ISAS - INTERNATIONAL SCHOOL
FOR ADVANCED STUDIES

Generalized KdV Structures in CFT and 2d Quantum Gravity

Thesis Submitted for the Degree of

Magister Philosophiae

Candidate:

Chuansheng Xiong

Supervisor:

Prof. Loriano Bonora

Academic Year 1989/90

TRIESTE

SISSA - SCUOLA
INTERNAZIONALE
SUPERIORE
I STUDI AVANZATI
TRIESTE
Strada Costiera 11

Acknowledgements

I would like to take this opportunity to express my deepest gratitude to my supervisor L.Bonora for his guidance in my master studies. His constant support for my studies, like providing references and discussing confusing points, is greatly acknowledged. I would also like to thank my colleagues for discussions and a lot of interesting seminars which are useful in some cases. Thank also goes to Dr. Gava, he has read the fourth Chapter of this thesis, and Dr. C. J. Zhu for many very useful discussions and comments on this thesis. At last but not least I also thank Dr. Zhu yuan for his kindly help of typesetting.

Index

Chapter 1
Introduction

A few years ago, the main interest of specialists in fundamental physics had concentrated on the theory of strings and superstrings[1]. This theory was considered as a good candidate for unified theory of all interactions of nature. Many works had been done on this subject. Its perturbation theory was successful, however, further investigations met some difficulties. The efforts to overcome these difficulties lead to the flourishing of the several subjects in this years, which are conformal field theory (CFT)[2–7] and quantum gravity[8–11], as well as integrable system[12–15]. One of their common features which is of great importance is the underlying KdV structure, due to this property, one can identify some of these theories.

On the one hand, it is well–known that the vacua of the string theory correspond to conformally–invariant field theories. The clasification of CFT (particularly, Rational Conformal Field Theory(RCFT)) is one of the outstanding problems in string theory as well as statistic physics. A promising approach to this problem is provided by the so–called W–algebra, which is a richer symmetry than conformal invariance, and includes the Virasoro algebra as subalgebra, and therefore, is also called the extended Virasoro algebra[13,16–25]. The classical version of the W–algebra is deeply related to Integrable model[13], and it seems to be the symmetry of matrix model and the theory of CFT coupled to quantum gravity.

On the other hand, after successful study of the string theroy in critical dimension, a natural and physically more important problem is the string theory in non–critical dimension, which is in fact a theory of some matter fields coupled to 2d–quantum gravity[8,26,27]. Its discretization[28–32] is believed to be equivalent to a certain matrix model in double scaling limit[33–37], in which the specific heat satifies the string equation that is nothing but the initial condition of the generalized KdV equation. So, the three subjects, W–algebra and 2d–quantum gravity, as well as integrable system (the generalized KdV equations) are closely related, which are the topics of This thesis.

In chapter 2, we will consider the classical W–algebra. We at first introduce Drinf'eld–Sokholov linear system, analyze its symmetries:gauge symmetry and W–symmetry, then, we introduce the Gelfand–Dickii Poisson brackets[13,38] and W–algebra, as well as KdV equation. In order to simplify the expression of W–algebra, we give two new sets of independent coordinates which are conformal tensors. Then, we present explicit formulae for gauge invariant functions and the covariant operators in these gauges.

In chapter 3, we will show that the classical version of null vector equation in CFT is equivalent to a certain DS–sytem, therefore, there exists a KdV structure in semi–classical conformal field theory.

In chapter 4, we mainly give a short introduction of 2d–quantum gravity. There are three ways to attach the problem. The first one is the so–called Matrix Model approach. The second one is path integral method. The third one is the topological field theory[39,40,41]. Utill now, we only can calculate the critical exponents of the specific heat, which is determined by the string equation. What is the exact relation to KdV equation, therefore, to Drinf'eld–Sokholov linear system, is still lack of clear explanations.

Chapter 2

W–Algebra

W–algebra becomes more and more important in theoretical physics due to the following four reasons, firstly, it serves as a useful tool in the investigation of Integrable systems, secondly, it provides a promising approach to classify all rational conformal field theories, thirdly, it is of mathematical interest itself, finally, it seems to be the fundamental symmetry of matrix model and some other theories.

Historically, the classical version of W–algebra stems from the study of Integrable System which has the infinite number of the conserved quantities. It is well–known that the above Integrable System admits a Lax pair representation, whose linearized form we will call the Drinf'eld–Sokholov[13,42] linear system. As such a system, Gelfand–Dickii constructed one Poisson bracket[43], which is called now the first Gelfand–Dickii Poisson bracket. Then, Alder[44] conjetured and constructed another different Poisson bracket, but the proof of its associativity was given by Gelfand–Dickii(unpublished, see[45]). This Poisson bracket is usually called the second Gelfand–Dickii Poisson bracket. With respect to the second Gelfand–Dickii Poisson bracket, the functions on phase space form a closed algebra, which is W–algebra. It is also called the extended Virasoro algebra, since it contains the Virasoro algebra as a subalgebra, in addition, it also contains a set of higher spin conserved currents. Its quantized version was cosidered by Zamolochikov at rank–

3[16], then it was generalized by several people[17,18]. In this thesis, we will only consider the classical case. We at first introduce the Drinf'eld–Sokholov linear system, then, define the Poisson brackets, deduce the generalized KdV equation, furthermore, we discuss DS–system in sl(2) and sl(n+1) cases in detail and give some comments on their relations.

§2.1 the DS–System

Let $\mathcal{G}$ be a simple finite dimensional Lie algebra with rank n, equipped with an invariant scalar product $<\ ,\ >$.$\mathcal{H}$ denotes Cartan sub-algebra, $\mathcal{N}_\pm$ denote the sets of elements with positive (negative) roots. We have the following Cartan decomposition

$$\mathcal{G} \equiv \mathcal{N}_- \oplus \mathcal{H} \oplus \mathcal{N}_+$$

Choosing the Cartan-Weyl basis, we have the following standard commutation relations

$$[H_i,\ H_j] = 0,$$

$$[H_i, E_{\pm\alpha}] = \pm\alpha_i E_{\pm\alpha},$$

$$[E_\alpha,\ E_\beta] = \begin{cases} H_\alpha, & \alpha+\beta = 0 \\ N_{\alpha\beta}E_{\alpha+\beta}, & \alpha+\beta \in \Gamma \\ 0, & \alpha+\beta \notin \Gamma \end{cases}$$

Here we adopt the normalization $< E_\alpha, E_{-\alpha} >= 1$

The Drinf'eld-Sokholov system has the following form

$$(\partial + L - I_+)\xi = 0 \tag{2.1a}$$

$$L \in \mathcal{H} \oplus \mathcal{N}_- \tag{2.1b}$$

where ξ is a (n+1)–dimensional column vector

$$\xi = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_{n+1} \end{pmatrix}$$

If we eliminate out all the components of the ξ but ξ_1, then we obtain a higher order differential equation for ξ_1

$$A\xi_1 = 0 \tag{2.2}$$

where the operator A has the form

$$A = \partial^{(n+1)} + \sum_{i=1}^{n} W_i(L)\partial^{n-i} \tag{2.3}$$

This equation has (n+1)–independent solutions, and by using these independent solutions $(\xi_1, \xi_2, \ldots, \xi_{(n+1)})$, we can re–express the Drinf'eld–Sokholov system as[13]

$$A\xi = det \begin{pmatrix} \xi & \xi_1 & \cdots & \xi_{n+1} \\ \xi' & \xi_1' & \cdots & \xi_{n+1}' \\ \vdots & \vdots & \ddots & \vdots \\ \xi^{(n+1)} & \xi_1^{(n+1)} & \cdots & \xi_{n+1}^{(n+1)} \end{pmatrix} = 0 \tag{2.4}$$

here

$$A\xi_i = 0, \qquad\qquad i = 1, 2, \ldots, (n+1)$$

So, all the functions W(L) can be expressed in terms of ξ_i's, therefore, the differential equation(2.2) is equivalent to the DS–system (2.1).

The DS–system (2.1) has a large symmetry, i.e. there exists a transformation group formed by some matrix g, such that the form of the linear system (2.1) is invariant under the following transformation

$$\begin{cases} \xi \longrightarrow \tilde{\xi} = g^{-1}\xi \\ L \longrightarrow \tilde{L} = g^{-1}(L - g' + I_+)g - I_+ \end{cases} \tag{2.5}$$

At first, it is easy to check that $\tilde{L} \in \mathcal{H} \oplus \mathcal{N}_-$, if

$$g = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ * & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ * & \cdots & * & 1 \end{pmatrix} \in \exp(\mathcal{N}_-) \tag{2.6}$$

Furthermore, due to the equivalence of the systems (2.1) and (2.2), we can show that the functions W(L) and ξ_1 are also invariant under this transformation, therefore, it preserves the W–algebra. We will call the transformation given by the eq.(2.5) and (2.6) as gauge symmetry. Obviously, its degree of freedom is $\frac{1}{2}n(n+1)$. Generally speaking, the elements of the lower triangular marix L can be considered as the coordinates of the system, their total number is $\left(\frac{(n+1)(n+2)}{2}-1\right)$. So, when we make a complete gauge fixing, only n–coordinates will survive, which is equal to the number of the degree of the freedom of the the system. Therefore, with the help of the gauge symmetry, we can choose appropriate gauge transformation such that the transformed matrix L just has the desired form.

There are two ordinary gauges, one is the diagonal gauge in which the matrix L lies on the Cartan subalgebra.

$$L=\begin{pmatrix} h_1 & 0 & \dots & 0 \\ 0 & h_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & h_{n+1} \end{pmatrix} \tag{2.7a}$$

where

$$\sum_{i=1}^{n+1} h_i = 0 \tag{2.7b}$$

The operator A can be easily written as

$$A=(\partial+h_{n+1})\dots(\partial+h_2)(\partial+h_1) \tag{2.8}$$

The other is the the DS–gauge, in which the matrix takes the following form

$$L=\begin{pmatrix} 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & 0 \\ W_n & \dots & W_2 & W_1 & 0 \end{pmatrix} \tag{2.9}$$

Then, the operator A has the simple form shown in equation(2.3).

Now, let us consider another kind of symmetry. In order to see it explicitly, let us choose the sl(2) case as an example . The most general form of L is

$$L=\begin{pmatrix} h_1 & 0 \\ h_2 & -h_1 \end{pmatrix} \tag{2.10}$$

with some gauge condition (since the degree of freedom of this system is 1, so, h_1 and h_2 are not independent). Now, we define

$$J \equiv L - I_+ = \begin{pmatrix} h_1 & -1 \\ h_2 & -h_1 \end{pmatrix} \tag{2.11}$$

One can show that there exists an infintesimal transformation

$$\begin{cases} g = 1 + R \\ R = aH + bE_+ + eE_- \end{cases} \tag{2.12}$$

Such that the form of J remains unchanged, i.e.

$$\begin{aligned} \delta J = R' + [J, R] \qquad\qquad & \in \mathcal{H} \oplus \mathcal{N}_- \\ =(a' - e - bh_2)H + (b' + 2a + 2bh_1)E_+ & + (e' - 2eh_1 + 2ah_2)E_- \end{aligned} \tag{2.13}$$

Provided

$$b' + 2bh_1 + 2a = 0 \tag{2.14}$$

So,

$$R = b(h_1 H - E_+) - \frac{1}{2} b' H + eE_- \tag{2.15}$$

and

$$\begin{cases} \xi_1 \longrightarrow \tilde{\xi}_1 = (1 + bh_1 - \frac{1}{2} b')\xi_1 - b\xi_2, \\ h_1 \longrightarrow \tilde{h}_1 = h_1 + (-\frac{1}{2} b'' - (bh_1)' - e - bh_2), \\ h_2 \longrightarrow \tilde{h}_2 = h_2 + (e' - 2eh_1 - b' h_2 - 2bh_1 h_2). \end{cases} \tag{2.16}$$

Generally, the field b depends on h_1, h_2, and relates to the field e (due to the gauge condition). The transformation (2.5) and (2.12) is not the same as the gauge transformation we showed before, since it doesn't preserve W(L) and ξ_1, therefore, it doesn't preserve the W–algebraic structure. However, it keeps the gauge unchanged. thus, it is the residual symmetry in the physical phase space spanned by the n–independent coordinates after complete gauge fixing. We will call it as W–symmetry.

As a example, let us consider the following system in the DS–gauge

$$\left(\partial + \begin{pmatrix} 0 & -1 \\ u & 0 \end{pmatrix} \right) \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0 \tag{2.17}$$

It is easy to prove that the transformation

$$R = \begin{pmatrix} \frac{1}{2}\epsilon' & -\epsilon \\ \epsilon u + \frac{1}{2}\epsilon'' & -\frac{1}{2}\epsilon' \end{pmatrix} \tag{2.18}$$

keeps the gauge unchanged, since

$$\delta J = \begin{pmatrix} 0 & 0 \\ 2u\epsilon' + u'\epsilon + \frac{1}{2}\epsilon''' & 0 \end{pmatrix} \tag{2.19}$$

This is exactly the infinitesimal conformal transformatiom

$$u \longrightarrow \tilde{u} = u + 2u\epsilon' + u'\epsilon + \frac{1}{2}\epsilon''' \tag{2.20}$$

For the same reason, for arbitrary intergers n, the transformation which preserves the forms of L, but does not belong to $\exp \mathcal{N}_-$, has the form

$$\begin{cases} g = 1 + R \\ R = \sum_{1 \leq i \leq n} b_i E_{i,i+1} + \sum_{1 \leq i \leq n+1} a_i H_i + r \end{cases} \tag{2.21}$$

where $r \in \mathcal{N}_-$ and

$$\delta J \in \mathcal{N}_- \oplus \mathcal{H}$$

Provided

$$b_i' + (h_i - h_{i-1}) b_i + a_i - a_{i-1} = 0, \qquad i = 1, 2, \ldots, n \tag{2.22}$$

Since we always can adjust the fields a_i's in order to satisfy the above conditions, so, these transformations have n–independent parameters.

We should notice that this particular type of gauge symmetry implies the existence of some Poisson brackets

$$\delta J = R' + [J, R] \equiv \int dx \{J, W(x)\} \epsilon(x) \tag{2.23}$$

where W(x) is the implicit generator of the transformation.

§2.2 Gelfand-Dickii Poisson Bracket Structure

In this section, we will mainly discuss how we can get W-algebra from the Gelfand-Dickii Poisson structure, which is based on the algebra of the differential operators. This algebra $\wp$ has been studied intensively for a long time. There are some very nice reviews on this subject, for examples[13,38,42,46]. Hereafter, we will adopt the convention in [13]. The algebra $\wp$ contains the following kind of objects

$$A = \sum_{-\infty}^{N} a_i(x)\partial_i \tag{2.24}$$

where $a_i(x)'s$ are one variable functions, and ∂_x is the derivative, and the integration operation is denoted by the operator ∂^{-1}

$$\partial^{-1}a(x) \equiv \int_{x_0}^{x} dx' a(x')$$

The algebra is equipped with the Leibnitz rules, and its generalized version for pseudo-differential operators

$$\begin{cases} \partial a = a\partial + a' \\ \partial^{-1}\partial = \partial\partial^{-1} = 1 \\ \partial^{-j-1}a = \sum_{v=0}^{\infty}(-1)^v \left(\binom{j+v}{v}\right) a^{(v)}\partial^{-j-v-1} \end{cases} \tag{2.25}$$

There are two subalgebras

$$\wp_+ = \{A = \sum_{0}^{N} a_i(x)\partial^i\}$$

and

$$\wp_- = \{A = \sum_{-\infty}^{-1} a_i(x)\partial_i\}$$

So, the algebra $\wp$ has a direct sum decomposition as a vector space.

$$\wp = \wp_+ \oplus \wp_-$$

One can define the inner scalar product on the algebra $\wp$, which is also called the trace on $\wp$ (and $a_1(x)$ is known as the residual of the operator A)

$$< A >= \int dx a_{-1}(x) \tag{2.26}$$

Then, let $X \in \wp_-$, and A takes the form

$$X = \sum_{i=0}^{\infty} \partial^{-i-1} \chi_i(x) \tag{2.27a}$$

$$A = \partial^{(n+1)} + \sum_{i=1}^{n} W_i \partial^{n-i} \equiv \sum_{i=-1}^{n} W_i \partial^{n-i} \tag{2.27b}$$

On the one hand, by some general mathematical theorems[38,42,46], the (n+1)–th root of the operator A defined by (2.27b) is well defined, that is to say, there exists a unique pseudo–differential operator L satisfying

$$\begin{cases} A = L^{(n+1)} \\ L = \partial + \alpha_1 \partial_1 + \alpha_2 \partial_2 + \dots \end{cases} \tag{2.28}$$

The operator L is very convenient object, from now on, we will often use the the fractional powers of the operator A (that is just powers of L). On the other hand, from (2.26), one gets

$$\begin{aligned} f_X(A) &= < AX > \\ &= \int dx \left(\chi_{n+1}(x) + \sum_{i=0}^{n-1} \chi_i(x) W_{n-i}(x) \right) \end{aligned} \tag{2.29}$$

It forms a functional space on which the Kirillov bracket is defined as[13]

$$\{f_X, f_Y\}(A) = < A, [\, df_X,\, df_Y] > \tag{2.30}$$

It is easy to show that

$$df_X = X \tag{2.31}$$

Therefore, the above bracket also can be written as[38]

$$\{f_X, f_Y\}_1(A) = < AXY > - < AYX > \tag{2.32}$$

This is the so-called first Gelfand-Dickii Poisson bracket. Some detailed mathematical analysis shows that there exists another Poisson structure[44] (in fact, the above equation is ill–defined for some special kind of operators, we will discuss this point in §2.7.2)

$$\{f_X, f_Y\}_2(A) =< (AX)_+AY > - < (XA)_+YA > \\ - \frac{1}{n+1}\int_0^{2\pi} dx((\partial_{-1}[A,X]_{-1})[A,Y]_{-1}) \tag{2.33a}$$

where

$$< (AX)_+AY > = \sum_{\substack{-1\le k\le n-i-1 \\ l\ge n-j}} (-1)^{2n-m}\binom{n-k-i-1}{j+l-n}\mathcal{J}^m_{ijlk}$$

$$\begin{aligned} < (XA)_+YA > = & \sum_{\substack{-1\le l\le n-i-1 \\ -1\le k\le n-i-1}} (-1)^{n-k+j} S_{ij}^{n-l,n-k}\mathcal{J}^m_{ijlk} \\ & + \sum_{-1\le l\le n-i-1,} \sum_{n-k\le i,j} \binom{n-l-j-1}{i+k-n}\mathcal{J}^m_{ijlk} \\ & + \sum_{\substack{-1\le l\le n-i-1 \\ n-j\le k\le n-i-1}} (-1)^{2n-m}\binom{n-k-i-1}{j+l-n}\mathcal{J}^m_{ijlk} \end{aligned} \tag{2.33b}$$

$$\int dx((\partial_{-1}[A,X]_{-1})[A,Y]_{-1}) = \sum_{\substack{-1\le l\le n-i \\ -1\le k\le n-j}} (-1)^{n-k+j+1}\binom{n-l}{i}\binom{n-k}{j}\mathcal{J}^m_{ijlk}$$

where, we define

$$\begin{cases} m = 2n-k-l-i-j-1, \\ S_{ij}^{lk} = \sum_{s\ge 0}(-1)^s\binom{l}{i+s+1}\binom{k+s}{j} \\ \mathcal{J}^m_{ijlk} = \int dx\,(x_iW_l)\partial^m(y_jW_k) \end{cases} \tag{2.33c}$$

This is the second Gelfand-Dickii Poisson bracket. Drinf'eld and Sokholov have proved that it is precisely the Poisson bracket defined by the eqs.(2.23).

It is straightforward to give the explicit formulas for the first few cases.

$$\{W_1(x), W_1(y)\} = -\left(\frac{1}{2}\binom{n+2}{3}\partial_x^3 + 2W_1(x)\partial_x + W_1'(x)\right)\delta(x-y), \tag{2.34a}$$

$$\{W_1(x), W_2(y)\} = \left(\binom{n+2}{4}\partial^4 - 3W_2\partial - 2W_2' + (n-1)\partial^2 W_1\right)\delta(x-y), \tag{2.34b}$$

$$\{W_1(x), W_3(y)\} = -\left(\frac{3}{2}\binom{n+2}{5}\partial^5 + \frac{1}{6}\binom{n-1}{2}\partial^3 W_1 - \frac{3}{2}(n-2)\partial^2 W_2\right.$$

$$+ 4W_3\partial + 3W_3'\Big)\delta(x-y), \tag{2.34c}$$

$$\{W_1(x), W_4(y)\} = \left(2\binom{n+2}{6}\partial^6 + \frac{1}{4}(n+4)\binom{n-1}{3}\partial^4 W_1 + 2(n-3)\partial^2 W_3\right.$$

$$\left. - \frac{1}{6}(n+8)\binom{n-2}{2}\partial^3 W_2 - 5W_4\partial - 4W_4'\right)\delta(x-y). \tag{3.34d}$$

Here, all the functions appearing in the right hand side are evaluated at the point x(and hereafter, without specific indication, all the functions and the derivatives appeared in the right hand side of the Poisson bracket are valued at the point x.), and the derivative ∂_x acts on fields and δ-functions as an operators in the following way

$$\partial_x^2 W_1(x)\delta(x-y) = W_1''(x)\delta(x-y) + 2W_1'(x)\delta_x'(x-y) + W_1(x-y)\delta_x''(x-y) \tag{3.35}$$

The first Poisson bracket is just the semi–classical Virasoro algebra. And all of the coordinates are not conformal tensors.

