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ANOMALIES IN ORDINARY AND SUPERSYMMETRIC FIELD THEORIES

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Introduction

The subject of this thesis is a general study of the anomalies in the framework of ordinary and supersymmetric field theories. In what follows, we shall see that the importance of such a treatment is motivated by the fact that anomalies control the construction of consistent field theories.

It is well-known that symmetric properties of a classical field theory are studied in terms of the Noether theorem, which states that "to every continuous transformation of coordinates which makes the variation of the corresponding action to vanish, and for which the transformation law of the field functions is also given, there corresponds a definite invariance, i.e. a combination of the field functions and their derivatives, which remains conserved. Here, one should remember that the coordinates can be in an internal space as well as in space-time. For example, they can be the coordinates in the abstract isospin space. Let us see how this theorem works in classical theories.

Suppose we have a general Lagrangian

$$\mathcal{L} = \mathcal{L}[\varphi(x), \partial_\mu \varphi(x)] ,$$

which is not an explicit function of the space-time coordinates, x_μ. If this theory, or equivalent this Lagrangian, is invariant under a continuous transformation, we shall get a vanishing variation of the Lagrangian, i.e.

$$\delta \mathcal{L} = 0 .$$

From the Noether theorem, we can get the following equation stemming from the invariance of the Lagrangian

$$\partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \delta \varphi - \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \partial_\nu \varphi - \mathcal{L} g^\mu{}_\nu \right) \Delta x^\nu \right] = 0, \qquad (0\text{-}1)$$

where the spinor and internal indices, $i\alpha$, are dropped for the sake of simplicity. Introducing a current operator, j^μ_a, we can rewrite this equation as

$$\partial_\mu j^\mu_a = 0 , \qquad (0\text{-}2)$$

where a stands for all indices which have not been summed over. After a some algebra, we can define a charge

$$Q = \int d^3x \, j_0 \, , \tag{0-3}$$

which satisfies a conservation equation

$$\dot{Q} = 0 . \tag{0-4}$$

This procedure shows how to see the symmetry properties of a classical field theory. The Lagrangian of the classical theory has the same symmetry properties as the theory itself. So we can study the symmetries of the Lagrangian in terms of the Noether theorem to know those of the theory. One has to remember here that this statement is not true in quantum field theories. As we discussed above, corresponding to a transformation, we can derive the variation of the action associated with our Lagrangian. If this variation vanishes, the Lagrangian is invariant and we can get a symmetry. If not, the Lagrangian is not invariant and we can not get a symmetry. We can consider two examples to see practically how we can derive the desired symmetry. The first one is about the translation of the space-time coordinates. The second one is about the U(1) invariance of a complex field. This is, in fact, an internal coordinate transformation.

First of all, let us consider the space-time translation invariance. The transformations of the coordinates are

$$x_\mu \longrightarrow x'_\mu = x_\mu + a_\mu \,, \qquad (0\text{-}5)$$

and the transformation law of the field functions is

$$\begin{aligned} \delta\varphi(x) &\equiv \varphi'(x') - \varphi(x) \\ &= \partial_\mu \varphi(x)\, \Delta x^\mu . \end{aligned} \qquad (0\text{-}6)$$

Using the equation (0-1), we get the well-known energy-momentum tensor as

$$T_{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu\varphi)} (\partial_\nu \varphi) - \mathcal{L}\, g^\mu{}_\nu \qquad (0\text{-}7)$$

which satisfies

$$\partial^\mu T_{\mu\nu} = 0 \,. \qquad (0\text{-}8)$$

From the zeroth component of this tensor, we get the conserved energy-momentum 4-vector

$$P_\mu = \int d^3x \, T^0{}_\mu . \tag{0-9}$$

Second, we consider the following transformation

$$\begin{aligned} \varphi &\longrightarrow e^{i\alpha} \varphi(x) \\ \bar{\varphi} &\longrightarrow e^{-i\alpha} \bar{\varphi}(x) \end{aligned} \tag{0-10}$$

for a complex scalar field. The infinitesimal transformation is

$$\begin{aligned} \delta\varphi(x) &= i\alpha\varphi(x) \\ \delta\bar{\varphi}(x) &= -i\alpha\bar{\varphi}(x) \end{aligned} \tag{0-11}$$

The Lagrangian of a free complex scalar field can be written as

$$\mathcal{L} = (\partial_\mu \bar{\varphi})(\partial^\mu \varphi) - \mu^2 \bar{\varphi}\varphi , \tag{0-12}$$

from which we can derive the conserved current as

$$j_\mu = -i[(\partial_\mu \bar{\varphi})\varphi - \bar{\varphi}(\partial_\mu \varphi)]$$

(O-13)

$$\partial^\mu j_\mu = 0$$

because α is a coordinate independent parameter and the Lagrangian is invariant under such a transformation. The corresponding charge can be defined as

$$Q = -i\int d^3x[(\partial_0 \varphi^+)\varphi - \varphi^+(\partial_0\varphi)] , \qquad \text{(O-14)}$$

which satisfies

$$\dot{Q} = 0 . \qquad \text{(O-15)}$$

In the classical field theory, it is a conserved quantity.

In quantum theories, instead of the Lagrangian, one has to consult the Ward Identities in order to investigate whether or not one can implement, at the quantum level, a symmetry exhibited by the classical Lagrangian. In chapter 1, we shall discuss this issue. In what follows, we shall briefly discuss the spontaneous breaking of a symmetry in order to re-examine the Goldstone theorem in the case that a global symmetry is broken by an anomaly. Let us consider a simple one-parameter group $G(\lambda)$ as an example. Suppose the Lagrangian is invariant under the transformation

$$\varphi^i(x) \xmapsto{\lambda} \varphi^{i\prime}(x) \qquad (0\text{-}16)$$

By Noether theorem, we can define a current, j^i_μ , which satisfies the equation

$$\partial^\mu j^i_\mu = 0. \qquad (0\text{-}17)$$

In general, if the integral

$$\int_{x_0 = \mathrm{const}} d^3x\, j_0(x) \qquad (0\text{-}18)$$

exist, we can define it as the conserved charge Q , and the charge should have zero eigenvalue acting on the vacuum,

$$Q\,|0\rangle = 0\,. \qquad (0\text{-}19)$$

If we suppose that the vacuum possesses the translational invariance, we have

$$\langle 0|QQ|0\rangle = \int d^3x \langle 0|j_0(x)Q|0\rangle$$

$$= \langle 0|j_0(x)Q|0\rangle \int d^3x \qquad (0\text{-}20)$$

$$= \langle 0|j_0(x)Q|0\rangle \times \text{infinity}$$

because the amplitude $\langle 0|j_0(x)Q|0\rangle$ is independent of the space-time coordinates. In fact

$$\langle 0|j_0(x)Q|0\rangle = \langle 0|e^{-iP_\mu a^\mu} j_0(x)Q\, e^{iP^\mu a_\mu}|0\rangle$$

$$= \langle 0|e^{-iP_\mu a^\mu} j_0(x) e^{iP_\mu a^\mu} Q|0\rangle \qquad (0\text{-}21)$$

$$= \langle 0|j_0(x+a)Q|0\rangle .$$

So, only if $Q|0\rangle=0$, we can have finite amplitude. But, if the charge, i.e. the integral, does not exist, the transformation operator can not be defined and nothing can ensure the amplitude invariant under this transformation. This is the so-called spontaneous breaking of symmetries. In this case, we can not find any sign of symmetry breaking in Lagrangian because the Lagrangian is still invariant.

Suppose a classical Lagrangian has a given symmetry and the associated current satisfies the equation

$$\partial^\mu j_\mu = 0 \, . \tag{0-22}$$

After the quantization procedure for this theory, the above equation becomes generally

$$\partial^\mu j_\mu = A + I \, . \tag{0-23}$$

On the right hand side, the second term is infinite and unacceptable. Fortunately, it can be removed by renormalization. But, if A is different from zero, this symmetry is broken even after the quantization procedure. What is important here is that a classically conserved transformation is no longer conserved. The breakdown of the symmetry is caused by the finite term A, which can not be removed by renormalization.

The renormalized current satisfies the equation

$$\partial^\mu j_\mu = A \, . \tag{0-24}$$

In this case, the quantized theory possesses different symmetries respect to the classical field. This is not desired because we require a definite symmetry group in the quantized theory. In the early days, when this behavior was noticed, one call it anomaly because this is an anomalous breakdown of symmetries. One has to pay attention to the fact that the anomaly comes from divergent graphs, but it appears as a finite operator. We shall see it again in chapter 1, where we shall derive the Ward Identities.

Now, we are going to see what the consequence of anomalies is and where they come from.

Anomalies appear as a loop effect in quantum field theories. Even though they are detected through the failure of a certain regularization scheme, such a fact is not sufficient to indicate the presence of the anomaly. It might happen that another regularization scheme exists which is compatible with the symmetry one thinks is broken by the anomaly. The anomaly, if it shows up, is a more intrinsic property of the theory. It is true that it is detected by means of some regularization scheme.

In order to see the possible consequences of the anomalies, it is advisable to classify them into two different categories those associated with global or local symmetries.

In the case of a global symmetry, the anomaly does not introduce inconsistences and we need not worry about them. Sometimes, they are even welcome to get results compatible with phenomenology. Practically, we should say that the anomalies associated with global symmetries do not destroy the theory. In the following, we will give some examples to show that, sometimes, the anomalies in global symmetry theories are necessary to render a theory consistent with observed results. On the other hand, the anomalies of a local symmetry theory are disastrous because the gauge symmetry is broken, which is crucial in the proof of the unitarity and renormalizability of the theory. In this case, the theory will lose its unitarity and its renormalizability. In this sense, the theory will be destroyed by such anomalies. But the global symmetries are not used to prove these properties.

Now let us have a look at the problem of the implementation of a classical symmetry in field theory and the effect of the regularization scheme on this problem. Generally the symmetries in a classical field theory are checked by nature and desired implementable. But, after quantization, many kinds of symmetries may get anomalies. The first result is, of course, the breakdown of the associated symmetry. Afterwards it will lead to different final results for local and global symmetries. As we said above, for global symmetries, the anomalies do not introduce difficulty. But for locally symmetric theories, they do destroy the consistence of the theories. This can be seen by analysis of the Ward Identity as we shall do in the first chapter. In fact, we can only see if the symmetry is broken or not in view of the appearance of anomalies. The fact is that anomalies spoil the Ward Identity, but anomalies are not necessary to appear when the Ward Identities are spoiled. Therefore, within a regularization procedure, symmetries may be broken by anomalies. In this sense, the implementation of a symmetry directly depends on the appearance of anomalies. On the other hand, the implementation depends also on the regularization scheme, which is unavoidable during quantization because we always get some infinite quantities, which have to be removed.

Now, let us consider some examples which show that the anomalies of global symmetries do not destroy the theories and necessary for a good agreement of known experimental facts.

The first example is the famous triangle (Adler-Bell-Jackiw) anomaly [1]. In the two-photon decay of the neutral pion, when one calculates the divergence of the current associated with the γ_5-transformation in the framework of perturbation theory, one gets, at the one-loop approximation, a finite quantity as follows

$$\partial^\mu j^5_\mu = \Delta \tag{0-25}$$

instead of a vanishing quantity as in the classical theory. This is checked by calculating the graph given in fig. 0-1. This behavior was found first by Adler when he calculated the lifetime of the neutral pion decay

$$\pi^0 \longrightarrow 2\gamma \; .$$

He could get the correct value, with anomaly, according to the experiments.

We have also some other examples. In the following, we shall consider the U(1) problem and θ-vacuum in QCD[2]. QCD is a gauge theory for the strong interaction.

In QCD, the fundamental state is degenerate and more

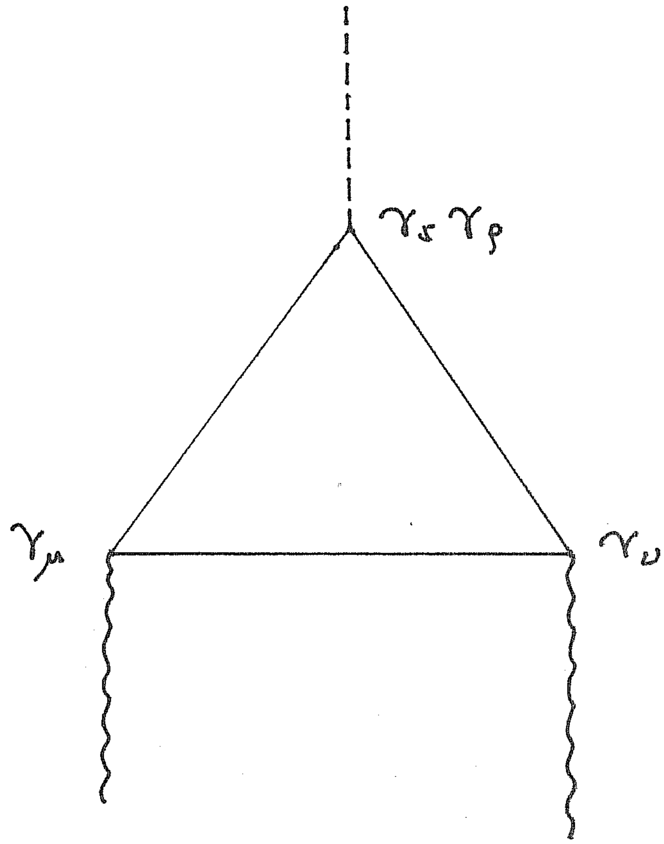


fig. 0-1

complicated. We have to distinguish them by a parameter θ, and denote the real vacuum by θ =0, i.e. $|\theta\rangle$. According to the Pecei and Quinn's convention[2], the transition amplitude of θ-vacuum is

$$\langle\theta|\theta\rangle = \sum_{q=-\infty}^{\infty}\int (dA_\mu)_q d\varphi \exp\{\int d^4x \mathcal{L}_E(A,\varphi)\}\exp(iq\theta), \quad (0\text{-}26)$$

where the integration over the gauge fields, A_μ, should be done according to the constraint

$$\frac{g^2}{32\pi^2}\int d^4x F^a_{\mu\nu}\tilde{F}^{a\mu\nu} = q. \quad (0\text{-}27)$$

If we remove the θ-relation from the θ-vacuum to the Lagrangian and use the normal vacuum, we shall get a new term in the Lagrangian like

$$i\theta \frac{g^2}{32\pi^2} F^a_{\mu\nu}\tilde{F}^{a\mu\nu}. \quad (0\text{-}28)$$

From the chiral U(1) transformation

$$\psi(x) \longrightarrow \exp\{i\gamma^5\sigma\}\psi(x)$$
$$\bar{\psi}(x) \longrightarrow \bar{\psi}\exp\{i\gamma^5\sigma\}$$

$$\varphi(x) \longrightarrow \exp(-2i\sigma)\,\varphi(x), \tag{0-29}$$

the Lagrangian acquires a term like

$$-2i\sigma \frac{g^2}{32\pi^2} F^a_{\mu\nu} \tilde{F}^{a\mu\nu}. \tag{0-30}$$

If we set $\theta = 2\sigma$, the transformed Lagrangian will have no θ-dependent term.

