ISAS - INTERNATIONAL SCHOOL
FOR ADVANCED STUDIES

Thesis submitted for the degree of
"Magister Philosophiae"

On One-Dimensional Dynamical Systems

CANDIDATE
WU He

SUPERVISOR
Prof. Giulio PIANIGIANI

Academic Year 1989/90

TRIESTE

SISSA - SCUOLA
INTERNAZIONALE
SUPERIORE
I STUDI AVANZATI
TRIESTE
Strada Costiera 11

Contents

Acknowledgements

I wish to express my deep gratitude to my supervisor Professor Giulio PIANIGIANI for his constant help , tireless guidence and carefully reading this manuscript. I am also very grateful to Professor Arrigo CELLINA for his generous and helpful encouragements. My sincere thanks to Professor Yunmei CHEN and Dr. Giovanni COLOMBO for their many kind helps. At last, I wish to say thanks to every staff and student at SISSA for their collaboration and helps.

Chapter 1

Introduction

One-dimensional dynamical systems has been an interesting subject of research for the last twenty five years. Mathematicians, physicists, ecologists and many others have all made important contributions to this field. In this thesis we will study one-dimensional dynamical systems in a topological view. First of all, we give some simple examples to explain what is a dynamical system. Take a scientific calculator and start striking one of the function keys over and over again. This iterative procedure is an example of a discrete dynamical system. For example, if we repeatedly strike the "sin" key, given an initial input x, we are computing the sequence of numbers of

$$\sin x,\ \sin(\sin x),\ \sin(\sin(\sin x)),\ldots$$

we will see that any initial x_0 leads to a sequence of iterate which tends to 0.

As another example, consider the so-called "tend" mapping $f(x) =$

$1-|2x-1|$ on the unit interval. This example looks like very simple, but its iterative properties is very complicated. The "tend" mapping has much to do with symbolic dynamics. As we known, symbolic dynamics is an important tool in the studying of dynamical systems. In this thesis, we will use symbolic dynamics to present several simple proofs for some well-known results in the dynamical systems.

The basic goal of the theory of dynamical systems is to understand the eventual or asymptotic behavior of an iterative process. For a continuous mapping f from the unit interval into itself, the set of periodic point, the nonwandering set, the topological entropy and the chaotic set of f are the most important subjects. In this thesis, our interest is to study those subjects.

For the convenience, now we introduce some notations and definitions.

Throughout this thesis I stands for the unit interval $[0,1]$, N stands for the set of all positive integers, $C^0(I)$ stands for the space of all continuous mappings $f: I \to I$, $[a,b]$ stands for the closed interval and $f[a,b]$ stands for the set $\{f(x) \mid x \in [a,b]\}$. Suppose that $f \in C^0(I)$ and $x_0 \in I$. The orbit of x_0 under f is defined as the set $\{x \mid x = f^n(x_0), n \in \mathrm{N}\}$ (we often denote it by $\mathrm{Orb}(x_0, f)$), where, for every positive integer n, f^n is the n-th iterate of f, $f^1 = f$ and $f^0 =$ identity mapping. A point x_0 is said to be a periodic point of period $p > 1$ ($p \in \mathrm{N}$) if $f^p(x_0)=x_0$ and $f^k(x_0) \neq x_0$ for $1 \leq k < p$. A point x_0 is said to be a periodic point of 1 (more generally, x_0 is said to be a fixed point of f) if $f(x_0) = x_0$. Clearly, if x_0 is a periodic point of period p, then its orbit consists of p points.

We denote by $P(f)$ the set of all periodic points of f , and denote by

$pp(f)$ the set of $n \in \mathbf{N}$ such that f has a periodic point of period n.

This thesis included four chapters. In chapter 1, we give an introduction and outline of this thesis. In chapter 2, we give a new and simple proof for the well-known Sarkovskii's theorem, state two typical Sarkovskii-like results in the cases circle and the so-called n-od (see 2.3 for the definitions), and give a proof [Bl4] for the stability of periodic orbits in the Sarkovskii's theorem. In chapter 4, we extend the Bowen's theorem [Bo] to the compact topological space, give a proof [Xi1] of that $\Omega(f|_{\Omega(f)}) = \overline{P(f)}$ where $\Omega(f)$ is the nonwandering set of f, $f|_{\Omega(f)}$ is the restriction of f to the nonwandering set, $\overline{P(f)}$ is the closure of the set of periodic points of f, and in the last section of chapter 3 we state some connections between the set of periodic points and the topological entropy of f. In chapter 4, we introduce the concept of chaos in the sense of Li- Yorke [LY], give a new proof about "period 3 implies chaos" , give a easy way to check if a mapping is chaotic due to [LMPY 2], and in the last part of chapter 3 we construct an example of a mapping with a chaotic set of full Lebesgue measure.

Chapter 2

Sarkovskii's Theorem

2.1 Introduction

The continuous mappings of the interval have been studied for many years. However some important and beautiful results have been found only in the last thirty years. For example, Sarkovskii's Theorem [Sa.] is amazing for its lack of hypotheses (the mapping is only assumed continuous) and its strong conclusion. It was proven by Sarkovskii [Sa] in 1964 but it was unknown in the English speaking world until Stefen [Ste] reproved it in 1977. Before this, in 1975 Li and Yorke [LY] proved a special case of the Sarkovskii's Theorem . In the early 1980's, Osikawa and Oono [OO], Block, Gukenheimer, Misiurewicz and Young [BGMY], Ho and Morris [HM], and Gawel [Ga] proposed a series of different proofs of the Sarkovskii's theorem.

Sarkovskii's Theorem and Sarkovskii ordering (see next section for the definition) is so remarkable that mathematicians try to generalize them to other spaces than the interval or the real line. The spaces with the most interesting Sarkovskii-like results seem to be the one-dimensional connected space. Block,Gukenheimer,Misiurewicz and Young [BGMY], Alseda [Al], Misisurewicz [Mi3] got such a result for the circle; Alseda ,Llibre,and Misiurewicz [ALM] gave such a result for the triod; Baldwin [Ba3] discovered such a result for the more general space: n-od. The n-od is the subspace of the plane which is most easily described as the set of all complex numbers z such that z^n is in the unit interval I, i.e. a central point 0 with n copies of I attached.

In this chapter we will propose a new and simple proof of Sarkovskii's Theorem in 2.2, state some related Sarkovskii-like results without proof in 2.3, and give a proof for the stability of periodic orbits in the Sarkovskii's Theorem in 2.4 due to Block [Bl4].

2.2 Sarkovskii's Theorem of the interval

Consider the following ordering of the natural number:

$3 \rhd 5 \rhd 7 \rhd \ldots$

$\rhd 2 \times 3 \rhd 2 \times 5 \rhd 2 \times 7 \rhd \ldots$

$\rhd 2^2 \times 3 \rhd 2^2 \times 5 \rhd 2^2 \times 7 \rhd \ldots$

$\ldots$

$\rhd 2^n \times 3 \rhd 2^n \times 5 \rhd 2^n \times 7 \rhd \ldots$

$\ldots$

$\ldots \rhd 2^n \rhd 2^{n-1} \rhd \ldots \rhd 2^3 \rhd 2^2 \rhd 2 \rhd 1$

That is, first list all odd numbers (except one) in the increasing order, followed by 2 times the odds, 2^2 times the odds, etc. This exhausts all the natural numbers with exception of the power of 2 which we list last in decreasing order. Such a ordering is called the Sarkovskii ordering of the natural numbers.

Theorem 2.2.1 *Let $f : I \to I$ be a continuous mapping, which has a periodic point of period n, if $n \rhd k$ in the Sarkovskii ordering, then f also has a periodic point of period k.*

Before proving this theorem, we note several consequences:

(i) If f has a periodic point whose period is not a power of two, then f necessarily has infinitely many periodic points with infinitely many periods. Conversely, if f has only finitely many periodic points, then all of them have periods which are powers of two.

(2) **Li-Yorke's Theorem[LY]** Period 3 is the "greatest" period in

the Sarkovskii ordering and therefore it implies the existence of all other periods.

(iii) There are mappings which have periodic points of period p and no periodic points of k if $k \rhd p$ in the Sarkovskii ordering. For example, let $f : I \to I$ be defined such that

$$f(0) = 2/4, f(1/4) = 1, f(2/4) = 3/4, f(3/4) = 1/4, f(1) = 0$$

and on each interval $[\frac{i}{4}, \frac{i+1}{4}] (i = 0, 1, 2, 3)$, assume that f is linear. It is easy to check that $3 \in pp(f)$ but $5 \notin pp(f)$.

To prove the Sarkovskii's Theorem, we need some lemmas. The following Lemma 2.2.2, Lemma 2.2.3 and Lemma 2.2.4 are very easy and very useful in the proof of Sarkovskii's Theorem. We just state them without proofs.

Lemma 2.2.2 *Let J be a closed subinterval of I. If $f(J) \supset J$, then there exists $x_0 \in J$ such that $f(x_0) = x_0$.*

Lemma 2.2.3 *Let J_1 and J_2 be closed subintervals of I. If $f(J_1) \supset J_2$, then there exists a closed interval $J \subset J_1$ such that $f(J) = J_2$ and for any closed subinterval $K \subset J$, $f(K) \neq J_2$.*

Lemma 2.2.4 *Let $J_0, J_1, ..., J_{n-1}$ be closed subintervals of I, if $f(J_i) \supset J_{i+1}$ for $i = 0, 1, ..., n-2$ and $f(J_{n-1}) \supset J_0$.*

Then there exists $x_0 \in J_0$ *such that* $f^n(x_0) = x_0$ *and* $f^k(x_0) \in J_k$ *for* $k = 0, 1, ..., n-1$.

Theorem 2.2.5 [Ste] *Let* $f \in C^0(I)$ *and let* $p \geq 3$ *be an odd integer. If* x_0 *is a periodic point of period* p *, then there exists a point* $y \in \{x_0, f(x_0), ..., f^{p-1}(x_0)\}$ *such that either*

(A)

$$\begin{aligned} f^{p-2}(y) &< f^{p-4}(y) < ... < f^3(y) < f(y) < y \\ &< f^2(y) < f^4(y) < ... < f^{p-1}(y), \end{aligned}$$

or

(B)

$$\begin{aligned} f^{p-2}(y) &> f^{p-4}(y) > ... > f^3(y) > f(y) > y \\ &> f^2(y) > f^4(y) > ... > f^{p-1}(y) \end{aligned}$$

holds.

PROOF: See [St] or [Ga].

Lemma 2.2.6 *Let* $f \in C^0(I)$ *and* $pp(f)$ *is not the whole* $\mathbf{N}$. *Then for every* $x \in I$*, we have*

(i) if $f(x) < x$*, and* $a = \max\{f(y) \mid y \in [f(x), x]\} \geq x$*, then* $f(z) < x$ *for every* $z \in [x, a]$*;*

(ii) if $f(x) > x$*, and* $b = min\{f(y) \mid y \in [x, f(x)]\} \leq x$*, then* $f(z) > x$ *for every* $z \in [b, x]$.

PROOF: (i) Assume that there exists a point $z \in [x, a]$ such that $f(z) > x$. Let $u \in [f(x), x]$ such that$f(u) = a$, then

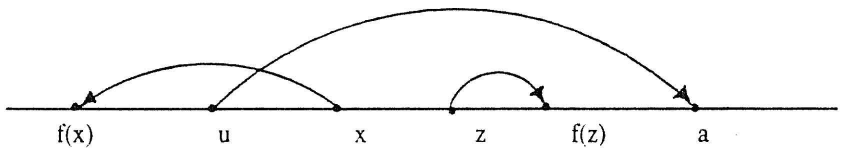

Figure 2.2.1

$$f([x, z]) \supset [u, x]$$

and

$$F([u, x]) \supset [u, x] \cup [x, z].$$

By Lemma 2.2.2, there exists a fixed point of f in $[u, x]$. By Lemma 2.2.4, for every integer $k > 1$, there exists a fixed point $y \in [x, z]$ such that $f^i(y) \in [u, x]$, for $i = 0, 1, ..., k - 1$. Clearly, the period of y is k. It follows that $pp(f) = \mathbf{N}$. A contradiction to $pp(f) \neq \mathbf{N}$.