On the other hand, from the equation(2.4), we know that all of the W–functions can be expressed in terms of the (n+1)–independent solutions to the higher order differential equation(2.2), this in fact gives one new gauge of the DS–system(2.1) in which the coordinates are $\xi_1, \xi_2, \ldots, \xi_{n+1}$. Therefore, we can consistently define the Poisson brackets among these coordinates so that the Poisson brackets among W–functions are in coincidence with the ones given in the above. Furthermore, we also can calculate the Poisson brackets between W–functions and ξ_i's, which are

$$\begin{aligned}&\{f_X(A), \xi_i(x)\}\\ &= -\sum_{\substack{k,j\\ s\geq 0}} (-1)^{n+j+s-k}\binom{n-k-s-1}{j}\partial^{n-k-s-j-1}(x_j W_k)\partial^s \xi_i(x)\\ &+ \frac{1}{n+1}\sum_{k<n-j}(-1)^{n+j-k}\partial^{n-k-j-1}(x_j W_k)\xi_i(x)\end{aligned} \tag{2.36}$$

In particular, if

$$f_X(A) = \int dx \epsilon(x) W_1(x) \tag{2.37}$$

then, we get conformal transformation law of ξ_i

$$\{f_X(A), \xi_i(x)\} = \epsilon\xi_i^{'}(x) + \Delta_i\epsilon^{'}\xi_i(x) \tag{2.38}$$

where, Δ_i is known as the conformal weight.

§2.3 The generalized KdV Equation

After one constructed the two basic Poisson structures, then, there are many subjects to be discussed. For example, we can try to find the conserved quantities, the involution equations, and Yang–Baxter relations , etc. Once we consider the involutions, we will see that these two Poisson brackets are related to each other. At first, one can show that with respect to two Poisson brackets, the conserved quantities are as follows

$$f_Y(A) = \frac{n+1}{n+k+1} < A^{\frac{k}{n+1}+1} > \equiv h_k(A) \tag{2.39}$$

One can prove that the tangent vector is

$$Y = (A^{\frac{k}{n+1}})_{(-n,-1)} \tag{2.40}$$

From the eqs.(2.32) and (2.33), one can show that

$$\{h_k, h_l\}_1(A) = 0 \tag{2.41}$$

$$\{h_k, h_l\}_2(A) = 0 \tag{2.42}$$

Now, suppose we choose the first Poisson bracket, and correspodingly we choose h_{n+k+1} as a hamiltonian. For simplicity, let X be time-independent, then we have

$$\begin{aligned} \dot{f}_X(A) &=< \dot{A}X >= \{f_X, h_{n+k+1}\}_1(A) \\ &=< [(A^{\frac{k}{n+1}+1})_+, A], X > \end{aligned} \tag{2.43}$$

In a straightforward manner, one can deduce the involution equation of the operator A

$$\dot{A} = [(A^{\frac{k}{n+1}+1})_+, A] \tag{2.44}$$

This is nothing but the generalized KdV equation in the Lax pair formalism. On the other hand, if we use the second Gelfand-Dickii Poisson bracket but choose $h_k(A)$ as a hamiltonian, we also get the same dynamic equation for A.

§2.4 DS–System in sl(2) Case

Till now, we have introduced the W–algebra through the the DS–linear system, which can be reconstructed from the differential operator A with the help of the Gelfand-Dickii second Poisson bracket. Then, when choosing the suitable hamiltonian, we get the dynamic equation of the operator A, which is just generalized KdV equation in Lax formalims. Now, we will mainly try to show the explicit algebraic structure. Firstly, we will introduce the covariant sl(2) operators, and give its explicit formulas. Then, we consider the general cases, focusing our attention on the algebra sl(n+1). Two new gauges will be introduced into the game, which are probably useful, at least they are much easier to discuss. With the help of the gauges, we will establish the decomposition of the operator A in the case sl(n+1) in terms of the ones for sl(2).

§2.4.1 Covariant Operator $d^{(n+1)}$

In the DS–linear system, the matrix L belongs to some algebra $\mathcal{G}$. In this section, we consider the case $\mathcal{G} = sl(2)$, denoted by $g = \{H, E_\pm\}$ with the following commutation relations

$$[H, E_\pm] = \pm 2E_\pm$$

$$[E_+, E_-] = H$$

Its (n+1)–dimensional representation ($n = 2j$) has the form

$$D(E_+) = \begin{pmatrix} 0 & \sqrt{2j} & 0 & \dots & 0 \\ 0 & 0 & \sqrt{2(j-1)} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots \\ 0 & 0 & 0 & \dots & \sqrt{2j} \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} \tag{2.45a}$$

$$D(E_-) = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ \sqrt{2j} & 0 & 0 & \dots & 0 \\ 0 & \sqrt{2(j-1)} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & \dots & \sqrt{2j} \end{pmatrix} \tag{2.45b}$$

$$D(H) = 2\begin{pmatrix} j & 0 & 0 & \dots & 0 \\ 0 & (j-1) & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & (j-1) & 0 \\ 0 & 0 & \dots & 0 & j \end{pmatrix} \tag{2.45c}$$

Define

$$a_k \equiv \sqrt{(2j-k+1)k} \qquad\qquad k = 1, 2, \dots, 2j \tag{2.46}$$

Then The matrices can be written as

$$(D(E_+))_{lk} = a_k \delta_{l,k+1} \equiv (\tilde{\epsilon}_+)_{lk} \tag{2.47a}$$

$$(D(E_-))_{lk} = a_k \delta_{k,l+1} \equiv (\tilde{\epsilon}_-)_{lk} \tag{2.47b}$$

$$D(H)_{lk} = 2(j-k+1)\delta_{lk} \equiv \tilde{\mathcal{H}}_{lk} \tag{2.47c}$$

So, the DS system has the form

$$(\partial + \theta^{'}\tilde{\mathcal{H}} - \tilde{\epsilon}_+)\xi = 0 \tag{2.48}$$

And the higher order differential equation is

$$(\partial - 2j\theta^{'})(\partial - 2(j-1)\theta^{'})\dots(\partial + 2j\theta^{'})\xi_1 = 0 \tag{2.49}$$

We have the (n+1)–th order differential operator in diagonal gauge

$$d^{(n+1)} = (\partial - 2j\theta^{'})(\partial - 2(j-1)\theta^{'})\dots(\partial + 2j\theta^{'}) \tag{2.50}$$

Now, one can make the gauge transformation

$$
\begin{aligned}
&g = \exp(\theta' \tilde{\epsilon}_-), \\
&\tilde{\xi} = g^{-1}\xi.
\end{aligned}
\tag{2.51}
$$

Then, the eq.(2.48) becomes

$$
(\partial + u\tilde{\epsilon}_- - \tilde{\epsilon}_+)\tilde{\xi} = 0 \tag{2.52}
$$

where

$$
u \equiv \theta'' - \theta'^2 \tag{2.53}
$$

If we define the set of new fields

$$
\eta_k \equiv a_1 a_2 \ldots a_k \tilde{\xi}_k
$$

Finally, we obtain the transformed the DS–system

$$
\left(\partial + \begin{pmatrix} 0 & -1 & 0 & \ldots & 0 \\ nu & 0 & -1 & \ldots & 0 \\ 0 & 2(n-1)u & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & nu & 0 \end{pmatrix} \right) \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \vdots \\ \eta_{n+1} \end{pmatrix} = 0 \tag{2.54}
$$

This is very convenient gauge,since our coordinate in this gauge is the stress–energy momentum tensor instead of θ'. Typically, when we discuss the covariant decomposition of the operator A, this gauge will become very useful. In fact, it just gives a new (n+1)–dimensional representation of sl(2), which has the following explicit form

$$
\hat{\rho} = \sum_{i=1}^{n+1} (n+2-2i)E_{ii} \tag{2.55a}
$$

$$
I_+ = \sum_{i=1}^{n} E_{i,i+1} \tag{2.55b}
$$

$$
I_- = \sum_{i=1}^{n} a_i^2 E_{i+1,i} \tag{2.55c}
$$

It is easy to check the algebraic relations

$$[\hat{\rho}, I_{\pm}] = \pm 2 I_{\pm} \tag{2.56a}$$

$$[I_+, I_-] = \hat{\rho} \tag{2.56b}$$

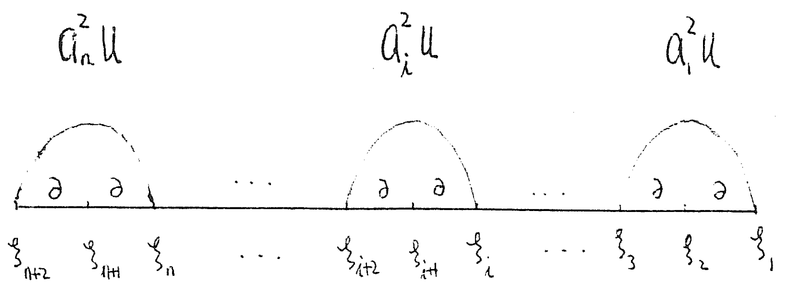
and the eq.(2.54) can be rewritten as follows

$$(\partial + uI_- - I_+)\eta = 0 \tag{2.57}$$

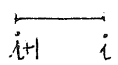
If we write the operator $d^{(n+1)}$ in the following way

$$d^{(n+1)} = \partial^{(n+1)} + \sum_{i=1}^{n} \omega_i^{[n+1]} \partial^{n-i} \tag{2.58}$$

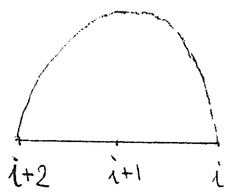
Then, all the functions $\omega_i^{[n+1]}$'s can be expressed in terms of the field u. In order to do this, we notice that the DS–system admits a graph representation as follows



where, from the right to the left, (n+2)–points on this straight line denotes $\xi_1, \xi_2, \ldots, \xi_{n+2}$, and



$= \partial$ (line segment)



$= a_i^2 u$ (curved line)

Taking summation over all of the different paths from ξ_1 to ξ_{n+2}, we get

$$\xi_{n+2} = d^{(n+1)} \xi_1 = 0$$

Thus, the formula of the operator $d^{(n+1)}$ is

$$d^{(n+1)} = \sum_{q=0}^{[\frac{n+1}{2}]} \sum_{p_1+\ldots+p_{q+1}=n-2q+1} \partial^{p_1} a^2_{p_1+1}$$

$$\cdot \partial^{p_2} a^2_{p_1+p_2+3} u \ldots \partial^{p_q} a^2_{p_1+\ldots+p_q+2q-1} u \partial^{p_{q+1}}$$

$$= \sum_{q=0}^{[\frac{n+1}{2}]} \sum_{p_1+\ldots+p_{q+1}=n-2q+1} \partial^{p_1} u \partial^{p_2} u \ldots \partial^{p_q} u \partial^{p_{q+1}} \tag{2.59}$$

$$\prod_{i=1}^{q} \left(n+2-2i-\sum_{l=1}^{i} p_l\right)\left(\sum_{l=1}^{i} p_l + 2i - 1\right)$$

$$= \sum_{\substack{0\le q\le [\frac{n+1}{2}] \\ j_1+\ldots+j_q=r-2q+1}} F^{[n+1]}_{j_1 j_2 \ldots j_q} u^{(j_1)} u^{(j_2)} \ldots u^{(j_q)} \partial^{n-r}$$

directly, we obtain the following explicit fomula

$$\omega^{[n+1]}_r = \sum_{q=1}^{[\frac{r+1}{2}]} \sum_{\sum_{i=1}^{q} j_i = r-2q+1} F^{[n+1]}_{j_1 j_2 \ldots j_q} u^{(j_1)} u^{(j_2)} \ldots u^{(j_q)} \tag{2.60}$$

where

$$u^{(l)} \equiv \frac{\partial^l u}{\partial x^l}, \qquad \forall l \ge 0$$

and

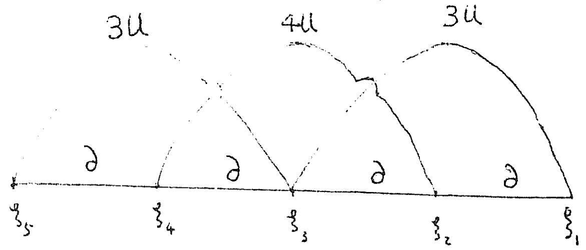
$$F^{[n+1]}_{j_1 j_2 \ldots j_q} \equiv \sum_{\sum_{i=1}^{q+1} p_i = n-2q+1} \prod_{i=1}^{q} \left[\left(n+2-2i-\sum_{l=1}^{i} p_l\right)\right. \tag{2.61}$$

$$\left.\cdot\left(\sum_{l=1}^{i} p_l + 2i - 1\right) \cdot \binom{p_1+p_2+\ldots+p_i-j_1-j_2-\ldots-j_{i-1}}{j_i}\right]$$

The first few ones are given below

$$\begin{aligned}
\omega^{[n+1]}_1 &= \binom{n+2}{3} u, \\
\omega^{[n+1]}_2 &= 2\binom{n+2}{4} u', \\
\omega^{[n+1]}_3 &= (3u'' + \frac{1}{3}(5n+12)u^2)\binom{n+2}{5}, \\
\omega^{[n+1]}_3 &= (4u''' + 2(5n+12)uu')\binom{n+2}{6}.
\end{aligned} \tag{2.62}$$

For example, let us consider the case n=3



$$d^{(4)} = \partial^4 + \partial^2(3u) + 4\partial u\partial + 3u\partial^2 + (3u)^2$$

$$= \partial^4 + 10u\partial^2 + 10u^{'} + 3u^{''} + 9u^2,$$

Now, we give the first few covariant operators below

$$\begin{aligned}
d^{(0)} =& 1, \\
d^{(1)} =& \partial, \\
d^{(2)} =& \partial^2 + u, \\
d^{(3)} =& \partial^3 + 4u\partial + 2u^{'}, \\
d^{(4)} =& \partial^4 + 10u\partial^2 + 10u^{'} + 3u^{''} + 9u^2, \\
d^{(5)} =& \partial^5 + 20u\partial^3 + 30u^{'}\partial^2 + 18u^{''}\partial + 64u^2\partial + 4u^{'''} + 64uu^{'},
\end{aligned} \tag{2.63}$$

§2.4.2 The Covariance of The Operator $d^{(n+1)}$

At first, let us consider the Schrödinger equation as a example[24]

$$(\partial^2 + u(x))\xi(x) = 0 \tag{2.64}$$

From the eqs(2.34a) and (2.38), it is easy to find the Poisson brackets as follows

$$\begin{aligned}
\{u(x), u(y)\} &= -(\frac{1}{2}\partial^3 + 2u\partial + u^{'})\delta(x-y) \\
\{u(x), \xi(y)\} &= \frac{1}{2}(\xi\partial + 3\xi^{'})\delta(x-y)
\end{aligned} \tag{2.65}$$

Using these relations, one can show that

$$
\begin{aligned}
&\int\int dx\, dy\, f(x)\, g(y)\{u(y), (\partial^2 + u(x))\xi\} \\
&= \int\int dx\, dy\, g(y)\{u(y), u(x)\}\xi(x) + \{u(y), \xi(x)\}u(x)) \\
&\quad + \int\int dx\, dy\, g(y)\, \{u(y), \xi(x)\}f^{''}(x), \\
&= -\int dx\, g(x)\left(\frac{3}{2}(\xi^{''} + u\xi)\partial + \frac{1}{2}(\xi^{''} + u\xi)^{'}\right)f(x).
\end{aligned}
\tag{2.66}
$$

This is equivalent to the following one

$$
\{u(y), (\partial^2 + u(x))\xi\} = -(\frac{3}{2}(\xi^{''} + u\xi)\partial + \frac{1}{2}(\xi^{''} + u\xi)
\tag{2.67}
$$

We see that $(\partial^2 + u(x))\xi(x)$ is a conformal primary field with the weight $\frac{3}{2}$. Since the Poisson brackets generate the conformal transformation in the following way

$$
\delta_\epsilon u(x) \equiv \int dy\, \{u(y), u(x)\}\epsilon(y)
\tag{2.68a}
$$

$$
\delta_\epsilon \xi(x) \equiv \int dy\, \{u(y), \xi(x)\}\epsilon(y)
\tag{2.68b}
$$

then, the Schrödinger equation is invariant under such transformation generated by $u(x)$ which forms the semi–classical Virasoro algebra.

Now, we turn our attention on the Boussinesque equation

$$
(\partial_3 + 4u\partial + 2u^{'} + V_2)\xi(x) = 0
\tag{2.69}
$$

or instead, define

$$
W_1(x) \equiv 4u(x)
\tag{2.70}
$$

Then, from the equation(2.34), we get the following algebra

$$
\begin{aligned}
\{W_1(x), W_1(y)\} &= -(2\partial + 2W_1\partial + W_1^{'})\delta(x-y), \\
\{W_1(x), V_2(y)\} &= -(3V_2\partial + 2V_2^{'})\delta(x-y), \\
\{W_1(x), \xi(y)\} &= (\xi\partial + 2\xi^{'})\delta(x-y), \\
\{V_2(x), \xi(y)\} &= \frac{1}{4}(\frac{2}{3}\xi\partial^2 - 2\xi'\partial + 4\xi^{''} + \frac{8}{3}W_1\xi)\delta(x-y), \\
\{V_2(x), V_2(y)\} &= \frac{1}{6}(\partial + 5W_1\partial^3 + \frac{15}{2}W_1^{'}\partial + \frac{9}{2}W_1^{''}\partial \\
&\quad + 4W_1^2\partial + W_1^{'''} + 4W_1W_1^{'})\delta(x-y).
\end{aligned}
\tag{2.71}
$$

a straightforward computation shows that

$$
\begin{aligned}
&\{W_1(x), (\partial_3 + 4u\partial + 2u^{'} + V_2)\xi(x)\} \\
&= -\Big(2(\xi^{'''} + W_1\xi^{'} + \frac{1}{2}W_1^{'}\xi + V_2\xi)\partial \\
&\quad + (\xi^{'''} + W_1\xi^{'} + \frac{1}{2}W_1^{'}\xi + V_2\xi)^{'}\Big)\delta(x-y).
\end{aligned}
\tag{2.72}
$$

Again, we see that $(\partial_3 + 4u\partial + 2u^{'} + V_2)\xi(x)$ is a conformal primary field with the weight 2, which will be denoted by $D^{(3)}\xi(x)$. Therefore, the Boussinesque equation is invariant under the conformal transformation generated by the field $W_1(x)$. In fact, this equation is also invariant under spin–3 transformation generated by $V_2(x)$, due to the fact that

$$
\begin{aligned}
&\{V_2(x), D^{(3)}\xi(y)\} \\
&= -\Big(\frac{5}{3}D^{(3)}\xi(y)\partial^2 - \frac{5}{2}(D^{(3)}\xi(x))^{'}\partial \\
&\quad + (D^{(3)}\xi(x))^{''} + \frac{2}{3}u(x)D^{(3)}\xi(x)\Big)\delta(x-y).
\end{aligned}
\tag{2.73}
$$

That is to say, the Boussinesque equation has W–algebraic symmetry which is generated by the conserved currents $W_1(x), V_2(x)$.