The last example is the B-number (baryon number) violation in GUT's. Here the graph which we have to consider is the vector current insertion in an axial theory. The graph is drawn in fig. 0-2. Where the current $j^B_\mu \sim \bar{q}\gamma_\mu G\, q$ is inserted in the axial theory from which a vertex with γ_5 can be found. G_a, G_b, G_c are the generators of internal symmetry group. The loop integral of this graph is the same as the one we find in computing the Adler's anomaly. From the internal group, we shall get a factor like

$$tr(G_a\{G_b, G_c\}). \tag{0-31}$$

At the beginning of chapter2, we shall see how to get this

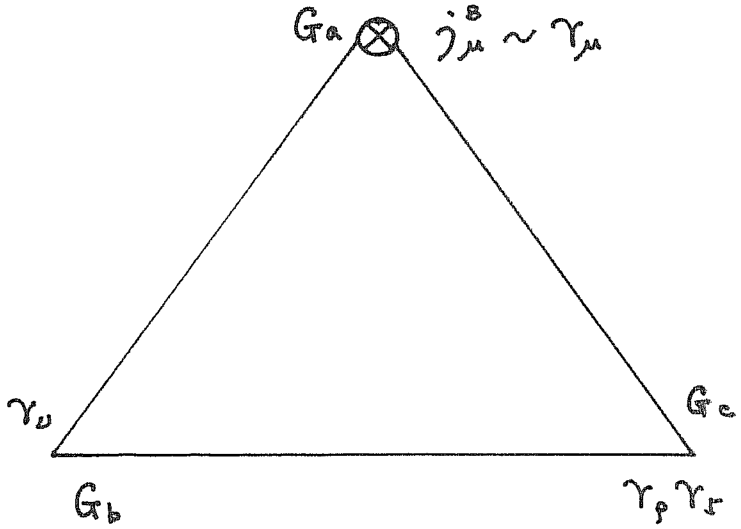


fig. 0-2

factor. Because the generators are

$$G_a = t_a \otimes \mathbb{1}_{color}$$

$$G_b = t_b \otimes \mathbb{1}_{color} \qquad (0\text{-}32)$$

$$G_c = \frac{1}{3}\, \mathbb{1}_{color} ,$$

where the generators t_a and t_b are the generators of the flavor group SU(n). Therefore the factor will be

$$\frac{1}{3}\, tr(\{t_a , t_b\}) \qquad (0\text{-}33)$$

If the three colors are taken into account, we should get the following factor

$$tr(\{t_a , t_b\}) . \qquad (0\text{-}34)$$

This result shows that the anomaly does not vanish. On the other hand, we can consider the anomaly of leptons in our theory. The graph which gives out anomalies is as before, but the current insertion are changed into the lepton current,

$$j^{l}_{\mu} = \bar{l}\gamma_{\mu} l . \tag{0-35}$$

The internal generators are

$$G_a = t_a ,$$

$$G_b = t_b , \tag{0-36}$$

$$G_c = \mathbb{1}$$

The corresponding anomaly has the same space part as the baryon anomaly. The internal The internal factor is

$$tr\{t_a , t_b\} . \tag{0-37}$$

Therefore the lepton anomaly looks exactly like the baryon anomaly. So the difference of the baryon number and the lepton number will be anomaly-free because the baryon and lepton anomalies are canceled by each other. This result means that in GUT's theories, a baryon can decay into leptons[3], what is conserved is the difference between lepton and baryon numbers.

In locally symmetric theories, the influence of an anomaly will be very different because, here, we have a gauge symmetry group. Since the Ward Identities associated with the gauge symmetries are violated due to the presence of an anomaly, and since they are crucial in proving the unitarity and the renormalizability of the theory, an anomaly jeopardizes the quantum consistency of the theory. This fact tells us that local symmetry anomalies should be somehow removed to have a consistent theory.

From the divergence equation

$$\partial^\mu j_\mu = \Delta , \tag{0-38}$$

one can think that the anomaly can be absorbed into the redefinition of the current j_μ , so that the redefined current is anomaly-free. But this redefinition will lead to the introduction of a non-local term in our Lagrangian, which is not desired. As we said above, anomalies can not be removed by a proper counter term in Lagrangian. Another approch is to cancel anomalies by other proper particles. When necessary, new particles can be introduced. For example, in the standard model, the anomalies of leptons are canceled by the anomalies of quarks. This is also the indirect evidence of introducing quarks. But sometimes, new particles introduced can not be found in nature, then this approach is not acceptable, either, unless we have a nutural explanation why we can not find them. If we take the internal group into account, we shall get a factor like

$$tr(\{G_a, G_b\} G_c). \tag{0-39}$$

If a proper representation of the group is chosen, this factor will vanish and the anomaly will be removed. For example, for real representation, this factor is zero, that is anomaly-free. For simple algebra, the only ones which have non-zero factors are SU(n), n 3, as well as SO(6). This approach is also used in superstring theories. For every theory, one should find out a proper method to remove the local symmetry anomaly, if it exists, so that the theory is free from it. If no way to remove it, the theory will be replaced by another one which is, or can be made, anomaly-free for local symmetries.

A supersymmetric theory may also have anomalies. But some new features will appear here. A very important fact pointed out by Ferrara and Zumino[4] is that the chiral current j^5_μ, the supersymmetric spinor current $S_{\mu\alpha}$ and the energy-momentum tensor $\theta_{\mu\nu}$ can be put into a supersymmetric multiplet. This is very convenient because we can get all anomalies after we derive out the anomaly from this superconformal current in a supersymmetric formalism. Of course, the anomalies are members of another supermultiplet.

Most recently, superstring theories have attracted a lot of attention. Because in superstring theories, we always work in higher dimensional space instead of the normal 4-dimensional space, some new features appear. For example, in higher dimensional field theories, anomalies can be canceled by proper counter terms introduced into Lagrangians. But this is impossible in 4-dimensional space according to the non-renormalization theorem.

In study of superstring theories, one can see that it is more important to study anomalies because, here, the gauge group can be fixed by the conditions for an anomaly-free theory. These conditions are composed of the vanishing trace

$$tr\,(G_a G_b G_c G_d G_e G_f + \text{permutations}) = 0 \qquad (0\text{-}40)$$

and one for gravitational anomaly-free[5]. Many groups are excluded by the conditions for the cancellation of anomalies. We have known that the type-I, open or Green-Schwarz superstring theories are anomaly-free with the gauge group SO(32)[6]; The Heterotic superstring theories are anomaly-free with the gauge group SO(32) or E E [7]. The first type of superstring theories is the combination of N=1 supergravity and the Yang-Mills theories. In this period, the superstring theories are the most hopeful because they possess many unexpected and wonderful features. Therefore, the study of anomalies in superstring theories is very mean- ingful and important because we can restrict ourselves in a smaller range.

In chapter 1 and 2, I present a general overview on anomalies in quantum field theories. In chapter 3, I tackle the problem of the R-symmetry chiral anomaly in global supersymmetric field theories. Finally, in chapter 4, we carry out the calculation of the divergence of the superconformal current in N=1 super Yang-Mills. We perform a two-loop analysis in the background Field Formalism in terms of the Heat-kernel expansion and check the well-known result that

$$\bar{D}^{\dot{\alpha}} J_{\alpha\dot{\alpha}}\Big|_{2\text{-}\overline{loop}} = -\frac{\beta^{(2)}}{3g} D_\alpha (W^2). \qquad (0\text{-}41)$$

Chapter 1 General features of anomalies in quantum field theories

First of all, let us see the exact definition of anomaly in terms of the generating functional, Γ (effective action or generating functional of the 1-particle irreducible diagrams).

One generally says that a symmetry of the classical action develops an anomaly whenever the symmetry is explicitly broken by quantum corrections.

One has to pay attention here to the fact that this breakdown is explicit, instead of spontaneous. The explicit breakdown of a symmetry can be seen from the Lagrangian and no conservation equation of a current can be derived. So, we can neither define a conserved charge. But, a spontaneous breakdown of a symmetry can not be seen directly from the Lagrangian of the theory. One can derive the vanishing divergence of a current in terms of the Noether theorem. But here, the integral

$$\int d^3x\, j_0 \qquad (1-1)$$

does not exist. Therefore, we can not define the associated charge, so the symmetry is broken.

It is known that anomalies break symmetries, but the inverse is not true. For example, let us consider a gauge theory. If the cut-off regularization scheme is employed, there is a term with non-zero mass dimension which appears in Lagrangian from cut-off. But this term is not gauge invariant. Then the gauge symmetry appears to be broken. But, anomalies may not appear.

In classical field theories, we can use the Noether theorem. But, in quantum field theories, the Noether theorem does not work, and is replaced by Ward Identities for studying symmetric properties of our theory. In other words, anomaly must break, if it exists, the Ward Identities and the associated symmetries.

As we said above, the spontaneous breakdown of a symmetry does not lead to an anomaly. But in the case of explicit breakdown, there are still two possibilities. If the breakdown can be switched off by changing the parameters of the transformation, it does not lead to anomaly, either. In the case of an anomaly, an explicitly breaking term appears, but it cannot be switched off by continuously varying its coefficient, as in the case of an usual explicit breaking term.

Consider a classical action

$$\mathcal{S} = \int d^4x \, \mathcal{L}_{classical} \; . \tag{1-2}$$

Under a conserved transformation, the variation of the action is zero, i.e.

$$\delta \mathcal{S} = 0 \; . \tag{1-3}$$

Taking quantum corrections into account, one can define the so-called effective action Γ,

$$\Gamma = \mathcal{S} + \Delta^{loop}\Gamma \; , \tag{1-4}$$

where $\Delta^{loop}\Gamma$ account for all loop corrections contributed by 1-PI diagrams with an arbitrary number of amputated external legs (amputated means putting on the condition $[\Box + m^2]$ to remove the propagators away from external legs).

A symmetry is said to be anomalous if

$$\delta \Gamma = \int d^4x \, A \neq 0 \qquad (1-5)$$

with $A \neq \delta \Gamma_{local}$, where Γ_{local} denotes a local monomial (or polynomial) in the fields. Because an action can be written as an integral

$$\delta S = \int d^4x \, \delta \Gamma = \int d^4x \, \delta \Gamma_{local} \qquad (1-6)$$

and

$$S = \int d^4x \, \Gamma \, . \qquad (1-7)$$

If one can write

$$A = \delta \Gamma_{local} \, , \qquad (1-8)$$

we can have

$$\delta \int d^4x \, (\Gamma - \Gamma_{local}) = 0 \, . \qquad (1-9)$$

The condition is that A cannot be written as the variation of a local polynomial in the fields.

In quantum field theories, we have two ways to see anomalies. One is using Ward Identities and considering the graph in fig.1-1. Another way is considering all one-loop corrections. We can see that the graph in fig.1-2 spoils the symmetries of the effective action.

Now, let us briefly discuss the definitions of generating functionals Z, W and Γ .

The generating functional of a general Green-function is defined as the vacuum-to-vacuum amplitude

$$W[J] \equiv \langle 0|0\rangle_J = N\int \mathcal{D}\varphi \, e^{iS[\varphi;J]} \qquad (1\text{-}10)$$

where

$$S[\varphi,J] = \int d^4x \left(\frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - \frac{1}{2}m^2\varphi^2 - V(\varphi) + J\varphi\right) \qquad (1\text{-}11)$$

By means of this definition, an n-point Green-function can be written as

$$G^{(n)}(x_1 \cdots x_n) \equiv \langle T[\varphi(x_1)\cdots\varphi(x_n)]\rangle$$

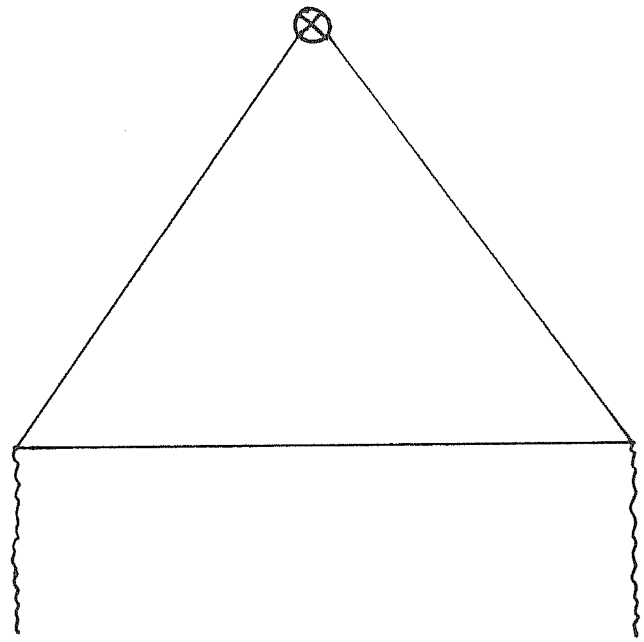


fig. 1-1

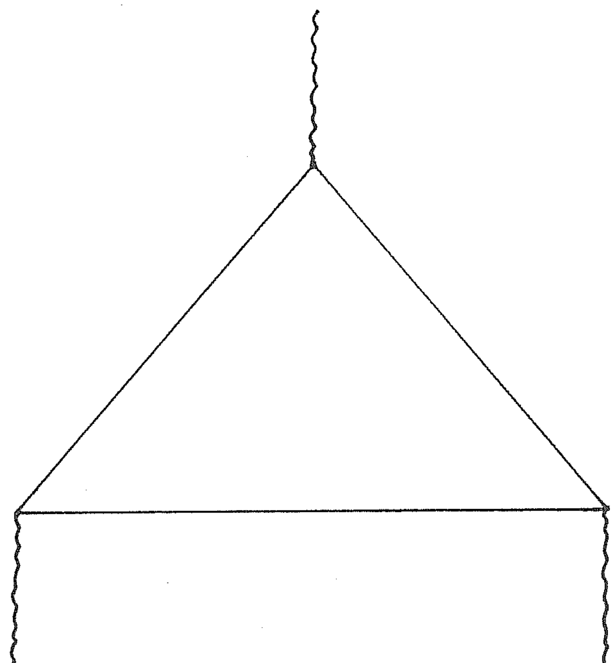


fig. 1-2

(1-12)

$$= (-i)^{n} \frac{\delta}{\delta J(x_1)} \cdots \frac{\delta}{\delta J(x_n)} W[J] \Big|_{J=0} .$$

The generating functional of connected Green-functions is defined as $Z[J]$ which satisfies

$$W[J] = e^{iZ[J]} . \qquad (1\text{-}13)$$

Therefore, the connected n-point Green-function can be written as

$$G_c^{(n)}(x_1 \cdots x_n) = (-i)^{n} \frac{\delta}{\delta J(x_1)} \cdots \frac{\delta}{\delta J(x_n)} Z[J] \Big|_{J=0}$$

(1-14)

$$= \langle T[\varphi(x_1) \cdots \varphi(x_n)] \rangle_c .$$

And the 1-PI Green-functions corresponding to $G^{(n)}$, $\Gamma^{(n)}$ is defined by

$$\Gamma^{(n)} = G^{(n)} \times (G^{(2)})^{-n} \qquad (1\text{-}15)$$

and

$$\Gamma^{(2)} = G^{(2)} \times (G^{(2)})^{-2} = (G^{(2)})^{-1} . \qquad (1\text{-}16)$$

We shall derive this expression in detail afterwards.

In terms of the generating functional of Green-functions without potential, we can write

$$W[J] = N \exp\{-i\int d^4x\, V(-i\frac{\delta}{\delta J(x)})\} W_0[J] . \qquad (1\text{-}17)$$

Including the quantum corrections, the effective action can be expressed as

$$\Gamma(\varphi_{cl}) = Z[J] - \int d^4x\, J(x)\varphi_{cl}(x) \qquad (1\text{-}18)$$

with the definition

$$J(\varphi_{cl}) = -\frac{\delta}{\delta\varphi_{cl}(x)}\Gamma(\varphi_{cl}) \qquad (1\text{-}19)$$

or

$$\varphi_{cl}(J) = \frac{\delta}{\delta J(x)} Z[J] . \qquad (1\text{-}20)$$

In fact, φ_{cl} is the classical fields.