The proof of (ii) is the same .#

Lemma 2.2.7 *Let $f \in C^0(I)$ and let $pp(f)$ be not the whole* $\mathbf{N}$. *Suppose that there exists a periodic orbit $\{x_1 < x_2 < ... < x_n\}$ with period n, then*

(1) if n is even, we have

$$f(J_1) \supset J_2 \quad \text{and} \quad f(J_2) \supset J_1.$$

where $J_1 = [x_1, x_{\frac{n}{2}}]$ *and* $J_2 = [x_{\frac{n}{2}+1}, x_n]$.

(2) if n *is odd, we have*

$$f(J_1) \supset J_2 \quad \text{and} \quad f(J_2) \supset J_1.$$

where $J_1 = [x_1, x_{\frac{n+1}{2}}]$ *and* $J_2 = [x_{\frac{n+1}{2}}, x_n]$.

PROOF: Case 1: n even. Let $f(x_i) = x_1$, and $a = max\{f(y) \mid y \in [x_1, x_i]\}$. Then there exists a point $y \in [x_1, x_i]$ such that $f(y) \geq x_i$. If not,

$$\{f(x_1), ..., f(x_i)\} \subset \{x_1, ..., x_{i-1}\},$$

this is impossible. By Lemma 2.2.6, we know that $f(y) \leq a$ for every $y \in f([x_1, a])$. Clearly, $f^j(x_1) \leq a$ for $j = 0, 1, ..., n-1$. In particular $x_n \leq a$.It follows that

$$f([x_1, x_i]) \supset [x_1, x_n]. \tag{2.1}$$

If $i \leq n/2$, then

Card$\{Orb(x_1, f) \cap [x_i, x_n]\}$ >Card$\{Orb(x_1, f) \cap [x_1, x_i]\}$ (where Card$\{A\}$ stands for the number of the elements of set A),then

$$f(Orb(x_1, f)) \cap [x_i, x_n] \neq \emptyset,$$

and since $f(x_i) = x_1$, we have

$$f([x_i, x_n]) \supset [x_1, x_i]. \tag{2.2}$$

Therefore from (2.1) , (2.2) and Lemma 2.2.4, it follows $pp(f) = \mathrm{N}$. A contradiction . Hence $i > n/2$.

Since Card$\{Orb(x_1, f) \cap J_1\} = n/2 =$ Card$\{Orb(x_1, f) \cap J_2\}$, then

(i) if $f(Orb(x_1,f)\cap J_2)= Orb(x_1,f)\cap J_1)$, then $f(J_2)\supset J_1$.

(ii) if $f(\mathrm{Orb}(x_1,f)\cap J_2)\neq$ $\mathrm{Orb}(x_1,f)\cap J_1)$, then $f(\mathrm{Orb}(x_1,f)\cap J_2)$ $\cap\mathrm{Orb}(x_1,f)\cap J_2)\neq\emptyset$. This together with $f(x_i)=x_1$ implies $f(J_2)\supset J_1$.

Let $f(x_j)=x_n$ and $b=min\{f(y)\mid y\in[x_j,x_n]\}$ and note that the conclusion in Lemma 2.2.4, By a similar arguments as in above, we can prove that $f(J_1)\supset J_2$.

Case 2: n odd. The proof is same as that in Case 1. #

Lemma 2.2.8 *Let $f\in C^0(I)$ and $n\in pp(f)$. Then*

(i) if $n>2$, then $2\in pp(f)$;

(ii) if $n\geq 3$ is odd, then $3\in pp(f^2)$;

(iii) if $n\geq 3$ is odd, then $n+2k\in pp(f)$, for every $k\in\mathbf{N}$.

PROOF: (i) Let $n\in pp(f)$ and $n>2$. Assume that $2\notin pp(f)$, by Lemma 2.2.7, it follows $2\in pp(f)$, a contradiction.

(ii) Let $n\in pp(f)$,$n\geq 3$ odd and let x be a periodic point of period n. Assume that $3\notin pp(f^2)$, then $3\notin pp(f)$. Since n is odd, the period of x under f^2 is also n. By using Lemma 2.2.7 for both f and f^2, it follows that $f^2(J_1)\supset J_2$ and $f(J_2)\supset J_1$. Hence $3\in pp(f)$, a contradiction.

(iii) Let $n\in pp(f)$ and $n\geq 3$ odd. Without loss of the generality, we may assume that n is the minimal number in $pp(f)$. Let x_1 be a periodic point of period n. By Lemma 2.2.7 we have $f(J_1)\supset J_2$, and $f(J_2)\supset J_1$,and from $Orb(x_1,f)$ is a periodic orbit of period n, we have $f^n(J_2)\supset J_2$, Hence, for every $k\in\mathbf{N}$ there exists a point $x_0\in I$ such that

$$f^{n+2k}(x_0)=x_0$$

with the following properties:

$x_0 \in J_1$, $f(x_0) \in J_0$, $f^2(x_0) \in J_1$, $f^3(x_0) \in J_0, \ldots, f^{2k-1}(x_0) \in J_0$, $f^{2k}(x_0) \in J_1$, $f^{2k+n}(x_0) \in J_1$.

Let m be the period of x_0, then m must be an odd number. Therefore $m \geq n$ (since n in the minimal number in the set $pp(f)$).

If $i < k$, $f^{2i+1}(x_0) \in J_0$, hence $m \geq (2(k-1)+1)+1 = 2k$, so $m \geq \frac{n+2k}{2}$.

As we know that m is a factor of $n + 2k$ and $n + 2k$ is odd, therefore $m = n + 2k$. The proof for this Lemma is finished. #

PROOF OF THEOREM 2.2.1: Obviously, $1 \in pp(f)$. In the following we assume that $k > 1$.

(1) If $n = 2^m$, then $n \in pp(f)$ implies $2^{m-r+1} \in pp(f^{2^{r-1}})$ for $0 < r < m$. By Lemma 2.2.8, $2 \in pp(f^{2^{r-1}})$. Hence $2^r \in pp(f)$.

(2) In the following, we assume that $n = 2^m \cdot p$ where $p \geq 3$ is odd.

(i) $k = 2^m \cdot q$ where $q > p$ is odd. Then $n \in pp(f)$ implies $p \in pp(f^{2^m})$. By Lemma 2.2.8, $q \in pp(f^{2^m})$, hence $k \in pp(f)$.

(ii) $k = 2^s \cdot q$ where $s > m$ and $q \geq 3$ is odd. Then $n \in pp(f)$ implies $p \in pp(f^{2^m})$. Since p is odd and $s > m$, clearly $p \in pp(f^{2^{s-1}})$. By Lemma 2.2.8(ii), $3 \in pp(f^{2^s})$, by Lemma 2.2.8(iii), $q \in pp(f^{2^s})$. Hence $k \in pp(f)$.

(iii) $k = 2^s$. If $s > m$, then $n \in pp(f)$ implies $p \in pp(f^{2^{s-1}})$; If $1 < s \leq m$, then $n \in pp(f)$ implies $2^{m-s+1} \cdot p \in pp(f^{2^{s-1}})$. By Lemma 2.2.8(i), $2 \in pp(f^{2^{s-1}})$. Therefore $k \in pp(f)$.

This completes the proof. #

2.3 Sarkovskii-like Results

We will introduce two typical Sarkovskii-like results here. One is for the continuous mapping of the circle, the other is for the continuous mapping of the n-od.

2.3.1 The case S^1

Let $f : S^1 \to S^1$ be a continuous mapping of a circle S^1 into itself. To study the dynamics of a circle mapping, it is helpful to *lift* the mapping to $\mathbf{R}$. That is, we define the **covering mapping** $\pi : \mathbf{R} \to S^1$ by

$$\pi(x) = e^{2\pi i x} = \cos(2\pi x) + i\sin(2\pi x).$$

$F : \mathbf{R} \to \mathbf{R}$ is called a **lift** of $f : S^1 \to S^1$ if

$$\pi \circ F = f \circ \pi.$$

Note that the lift F of f is not unique, but if F and F_1 are two lifts of the same mapping f, then $F = F_1 + k$ for some integer k. Hence there exists an integer m such that $F(x+1) = F(x) + m$ for all x. We call this m the **degree** of f and denote it by deg(f).

For $n \in \mathbf{N}$, we denote that

$$\mathcal{S}(n) = \{k \in \mathbf{N} \mid n \rhd k, or\ k = n\}.$$

and

$$\mathcal{S}(2^\infty) = \{2^k \mid k = 0, 1, 2, ...\}.$$

Theorem 2.3.1[Bl4] *Let f be a continuous mapping from S^1 into itself. If $1 \in pp(f)$, then either*

(a) there exists positive integers m and k such that

$$pp(f) = \{n \in \mathbf{N} \mid n \geq m\} \cup \mathcal{S}(k)$$

or

(b) there exists a positive integer k such that

$$pp(f) = \mathcal{S}(k) \text{ or } pp(f) = \mathcal{S}(2^\infty).$$

Theorem 2.3.2 *Let f be a continuous mapping from S^1 into itself, then*

(i) If $|deg(f)| \geq 2$ then $pp(f) = \mathbf{N}$ with one exception: if $deg(f) = -2$ it is possible also that $pp(f) = \mathbf{N} \setminus \{2\}$. (see [BGMY] and [J.])

(ii)If $deg(f) = 0$ then either $pp(f) = \mathcal{S}(n)$ for some $n \in \mathbf{N}$ or $pp(f) = \mathcal{S}(2^\infty)$.(see [BGMY] and [J.])

(iii) If deg(f) $= -1$ *then either* $pp(f) = \mathcal{S}(n)$ *for some* $n \in \mathbf{N}$ *or* $pp(f) = \mathcal{S}(2^\infty)$.*(see [BGMY] and [J.])*

(v) If deg(f) $= 1$, *there exist* $a, b \in \mathbf{R}$ *and* $l, r \in \mathbf{N} \cup \{2^\infty\}$ *with* $a \leq b$ *such that*

$$pp(f) = M(a,b) \cup S(a,l) \cup S(b,r), \tag{2.3}$$

where

$M(a,b) = \{n \in \mathbf{N} \mid$ *there exists* $k \in \mathbf{N} \cup \{0\}$ *such that* $a < \frac{k}{n} < b\}$;

$$S(a,l) = \begin{cases} \emptyset & \text{if } a \text{ is irrational} \\ \{n \cdot s \mid s \in \mathcal{S}(l)\} & \text{if } a = \frac{k}{n} \text{ for } k \text{ and } n \text{ coprime} \end{cases} \tag{2.4}$$

and $S(b,r)$ *analogously.*

Moreover, for every set $A \subset \mathbf{N}$ *of the form (2.3), there exists a* C^∞ *mapping* $f : S^1 \to S^1$ *of degree one, with* $pp(f) = A$.*(see [M.])*

2.3.2 n-od

Let I be the unit interval $[0,1]$ and R be the set of real numbers. The n-od is the subspace of the plane which is easily described as the set of all complex numbers $\{z \mid z^n \in I\}$, i.e. a central point 0 with n copies of I attached. The 3-od is generally refered to as the triod. We denote by X_n the n-od. A **branch** of the n-od will be any component of $X_n - \{0\}$.

A **graph** is any subset of R^n which is the union of finitely many compact straight line segments and a **tree** is any connected graph which contains no homeomorphic copy of the circle.

Definition 2.3.3 *We define partial orderings $\leq_n$ for all positive integers n. The ordering $\leq_1$ is defined in the following way (Sarkovskii ordering):*

$$2^i \leq_1 2^{i+1} \leq_1 2^{j+1}(2m+1) \leq_1 2^j(2k+3) \leq_1 2^j(2k+1)$$

for all integers $i, j \geq 0$ and $k, m > 0$.