We have shown that $d^{(2)}, d^{(3)}$ are covariant operators, i.e. which map the conformal tensors with the weights $(-\frac{1}{2})$ and (-1) to the conformal tensors with the weights $\frac{3}{2}$ and

2 respectively. Generally, if $\Psi(x)_{\frac{-n}{2}}$ is a conformal tensor with the weight $\frac{-n}{2}$, then, $\left(d^{(n+1)}\Psi(x)_{\frac{-n}{2}}\right)$ will be another tensor with the weight $(1+\frac{n}{2})$. This can be done very easily in the linearized form. Consider the DS–system(2.57). Define

$$J \equiv uI_- - I_+ \tag{2.74}$$

Then, under the infinitesimal conformal transformation

$$\begin{aligned} g =& 1 + R, \\ R =& \epsilon(uI_- - I_+) + \frac{1}{2}\epsilon^{'}\hat{\rho} + \frac{1}{2}\epsilon^{''}I_-. \end{aligned} \tag{2.75}$$

J transforms in the following way

$$\begin{aligned} \delta_\epsilon J =& R^{'} + [J, R] = \delta_\epsilon u I_- \\ \delta_\epsilon u =& 2\epsilon^{'} u + \epsilon u^{'} + \frac{1}{2}\epsilon^{'''} \end{aligned} \tag{2.76}$$

and

$$d^{(n+1)}[u] \longrightarrow d^{(n+1)}[u + \delta_\epsilon u] \tag{2.77}$$

This means that the set of the linear equations remain unchanged, and the transformed covariant operator has the same form as the untransformed one but different ingredients. This shows its covariance. In fact, if we make the finite conformal transformation

$$x \longrightarrow f(x) \tag{2.78}$$

Then, we will have

$$d^{(n+1)}[u] \longrightarrow (f^{'})^{(1+\frac{n}{2})}\left(d^{(n+1)}[\tilde{u}]\right)(f^{'})^{\frac{n}{2}} = d^{(n+1)}[u(x,t)] \tag{2.79}$$

where

$$\tilde{u} \equiv u(f(x)) \tag{2.80}$$

Obviously, the field $\left(d^{(n+1)}[\tilde{u}]\Psi_{\frac{-n}{2}}(f(x))\right)$ is a conformal tensor. Therefore we see that $d^{(n+1)}[u]$ is a tensorial operator when acting on $\Psi_{\frac{-n}{2}}(f(x))$. The conformal weight of Ψ is determined by the central charge of the Virasoro algebra.

§2.5 The Representation of The Operator $D^{(n+1)}$ in The sl(n+1) case

Now, we turn our attention on the algebra sl(n+1). Consider its (n+1)–dimensional matrix representation. The DS–system reads

$$(\partial + L - I_+)\xi = 0 \tag{2.81}$$

where

$$I_+ = \sum_{i=1}^{n} E_{i,i+1} \tag{2.82}$$

and L is a lower triangular matrix. In the following, we will show its properties in several different gauges.

§2.5.1 Diagonal Gauge

In the diagonal gauge, L is a diagonal matrix. The the DS–system takes the following form.

$$\left(\partial + \begin{pmatrix} h_1 & -1 & 0 & \dots & 0 \\ 0 & h_2 & -1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & h_n & -1 \\ 0 & 0 & 0 & \dots & h_{n+1} \end{pmatrix}\right) \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \vdots \\ \xi_{n+1} \end{pmatrix} = 0 \tag{2.83}$$

where

$$\sum_{i=1}^{n+1} h_i = 0 \tag{2.84}$$

The operator $D^{(n+1)}$ can be easily written as

$$D^{(n+1)} = (\partial + h_{n+1}) \dots (\partial + h_2)(\partial + h_1) \tag{2.85}$$

§2.5.2 Drinf'eld–Sokholov Gauge

Now, if we choose

$$L = \begin{pmatrix} 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & 0 \\ W_n & \dots & W_2 & W_1 & 0 \end{pmatrix} \tag{2.86}$$

Then, the operator $D^{(n+1)}$ has the simple form

$$D(n+1) = \partial^{(n+1)} + \sum_{i=1}^{n} W_i \partial^{n-i} \tag{2.87}$$

This gauge is the so-called Drinf'eld–Sokholov gauge(the DS–gauge), which is related to the Diag. gauge by some gauge transformation. Due to the gauge invariance of the operator $D^{(n+1)}$, one can express the functions W_i's in terms of the coordinates h_i's, vice versa.

$$\begin{aligned} W_1 &= \sum_{l=1}^{n} (n-l+1) h_l' + \frac{1}{2} \sum_{l \neq k} h_l h_k, \\ W_2 &= \sum_{l=1}^{n-1} \binom{n-l+1}{2} h_2'' + \sum_{k \leq l} ((n-l+1) h_l' h_k + (n-k) h_l h_k') + \sum_{l>k>m} h_l h_k h_m \end{aligned} \tag{2.88}$$

Generally, we have

$$W_r = \sum_{l=1}^{n-r+1} \binom{n-l+1}{r} h_l^{(r)} + \sum_{\substack{l,k \\ j_1+j_2=r-1}} \binom{l}{j_1} \binom{l+k-j_1}{j_2} h_{n-l+1}^{(j_1)} h_{n-k-l}^{(j_2)} + \dots \tag{2.89}$$

W–algebra are given by the second Gelfand–Dickii Poisson bracket, which has been shown before(see the equation(2.34)). We see that W_1 is the energy momentum tensor which forms the semi-classical Virasoro algebra, but the other brackets are complicated and other conserved currents are not conformal tensors.

§2.5.3 Covariant Gauge I

In order to simplify the W–algebraic structure, it is better to look for a set of conformal tensors(primary fields) as the coordinates of the the DS–system. We will call such kind of gauges as the covariant gauge. One of them is as follows[25]

$$L = \sum_{I=1}^{n} N_I (I_-)^I X_I \tag{2.90}$$

where the set of fields $\{X_I\text{'s}\}$ are choosen to be the n–independent coordinates, and $\{N_I's\}$ are normalization constants satisfying the relations

$$N_I \sum_{i=1}^{n-I+1} ((I_-)^I)_{i+I,i} = 1 \qquad I = 1, 2, \ldots, n. \tag{2.91}$$

In this gauge, if we write the operator $D^{(n+1)}$ in the following manner

$$D^{(n+1)} = \partial^{(n+1)} + W_1 \partial^{n-1} + W_2 \partial^{n-2} + \ldots + W_n \tag{2.92}$$

Then, these W–functions have the following expressions

$$W_2 = \frac{1}{2}(n-1)X_1' + X_2 = 2\binom{n+2}{4}u' + X_2, \tag{2.93a}$$

$$W_3 = (3u'' + \frac{1}{3}(5n+12)u^2)\binom{n+2}{5} + \frac{1}{2}(n-2)X_2' + X_3, \tag{2.93b}$$

$$\begin{aligned} W_4 =& (4u''' + 2(5n+12)uu')\binom{n+2}{6} + \frac{2}{7}\binom{n-2}{2}X_2'' \\ &+ \frac{1}{21}(7n+20)\binom{n-2}{2}uX_2 + \frac{1}{2}(n-3)X_3' + X_4. \end{aligned} \tag{2.92c}$$

Generally, we obtain

$$\begin{aligned} W_r =& \sum_{q\geq 1} \sum_{(t_1=n-q+1)} \sum_{(t_2=r-q+1)} \prod_{l=1}^{q} \binom{m_l}{j_l} \prod_{l=1}^{q} c_{k_l}^{i_l} X_{i_1}^{(j_1)} X_{i_2}^{(j_2)} \ldots X_{i_q}^{(j_q)} \\ =& X_r + \omega_r^{[r+1]} + f_r(X_1, X_2, \ldots, X_{r-1}) \end{aligned} \tag{2.94a}$$

where

$$\begin{cases} t_1 = \sum_{l=1}^{q+1} p_l + \sum_{l=1}^{q} i_l, & t_2 = \sum_{l=1}^{q} j_l + \sum_{l=1}^{q} i_l, \\ k_l = \sum_{s=1}^{l} p_s - \sum_{s=1}^{l-1} i_s + l, & m_l = \sum_{s=1}^{l} p_s - \sum_{s=1}^{l-1} j_s. \end{cases} \tag{2.94b}$$

and

$$c_{k_l}^{i_l} \equiv a_{k_l}^2 a_{k_l+1}^2 \cdots a_{k_l+i_l-1}^2 \tag{2.94c}$$

These formulas establish the connection between two sets of coordinates $\{W_i\text{'s}\}$ and $\{X_i\text{'s}\}$. On the other hand, if we define the rank of the field in the following way

$$Rank[X_i] \equiv i+1 \tag{2.95}$$

then, from the last step of the above equation, we see that on the right hand side, the function f_r only involves the fields whose ranks are less than (r+1). So, we can say that it represents a decomposition of the field $\{W_r\}$ in terms of the fields $\{X_1, X_2, \ldots, X_r\}$ and $\omega_r^{[r+1]}$. Now, our aim is to prove that all the fields $\{X_2, X_3, \ldots, X_n\}$ are conformal tensors whose conformal weights are equal to their ranks. In order to do this, we go back to the linearized system again. In the same way as before, we define

$$J \equiv L - I_+$$

then, make the following infinitesimal transformation

$$\begin{aligned} g =& 1 + R, \\ R \equiv& \epsilon(x)J(x) + \frac{1}{2}\epsilon'(x)\hat{\rho} + \frac{1}{2}\epsilon''(x)I_- \end{aligned} \tag{2.96}$$

The matrix J transforms like

$$\delta_\epsilon J = \sum_{I=1}^{n} N_I(I_-)^I\Big((I+1)\epsilon'(x)X_I(x) + \epsilon(x)X_I'(x)\Big) + \frac{1}{2}\epsilon'''(x)I_- \tag{2.97}$$

Obviously, one can extract the transformation laws of the coordinates

$$\begin{aligned} \delta_\epsilon X_1(x) =& 2\epsilon'(x)X_1(x) + \epsilon(x)X_1'(x) + \frac{1}{2}\binom{n+2}{3}\epsilon'''(x), \\ \delta_\epsilon X_I(x) =& (I+1)\epsilon'(x)X_I(x) + \epsilon(x)X_I'(x) \qquad I = 2, 3, \ldots, n. \end{aligned} \tag{2.98}$$

The first equation tells us that this transformation is a conformal transformation, and the second one shows that all the fields $\{X_2, X_3, \ldots, X_n\}$ are conformal tensors whose conformal weights are equal to their ranks , i.e.

$$[X_I(x)]_{conf.weight} = I + 1 \tag{2.99}$$

Now, our coordinates are stress tensor and primary fields, and the the DS–system has very simple form(the matrix L only contains I_-,which belongs to subalgebra sl(2)). However, if we try to express $D^{(n+1)}$ in terms of $d^{(n+1)}$, it is still very complicated. In order to do this, we will introduce another covariant gauge in next subsection.

§2.5.4 Covariant Gauge II

In the last subsection, we have introduced one set of coordinates which are conformal tensors. In order to show how we can decompose $D^{(n+1)}$ into pieces which are covariant operators we discussed in the first section, we consider some examples.

<u>*Example.1*</u> **The algebra is sl(4)**

In this case, from the definitions (2.90) and (2.91), it is easy to see that the DS–system (2.81) has the following form

$$J = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 3u & 0 & -1 & 0 \\ \frac{1}{2}X_2 & 4u & 0 & -1 \\ X_3 & \frac{1}{2}X_2 & 3u & 0 \end{pmatrix} \tag{2.100}$$

where

$$J = L - I_+$$

and the operator $D^{(4)}$ reads

$$D^{(4)} = d^{(4)} + X_2\partial + \frac{1}{2}X_2' + X_3 \tag{2.101}$$

now, we introduce new coordinates in the following way

$$v_2^2 \equiv X_2, \qquad\qquad v_3^2 \equiv X_3 \tag{2.102}$$

Surely, they are conformal tensors. From the eq(2.33), the Poisson brackets can be written down

$$
\begin{aligned}
\{u(x),\ u(y)\} &= -\,(\frac{1}{2}\partial^3 + 2u\partial + u')\delta(x-y) \\
\{u(x), X_2(y)\} &= -\,(3X_2\partial + 2X_2')\delta(x-y) \\
\{X_2(x), u(y)\} &= -\,(3X_2\partial + X_2')\delta(x-y) \\
\{u(x), X_3(y)\} &= -\,(4X_3\partial + 3X_3')\delta(x-y) \\
\{X_3(y), u(y)\} &= -\,(4X_3\partial + X_3')\delta(x-y)
\end{aligned}
\tag{2.103}
$$

and, the operator $D^{(4)}$ can be rewritten as

$$
D^{(4)} = d^{(4)} + v_2 d^{(1)} v_2 + v_3 d^{(0)} v_3 \tag{2.104}
$$

<u>*Example.2*</u> **The algebra is sl(5)**

In this case, the matrix J takes the form

$$
J = \begin{pmatrix}
0 & -1 & 0 & 0 & 0 \\
4u & 0 & -1 & 0 & 0 \\
\frac{2}{7}X_2 & 6u & 0 & -1 & 0 \\
\frac{1}{2}X_3 & \frac{3}{7}X_2 & 6u & 0 & -1 \\
X_4 & \frac{1}{2}X_3 & \frac{2}{7}X_2 & 4u & 0
\end{pmatrix} \tag{2.105}
$$

In the same fashion as before, we get

$$
\begin{aligned}
\{u(x), u(y)\} &= -\,(\frac{1}{2}\partial^3 + 2u\partial + u')\delta(x-y) \\
\{u(x),\ X_2(y)\} &= -\,(3X_2\partial + 2X_2')\delta(x-y) \\
\{X_2(x), u(y)\} &= -\,(3X_2\partial + X_2')\delta(x-y) \\
\{u(x), X_3(y)\} &= -\,(4X_3\partial + 3X_3')\delta(x-y) \\
\{X_3(y), u(y)\} &= -\,(4X_3\partial + X_3')\delta(x-y) \\
\{u(x), X_4(y)\} &= -\,(5X_3\partial + 4X_4')\delta(x-y) \\
\{X_4(y), u(y)\} &= -\,(5X_3\partial + X_4')\delta(x-y)
\end{aligned}
\tag{2.106}
$$

It is not so difficult to show that

$$
D^{(5)} = d^{(5)} + v_2 d^{(2)} v_2 + v_3 d^{(1)} v_3 + v_4 d^{(0)} v_4 \tag{2.107}
$$

Here,

$$
\begin{aligned}
v_2^2 &\equiv X_2, \qquad\qquad v_3^2 \equiv X_3 \\
V_4^2 &\equiv X_4 + \frac{9}{7}uX_2 - \frac{3}{14}X_2'' + \frac{1}{4}\frac{X_2'^{\,2}}{X_2}
\end{aligned}
\tag{2.108}
$$

Using the above Poisson brackets, one can easily show that $(\frac{9}{7}uX_2 - \frac{3}{14}X_2'' + \frac{1}{4}\frac{X_2'^{\,2}}{X_2})$ is the rank–5 conformal tensor,therefore v_4 is also a conformal tensor with weight 5.

<u>*Example.3*</u> **The case sl(7)**

In this case, we have the following matrix

$$
J = \begin{pmatrix}
0 & -1 & 0 & 0 & 0 & 0 & 0 \\
6u & 0 & -1 & 0 & 0 & 0 & 0 \\
\frac{5}{42}X_2 & 10u & 0 & -1 & 0 & 0 & 0 \\
\frac{1}{6}X_3 & \frac{5}{21}X_2 & 12u & 0 & -1 & 0 & 0 \\
\frac{3}{11}X_4 & \frac{1}{3}X_3 & \frac{2}{7}X_2 & 12u & 0 & -1 & 0 \\
\frac{1}{2}X_5 & \frac{5}{11}X_4 & \frac{1}{3}X_3 & \frac{5}{21}X_2 & 10u & 0 & -1 \\
X_6 & \frac{1}{2}X_5 & \frac{3}{11}X_4 & \frac{1}{6}X_3 & \frac{5}{42}X_2 & 6u & 0
\end{pmatrix}
\tag{2.109}
$$

It is tidious but straightforward to do the calculation, some Poisson brackets are in the following

$$
\begin{aligned}
\{u(x),\ u(y)\} &= -\,(\frac{1}{2}\partial^3 + 2u\partial + u')\delta(x-y) \\
\{u(x), X_I(y)\} &= -\,((I+1)X_I\partial + IX_I')\delta(x-y), \qquad I = 2,\ldots,6. \\
\{X_I(x), u(y)\} &= -\,((I+1)X_I\partial + X_I')\delta(x-y), \qquad I = 2,\ldots,6.
\end{aligned}
\tag{2.110}
$$

and, the operator $D^{(7)}$ can be written in the following manner

$$
D^{(7)} = d^{(7)} + v_2 d^{(4)} v_2 + v_3 d^{(3)} v_3 + v_4 d^{(2)} v_4 + v_5 d^{(1)} v_5 + v_6^2 \tag{2.111}
$$

where

$$
\begin{aligned}
X_2 &= v_2^2, \qquad\qquad X_3 = v_3^2, \\
X_4 &= v_4^2 - \frac{54}{7} u v_2^2 + \frac{3}{7} {v_2'}^2 - \frac{18}{7} v_2 v_2'', \\
X_5 &= v_5^2 - \frac{16}{3} u v_3^2 - \frac{5}{3} {v_3'}^2 + \frac{4}{3} v_3 v_3'' - \frac{125}{42^2} v_2^4, \\
X_6 &= v_6^2 - \frac{25}{11} u v_4^2 + \frac{5}{11} v_4 v_4'' - \frac{6}{11} {v_4'}^2 - \frac{5}{126} v_2^2 v_3^2 + \frac{1}{33}(2 v_2 v_2^{(4)} - 16 v_2' v_2''' \\
&\quad + 15 {v_2''}^2 - 6u'' v_2^2 + 20 u' v_2 v_2' - 142 u v_2 v_2'' + 176 u {v_2'}^2 + 225 u^2 v_2^2).
\end{aligned}
\tag{2.112}
$$

Now, let us consider the general case sl(n+1). We introduce a set of new fields in the following way

$$
\begin{aligned}
v_2^2 &\equiv X_2, \qquad\qquad v_3^2 \equiv X_3, \\
v_4^2 &\equiv X_4 + \binom{n-2}{2}\left(\frac{9}{7} u X_2 - \frac{3}{14} X_2'' + \frac{1}{4}\frac{{X_2'}^2}{X_2}\right) \\
v_5^2 &\equiv X_5 + \binom{n-3}{2}\left(\frac{16}{9} u X_3 - \frac{2}{9} X_3'' + \frac{1}{4}\right)\frac{{X_3'}^2}{X_3}\Bigg) + \beta_1 X_2^2 \\
v_6^2 &\equiv X_6 + \binom{n-4}{2}\left(\frac{25}{11} u v_4^2 - \frac{5}{11} v_4 v_4'' + \frac{6}{11} {v_4'}^2\right) - \beta_2 X_2 X_3 \\
&\quad + \binom{n-4}{4}\left(\frac{21}{176}\frac{{X_2'}^4}{{X_2}^3} - \frac{9}{44}\frac{{X_2'}^2 X_2''}{{X_2}^2} - \frac{3}{44}\frac{{X_2''}^2}{X_2} + \frac{2}{11}\frac{X_2' X_2'''}{X_2} - \frac{1}{33} X_2^{(4)} \right. \\
&\quad \left. - 10 u' X_2' + 71 u X_2'' - \frac{159}{2} u \frac{{X_2'}^2}{X_2} + 6 u'' X_2 - 225 u^2 X_2 \right)
\end{aligned}
\tag{2.113}
$$

where the coefficient is

$$
\beta_1 = \frac{a_1^4 a_2^4}{N_2^2}, \qquad \beta_2 = \frac{a_1^2 a_2^2 a_3}{N_2 N_3}
$$

here N_2,N_3, a_1, a_2 and a_3 are constants given before. Recusively, we define

$$
v_r^2 \equiv X_r + f_r(X_1, X_2, \ldots, X_{r-1}) - \sum_{i=2}^{r-1} \sum_{l=0}^{r-i-1} \binom{n-i-l-1}{n-r} v_i v_i^{(r-i-l-1)} \omega_l^{[n-i]} \tag{2.114}
$$

Then, the operator $D^{(n+1)}$ has the following decoposition

$$D^{(n+1)} = d^{(n+1)} + \sum_{i=2}^{n} v_i d^{(n-i)} v_i \tag{2.115}$$

This relation can be checked directly, since

$$\begin{aligned} W_r =& X_r + \omega_r^{[n+1]} + f_r(X_1, X_2, \ldots, X_{r-1}) \\ =& \omega_r^{[n+1]} + \sum_{i=2}^{r} \sum_{l=0}^{r-i-1} \binom{n-i-l-1}{n-r} v_i v_i^{(r-i-l-1)} \omega_l^{[n-i]} \end{aligned} \tag{2.116}$$

Now, what we should do is to prove the conformal tensorial properties of the fields $\{v_i\text{'s}\}$. From the construction, we see that v_2, v_3 are tensors. It is not so difficult to show that v_4, v_5, v_6 are also tensors with ranks $\frac{5}{2}$, 3 and $\frac{7}{2}$ respectively. As to v_r for arbitrary integer r, we also can show that it is a conformal tensor with rank $(\frac{r+1}{2})$. Since in the first section of this chapter, we have said that $d^{(n+1)}$ is a tensorial operator when acting on the primary field with the weight $\frac{-n}{2}$ (see the eq(2.79)), on the other hand, we know that $D^{(n+1)}$ is also a tensorial operator, therefore, in such kind of the decomposition, v_r must be a conformal tensor with rank $(\frac{r+1}{2})$. Otherwise, suppose that v_r is not a tensor, which satisfies the relation

$$\{v_r(x), u(y)\} = -(b_1\partial^{r+2} + b_2\partial^{r+1} + \ldots + b_{r+1}\partial^2 + a_1 v_r \partial + a_2 v_r' + g(x))\delta(x-y) \tag{2.117}$$

then, under the infinitesimal conformal transformation, the transformation law of v_r will be deduced from this Poisson, which is

$$\begin{aligned} \delta_\epsilon v_r(x) =& -\int dy\, \{v_r(x), u(y)\}\epsilon(y) \\ =& a_1 \epsilon' v_r + a_2 \epsilon v_r' + g\epsilon + b_1 \epsilon^{(r+2)} + \ldots + b_{r+1}\epsilon^2 \end{aligned} \tag{2.118}$$

We see that the higher derivatives of $\epsilon(x)$ are b–type of terms. Noting that under the infinitesimal conformal transformation,

$$d^{(n+1)}[u] \longrightarrow (1 + \frac{n}{2})\epsilon' d^{(n+1)}[\tilde{u}] + \frac{n}{2} d^{(n+1)}[\tilde{u}]\epsilon' \tag{2.119}$$

where $\tilde{u}$ is defined as $2\epsilon' u + \epsilon u'$ (see eq(2.80)). Since $(D^{(n+1)}\Psi_{-\frac{n}{2}})$ is a conformal tensor with the conformal weight $(1+\frac{n}{2})$, so, its infinitesimal change would be like

$$\delta_\epsilon(D^{(n+1)}\Psi_{-\frac{n}{2}}) = (1+\frac{n}{2})\epsilon'(D^{(n+1)}\Psi_{-\frac{n}{2}}) + \epsilon(D^{(n+1)}\Psi_{-\frac{n}{2}})' \tag{2.120}$$

Substituting (2.115), (2.119) and (2.80) into the above eqation, and comparing the derivatives of ϵ on both sides, we only have one possibility—v_r is a conformal tensor with the weight $(\frac{r+1}{2})$.