Let us consider the derivative of the effective action respect of φ_{cl},

$$\frac{\delta}{\delta\varphi_{cl}}\Gamma(\varphi_{cl}) = \frac{\delta}{\delta\varphi_{cl}}Z - J - \int d^4y \frac{\delta J(y)}{\delta\varphi_{cl}(x)}\varphi_{cl}(y)$$

$$= \frac{\delta Z}{\delta J}\frac{\delta J}{\delta\varphi_{cl}} - J - \int d^4y \frac{\delta J}{\delta\varphi_{cl}}\varphi_{cl}$$

$$= \varphi_{cl}\frac{\delta J}{\delta\varphi_{cl}} - J - \int d^4y\, \delta^4(x-y)\frac{\delta J}{\delta\varphi_{cl}}\varphi_{cl} \qquad (1\text{-}21)$$

$$= -J(x).$$

The second order of the derivative is

$$\frac{\delta^2\Gamma(\varphi_{cl})}{\delta\varphi_{cl}(x_1)\delta\varphi_{cl}(x_2)} = -\frac{\delta J(x_2)}{\delta\varphi_{cl}(x_1)} = -\delta(x_1-x_2)\frac{\delta J(x_1)}{\delta\varphi_{cl}(x_1)} \qquad (1\text{-}22)$$

or we can write

$$\frac{\delta J(x_2)}{\delta\varphi_{cl}(x_1)} = -\frac{\delta^2\Gamma(\varphi_{cl})}{\delta\varphi_{cl}\,\delta\varphi_{cl}} = \left(\frac{\delta^2 Z[J]}{\delta J(x_1)\delta J(x_2)}\right)^{-1} \qquad (1\text{-}23)$$

From these expressions, we can derive the two-point 1-PI Green-function as

$$\Gamma^{(2)}(x_1, x_2) \equiv \frac{\delta^2\Gamma(\varphi_{cl})}{\delta\varphi_{cl}(x_1)\delta\varphi_{cl}(x_2)}\Big|_{J=0}$$

$$= -\left(\frac{\delta^2 Z[J]}{\delta J\,\delta J}\right)^{-1} = \left(G_c^{(2)}\right)^{-1}, \qquad (1\text{-}24)$$

as we pointed out above.

To see something more about the effective action, let us consider the definition

$$\Gamma(\varphi_{cl}) = Z(J) - \int d^4x\, J(x)\, \varphi_{cl}(x) \qquad (1\text{-}25)$$

At the tree-level, i.e. the classical approximation, we have

$$\Gamma(\varphi_{cl}) = S(\varphi_{cl}) \qquad (1\text{-}26)$$

where $S(\varphi_{cl})$ is the classical action, because there is no quantum corrections here. One can write

$$Z[J]_{tree} = S(\varphi_{cl}) + \int d^4x\, J(x)\, \varphi_{cl}(x) \qquad (1\text{-}27)$$

From the above equation of the connected n-point Green-function, we can write , at tree-level,

$$Z[J]_{tree} = \sum \frac{1}{n!} \int dx_1 \cdots dx_n\, G_c^{(n)\,tree}\, J(x_1) \cdots J(x_n) \qquad (1\text{-}28)$$

or

$$G_c^{(n)tree}(x_1 \cdots x_n) = \frac{\delta}{\delta J(x_1)} \cdots \frac{\delta}{\delta J(x_n)} Z[J]_{tree}\Big|_{J=0} . \qquad (1\text{-}29)$$

Similarly, we can also write

$$\Gamma(\varphi_{cl}) = \sum \frac{1}{n!} \int dx_1 \cdots dx_n \Gamma^{(n)} \varphi_{cl}(x_1) \cdots \varphi_{cl}(x_n) \qquad (1\text{-}30)$$

where $\Gamma^{(n)}$ is 1-PI n-point Green function.

Now, let us consider the derivative of the generating functional of general Green-functions with respect to $J(x)$,

$$\frac{\delta W[J]}{\delta J} = \frac{\delta}{\delta J} exp\{iZ(J)\}$$

$$= \int d^4x_2 \frac{\delta e^{iZ(J)}}{\delta J(x_2)} \delta(x_1 - x_2) \qquad (1\text{-}31)$$

$$= \int dx_2\, i\, e^{iZ(J)} \frac{\delta Z(J)}{\delta J(x_2)} \delta(x_1 - x_2)$$

Therefore,

$$\frac{1}{i} \frac{\delta W[J]}{\delta J} = W[J] \cdot \frac{\delta Z[J]}{\delta J(x_1)} . \qquad (1\text{-}32)$$

Put $J(x)=0$, we shall get

$$G^{(1)}(x_1) = W(0)\frac{\delta Z(J)}{\delta J(x_1)}\Big|_{J=0} \tag{1-33}$$

Removing W(O) away, we'll get

$$G_c^{(1)}(x_1) = \frac{\delta Z(J)}{\delta J}\Big|_{J=0} . \tag{1-34}$$

Furthermore, the second order of the derivative is

$$\frac{1}{i}\frac{\delta W[J(x)]}{\delta J(x_2)\delta J(x_1)} = \frac{\delta W[J(x)]}{\delta J(x_2)}\frac{\delta Z(J(x))}{\delta J(x_1)} +$$

$$+ W[J(x)]\frac{\delta Z[J(x)]}{\delta J(x_2)\delta J(x_1)} \tag{1-35}$$

$$= i W[J]\frac{\delta Z(J)}{\delta J}\frac{\delta Z(J)}{\delta J} + W(J)\frac{\delta^2 Z(J)}{\delta J\delta J} .$$

Putting $J(x)=0$, we'll get

$$G^{(2)}(x_1,x_2) = W(0)\left\{G_c^{(1)}G_c^{(1)} + \frac{1}{i}\frac{\delta^2 Z(J)}{\delta J\delta J}\right\} , \tag{1-36}$$

where the second term is the connected part,

$$G_c^{(2)}(x_1,x_2) = \frac{\delta^2 Z[J(x)]}{\delta J(x_1)\delta J(x_2)} . \tag{1-37}$$

From the definition of the generating functional of general Green functions and the effective action, we can write

$$\exp\{i\Gamma(\varphi_{cl})\} = N\int \mathcal{D}\varphi \exp\{iS(\varphi) + i\int d^4x\, J(\varphi - \varphi_{cl})\}. \tag{1-38}$$

Here, we can define

$$\varphi_a = \varphi - \varphi_{cl} \tag{1-39}$$

where φ_{cl} plays the rule of a background field and φ_a, in fact, is the quantum field. On the other hand, the effective action can be written as

$$\Gamma(\varphi_{cl}) = S(\varphi_{cl}) + \Gamma^{loop}(\varphi_{cl}), \tag{1-40}$$

that is, the summation of the classical action and all loop corrections. Then, we have

$$\exp\{i\Gamma^{loop}(\varphi_{cl})\} = N\int \mathcal{D}\varphi \exp\{i[S(\varphi) -$$

$$- S(\varphi_{cl})] + i\int d^4x\, J(\varphi - \varphi_{cl})\} \tag{1-41}$$

$$= N\int \mathcal{D}\varphi_a \exp\{i[S(\varphi_a + \varphi_{cl}) - S(\varphi_{cl})] - i\int d^4x \frac{\delta\Gamma}{\delta\varphi_{cl}}\varphi_a\}$$

Here $\Gamma^{loop}(\varphi_{cl})$ generates all 1-PI diagrams with, at least, one-loop.

Since $\frac{\delta\Gamma}{\delta\varphi_{cl}}$ is nothing but the equation of motion, this term does not contribute for the on-shell divergences. From the above expression, we can see that the only integrated quantity is the quantum field. So, the only non-vanishing propagator is $\langle \qquad \rangle$, and the classical field can be only a parameter in vertices or in external lines.

Now, let us consider the Ward Identities without anomalies. After this, it is easy to study the Ward Identities with anomalies. Suppose that we start from a classical action which possesses some symmetry. We shall derive the Ward Identities corresponding to this symmetry of the classical action.

The motivation is that Ward Identities are more essential in studying the symmetric properties of a quantum theory. From the Ward Identity, we can directly see the breaking of a symmetry and the anomaly.

Let us consider a classical action, $S(\varphi)$ and a transformation of fields,

$$\delta \varphi = \varepsilon q_i \varphi(x_i) , \qquad (1-42)$$

where δ is the variance of the field under this transformation, ε is a parameter of the transformation, and q_i is the charge of the field $\varphi(x)$. By definition,

$$S(\varphi) = \int d^4x \mathcal{L}(\varphi)$$

$$(1-43)$$

$$\delta S(\varphi) = \int d^4x [\delta\mathcal{L} + \partial_\mu(\delta x^\mu)\mathcal{L}] .$$

Since we are interested only in internal symmetries, $\delta x^\mu = 0$ is always assumed. Then, there should be

$$\delta S(\varphi) = \int d^4x \, \delta\mathcal{L}(\varphi) . \qquad (1-44)$$

To study the invariance of the action, we need only study the invariance of the Lagrangian,

$$\delta \mathcal{L} = \frac{\partial\mathcal{L}}{\partial \partial_\mu \varphi} \delta \partial_\mu \varphi + \frac{\partial \mathcal{L}}{\partial \varphi} \delta\varphi . \qquad (1-45)$$

For internal symmetry,

$$\varphi'(x) = \varphi(x) + \delta\varphi(x)$$

$$(\partial_\mu \varphi)' = \partial_\mu [\varphi(x) + \delta\varphi(x)] \tag{1-46}$$

therefore

$$\delta(\partial_\mu \varphi) = \partial_\mu (\delta\varphi) \ . \tag{1-47}$$

So

$$\delta \mathcal{L} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi} \delta\varphi\right) - \left(\partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi} - \frac{\partial \mathcal{L}}{\partial \varphi}\right) \delta\varphi . \tag{1-48}$$

Here, the second term is nothing but the equation of motion. If the equation of motion and the classical invariance of the Lagrangian are assumed, we get

$$\partial^\mu j_\mu = 0 ,$$

$$j_\mu = \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi} \delta\varphi . \tag{1-49}$$

The Ward Identities are generally used for local symmetries. But, it can be also used for global symmetries. In order to do this, we use a trick that the parameter, ε , is assumed to be x-dependent at the beginning. At the end of the calculation, we put it back to a constant. Now, let us do this derivation.

From the above expression, we have

$$\delta \mathcal{L} = \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi} \delta \varphi \,, \tag{1-50}$$

where the equation of motion term is removed. Since

$$\delta \varphi = q \, \varphi(x) \, \varepsilon(x) \,, \tag{1-51}$$

we have

$$\begin{aligned} \delta_{local} \mathcal{L} &= q \, \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi} \varphi(x) \, \varepsilon(x) \right] \\ &= \varepsilon(x) \cdot \partial_\mu j^\mu(x) + j^\mu \cdot \partial_\mu \varepsilon(x) \\ &= \delta_{global} \mathcal{L} + j^\mu \cdot \partial_\mu \varepsilon(x) \,. \end{aligned} \tag{1-52}$$

To derive Ward Identities, let us start from n-point connected Green-functions

$$G_c^{(n)}(x_1, \cdots, x_n) = \langle T [\Pi \, \varphi(x_i)] \rangle$$

$$= N\int \mathcal{D}\varphi \prod_i \varphi(x_i)\, e^{i\int d^4x \mathcal{L}} . \tag{1-53}$$

The transformation is as follows

$$\varphi(x_i) \xrightarrow{\text{class. rigor}} \varphi'(x_i) = \varphi(x_i) + i\,\varepsilon(x_i)\, q_i\, \varphi(x_i) \tag{1-54}$$

under which, we have

$$G_c^{(n)\prime}(x_1 \cdots x_n) = \langle T[\prod_i \varphi'(x_i)]\rangle$$

$$= N\int \mathcal{D}\varphi' \prod_i \varphi'(x_i)\, e^{i\int d^4x \mathcal{L}'} . \tag{1-55}$$

The delicate point here is the invariance of the measure. Here we are going to assume that

$$\mathcal{D}\varphi' = \mathcal{D}\varphi + i\int d^4x\, \varepsilon(x)\, \mathcal{A}(x), \tag{1-56}$$

as treated in the Fujikava's approach. However, for the time being, we are interested in the Ward Identities without anomaly, i.e. $\mathcal{A}$ is zero.

Since we can write

$$\prod_i [\varphi(x_i) + i\varepsilon(x_i) q_i \varphi(x_i)$$

(1-57)

$$= \prod_i \varphi(x_i) + \sum_j i\varepsilon(x_j) q_j \prod_i \varphi(x_i) + O(\varepsilon^2),$$

we can get

$$G_c^{(n)\prime} = N\int \mathcal{D}\varphi [1 + \sum_j i\varepsilon(x_j) q_j] \prod_i \varphi(x_i) [1 + i\int d^4x\, \delta_{local}\mathcal{L}] e^{i\int d^4x \mathcal{L}}$$

$$= N\int \mathcal{D}\varphi [1 + \sum_j i\varepsilon(x_j) q_j] \prod \varphi(x_i) \times$$

$$\times [1 + i\int d^4x \{\varepsilon(x_i) \partial_\mu j^\mu(x_i) + j^\mu \cdot \partial_\mu \varepsilon\}] e^{i\int d^4x \mathcal{L}} \quad (1-58)$$

$$= G_c^{(n)} + N\int \mathcal{D}\varphi\, e^{i\int d^4x \mathcal{L}} \prod \varphi(x_i) \int d^4x\, i[$$

$$\times [\Sigma\, \delta(x - x_j) q_j \varepsilon(x_j) + \varepsilon \cdot \partial_\mu j^\mu + j^\mu \cdot \partial_\mu \varepsilon].$$

As before, in the case of global symmetries,

$$\partial^\mu j_\mu = \delta_{global}\mathcal{L} + (\text{eq. of motion})\, \delta\varphi. \quad (1-59)$$

Considering a general case, $\delta_{global}\mathcal{L} \neq 0$ is assumed, that is the Lagrangian may contain terms which explicitly break the classical symmetries. As an example, let us take the chiral symmetry in QED into account, where the first term in Lagrangian, $\bar{\psi}\gamma_\mu\gamma_5\psi$ preserves the chiral symmetry, but not the mass term.

Generally to say, the spontaneous breakdown of symmetries does not introduce anomalies. Neither does the explicit breakdown which can be switched off by changing the transformation parameter. Anomalies are introduced by the explicit breakdown of symmetries which can not be switched off by changing the parameters.