If $n > 1$, we define the partial ordering $\leq_n$ is defined as follows: let $m, k \in \mathbf{N}$.

Case 1 $k = 1$. *Then $m \leq_n k$ iff $m = 1$.*

Case 2 k *is divisible by n. Then $m \leq_n k$ iff either $m = 1$ or m is divisible by n and $\frac{m}{n} \leq_1 \frac{k}{n}$.*

Case 3 $k > 1$, *k is not divisible by n. Then $m \leq_n k$ iff either $m = 1, m = k$ or $m = ik + jn$ for some integers $i \geq 0, j \geq 1$.*

We give diagrams of some typical cases to illuminate the description given above. See Figures 2.3.1 and 2.3.2 for diagrams of the $\leq_3$ and $\leq_4$ orderings. One can easy check that the ordering $\leq_2$ is the same as the ordering $\leq_1$,i.e.,the Sarkovskii ordering.

In the case $n = 3$, the $\leq_3$ partial ordering combines the information contained in two linear ordering developed in [ALM], and called there the "red" and "green" orderings.

A set Z of positive integers is an initial segment of $\leq_n$ if whenever k is a member of Z and $m \leq_n k$ then m is also a member of Z, i.e. Z is closed under $\leq_n$-predecessors. We can now state the Sarkovskii-like result in the case n-od:

Theorem 2.3.4 *Let f be a continuous mapping on the n-od. Then pp(f) is a nonempty finite union of initial segments $\{Z_p \mid 1 \leq p \leq n\}$ of $\{\leq_p \mid 1 \leq p \leq n\}$. Conversely, if A is a nonempty finite union of initial segments $\{Z_p \mid 1 \leq p \leq n\}$ of $\{\leq_p \mid 1 \leq p \leq n\}$, then there is a continuous mapping f on the* **n-od** *which fixes the central point* 0 *such that $P(f) = A$.*

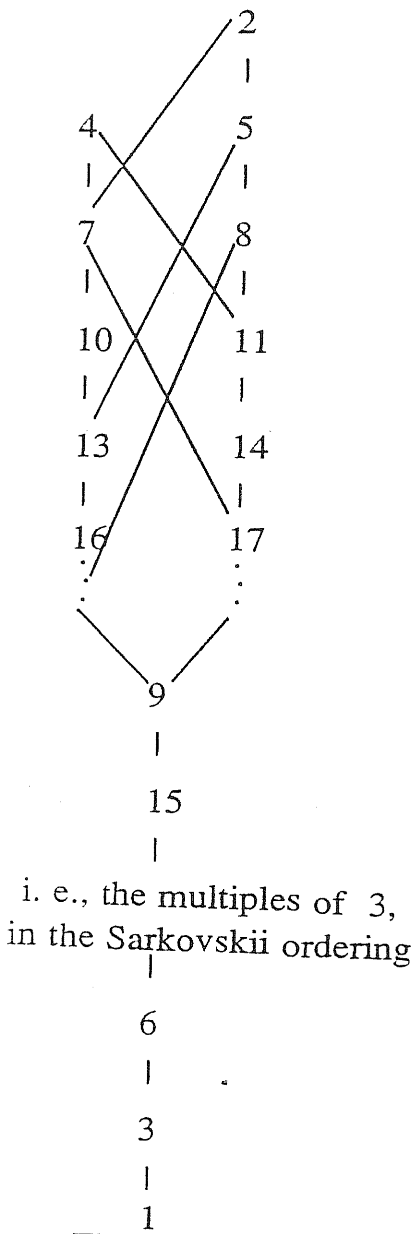

Figure 2.3.1

The ordering $\leq_3$

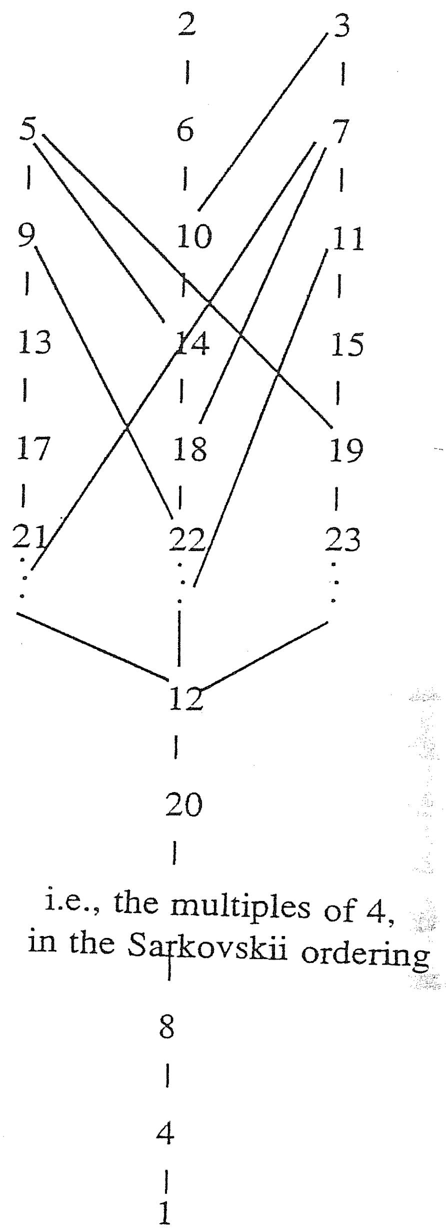

Figure 2.3.2

The ordering $\leq_4$

2.4 The stability of periodic orbits in the Sarkovskii's Theorem

A property of a mapping $f \in C^0(I)$ is said to be stable in the space $C^0(I)$, if there exists a $\epsilon > 0$ such that for every $g \in C^0(I)$ and $\|f - g\| = max_{x \in I}|f(x) - g(x)| < \epsilon$, g also has the given property. Studying the stability of some properties of a mapping is important from a physical point of view, because when one uses a mapping as a model of some phenomena there will normally be some possibility for error. Butler and Pianigiani [BP] discussed the stability of periodic orbits in the Sarkovskii's Theorem. They proved that if $f \in C^0(I)$ has a periodic point of period 3, then there exists a neighborhood U of f in $C^0(I)$ such that for every $g \in U$ and every positive integer k with $3 \rhd k$ in the Sarkovskii ordering, $k \in pp(f)$. Later, Block [Bl4] got a general result for the stability of periodic orbits in the Sarkovskii's Theorem. In this section, we will prove that results.

Theorem 2.4.1 *Let f be a continuous mapping from I into itself, and suppose that $n \in pp(f)$. Then there is a neighborhood U of f in $C^0(I)$ such that $k \in pp(g)$* for every $g \in U$ and $k \in \mathbf{N}$ with $n \rhd k$ in the Sarkovskii ordering.

For proving this Theorem, we will use a Theorem of Stefan (see Theorem

2.2.5) and Sarkovskii's Theorem. We present a few technical Lemmas.

Lemma 2.4.2 *Let $f \in C^0(I)$ and let k be an odd positive integer with $k \geq 3$. Suppose that there is a point $y \in I$ such that the following inequalities hold:*

(i)

$$\begin{aligned} f^{k-2}(y) &< f^{k-4}(y) < ... < f^3(y) < f(y) < y \\ &< f^2(y) < f^4(y) < ... < f^{k-1}(y), \end{aligned}$$

(ii) $y < f^k(y)$.

Then $k \in pp(f)$.

PROOF: Let $k \geq 3$ be any odd number, and let y satisfies (i) and (ii). Since $f(y) < y$ and $f(f(y)) > f(y)$, we have $f[f(y), y] \supset [f(y), y]$. Hence f has a fixed point $e \in (f(y), y)$. Set $I_1 = [e, y], I_3 = [y, f^2(y)], ...,$ and $I_k = [f^{k-3}(y), f^{k-1}(y)]$. Also, set $I_2 = [f(y), e]$,$I_4 = [f^3(y), f(y)], ...,$ and $I_{k-1} = [f^{k-2}(y), f^{k-4}(y)]$. From (i) and (ii), we have $f(I_i) \supset I_{i+1}$ for $i = 1, ..., k-1$, and $f(I_k) = I_1$.

Hence, by Lemma 2.2.3, there are closed intervals $J_1, ..., J_k$ with $J_i \subset I_i$ for $i = 1, ..., k$ such that $f(J_k) = I_1$ and $f(J_i) = J_{i+1}$ for $i = 1, ..., k-1$. It follows that

$$f^k(J_1) = I_1 \supset J_1.$$

By Lemma 2.2.4, f^k has a fixed point $z \in J_1$. Suppose that $f^i(z) = z$ for some $i < k$. Since $f^i(z) \in J_{i+1}$, we have $f^i(z) \neq z$ for $i = 1, ..., k-1$.Thus , z is a periodic point of f of period k.The proof of this lemma is completed.#

Lemma 2.4.3 *Let $f \in C^0(I)$ and suppose that $n \in pp(f)$ where n is odd and $n \geq 3$. Then there is a neighborhood U of f in $C^0(I)$ with the property that if $g \in U$ then $(n+2) \in pp(g)$.*

PROOF: By the Sarkovskii's Theorem it is sufficient to prove this Lemma in the case that $j \notin pp(f)$ for all $j \in \{3, 5, ..., n-2\}$.

Since $n \in pp(f)$, there is a periodic point $x \in I$ such that $Orb(f, x) = \{x_1, ..., x_n\}$ with $x_1 < x_2 < ... < x_n$. By Stefan's Theorem (Theorem 2.2.5) we may assume that (A) in Theorem 2.2.5 holds. Then by setting $m = \frac{n+1}{2}$ and $z = x_m$, we have

(i)

$$\begin{aligned} f^{k-2}(z) &< f^{k-4}(z) < ... < f^3(z) < f(z) < z \\ &< f^2(z) < f^4(z) < ... < f^{k-1}(z), \end{aligned}$$

(ii) $z = f^k(z)$.

Since $f(z) < z$ and $f(f(z)) > z$, there is a point $b \in (f(z), z)$ with $f(b) = z$. Also, since $f(b) = z > b$ and $f(z) < b$, there is a $y \in (b, z)$ with $f(y) = b$. Thus $f^2(y) = z$. It follows that y satisfies the condition of Lemma 2.4.2 with $k = n+2$. Since each inequality in Lemma 2.4.2 is strict , there is a neighborhood U of f satisfies the condition of Lemma 2.4.2 with $k = n+2$ for all $g \in U$. Therefore $n+2 \in pp(g)$ for all $g \in U$. #

Lemma 2.4.4 *Let $f \in C^0(I)$. Suppose $n \in pp(f)$ and $n = 2^i \cdot p$ where $i \geq 0$ is an integer, and p is an odd integer with $p \geq 3$. Then there is a neighborhood U of f in $C^0(I)$ such that $k \in pp(g)$ for every $g \in U$ and* *every positive integer k with $n \rhd k$ in the Sarkovskii ordering.*

PROOF: Note that $n \in pp(f)$ implies $p \in pp(f^{2^i})$. By Lemma 2.4.3, there is a neighborhood U_1 of f^{2^i} with the property that if $g \in U_1$ then $(p+2) \in pp(g)$. Since the mapping g to g^{2^i} is continuous, there is a neighborhood U of f such that if $g \in U$ then $g^{2^i} \in U_1$. Hence, $(p+2) \in pp(g^{2^i})$. This implies that $(p+2) \cdot 2^j \in pp(g)$, where j is some integer with $0 \leq j \leq i$. From the Sarkovskii's Theorem, it follows that $k \in pp(g)$ for every positive integer k with $n \rhd k$. #

Lemma 2.4.5 *Let $f \in C^0(I)$ and suppose $4 \in pp(f)$. Then there is a neighborhood U of f in $C^0(I)$ such that if $g \in U$ then $2 \in pp(g)$.*

PROOF: Since $4 \in pp(f)$, f has a periodic point $\{x_1, x_2, x_3, x_4\}$ of period 4 with $x_1 < x_2 < x_3 < x_4$. If $3 \in pp(f)$ then the conclusion of this Lemma follows from Lemma 2.4.3. Hence we assume that $3 \notin pp(f)$. From Block [B13] we know that if $f(\{x_1, x_2\}) \neq \{x_3, x_4\}$, then $3 \in pp(f)$(it is easy to check this result).It implies that $f(\{x_1, x_2\}) = \{x_3, x_4\}$. Clearly $f(\{x_3, x_4) = \{x_1, x_2\}$.

