in $\mathcal{M}_0(M)$ and let $\mathcal{E}$be the family of all subsets of Msuch that $\overline{E}$ is *compact* and $\mathrm{cap}_{\mu^*}(\overline{E}) = \mathrm{cap}_{\mu^*}(\mathrm{int}E)$. Then $\mathcal{E}$ is a rich family and

$$\mathrm{cap}_m u(E) = \lim_h \mathrm{cap}\mu_h(E) \tag{3.12}$$

for every $E \in \mathcal{E}$.

PROOF

The fact that $\mathcal{E}$ is a rich family follows from Lemma 1. in Chapter 1. Since $\mathrm{int}E$ is open we have, by Remark 3.10,

$$\mathrm{cap}_{\mu^*}(\mathrm{int}E) = \mathrm{cap}_\mu(\mathrm{int}E) \leq \mathrm{cap}_\mu(E) \leq \mathrm{cap}_\mu(\overline{E}) \leq \mathrm{cap}_{\mu^*}(\overline{E}).$$

Hence $\mathrm{cap}_\mu(\mathrm{int}E) = \mathrm{cap}_m u(\overline{E})$; now the conclusion follows from Proposition 3.14.

The next caracterization of the γ–convergence holds true which can be proved as in ([DM] theorem 6.3).

PROPOSITION 3.17

The following are equivalent:

(a) $(\mu_h)_h$ is a sequence in $\mathcal{M}_0(M)$ which γ– converges to μ in $\mathcal{M}_0(M)$.

(b) The inequalities

$$\mathrm{cap}_\mu(K) \leq \liminf_h \mathrm{cap}_{\mu_h}(U)$$

$$\mathrm{cap}_\mu(U) \geq \limsup_h \mathrm{cap}_{\mu_h}(K)$$

hold true for every K, U compact and open sets in M.

(c) For every open set $U \subset M$

$$\mathrm{cap}_\mu(U) = \sup\left\{\liminf_h \mathrm{cap}_{\mu_h}(K) : K \subset U,\ K \quad \text{compact}\right\} =$$

$$= \sup\left\{\limsup_h \mathrm{cap}_{\mu_h}(K) : K \subset A,\ K \quad \text{compact}\right\}.$$

(d) The family of all $E \in M$ such that

$$\mathrm{cap}_{\mu}(E) = \lim_h \mathrm{cap}_{\mu_h}(E)$$

is dense in $\mathcal{P}(\mathcal{M})$.

Using the same methods in [DM], the above characterization is can be used to prove the

PROPOSITION 3.18

For every $\mu \in \mathcal{M}_0(M)$ let $\mathcal{H}$ be the family of all subsets of M such that

$$\mathrm{cap}_{\mu^*}(V \cap \mathrm{int}E) = \mathrm{cap}_{\mu^*}(V \cap \overline{E})$$

for every open set $V \subset M$. Then $\mathcal{H}$ is rich in $\mathcal{P}(\mathcal{M})$ and $(\mu_h^E)_h$ γ–converges to μ^E for every $E \in \mathcal{H}$ and for every $(\mu_h)_h$ γ–converging to μ in $\mathcal{M}_0(M)$.

Now we are in condition to give the desired result of this Chapter

THEOREM 3.19

Let $(\mu_h)_h$ be a sequence in $\mathcal{M}_0(M)$ and let $\mathcal{H}$ be the rich family introduced in the Proposition 3.18; then the following are equivalent:

(a) $(\mu_h)_h$ γ–converges to $\mu \in \mathcal{M}_0(M)$.

(b) The sequence of functionals $(\Psi_D + \Phi^E_{\mu_h})$ Γ–converges in $L^2(D)$ to the functional $(\Psi_D + \Phi^E_{\mu}$ for every open submanifold D of M and for every $E \in \mathcal{H}$ with $E \subset D$.

CHAPTER 4

First of all we recall the functionals defined in Chapter 2. Let

$$F_h(u) = \begin{cases} \Psi(u) + \Phi_h(u) & u \in H_0^1(M), \\ +\infty & \text{otherwise in } L^2(M), \end{cases} \tag{4.1}$$

where

$$\Phi_h(u) = \int_M u^2 d\mu_h$$

and

$$\Psi(u) = \int_M [|\nabla u|^2 + u^2] dV_g,$$

where $\mu_h \in \mathcal{M}_0(M)$ and let

$$F(u) = \int_M [|\nabla u|^2 + u^2] \mathrm{d}V_g + \int_M u^2 \mathrm{d}\mu,$$

if $u \in H_0^1(M)$, and $F(u) = +\infty$ if $u \notin H_0^1(M)$ be the Γ–limit in $L^2(M)$ of $(F_h)_h$ (theorem 2.2).