From the above expression, we can write the variance of the classical action when ε is treated local

$$\delta S = \int d^4x \, \delta_{local} \mathcal{L}$$

$$= \int d^4x [\varepsilon(x) \partial_\mu j^\mu + j^\mu \cdot \partial_\mu \varepsilon(x)]. \qquad (1\text{-}60)$$

The first term of the integrand is the variance of Lagrangian for global ε, which can be written as

$$\delta_{global} \mathcal{L} = \partial_\mu j^\mu = \varepsilon \bar{\delta} \mathcal{L}, \qquad (1\text{-}61)$$

where the equation of motion term has been dropped because it does not contribute at the end. In terms of the new symbol, one can write

$$\delta S = \int d^4x \, [\, \varepsilon(x) \, \bar{\delta}\mathcal{L} + j^\mu \cdot \partial_\mu \varepsilon(x) \,] \qquad (1\text{-}62)$$

and

$$G_c^{(n)\prime} = N \int \mathcal{D}\varphi \, [\, 1 + i \sum \varepsilon(x_i) \, q_i \,] \prod \varphi(x_i) \times$$
$$\times [\, 1 + i \int d^4x \, \{ \varepsilon(x) \, \bar{\delta}\mathcal{L} + j^\mu \cdot \partial_\mu \varepsilon(x) \}] \, e^{i\int d^4x \mathcal{L}} . \qquad (1\text{-}63)$$

If $\bar{\delta}\mathcal{L} = 0$, a classical symmetry appears. Otherwise, the eventually explicit breakdown is present. For example, there exists a mass term in the Lagrangian of a chiral theory. To understand the second term, the above expression can be rewritten as

$$G_c^{(n)\prime} = G_c^{(n)} + N \int \mathcal{D}\varphi \, e^{i\int d^4x \mathcal{L}} \prod \varphi(x_i) \int d^4x \times$$
$$\times [\, \sum \delta(x - x_i) \, q_i + \bar{\delta}\mathcal{L} + j^\mu \cdot \partial_\mu \,] \, i \, \varepsilon(x) . \qquad (1\text{-}64)$$

Here we used the trick

$$\varepsilon(x) = \varepsilon\,\delta(x-y), \tag{1-65}$$

where ε is a constant. Since the Green function is nothing but $\langle T\varphi\cdots\varphi\rangle = \langle f|i\rangle$, the scattering amplitude which must be preserved, i.e. $G_c^{(n)\prime} = G_c^{(n)}$, the second term should be zero. This condition is equivalent to

$$\sum \varphi_i\, \delta(x-x_i)\langle T[\prod\varphi(x_i)]\rangle_c + \langle T\tilde{\delta}\mathcal{L}\prod\varphi_i\rangle - \tag{1-66}$$
$$-\partial_\mu\langle T j^\mu \prod \varphi_i\rangle_c = 0 \quad .$$

This is nothing but the Ward Identities. In above expression, the second term is a n-point Green function with the insertion of $\tilde{\delta}\mathcal{L}$, that is the variation of the term which explicitly break the classical symmetries. Since this term is not interesting for us, we always assume it vanishing. On the other hand, the third term is a n-point Green function with the current insertion. It is this term that we use to study anomalies.

Now, a remark is available, from which we can see how to get the information about anomaly. The Ward Identities offer a method for checking whether the quantum theory possesses a symmetry. If the classical symmetry is implementable in the quantum theory, the third term will exactly cancel the first one, and the Ward Identity is preserved. If not, the third term will give an extra term except canceling the first term, and the Ward Identity will be spoiled. In this case, the divergence of the corresponding current is a finite term, called anomaly. It can be writen as

$$\partial_\mu j^\mu = A . \tag{1-67}$$

In the Ward Identity, the contribution to the third term is of the form like

$$\langle T[A \Pi \varphi(x_i)] \rangle . \tag{1-68}$$

Therefore, when we calculate anomalies, we always evaluate the graph in fig. 1-3 and construct it with q_μ. If this construction vanishes, the term $\langle A \Pi \varphi(x_i) \rangle$ does not contribute and the Ward Identities are respected. Otherwise, $\langle A \Pi \varphi(x_i) \rangle \neq 0$, we shall get anomaly. This is why we probe anomaly just by computing the construction.

In fact, we can have a look at the third term,

$$\partial_\mu \langle T[j^\mu(x) \Pi \varphi(x_i)] \rangle \sim \langle \partial^\mu j_\mu \Pi \varphi_i \rangle + \tag{1-69}$$

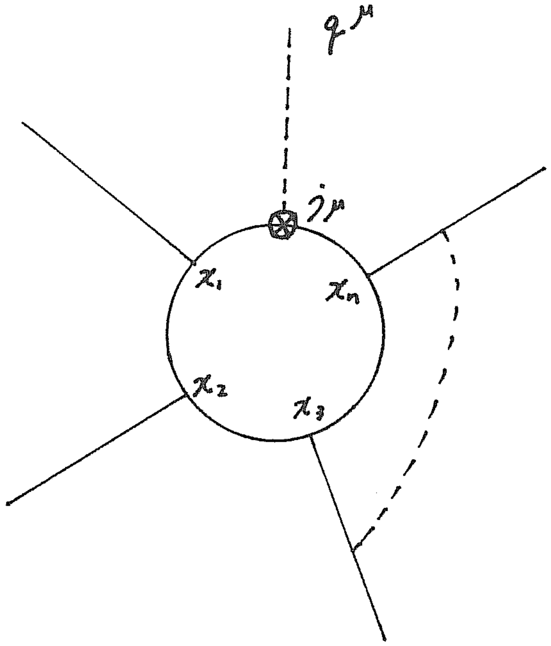


fig. 1-3

$$+\delta(x_i^0 - x^0)[j_0, \varphi] \, ,$$

where the commutator is

$$[j_0 , \varphi] = i\varepsilon q_i \varphi_i \delta(\vec{x}_i - \vec{x}) \qquad (1\text{-}70)$$

from the quantization of the theory.

From the Ward Identities, we can see that

$$\sum q_j \delta(x - x_j) \qquad (1\text{-}71)$$

is the sum of the conserved charge. The appearance of anomalies will destroy the conservation of the charge.

Another point should be mentioned here. From this expression, we can also derive the Goldstone theorem provided that no anomaly appears. The anomaly will also make the expected Goldstone particle disappear. It is in this mechanism that the U(1) anomaly in QCD explains why the expected Goldstone particle does not exist. We shall derive the Ward Identities with and without anomaly in the following.

Let us write down the Ward Identity again

$$\sum q_j \, \delta(x - x_j) G_c^{(n)} - \partial_\mu \langle T[j^\mu \Pi \varphi_i] \rangle_c = 0 , \qquad (1\text{-}72)$$

where the explicit breaking term of the classical symmetry is dropped. Taking the Fourier transformation in x for the Ward Identity gives

$$\sum q_j \, e^{ipx_j} G_c^{(n)}(x_j) - \int d^4x \, e^{ipx} \partial^\mu \langle T j_\mu \Pi \varphi_i \rangle = 0 \qquad (1\text{-}73)$$

or

$$\sum q_j \, e^{ipx_j} G_c^{(n)}(x_j) + i \int d^4x \, e^{ipx} p^\mu \langle T j_\mu \Pi \varphi_i \rangle = 0 \qquad (1\text{-}74)$$

Now, it is important to note that in the limit of zero momentum, the last term receives contribution only from the eventual zero mass states present in the theory. Moreover, we can use the arguments of Lorenz covariance

$$\langle 0 | j_\mu(0) | G \rangle = i F_G P_\mu , \qquad (1\text{-}75)$$

F_G measures the coupling of the massless state G to the vacuum through the current j_μ,

$$F_G \langle 0 | \Pi \varphi_i | G \rangle = \Sigma q_i \langle \Pi \varphi_i \rangle + \int d^4x \langle T[\bar{\delta}\mathcal{L} \, \Pi \varphi_i] \rangle \quad . \tag{1-76}$$

Now, if (a) $\bar{\delta}\mathcal{L} = 0$ (as we assumed above), (b) $(\Sigma q_i) \langle \Pi \varphi_i \rangle \neq 0$, i.e. there exist a charged order parameter which acquires a non-vanishing vacuum expectation value, we shall have the following expression

$$F_G \langle 0 | \Pi \varphi_i | G \rangle = \Sigma q_i \langle \Pi \varphi_i \rangle \neq 0 \quad . \tag{1-77}$$

Therefore, we shall have $F_G \neq 0$. Since this fact, the spontaneous breaking of a symmetry implies that, via Ward Identity, there exist massless state G coupled to the current j_μ . From the above analysis, we can get the conclusion that in the absence of anomaly the spontaneous breaking of a con- tinuous symmetry leads to that there exist massless spin-0 particles. This result can be derived from the Ward Identi- ties associated with the invariance.

Now we are going to see the dilatation anomaly. The transformation law can be written as

$$x'_\mu = e^\lambda x_\mu = (1 + \lambda + \cdots) x_\mu \tag{1-78}$$

where

$$\lambda > 0, \quad \text{for dilatation,}$$

$$\lambda < 0, \quad \text{for contraction.} \tag{1-79}$$

This transformation gives a factor e^{λ} to a line, a factor $e^{2\lambda}$, to an area and a factor $e^{3\lambda}$ to a volume. That is to say, the transformation law of a quantity depends on its dimension. Suppose the dimension of the fields is

$$[\phi] = L \tag{1-80}$$

then

$$\varphi'(x') = e^{\lambda L}\varphi(x) = (1+\lambda L)\varphi(x)$$

$$\partial'\varphi'(x') = e^{\lambda(L-1)}\partial\varphi(x) = (1+\lambda[L-1])\partial\varphi(x). \tag{1-81}$$

In the case of $\mathcal{L}_{KG}$, we have

$$[\varphi] = L = -1, \tag{1-82}$$

so

$$\delta\mathcal{L} = \lambda L\,\varphi(x)\frac{\partial\mathcal{L}}{\partial\varphi} + \lambda(L-1)\partial_\mu\varphi\frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi} +$$

$$+\lambda x_\mu\frac{\partial\mathcal{L}}{\partial x_\mu} = \lambda L\varphi\,\partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi} + \qquad (1\text{-}83)$$

$$+\lambda L\;\partial_\mu\varphi\frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi} - \lambda\,\partial_\mu\varphi\frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi} + \lambda x_\mu\frac{\partial\mathcal{L}}{\partial x_\mu}$$

Therefore

$$\partial_\mu\left[L\frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi}\varphi - x^\nu T_\nu{}^\mu\right] = 0 \qquad (1\text{-}84)$$

we can define the canonical current associated with the dilatation symmetry, D_μ , such that

$$\partial^\mu D_\mu = 0 \;. \qquad (1\text{-}85)$$

It can be also expressed as the improved quantity in terms of the improved energy-momentum tensor. Here one should notice that D_μ is associated with $T_\mu{}^\nu$.

We note that the dilation anomaly depends on the appearance of a dimensional term in Lagrangian. Since the dimensional regularization scheme introduces a mass scale term, the scale invariance will be spoiled by this scheme.

Now we are going to calculate the dilatation anomaly for

$$\mathcal{L}_{K\cdot G} = \frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}m^2\varphi^2 - \frac{\lambda}{4!}\varphi^4 . \tag{1-86}$$

then

$$S = \int d^4x \mathcal{L} = \int d^4x \mathcal{L}_{K\cdot G} . \tag{1-87}$$

From the first term, in mass scale, we have

$$[\varphi] = X , \qquad 2X = n-2 , \tag{1-88}$$

The mass term is same as above. From the interaction term, we have

$$[\lambda] = Y , \quad Y = n - 4X \neq 0 . \tag{1-89}$$

In order to keep $[\lambda]=0$, we introduce a parameter $[\mu]=1$ and rewrite the Lagrangian as follows

$$\mathcal{L}_{KG} = \frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}m^2\varphi^2 - \mu^{4-n}\frac{\lambda}{4!}\varphi^4 , \tag{1-90}$$

so

$$\Upsilon = n - 2n + 4 = 4 - n \,. \qquad (1\text{-}91)$$

From this we get

$$\Theta_{\mu\nu} = (\partial_\mu \varphi)(\partial_\nu \varphi) - \eta_{\mu\nu} \mathcal{L}$$

$$+ \frac{1}{6}(\Box \eta_{\mu\nu} - \partial_\mu \partial_\nu)\varphi^2 \qquad (1\text{-}92)$$

$$\partial^\mu \mathcal{D}_\mu^{(imp)} = \Theta_\mu{}^\mu = -\frac{n-4}{3}\mathcal{L} + \cdots\cdots ,$$

that is

$$\Theta_\mu{}^\mu \sim -\frac{n-4}{3}\mathcal{L} \,. \qquad (1\text{-}93)$$

Here we can see the anomaly because, generally

$$\mathcal{L} \sim \frac{1}{(4-n)^a} \,. \qquad (1\text{-}94)$$

Let us go on to see the connection between $\Theta_\mu{}^\mu$ and the renormalization group equation. Renormalization group equation says that any physically measured quantity is independent of the parameter μ . So, $(\mathcal{O}_\mu^\mu)_{anomaly}$ is proportional to the renormalization group equation. We renormalize the Lagrangian as follows

$$\mathcal{L} = \frac{1}{2}(\partial\varphi_R)^2 - \frac{1}{2}m_R^2\varphi_R^2 - \frac{\lambda_R}{4!}\mu^\varepsilon\varphi_R^4 + \tag{1-95}$$

$$+ K_\varphi\frac{1}{2}(\partial\varphi_R)^2 - K_m\frac{1}{2}m_R^2\varphi_R^2 - K_v\frac{\lambda_R}{4!}\mu^\varepsilon\varphi_R^4 .$$

where

$$K_{\varphi,m,v} = \sum_n k^n_{\varphi,m,v}\frac{\lambda_R}{\varepsilon^n} ,$$

$$\varphi_B = (1+K_\varphi)^{1/2}\varphi_R = Z_\varphi^{1/2}(\lambda_R,\varepsilon)\varphi_R ,$$

$$m_B^2 = (1+K_m)^{1/2}(1+K_\varphi)^{-1}m_R^2 = Z_m Z_\varphi^{-1} m_R^2 , \tag{1-96}$$

$$\lambda_B = \mu Z_\lambda \lambda_R , \qquad \varepsilon = 4-n$$

$$Z_\lambda = Z_v \cdot Z_\varphi^{-2} , \qquad Z_v = (1+K_v) .$$

We consider n-point function, bare and renormalized, $\Gamma_B^{(n)}$, $\Gamma_R^{(n)}$, which satisfy

$$\Gamma_R^{(n)} \sim \langle \varphi_R \cdots \varphi_R \rangle \langle \varphi_R \varphi_R \rangle^{-1} \cdots \langle \varphi_R \varphi_R \rangle^{-1}$$

$$\sim Z_\varphi^{-n/2} \cdot Z_\varphi^{n} \langle \varphi_B \cdots \varphi_B \rangle \langle \varphi_B \varphi_B \rangle^{-1} \cdots \langle \varphi_B \varphi_B \rangle^{-1} \qquad (1-97)$$

$$= Z_\varphi^{n/2} \Gamma_B^{(n)},$$

or

$$\Gamma_B^{(n)}(m_B, \lambda_B) = Z_\varphi^{-n/2}(\lambda_R, \varepsilon) \Gamma_R^{(n)}(m_R, \lambda_R, \mu) \qquad (1-98)$$

Taking the derivative with respect to μ, we get

$$\frac{d\Gamma_B^{(n)}}{d\mu} = \frac{\partial Z_\varphi^{-n/2}}{\partial \mu} \Gamma_R^{(n)} + Z_\varphi^{-n/2} \frac{\partial \Gamma_R^{(n)}}{\partial \mu}$$

$$= -\frac{n}{2} Z_\varphi^{-\frac{n}{2}-1} \Gamma_R^{(n)} \frac{dZ_\varphi}{d\mu} + \Big(\frac{\partial}{\partial \mu} + \qquad (1-99)$$

$$+ \frac{\partial m_R}{\partial \mu} \frac{\partial}{\partial m_R} + \frac{\partial \lambda_R}{\partial \mu} \frac{\partial}{\partial \lambda_R}\Big) \Gamma_R^{(n)} \cdot Z_\varphi^{-n/2} = 0 .$$

which can be rewritten as

$$\Big(\mu \frac{\partial}{\partial \mu} + \mu \frac{\partial \lambda_R}{\partial \mu} \frac{\partial}{\partial \lambda_R} + \mu \frac{\partial m_R}{\partial \mu} \frac{\partial}{\partial m_R} - \frac{n}{2} \frac{d}{d\mu} \ln Z_\varphi\Big) \Gamma_R^{(n)} = 0 . \qquad (1-100)$$

It becomes

$$\left(\mu\frac{\partial}{\partial\mu}+\beta(\lambda_R)\frac{\partial}{\partial\lambda_R}-n\gamma(\lambda_R)+\gamma_m(\lambda_R)\frac{\partial}{\partial m_R}\right)\Gamma_R^{(n)}=0 \qquad (1\text{-}101)$$

after introducing the definitions

$$\beta(\lambda_R)=\mu\frac{\partial\lambda_R}{\partial\mu},$$

$$\gamma_\mu(\lambda_R)=\mu\frac{\partial m_R}{\partial\mu}, \qquad (1\text{-}102)$$

$$\gamma(\lambda_R)=\frac{1}{2}\frac{d}{d\mu}\ln Z_\varphi,$$

where γ and γ_m are called anomalous dimensions. If we put $(\theta_\mu^\mu)_{anomaly}$ into the generating functional

$$Z=\int\mathcal{D}\varphi(x)\,e^{\frac{i}{a}[S+\int d^4x J\varphi]} \qquad (1\text{-}103)$$

,

we have

$$\frac{1}{Z}\frac{\partial Z}{\partial a}=-\frac{1}{a^2}\langle\int d^4x\mathcal{L}+\int d^4x J\varphi\rangle, \qquad (1\text{-}104)$$

where a is the loop counting operator because $a^{L-1}=a^{I-V}$.