Set $I_1 = [x_1, x_2]$, and $I_2 = [x_3, x_4]$. Thus $f(I_1) \supset I_2$ and $f(I_2) \supset I_1$. By Lemma 2.2.3, there is a closed interval $J \subset I_2$ with $f(J) = I_1$. Let $z \in \{x_1, x_2\}$ with $f(z) = x_4$. Therefore there is a $x \in J$ such that $f(x) = z$. Clearly, $x \neq x_4$ and $f^2(x) > x$. Similarly, there is a point $y \in J$ such that $f^2(y) < y$. Hence, there is a neighborhood U of f such that if $g \in U$, then $g^2(x) > x$, $g^2(y) < y$ and $g(z) < x_3$ for all $z \in J$. Then $2 \in pp(g)$ for all

$g \in U$.#

Lemma 2.4.6 *Let $f \in C^0(I)$ and let $2^k \in pp(f)$ for some positive integer k. Then there is a neighborhood U of f in $C^0(I,I)$ such that if $g \in U$ then $2^i \in pp(g)$ for every integer i with $0 \leq i \leq k-1$.*

PROOF: We may assume that $k \geq 2$. Clearly, $4 \in pp(f^{2^{k-2}})$. By Lemma 2.4.5, $2 \in pp(g)$ for every g in some neighborhood U_1 of $f^{2^{k-2}}$. There is a neighborhood U of f such that if $g \in U$ then $g^{2^{k-2}} \in U_1$ by the continuity of the mapping from g to $g^{2^{k-2}}$. Let $g \in U$. Since $2 \in pp(g^{2^{k-2}})$,we have $2^{k-1} \in pp(g)$.By the Sarkovskii's Theorem, it follows that $2^i \in pp(g)$ for every integer i with $0 \leq i \leq k-1$.#

The Theorem 2.4.1 follows now immediately from Lemma 2.4.4 and Lemma 2.4.6.

Chapter 3

Topological Entropy

3.1 Introduction

The concept of topological entropy was originally introduced by Adler, Konheim and McAndrew [AKM] in 1965. We recall the definition of topological entropy.(for details see [AKM])

Suppose that X is a compact topological space and $f : X \to X$ is a continuous mapping from X into itself.

For α an open covering of X let $N(\alpha)$ denotes the minimum number of members of a subcovering of α, and let $H(\alpha) = logN(\alpha)$ and $f^{-1}(\alpha) = \{f^{-1}(A) \mid A \in \alpha\}$. For α and β two open coverings of X let $\alpha \bigvee \beta = \{A \bigcap B \mid A \in \alpha, B \in \beta\}$. By elementary analysis one can find that the limit

$$ent(f,\alpha) = \lim_{n\to\infty} \frac{1}{n} H(\bigvee_{i=0}^{n-1} f^{-1}(\alpha)) \tag{3.1}$$

exists and is nonnegative. Define the topological entropy of f by

$$ent(f) = \sup_{\alpha} ent(f,\alpha) \tag{3.2}$$

where the superemum is over all open coverings α of X. If α and β are open coverings of X, and α is a refinement of β (i.e., every member of α is a subset of some member of β), then we write that $\alpha \succ \beta$ or $\beta \prec \alpha$. It is easy to show that if $\alpha \succ \beta$, then $ent(f) \geq ent(f,\beta)$.

The main properties of topological entropy are in the following two theorems:

Theorem 3.1.2 *If n is a positive integer, and f is a continuous mapping from X into itself, then $ent(f^n)$=$n \cdot ent(f)$*

Theorem 3.1.3 *Suppose that X, Y are two compact topological spaces , f is a continuous mapping from X into itself and g is a continuous mapping from Y into itself. If there exists a continuous mapping $h : X \longrightarrow Y$ with $h(X) = Y$ and $g \circ h = h \circ f$, then $ent(f) \geq ent(g)$. if h is a homeomorphism , then $ent(f) = ent(g)$.*

See Adler, Konheim,and McAndrew [AKM] for the proofs of Theorems 3.1.2 and 3.1.3.

A subset Y of X is called to be invariant (in respect to f), if $f(Y) \subset Y$.

The following lemma comes from [AKM].

Lemma 3.1.4 *Let X be a compact topological space and let $f \in C^0(X)$.If $Y \subset X$ is invariant, then $ent(f) \geq ent(f|_Y)$.*

The calculation of topological entropy is generally very difficult, the following famous result is due to Bowen [Bo]. Before stating this result, we give the definition of nonwandering set of a continuous mapping f.

A point $x \in X$ is said to be nonwandering, if for any neighborhood U of x there exists some $n \geq 0$ such that $U \bigcap f^n(U) \neq \emptyset$. Denote by $\Omega(f)$ the set of all nonwandering points of f. Obviously, $\Omega(f)$ is closed ,nonempty, and invariant subset, and $\Omega(f^n) \subset \Omega(f)$ for any integer $n \geq 1$. Since $P(f) \subset \Omega(f)$, it follows that $\overline{P(f)} \subset \Omega(f)$.

Theorem 3.1.5 *Let X be a compact metric space and let $f \in C^0(X)$. Then*

$$ent(f) = ent(f|_{\Omega(f)})$$

where $\Omega(f)$ is the nonwandering set of f.(see [Bo])

This result is very useful for calculating and estimating topological entropy. Ones obtained a series results for some special kinds of mappings by Theorem 3.1.5. Recently this result was extended to the compact topological space with a proof simpler than the one given by Bowen [Bo]. We will give the proof in 3.2, and in 3.3 we introduce an important result about the nonwandering set due to [Xi1] which said that $\Omega(f|_{\Omega(f)}) = \overline{P(f)}$, where $\Omega(f)$ is the nonwandering set of f, $f|_{\Omega(f)}$ is the restriction of f to the set $\Omega(f)$, $\overline{P(f)}$ is the closure of $P(f)$ of the periodic points set of f. In 3.4 we will state some connections between the periodic set $P(f)$ and the topological entropy $ent(f)$ due to [BF], [Ste], [BGMY] and [Mi3].

3.2 The extension of Bowen's Theorem

In the following of this section X will be a compact topological space and $f : X \to X$ is a continuous mapping.

Theorem 3.2.1 *Suppose $f : X \to X$ is a continuous map, where X is a compact topological space. Then*

$$ent(f) = ent(f|_{\Omega(f)}) \tag{3.3}$$

where $\Omega(f)$ is the nonwandering set of f.

If α is an open covering of X and $k > 0$ is an integer. We define let $k\alpha = \{A_1 \cup A_2 \cup ... \cup A_k \mid A_i \in \alpha, 1 \leq i \leq k\}$. It is easy to see that $k\alpha$ is also an open covering of X. It is obvious that:

(1) $N(\alpha) \leq kN(k\alpha)$;

(2) $f^{-1}(k\alpha) = kf^{-1}(\alpha)$;

(3) $\bigvee_{i=0}^{n-1} k\alpha_i \succ k^n \bigvee_{i=0}^{n-1} \alpha_i$ provided $\alpha_0, ..., \alpha_{n-1}$ are open coverings of X.

Before proving Theorem 3.2.1, we need some lemmas.

Lemma 3.2.2

$$ent(f, k\alpha) \geq ent(f, \alpha) - \log k. \tag{3.4}$$

PROOF:

$$\begin{aligned}
ent(f, k\alpha) &= \lim_{n\to\infty} \frac{1}{n} H(\bigvee_{i=0}^{n-1} f^{-i}(k\alpha)) && \text{(by definition)} \\
&= \lim_{n\to\infty} \frac{1}{n} H(\bigvee_{i=0}^{n-1} kf^{-i}(\alpha)) && \text{(by (2))} \\
&\geq \lim_{n\to\infty} \frac{1}{n} H(k^n \bigvee_{i=0}^{n-1} f^{-i}(\alpha)) && \text{(by (3))} \\
&\geq \lim_{n\to\infty} \frac{1}{n} \log \frac{N(\bigvee_{i=0}^{N-1} f^{-i}(\alpha)}{k^n} && \text{(by (1))} \\
&= \lim_{n\to\infty} \frac{1}{n} H(\bigvee_{i=0}^{n-1} f^{-1}(\alpha)) - \log k \\
&= ent(f, \alpha) - \log k. \quad \#
\end{aligned}$$

Lemma 3.2.3 *If the open covering α of X satisfies the condition $\Omega(f) \subset A$ for some $A \in \alpha$, then $ent(f, \alpha) = 0$.*

PROOF: For each $x \in X \backslash \Omega(f)$, there is a open set $A_x \in \alpha$ such that $x \in A_x$. Now take an open neighborhood $B_x \subset A_x$such that $B_x \cap\ (\cup_{i=1}^{\infty} f^i(B_x)) = \emptyset$. Then $\alpha' = \{A\} \cup \{B_x \mid \ x \in X \setminus \Omega(f)\}$ is an open covering of X which is a refinement of α. Choose an arbitrary finite subcovering $\beta =$

$\{A, B_{x_1}, ..., B_{x_l}\}$ of α'. Obviously, $\beta \succ \alpha$ and hence it is sufficient to prove that $ent(f, \beta) = 0$.

For an integer $n > 0$ define a mapping $\xi : \beta^n \longrightarrow \bigvee_{i=0}^{n-1} f^{-i}(\beta)$ such that $\xi(C_0, ..., C_{n-1}) = \cap_{i=0}^{n-1} f^{-i}(C_i)$ for every $(C_0, ..., C_{n-1}) \in \beta^n$.It is easy to see that ξ is surjective. If for $(C_0, ..., C_{n-1}) \in \beta^n$ the number of the components C_i different from A is greater than l,then there exist j and j' with $j < j'$ such that $C_j = C_{j'} = B_{x_q}$ for some q. In this case we have $\xi(C_0, ..., C_{n-1}) = \cap_{i=0}^{n-1} f^{-i}(C_i) = \emptyset$, because $x \in \cap_{i=0}^{n-1} f^{-i}(C_i)$ would imply $f^j(x), f^{j'}(x) \in B_{x_q}$ and hence $B_{x_q} \cap f^{j-j'}(B_{x_q}) \neq \emptyset$, a contradiction with the choosing of B_{x_q}. Therefore, in $\bigvee_{i=0}^{n-1} f^{-i}(\alpha)$ the number of nonempty members is not greater than the cardinality of the set Γ consisting of the points $(C_0, ..., C_{n-1}) \in \beta^n$, which has at most l components different from A. By the inductive method and using the basic properties of permutation we can prove that

$$N(\bigvee_{i=0}^{n-1} f^{-i}(\beta)) \leq (n+1)^l l^l. \tag{3.5}$$

and hence

$$\begin{aligned} ent(f, \beta) &= \lim_{n\to\infty} \frac{1}{n} H(\bigvee_{i=0}^{n-1} f^{-i}(\beta)) \\ &= \lim_{n\to\infty} \frac{1}{n} \log N(\bigvee_{i=0}^{n-1} f^{-i}(\beta)) \\ &\leq \lim_{n\to\infty} \frac{1}{n}(l \log(n+1) + l \log l) \\ &= 0 \end{aligned}$$

The lemma is proved.#

Suppose α is an open covering of X. For an invariant closed subset Y of f, denote by $\alpha|_Y = \{A \cap Y \mid A \in \alpha\}$.It is obvious that:

(i) $f^{-1}(\alpha)|_Y=(f|_Y)^{-1}(\alpha|_Y)$;

(ii) If β is also an open covering of X, then $(\alpha \vee \beta)|_Y=(\alpha|_Y) \vee (\beta|_Y)$;

(iii) If Y_1 is also an invariant closed subset of f and $Y_1 \subset Y$, then $N(\alpha|_{Y_1})\leq N(\alpha|_Y)$.