Let us consider the following problem,

$$\begin{aligned} -\Delta_g u + \mu u &= \lambda u \qquad \text{in } M \\ u &= 0 \qquad \text{on} \partial M, \end{aligned} \tag{4.2}$$

where Δ_g is the Laplace-Beltrami operator. In analogy with the case of an open set in R^n (see e.g. Dal Maso-Mosco [1,2,3]), we say that (4.2) is a Relaxed Dirichlet Problem.

DEFINITION 4.1

We say that u is a weak solution to the problem (4.2), if

$$\int_M \langle \nabla u_h, \nabla z \rangle_g \mathrm{d}V_g + \int_M uz \mathrm{d}\mu_h + \lambda \int_M v^2 \mathrm{d}V_g = \int_M fz \mathrm{d}V_g$$

for every $z \in H_0^1(M) \cap L^2(M,\mu)$.

REMARK 4.2

It can be proved, see Dal Maso-Mosco [3, proposition 2.1] that for any $\mu \in \mathcal{M}_0(M)$ and for every $\lambda > 0$ there exists an unique weak solution u_h to (4.2) belonging to $H_0^1(M) \cap L^2(M,\mu)$;

moreover it can be proved by variational methods that u_h is the unique solution of the minimization problem

$$\min_{v\in H_0^1(M)}\left[F_h(v)+\int_M v^2\mathrm{d}\mu-\lambda\int_M v^2\mathrm{d}V_g\right].$$

DEFINITION 4.3

Let $\mu\in\mathcal{M}_0(M)$; then the ***Resolvent Operator*** for Relaxed Dirichlet Problem (4.2) is defined as

$$R_\lambda^\mu : L^2(M)\longrightarrow L^2(M) \tag{4.3}$$

that associates to every $f\in L^2(M)$ the weak solution u to (4.2).

Let $(\mu_h)_h$ be a sequence in $\mathcal{M}_0(M)$; consider the problems

$$\begin{aligned} -\Delta_g u+\mu_h u=\lambda u \qquad & \text{in } M\\ u=0 \qquad & \text{on } \partial M. \end{aligned} \tag{4.4}$$

The result of this chapter is the following

THEOREM 4.4

Let $(\mu_h)_h$ be a sequence in $\mathcal{M}_0(M)$; if $(\mu_h)_h$ γ–converges to a measure $\mu\in\mathcal{M}_\prime(\mathcal{M})$ then

$$\lambda_h^i\longrightarrow\lambda^i, \qquad \text{as } h\to+\infty \tag{4.5}$$

where λ_h^i is the i-th eigenvalue of the problem (4.4) and λ^i is the i-th eigenvalue of the problem (4.2) (counted according their multiplicity).

PROOF

By a general result in Attouch [1] (theorem 3.26), we get from the Γ–convergence of the functional F_h to F, that the resolvent operator $R_h^{\mu_h}$ converges strongly to the resolvent operator $R\mu$ in $L^2(M)$. The strong convergence of the resolvent operators is sufficient to give the convergence of the eigenvalues [Dunford-Schwarz [1], Lemma XI. 9.5]

APPENDIX

LEMMA A.1

Let $(W_i)_{i\in I}$ be a finite or countable open cover of a compact Riemannian manifold with boundary $\overline{M}$; then there exists a family of open sets in $\overline{M}$, $\mathcal{U} = (U_i)_{i\in I}$ such that $U_i \subset W_i$ for every $i \in I$, $U_i \cap U_j = \emptyset$ for every $i \neq j$ and

$$\bigcup_{i\in I} U_i = \overline{M} \setminus \left[\bigcup_{i\in I} \partial U_i\right].$$

PROOF For every W_i, there exists an open set V_i, such that $(V_i)_i$ is an open cover of M,

$$\overline{V_i} \subset W_i.$$

Now define

$$U_1 = V_1$$

$$U_2 = V_2 \setminus \overline{U_1}$$

$$\cdot$$

$$\cdot$$

$$\cdot$$

$$U_i = V_i \setminus \left[\,\overline{U_1 \cup \ldots \cup U_{i-1}}\,\right].$$

It is a matter of fact that the U_i's are open and pairwise disjoint.