Until now, we have introduced a new covariant gauge in which the operator $D^{(n+1)}$ decomposes into pieces of the covariant operators $d^{(n+1)}$. However, in this gauge, if we want to write down the matrix L, furthermore, the explicit form of the DS–system, that is really not an easy job.

§2.6 The embedding of sl(2)–system into sl(n+1)–case

In the last paragraph, we see that $D^{(n+1)}$ decomposes into pieces of the covariant operators $d^{(n+1)}$. This gives us some hints, the DS–system in sl(n+1) probably can be expressed in terms of the ones in sl(2) case, they can be transformed to each other by W–algebaic transformation. In this subsection, we will take a example to show this property.

We know that sl(2)–system(in (n+1)–dimensional representation) is a sub–system of the sl(n+1) case, i.e. if we set all v fields in (2.81) equal to zero, then we obtain (2.57)

$$(\partial + uI_- - I_+)\xi = 0$$

We know that this system only has one parameter family of sl(2) gauge symmetry, which corresponds to the Virasoro symmetry, We will denote such a sl(2) subalgebra as $sl_m(2)$. Now, if we make the transformation which belongs to sl(n+1) rather than $sl_m(2)$, surely, the transformed matrix $\tilde{L}$ will be a lower triangular one. Since sl(n+1) is no–direct sum of n–sl(2) algebras whose representations are (n+1)–dimensional, so it is resonable to conjecture that appropriately adjusting n–parameters transformation will give us the covariant gauge I, or II. In order to shed some light on this, let us take sl(3)–case as a example.

In covariant gauge I, the the DS–system is

$$(\partial + J_2)\xi = 0 \tag{2.121}$$

where

$$J_2 \equiv \begin{pmatrix} 0 & -1 & 0 \\ 2u & 0 & -1 \\ X_2 & 2u & 0 \end{pmatrix} \tag{2.122}$$

set

$$X_2 = 0 \tag{2.123}$$

then, we get $sl_3(2)$ system(i.e. the matrices belong to 3–dimensional representation of sl(2)).

$$(\partial + J_2)\xi = 0 \tag{2.124}$$

where

$$J_3 \equiv \begin{pmatrix} 0 & -1 & 0 \\ 2u & 0 & -1 \\ 0 & 2u & 0 \end{pmatrix} \tag{2.125}$$

It is straightforward to check that there does exist a transformation

$$g = \begin{pmatrix} -\sigma^{'} & \sigma & 0 \\ -(\sigma^{''} + 2u\sigma) & 0 & \sigma \\ \frac{1}{2}X_2\sigma & -(\sigma^{''} + 2u\sigma) & \sigma^{'} \end{pmatrix} \tag{2.126}$$

$$= \sigma I_+ - \frac{1}{2}(\sigma^{''} + 2u\sigma)I_- + \frac{1}{4}d(3)\sigma {I_-}^2$$

such that

$$g^{-1}g^{'} + g^{-1}J_2\, g = J_3 \tag{2.127}$$

provided

$$\begin{cases} \sigma^{-3} = const.X_2, \\ (d(3) - \frac{1}{2}X_2)\sigma = 0. \end{cases} \tag{2.128}$$

This is not the gauge transformation (which connects the different gauges of the same DS–system),but the transformation generated by W–algebra. The condition (2.128) only requires that the field X_2 should be a rank–3 conformal tensor, this is just what we need. On the other hand, this means that given two systems, the relevant transformation is completely determined, or in other words, the embedding way of sl(2) into sl(3) is unique, up to the gauge transformation.

§2.7 The Property of W–Algebra

In this subsection, we will show some explicit W–algebraic structures in the *covariant gauge II*.

§2.7.1 Examples

Example.1 **The sl(3) case**

From the eq.(2.115), we see that the operator $D^{(n+1)}$ can be re–expressed in terms of $d^{(i)}$ (i=0, 1, ..., n-2, n+1) and a set of conformal tensors, so, for a particular case n=2, we have

$$\begin{aligned} D^{(3)} &= d^{(3)} + V_2 \equiv \partial^3 + W_1\partial + W_2 \\ V_2 &\equiv v_2^2, \qquad\qquad W_1 \equiv 4u \\ W_2 &\equiv 2u' + V_2 = \frac{1}{2}W_1' + V_2 \end{aligned} \tag{2.129}$$

From the second Gelfand–Dickii Poisson bracket(2.33), we easily obtain the following algebra

$$\begin{aligned} \{W_1(x), W_1(y)\} &= -\,(2\partial + 2W_1\partial + W_1')\delta(x-y) \\ \{W_1(x), V_2(y)\} &= -\,(3V_2\partial + 2V_2')\delta(x-y) \\ \{V_2(x), W_1(y)\} &= -\,(3V_2\partial + V_2')\delta(x-y) \\ \{V_2(x), V_2(y)\} &= \frac{1}{6}d^{(5)}\delta(x-y) \end{aligned} \tag{2.130}$$

Example.2 **sl(4) case**

Again, from the eq.(2.115), in this case n=3, we have

$$\begin{aligned} D^{(4)} =& d^{(4)} + V_2 d(1) V_2 + V_3^2, \\ \equiv& \partial^4 + W_1\partial^2 + W_2\partial + W_3, \\ V_2 \equiv& v_2^2, \qquad W_1 \equiv 10u, \\ V_3 \equiv& v_3^2, \qquad W_2 \equiv 10u' + V_2, \\ W_3 \equiv& 3u'' + 9u^2 + \frac{1}{2}V_2' + V_3. \end{aligned} \tag{2.131}$$

Then, doing the same thing as in the first example, one will get the W_4 algebra

$$\begin{aligned} \{W_1(x), W_1(y)\} =& -(5\partial + 2W_1\partial + W_1')\delta(x-y), \\ \{W_1(x), V_2(y)\} =& -(3V_2\partial + 2V_2')\delta(x-y), \\ \{V_2(x), W_1(y)\} =& -(3V_2\partial + V_2')\delta(x-y), \\ \{V_2(x), V_2(y)\} =& (d^{(5)} - 4v_3 d^{(1)} v_3)\delta(x-y), \\ \{W_1(x), V_3(y)\} =& -(4V_3\partial + 3V_3')\delta(x-y), \\ \{V_3(x), W_1(y)\} =& -(4V_3\partial + V_3')\delta(x-y), \\ \{V_3(x), V_3(y)\} =& (-\frac{1}{20}d^{(7)} - \frac{3}{5}v_3 d^{(3)} v_3 - \frac{8}{25}\tilde{v}_3 d^{(1)}\tilde{v}_3)\delta(x-y), \\ \{V_2(x), V_3(y)\} =& (\frac{7}{5}V_2\partial^3 + \frac{14}{5}V_2'\partial^2 + 2V_2''\partial + \frac{1}{2}V_2''' \frac{26}{25}V_2W_1\partial \\ &+ \frac{17}{25}W_1V_2' + \frac{27}{25}V_2W_1')\delta(x-y), \\ \{V_3(x), V_2(y)\} =& (\frac{7}{5}V_2\partial^3 + \frac{7}{5}V_2'\partial^2 + \frac{3}{5}V_2''\partial + \frac{1}{10}V_2''' \frac{26}{25}V_2W_1\partial \\ &+ \frac{1}{2}V_2W_1' + \frac{9}{25}W_1V_2')\delta(x-y), \end{aligned} \tag{2.132}$$

where

$$\tilde{v}_3^2 \equiv V_3W_1 - \frac{5}{4}V_3'' + \frac{45}{32}\frac{V_3'^2}{V_3}. \tag{2.133}$$

It can be easily checked that $\tilde{v}_3^2$ is a conformal tensor with conformal weight 6. Generally, for any tensor V_I with the conformal weight (I+1), we can construct new tensors like

$$V_I W_1 - \frac{c}{I+1} V_I^{''} + \frac{(2I+3)c}{2(I+1)^2} \frac{{V_I^{'}}^2}{V_I},$$
$$V_I W_1^{'} - \frac{2}{I+1} V_I^{'} W_1 - \frac{c}{I+1} V_I^{'''} \frac{3(I+2)c}{2(I+1)^2} \left(\frac{V_I^{'} V_I^{''}}{V_I} - \frac{{V_I^{'}}^3}{{V_I}^2} \right). \tag{2.134}$$

and so on. Where c is the central charge of the Virasoro algebra generated by W_1.

§2.7.2 The Properties of W–Algebra

<u>*Property.1*</u> **Covariance**

From the previous results, using the Poisson brackets given above, one can show that the Poisson brackets are covariant under the conformal transformation. That is to say, if we denote the bracket in this manner

$$\{A(x), B(y)\} = \hat{C}(x)\delta(x-y) \tag{2.135}$$

where, A(x) and B(x) are fields with weights $[A]_{conf.}$ and $[B]_{conf.}$ respectively , $\hat{C}(x)$ is an operator. Then, this operator is a tensorial operator with weight $([A]_{conf.}+[B]_{conf.})$

<u>*Property.2*</u> **Antisymmetry**

Before discussion the antisymmetry, we recall that

$$\partial_x \delta(x-y) = -\partial_y \delta(x-y)$$
$$f(x)\delta(x-y) = f(y)\delta(x-y) \tag{2.136}$$
$$f(y)\partial_x \delta(x-y) = (f(x)\partial_x + f(x)^{'})\delta(x-y)$$

So, we can introduce one operation as

$$\begin{cases} f^* \equiv f, & \forall f \\ \partial^* \equiv -\partial, & \\ (\hat{A}\hat{B})^* \equiv \hat{B}^* \hat{A}^* & \end{cases} \tag{2.137}$$

So, if the fields A(x) and B(y) satisfy (2.135), then, after this operation, we get

$$\{B(x), A(y)\} = -\hat{C}'(x)\delta(x-y) \tag{2.138}$$

This is just equivalent to

$$\{B(y), A(x)\} = -\{A(x), B(y)\} \tag{2.140}$$

This is exactly the antisymmetric property.

Property.3 **Jacobi Identity**

These Poisson brackets satisfy the Jacobi identity. this is guaranteed by the construction of the Gelfand–Dickii brackets in a rigorous way, so, we avoid the calculation here, although we can prove this from the above brackets.

Chapter 3
The KdV Structure in CCFT

§3.1 Introduction to CFT

Recent years CFT is one of the fascinating subject in theoreticel physics. On the one hand, it is the vacuum of the string theory, on the other hand, it is also the continuum limit of the lattice medel in two dimensions. It has been studied intensively[2,6].

§3.1.1 Conformal Symmetry

A conformal field theory(CFT) is a quantum field theory, which possesses the conformal symmetry. The conformal transformation is one special kind of the general coordinate transformation. In 2–dimensions, it is just the holomorphic and anti–holomorphic cooordinate transformations as follows

$$z \longrightarrow f(z), \qquad \bar{z} \longrightarrow \bar{f}(\bar{z})$$

Its infinitesimal generators are

$$L_n = -z^{n+1}\partial_z, \qquad \bar{L}_n = -\bar{z}^{n+1}\partial_{\bar{z}}$$

At the quantum level, they satisfy the commutation relation

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0} \tag{3.1}$$

This is the well–known Virasoro algebra. Here, c is the central charge, which indicates a class of theories rather than one particular theory. For example, c=1 both in the theory of a single free boson and in the theory of two free fermions.

One way to study 2–dimensional CFT is the lagrangian formalism. In this way, one can follow the standard manner developed in quantum field theory to calculate the physical quantities. However, on the one hand, not all of the CFT are realized in lagrangian formalism. On the other hand, CFT can be obtained without reference to the lagrangian formalism. Indeed, one can construct CFT only from the following requirements

(1) Field Content: There is a set of fields $\{A_i\}$ where the index i specifies the different fields. This set of fields in general is infinite and contains in particular the derivatives of the fields A_i's. It is closed under the Operator Product Expansion (OPE).

(2) Primary Fields: There is a subset of fields $\{\Phi_i\}$, called the primary fields, that under the conformal transformation, transform like

$$\Phi(z,\bar{z}) \longrightarrow \left(\frac{\partial f}{\partial z}\right)^{\Delta} \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{\bar{\Delta}} \Phi(f(z), \bar{f}(\bar{z})) \tag{3.2}$$

where Δ is the conformal weight of the primary field $\Phi(z)$. If the number of fields in this subset is finite, then the CFT is called rational conformal field theory(RCFT).

(3) The rest of the fields in $\{A_i\}$ can be expressed as the linear combinations of the primary fields and their derivatives.

(4) The theory is conformal invariant and modular invariant.

(5) And finally, CFT can be unitary or non–unitary.For example, Lee–Yang model doesn't have unitarity.

In a conformal field theory, physical quantities are correlation functions. We will see that they are determined by the above requirements.

§3.1.2 Conformal Ward Identity

For any quantum field theory with an exact symmetry, there is an associated conserved current. Consider a CFT on the flat Euclidean plane($g_{\mu\nu} = \delta_{\mu\nu}$), in the complex coordinates $z = x + iy$, the line element is

$$ds^2 = g_{\mu\nu} dx^\mu \, dx^\nu = dz \, d\bar{z}$$

So, the components of the metric tensor $(g^{ab}(z,\bar{z}))$ referred to the complex coordinates are

$$g^{zz} = g^{\bar{z}\bar{z}} = 0, \qquad g^{z\bar{z}} = g^{\bar{z}z} = \frac{1}{2}$$

and the components of the stress–energy tensor $T^{ab}(z,\bar{z})$ are

$$T_{zz} = \frac{1}{4}(T_{11} - T_{22} - 2iT_{12}),$$

$$\bar{T}_{\bar{z}\bar{z}} = (T_{zz})^*, \qquad T_{z\bar{z}} = \frac{1}{4}T^\mu_\mu$$

The conformal symmetry acquires that the stress–energy tensor $T^{ab}(z,\bar{z})$ should be traceless , that means

$$T_{z\bar{z}} = 0$$

and conserved, that is to say

$$\partial^\mu T_{\mu\nu} = 0 \Longrightarrow \begin{cases} \partial_{\bar{z}} T_{zz} + \partial_z T_{\bar{z}z} = 0, \\ \partial_z T_{\bar{z}\bar{z}} + \partial_{\bar{z}} T_{z\bar{z}} = 0. \end{cases}$$

So, we have the holomorphic and anti–holomorphic parts

$$\begin{cases} \partial_{\bar{z}} T_{zz} = 0, & T(z) \equiv T_{zz}(z) \\ \partial_z \bar{T}_{\bar{z}\bar{z}} = 0, & \bar{T}(\bar{z}) \equiv \bar{T}_{\bar{z}\bar{z}}(\bar{z}) \end{cases}$$

For holomorphic sector, consider general correlation functions

$$< X > \equiv < A_1(z_1) A_2(z_2) \ldots A_n(z_n) > \tag{3.3}$$

then

$$\delta_\epsilon < X > = \oint_{C_{\{z_1, z_2, \ldots, z_n\}}} dz \epsilon(z) < T(z) A_1(z_1) A_2(z_2) \ldots A_n(z_n) > \tag{3.4}$$

This is the integrated conformal Ward identities. where the integral contour surrounds the points $\{z_1, z_2, \ldots, z_n\}$. From the above equation, one can derive the conformal transformation laws of the fields $\{A_i\}$. For example

$$\delta_\epsilon T(z) = 2\epsilon' T(z) + \epsilon T' + \frac{c}{12}\epsilon''' \tag{3.5}$$

that is equivalent to the Operator Product Expansion(OPE)

$$T(z)T(w) = \frac{c}{2}\frac{1}{(z-w)^4} + \frac{1}{(z-w)^2}2T(w) + \frac{1}{z-w}\partial T(w) + \text{regular terms} \tag{3.6}$$

On the other hand, as we knew in the last chapter, the transformation law implies the commutation relation

$$\delta_\epsilon A(z) = [T_\epsilon, A(z)] \tag{3.7}$$

where

$$T_\epsilon = \oint_{C_0} d\zeta\ \epsilon(\zeta)T(\zeta) \tag{3.8}$$

That is to say, the conformal transformation is generated by the stress–energy tensor, which can be expanded as

$$T(z) = \sum_{n\in Z} L_n z^{-n-2} \tag{3.9}$$

If one consider the correlation functions of primary fields, then

$$\begin{aligned} &< T(z)\Phi_1(w_1)\Phi_2(w_2)\ldots\Phi_n(z_n) > \\ &= \sum_{i=1}^{n}\left(\frac{\Delta_i}{(z-w_i)^2} + \frac{1}{z-w_i}\frac{\partial}{\partial w_i}\right) < \Phi_1(w_1)\Phi_2(w_2)\ldots\Phi_n(z_n) > \end{aligned} \tag{3.10}$$

This is the unintegrated conformal Ward identities, which state that the correlation functions are meromorphic functions of z with singularities at the posotions of the inserted operators. The singular terms in the operator product can be written out and can be expressed in three equivalent ways

$$\begin{cases} \delta_\epsilon \Phi(z) = \epsilon\partial_z\Phi(z) + \Delta\epsilon'\Phi(z), \\ T(z)\Phi(w) = \frac{\Delta}{(z-w)^2}\Phi(w) + \frac{1}{z-w}\partial\Phi(w) + \ldots, \\ [L_n, \Phi(z)] = z^{n+1}\partial_z\Phi(z) + \Delta(n+1)z^n\Phi(z) \end{cases} \tag{3.11}$$

Define

$$L_{-k}(z) = \oint_{C_z} d\zeta \frac{T(\zeta)}{(\zeta - z)^{k-1}} \tag{3.12}$$

then, the derivatives of primary fields and their products with the stress tensor T(z) are expressible in terms of this set of operators acting on the primary fields.

$$\begin{aligned} L_0(z)\Phi(z) =& \Phi(z), \\ L_{-1}(z)\Phi(z) =& \partial_z \Phi(z), \\ & \vdots \\ L_{-k}(z)\Phi(z) =& \Phi^{(-k)}(z), \qquad \forall k \geq 0 \end{aligned} \tag{3.13}$$

These are known as secondary fields or the descendant fields of $\Phi(z)$, under the infinitesimal conformal transformation, they transform like

$$\begin{aligned} \delta_\epsilon \Phi^{(-k)}(z) =& \epsilon \partial_z \Phi^{(-k)}(z) + (\Delta + k)\epsilon' \Phi^{(-k)}(z) \\ &+ \sum_{l=1}^{k} \frac{k+l}{(l+1)!} \left[\frac{d^{l+1}\epsilon(z)}{dz^{l+1}} \right] \Phi^{(-k-l)}(z) \\ &+ \frac{c}{12} \frac{1}{(k-2)!} \left[\frac{d^{k+1}\epsilon(z)}{dz^{k+1}} \right] \Phi(z) \end{aligned} \tag{3.14}$$

A primary field together with all of its descendants is known as a *conformal family*, denoted by $[\Phi_i]$.