Lemma 3.2.4 *Let α be an open covering of X, then*

$$ent(f,\alpha) \leq H(\alpha|_{\Omega(f)}).$$

PROOF: Let $k = N(\alpha|_{\Omega(f)})$ and let $\{A_1 \cap \Omega(f), ..., A_k \cap \Omega(f)\}$ is a subcovering of $\alpha|_{\Omega(f)}$, where A_1, ..., A_k are in α. Then $A_1 \cup ... \cup A_k \in k\alpha$ and $\Omega(f) \subset A_1 \cup ... \cup A_k$. By Lemma 3.2.2 and Lemma 3.2.3 we have that

$$0 = ent(f,k\alpha) \geq ent(f,\alpha) - \log k. \tag{3.6}$$

The proof is completed. #

PROOF OF THEOREM 3.2.1 : Let α be any open covering of X and let $m > 0$ be any integer, we have that

$$\begin{aligned} ent(f,\alpha) &= \lim_{n\to\infty} \frac{1}{n} H(\bigvee_{i=0}^{n-1} f^{-i}(\alpha)) = \lim_{n\to\infty} \frac{1}{mn} (\bigvee_{i=0}^{mn-1} f^{-i}(\alpha)) \\ &= \lim_{N\to\infty} \frac{1}{mn} H(\{A_0 \cap f^{-1}A_1 \cap ... \cap f^{-(mn-1)}A_{mn-1} \mid A_j \in \alpha\}) \\ &= \lim_{N\to\infty} \frac{1}{mn} H(\{(A_0 \cap f^{-1}A_1 \cap ... \cap f^{-(m-1)}) \\ &\quad \cap f^{-m}(A_m \cap f^{-1}A_{m+1} \cap ... \cap f^{-(m-1)}) \end{aligned}$$

$$
\begin{aligned}
&\cap \ldots \\
&\cap f^{-(n-1)m}(A_{(n-1)m} \cap \ldots \cap f^{-(m-1)} A_{nm-1}) \mid A_j \in \alpha\}) \\
= {} & \lim_{n\to\infty} \frac{1}{mn} H(\bigvee_{i=0}^{n-1} (f^m)^{-i}(\bigvee_{j=0}^{m-1} f^{-j}(\alpha))) = \frac{1}{m} ent(f^m, \bigvee_{j=0}^{m-1} f^{-j}(\alpha))
\end{aligned}
$$

By Lemma 3.2.4, we have that

$$
\begin{aligned}
ent(f,\alpha) &\le \frac{1}{m} H((\bigvee_{j=0}^{m-1} f^{-j}(\alpha))|_{\Omega(f^m)}) \le \frac{1}{m} H((\bigvee_{j=0}^{m-1} f^{-j}(\alpha))|_{\Omega(f)}) \\
&= \frac{1}{m} H(\bigvee_{j=0}^{m-1} (f|_{\Omega(f)})^{-j}(\alpha|_{\Omega(f)})).
\end{aligned}
$$

Therefore,

$$
ent(f,\alpha) \le ent(f|_{\Omega(f)}, \alpha|_{\Omega(f)}). \tag{3.7}
$$

It follows that

$$
ent(f) \le ent(f|_{\Omega(f)}). \tag{3.8}
$$

Then the Theorem follows from Lemma 3.1.3.

3.3 $\Omega(f|_{\Omega(f)}) = \overline{P(f)}$

Let $f \in C^0(X)$ where X is a compact metric space. Suppose that $\Omega(f)$ is the nonwandering set of f, $f|_{\Omega(f)}$ is the restriction of f to the nonwandering set $\Omega(f)$, and $\overline{P(f)}$ is the closure of the set of periodic points of f. If we put $\Omega_1(f) = \Omega(f)$, then we can define $\Omega_n(f) = \Omega(f|_{\Omega_{n-1}(f)})$

inductively. So clearly $\Omega_1(f) \supset \Omega_2(f) \supset ...$ is a decreasing sequence of closed subsets of X and the intersection $\cap_{n=1}^{\infty}\Omega_n(f)$ is denoted by $\Omega_\infty(f)$ and called the **center** of f. It is easy to find an example such that $\Omega_2(f) \neq \Omega(f)$ (cf.[Wa] or [Ni]). If $X = I$, Nitecki [Ni] proved that $\Omega_2(f) = \overline{P(f)}$ for f piecewise monotone, Zhou [Zh] proved the same result holds if $ent(f) = 0$. Xiong [Xi1] proved this result for any continuous mapping on the closed interval. From $\Omega_2(f) = \overline{P(f)}$, it is easy to see that for all $j \geq 2$, we have $\Omega_j(f) = \Omega_2(f) = \overline{P(f)}$. Hence $\Omega_\infty(f) = \overline{P(f)}$.

Theorem 3.3.1[Xi1] *Let f be a continuous mapping from I into itself. Then*

$$\Omega(f|_{\Omega(f)})=\overline{P(f)}$$

To prove Theorem 3.3.1, we need some notations and lemmas. Let $f \in C^0(I)$. For every positive integer n, we denote

by $A(+, f, n)$ the set of subintervals of I satisfying the following conditions:

$J \in A(+, f, n)$ iff $f^n(x) > x$ for all $x \in J$;

by $A(-, f, n)$ the set of subintervals of I satisfying the following conditions:

$J \in A(-, f, n)$ iff $f^n(x) < x$ for all $x \in J$;

by $SA(+, f)$ the set of subintervals of I satisfying the following conditions:

$J \in SA(+, f)$ iff $f^n(x) > x$ for all $x \in J$ and for all positive integer n;

by $SA(-, f)$ the set of subintervals of I satisfying the following conditions:

$J \in SA(-, f)$ iff $f^n(x) < x$ for all $x \in J$ and for all positive integer n.

Lemma 3.3.2 *Let $f \in C^0(I)$ and $J \subset I$ be an interval. Then the following statements are equivalent:*

(1) $J \cap P(f) = \emptyset$;

(2) $J \in \cap_{n=1}^{\infty}(A(+,f,n) \cup A(-,f,n))$;

(3) $J \in SA(+,f) \cup SA(-,f)$.

PROOF: It is immediately seen that "(3)$\Rightarrow$ (1)" and "(1) $\Rightarrow$(2)" by the related definitions. In the following, we shall prove that "(2) $\Rightarrow$ (3)".

Let $J \in \cap_{n=1}^{\infty}(A(+,f,n) \cup A(-,f,n))$ but $J \notin SA(+,f) \cup SA(-,f)$. Since $J \notin SA(+,f)$, there exist $x_1 \in J$ and $n_1 > 0$ such that $f^{n_1}(x_1) \in J$ and $f^{n_1}(x_1) < x_1$. Hence $J \in A(-,f,n_1)$, for some positive integer k, $J \in A(-,f,k \cdot n_1)$, then from $f^{n_1}(x_1) \in J$, $f^{n_1(k+1)}(x_1) = f^{n_1 \cdot k}(f^{n_1}(x_1))$ $< f^{n_1}(x_1) < x_1$ follows that $J \in A(-,f,n_1(k+1))$. Hence

$$J \in \cap_{k=1}^{\infty} A(-,f,kn_1).$$

Similarly, there exists a positive integer n_2 such that

$$J \in \cap_{k=1}^{\infty} A(+,f,kn_2).$$

Hence

$$J \subset A(+,f,n_1n_2) \cap A(-,f,n_1n_2) = \emptyset,$$

but from the definitions of $A(+,f,.)$ and $A(-,f,.)$, clearly we have

$$A(+,f,n_1n_2) \cap A(-,f,n_1n_2) = \emptyset.$$

A contradiction. #

Lemma 3.3.3 *Let $f \in C^0(I)$. If an interval $J \subset I$ is open and $J \cap P(f) = \emptyset$, then for every positive integer n, $J \cap f^n(J \cap \Omega(f)) = \emptyset$.*

PROOF: Assume that this lemma is false. Then there exists a point $x \in J \cap \Omega(f)$ and a positive integer n such that $f^n(x) \in J$ and for $0 < k < n$, $f^k(x) \notin J$. Clearly $f^i(x) \neq f^j(x)$ for $0 \leq i \neq j \leq n$.

Without loss of the generality, we can assume that $x < f^n(x)$. By Lemma 3.3.2 which implies $J \in SA(+, f)$. We can now choose the neighborhood U_i of $f^i(x)$ for each $i = 0, 1, ..., n$ such that $U_i \neq U_j$ for $0 \leq i \neq j \leq n$, and $U_n \subset J$. Set $V_0 = \cap_{i=0}^{n} f^{-i}(U_i)$, then V_0 is an neighborhood of x. Since $x \in \Omega(f)$, there exists $l > 0$ such that $V_0 \cap f^l(V_0) \neq \emptyset$. Obviously $l > n$. Since $f^n(V_0) \subset U_n$ and $f^{l-n}(U_n) \cap V_0 \neq \emptyset$, there exists $z \in U_n$ such that $f^{l-n}(z) \in V_0$. Hence $z > f^{l-n}(z)$. Since $J \in SA(+, f)$ we have a contradiction. #

Lemma 3.3.4 *Let $f \in C^0(I)$. Then for every $x \in \Omega(f)$ and every connected component C of $I \setminus \overline{P(f)}$, there is at most one nonnegative integer n such that $f^n(x) \in C$.*

PROOF: Assume that there exists a positive integer l such that $\{f^n(x), f^{n+l}(x)\} \subset C$. Set $y = f^n(x)$. By the definition of nonwandering set, we have that $y \in \Omega(f)$. But

$$f^l(y) \in C \cap f^l(C \cap \Omega(f)).$$

This is a contradiction. #

Proof of theorem 3.3.1: By the definitions of $P(f)$ and $\Omega(f)$, we have $\overline{P(f)} \subset \Omega(f|_{\Omega(f)})$. From Lemma 3.3.3 it follows that for every connected component C of $I \setminus \overline{P(f)}$, and for every nonnegative integer n,we have

$$C \cap f^n(C \cap \Omega(f)) = \emptyset.$$

which implies

$$C \cap (f|_{\Omega(f)})^n(C) = \emptyset.$$

Hence

$$C \cap \Omega(f|_{\Omega(f)}) = \emptyset.$$

By the definition of $\Omega(f)$, we have

$$\Omega(f|_{\Omega(f)}) \subset \overline{P(f)}.$$

This completes the proof. #

From Theorem 3.3.1 and Theorem 3.1.4[Bo], we can easily obtain the following Corollary:

Corollary 3.3.5 *Let f and I be the same as Theorem 3.3.1, then*

$$ent(f) = ent(f|_{\overline{P(f)}}).$$

3.4 Some relations between $P(f)$ and $ent(f)$

Bowen and Franks[BF] estimated the topological entropy of $f \in C^0(I)$.

Theorem 3.4.1 [BF] *Let $f \in C^0(I)$ and let f have a periodic point of period $n = 2^d \cdot p$, where d is an nonnegative integer and $p \geq 3$ is an odd integer, then*

$$ent(f) > \frac{1}{n} \log 2.$$

Under the same assumptions of Theorem 3.4.1, Stefan [Ste] got the following stronger result:

Theorem 3.4.2 [Ste] *If f satisfies the assumptions of Theorem 3.4.1, then*

$$ent(f) > \frac{1}{2^{d+1}} \log 2.$$

One may ask: can we get the best lower bound for the topological entropy of mappings in $C^0(I)$? The best possible lower bound for topological entropy of f in $C^0(I)$ is the following Theorem given by Block, Guckheimer, Misiurewicz and Yorke [BGMY]:

Theorem 3.4.3[BGMY] *Let $f \in C^0(I)$ and let f have a periodic point of period $2^n \cdot p$, where n is a nonnegative integer and $p \geq 3$ is is an odd integer , then*

$$ent(f) \geq \frac{1}{2^n} \log \lambda_p$$

where λ_p is the largest positive root of the polynominal $x^p - 2x^{p-2} - 1$.