Since the second term on the right hand side is not changed due to

$$\varphi \longrightarrow Z_\varphi^{-1/2}\varphi \quad ,$$

(1-105)

$$J \longrightarrow Z_\varphi^{1/2} J \ .$$

which keep $J\varphi$ invariant, we do not consider it. Then we have

$$(n-4)\frac{\partial Z}{\partial a} = -(n-4)\frac{1}{a^2}\langle \int d^4x \mathcal{L} \rangle Z$$

$$= (n-4)\left[\frac{\partial m_R}{\partial a}\frac{\partial}{\partial m_R} + \frac{\partial \lambda_R}{\partial a}\frac{\partial}{\partial \lambda_R} + \frac{\partial J_R}{\partial a}\frac{\partial}{\partial J_R}\right] * \qquad (1\text{-}106)$$

$$* Z(m_R, \lambda_R, J_R, a) = -\frac{1}{a^2}\langle \theta_\mu{}^\mu \rangle Z .$$

Here we used the following formulas

$$(n-4)\frac{\partial m_R}{\partial a} \longrightarrow -\gamma_m m_R \ ,$$

$$(n-4)\frac{\partial \lambda_R}{\partial a} \longrightarrow \beta(\lambda_R) \ , \qquad (1\text{-}107)$$

$$(n-4)\frac{\partial J_R}{\partial a} \longrightarrow \gamma \ .$$

In terms of the Lowenstein's result, we get

$$\Theta_{\mu}{}^{\mu}\Big|_{\text{on-shell}} = \frac{\beta(\lambda_R)}{2\lambda_R} F^{\mu\nu}F_{\mu\nu} \tag{1-108}$$

for a gauge theory[8].

In what follows, we give a short summary on gravitational anomalies. Gravity has a non-Abelian structure similar to Yang-Mills theories. This can be seen by analyzing the general coordinate transformations which are at the basis of general covariance or, if one wishes, by considering the local Lorentz invariance of the Einstein-Cartan formalism. The non-Abelian structure of the general coordinate transformations can be seen by considering that the commutator of two transformations, characterized by parameters $\xi^1_\mu(x)$ and $\xi^2_\mu(x)$ gives a third transformation with parameter , $\xi^3_\mu(x)$ given by

$$\xi^3_\mu(x) = \xi^{2\nu}(x)\partial_\nu \xi^1_\mu(x) - \xi^{1\nu}\partial_\nu \xi^2_\mu \,. \tag{1-109}$$

If one considers a theory where chiral fields are coupled to a gravitational background field, gravitational anomalies can be induced through the non-conservation of the energy-momentum tensor (for the effective action to be invariant under general coordinate transformations, the energy-momentum tensor of the theory should be covariantly conserved). The structure of the purely gravitational anomalies implies that they can exist only in the space-times of dimension D=4n+2, that is, 2, 6, 10, etc. Indeed, by considering supergravity theories in ten-dimensions, Witten and Álvarez-Gaume[7] have computed the expressions for the gravitational anomaly coming from loops chiral 1/2, chiral 3/2 and self-dual tensor fields. These expressions have actually been crucial for the anomaly cancellation mechanism proposed by Green and Schwarz in the context of the type-I open superstring [8].

To conclude this chapter, we give some words about the topological origin of anomalies. The basis for understanding the topological origin of the anomaly is through the Atiyah-Singer theorem[9], which states that the index of some elliptic operator is a topological invariance which which depends on the basis manifold and the bundles connected by the elliptic operator. Just to fix the ideas, one can consider the axial anomaly problem, where the elliptic operator is the Dirac Operator, $\not{D}(A)$, where A is a gauge field background configuration defined on a 2n- dimensional space-time. The operator $\not{D}$ connects the space of positive chirality spinor to the space of negative chiralities. The index of $\not{D}$ is

$$\mathrm{Ind}(\not{D}) = \dim[\ker(\not{D})] - \dim[\ker(\not{D}^{+})] \qquad (1\text{-}110)$$

This is a topological invariance which moreover can be written as the integral of some characteristic class which measures the topological twisting of the manifold and the spaces of "plus" or "minus" chiralities. By computing the index of the Dirac operator, one can obtain the chiral anomaly expression and the index theorem, which illustrates the topological origin of the anomaly, also provides the choice of the particular characteristic class which determines the index of the given elliptic operator. This is the analytic way of computing the index, which is a topological invariance. In this sense, one can say that the index theorem provides an interesting bridge between analytical methods and topology. Then it helps in pointing out the basic topological nature of anomalies.

Chapter 2 Technical considerations on anomalies

2.1 Calculation in the usual approach

In the previous chapter we introduced the general idea of what an anomaly means and through a series of examples we discussed for a given theory, which currents (global or local) may develop an anomaly. We ever saw that the gauge anomaly can be eliminated through its group theoretical factor $tr(\{ G_a , G_b \} G_c)$. We used, in short, a few formulae about anomalies to discuss general results. In this chapter, we shall devote our attention to more technical details and derive the results previously adopted. Issues like anomaly power counting rule, non-renormalization for the anomaly and explicit evaluation of an anomalous graphs will be touched here.

First of all, let us see what kind of graphs may lead to anomalies. Suppose we have a graph with n-external vector legs drawn in fig.2-1. The anomalies are always proportional to an $\epsilon_{\mu_1 \mu_2 \cdots \mu_{2D}}$. Since this graph contains

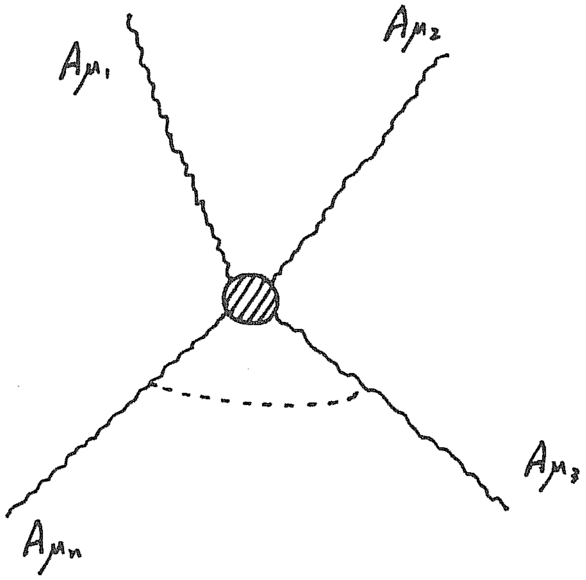


fig. 2-1

n external lines and n-1 independent external momenta, covariant arguments indicate that the final expression of the graph will be

$$\epsilon_{\mu_1\mu_2\cdots\mu_{2D}} A^{\mu_1} A^{\mu_2} \cdots A^{\mu_n} (p_1)^{\mu_{n+1}} \cdots (p_{n-1})^{\mu_{2D}} \tag{2-1}$$

This means that the number of vector lines and independent external momenta have to be such that they allow for a complete saturation of the 2D indices of the tensor, $\epsilon_{\mu_1\mu_2\cdots\mu_{2D}}$. So we have

$$n + (n-1) \geq 2D \Rightarrow n \geq D + \frac{1}{2} \ . \tag{2-2}$$

Since both n and D are integers, we get

$$n \geq D + 1 \ . \tag{2-3}$$

This formula gives, in 2D-dimensional field theories, the lowest graph carrying anomalies. Since the higher order graphs may also carry anomalies and be expressed in terms of the lowest order graph, in fact, we need only calculate the lowest order graphs.

Now, let us see some examples. In 2-dimensional space, D=1, n $\geqslant$ 2. The lowest order graph is drawn in fig.2-2. In 4-dimensional space, D=2, n $\geqslant$ 3. The lowest order graph is a triangle, drawn in fig.2-3. In 6-dimensional space, D=3, n $\geqslant$ 4. The lowest order graph is a square, like the graph in fig.2-4.

In what follows, we are going to derive the triangle anomaly in 4-dimensional space. First of all, we consider the group theory factor of the anomaly.

The anomaly of a certain current, let us say the gauge current, comes as a one-loop corrections to its divergence through the computation of the graph given in fig.2-5. The fermions which circulate inside the loop, and appear in the definition of the classical conserved current, have the following coupling to the gauge fields,

$$g\bar{\Psi}\gamma^{\mu}G^{a}\Psi A_{\mu a} \qquad (2\text{-}4)$$

where G^{a} denotes the generators of the gauge group in the representation, R, of the chiral fermions. Rendering explicit the indices of the representation, R, the vertex reads

$$g\bar{\Psi}_{i\alpha}\gamma^{\mu}_{\alpha\beta}(G^{a})_{ij}\Psi_{j\beta}A_{\mu a} \qquad (2\text{-}5)$$

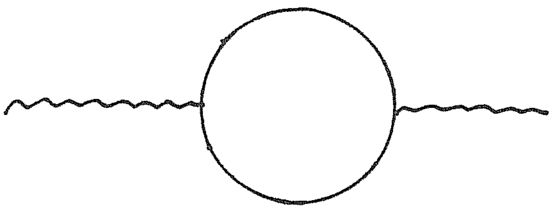


fig. 2-2

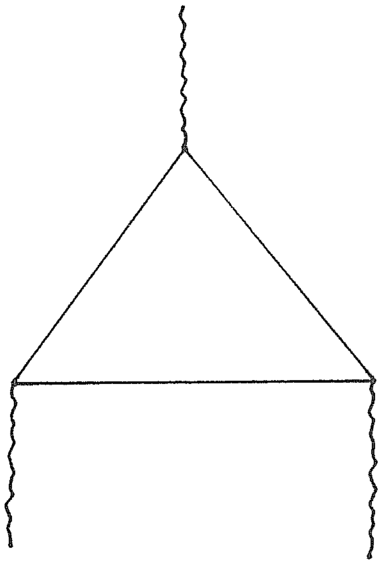


fig. 2-3

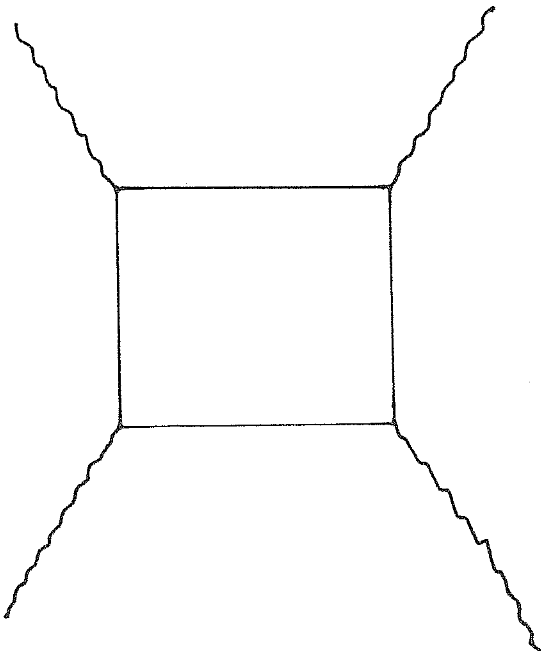


fig. 2-4

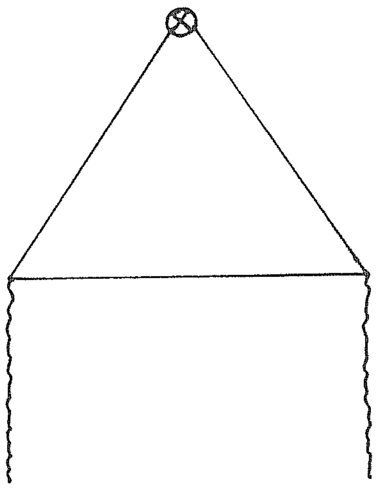


fig. 2-5

where i=1, 2, ..., dim. R. So, the corresponding Feynman rule becomes like the graph in fig. 2-6. (The derivation of this vertex comes basically from the application of the functional derivatives

$$\frac{\delta^3}{\delta\bar{\psi}(y_1)\,\delta\psi(y_2)\,\delta A(y_3)} \qquad (2\text{-}6)$$

to the classical action).

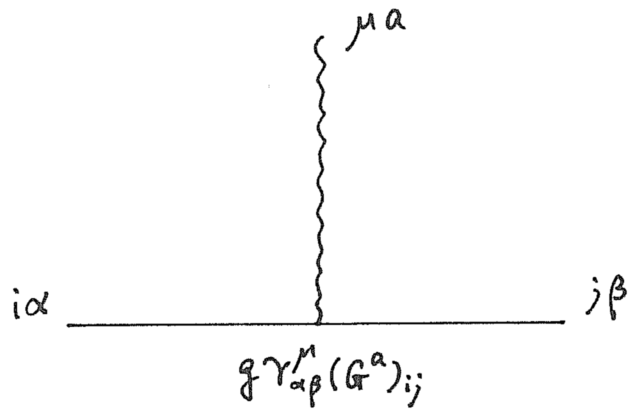
Now, let us come back to the anomalous graph and insert in it the vertex Feynman rule as in the graph in fig. 2-7.