As regard the union, a first fact is that, being the (u_i) disjoint, the boundary of U_i must have empty intersection with every U_k, for any $i, k \in I$. On one hand, we get immediately from the above fact, that $\bigcup_{i\in I} U_i \subset \overline{M} \setminus \bigcup_{i\in I} \partial U_i$. To prove the opposite inclusion, let $x \in \overline{M} \setminus \bigcup_{i\in I} \partial U_i$; since $(V_i)_{i\in I}$ is an open cover of M, then $x \in V_k$, for some $k \in I$. By definition, either $x \in U_k$ or $x \in \overline{U_1} \cup \ldots \cup \overline{U_{k-1}}$; in the first case the proof is done, in the second one we have to consider that, by hypothesis, x does not belong to ∂U_i, for every $i \in I$, hence x is in U_j, for some $j \in I$. The proof is then achieved.

Under the same assumptions of Lemma A.1,We know that for every W_i, there exists an open set V_i, such that $(V_i)_i$ is an open cover of M,

$$\overline{V_i} \subset W_i.$$

Let $\zeta_i \in C_c^\infty(W_i)$ such that

$$\zeta_i = 1 \text{ on } V_i$$

$$0 \leq \zeta_i \leq 1 \text{ on } W_i.$$

Now set

$$U_i(\rho) = \{x \in W_i : \ \zeta_i(x) > \rho\}$$

for all $i \in I$ and for every $\rho \in (0,1)$; we have

$$V_i \subset U_i(\rho) \subset W_i, \qquad \forall i \in I$$

(actually $\overline{U_i(\rho)} \subset W_i$).

LEMMA A.2

Under the same assumptions of the Lemma A.1, for every $\varepsilon > 0$ and for every Borel measure σ, there exists an open cover $(U_i)_i$ of M such that

$$(A.1) \qquad \sigma(U_i \cap U_j) < \varepsilon, \quad \forall i,j \in I, \ i \neq j.$$

PROOF

Let $U_i(\rho)$ as in the construction above. Let us consider the real function, defined on $(0,1)$,

$$f(\rho) = \sigma(U_i(\rho))$$

for every $0 < \rho < 1$. This function f is positive and increasing on $(0,1)$, so it has a countable set of discontinuity points $(\rho_n)_n$ in $(0,\rho_1)$. From now on ,we consider ρ_1 as a point of continuity to f. For a given $\varepsilon > 0$, if ρ is sufficiently close to ρ_1,$\rho < \rho_1$, we have

$$\sigma\left(U_i(\rho) \setminus \overline{U_i(\rho_1)}\right) \leq \sigma(U_i(\rho)) - \sigma(U_i(\rho)) < \varepsilon,$$

hence $\sigma(\partial U_i(\rho)) = 0$.

Let us define, for $0 < \rho_2 < \rho_1$,

$$U_1 = U_1(\rho_2)$$

$$U_2 = U_2(\rho_2) \setminus \overline{U_1(\rho_1)}$$

$$\vdots$$

$$U_i = U_i(\rho_2) \setminus \left[\overline{U_1(\rho_1) \cup \ldots \cup U_{i-1}(\rho_1)}\right].$$

So the U_i are open set in M; now prove that

$$\bigcup_{i \in I} U_i = M.$$

Let $x \in M$, hence $x \in U_k(\rho_2)$, for some $k \in I$. If $x \in U_k$, the proof is done; otherwise let us suppose that $x \notin U_k$, hence $x \in \overline{U}_j(\rho_1)$ for some $j \leq k-1$. Since $\overline{U}_j(\rho_1) \subset U_j(\rho_2)$, we have, as before, two excluding alternatives: either $x \in U_j$ or $x \notin U_j$. We have that the set of the indices I is finite, so in a finite number of steps we find that either $x \in U_l$, for some $l \in I$, or $x \in \overline{U}_1(\rho_1) \subset U_1(\rho_2) = U_1$; the proof is then achieved.

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