§3.1.3 Null Vector and Kac Determinant

For a physical model, the energy–momentum tensor $T(z)$ should be unitary and regular everywhere. This requires the vacuum satisfying the following conditions

$$\begin{cases} L_n \mid 0 >= 0, & n \geq -1 \\ < 0 \mid L_n = o, & n \leq 1 \\ L_n^\dagger = L_{-n}, & n \in Z \end{cases} \tag{3.15}$$

The field operators acting on this vacuum yield states of the Hilbert space. Particularly, the primary field creats a highest weight state

$$| \Delta >= \Phi(0) \mid 0 >$$

which satisfies

$$\begin{cases} L_0 \mid \Delta >= \Delta \mid \Delta >, \\ L_n \mid \Delta >= 0, \qquad\qquad n > 1 \end{cases}$$

All of the descendants will creat the states $\{| \Delta >, L_{-k_1}^{\lambda_1} L_{-k_2}^{\lambda_2} \ldots L_{-k_n}^{\lambda_n} \mid \Delta >\}$ (for arbitrary positive integers $\lambda_1, \lambda_2, \ldots, \lambda_n; k_1, k_2, \ldots, k_n$) in Hilbert space of the states of the given CFT. For any primary field, there is a subspace (corresponding to a conformal family, called Verma module) which spans a representation of the Virasoro algebra. But for the total Verma module obtained in this way, we are not guaranteed however that all the states are linearly independent. That depends on the structure of the Virasoro algebra for given values Δ and c. A linear combination of states is called a null vector state if it satisfies

$$\begin{cases} L_n \mid \chi >= 0, & n > 0 \\ L_0 \mid \chi >= (\Delta + k) \mid \chi >, & \\ < \chi \mid \chi >= 0, & \\ < \chi \mid \Delta >= 0, & \forall \mid \Delta > \end{cases} \tag{3.16}$$

Where k is an integer, and is called the level of the null vector. The irreducible representation of the Virasoro algebra with the highest weight $\mid \Delta >$ is constructed from the above Virma module by removing all null vectors (and their descendants). The vacuum is level–0 null vector. From (3.1) and (3.16), one can show that the level–2 null vector is

$$| \chi_2 >= (L_{-2} - \frac{3}{2(2\Delta + 1)} L_{-1}^2) \mid \Delta > \tag{3.17}$$

privoded

$$8\Delta^2 + (c - 5)\Delta + \frac{1}{2}c = 0 \tag{3.18}$$

The level–3 null vector is

$$| \chi_3 >= (L_{-3} - \frac{2}{\Delta + 2} L_{-1} L_{-2} + \frac{1}{(\Delta+1)(\Delta+2)} L_{-1}^3) \mid \Delta > \tag{3.19a}$$

if

$$3\Delta^2 - 7\Delta + 2 + c(\Delta + 1) = 0 \tag{3.19b}$$

Generally, at level N (arbitary integer), one should calculate the Kac–determinant of the matrix, which is the inner products of the states at the given level of the form

$$< \Delta \mid L_{m_l} \ldots L_{m_1} L_{-k_1} \ldots L_{-k_n} \mid \Delta >$$

This is a $P(N) \times P(N)$ matrix(where $P(N)$ is the partition of N). The vanishing of the determinant means the existence of null vectors. Kac has given the formula

$$det M_N(c, \Delta) = const. \prod_{pq \leq N} \left(\Delta - \Delta_{pq}(c) \right)^{P(N-pq)} \tag{3.20}$$

The null vetors occur, if the weight of the primary state takes the following values

$$\Delta_{pq}(m) = \frac{[(m+1)p - mq]^2 - 1}{4m(m+1)} \tag{3.21}$$

where

$$m = -\frac{1}{2} \pm \frac{1}{2} \sqrt{\frac{25 - c}{1 - c}} \tag{3.22}$$

Once we set null vectors equal to zero, then, all the correlation functions between null vectors and other fields will vanish, which results in a set of differantial equetions to determine physical correlation functions among primary fields[4]. For example, level–2 null vetor gives the following order–2 differential equations

$$\left[\frac{3}{2(2\Delta + 1)} \partial_{z_1}^2 + \sum_{i \neq 1} \left(\frac{\Delta_i}{(z_1 - z_i)^2} + \frac{1}{z_1 - z_i} \frac{\partial}{\partial z_i} \right) \right]$$
$$< \Phi_1(z_1) \Phi_2(z_2) \ldots \Phi_n(z_n) >= 0, \qquad \forall n \geq 2 \tag{3.23}$$

The solutions to the above equations are expressible in terms of standard hypergeometric functions. This is the key point to solve a CFT model, since we will see that all the n–point functions can be constructed from the so–called *Conformal Block* ,which generally is not easy to calculate, only at some special values of Δ and c, we can compute it by using the differential equations derived from null states.

§3.1.4 Fusion Algebra

For any given CFT, the field content $\{A_i\}$ contains all the conformal families, which form a closed algebra under the multiplication(OPE). That is *fusion algebra*,

$$[\Phi_i] \cdot [\Phi_j] = \sum_k N_{ij}^k [\Phi_k] \tag{3.24}$$

where the constants N_{ij}^k's are integers that can be interpreted as the the multiplicity of the conformal family $[\Phi_k]$ in the product $[\Phi_i] \cdot [\Phi_j]$, and which also satisfy

$$N_i N_j = N_j N_i, \qquad\qquad N_i N_j = N_{ij}^k N_k \tag{3.25}$$

where we use a matrix notation

$$(N_i)_j^k \equiv N_{ij}^k$$

They can be simutaneously diagonalized and their eigenvalues form one dimensional representations of the fusion algebra[5]. For RCFT, matrices are of finite dimensions.

§3.1.5 Duality and Others

As we saw in the previous subsection, each element of the Fusion algebra is a conformal family. Now, let us consider the product of arbitrary members of conformal families, OPE between two primay fields is

$$\Phi_i(z)\Phi_j(w) = \sum_{p\{k\}} C_{ijp}^{\{k\}} (z-w)^{(\Delta_p - \Delta_i - \Delta_j + \sum_l k_l)} \Phi_p^{\{k\}}(w) \tag{3.26}$$

Non–zero 3–point functions are completely determined by the coefficients $C_{ijp}^{\{k\}}$

$$< \Phi_i(\infty)\Phi_j(z,\bar{z})\Phi_p(0) >= C_{ijp} z^{\Delta_i-\Delta_j-\Delta_p} \bar{z}^{\bar{\Delta}_i-\bar{\Delta}_j-\bar{\Delta}_p} \tag{3.27}$$

For 4–point function

$$< \Phi_i(z_1)\Phi_j(z_2)\Phi_l(z_3)\Phi_m(z_4) >$$

there are two ways to perform OPE, one is $z_1 \longrightarrow z_2, z_3 \longrightarrow z_4$; the other is $z_1 \longrightarrow z_3, z_2 \longrightarrow z_4$. As a multiplication, OPE is of associativity, this means that two different ways should give the same results. So, we have the crossing symmetry or *duality*

$$< \Phi_i(z_1)\Phi_j(z_2)\Phi_l(z_3)\Phi_m(z_4) >$$

$$= \sum_p C_{ijp} C_{lmp}$$

$$(3.28)$$

$$= \sum_q C_{ilq} C_{jmq}$$

This equation gives an infinte number of equations for the coefficients $C_{ijp}^{\{k\}}$. Here, the intermediate states involve many conformal families, if we only consider one of them, for example, $[\Phi_p]$, then we call the following amplitude as the *conformal block*

$$\mathcal{F}_{ij}^{lm}(p \mid x)\bar{\mathcal{F}}(p \mid \bar{x}) = \tag{3.29}$$

Any correlation function can be built from them. Particularly, one conformal block in the first way of OPE can be expressed in terms of ones in the second way of OPE, the coefficients in this transformation is called fusion matrix.

The spin structures of primary fields are determined by monodromy matrices, which are defined as

$$\begin{cases} z \longrightarrow z\exp(2\pi i) \\ \Phi_i(z) \longrightarrow M_{ij}\Phi_j(z) \end{cases}$$

Suppose that in the operator product, we make half monodromy such that the positions of two operators are exchanged, then we get the *braiding matrix*.

The braiding matrices and fusion matrices are not independent. Starting with them, one can discuss the quantum group structure of CFT.

Till now, we only consider CFT on the complex plane, if we work on Riemann surface, besides the conformal symmetry, there is the modular invariance which gives additional constraints on the partion function. We have no space to discuss these subjects.

§3.2 The Semi–classical Limit of CFT

For a CFT which has Virasoro symmetry, the central charge c is very important index. When $c \longrightarrow \infty$, we call theis theory as the semi–Classical limit of the given CFT(CCFT).

In this section, we mainly consider how we can get the classical limit of some simple null vectors. Then, by a strong arguement, we write out the most general form of the classical null vectors.

§3.2.1 The Level–2 Null Vector

From the definition of the secondary fields (3.12), one can rewrite OPE as the following form

$$T(z)\Phi(w) = \sum_{n\in Z} \frac{L_n(z)\Phi(z)}{(z-w)^{n+2}} \tag{3.30}$$

In order to calculate the explicit form of the descendant fields, we introduce the following normal product

$$\begin{aligned} \ddagger T(z)\Phi(w)\ddagger \equiv \sum_{n>0} \Big(z^{n-2} L_{-n}\Phi(w) + z^{-n-2}\Phi(w)L_n \Big) \\ + \frac{1}{2} z^{-2} (L_0\Phi(w) + \Phi(w)L_0) \end{aligned} \tag{3.31}$$

By making use of the commutation relations between L_n's and $\Phi(w)$(3.11), we can establish the connection of these two kinds of the operator product, which is

$$\begin{aligned} T(z)\Phi(w) &= \sum_{n\in Z} L_n z^{-n-2}\Phi(w) \\ &= \ddagger\, T(z)\Phi(w)\,\ddagger + \sum_{n>0} z^{-n-2}[L_n, \Phi(w)] + \frac{1}{2z^2}[L_0, \Phi(w)] \\ &= \ddagger\, T(z)\Phi(w)\,\ddagger + \frac{w(z+w)}{2z^2(z-w)}\partial\Phi(w) + \frac{z^2+2zw-w^2}{2z^2(z-w)^2}\Delta\Phi(w) \\ &= \frac{\Delta}{(z-w)^2}\Phi(w) + \frac{1}{z-w}\partial\Phi(w) \\ &\quad + \ddagger T(w)\Phi(w)\,\ddagger - \frac{\Delta}{2w^2}\Phi(w) - \frac{3}{2w}\partial\Phi(w) \\ &\quad + (z-w)\Big(\ddagger\, T'(w)\Phi(w)\,\ddagger + \frac{\Delta}{w^3}\Phi(w) + \frac{2}{w^2}\partial\Phi(w) \Big) \\ &\quad + \qquad \cdots\cdots \end{aligned} \tag{3.32}$$

This Laurant series results in the following expressions

$$
\begin{aligned}
&L_n(z)\Phi(z) = 0, \qquad\qquad n \geq 1\\
&L_0(z)\Phi(z) = \Delta\Phi(z),\\
&L_1(z)\Phi(z) = \partial_z\Phi(z),\\
&L_2(z)\Phi(z) = \ddagger T(z)\Phi(z) \ddagger - \frac{\Delta}{2z^2}\Phi(z) - \frac{3}{2z}\partial\Phi(z)\\
&L_3(z)\Phi(z) = \ddagger T'(z)\Phi(z) \ddagger + \frac{\Delta}{z^3}\Phi(z) + \frac{2}{z^2}\partial\Phi(z)\\
&\quad\vdots
\end{aligned}
\tag{3.33}
$$

This leads to the following level–2 null vector equation

$$\left((z\partial_z)^2 + 2\Delta z\partial_z + \frac{\Delta(2\Delta+1)}{3}\right)\Phi(z) = \frac{2(2\Delta+1)}{3}z^2 \ddagger T(z)\Phi(z)\ddagger \tag{3.34}$$

The CFT on the complex plane coordinatized by z can now be transformed to a cylinder by the conformal transformation

$$z \longrightarrow w = \ln z, \qquad or \qquad z = e^w \tag{3.35}$$

The primary field transforms tensorially

$$\Phi(z) \longrightarrow \Phi_{cyl}(w)\exp(-\Delta w) \tag{3.36}$$

but, the stress–energy tensor T(z) picks up an anomalous piece which is propertional to the Schwartzian derivative

$$T_{cyl}(w) = z^2T(z) - \frac{c}{24} \tag{3.37}$$

So, the null vector equation on the cylinder is

$$\left(\partial_w^2 + \frac{\Delta(1-\Delta)}{3}\right)\Phi_{cyl}(w) = \frac{2(2\Delta+1)}{3} \ddagger \left(T_{cyl}(w) + \frac{c}{24}\right)\Phi(w)\ddagger \tag{3.38}$$

In the classical limit, the central charge goes to infinity, therefore

$$\Delta = -\frac{1}{2} - \frac{9}{2c} + o(\frac{1}{c^2}) \tag{3.39}$$

On the other hand, in the semi–classical limit, we can simply replace the normal products of operators by the ordinary field products. So, the null vector equation (3.38) becomes

$$\left(\partial_w^2 + \left[\lim_{c\to\infty} \frac{6}{c} T_{cyl}(w)\right]\right) \Phi_{cyl}(w) = 0 \tag{3.40}$$

Since the stress–energy tensor $T_{cyl}(w)$ obeys on the transformation law

$$\delta_\epsilon T_{cyl}(w) = 2\epsilon' T_{cyl}(w) + \epsilon T'_{cyl}(w) + \frac{c}{12}\epsilon'''$$

so, in the classical limit, $T_{cyl}(w)$ is divergent, we can remove this divergence by simply redefining

$$u(w) \equiv \frac{6}{c} T_{cyl}(w) \tag{3.41}$$

This new field obeys the following transformation law

$$\delta_\epsilon u(w) = 2\epsilon' u(w) + \epsilon u'(w) + \frac{1}{2}\epsilon''' \tag{3.42}$$

We recognize that it corresponds to the classical Virasoro algebra

$$\{u(w), u(\tilde{w})\} = -(\frac{1}{2}\partial_w^3 + 2u(w)\partial_w + u'(w))\delta(w-\tilde{w}) \tag{3.43}$$

In terms of this new field, the null vector equation (3.40) can be rewritten as

$$(\partial_w^2 + u(w))\Phi_{cyl}(w) = 0 \tag{3.44}$$

This theory lives on cylinder, particularly, on the circle $w = ix$. Noting that

$$\begin{cases} u(w) = -u(x), \\ \partial_w^2 = -\partial_x^2, \\ \Psi(x) \equiv \Phi_{cyl}(w) \end{cases}$$

then, we obtain the Schrödinger equation

$$(\partial_x^2 + u(x))\Psi(x) = 0 \tag{3.45}$$

§3.2.2 The Level–3 Null Vector

Substituting the explicit expressions of the descendants in (3.33) into the eq(3.19a), then the straightforward computation gives us the following level–3 null vector equation

$$\begin{gathered}(z^3\partial^3 + 3(\Delta+1)z^2\partial^2 + 2(\Delta+1)^2 z\partial + \Delta^2(\Delta+1))\Phi_\Delta(z) \\ -2(\Delta+1)z^3 \ddagger T(z)\partial\Phi_\Delta(z) \ddagger + \Delta(\Delta+1)z^3 \ddagger T'(z)\Phi_\Delta(z)\ddagger = 0\end{gathered} \tag{3.46}$$

where the contant c and Δ satisfy the eqation (3.19b). Again, transforming to cylinder

$$\begin{cases} \Phi_{cyl}(w) = \Phi_\Delta(z)z^\Delta \\ T_{cyl}(w) = z^2T(z) - \frac{c}{24} \\ T'_{cyl}(w) = 2z^2T(z) + z^3T'(z) \end{cases} \tag{3.47}$$

then, the eq(3.46) becomes

$$\begin{aligned}&\left(\partial_w^3 - (1+\frac{\Delta+1}{12}c)\partial_w\right)\Phi_{cyl}(w) - 2(\Delta+1)\ddagger T_{cyl}(w)\partial\Phi_{cyl}(w)\ddagger \\ &+ \Delta(\Delta+1)\ddagger T'_{cyl}(w)\Phi_{cyl}(w)\ddagger = 0.\end{aligned} \tag{3.48}$$

In the classical limit, from (3.19b), we have

$$\Delta = -1 - \frac{12}{c} + o(\frac{1}{c^2}) \tag{3.49}$$

Again, we remove the divergence of $T_{cyl}(w)$ by simply making redefinition (3.41). The new field u(w) satisfies the Virasoro relation (3.43). In terms of u(w), we can write out the null vector equation on the circle $w = ix$

$$(\partial^3 + 4u\partial + 2u')\Psi_{-1}(x) = 0 \tag{3.50}$$

where

$$\Psi_{-1}(x) \equiv \Phi_{cyl}(w)$$

From these two simple examples, we see that the classical limits of null vectors takes the forms

$$d^{(n+1)}[u]\Psi_{-\frac{n}{2}}(x) = 0 \tag{3.51}$$

and u(w) is the classical version of the stress–energy tensor, which satisfies (3.43)(f0r n=1,2).

Generally, a higher level null vector has the form like

$$\chi_{n+1}(z) = \hat{O}(z)\Phi_{\Delta}(z) \tag{3.52}$$

where $\hat{O}(z)$ is an operator which contains $L_k(k = 1, 2, \ldots, n + 1)$. So, in the classical limit, it will be an order (n+1) differential operator. On the other hand, since the conformal transformation law of primary fields have the same form both at quantum level and at the classical level, therefore, the classical null vector is conformal tensor, which is yielded by an order (n+1) differential operator acting on another tensor. For the minimal model of CFT representation), the only involving symmetry is the Virasoro algebra, so from the analysis in the last chapter, we know that the only possible candidate for level–(n+1) null vectors is

$$\chi_{n+1}(x) = d^{(n+1)}\Psi_{-\frac{n}{2}}(x) \tag{3.53}$$

If the CFT has W–algebaic symmetry, then, in its classical limit, the null vector will be of the following form

$$\chi_{n+1}(x) = D^{(n+1)}\Psi_{-\frac{n}{2}}(x) \tag{3.54}$$

Once we set them to zero, we get the null vector equations, which are higher order differential equations for $\Psi_{-\frac{n}{2}}(x)$.

§3.3 CFT as Integrable Model

In the previous section, we have shown that the null vector equation of CCFT has the following universal form

$$Q\Psi(x) = o \tag{3.55}$$

where

$$Q = \begin{cases} D^{(n+1)}, & sl(n+1) \\ d^{(n+1)}, & sl(2) \end{cases} \tag{3.56}$$

We can linearize these equations, what we obtain are just the DS–system (2.57). So, following the standard way, we can define the Gelfand–Dickii Poisson brackets, reconstruct the W–algebra, and solve the Hamiltonian involution system. The involution equation of the operator Q is just the generalized KdV equation

$$\dot{Q} = [P, Q] \tag{3.57}$$

Here, P is the positive parts of the certain fractional powers of Q. Therefore, we can say that the classical conformal field theory(CCFT) in this sense contains a certain DS–system.

§3.4 CCFT as Constrained Involution System

In this section, we will consider several simple null vector equations. At first, we introduce time coordinate by making one parameter family of conformal transformation, then, we will see that the consistent condition of the involution of the constrained condition is just the generalized KdV equation.

§3.4.1 Schrödinger Equation

At first, let us choose the simplst case as a example,

$$D^{(2)}\Psi_{-\frac{1}{2}}(x) = (\partial^2 + u(x))\Psi_{-\frac{1}{2}}(x) = 0 \tag{3.58}$$

From the analysis in §2.4.2, we know that this equation is conformally invariont. therefore if we make one perameter family of the conformal transformations[24]

$$x \longrightarrow f_t(x)$$

then, the fields and the operator transform according to

$$\begin{cases} \Psi_{-\frac{1}{2}}(x) \longrightarrow (f_t'(x))^{-\frac{1}{2}}\Psi_{-\frac{1}{2}}(f_t(x)) \equiv \Psi_{-\frac{1}{2}}(x,t), \\ u(x) \longrightarrow (f_t'(x))^2 u(x) + \frac{1}{2}S_z \equiv u(x,t), \\ D^{(2)}[u(x)] \longrightarrow (f_t'(x))^{\frac{3}{2}} D^{(2)}[u(x)](f_t'(x))^{\frac{1}{2}} = D^{(2)}[u(x,t)]. \end{cases} \tag{3.59}$$

where S_z is the Schwarzian derivative

$$S_z = \frac{f_t'''(x)}{f_t'(x)} - \frac{3}{2}\left(\frac{f_t''(x)}{f_t'(x)}\right)^2 \tag{3.60}$$

The transformed Schödinger equation takes the following form

$$D^{(2)}[u(x,t)]\Psi_{-\frac{1}{2}}(x,t) = (\partial_x^2 + u(x,t))\Psi_{-\frac{1}{2}}(x,t) = 0 \tag{3.61}$$

That means the transformed primary field still be a solution of the transformed equation.