Here two remark are in order:

(i) Notice that a certain internal line connects a ψ_i with a $\bar{\psi}_i$, simply due to the fact that the fermionic kinetic term is diagonal, namely

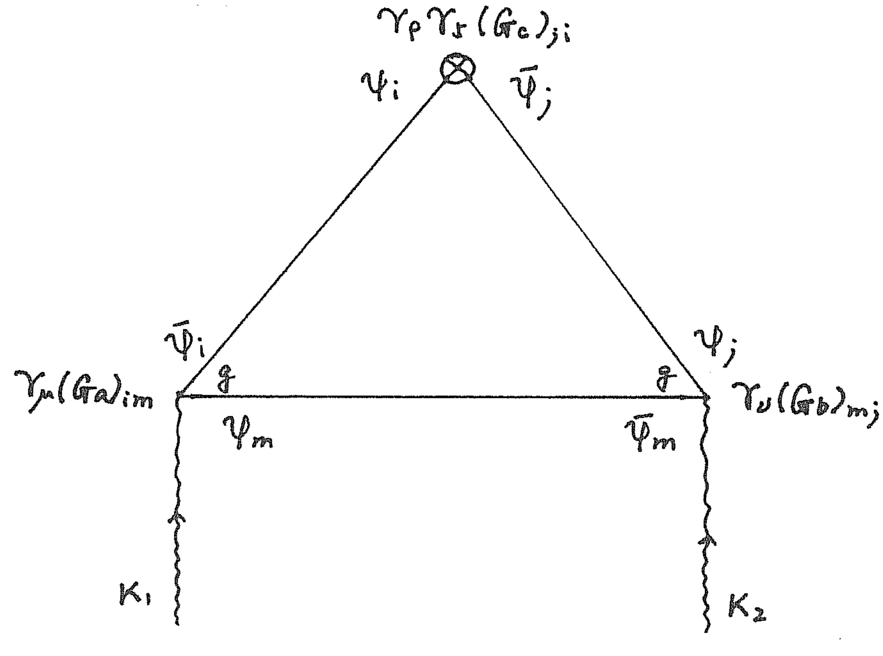
$$\bar{\psi}_i\, i\gamma^\mu \partial_\mu \psi_i \qquad (2\text{-}7)$$

(ii) Notice also the order of the matrix indices of the generators G_a : the first one refers to the spinor $\bar{\psi}$, and the second to the spinor ψ.



μa
$i\alpha$
$j\beta$
$g\gamma^{\mu}_{\alpha\beta}(G^{a})_{ij}$

fig. 2-6



$\gamma_\rho\gamma_5(G_c)_{ji}$
ψ_i
$\bar{\psi}_j$
$\bar{\psi}_i$
ψ_j
g
g
$\gamma_\mu(G_a)_{im}$
$\gamma_\nu(G_b)_{mj}$
ψ_m
$\bar{\psi}_m$
K_1
K_2

fig. 2-7

Since the gauge group generators appearing in the vertices can pass freely through the γ-matrices of the vertices and the propagator, the overall group theoretical factor of the triangle graph turns out to be

$$(G_a)_{im}(G_b)_{mj}(G_c)_{ji} = tr(G_a G_b G_c). \qquad (2\text{-}8)$$

Do not forget the symmetric graph drawn in fig.2-8 which gives the factor

$$tr(G_b G_a G_c). \qquad (2\text{-}9)$$

Putting them together, we finally get the following group theoretical factor

$$tr(\{G_a, G_b\} G_c), \qquad (2\text{-}10)$$

where $\{G_a, G_b\}$ is a anticommutator. This factor is very useful for canceling anomalies.

Now, we are going to calculate the space-time part of the anomalous triangle graph. According to the graph drawn in fig.2-9, we have the expression in momentum space

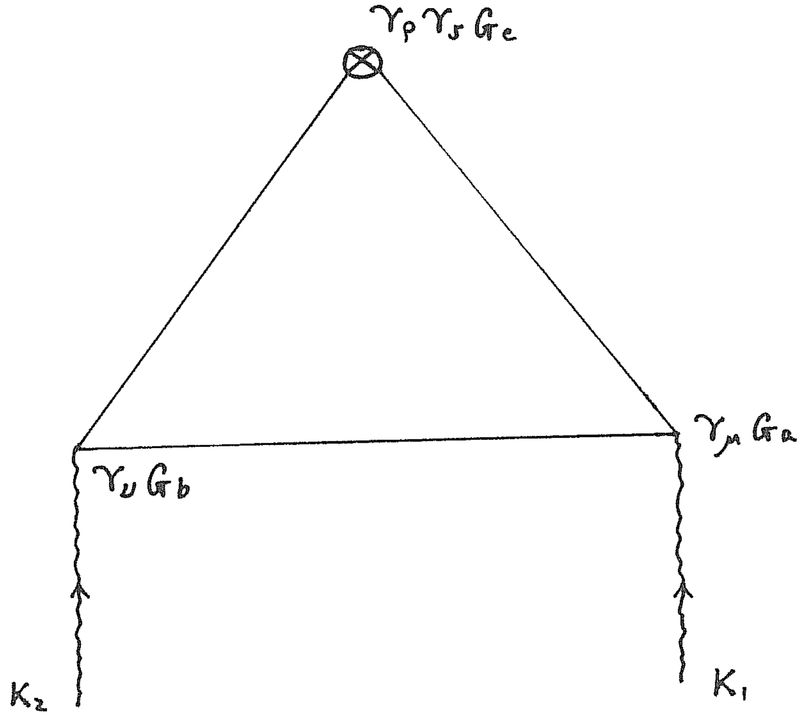

$\gamma_\rho \gamma_5 G_c$
$\gamma_\mu G_a$
$\gamma_\nu G_b$
K_2
K_1

fig. 2-8

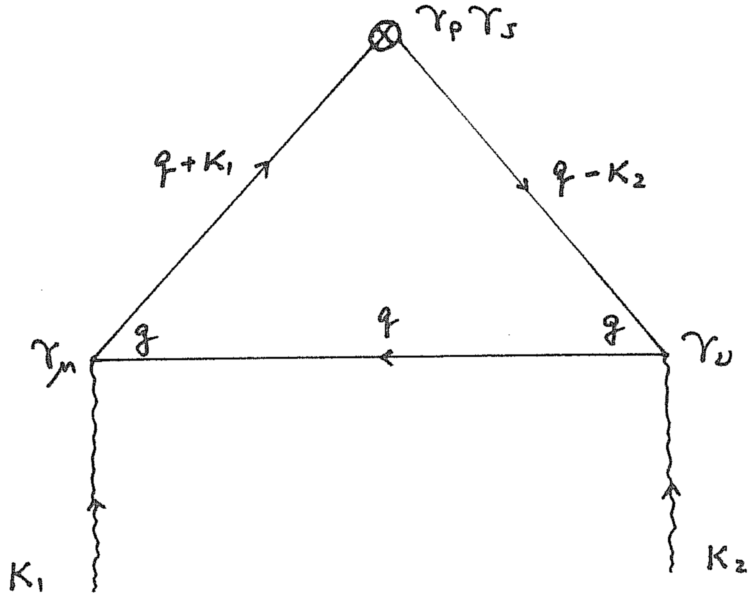

$\gamma_\rho \gamma_5$
$q+K_1$
$q-K_2$
q
q
q
γ_μ
γ_ν
K_1
K_2

fig. 2-9

$$-g^2 \mathrm{tr}\int \frac{d^4q}{(2\pi)^4}\Big[\frac{\not q+m}{q^2-m^2}\gamma_\mu\frac{\not q+\not K_1+m}{(q+K_1)^2-m^2}\gamma_\rho\gamma_5\frac{\not q-\not K_2+m}{(q-K_2)^2-m^2}\gamma_\nu\Big] \quad (2\text{-}11)$$

where the minus sign comes from the fermion loop and the trace is over the spinorial space. Each factor of i comes from the fermionic propagator $\frac{i}{\not q-m}$, from the vertex and insertion.

Before concentrating on the evaluation of this integral, let us compute an auxiliary integral whose result will be useful for the case we are studying. It is

$$I_{1\mu\nu}=\int\frac{d^4p}{(2\pi)^4}\mathrm{tr}\Big[\frac{\not q+\not p-\not K_1+m}{(q+p-K_1)^2-m^2}\gamma_\mu\frac{\not q+\not p+m}{(q+p)^2-m^2}\times$$
$$\times\gamma_5\frac{\not p+m}{p^2-m^2}\gamma_\nu\Big], \quad (2\text{-}12)$$

where

$$q\equiv K_1+K_2. \quad (2\text{-}13)$$

This integral is apparently linear divergent, however the trace of the γ-matrices will be vanishing for the pieces which diverge. Indeed the trace in the numerator is equal to

$$4m\,\epsilon_{\mu\nu\rho\sigma} K_1^{\rho} K_2^{\sigma} \tag{2-14}$$

and the remaining integral is convergent

$$4m\,\epsilon_{\mu\nu\rho\sigma} K_1^{\rho} K_2^{\sigma} \int \frac{d^4p}{(2\pi)^4} \Big\{[(p+K_2)^2-m^2][(p+K_1+K_2)^2-m^2]\times \tag{2-15}$$

$$\times[p^2-m^2]\Big\}.$$

Feynman parametrization of the momentum integral:

$$\frac{1}{ABC} = 2\int_0^1 dx \int_0^1 dy \int_0^1 dz \frac{\delta(1-x-y-z)}{(Ax+By+Cz)^3} \quad , \tag{2-16}$$

or

$$\frac{1}{ABC} = 2\int_0^1 x\,dx \int_0^1 dy [(A-B)xy+(B-C)x+C]^{-3}. \tag{2-17}$$

So the parametric integrals become

$$I_2 = 2\int_0^1 x\,dx\,dy \int \frac{d^4p}{(2\pi)^4} \{ p^2 +$$

$$+ 2p[x(K_1+K_2)-xyK_1] + x(K_1+K_2)^2 \}^{-1}. \tag{2-18}$$

Now, we are ready to compute the momentum integral, which is done by means of the formula

$$\int \frac{d^4p}{(4\pi)^2}(p^2+2p\cdot K+M^2)^{-3}=\frac{1}{2}(4\pi)^{-2}(M^2-K^2)^{-1}. \qquad (2\text{-}19)$$

Then

$$I_2=\frac{1}{(2\pi)^4}\int_0^1 x\,dx\,dy[x(K_1+K_2)^2-$$

$$-xyK_1(K_1+2K_2)-m^2-x^2(K_1+K_2)^2- \qquad (2\text{-}20)$$

$$-x^2y^2K_1^2+2x^2yK_1\cdot(K_1+K_2)]^{-1}$$

or

$$I_2=\frac{1}{(4\pi)^2}\int_0^1 x\,dx\,dy[x(1-x)(K_1+K_2)^2-$$

$$-xy(1+xy-2x)K_1^2- \qquad (2\text{-}21)$$

$$-2xy(1-x)K_1\cdot K_2-m^2]^{-1}.$$

Now, for future purposes the case of importance will be that for which $K_1^2=K_2^2=0$ and $(K_1+K_2)^2=2K_1\times K_2$ (this will be the on-shell condition for the external fields).

Then the integral I_2 reduces to

$$I_2 = \frac{1}{(4\pi)^2}\int_0^1 x\,dx \int_0^1 dy\left[(K_1+K_2)^2 x(1-x)(1-y)-m^2\right]^{-1}. \quad (2\text{-}22)$$

Then the final parametric representation for the loop integrals $I_{1\mu\nu}$ is

$$I_{1\mu\nu} = 4m\,\epsilon_{\mu\nu\rho\sigma}K_1^{\rho}K_2^{\sigma}(4\pi)^{-2} \times$$
$$\times \int_0^1 x\,dx\,dy\left[(K_1+K_2)^2 x(1-x)(1-y)-m^2\right]^{-1}. \quad (2\text{-}23)$$

In this form, it already suites for our purposes.

Let us now come back to our anomaly triangle graph. Before computing the integral on q, let us convert this integral into the divergence of the current. Momentum conservation of this graph gives that the current insertion carries a momentum K_1+K_2, so that we have actually to calculate is

$$(K_1+K_2)^{\rho}\,\hat{T}_{\mu\nu\rho}(K_1, K_2; m), \quad (2\text{-}24)$$

which is given by

$$(K_1+K_2)^\rho \hat{T}_{\mu\nu\rho}(K_1,K_2;m) = g^2 \,\mathrm{tr}\int \frac{d^4q}{(2\pi)^4}\frac{\not{q}+m}{q^2-m^2}\times$$

$$\times\gamma_\mu \frac{\not{q}+\not{K}_1+m}{(q+K_1)^2-m^2}(\not{K}_1+\not{K}_2)\gamma_5\frac{\not{K}_2-\not{q}+m}{(K_2-q)^2-m}\gamma_\nu \tag{2-25}$$

Now, the trick is to rearrange central terms inside the trace

$$(\not{q}+\not{K}_1+m)(\not{K}_1+\not{K}_2)\gamma_5(\not{K}_2-q+m)=$$

$$=(\not{q}+\not{K}_1+m)(\not{q}+\not{K}_1-m+K_2+m-\not{q})\gamma_5(\not{K}_2-\not{q}+m)$$

$$=2m(\not{q}+\not{K}_1+m)\gamma_5(\not{q}-\not{K}_2+m)+ \tag{2-26}$$

$$+[(q+K_1)^2-m^2]\gamma_5(\not{q}-\not{K}_2+m)+$$

$$+(\not{q}+\not{K}_1+m)\gamma_5[(q-K_2)^2-m^2] .$$

Inserting this rearrangement into our expression for the divergence, we obtain

$$(K_1+K_2)^\rho \hat{T}_{\mu\nu\rho}(K_1,K_2;m)=$$

$$= 2m\, I_{1\mu\nu}(K_1+K_2) +$$

(2-27)

$$+\, tr \int \frac{d^4q}{(2\pi)^4} \frac{\not{q}+m}{q^2-m^2} \gamma_\mu \gamma_5 \frac{\not{q}-\not{K}_2+m}{(q-K_2)^2-m^2} +$$

$$-\, tr \int \frac{d^4q}{(2\pi)^4} \frac{\not{q}+m}{q^2-m^2} \gamma_\mu \frac{\not{q}+\not{K}_1+m}{(q+K_1)^2-m^2} \gamma_5 \gamma_\nu ,$$

where the last term can be written as

$$tr \int \frac{d^4q}{(2\pi)^4} \frac{\not{q}+\not{K}_1+m}{(q+K_1)^2-m^2} \gamma_5 \gamma_\nu \frac{\not{q}+m}{q^2-m^2} \gamma_\mu . \qquad (2\text{-}28)$$

Here we have to add a remark. Notice that the last two terms are both divergent. They then need to be regularized. The safest method to be used here is by introducing a mass regulator. Dimensional regularization would not be convenient here due to the presence of the γ_5 -matrix which is a particularity of the 4-dimensional space-time. Upon introduction of a regulator field with mass M (M goes to infinity at the end of the calculations) our expression for $(K_1+K_2)^\rho\, \hat{T}_{\mu\nu\rho}\,(K_1\,, K_2\,; m)$ becomes

$$(K_1+K_2)^\rho [\hat{T}_{\mu\nu\rho}(K_1,K_2;m) - \hat{T}_{\mu\nu\rho}(K_1,K_2;m)] =$$

$$= 2m\, I_{1\mu\nu}(K_1, K_2; m) - 2M\, I_{1\mu\nu}(K_1, K_2; M) +$$

(2-29)

$$+ \Big[tr \int \frac{d^4q}{(2\pi)^4} \frac{\not{q}+m}{q^2-m^2} \gamma_\mu \gamma_5 \frac{\not{q}-\not{K}_2+m}{(q-K_2)^2-m^2} \gamma_\nu -$$

$$- tr \int \frac{d^4q}{(2\pi)^4} \frac{\not{q}+\not{K}_1+m}{(q+K_1)^2-m^2} \gamma_5 \gamma_\nu \frac{\not{q}+m}{q^2-m^2} \gamma_\mu \Big] - [m \to M] .$$

Now that the two integrals have been regulated by the large mass M, we can apply the following argument: each integral is a pseudotensor, due to the presence of γ_5. However, they individually depend only on one four-vector: the first on K_2 and the second on K_1. So they are both zero since they are Lorentz pseudotensors depending only on one vector. Notice that before regularization, this argument could not be applied as the integrals were divergent and did not have any sense. Therefore

$$(K_1+K_2)^\rho [\hat{T}_{\mu\nu\rho}(K_1, K_2; m) - \hat{T}_{\mu\nu\rho}(K_1, K_2; M)] =$$

(2-30)

$$= 2m\, I_{1\mu\nu}(K_1, K_2; m) - 2M\, I_{1\mu\nu}(K_1, K_2; M) .$$

We can see the relevance of the integral $I_{1\mu\nu}$ which we defined previously. We had

$$I_{1\mu\nu} = 4im\epsilon_{\mu\nu\rho\sigma} K_1^\rho K_2^\sigma \frac{1}{(4\pi)^2}\int_0^1 x\,dx\,dy\,\times \tag{2-31}$$

$$\times\left[(K_1+K_2)^2 x(1-x)(1-y)-m^2\right]^{-1}.$$

Then

$$I_{1\mu\nu}(0,0;m) = -\frac{2}{m}\frac{1}{(4\pi)^2}\epsilon_{\mu\nu\rho\sigma}K_1^\rho K_2^\sigma . \tag{2-32}$$

So an expansion of $I_{1\mu\nu}(K_1, K_2; m)$ around the point $K_1 = K_2 = 0$ gives

$$I_{1\mu\nu}(K_1, K_2; m) = -\frac{2}{(4\pi)^2}\frac{1}{m}\epsilon_{\mu\nu\rho\sigma}K_1^\rho K_2^\sigma + \tag{2-33}$$

+ higher powers of the inverse of m.