Misiurewicz [Mi] proved the following well-known result:

Theorem 3.4.4 [Mi1] *If $f \in C^0(I)$ and the period of every periodic point of f is a power of 2, then $ent(f) = 0$.*

From Theorem 3.4.1 [BF] and Theorem 3.4.4 [Mi1], it follows immediately that

Theorem 3.4.5 *Let $f \in C^0(I)$, then $ent(f) = 0$ iff the period of every periodic point of f is a power of 2.*

In fact, before [BGMY] and [Mi1] Jonker and Rand [JR] proved the same results as in Theorem 3.4.2 and Theorem 3.4.3 hold for the smooth mappings on the interval.

The history of Theorem 3.4.4 is in the following: In 1978, Block [Bl2] considered this problem; In 1979, Misiurewicz [Mi1] gave an outline of a proof of this theorem; In 1982, Bloch declared that this result is one corollary of his theorem in [Blo], but his theorem included some serious mistakes (see a counterexample in [CX]); In 1985, Zhou got a new complete proof but it's complicated ;In 1989, Xiong [Xi2] gave a new and simple proof for it.

Chapter 4

Chaos

4.1 Introduction

The simplest non-trivial dynamical system that exhibits a“chaotic behavior” is the one governed by an continuous mapping $f : I \to I$ where I is a closed interval. Li and Yorke introduced the concept of “chaos”in [LY]. Following them, there are many papers on the “chaotic behavior”. The definition of “chaos” is not unique. In this thesis we only consider Li-Yorke chaos. In 4.2 we give a new proof of that positive topological entropy implies chaos; In 4.3 we will give some characterizations of chaos due to [LMPY2],in 4.4 we will give an example [Mi2] of a mapping which is chaotic everywhere, i.e., the chaotic set with full Lebesgue measure . However, there are examples (see [CX], [Sm3] and [MiS]) which are chaotic but their topological entropy is 0.These examples show that positive entropy (or the existence of a periodic point whose period is not a power of 2) is not equivalent to the chaocity of a mapping f.

4.2 Li-Yorke Chaos

The following definition of chaos was introduced by Li and Yorke [LY]: a continuous mapping $f : I \to I$ of a closed interval I is said to be chaotic if, there are points in $P(f)$ of arbitrarily large period and, there is an uncountable set $S \subset I$ such that no point in S is even asymptotically periodic. More generally we define the Li-Yorke chaos as follows:

Definition 4.2.1 *A continuous mapping $f : I \to I$ of a closed interval I is chaotic , if there is an uncountable set $S \subset I \setminus P(f)$ such that:*

(i) for every $x, y \in S$ with $x \neq y$,

$$\limsup_{n\to\infty} |f^n(x) - f^n(y)| > 0, \tag{4.1}$$

$$\liminf_{n\to\infty} |f^n(x) - f^n(y)| = 0; \tag{4.2}$$

and

(ii) for every $x \in S$ and every $p \in P(f)$,

$$\limsup_{n\to\infty} |f^n(x) - f^n(p)| > 0, \tag{4.3}$$

Such an S is called a chaotic set of f .

In their well-known paper " Period 3 implies chaos", Li and Yorke[LY] proved the following result:

Theorem 4.2.2: *Let $f \in C^0(I)$. If f has a periodic point of period 3, then f is chaotic .*

We will give a simple proof by using the method of symbolic dynamics (cf. [W]). Before giving the proof, we give a new proof of a theorem in [Bl2] which will be used in the proof of Theorem 4.2.2.

Theorem 4.2.3 [Bl2] *Let $f \in C^0(I)$. The following statements are equivalent:*

(1) f has a periodic point whose period is not a power of 2;

(2) There are disjoint closed intervals I_0 and I_1, and a positive integer n, such that

$$f^n(I_0) \cap f^n(I_1) \supset I_0 \cup I_1.$$

PROOF: We claim that: If $p \in pp(f)$ and $p \geq 3$ is odd, then there exists two closed intervals I_0 and I_1 such that $f^2(I_0) \cap f^2(I_1) \supset I_0 \cup I_1$ and $I_0 \cap I_1 = \emptyset$.

Now we prove above claim. By Stefan's Theorem (Theorem 1.2.5), without loss of the generality we may assume that there is a point $x \in I$ such that

$$f^{p-2}(x) < f^{p-4}(x) < ... < f^3(x) < f(x) < x <$$
$$< f^2(x) < f^4(x) < ... < f^{p-1}(x)$$

and $f^p(x) = x$.

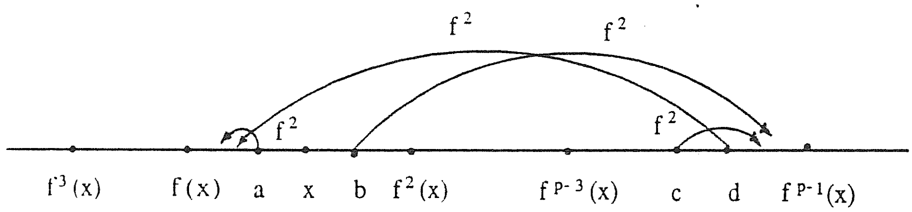

Figure 4.2.1

It is easy to see that

$$f^2([f(x), x]) \supset [f^3(x), f^2(x)]$$

Hence from the continuity of f^2, there exists $a \in (f(x), x)$ such that $a > f^2(a)$.From the figure 4.2.1, clearly

$$f^2([f^{p-3}(x), f^{p-1}(x)]) \supset [f(x), f^{p-1}(x)]$$

Hence there exists $d \in (f^{p-3}(x), f^{p-1}(x))$ such that $f^2(d) < a$.

Clearly

$$f^2([f^{p-3}(x), d]) \supset [a, f^{p-1}(x)]$$

So that there exists $c \in (f^{p-3}(x), d)$ such that $f^2(c) > d$.

Hence

$$f^2([a, f^{p-3}(x)]) \supset [a, f^{p-1}(x)].$$

So that there exists $b \in (a, f^{p-3}(x))$ such that $f^2(b) > d$.

Set $I_0 = [a, b]$ and $I_1 = [c, d]$. From above inequalities we have that

$$f^2(I_0) \cap f^2(I_1) \supset I_0 \cup I_1$$

and $I_0 \cap I_1 = \emptyset$. The proof of the claim is completed.

If f has a periodic point of period $2^n \cdot p$, where n is a positive integer and $p \geq 3$ is an odd integer, then f^{2^n} has a periodic point with period p. By the above claim there exist closed intervals I_0 and I_1 such that

$$f^{2^{n+1}}(I_0) \cap f^{2^{n+1}}(I_1) \supset I_0 \cup I_1$$

and $I_0 \cap I_1 = \emptyset$. Therefore ,the proof of "(1) $\Rightarrow$ (2)" is finished.

By Lemmas 2.2.2, 2.2.3 and 2.2.4, the proof of "(2) $\Rightarrow$(1)" is immediate. #

For proving Theorem 4.2.2, we use the method of symbolic dynamics. Σ_2 is called the *sequence space* on the two symbols 0 and 1, i.e., $\Sigma_2 = \{\mathrm{w} = (w_i)_{i=0}^{\infty} \mid w_i = 0 \text{ or } 1\}$. We may make Σ_2 into a metric space as follows. For any two sequences $(w_i)_{i=0}^{\infty}$ and $(v_i)_{i=0}^{\infty}$, define the distance between them by

$$d(\mathrm{w}, \mathrm{v}) = \sum_{i=0}^{\infty} \frac{|w_i - v_i|}{2^i}.$$

Obviously, this infinite series is convergence, and d is a metric of the space Σ_2. The shift mapping $\sigma : \Sigma_2 \to \Sigma_2$ is defined by $\sigma((w_i)_{i=0}^{\infty}) = ((w_i)_{i=1}^{\infty})$. We denote by $P(\sigma)$ the set of periodic points of σ. It is easy to check that:

(i) $\sigma : \Sigma_2 \to \Sigma_2$ is continuous;

(ii) $P(\sigma)$ is dense in Σ_2;

(iii) There exists a dense orbit of σ which is dense in Σ_2.

If $f \in C^0(I)$ has a periodic point of period 3, by Theorem 4.2.3 there exists disjoint closed intervals I_0 and I_1, and a positive integer n such that

$$f^n(I_0) \cap f^n(I_1) \supset I_0 \cup I_1.$$

For any $\mathrm{w} = (w_0, w_1, w_2, ...) \in \Sigma_2$, we define

$$\begin{aligned} I_{w_0 w_1 \ldots w_k} &= \{x \in I \mid \ x \in I_{w_0}, f^n(x) \in I_{w_1}, ..., (f^n)^k(x) \in I_{w_k}\} \\ &= I_{w_0} \cap (f^n)^{-1}(I_{w_1}) \cap ...(f^n)^{-k}(I_{w_k}) \end{aligned}$$

Note that the $I_{w_0 w_1 \ldots w_k}$ form a nested sequence of nonempty closed intervals as $n \to \infty$. It follows that

$$\cap_{k \geq 0} I_{w_0 w_1 \ldots w_k} \neq \emptyset.$$

Therefore $\cap_{k \geq 0} I_{w_0 w_1 \ldots w_k}$ is a closed interval or a single point. Note that if $x \in \cap_{k \geq 0} I_{w_0 w_1 \ldots w_k}$, then $x \in I_{w_0}, f^n(x) \in I_{w_1}$,etc. Hence the mapping $\phi : I_0 \cup I_1 \to \Sigma_2$ given by $\phi(x) = (w_0, w_1, w_2, ...)$ is surjective.

Note that $\mathrm{w} \neq \mathrm{v}$ implies that $(\cap_{k \geq 0} I_{w_0 w_1 \ldots w_k}) \cap (\cap_{k \geq 0} I_{v_0 v_1 \ldots v_k}) = \emptyset$. and Σ_2 is uncountable, hence there is a $\mathrm{w} \in \Sigma_2$ such that $\cap_{k \geq 0} I_{w_0 w_1 \ldots w_k}$ is a single point. If not, there are uncountable pairwise disjoint intervals in I, this is impossible (because the set of rational numbers is countable).

Another easy and useful fact is that $\phi \circ f^n = \sigma \circ \phi$. That is to say that ϕ is a semi-conjugacy between f^n and σ.

Now we can give the proof of Theorem 4.2.2.

PROOF OF THEOREM 4.2.2 [W]: Let $\mathbf{u} = (u_i)_{i=0}^{\infty} \in \Sigma_2$ such that $\cap_{k \geq 0} I_{u_0 u_1 \dots u_k}$ is a single point. For any $\mathbf{w} = (w_0, w_1, w_2, \dots) \in \Sigma_2$, we define

$$W_{\mathbf{w}} = u_0 w_0 u_0 u_1 w_0 w_1 u_0 u_1 u_2 w_0 w_1 w_2 \dots u_0 u_1 \dots u_k w_0 w_1 \dots w_k \dots .$$

For every $\mathbf{w} \in \Sigma_2$, we pick only one point $x(\mathbf{w}) \in \cap_{k \geq 0} I_{w_0 w_1 \dots w_k}$ and set $X = \{x(W_{\mathbf{w}}) \mid \mathbf{w} \in \Sigma_2 \setminus P(\sigma)\}$.

We claim that X is a chaotic set of f.