Now, let us view t as time coordinate, then analyze the dynamics of this theory. Directly calculating gives us

$$\begin{aligned} \dot{\Psi}_{-\frac{1}{2}}(x,t) &\equiv \frac{\partial}{\partial t}\Psi_{-\frac{1}{2}}(x,t) \\ &= (-\frac{1}{2}\sigma'(x,t) + \sigma(x,t)\partial_x)\Psi_{-\frac{1}{2}}(x,t) \\ &= (-2\partial_x\sigma + 3\sigma\partial_x + \frac{3}{2}\sigma')\Psi_{-\frac{1}{2}}(x,t) \end{aligned} \tag{3.62}$$

where

$$\sigma(x,t) \equiv \frac{\dot{f}_t(x)}{f_t'(x)} \tag{3.63}$$

We choose

$$\sigma(x,t) = \frac{1}{2}u(x,t)$$

then, from eqs(3.58) and (3.62), we obtain the time involution equation of the primary field Psi, or its dynamical equation

$$\dot{\Psi}_{-\frac{1}{2}}(x,y) = P\Psi_{-\frac{1}{2}}(x,t) \tag{3.64}$$

where we adopt the convetions

$$\begin{cases} Q = \partial_x^2 + u(x,t), \\ P \equiv (Q^{\frac{3}{2}})_+ \end{cases} \tag{3.65}$$

and generally, we define

$$P_k \equiv (Q^{\frac{k}{2}})_+ \qquad\qquad \forall k \geq 3. \tag{3.66}$$

In fact, since the involution dynamics and the infinitesimal transformation law are closely related, i.e.

$$\delta_\epsilon \Psi_{-\frac{1}{2}}(x) = \dot{\Psi}_{-\frac{1}{2}}(x, t=0)\delta t = -\frac{1}{2}\epsilon'(x)\Psi_{-\frac{1}{2}}(x) + \epsilon(x)\Psi'_{-\frac{1}{2}}(x) \tag{3.67}$$

if we define

$$\delta t \dot{f}_t(x) \mid_{t=0} \equiv \epsilon(x)$$

then, we can simply get $\dot{\Psi}$ directly from this infinitesimal transformation law by making the replacement

$$\begin{cases} \delta_\epsilon \Psi_{-\frac{1}{2}}(x) \longrightarrow \dot{\Psi}, \\ \epsilon \longrightarrow \sigma(x,t) = \frac{\dot{f}_t(x)}{f_t'(x)} \end{cases} \tag{3.68}$$

Thus, we have introduced the dynamics through one parameter family of field–dependent conformal transformations.

$$\begin{cases} \dot{\Psi}_{-\frac{1}{2}}(x,t) = P\Psi_{-\frac{1}{2}}(x,t), & \text{dynamical equation,} \\ Q\Psi_{-\frac{1}{2}}(x,t) = 0, & \text{constrained condition} \end{cases} \tag{3.69}$$

§3.4.2 The Boussinesque Equation

Now, we turn our attention to the CCFT with W-algebra symmetry. let

$$\begin{cases} Q = \partial^3 + 4u\partial + 2u' + V_2, \\ Q\Psi_{-1}(x) = 0 \end{cases} \tag{3.70}$$

As we know, this equation is invariant not only under the conformal trasformation, but also under the spin-3 transformation. The infinitesimal spin-3 transformation is defined as

$$\begin{aligned} \delta x =& \epsilon(x), \\ \delta_\epsilon \Psi_{-1}(x) =& \int dy \epsilon(y) \{\Psi_-(x), V_2(y)\} \\ =& -(\frac{1}{6}\epsilon'' + \epsilon\partial_x^2 + \frac{8}{3}\epsilon u - \frac{1}{2}\epsilon' \partial)\Psi_{-1}(x) \end{aligned} \tag{3.71}$$

Since the involution of $\Psi_{-1}(x)$ has the same structure as this transformation law, so, for a finite one parametor family of spin-3 transformation

$$\begin{cases} x \longrightarrow f_t(x), \\ \sigma(x,t) \equiv \frac{\dot{f}_t(x)}{f_t'(x)} \end{cases}$$

directly, we have

$$\dot{\Psi}_{-1}(x,t) = -(\frac{1}{6}\sigma'' + \sigma\partial^2 + \frac{8}{3}\sigma u - \frac{1}{2}\sigma' \partial)\Psi_{-1}(x,t) \tag{3.72}$$

Now we choose

$$\sigma(x,t) = -1$$

That means

$$(\partial_t + \partial_x) f(x,t) = 0$$

This tell us that $f(x,t)$ only depends on $(x-t)$. Accordingly, we have

$$\dot{\Psi}_{-1}(x,t) = (\partial^2 + \frac{8}{3}u(x,t))\Psi_{-1}(x,t) \equiv P\Psi_{-1}(x,t) \tag{3.73}$$

it is easy to prove

$$P = (Q^{2/3})_+ \tag{3.74}$$

Then, we have a constrained system once again

$$\begin{cases} \dot{\Psi}_{-1}(x,t) = P\Psi_{-1}(x,t) \\ Q\Psi_{-1}(x,t) = 0 \end{cases} \tag{3.75}$$

Thus, we may conclude that CCFT can be considered as a certain constrained involutive system, which has the following universal form

$$\begin{cases} \dot{\Psi}_{-\frac{n}{2}}(x,t) = P\Psi_{-\frac{n}{2}}(x,t) \\ Q\Psi_{-\frac{n}{2}}(x,t) = 0 \end{cases} \tag{3.76}$$

Here, Q is some higher differential operator, and P is the positive parts of the certain fractional powers of Q.

§3.4.3 The KdV Structure in CCFT

Till now, we have shown that CCFT is equivalent to some constrained involutive system. Now, the problem is the consistency. Since if the constrained system is well-defined, the dynamics and the constrain condition should be compatible. This compatibility will give us some consistent condition. Now, we will see that this condition is just the KdV equation.

Take the derivative of the constrained condition in (3.84) with respect to the time parameter, we get

$$\dot{Q}\Psi_{-\frac{n}{2}}(x,t) + Q\dot{\Psi}_{-\frac{n}{2}}(x,t) = 0 \tag{3.77}$$

From eq(3.84), we see that this equation leads to

$$(\dot{Q} + [Q,P])\Psi_{-\frac{n}{2}}(x,t) = 0 \tag{3.78}$$

Since the operator Q annihilates the field $\Psi_{-\frac{n}{2}}(x,t)$, therefore, its any polynomal also annihilate $\Psi_{-\frac{n}{2}}(x,t)$, but all of the them have higher ranks than that of Q. On the other hand, $\dot{Q}$ has less rank than Q. Furthermore, since

$$[Q,P] = [Q,(Q^{\frac{m+1}{n+1}})_+] = [(Q^{\frac{m+1}{n+1}})_-,Q] \tag{3.79}$$

so $[P, Q]$ also has lower order than Q has. Finally, we find that the order of the operator $(\dot{Q} + [Q, P])$ is less than that of Q, however, the simplest annihilator of $\Psi_{-\frac{n}{2}}(x, t)$ should be Q. Thus we get

$$\dot{Q} = [P, Q] \tag{3.80}$$

This is nothing but KdV equation. For example, if

$$Q = \partial^3 + 4u\partial + 2u' + V_2 \tag{3.81}$$

then, this KdV equation reduces

$$\begin{cases} \dot{u} = \frac{1}{2}V_2' \\ \dot{V}_2 = -\frac{2}{3}u''' - \frac{32}{3}uu' \end{cases} \tag{3.82}$$

which is just the Boussinesque equation.

We summarize in a few words, for null vector equation, when we make one parameter family of the transfirmation , the dynamies for the primary field appears, and the consistent involution of the constrain gives us the condition which is nothing but the KdV equation.

$$\begin{cases} \dot{\Psi} = P\Psi \\ Q\Psi = 0 \end{cases} \tag{3.83}$$

and

$$\dot{Q} = [P, Q] \tag{3.84}$$

A few comments should be mentioned. Firstly, when we choose different field-dependent gauge transformation $\sigma(x, t)$, we will introduce the different dynamics for $\Psi(x, t)$. Secondly, since the null vector equation gives a set of differential equations for correlation functions, we can in principle discuss their properties with the help of such a kind of KdV structure. Furthermore, we also can consider the quantized version of the classical null vector equations.

Chapter 4

Introduction to 2–Dimensional Quantum Gravity

In this chapter, we will give a relatively self–contained but far from complete introduction to 2–dimensional quantum gravity. There are two motivations to study 2–dimensional quantum gravity. One of them stems from the investigation of the off–critical string theory. Another reason is to shed some light on 4–dimensional quantum gravity. It is well–known that a first quantized string propagating in R^d–spacetime can be most elegantly discribled as a theory of d–free bosons coupled to 2d–quantum gravity[8]. The bosonic matter system has the conformal invariance, therefore, the theory can be considered as a certain CFT coupled to 2d–quantum gravity. Since a physical theory should be anomaly free, the total anomaly of matter sector and gravity sector should be zero. In this sense, we say that matter and gravity couple together. In critical dimension $d = 26$(or in superstring case, $d = 10$), the matter part is the anomaly free, so, the matter and the gravity essentially decouple, we can consider them separately. However, if the target space has non–critical dimension, the non–zero anomaly of matter sector should be compensated by the anomaly of gravity, so, dealing 2d–quantum gravity is an unavoidable step. The investigations of 2d–gravity are going on three different directions. The first one is the path integral formalism, in which by making suitable gauge fixing, we can get some critical exponents. Instead, the discretized version of gravity admits a matrix model representation, so, one can use matrix approach to study the perturbative and non–perturbative properties of gravity. Finally, the topological field theory is also a powerful tool to attack gravity.

§4.1 Path Integral Formalism

Considerable progress in this direction was made by Polyakov[8], and later KPZ[10]. They quantized the theory in light–cone gauge, in which they discovered a rich symmetry structure that is sl(2,R) Virasoro–Kac–Moody algebra. This symmetry survives the quantization of the theory and gives the exact solution of 2d–quantum gravity.

After the success in light–cone gauge formalism, Distler and Kawai[26] and David[27] proposed a conformal gauge method. This is based on the fact that 2d–gravity can be represented as the Liouville action, whose free part is a conformal theory, and the Liouville interaction term can be treated as a marginal deformation of the free action. In this way, we can derive the gravitational anomalous dimensions in a much easier manner. But, the gravitational dressed correlation functions are still not so easy to calculate.

These two methods both are based on path integral formalism. Now, for the sake of simplicity, we briefly review the second procedure.

§4.1.1 Conformal Gauge

Let Σ be a smooth two dimentional surface of genus h (no complex structure given), and let g be a metric on Σ. The space of metrics is an infinite dimensional Riemannian manifold, which will be denoted by MET_h. The inner product on its tangent vector space can be defined as

$$\| \delta g \|_g^2 = \int d^2\xi \sqrt{g}(Ag^{ab}g^{cd} + Bg^{ac}g^{bd})\delta g_{ab}\delta g_{cd} \tag{4.1}$$

Where A and B are non-negative constants. This determines a metric on MET_h, and thus formally, a Riemannian measure denoted as $\mathcal{D}g$, which is g-dependent. On the other hand, if there is matter fields living on Σ, in the same fashion, one can define the functional measure $\mathcal{D}_g X$ as

$$\begin{aligned} &\int \mathcal{D}_g X \delta X \exp(-\| \delta X \|_g^2) = 1, \\ &\| \delta X \|_g^2 = \int d^2\xi \sqrt{g} \delta X \cdot \delta X. \end{aligned} \tag{4.2}$$

Consider a general action (which describibles matter fields X^μ couple to 2d-gravity)

$$Z = \int_\Sigma \frac{\mathcal{D}g\mathcal{D}_g X}{Vol.(Diff)} \exp(-S_M(X;g) - S_c) \tag{4.3}$$

Where S_M is the matter action, and S_c is the counter-term. The factor divided out is the volume of the symmetry group which are the diffeomorphisms of the Reimann surface. The matter action and the measures are totally reparametrization invariant, furthermore, the matter action is also invariant under the Weyl rescaling of the metric g

$$\begin{cases} g \longrightarrow e^\sigma g, \\ S_M(X;g) \longrightarrow S_M(X;e^\sigma g) = S_M(X;g). \end{cases} \tag{4.4}$$

but, this doesn't work for the measures. This is the crucial point of the theory which we should carefully analyze. In order to see this, we make gauge fixing as we usually do in quantum field theory, i.e. we parametrize g by a reference metric $\hat{g}$ and the Liouville field Φ.

$$Z = \int \frac{[d\tau]}{Minimal\ Vol.} \mathcal{D}_g X \mathcal{D}_g \Phi \mathcal{D}_g b \mathcal{D}_g c \exp(-S_M(X;g) - S_{gh}(b,c;g) - S_c) \tag{4.5}$$

Where $S_{gh}(b,c;g)$ is the ghost action, which is also Weyl and reparametrization invariant. On the other hand, the measures of the matter and ghost fields transform according to

$$\mathcal{D}_{\hat{g}e^\Phi} X = \mathcal{D}_{\hat{g}} X \exp(\frac{d}{48\pi} S_L(\hat{g};\Phi)) \tag{4.6a}$$

$$\mathcal{D}_{\hat{g}e^\Phi} b \mathcal{D}_{\hat{g}e^\Phi} c = \mathcal{D}_{\hat{g}} b \mathcal{D}_{\hat{g}} c \exp(-\frac{26}{48\pi} S_L(\hat{g};\Phi)) \tag{4.6b}$$

Where S_L is the Liouville action

$$S_L(\hat{g};\Phi) = \int d^2\xi\sqrt{\hat{g}}\left(\frac{1}{2}\hat{g}^{ab}\partial_a\Phi\partial_b\Phi + \hat{R}\Phi + \mu_0 e^{\Phi}\right) \tag{4.7}$$

Here μ_0 is the bare cosmological constant, and $\hat{R}$ is the scalar curvature of the reference metric $\hat{g}$

$$(\frac{1}{8\pi})\int d^2\xi\sqrt{\hat{g}}\hat{R} = 1 - h \tag{4.8}$$

Where h is the genus of the Reimann surface. The norm of the Liouville field Φ is induced by (4.1)

$$\| \delta\Phi \|_g^2 = \int d^2\xi\sqrt{\hat{g}}e^{\Phi}(\delta\Phi)^2 \tag{4.9}$$

So, it determines a functional measure for Φ (denoted by $\mathcal{D}_g\Phi$) , which is obviously Φ–dependent. Due to this fact, the path integral over Φ is quite difficult to perform. It would be nice if we had a new measure like

$$\| \delta\Phi \|_{\hat{g}}^2 = \int d^2\xi\sqrt{\hat{g}}(\delta\Phi)^2 \tag{4.10}$$

Several authors have shown that this measure can be obtained from (4.9) by the Weyl rescaling transformation (4.4). The Jacobian of this transformation is just the exponential of the local action of the Liouville type[47,48]. Finally, the total measure transforms like

$$\mathcal{D}_{\hat{g}e^{\Phi}}X\mathcal{D}_{\hat{g}e^{\Phi}}b\mathcal{D}_{\hat{g}e^{\Phi}}c\mathcal{D}_{\hat{g}e^{\Phi}}\Phi = \mathcal{D}_{\hat{g}}X\mathcal{D}_{\hat{g}}b\mathcal{D}_{\hat{g}}c\mathcal{D}_{\hat{g}}\Phi\exp(-S(\Phi;\hat{g})) \tag{4.11}$$

where $S(\Phi;\hat{g})$ is a Liouville type action, choosing suitable convention, which has the form

$$\begin{aligned} S(\Phi;\hat{g}) =& \frac{1}{8\pi}\int d^2\xi\sqrt{\hat{g}}(\hat{g}^{ab}\partial_a\Phi\partial_b\Phi - \mathcal{Q}\hat{R}\Phi + 4\mu\exp\alpha\Phi) \\ =& \frac{1}{2\pi}\int d^2z(\partial\Phi\bar{\partial}\Phi - \frac{1}{4}\mathcal{Q}\sqrt{\hat{g}}\hat{R}\Phi + \mu\sqrt{\hat{g}}\exp\alpha\Phi) \end{aligned} \tag{4.12}$$

here, S can also have other finite counter–terms. The coefficients are fixed by the requirements of the anomaly free condition and the dimension analysis.

$$\mathcal{Q} = \sqrt{\frac{(25-d)}{3}} \tag{4.13a}$$

and

$$\alpha_{\pm} = -\frac{1}{2}Q \pm \sqrt{Q^2 - 8} = (-\frac{1}{2\sqrt{3}})(\sqrt{25-d} \mp \sqrt{1-d}) \tag{4.13b}$$

Finally, we can get the totally renormalized action and the stress tensor

$$S_{total} = S_M + S_{gh} + S_0 + \delta S_f \tag{4.14a}$$

$$T_{total} = T_M + T_{gh} + T_\phi + \delta T_f \tag{4.14b}$$

$$Z(\beta_i) = \int \mathcal{D}_{\hat{g}} X \mathcal{D}_{\hat{g}} b \mathcal{D}_{\hat{g}} c \mathcal{D}_{\hat{g}} \Phi e^{-S_{total}} \tag{4.14c}$$

where S_0 is the free part of the Liouville action S

$$S_0 = \frac{1}{2\pi}\int d^2z(\partial\Phi\bar{\partial}\Phi - \frac{1}{4}Q\sqrt{\hat{g}}\hat{R}\Phi)$$

the stress tensor obtained from this action is

$$T_\phi(z) = -\frac{1}{2}(\partial\Phi\partial\Phi + Q\partial^2\Phi)$$

The last term in (4.14a) is the finite counter–term, which are renormalized vertices. Since, in CFT, a vertex is in fact a primary field, so, the possible finite counter–terms are of the following form

$$\delta S_f = \sum_i \beta_i \int d^2\xi\sqrt{\hat{g}}\Psi_i^M(\xi)\Psi_i^{gh}(\xi)e^{(\alpha_i\Phi)} \tag{4.15}$$

where, we denote the matter primary field by $\Psi_i^M(\xi)$, the ghost one by $\Psi_i^{gh}(\xi)$ and the liouville ghost screening factor by $\exp(\alpha_i\Phi)$. with the total conformal weight is equal to 1

$$\Delta_i^M + \Delta_i^{gh} + \Delta_i^\Phi = 1 \tag{4.16}$$

and

$$\Delta_i^\Phi = -\frac{1}{2}{\alpha_i}^2 - \frac{1}{2}\alpha_i Q \tag{4.17}$$

Now, all the terms in δS_f are marginal operators. On the other hand, since Φ is dummy integration field, the partition function is is invariant under its shift, this invariance will help us to get some important results.

§4.1.2 String Susceptibility

In a theory containing gravity, the physical observables should integrate over all the Riemann surface. For Fadeev–Popov ghost independent operators, we have

$$\Psi_i^{gh}(\xi) = 1, \qquad \Delta_i^{gh} = 0$$

therefore, from the eq(4.17), we obtain the screening charge as follows

$$\alpha_i = -\frac{1}{\sqrt{12}} \frac{[25 - d - \sqrt{(25-d)(1-d+24\Delta_i^M)}]}{\sqrt{25-d}} \tag{4.18}$$

One of the special case is that the matter primary field is the identity operator, also called puncture operator,denoted by $\mathcal{P}$, we have

$$\Psi_i^M(\xi) = 1, \qquad \Delta_i^M = 0$$

this leads to

$$\alpha_0 = -\frac{1}{\sqrt{12}}(\sqrt{25-d} - \sqrt{1-d}) \tag{4.19}$$

For unitary models, all the conformal weights are non–negative, therefore, we have α_0 is the minimal screening charge of the primary fields. However, in non–unitary models, the conformal weight can be negative, so, the minimal screening charge corresponds to the primary field of the most negative conformal weight. Now, choose the following finite counterterm

$$\delta S_f = \mu \int d^2\xi \sqrt{\hat{g}} \exp(\alpha_0 \Phi) \tag{4.20}$$

where, μ is in fact the renormalized cosmological constant. The partition function

$$Z(\mu) = \int \mathcal{D}_{\hat{g}} X \mathcal{D}_{\hat{g}} b \mathcal{D}_{\hat{g}} c \mathcal{D}_{\hat{g}} \Phi \exp -(S_M + S_{gh} + S_0 + \mu \int d^2\xi \sqrt{\hat{g}} \exp(\alpha_0 \Phi)) \tag{4.21}$$

for small μ, has the form

$$Z(\mu) \approx \mu^{-\Gamma+2} \tag{4.22}$$

under the following translation

$$\begin{cases} \Phi \longrightarrow \Phi + \frac{\rho}{\alpha_0} \\ S_0 \longrightarrow S_0 - \mathcal{Q} \int d^2z \sqrt{\hat{g}} \hat{R} \frac{\rho}{\alpha_0} = S_0 - \mathcal{Q}(1-h)\frac{\rho}{\alpha_0} \end{cases} \tag{4.23}$$

due to the translational invariance of the partition function, we have

$$Z(\mu) = Z(\mu e^{\rho}) \exp((1-h)\frac{\mathcal{Q}}{\alpha_0}\rho) \tag{4.24}$$

Now, we introduce the critical exponent in the following way

$$(2-\Gamma) \equiv (2-\gamma_{str.})(1-h) \tag{4.25}$$

Therefore, from eq.(4.22), one can easily obtain

$$\gamma_{str.} = \frac{1}{12}\left(d - 25 - \sqrt{(25-d)(1-d)}\right) + 2 \tag{4.26}$$

This is the so–called string susceptibility, which accounts for the contribution of the identity operator to the free energy, and indicates the singularity of the free energy in the infrared limit($\mu \longrightarrow 0$).