This then shows that

$$\lim_{M\to\infty} M I_{1\mu\nu}(K_1, K_2, M) = -\frac{2}{(4\pi)^2}\epsilon_{\mu\nu\rho\sigma}K_1^\rho K_2^\sigma . \tag{2-34}$$

So we see that a finite part persists from the effects of the regulator field of mass . This is the momentum space version of the anomaly developed by the divergence of the current

$$(K_1 + K_2)^\rho \hat{T}_{\mu\nu\rho}(K_1, K_2; m) =$$

$$= 2m I_{1\mu\nu}(K_1, K_2; m) - \frac{4}{(4\pi)^2} \epsilon_{\mu\nu\rho\sigma} K_1^\rho K_2^\sigma . \tag{2-35}$$

Finally, remember that the graphs contributing are the one calculated above and another obtained from it by exchanging K_1 with K_2 and μ with ν, i.e.,

$$(K_1 + K_2)^\rho \hat{T}_{\nu\mu\rho}(K_2, K_1; m) =$$

$$= 2m I_{1\nu\mu}(K_2, K_1; m) - \frac{4}{(4\pi)^2} \epsilon_{\nu\mu\rho\sigma} K_2^\rho K_1^\sigma . \tag{2-36}$$

By adding the two graphs together, we get the following expression for the anomaly of the divergence of the current

$$-\frac{8}{(4\pi)^2} \epsilon_{\mu\nu\rho\sigma} K_1^\rho K_2^\sigma . \tag{2-37}$$

Finally, the operator form for the anomaly is obtained by attaching the field operators A_μ and A_ν (Keeping in mind that the on-shell conditions were already used in calculating the integral $I_{\mu\nu}(K_1, K_2; m)$). In so doing we get the celebrated expression for the anomaly:

$$\text{Anomaly} \equiv \Delta = -\frac{8}{(4\pi)^2} g^2 \epsilon_{\mu\nu\rho\sigma} A^\mu A^\nu K_1^\rho K_2^\sigma \qquad (2\text{-}38)$$

or

$$\Delta = \frac{2}{(4\pi)^2} g^2 \epsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma} . \qquad (2\text{-}39)$$

With this result, one can calculate the π^0 ---- 2γ decay procedure and the result is as follows. Using the smooth extrapolation in the neighborhood of q =0 with

$$\mathcal{T}(q^2) = \left(1 + \frac{q^2}{12 m^2} + \cdots\right) \mathcal{T}(0) . \qquad (2\text{-}40)$$

and the quantity, g/m=1/f , where f $\sim$ 93 Mev, the life-time from the leading term of the expansion (2-40) is

$$\Gamma = \frac{1}{2 m_\pi} \sum_{\epsilon_1, \epsilon_2} \int \frac{d^3k_1}{4(2\pi)^6} \frac{d^3k_2}{\omega_1 \cdot \omega_2} |\mathcal{T}|^2 (2\pi)^4 \delta(q - k_1 - k_2)$$

(2-41)

$$= \frac{\alpha^2 m_\pi^3}{64\pi^3 f_\pi^2} = 7.63 \ eV$$

This result[10] is quite near the experimental rate

$$\Gamma^{exp} = (7.37 \pm 1.5) \ eV. \qquad (2\text{-}42)$$

Now, two important remarks are worthwhile.

1). Notice that though we have done our calculations by considering a mass m for the fermion, the expression for the anomaly is mass independent and indeed chiral fermions, which are massless, are the responsible for the appearance of the anomalies. The mass we have used for our calculations has been introduced as a sort of infrared regulator(the triangle loop graph without mass in the propagators has infrared divergencies). But the limit $m \longrightarrow 0$ at the end of the calculations does not cure the anomaly, which is really an effect of the zero mass fermions.

2). The above expression for the anomaly can be shown to be a total derivative:

$$\epsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma} = \epsilon_{\mu\nu\rho\sigma} (\partial^\mu A^\nu - \partial^\nu A^\mu) F^{\rho\sigma} =$$

$$= 2\,\epsilon_{\mu\nu\rho\sigma}(\partial^\mu A^\nu)F^{\rho\sigma} \tag{2-43}$$

$$= 2\,\epsilon_{\mu\nu\rho\sigma}\,\partial^\mu(A^\nu F^{\rho\sigma})$$

as

$$\epsilon_{\mu\nu\rho\sigma}\,\partial^\mu F^{\rho\sigma} = \epsilon_{\mu\nu\rho\sigma}(\partial^\mu\partial^\rho A^\sigma - \partial^\mu\partial^\sigma A^\rho) \equiv 0\,. \tag{2-44}$$

So the anomaly is a total derivative:

$$\Delta = \frac{4}{(4\pi)^2}\, g^2\, \epsilon_{\mu\nu\rho\sigma}\,\partial^\mu(A^\nu F^{\rho\sigma})\,. \tag{2-45}$$

2.2 Path integral and anomalies: the Fujikava's approach

In our derivation of the Ward identities, we warned the non-trivial fact of the invariance of the path integral measure under the classical symmetry transformations. This might not be the case in several interesting physical situations. Indeed, as we shall discuss in this section, the origin and interpretation of the anomalies will be traced back in the non-invariance of the functional measure under the transformations of the classical symmetries.

Just to conclude, we could summarize by saying that in the path integral quantization it is in the functional measure that we locate the anomaly, which is equivalent to studying classical current insertions into connected Green's functions.

As an application of Fujikava's approach[11], we shall derive the axial U(1) and the non-Abelian anomalies.

1). The chiral U(1) anomaly

The anomaly associated with the global chiral U(1) symmetry plays an important rule in the general discussion of the chiral anomaly. This anomaly also deserves sufficient attention in view of the non-renormalization theorem they obey. Phenomenologically this anomaly has application in connection with the QCD θ -vacuum problem.

To derive the chiral U(1) anomaly, we shall start from a QCD-like Lagrangian:

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi - \frac{1}{2g^2} tr F^{\mu\nu} F_{\mu\nu} , \qquad (2-46)$$

where

$$F_{\mu\nu} = [D_\mu, D_\nu] \,, \quad D_\mu \equiv \partial_\mu - i A_\mu \,. \tag{2-47}$$

and

$$A_\mu \equiv A_\mu^a T_a \,, \quad [T_a, T_b] = i f_{abc} T_c \,, \tag{2-48}$$

and

$$\mathrm{tr}(T_a T_b) = \frac{1}{2} \delta_{ab} \tag{2-49}$$

where delta function is in the adjoint representation. Notice that we follow Fujikava's conventions for the γ - matrices. After Vick rotation to the Euclidean theory, we have

$$\gamma_\mu^+ = -\gamma_\mu \,. \tag{2-50}$$

We start then now by defining the path integral of the theory according to

$$W = \int \mathcal{D}\bar{\Psi}\mathcal{D}\Psi\mathcal{D}A_\mu \exp\{\int d^4x \mathcal{L}\} \,, \tag{2-51}$$

where gauge fixing and ghost terms are supposed to be included above, however, they are not relevant to the anomaly analysi.

Let us now specify our chiral U(1) transformation

$$\psi(x) \longrightarrow \psi'(x) = e^{i\alpha(x)\gamma_5}\psi(x),$$

$$\bar{\psi}(x) \longrightarrow \bar{\psi}'(x) = \bar{\psi}(x)\, e^{i\alpha(x)\gamma_5} \qquad (2\text{-}52)$$

with

$$\bar{\gamma}_5 = -\gamma_5 \qquad (2\text{-}53)$$

and see how the Lagrangian changes under an infinitesimal γ_5 - transformation.

$$\delta\mathcal{L} = \delta\partial_\mu\bar{\psi}\frac{\partial\mathcal{L}}{\partial\partial_\mu\bar{\psi}} + \frac{\partial\mathcal{L}}{\partial\partial_\mu\psi}\delta\partial_\mu\psi + \frac{\partial\mathcal{L}}{\partial\bar{\psi}}\delta\bar{\psi} +$$

$$+ \frac{\partial\mathcal{L}}{\partial\psi}\delta\psi = \bar{\psi}\, i\gamma^\mu\partial_\mu(i\alpha\gamma_5\psi) +$$

$$+ (\bar{\psi}\, i\alpha\gamma_5)(i\gamma^\mu D_\mu - m)\psi + \qquad (2\text{-}54)$$

$$+(\bar{\psi}\gamma^{\mu}A_{\mu} - m\bar{\psi})(i\alpha\gamma_5\psi).$$

Since

$$\frac{\partial\mathcal{L}}{\partial\partial_{\mu}\bar{\psi}} = 0, \quad \delta\psi = i\alpha\gamma_5\psi, \tag{2-55}$$

so we have

$$\delta\mathcal{L} = -(\partial_{\mu}\alpha)\bar{\psi}\gamma^{\mu}\gamma_5\psi - 2im\alpha\bar{\psi}\gamma_5\psi. \tag{2-56}$$

The term multiplying $\partial_{\mu}\alpha$ defines the symmetry current

$$J_{\mu}^{(5)} \equiv \bar{\psi}\gamma^{\mu}\gamma_5\psi. \tag{2-57}$$

The term multiplying m gives the explicit breaking of the classical current

$$\partial^{\mu} j_{\mu}^{(5)} = 2im\bar{\psi}\gamma_5\psi. \tag{2-58}$$

The whole idea of the Fujikava's approach comes now. It consists in expressing Jacobian factor for the classical transformation as

$$\gamma^\mu D_\mu(A)\varphi_n(x) = \lambda_n \varphi_n(x) \qquad (2\text{-}62)$$

and

$$\psi(x) = \sum_n a_n \varphi_n(x) = \sum_n a_n \langle x | \varphi_n \rangle$$

$$(2\text{-}63)$$

$$\bar{\psi}(x) = \sum_n \bar{b}_n \varphi_n^+(x) = \sum_n \bar{b}_n \langle \varphi_n | x \rangle ,$$

with the coefficients a_n and $\bar{b}_n$ being elements of a Grassman algebra and the basis vectors $\varphi_n(x)$ being an orthogonal basis

$$\int d^4x\, \varphi_m^+(x)\varphi_n(x) = \delta_{mn} \qquad (2\text{-}64)$$

The expansions of $\bar{\psi}(x)$ and $\psi(x)$ in terms of the $\varphi_n(x)$ should be viewed as a coordinate transformation in the field space, then the volume element

$$\mathcal{D}\bar{\psi}\mathcal{D}\psi \qquad (2\text{-}65)$$

is to be re-expressed in terms of the new coordinates a_n and $\bar{b}_n$ according to

$$\mathcal{D}\bar{\psi}'\mathcal{D}\psi' = \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\{i\int d^4x\, \alpha(x) \mathcal{A}(x)\}, \qquad (2\text{-}59)$$

since one knows that the anomaly appears as a local polynomial in the fields and as an explicit breaking term introduced into the effective action by quantum corrections. This is the motivation for assuming that the non-invariance of the measure comes with the exponential of the anomaly. The reason for the appearance of the parameter $\alpha(x)$ is that the anomaly is interpreted as the variation of the effective action Γ under the classical symmetry:

$$\delta_{class.\,sym}\Gamma \sim \alpha(x)\mathcal{A}(x). \qquad (2\text{-}60)$$

The problem which then remains is the evaluation of the Jacobian factor

$$\exp\{i\int d^4x\, \alpha(x)\mathcal{A}(x)\}. \qquad (2\text{-}61)$$

Its explicit calculation becomes more transparent if one expands the fields $\bar{\psi}(x)$ and $\psi(x)$ in terms of the eigenstates of the Dirac's operator in the presence of a background $A_\mu(x)$. Denoting them by $\varphi_n(x)$, we have

$$\mathcal{D}\bar{\psi}\mathcal{D}\psi = \frac{\prod da_n d\bar{b}_n}{\det\langle\varphi_n|x\rangle \det\langle x|\varphi_n\rangle} . \qquad (2\text{-}66)$$

One should note that the Jacobian determinants in the denominator come because ψ and $\bar{\psi}$ are Grassman fields and so can be thought of as $\frac{\partial}{\partial\bar{\psi}}$.