For any two $\mathbf{w}, \mathbf{v} \in \Sigma_2$ and $\mathbf{w} \neq \mathbf{v}$. Then there exists $r \in \mathbf{N}$ such that $w_r \neq v_r$. Therefore for any $m > r$, we have

$$|\phi \circ f^{n[r(r+1)-1]}(x(W_{\mathbf{w}})) - \phi \circ f^{n[r(r+1)-1]}\ (x(W_{\mathbf{v}}))| > \tfrac{1}{2^{r(r+1)-1}}$$

It follows

$$\limsup_{k \to \infty} |f^{n[k(k+1)-1]}(x(W_{\mathbf{w}})) - f^{n[k(k+1)-1]}\ (x(W_{\mathbf{v}}))| > 0$$

Hence (4.1) in Definition 4.2.1 is satisfied.

Since $\cap_{k \geq 0} I_{u_0 u_1 \dots u_k}$ is only one point, and $\phi \circ f^{nk(k+1)}(x(W_{\mathbf{w}}))$ converges to $\mathbf{u}$ for every $\mathbf{w} \in \Sigma_2$, it follows that

$$\lim_{k \to \infty} |f^{nk(k+1)}(x(W_{\mathbf{w}})) - f^{nk(k+1)}\ (x(W_{\mathbf{v}}))| = 0.$$

Hence (4.2) in Definition 4.2.1 is satisfied.

If $p \in P(f)$, then $\phi(p) \in P(\sigma)$. Clearly, for every $\mathbf{w} \in \Sigma_2 \setminus P(\sigma)$, $\limsup_{k \to \infty} |\phi \circ f^{nk}(x(W_{\mathbf{w}})) - \phi \circ f^{nk}(p)| > 0$.

Hence the (4.3) in Definition 4.2.1 is satisfied. #

By Sarkovskii's theorem and Theorem 4.2.1, it is easy to get the following theorem which was proved firstly by Oono[O.].

Theorem 4.2.4 *Let I and f be as above. If f has a periodic point whose period is not a power of 2, then f is chaotic.*

PROOF: Since f has a periodic point x with period $n \neq 2^i$ for every $i = 1, 2, ...$, then there exists a positive integer l and an odd integer $p > 1$ such that $2^l p$ is the period of x. By Sarkovskii's theorem we have that $f^{2^{l+1}}$ has a period point of period 3. By Theorem 4.2.2, $f^{2^{l+1}}$ is chaotic in the sense of Li-Yorke. It is easy to check that the chaotic set of $f^{2^{l+1}}$ is also a chaotic set of f in the sense of Li-Yorke.The proof of Theorem 4.2.4 is finished. #

4.3 No division implies chaos

From section 4.1, we know that if $f \in C^0(I)$ has a periodic point whose period is not the power of 2, then f is chaotic in the sense of Li-Yorke. In general, it is not easy to see if f has such a periodic point. Li, Misiurewicz, Pianigiani, and Yorke [LMPY 2] introduced the notation of " no division" to give a easy way to check the chaotic behavior of f.In the following we recall the definition of " no division" and sketch the main results in [LMPY 2].

Definition 4.3.1: *Let $f \in C^0(I)$ and $x_0 \in I$. $(x_0, x_1, ..., x_n)$ is called a trajectory if $x_{i+1} = f(x_i) \quad$ for $i = 0, 1, ..., n-1$.*

For such a trajectory $(x_0, x_1, ..., x_n)$ we say that there is no division if there is no $a \in I$ such that either

$x_j < a$ for all j even and $x_j > a$ for all j odd ,

or

$x_j > a$ for all j even and $x_j < a$ for all j odd.

By the definition, we have the following examples:

Example 4.3.2 Let $f \in C^0(I)$,and let $(x_0, x_1, ..., x_n)$ be a trajectory such that $x_0 < x_1 < ... < x_n$. If $n > 2$, then there is no division for the trajectory $(x_0, x_1, ..., x_n)$.

Example 4.3.3 Let $f \in C^0(I)$,and let $(x_0, x_1, ..., x_n)$ be a trajectory. If $n > 1$ is odd and $x_n \leq x_0 < x_1$ or $x_n \geq x_0 > x_1$, then there is no division for the trajectory $(x_0, x_1, ..., x_n)$.

Theorem 4.3.4 *Let $(x_0, x_1, ..., x_n)$ be a trajectory . If $x_n \leq x_0 < x_1$ or $x_n \geq x_0 > x_1$ for some integer $n \geq 3$, and there is no division for trajectory*

$(x_0, x_1, ..., x_n)$. *Then there exists a periodic point of period*

(i) n *if* n *is odd;*

(ii) $n-1$ *if* n *is even.*

PROOF: We only consider the case $x_n \leq x_0 < x_1$. The proof of the other case is similar to it. Let $X = (x_0, x_1, ..., x_n)$ and $S = \{x \in X \mid x > x_0, f(x) < x\}$. From $x_n \leq x_0 < x_1$, it is easy to see that $f(\max X) < \max X$ and $x_0 < f(x_0)$. So S is non-empty. Let $x_i = \min S$ and $x_j = \max\{x \in X \mid x < x_i\}$. Clearly $j \neq i \neq 0$. By the definition of x_i and x_j, we have that $f(x_i) < x_i$, $f(x_j) > x_j$ and $(x_j, x_i) \cap X = \emptyset$. Set $J = [x_j, x_i]$, we have $J \subset f(J)$ implies $f^k(J) \subset f^{j+1}(J)$ for any $k \in \mathbf{N}$, hence $\cup_{k=0}^{n-2} f^k(J) = f^{n-2}(J)$.

Since $x_n = f^{n-j}(X_j) \in f^{n-j}(J)$, $x_j \in J \subset f^{n-j}(J)$ and $x_n \leq x_0 < x_j$, we have that $x_0 \in f^{n-j}(J)$. Hence, every element of X except perhaps x_{j-1}(or x_{n-1} if $j = 0$) is in $\cup_{k=0}^{n-2} f^k(J) = f^{n-2}(J)$.

By similar arguments for x_i, we get that every element of X except perhaps x_{i-1} is in $f^{n-2}(J)$. Since $j \neq i$, we have $X \subset f^{n-2}(J)$.

Since there is no division, there exists an closed interval $K = [c, d]$ such that $c, d \in X$, $(c, d) \cap X = \emptyset$ and c and d lie on the same side of J but $f(c)$ and $f(d)$ lie on the opposite sides of J. Therefore $J \subset f(K)$ and $K \cap J$ contains at most one point . Hence we have

(1) $J \subset f(J)$,

(2) $J \subset f(K)$,

(3) $K \subset f^{n-2}(J)$.

From the above relations, the proof follows easily.

4.4 Chaos almost everywhere

In recent years, the "size" of the chaotic set has been studied in the one-dimensional dynamical systems. Smital gave an example of a chaotic set of full outer Lebesgue measure[Sm1] and an example of a chaotic set of positive Lebesgue measure [Sm2]. Kan [K] gave a example of a chaotic set with Lebesgue measure $\frac{1}{8}$. It is surprised that Misiurewicz [Mi2] and Bruckner and Hu [BH] discovered the examples of chaotic sets with full Lebesgue measure. Here we will construct one example of chaotic set of full Lebesgue measure due to above two examples.

It is reasonable to conjecture that there exists a chaotic set of full Lebesgue measure for some smooth mappings on the interval. But up to now, no such examples are constructed for C^1 mappings.

Based on the idea of [Mi2] and [BH] we will give an example in $C^0(I)$ with a chaotic set of full Lebesgue measure.

Theorem 4.4.1 *There exists a continuous mapping of the unit interval $I = [0,1]$ onto itself for which there exists a chaotic set of Lebesgue measure*

1.

The idea of the construction is as follows. For a standard "tend" mapping we construct (using symbolic dynamics) a sequence of chaotic sets. Their union is also a chaotic set and is dense. Each of them supports a non-atomic measure.A weighted average of these measures is a probabolistic non-atomic measure, positive on nonempty open sets. Then we transport everything (the mapping and the chaotic set) by an homeomorphism which sends this measure to the Lebesgue measure.

We denote by I the unit interval $[0,1]$, by J the right-open interval $[0,1)$.

Define the "tend"mapping $f : I \to I$ as follows:

$$f(x) = 1 - |2x - 1|.$$

Recall that Σ_2 is the space of all 0-1 sequences, σ is the shift on Σ_2, i.e.,$\sigma(x_i) = x_{i+1}$ if $\mathrm{x} = (x_i)_{i=0}^{\infty}$.

It is clear that for every $\mathrm{x} = (x_n)_{n=0}^{\infty} \in \Sigma_2$ there exists a unique point $p(x) \in I$ such that for all $n \in \mathbb{N}$,

$$f^n(p(x)) = \begin{cases} [0,1/2] & \text{if } x_n = 0, \\ [1/2,1] & \text{if } x_n = 1. \end{cases} \tag{4.4}$$

The mapping $p : \Sigma_2 \to I$ defined in such a way is continuous and $p \circ \sigma = f \circ p$.

For $\mathbf{x} = (x_0, x_1, x_2, ...) \in \Sigma_2$ and $A \subset \mathbf{N}$ such that $\mathbf{N} \setminus A$ is infinite, we define $F(A, \mathbf{x}) = (\underline{y_0}, \underline{y_1}, \underline{y_2}, ...) \in \Sigma_2$ as follows: we take the unique bijection $b : \mathbf{N} \setminus A \to \mathbf{N}$ such that b is order preserving and set

$$\underline{y_n} = \begin{cases} \underline{1_n} & \text{if } n \in A \\ \underline{x_{b(n)}} & \text{if } n \notin A \end{cases} \tag{4.5}$$

where $\underline{a_n} = (a_n, a_n, ..., a_n)$ is a block with n same elements a_n.
Fix an irrational number α. For $t \in J$ and $n \in \mathbf{N}$ we define

$$a(t,n) = \begin{cases} 0 & \text{if } \{t + n\alpha\} \in [0,1/2) \\ 1 & \text{if } \{t + n\alpha\} \in [1/2,1) \end{cases} \tag{4.6}$$

where $\{s\}$ is the fractional part of s. Then define the mapping $a : J \to \Sigma_2$ by $a(t) = (a(t,n))_{n=0}^{\infty} \in \Sigma_2$. In other words, $a(t)$ is the code of the point t, corresponding to the mapping $R_\alpha : t \to t + \alpha (mod\ 1)$ and the partition of J into the intervals $[0, \frac{1}{2})$ and $[\frac{1}{2}, 1)$.

The following fact is simple and well-known.

Lemma 4.4.2 *If $t, s \in J$ and $t \neq s$ then $a(t) \neq a(s)$.*

For each finite sequence $\mathrm{w} = (w_i)_{i=0}^{n-1} \in \{0,1\}^n$ and $t \in J$ we define $a_{\mathrm{w}}(t) \in \Sigma_2$ by setting $a_{\mathrm{w}}(t) = (y_i)_{i=0}^{\infty}$, where $y_i = w_i$ for $i = 0, 1, ..., n-1$ and $y_i = a(t, i-n)$ for $i = n, n+1, ...$ For w as above and a set A *subset*N such that $\mathrm{N} \setminus A$ is infinite, we define a mapping $G_{A,\mathrm{w}} : J \to I$ by $G_{A,\mathrm{w}}(t) = p(F(A, a_{\mathrm{w}}(t)).\#$

Lemma 4.4.3 *The mapping $G_{A,\mathrm{w}}$ is a Borel mapping.*

PROOF: The mappings $p, F(A, .)$ and the mapping which sends $a(t)$ to $a_{\mathrm{w}}(t)$, are continuous. Therefore $G_{A,\mathrm{w}}$ is continuous. Since the inverse image of a cylinder is an intersection of intervals, it is easy to see that the mapping $G_{A,\mathrm{w}}$ is Borel. #

For each positive integer n, define $A_n = \{n^2+1, n^2+2, ..., n^2+n\}$ and $B_n = \cup_{i=1}^{\infty} A_{i2^{n-1}}$. Clearly, the set $\mathrm{N} \setminus B_n$ is infinite.