Now, suppose we choose another kind of the finite counterterm

$$\delta S_f = \beta \int d^2\xi \sqrt{\hat{g}} \Psi_i^M(\xi) e^{(\alpha_i \Phi)} \tag{4.27}$$

then, making use of the translational invariance of the partition function, we get

$$Z(\beta) \sim \beta^{(h-1)\frac{\mathcal{Q}}{\alpha_i}} \tag{4.28}$$

therefore, from the eq.(4.22), one obtain

$$\gamma_{str.} = \frac{\mathcal{Q}}{\alpha_i} + 2 = \frac{1}{12\Delta_i^M}\left(25 - d - \sqrt{(25-d)(1-d+24\Delta_i^M)}\right) + 2 \tag{4.29}$$

Generally, the finite counterterm can contains all of the possible gravitational dressed physical primary fields. So, the string susceptibility is determined by the most singular term, i.e. the term of the minimal screening charge.

For the unitary model, the most singular term of the free energy comes from the identity operator(which minimizes $\gamma_{str.}$). So, for unitary minimal BPZ series

$$d = 1 - \frac{6}{m(m+1)} \qquad m = 3, 4, \ldots \tag{4.30}$$

one get the string susceptibility

$$\gamma_{str.} = -\frac{1}{m}, \qquad m = 3, 4, \ldots \tag{4.31}$$

For non–unitary BPZ series,

$$d = 1 - \frac{6(p-q)^2}{pq} \qquad (p, q)\ are\ relative\ prime \tag{4.32}$$

the primary field with the conformal weight

$$\Delta(p, q) = -\frac{(p-q)^2}{4pq} + \frac{1}{4pq}$$

contributes the most singular term. This leads to

$$\gamma_{str.} = -\frac{2}{p+q-1} \tag{4.33}$$

Suppose

$$\begin{cases} p = 2, \\ q = 2m - 1 \end{cases}$$

We get the sigularity of Yang–Lee model

$$\gamma_{str.} = -\frac{1}{m}$$

We see that it has the same value as the unitary minimal model. This fact reminds us that different theories probably have the same critical exponents.

We should remark that here, the deduction is only a formal one, since the integration involved in the eq.(4.14c) is difficult to perform and renormalization is also a big problem. The usual way to do this is to consider the behaviour of the partition function of fixed area. However, we can formally define the expectation value of the identity operator

$$< \mathcal{P} >= \int \mathcal{D}_{\hat{g}} X \mathcal{D}_{\hat{g}} b \mathcal{D}_{\hat{g}} c \mathcal{D}_{\hat{g}} \Phi \exp(-S_{total}) \int d^2\xi \sqrt{\hat{g}} \exp(\alpha_0 \Phi)$$

By making use of the translational invariance, one finds that for small μ, the area behaves like

$$< \mathcal{P} > \sim \mu^{-1}$$

so, $(< \mathcal{P} > \mu)$ is like a scalar under the rescaling of μ, therefore, we say that area and the renormalized cosmological constant are conjugate to each other, and the fixed area behaviour of the partition function can be describled in terms of the renormalized cosmological constant. The small μ region corresponds to the surfaces of large areas.

§4.2 Introduction to Matrix Model

Another powerful tool to attack 2d–quantum gravity is the matrix model approach. As we know, for a partition function of the off–critical string theory, the most difficult problem is how to perform the functional integral over the metric tensor g, in the previous section, we treated this by making gauge fixing of the reparametrization and introducing a set of ghost fields, then, by using the translational invariance of the partition function, we obtained the string susceptibility. In fact, instead, one can discretize the world sheet of the string(or two dimensional space in the case of pure gravity), so, every Reimann surface with a given topology corresponds to a certain triangulation(or polygons). Therefore, integration over MET_h reduces to counting inequivalent triangulations of the Riemann surface in pure gravity case. Fortunately, this counting problem can be solved in terms of random matrix method[39,49], finally, we obtain the string equation and the string susceptibility.

§4.2.1 Discretization of Riemann Surface

For the sake of simplicity, we only consider pure gravity[39]. In this case, we would like to compute the integral

$$F(h) = \int_{MET_h} \mathcal{D}g e^{-S}$$

where

$$S = \mu_0 \int_\Sigma d^2\xi\sqrt{g} + \frac{\theta}{2\pi}\int_\Sigma d^2\xi\sqrt{g}R$$

The first term is the area of the Riemann surface, and the second one is a topological invariant, which is the Euler characteristic $\chi(\Sigma)$. μ_0 is the bare cosmological constant. Now, let $MET_{A,h}$ be the subspace of metrics of total area A on a surface with genus h, denotes its volume by Vol(h,A), then, one can perform the integration in two steps

$$F(h) = \int dAF(h,A) \tag{4.33}$$

and

$$F(h,A) = \int_{MET_{A,h}} \mathcal{D}g e^{(-\mu_0 A - \theta\chi(\Sigma))} = Vol(h,A)e^{(-\theta A - \mu_0\chi(\Sigma))} \tag{4.34}$$

In order to perform this integral, one can discretize Σ. Imagine a surface(we will call it triangulation) made from adjacent equilateral triangles with area ϵ embedded into some auxiliary Euclidian space of sufficiently high dimension, we can say that it consists of some number of discrete points (the vertices of triangles) and definite couples of points (the neighbours connected by edges of a triangle) are equally seperated. We say two triangulations are different if they have different configurations in the auxiliary space. Two different triangulations with the same topology can be related by a sequence of certain flips of links (edges of triangles). Denote the number of triangles by n, then the total area of the triangulation is $A = n\epsilon$. For a fixed A, increasing the number of triangles, and appropriately choosing the type of the triangulation, we can obviously approximate a given Riemann surface. So, every triangulation determines a metric on Σ. Thus, counting different triangulations of Σ becomes approximation to computing the integration over the space MET_h. In fact, the triangulation of Riemann surface is just dual diagram of the certain Feynman graph of the matrix field theory with cubic interaction, therefore, we only need enumerate Feynman graphs. Using matrix method, one prove that the number of isomorphism classes of triangulations of a genus h surface with n triangles has the following large n behavior

$$V(h,n) \sim e^{\alpha n} \cdot n^{\gamma(2-2h)-1} \cdot b_h(1 + o(\frac{1}{n})) \tag{4.35}$$

and

$$Vol_\epsilon(h,A) \sim \frac{1}{\epsilon} V(h,n) = \frac{1}{\epsilon} e^{\alpha A/\epsilon} \left(\frac{A}{\epsilon}\right)^{\gamma(2-2h)-1} b_h \tag{4.36}$$

Here ϵ takes the role of the cutoff, and the corresponding cutoff version of F(h,A) is

$$F_\epsilon(h,A) = Vol_\epsilon(h,A) \exp(-\theta A - \mu_0(2-2h))$$

$$= \frac{1}{\epsilon} e^{\alpha A/\epsilon} \left(\frac{A}{\epsilon}\right)^{\gamma(2-2h)-1} b_h \exp(-\mu_0 A - \theta(2-2h)) \tag{4.37}$$

by choosing

$$\mu_0 = \alpha/\epsilon, \qquad \theta = \gamma \ln(A_0/\epsilon)$$

we get renormalized function F(h,A)

$$F(h,A) = \frac{1}{A}\left(\frac{A}{A_0}\right)^{\gamma(2-2h)-1} b_h \tag{4.38}$$

where A_0 is arbitrary constant with dimensions of area. Formally performing the integration over A, we have

$$F(h) = \int_0^\Lambda dA F(h,A) = \left(\frac{\Lambda}{A_0}\right)^{\gamma(2-2h)} \tilde{b}_h \tag{4.39}$$

Comparing with the result obtained in the previous section, we see that the ratio $\frac{A}{A_0}$ takes the role of the renormalized cosmological constant, and γ is related to the string anomalous dimension

$$\gamma = 2 - \gamma_{str}.$$

which is a universal constant, for example, if we use squares instead of triangles, we will get the same results. In fact, we can use any other polygon to replace triangle, then, after renormalization, the function F(h,A) has the same form as before, but γ will change discretely.

§4.2.2 The Graphic Enumeration

At first, let us consider a simple example, a matrix field theory with the action

$$V(M, g_0) = \frac{1}{2}Tr(M^2) + \frac{g_0}{N}Tr(M^4) \tag{4.40}$$

where M is $N \times N$ Hermitian matrix, which forms a N^2–dimensional real vectorspace. This real vector space carries a representation of the SU(N)

$$M \longrightarrow M^U = UMU^{-1}$$

The lebesgue measure

$$dM = \prod_{i=1}^{N} dM_{ii} \prod_{1\le i<j\le N} d(ReM_{ij})d(ImM_{ij})$$

is invariant under such a unitary transformation. The partition function is defined as usual

$$Z_N(\beta, g_0) = \int dM \exp(-\beta V(M, g_0)) \tag{4.41}$$

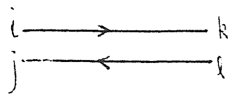
where β can be considered as the renormalization constant which gets into the game by repacing M by $\beta^{\frac{1}{2}}M$, and g_0 is bare coupling constant, whose renormalized version is denoted by g. The action becomes βV(M,g) with

$$V(M, g) = \frac{1}{2}Tr(M^2) + g(\frac{\beta}{N})Tr(M^4) \tag{4.42}$$

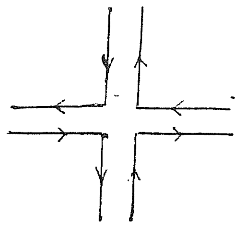
The propagator reads

$$< M_{ij}M^*_{kl} >= \frac{\int dM \exp\left(-\frac{\beta}{2}Tr(M^2)\right) M_{ij}M^*_{kl}}{\int dM \exp\left(-\frac{\beta}{2}Tr(M^2)\right)} = \frac{1}{\beta}\delta_{ik}\delta_{jl} \tag{4.43}$$

The Feymann rules are



$= \frac{1}{\beta}\delta_{ik}\delta_{jl}$



$= 4! \, \frac{g\beta^2}{N}$

Here, since a matrix has two indices, we should use a double line to denote it.

Following the usual way, we define the free energy

$$F_N(\beta, g) \equiv -\ln Z_N(\beta, g) \tag{4.44}$$

which generates all the connected graphs. It can be expanded in powers of the coupling constant g

$$F_N(\beta, g) = -\sum_{k=0}^{\infty} \frac{(-g\beta/N)^k}{N^k k!} < (Tr(M^4))^k >_c \tag{4.45}$$

where the index c means the connected graphs. For any given graph with V vertices B propagators(bonds) and I index loops, we have (for quartic interaction)

$$B = 2V$$

Since each closed loop corresponds to a dummy index running from 1 to N, then, gives a factor N, and each propagator contributes a factor $\frac{1}{\beta}$, and for fixed $\frac{\beta}{N}$, each vertex contributes a factor $\cancel{N}$ β. So, this graph contributes an overall N and β dependent factor which is

$$N^I \beta^{V-B} = N^{I+V-B}\left(\frac{\beta}{N}\right)^{V-B} = N^{\chi}\left(\frac{N}{\beta}\right)^V \tag{4.46}$$

where we have used Euler's formula

$$I + V - B = \chi = 2(1-h)$$

and h indicates the topology of the diagram. So, finally, we get the topological expansion of the free energy

$$-\ln Z_N(\beta) = F_N(\beta) = regular + \sum_{graph} N^{\chi} \left(\frac{\beta}{N}\right)^{V} F_{graph}[V] \tag{4.47}$$

where the factor $F_{graph}[V]$ is given by the products of the vertex weights derived by the symmetric factors. The regular term is constant which doesn't give any information about the theory. For general fixed value of $\frac{\beta}{N}$, in the limit $N \longrightarrow \infty$, only zero genus terms survive, this is so–called spheric limit. Now, define

$$F_{\chi} \equiv \sum_{\substack{graph \\ \chi\ fixed}} \left(\frac{\beta}{N}\right)^{A} F_{graph}[V] \tag{4.48}$$

In order to get the contributions of higher genus, we should carefully tune β such that F_{χ} is divergent, then , for convenient, tune the coupling constant so that the singularity occurs at the point

$$\frac{N}{\beta} = 1$$

therefore, the scaling laws arise, that is to say, F_{χ} has the following power form

$$F_{\chi} \sim \left(\frac{N}{\beta} - 1\right)^{2-\Gamma} \tag{4.49}$$

Define

$$2 - \Gamma \equiv \frac{\chi}{2}(2 - \gamma_{str.})$$

then, we can express the free energy as

$$\begin{aligned} F_N(\beta) &= \text{regular} + \sum_{\chi} N^{\chi} \left(\frac{N}{\beta} - 1\right)^{\frac{\chi}{2}(2-\gamma_{str.})} a_{\chi} \\ &\sim \sum_{\chi} \left[N \left(\frac{N}{\beta} - 1\right)^{\frac{2-\gamma_{str.}}{2}} \right]^{\chi} a_{\chi} \end{aligned} \tag{4.50}$$

Thus, when we choose the double limit

$$\begin{cases} N \longrightarrow \infty \\ \frac{N}{\beta} \longrightarrow 1 \end{cases} \tag{4.51a}$$

and

$$N\left(\frac{N}{\beta}-1\right)^{\frac{2-\gamma_{str.}}{2}} \equiv \frac{1}{g_{str.}} \tag{4.51b}$$

keeping fixed in this limit, the free energy has the following critical behavier

$$F(g_{str.}) \sim \sum_h g_{str.}^{(2h-2)} a_h \tag{4.52}$$

where $\gamma_{str.}$ is the string susceptibility, and $g_{str.}$ is the string coupling constant, or cosmological constant (from the genus dependences in (4.22) and (4.52), one can see that they are related), and the factor a_h is $g_{str.}$ independent constant.

§4.2.3 The Orthogonal Polynomial

A powerful mathematical tool for computing $F_N(g)$ is the orthogonal polynomial[51–54]. Now consider the original integral, at first, we can integrate out the angular variables and keep an integral over the N–eignvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_N\}$ of the matrix M.

$$Z_N(\beta, g) = \frac{\Omega_N}{(2\pi)^N} \int_{-\infty}^{+\infty} \prod_{1 \le i \le N} d\lambda_i \Delta_N^2(\lambda) \exp\left(-\beta \sum_{i=1}^{N} V(\lambda_i)\right) \tag{4.53}$$

where

$$V(\lambda) = \frac{1}{2}\lambda^2 + g\lambda^4 \tag{4.54}$$

and

$$\Delta_N(\lambda) = \prod_{1 \le i \le N} (\lambda_i - \lambda_j) = det \parallel \lambda_i{}^{j-1} \parallel \tag{4.55}$$

It is Vandermonde determinant. The quantity Ω_N is related to the volume of the unitary group.

Disgarding the irrelevant constant, we define

$$d\,\mu(\lambda) \equiv \exp\left(-\beta V(\lambda)d\,\lambda\right. \tag{56a}$$

$$\bar{Z}_N(\beta, c_i) \equiv Z_N(\beta, c_i)\frac{(2\pi)^N}{\Omega_N}$$

$$= \int_{-\infty}^{+\infty} \prod_{1\le i\le N} d\,\mu(\lambda_i)\Delta_N^2(\lambda) \tag{56b}$$

The set of the polynomials $P_n(\lambda)$ can be defined as

$$\int_{-\infty}^{+\infty} d\,\mu(\lambda)P_n(\lambda)P_m(\lambda) = h_n\delta_{nm} \tag{4.57}$$

where

$$P_n(\lambda) = \lambda^n + \text{lower powers of } \lambda$$

It is easy to show

$$\Delta_N(\lambda) = det \parallel \lambda_i^{j-1} \parallel = det \parallel P_{j-1}(\lambda_i) \parallel$$

and the polynomials obey the following recurssion relations

$$\lambda P_n(\lambda) = P_{n+1}(\lambda) + R_n P_{n-1}(\lambda) \tag{4.58}$$

and

$$h_n = R_n h_{n-1} \tag{4.59}$$

therefore we can formally perform the integration and obtain

$$\bar{Z}_N(\beta, g) = N!h_0h_1\ldots h_{N-1}$$

and

$$F_N(\beta) = regular + \sum_{i=1}^{N-1}(N-k)\ln R_k \tag{4.60}$$

Since we always can rescale matrix field M, that is equivalent to rescale λ, so we can choose

$$V(\lambda) = \lambda^2 + 4g\lambda^4 \tag{4.61}$$

such that

$$V' = 2\lambda + 16g\lambda^3 \tag{4.62}$$

Using the recurssion relation repeatedly, we get

$$V' P_{n-1} = 16gP_{n+2} + 2P_n + 16g(R_{n-1} + R_n + R_{n+1})P_n + 2R_{n-1}P_{n-2}$$

$$+16g(R_nR_{n-1} + R_{n-1}^2 + R_{n-1}R_{n-2})P_{n-2} + 16gR_{n-1}R_{n-2}R_{n-3}P_{n-4} \tag{4.63}$$

therefore, we can directly obtain the non-zero elements of the matrix V'

$$\begin{aligned}
V'_{n+3,n} &= 16gh_{n+3} \\
V'_{n-3,n} &= 16gR_nR_{n-1}R_{n-2}h_{n-3} \\
V'_{n+1,n} &= [2 + 16g(R_n + R_{n+1} + R_{n+2})]h_{n+1} \\
V'_{n-1,n} &= R_n[2 + 16g(R_n + R_{n-1} + R_{n+1})]h_n
\end{aligned} \tag{4.64}$$

and

$$\begin{aligned}
nh_n &= \int_{-\infty}^{+\infty} d\mu(\lambda)\lambda P'_n(\lambda)P_n(\lambda) \\
&= \int_{-\infty}^{+\infty} d\mu(\lambda)P'_n(\lambda)(P_{n+1} + R_nP_{n-1}) \\
&= R_n \int_{-\infty}^{+\infty} d\mu P'_n(\lambda)P_{n-1} \\
&= R_n\beta \int_{-\infty}^{+\infty} d\mu P_n V' P_{n-1} \\
&= R_n\beta h_n[2 + 16g(R_{n-1} + R_n + R_{n+1})]
\end{aligned} \tag{4.65}$$

Thus, we get the equation satisfied by R_n

$$R_n[2 + 16g(R_{n-1} + R_n + R_{n+1})] = \frac{n}{\beta} \tag{4.66}$$

Now, define the continuum parameters

$$x \equiv \frac{n}{\beta}, \qquad \epsilon \equiv \frac{1}{\beta}, \qquad R(x) \equiv R_n$$

then, eq(4.66) can be written as

$$2R(x) + 16gR(x)[R(x-\epsilon) + R(x) + R(x+\epsilon)] = x \tag{4.67}$$

In the limit $\beta \longrightarrow \infty$, it becomes

$$2R_0(x) + 48gR_0^2(x) = x \tag{4.68}$$

This equation, when the coupling contant takes the value $g = -\frac{1}{48}$, has the following singularity (at point x=1, where we say a function is regular, if it is contineous and infinitely differentiable, otherwise, it is called singular at that point)

$$x = 2R_0(x) - R_0^2(x) = 1 - [1 - R_0(x)]^2 \tag{4.69}$$

In other words

$$R_0(x) = 1 - (1-x)^{\frac{1}{2}} \tag{4.70}$$

So, in spheric limit(N, $\beta \longrightarrow \infty$, but X=$\frac{N}{\beta} < 1$ fixed)

$$\begin{aligned} F_N(\beta) &= \sum_{k=0}^{N-1} (N-k) \ln R_k \\ &= \beta^2 \int_0^X dx (X-x) \ln R(x) \\ &= N^2 (1-X)^{\frac{5}{2}} + \ldots \end{aligned} \tag{4.71}$$

we see that the scaling laws will arise from the singular behaviour of R(x) near the point X=1, when β equals to N, and the leading term was contributed by the neighbour of the point X[33,34,35]. In this case, it is appropriate to introduce scaling variables

$$t \equiv (1-x)\beta^{\frac{4}{5}}, \qquad \tau \equiv (1-X)\beta^{\frac{4}{5}} \tag{4.72a}$$

and

$$f(t) \equiv (1-R)\beta^{\frac{4}{5}} \tag{4.72b}$$

Obviously, in the double scaling limit, we have

$$f(\tau) = \tau^{\frac{1}{2}} \tag{4.73}$$

and

$$\begin{aligned} F_N(\beta) &= \int_1^{\tau} dt(\tau - t) f(t) \\ &= \int_0^{\tau} dt(\tau - t) f(t) + regular \end{aligned} \tag{4.74}$$