Lemma 4.4.4 *For each finite 0-1 sequence* w *and each positive integer n, the set $G_{B_n,\mathrm{w}}(J)$ is a chaotic set of f.*

PROOF: For every $t \in I$, denote that $F(B_n, a_{\mathrm{w}}(t)) = (t_i)_{i=0}^{\infty}$. From Lemma 4.4.2 and the definition of $F(B_n, a_{\mathrm{w}}(t))$, it follows that if $s, t \in I$ and $s \neq t$, then $F(B_n, a_{\mathrm{w}}(s)) \neq F(B_n, a_{\mathrm{w}}(t))$. However there exist arbitrarily long finite sequences of consecutive integers $k, k+1, ..., k+K$ and $l, l+1, ..., l+L$ such that $s_i = t_i = 1$ for $i = k, k+1, ..., k+K$ and $s_i = 1$ with $t_i = 0$ (or $s_i = 0$ with $t_i = 1$) for $i = l, l+1, ..., l+L$, and such that s_{l-1}, t_{l-1} are either both 0 or 1. It implies that $|f^k(G_{B_n,\mathrm{w}}(t) - f^k(G_{B_n,\mathrm{w}}(s))| \leq \frac{1}{2^K}$ and $|f^l(G_{B_n,\mathrm{w}}(t) - f^l(G_{B_n,\mathrm{w}}(s))| \geq 1 - \frac{1}{2^{L-1}}$.

It follows that $G_{A,\mathrm{w}}$ is a chaotic set of f.#

As we know that the set of all finite 0-1 sequences is countable, so we can write a sequence $(\mathrm{w}(n))_{n=1}^{\infty}$ consisting of all of the finite $0-1$ sequences. Set $F_n(t) = F(B_n, a_{\mathrm{w(n)}}(t))$, $G_n = G_{B_n,\mathrm{w}(n)}$, $S_n = G_n(J)$ and $S = \cup_{n=1}^{\infty} S_n$.

By the same arguments as in the proof of Lemma 4.4.4, we have the following:

Lemma 4.4.5 *The set S is a chaotic set of f.*

Denote the Lebesgue measure on the unit interval I by λ. By Lemma 4.4, the image of λ under G_n is a Borel measure. Call this measure μ_n. Since G_n is Borel and $S_n = G_n(J)$, we have that $\mu_n(I) = \mu_n(S_n) = 1$. Because G_n is injective and λ is non-atomic, μ_n is also non-atomic. Therefore the measure $\mu = \sum_{n=1}^{\infty} 2^{-n}\mu_n$ is a non-atomic Borel measure and $\mu(I) = \sum_{n=1}^{\infty} 2^{-n}\mu_n(I) = \sum_{n=1}^{\infty} 2^{-n}\mu_n(S_n) = \sum_{n=1}^{\infty} 2^{-n} = 1$.

Lemma 4.4.6 *If U is an open non-empty set , then $\mu(U) > 0$.*

PROOF: If U is open and non-empty, so is the set $p^{-1}(U)$. Hence it contains some cylinder $C = \{(y_i)_{=0}^{\infty} \in \Sigma_2 \mid y_i = w_i \text{ for } i = 0, 1, ..., j\}$, where $\mathrm{w} = (w_i)_{i=0}^{\infty}$ is some 0-1 sequence. If m is large enough then the smallest element of B_m is large than j. Therefore there exists n such that w(n) begins by w [and consequently, $\{(y_i)_{i=0}^{\infty} \in \Sigma_2 \mid y_i = w_{i,n} \text{ for } i = 0, 1, ..., l\} \subset C$, where $\mathrm{w}(n) = (w_{i,n})_{i=0}^{\infty}$] and the smallest element of B_n is large than j. Then we have $F_n(J) \subset p^{-1}(U)$, and consequently $S_n \subset U$. Hence $\mu(U) \geq 2^{-n}$. #

Define a mapping $q : I \to I$ by $q(t)$=$\mu([0.t])$. By the Lemma 4.4.6, q is a strictly increasing mapping. Since μ is non-atomic, q is continuous. From the definition of q, we have that $q(0) = 0$ and $q(1) = \mu(I) = 1$. Therefore the following lemma holds.

Lemma 4.4.7 *The mapping q is a homeomorphism from I to itself.*

If $0 \leq c < d \leq 1$ and ν is the image of μ under q, then $\nu([q(c), q(d)])$= $\mu([c, d])$=$\mu([0, d]) - \mu([0, c])$ =$q(d) - q(c)$. From the above Lemma, we have that q maps I onto itself, so it follows that $\nu = \lambda$. Hence The image of μ under q is λ.

Now we define a mapping $g : I \to I$ by $g = q \circ f \circ q^{-1}$. Since the mappings q, f, and q^{-1} are continuous, g is continuous. By Lemma 4.4.7, q is a conjugacy between f and g. Therefore $pp(f) = pp(g)$. By Lemma 4.4.8 and $\mu(S) = 1$, we obtain that $\lambda(q(S)) = 1$.

By Lemma 4.4.5 and Lemma 4.4.7, it follows that the set $q(S)$ is a chaotic set of g.

Remark 4.4.8: We have in fact proved that for every $x, y \in q(S)$ with $x \neq y$,

$$limsup_{n\to\infty}|g^n(x) - g^n(y)| = 1.$$

which is stronger than (ii) in Definition 4.1.1.

References

[**AKM**] Adler,R., Konheim, A., and McAndrew, M, *Topological Entropy*,Trans. AMS **114**(1965), 309-319

[**ALS**] Alseda,L., Llibre,J.,and Serra,R., *Trans. AMS* **286***(1984)*, 595-627

[**ALM**] Alseda,L., Llibre,J.,Misiurewicz,M., *Periodic orbits of maps of Y*, Trans. AMS, **313**(1989),475-538.

[**Ba1**] Baldwin, S., *Generalizations of a Theorem of Sarkovskii on Orbits of continuous real-valued functions, Discret Mathematics 67*(1987), 111-127.

[**Ba2**] Baldwin, S. *Some limitations toward extending Sarkovskii's Theorem to connected linearly ordered spaces*, Preprint, 1987.

[**Ba3**] Baldwin,S., *An extension of Sarkovskii's Theorem to the n-od* , preprint, 1990

[**Be1**] Bernhardt, C. Simple permutation with order a power of two, Ergodic Theory and Dynamical Systems 4(1984), 179-186.

[**Be2**] Bernhardt, C. *The on permutaions induced by continuous maps of real line*, Ergodic Theory and Dynamical systems 7(1987), 155-60.

[**Bl1**] Block,L., *An example where topological entropy is continuous* Trans. AMS **231** (1977) 201-214

[**Bl2**] Block,L., *Homoclinic points of mappings of the interval* Proc. AMS **72** (1978) 576-580

[**Bl3**] Block,L., *Simple periodic points of mappings of the interval* Trans.

AMS **254**(1979), 391-398

[Bl4] Block,L.,*Stability of periodic points in the theorem of Sarkovskii*, Proc. AMS **81**(1981) 333-336

[BGMY]Block, L. Guckenhenimer, J. Misiurewicz, M. & Young, L. S. *Periodic points and topological entropy of one-dimensional maps* , Lecture Notes in the Mathematics 819(1980), 18-34.

[Bo] Bowen,R., *Topological entropy and Axiom A*, Proc. Symp. Pure Math. **14**(1970), 23-34

[BoF] Bowen, R., and Franks,J.,*The periodic points of maps of the disc and interval*, Topology **15** (1976) 337-342

[BP] Bulter G.J., and Pianigiani, G., *Periodic points and chaotic functions in the unit interval*, Bull. Austra. Math. Soc. **18**(1978), 255-265

[CP] Collect,P., and Eckmann,J.-P., *Iterated maps of the interval as dynamical systems*, Progress in Physics 1 (Birkhuser, 1980)

[CX] Chu,H.,and Xiong,J., *A counterexample in dynamical systems of the interval*, Proc. AMS, **97**(1986), 361-366

[D] Devaney, R.L., *An introduction to chaotic dynamical systems*, Benjamin/Cummings Publishing Co. (1986)

[Gu] Guckenheimer,J., *The growth of topological entropy for one-dimensional maps*, Comm. Math. Phys. **70**(1979) 133-160

[Ga] Gawel,B., *On the theorem of Sarkovskii and Stefan on cycles*, Proc. AMS **107**(1989), 125-132

[I] Imrich,W, *Periodic points of small periods of continuous mappings of trees*, Ann.Discr.Math. 27(1985).443-446

[JaS] Jankova, K.,Smital,J., *A characterization of chaos* Bull. Austral.

Math. Soc. **34**(1986),283-292

[**K**] Kan,I., *a chaotic function possessing a crambled set of positive Lebesgue measure*, Proc. AMS **92**(1984), 45-49

[**LMPY1**] Li,T.Y.,Misiurewicz,M., Pianigiani,G.,and Yorke,J., *Odd chaos*, Phys. Lett. **87**(1982),271-273

[**LMPY2**] Li,T.Y.,Misiurewicz,M., Pianigiani,G.,and Yorke,J., *No division implies chaos*, Trans. AMS, **273**(1982), 191-199

[**LY1**] Li,T.Y.,and Yorke,J.A., *periodic three implies chaos*, Amer. Math. Monthly **82**(1975), 985-992

[**Ma**] May,R., *Simple mathematical models with very complicated dynamics*, Nature **261**(1976) 459-467

[**MT**] Minc, P. & Trasue, W. R. R. *Sarkovskii's Theorem for hereditarily decomposable chainable continuo*, to appear in transactions A. M. S.

[**MN**] Misiurewicz, M. & Nitecki, Z. *Combinatorial patterns for maps of an interval*, Preprint, 1988.

[**Mi1**] Misiurewicz, M., *Horseshoes for mappings of the interval*, Bull. Polon. Sci. Ser. Sci. Math. **27** (1979), 167-169

[**Mi2**] Misiurewicz, M., *Chaos almost everywhere*, Iteration Theory and Its Functional Equations(editor Liedl et al.), Lecture Notes in Math. 1163 (Springer 1985)

[**Mi3**] Misiurewicz, M., Ergod. Th. and Dynam. Sys. **2**(1982), 125-129

[**MiS**] Misiurewicz, M.,and Smital,J., *Smooth chaotic maps with zero topological entropy*,Ergod. Th. and Dynam.Sys. 8 (1988), 421-424

[**Ni**] Nitecki, Z., Proc. AMS, **80**(1980), 511-514

[**Sa**] Sarkovskii, A. N. *Coexistence of cycles of a continuous map of the line* *into itself*(Russian), Ukraine Math Z. 16(1964), 61-71.

[Sc] Schirmer, H. *A topologists view of Sharkovsky's Theorem*, Houston, Journal of Mathematics 11 (1985), 385-395.

[Sm1] Smital,J., *A chaotic function with extremal properties*, Proc. AMS **87** (1983),54-56

[Sm2] Smital,J., *Achaotic function with a crambled set of positive Lebesgue measure* Proc. AMS **92** (1984), 50-54

[Sm3] Smital,J., *Chaotic functions with zero topological entropy*, Trans. AMS **297** (1986),269-282

[Ste] Stefan, P. *A Theorem of Sarkovskii on the existence of periodic orbits of continuous endomorphisms of the real line*, Comm. Math. Physics 54 (1977), 237-248.

[Str] Straffin, P. D. *Periodic points of continuous functions*, Math. Mag. 51(1978), 99-105.

[W] Wu, H., *On topological entropy and chaotic set* J. of China University of Sci. and Tech. **1**(1989), 118-122

[Wa] Waters, P., *An Introduction to Ergodic Theory*, Springer-Verlag (1982)

[Xi1] Xiong, J.C., $\Omega(f|_{\Omega(f)}) = \overline{P(f)}$ *for every continuous self-map* f *of the interval*, Kexue Tongbao **28**(1983), 21-23

[Xi2] Xiong, J.C., *A simple proof for Misiurewicz's theorem*, J. of China University of Sci. and Tech. **1**(1989),21-24

[Zh] Zhou, Z.L., Annals of Math. ,4(1983), 732-736