Discarding the irrelevant term, the free energy and the specific heat have the following forms in the spheric approximation

$$\begin{aligned} F(\tau) &= \int_0^{\tau} dt(\tau - t) f(t) \\ f(\tau) &= F''(\tau) = \tau^{\frac{1}{2}} \\ F(\tau) &= \frac{4}{15}\tau^{\frac{5}{2}} \end{aligned} \tag{4.75}$$

Comparing with the previous result, we find the scaling variable τ and the string coupling constant are related with each other, we can define

$$g_{str.}^2 \equiv \tau^{-\frac{5}{2}} \tag{4.76}$$

therefore, the string anomalous dimension is

$$\gamma_{str.} = -\frac{1}{2} \tag{4.77}$$

In fact, if we consider the general potential like

$$V(M, g_{2k}) = \sum_{k=1}^{n} g_{2k} Tr(M^{2k}) \tag{4.78}$$

then, when we choose the coupling constants like

$$g_{2k} = (-1)^{k-1} \frac{n!(k-1)!}{(n-k)!(2k)!} \tag{4.79}$$

the function R(x) has the following singularity at the point x=1

$$R(x) = 1 - (1-x)^{\frac{1}{k}} \tag{4.80}$$

Define

$$\tau \equiv (1 - \frac{N}{\beta})\beta^{\frac{2k}{2k+1}} \tag{4.81}$$

we obtain

$$F(\tau) = \frac{k^2}{(k+1)(2k+1)}\tau^{(2+\frac{1}{k})} \tag{4.82}$$

and the specific heat is

$$f(\tau) = \tau^{\frac{1}{k}} \tag{4.83}$$

the string anomalous dimension is

$$\gamma_{str.} = -\frac{1}{k} \tag{4.84}$$

§4.2.4 The String Equation

In the previous subsection, we only consider spheric limit, i.e. in order to get the equation(4.68) from (4.67), we have ignored all the ϵ terms in the Taylor expansion of R($x \pm \epsilon$). In fact, carefully calculation will pick up ϵ^2 terms.

$$x = 2R - R^2 - \frac{1}{3}R_{xx}\beta^{-2} \tag{4.85}$$

or equivalently

$$1 - x = (1-R)^2 + \frac{1}{3}R_{xx}\beta^{-2} \tag{4.86}$$

where the subscripts mean the derivatives with respect to x, hereafter we will also use the similar notation for R_{tt}, $R_{\tau\tau}$. Since in the double scaling limit

$$R_{xx} = R_{\tau\tau}\beta^{\frac{8}{5}}, \qquad f''(\tau) = -R_{\tau\tau}\beta^{\frac{2}{5}}$$

thus, from the eqs.(4.72), and (4.86), we obtain the Painleve equation

$$f^2 - \frac{1}{3}f^{''} = \tau \tag{4.87}$$

In the limit $\tau \longrightarrow \infty$, f($\tau$) recovers the spheric approximation (4.83).

In fact, there is another way to derive the Painleve equation, which was developed by Douglas and Shenker[34]. At first, we normalize the orthogonal polynomials

$$\bar{P}_n(\lambda) \equiv P_n(\lambda)/\sqrt{h_n}$$

then, define

$$\tilde{Q}_{mn} \equiv \int_{-\infty}^{+\infty} d\mu(\lambda)\bar{P}_m(\lambda)\lambda\bar{P}_n(\lambda)$$

$$= \delta_{n,m+1}\sqrt{R_n} + \sqrt{R_{n+1}}\delta_{n,m-1} \tag{4.88a}$$

$$\tilde{P}_{mn} \equiv \int_{-\infty}^{+\infty} d\mu(\lambda)\bar{P}_m(\lambda)\frac{\partial}{\partial\lambda}\bar{P}_n(\lambda)$$

$$= \beta\bar{V}^{'}_{mn} - \tilde{P}_{nm} \tag{4.88b}$$

Obviously, only two pseudo–diagonal lines of the matrix $\tilde{Q}$ are non–zero. Since $\lambda^n = \lambda^{x\beta}$ is an eigenfunction of the operator e^{∂_x}, so, we can reexpress $\tilde{Q}$ in terms of this operator

$$\tilde{Q} = \sqrt{R(x+\epsilon)}e^{\epsilon\partial_x} + \sqrt{R(x)}e^{-\epsilon\partial_x}$$

$$= e^{\epsilon\partial_x}\sqrt{R}(x) + \sqrt{R(x)}e^{-\epsilon\partial_x}$$

Since in the double scaling limit, the leading singular term of the free energy is contributed by the neighbour of the point X=1, it would be also true for the expectation value of λ, therefore, we could replace the integral variable x by the scaling parameter, and the expectation value of λ wouldbe behave like

$$< \lambda >= \hat{\tilde{Q}}(\tau)e^{-F(\tau)}$$

where

$$\hat{\tilde{Q}} = e^{\beta^{-1/5}\partial_\tau}\sqrt{\Big(1-\beta^{-\frac{2}{5}}f(\tau)\Big)} + \sqrt{\Big(1-\beta^{-\frac{2}{5}}f(\tau)\Big)}\,e^{\beta^{-1/5}\partial_\tau}$$

$$= 2 + \beta^{-\frac{2}{5}}(\partial_\tau^2 - f(\tau)) + o(\beta^{-\frac{3}{5}}) \tag{4.89}$$

The unwanted constant can be removed away by re–adjusting the potential

$$M \longrightarrow M+2 \Rightarrow \lambda \longrightarrow \lambda = \tilde{\lambda} + 2$$

Since when we compute the partition function, M(or λ) is a dummy variable, so, we can make this shift freely, which doesn't influence the free energy. However, using this shift, we can define a new operator

$$Q \equiv \alpha\beta^{\frac{2}{5}}\int_{-\infty}^{+\infty} d\,\mu(\tilde{\lambda})\bar{P}_m(\tilde{\lambda})\tilde{\lambda}\bar{P}_n(\tilde{\lambda})$$

$$= \alpha(\partial_\tau^2 - f(\tau)) + o(\beta^{-\frac{1}{5}}) \tag{4.90}$$

In the double scaling limit, which is the Schödinger operator.

On the other hand, from the eqs.(4.64) and (4.88b), we get

$$\frac{\partial}{\partial\lambda}\bar{P}_n = n\bar{P}_{n-1} - \frac{1}{3}\beta(R_nR_{n-1}R_{n-2})^{-\frac{1}{2}}\bar{P}_{n-3} \tag{4.91}$$

therefore, the operator form of the matrix $\tilde{P}$ is

$$\tilde{P} = x\beta e^{-\epsilon\partial_x} - \frac{1}{3}\beta\Big(R(x)R(x-\epsilon)R(x-2\epsilon)\Big)^{-\frac{1}{2}}e^{-3\epsilon\partial_x}$$

$$= x\beta e^{-\epsilon\partial_x} + -\frac{\beta}{3\sqrt{R(x)}}e^{-\epsilon\partial_x}\frac{1}{\sqrt{R(x)}}e^{-\epsilon\partial_x}\frac{1}{\sqrt{R(x)}}e^{-\epsilon\partial_x} \tag{4.92}$$

$$\sim \frac{2}{3}\beta + \beta^{\frac{3}{5}}(\partial_\tau^2 - \frac{1}{2}f(\tau)) + \beta^{\frac{2}{5}}\Big(\frac{4}{3}\partial_\tau^3 - 2f(\tau)\partial_\tau - f'(\tau)\Big) + o(\beta^{\frac{1}{5}})$$

In order to eliminate the unwanted terms, it is natural to define

$$P \equiv \frac{1}{\alpha}\beta^{-\frac{2}{5}}\int_{-\infty}^{+\infty} d\,\mu(\tilde{\lambda})\bar{P}_m(\tilde{\lambda})\frac{\partial}{\partial\tilde{\lambda}}\bar{P}_n(\tilde{\lambda})$$

$$= \frac{1}{\alpha}\Big(\frac{4}{3}\partial_\tau^3 - 2f(\tau)\partial_\tau - f'(\tau)\Big) + o\Big(\beta^{-\frac{1}{5}}\Big) \tag{4.93}$$

Again, in the double scaling limit, P is an differential operator with order–3. Furthermore, from the definitions of P and Q, we have

$$[P,Q]_{mn} = \int_{-\infty}^{+\infty} d\,\mu(\tilde{\lambda})\bar{P}_m(\tilde{\lambda})[\frac{\partial}{\partial\tilde{\lambda}},\tilde{\lambda}]\bar{P}_n(\tilde{\lambda}) = \delta_{mn} \tag{4.94}$$

that is to say, as operators

$$[P,Q] = 1 \tag{4.95}$$

This is the so–called string equation, which leads to the Painleve equation. Now, if we introduce new variables

$$\tau \equiv \left(\frac{4}{3}\right)^{1/5} x, \qquad u(x) \equiv -\left(\frac{4}{3}\right)^{2/5} f(\tau) \tag{4.96}$$

and set

$$\alpha = \left(\frac{4}{3}\right)^{1/5}$$

then, the operators P and Q have the standaed forms

$$Q = \partial_x^2 + u(x), \tag{4.97a}$$

$$P = \partial_x^3 + \frac{3}{2}u(x)\partial_x + \frac{3}{4}u^{'}(x) = (Q^{\frac{3}{2}})_+. \tag{4.97b}$$

The string equation, or equivantly, the Painleve equation determines the perturbative and nonperturbative property of the specific heat. But, unfortunately, till now, we don't know how to get the exact solutions of these equations[35].

Till now, we only consider 1–matrix model. We can generalize the above discussion to multimatrix model[55–59]. Furthermore, we also can investigate the symmetries of matrix model[60,61,62], as well as d=1 string theory (the target space is 1–dimensional)[63–66]. But, there is still a long way to well–understanding of gravity.

§4.2.5 The Reconstruction of The Theory

Till now, we know that in the double scaling limit, matrix model is a field theory in which the physical quantity(the specific heat) satisfies the string equation, which defines the theory. In turn, we can view the string equation as the equation of motion deduced from a certain action. Now, we construct this underlying action[67].

As we know in the first chapter, the fractional power of the operator Q is well–defined. Define

$$R_l = \frac{1}{2} Res(Q^{l-\frac{1}{2}}) \tag{4.98}$$

which is half of the coefficient of the term ∂^{-1}. It is easy to see that this residue satisfies a simple recursion relation

$$R'_{l+1} = \frac{1}{4} R'''_l + u R'_l + \frac{1}{2} R_l \tag{4.99}$$

The first few R_l's are given below

$$R_0 = \frac{1}{2}, \qquad R_1 = \frac{1}{4} u, \qquad R_2 = \frac{3}{16} u^2 + \frac{1}{16} u'',$$

$$R_3 = \frac{1}{64}(u^{(4)} + 10uu'' + 5u'^2 + 10u^3) \tag{4.100}$$

$$R_4 = \frac{1}{256}(u^{(6)} + 14uu^{(4)} + 28u'u''' + 21u''^2 + 70uu'^2 + 70u^2u'' + 35u^4)$$

Straightforward computation show us

$$[(Q^{l-\frac{1}{2}})_-, Q] = -4R'_l \tag{4.101}$$

On the other hand, from the string equation

$$[(Q^{l-\frac{1}{2}})_+, Q] = 1$$

Since

$$[(Q^{l-\frac{1}{2}})_+, Q] = -[(Q^{l-\frac{1}{2}})_-, Q]$$

we have

$$4R'_l = 1 \tag{4.102}$$

After integration, we get

$$4R_l = x \tag{4.103a}$$

For convenience, now we rescale x and u, such that

$$(l+\frac{1}{2})R_l[u] = x \tag{4.103b}$$

Since

$$\frac{\delta}{\delta u}\int dx R_{l+1}[u] = \frac{\delta}{\delta u}\frac{1}{2}Tr(Q^{l+\frac{1}{2}}) = (l+\frac{1}{2})\cdot\frac{1}{2}Res(Q^{l-\frac{1}{2}})$$
$$= (l+\frac{1}{2})R_l[u] = x \tag{4.104}$$

Thus, we can write out the action as follows

$$S = \int dx\Big(R_{l+1}[u] - xu\Big)$$
$$= \int dx\Big(R_{l+1}[u] + t_0 R_1\Big) \tag{4.105}$$
$$= Tr\Big(Q^{l+\frac{1}{2}} + t_0 Q^{\frac{1}{2}}\Big)$$

where we define

$$t_0 \equiv -4x \tag{4.106}$$

For the matrix model with the quartic interaction, the correponding action is

$$S = \int dx\left[\frac{1}{64}(u^{(4)} - 10uu'' - 5u'^2 + 10u^3) + xu\right] \tag{4.107}$$

For general potential

$$S = \frac{1}{2}Tr\Big(\sum_{l=1} t_{(l)}Q^{l+\frac{1}{2}} + t_0 Q^{\frac{1}{2}}\Big) \tag{4.108}$$

the string equation is

$$x = \sum_{l=0}(l+\frac{1}{2})t_{(l)}R_l[u] \tag{4.109}$$

and the operator Q, equivalently, the specific heat $u(x)$ satisfies following KdV equation

$$\frac{\partial Q}{\partial t_{(l)}} = [(Q^{l+\frac{1}{2}})_+, Q] \tag{4.110}$$

References

[1] For a review, see Superstring Theory vol. I and II, M. Green, J. Schwarz and E. Witten, Cambridge University Press (1987)

[2] A. Belain, A. Polyakov and A. B. Zamolochikov, Nucl. Phys. B**241**(1984)333

[3] D. Friedan, Z. Qiu and S. Shenker, Phys. Rev. Lett.**52**(1984)1575

[4] G. Moore and N. Seiberg, Phys. Lett. **B212**(1988)451; Nucl. Phys. **B313**(1989)16; Commun. Math. Phys. **123**(1989)177

[5] E. Verlinde, Nucl. Phys. B**300**[FS22](1988)360

[6] There are many nice reviews on CFT, for examples, D. Friedan, in Unified String Theories, M. Green and D. Gross eds. World Scientific, Singapore (1986)

P. Goddard and D. Olive, Int. J. Mod. Phys. A**1**(1986)303

P. Ginsparg, Les Houches Lecture Notes (1988)

J. Cardy, Les Houches Lecture Notes (1988)

[7] M. Bershadsky and Ooguri, Commun. Math. Phys. **126**(1989)49

[8] A. M. Polyakov, Phys. Lett. B**103**(1981)207

[9] A. M. Polyakov, Mod. Phys. Lett. A**2**(1987)893

[10] V. Knizhnik, A. Polyakov and A. B. Zamolochikov, Mod. Phys. Lett. A**3**(1988)819

[11] J. -L. Gervais, Nucl. Phys. B**(Proc.Supp.)5B**(1988)119

[12] L. Fadeev and L. Takhtajan, Hamiltonian Methods in the Theory of Solitons, Spinger Verlag (1987)

[13] O. Babelon, Lecture Notes in SISSA (1989)

[14] A. Das, Integrable Models, World Scientific, Singapore (1989)

[15] B. Kupershmidt and G. Wilson, Invent. Math. **62**(1981)437

[16] A. B. Zamolochikov, Theor. Math. Phys. **65**(1986)129

[17] V. Fateev and A. B. Zamolochikov, Nucl. Phys. B**280**(1987)644

[18] V. Fateev and S. Luk'yanov, Int. J. Mod. Phys. A3(1988)507

S. Luk'yanov, Funct. Anal. Appl. **22**(1988)466

[19] I. Bakas, Phys. Lett. B**219**(1989)283; Commun. Math. Phys. **123**(1989)627

[20] A. Bilal and J. -L. Gervais, Phys. Lett. B**206**(1988)412; Nucl. Phys. B**314**(1989)646; Nucl. Phys. B**318**(1989)579

[21] O. Babelon, Phys. Lett. B**215**(1988)523

[22] O. Babelon and L. Bonora, preprint

[23] P. Mathiu, Phys. Lett. B**208**(1988)101

[24] A. Das and S. Roy, Rochester preprint UR–1151(1989)

[25] J. Balog, L. Fehér, L. O'Raifeartaigh, P. Forgács and A. Wipf, Phys. Lett. **B244**(1990)435

[26] Distler and Kawai, Nucl. Phys. B**231**(1989)509

[27] F. David, Mod. Phys. Lett. A**3**(1988)207

[28] V. Kazakov and A. Migdal, Nucl. Phys. B**311**(1988/89)171

[29] F. David, Nucl. Phys. B**257[FS14]**(1985)45, 543

[30] V. Kazakov, Phys. Lett. B**150**(1985)28

[31] V. Kazakov, I. Kostov and A. Migdal, Phys. Lett. B**157**(1985)295

[32] J. Ambjorn, B. Dorhuus and J. Fröhlich, Nucl. Phys. B**257[FS14]**(1985)433

[33] E. Brezin and V. Kazakov, Phys. Lett. B**236**(1990)144

[34] M. Douglas and S. Shenker, Nucl. Phys. **B335**(1990)635

[35] D. Gross and A. Migdal, Phys. Rev. Lett.**64**(1990)717; Nucl. Phys. **B340**(1990)333

[36] E. Witten, Trieste Summer School Lecture notes (1990)

[37] There are some reviews, F. David, E. Brezin and S. Wadia, Trieste Summer School Lecture notes (1990)

[38] I. Gelfand and L. Dickii, Russian Math. Surveys **30**(5)(1975)77

[39] E. Witten, IASSNS–HEP–90–45; Nucl. Phys. **B340**(1990)281

[40] R. Dijkgraaf, E. Verlinde and H. Verlinde, IAS preprint (1990);
H. Verlinde, Nucl. Phys. **B337**(1990)652;
E. Witten and R. Dijkgraaf, Nucl. Phys. **B342**(1990)486

[41] Distler, Nucl. Phys. **B342**(1990)523

[42] Drinf'eld and Sokholov, J. Sov. Math. **30**(1984)1975

[43] I. Gelfand and L. Dickii, Funct. Anal. Appl. **10**(1979)259

[44] M. Alder, Invent. Math. **50**(1979)219

[45] I. Gelfand and I. Dorfman, Funct. Anal. Appl. **14**(1981)223

[46] G. Segal and G. Wilson, Publ. IHES **61**(1985)1

[47] E. D'Hoker and P. Kurzepa UCLA/90/tep/15

[48] N. Mavromatos and J. Miramontes, Mod. Phys. Lett. A**4**(1989)1847

[49] D. Bessis, C. Itzykson and J. -B. Zuber, Adv. Appl. Math. **1**(1980)109

[50] E. Brezin, C. Itzykson, G. Parisi and J. Zuber, Commun. Math. Phys. **59**(1978)35

[51] D. Bessis, Commun. Math. Phys. **69**(1979)147

[52] C. Itzykson and J. -B. Zuber, J. Math. Phys. **21**(1980)411

[53] M. Mehta, Commun. Math. Phys. **79**(1981)327

[54] S. Chedha, G. Mahoux and M. Mehta, J. Phys. A: Math. Gen. **14**(1981)579

[55] M. Douglas, Phys. Lett. B**238**(1990)176

[56] E. Brezin, M. Douglas, V. Kazakov and S. Shenker, Phys. Lett. B**237**(1990)43; E. Brezin, E. Marinari and G. Parisi, Phys. Lett. B**242**(1990)35;

[57] C. Crnkovic, P. Ginsparg and G. Moore, Phys. Lett. **B237**(1990)196

[58] T. Banks, M. Douglas, N. Seiberg and S. Shenker, Phys. Lett. **B238**(1990)279

[59] P. Di Francesco and D. Kutasov, Nucl. Phys. **B342**(1990)589

[60] G. Sotkov, M. Stanishkov and C. J. Zhu, SISSA preprint, ISAS–EP–90–75

[61] K. W. Xu and C. J. Zhu, ICTP preprint (1990)

[62] Y. X. Cheng, Osaka preprint, OU–HEP–142; Hoseong La, Pennsylvanian preprint, UPR–0432T

[63] E. Brezin, V. Kazakov and A. B. Zamolochikov, Nucl. Phys. **B338**(1990)

[64] G. Parisi, Phys. Lett. B238(1990)209; Phys. Lett. **B238**(1990)213; E. Marinari and G. Parisi, Phys. Lett. **B240**(1990)375

[65] D. Gross and N. Miljkovic, Phys. Lett. **B238**(1990)217

[66] P. Ginsparg and J. Zinn–Justin, Phys. Lett. **B240**(1990)333

[67] P. Ginsparg, M. Goulian, M. Plesser and J. Zinn–Justin, Nucl. Phys. **B342**(1990)359