The Jacobian for the transformation from $\psi(x)$ and $\bar{\psi}(x)$ to a_n and $\bar{b}_n$ in the previous equation becomes the unity, the reason being that the Dirac operator $\not{D}$ is Hermitian. Then

$$\mathcal{D}\bar{\psi}\mathcal{D}\psi = \prod_n da_n d\bar{b}_n . \qquad (2\text{-}67)$$

The problem now is to see how the new volume element

$$\prod_n da_n d\bar{b}_n \qquad (2\text{-}68)$$

changes under the chiral transformation. This can be basically worked out if we set the transformation of the coefficients a_n and b_n . To do this, let us start from

$$\psi'(x) = e^{i\alpha(x)\gamma_5}\psi(x)$$

(2-69)

$$= \sum_n a_n' \varphi_n(x) = \sum_n a_n e^{i\alpha(x)\gamma_5} \varphi_n(x) .$$

Now, invoking the orthonormality of the basis $\varphi_n(x)$, one can derive that

$$a_m' = \sum_n \left[\int d^4x\, \varphi_m^+(x) e^{i\alpha(x)\gamma_5} \varphi_n(x) \right] a_n . \qquad (2-70)$$

Analogously we can find that

$$\bar{b}_n' = \sum_m \left[\int d^4x\, \varphi_n^+(x) e^{i\alpha(x)\gamma_5} \varphi_m(x) \right] \bar{b}_m . \qquad (2-71)$$

What we have next to do is to work out the Jacobian of the chiral transformation taken

$$(a_n , \bar{b}_n) \text{ to } (a_n' , \bar{b}_n'), \qquad (2-72)$$

remaining once again that the $d a_n$ and $d\bar{b}_n$ behave as $\frac{\partial}{\partial a_n}$ and $\frac{\partial}{\partial \bar{b}_n}$ due to their Grassmannian character. We have

$$\prod_m d a_m' d\bar{b}_m' = \det \left[\int d^4x\, \varphi_m^+(x) \times e^{i\alpha(x)\gamma_5} \varphi_m(x) \right]^{-1} \prod_n d a_n d\bar{b}_n . \qquad (2-73)$$

Recalling now that det(M)=exp{tr[ln(M)]}, we can rewrite the previous formula as

$$\prod_m da'_m d\bar{b}'_m = \exp\{-i\sum_n \int d^4x\,\alpha(x)\varphi_n^+(x)\gamma_5\varphi_n(x)\}\alpha \tag{2-74}$$

$$\times \prod_n da_n d\bar{b}_n \,.$$

To understand this last step, we have to use that small and the orthogonality of the basis φ_n:

$$\exp\{tr\,ln\int d^4x\,\varphi_m^+(x)\,e^{i\alpha(x)\gamma_5}\varphi_n(x) =$$

$$= \exp\{tr\,ln\int d^4x\,\varphi_m^{-1}(1+i\gamma_5\alpha)\varphi_n =$$

$$= \exp\{tr\,ln(\delta_{mn} + \int d^4x\,\varphi_m^+ i\alpha\gamma_5\varphi_n) = \tag{2-75}$$

$$= \exp\{tr\int d^4x\,\varphi_m^+ i\alpha\gamma_5\varphi_n\} =$$

$$= \exp\{\sum_n i\int d^4x\,\alpha(x)\varphi_n^+\gamma_5\varphi_n\}.$$

Recalling once again that the change in the volume element is in the Fujikava's approach given by

$$\mathcal{D}\bar{\psi}\mathcal{D}\psi = \exp\{-i\int d^4x\,\alpha(x)\mathcal{A}(x)\}, \qquad (2\text{-}76)$$

we identify the anomaly as being given by

$$\mathcal{A} = \sum_n \varphi_n^+(x)\gamma_5\varphi_n(x). \qquad (2\text{-}77)$$

At this point, it is already interesting to emphasize that the anomaly can be written in terms of the eigenvectors of the Dirac operator in the presence of a Yang-Mills background. As it is written in the previous formula, the anomaly expression, being given by an infinite series, may not converge and then needs a regularization. This is achieved by introducing a convergence factor in the series of the following form

$$\mathcal{A} = \sum_n \varphi_n^+(x)\gamma_5\varphi_n(x) = \qquad (2\text{-}78)$$

$$= \lim_{M\to\infty} \sum_n \varphi_n^+(x)\gamma_5 e^{-\frac{\lambda_n^2}{M^2}}\varphi_n(x).$$

As φ_n is an eigenstate of the Dirac operator, the regularized expression for the series giving the anomaly becomes

$$\mathcal{A} = \lim_{M\to\infty} \sum_n \varphi_n^+(x)\gamma_5 e^{-\not{D}^2/M^2}\varphi_n(x). \qquad (2\text{-}79)$$

We could now remark that the above expression corresponds to a local version of the Atiyah-Singer index theorem, on which we shall elaborate better at the end of this section. It is worthwhile however to keep in mind this point, since it represents an interesting marriage between analytical and topological considerations about anomalies.

Coming back to our series defining the anomaly, the problem now is to evaluate its sum. Having already regularized it through the exponential convergence factor, we may take the basis φ_n to be given by plane waves , exp{iKx}. This is a continuous basis and the sum over n should be convergent into an integration over the wave number K. One therefore obtains

$$\sum_n \varphi_n^+(x)\gamma_5\varphi_n(x) = \lim_{M\to\infty} tr \int \frac{d^4k}{(2\pi)^4} \times \\ \times\, e^{-ikx}\gamma_5\, e^{-\not{D}^2/M^2} e^{ikx} , \qquad (2\text{-}80)$$

where we should use that $\not{D}^2 = D^\mu D_\mu - \frac{i}{4}[\gamma^\mu,\gamma^\nu]F_{\mu\nu}$ and the trace is understood to be in the Dirac and internal spaces.

Manipulating the exponentials, the above equation can be rewritten as

$$\sum_n \varphi_n^+(x)\gamma_5\varphi_n(x)=\lim_{M\to\infty} tr\int\frac{d^4k}{(2\pi)^4}\gamma_5\, exp\{$$

(2-81)

$$-(ik_\mu+D_\mu)(ik^\mu+D^\mu)+\frac{i}{4M^2}[\gamma^\mu,\gamma^\nu]F_{\mu\nu}\} .$$

Now, a trick comes in. One re-scales K_μ according to

$$K_\mu \longrightarrow K_\mu' = \frac{K_\mu}{M} . \qquad (2-82)$$

(One should think of M as being the modulus of the cut-off M, in the sense that the mass dimension has been factorred out) and expands the exponential in the integrand, which gives us

$$\sum_n \varphi_n^+(x)\gamma_5\varphi_n(x)= tr\,\gamma_5\frac{1}{2!}\times$$

(2-83)

$$\times\left(\frac{i}{4}[\gamma^\mu,\gamma^\nu]F_{\mu\nu}\right)^2\int\frac{d^4k}{(2\pi)^4}e^{-K^\mu K_\mu} .$$

Here the basic facts we used were: first, isolate the factor $e^{-K^\mu K_\mu}$; then expand the remaining expotential, taking into account that tr(γ_5 [γ_μ, γ_ν])=0, and the limit when M goes to infinity. All we are left with is the calculation of

$$tr\,(\gamma_5[\gamma^\mu,\gamma^\nu][\gamma^\rho,\gamma^\sigma])F_{\mu\nu}F_{\rho\sigma} \qquad (2-84)$$

and the integral

$$\int \frac{d^4k}{(2\pi)^4} e^{-k^\mu k_\mu} . \tag{2-85}$$

With these results, the final expression we get for the axial current anomaly is

$$\mathcal{A} = \frac{1}{32\pi^2} tr\, \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta} . \tag{2-86}$$

As promised, the chiral variation of the volume element was the source for the above anomaly. Reconsidering our analysis of the Ward identities in the presence of an anomaly, the modified chiral Ward identity for the theory considered here will read

$$\partial^\mu \langle T(\bar{\psi}\gamma_\mu\gamma_5\psi)\rangle - 2im\langle T(\bar{\psi}\gamma_5\psi)\rangle - \frac{i}{32\pi^2}\langle T(tr\, F^{\mu\nu}\tilde{F}_{\mu\nu})\rangle = 0 , \tag{2-87}$$

where

$$\tilde{F}_{\mu\upsilon} = \epsilon_{\mu\upsilon\alpha\beta} F^{\alpha\beta} . \qquad (2\text{-}88)$$

To conclude this section we would like to comment on our results in the light of the Atiyah-Singer theorem[9]. The index of a compact and elliptic operator $\mathcal{D}$ (functional analysis book), which is a topological invariant quantity, is defined as the difference between the Kernels of the operator and its adjoint:

$$\text{index of } D = \text{Kernel of } D - \text{Kernel of } D^{+} . \qquad (2\text{-}89)$$

In other words, the index is the difference between the number of independent zero modes (zero modes means eigen-state of zero eigenvalue) of $\mathcal{D}$ and $\mathcal{D}^{+}$ since

$$\text{Kernel } D = \{ \varphi_n \,;\, D\varphi_n = 0 \} . \qquad (2\text{-}90)$$

In reference [9], Atiyah and Singer tackled the problem of calculating the index of the Dirac operator in the presence of a Yang-Mills background and obtained, as the topological invariant answer, the anomaly expression

$$\frac{1}{32\pi^2} tr(F\tilde{F}) . \qquad (2\text{-}91)$$

Then their work motivates a possible topological interpretation for the anomalies since they can be viewed as the index of the Dirac operator.

2). The non-Abelian anomaly

We wish, in this section, to discuss the possible anomalies of non-Abelian transformations. In this case, contrary to the previous chiral U(1) anomaly we shall see that two different forms (consistent and covariant) of anomalies will appear which correspond to the two different specifications of the determinant of the Dirac operator (det $\not{D}$). These two alternative ways of expressing a non-Abelian anomaly are equally relevant, however, they are used for different purposes.

In the present section, we are going to discuss and sketch the Abelian features of the derivation of the consistent and covariant expressions for the non-Abelian anomalies. For this purpose we start with a Lagrangian describing the coupling of chiral fermions to gauge fields according to

$$\mathcal{L} = \bar{\psi}_L i \gamma^\mu D_\mu \psi_L , \qquad (2-92)$$

where

$$D_\mu \equiv \partial_\mu - i A_\mu^a T_a ,$$

$$(2-93)$$

$$[T_a, T_b] = i f_{abc} T_c ,$$

and in the adjoint representation

$$tr(T_a T_b) = \frac{1}{2}\delta_{ab} , \quad T_a = T_a^\dagger \qquad (2-94)$$

and

$$\psi_L = \frac{1-\gamma_5}{2}\psi . \qquad (2-95)$$

Then in terms of a four-component spinor ψ, we can write

$$\mathcal{L} = \bar{\psi} i \gamma^\mu D_\mu \frac{1-\gamma_5}{2}\psi$$

$$= \bar{\psi} i \gamma^\mu (\partial_\mu - i A_\mu) \frac{1-\gamma_5}{2}\psi$$

$$= \bar{\psi} \gamma^\mu (i\partial_\mu + A_\mu) \frac{1-\gamma_5}{2}\psi \qquad (2-96)$$

$$= \bar{\psi} i \gamma^{\mu} \partial_{\mu} \frac{1-\gamma_5}{2} \psi +$$

$$+ \bar{\psi} \gamma^{\mu} \frac{1}{2} (A_{\mu} - A_{\mu} \gamma_5)$$

The relevant point we would like to call the attention to is the appearance of two vertices involving the gauge field: one with and another without a γ_5. This motivates a more general treatment, where one introduces a derivative with two different vector fields for the two different vertices (with and without γ_5). For this purpose, we follow Fujikava and start with

$$\mathcal{L} = \bar{\psi} i \gamma^{\mu} D_{\mu} \psi ,$$

$$D_{\mu} \equiv \partial_{\mu} - i V_{\mu}^{a} T_a - i A_{\mu}^{a} T_a \gamma_5 . \tag{2-97}$$

Notice that the artifact of introducing V_{μ} and A_{μ} will be shown to have also some technical motivation when we shall discuss the consistent anomalies. However, at the very end of the calculations we will set $V_{\mu} = - A_{\mu}$ as one can already see from the beginning of this section.

Before starting to separate our calculations for the consistent and covariant cases, it is worthwhile to remark that the basic operator $\not{D}$ is not Hermitian in the Euclidean sense: $(\varphi_1, \not{D}\varphi_2) \equiv (\not{D}\varphi_1, \varphi_2)$, where

$$(\varphi, \chi) \equiv \int d^4x \, \varphi^+ \chi . \qquad (2\text{-}98)$$

Using this definition of the scalar product, one can really check that

$$\not{D}^+ = \gamma^\mu(\partial_\mu - iV_\mu + iA_\mu) \neq \not{D} , \qquad (2\text{-}99)$$

where we used the following properties

$$\gamma_\mu^+ = -\gamma_\mu , \ \gamma_5^+ = -\gamma_5 , \ T_a^+ = T_a . \qquad (2\text{-}100)$$

To circumvent the non-hermiticity of $\not{D}$, there appear two possible ways: they consists in either making an analytical continuation in A_μ or using the so-called polar-decomposition of the operator $\not{D}$. These two different ways of handling the problem will lead to the consistent or covariant expressions for the anomaly. Let us now discuss each case separately.

1). The consistent anomaly

In this case, the hermiticity of $\not{D}$ is achieved by performing an analytic continuation in A_μ,

$$A_\mu \longrightarrow i A_\mu \tag{2-101}$$

so that $\not{D}^+ = \not{D}$.

If one now re-expresses the fields ψ and $\bar{\psi}$ in terms of the Grassman coefficients a_n and $\bar{b}_n$, with basis vectors $\varphi_n(x)$ and such that

$$\not{D}\, \varphi_n(x) = \lambda_n \varphi_n(x)\ , \tag{2-102}$$

the derivation of the Jacobian factor follows exactly the same patterns as in the derivation of the chiral U(1) anomaly. All one is left with is the evaluation of the Jacobian factor

$$\lim_{M\to\infty} \mathrm{tr} \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x} \gamma_5 T_a e^{-\not{D}^2/M^2} e^{ik\cdot x} . \tag{2-103}$$

Taking correctly into account the traces over the group generators and the expansion in powers of M, one obtains the following result for the anomaly

$$\mathcal{A} = \frac{1}{24\pi^2} \epsilon^{\mu\upsilon\alpha\beta} tr[T_a \partial_\mu (V_\upsilon \partial_\alpha V_\beta - \tag{2-104}$$

$$- \frac{i}{2} V_\upsilon V_\alpha V_\beta)] ,$$

if, at the end of the calculation, one sets $A_\mu = - V_\mu$.

Before proceeding to the investigation of the covariant anomaly, it would be interesting to remark that the consistent expression for the non-Abelian anomaly, will be responsible, as we shall discuss in another section for the integration of the anomalous Ward-Identities in the form of the Wess-Zumino term which has many relevant phenomenological applications.

2). The covariant anomaly

The second way to overcome the non-hermiticity of the original $\not{D}$ is by considering the following eigenvalue equations,

$$\not{D}^{+}\not{D}\,\varphi_n(x) = \lambda_n^2\,\varphi_n(x)$$

(2-105)

$$\not{D}\not{D}^{+}\phi_n(x) = \lambda_n^2\,\phi_n(x)\,.$$

Here we have used the so-called polar decomposition of the operator $\not{D}$ (functional analysis book).

In view of the above equations, one uses the different basis vectors φ_n and ϕ_n to reparametrize the fields $\psi\,(x)$ and $\bar{\psi}\,(x)$:

$$\psi\,(x) = \sum_n a_n \varphi_n(x) = \sum_n a_n \langle x | \varphi_n \rangle\,,$$

(2-106)

$$\bar{\psi}\,(x) = \sum_n \bar{b}_n\,\phi_n(x) = \sum_n \bar{b}_n \langle \phi_n | x \rangle.$$

With such a change of basis in the field space, the volume elements $\mathcal{D}\bar{\psi}\mathcal{D}\psi$ and $da_n\,d\bar{b}_n$ are related by

$$\mathcal{D}\bar{\psi}\mathcal{D}\psi = [\det\langle x|\varphi_n\rangle]^{-1}\times$$

(2-107)

$$\times[\det\langle\phi_n|x\rangle]^{-1}\prod_n da_n\,d\bar{b}_n\,.$$

On the other hand, using the inverse matrices, the above relation can be rewritten as

$$\mathcal{D}\bar{\psi}\mathcal{D}\psi = [\det\langle\varphi_n|x\rangle][\det\langle x|\varphi_n\rangle] \times \tag{2-108}$$

$$\propto \prod_n d\bar{b}_n da_n .$$

Having changed the volume element, the Jacobian giving rise to anomaly will become

$$-i\sum_n tr[\varphi_n^\dagger \gamma_5 T_a \varphi_n + \phi_n^\dagger \gamma_5 T_a \phi_n] , \tag{2-109}$$

analogously to the case of the U(1) chiral anomaly, we have studied. Introducing the convergence factors

$$e^{-D\!\!\!/^\dagger D\!\!\!//M^2} \quad \text{and} \quad e^{-D\!\!\!/ D\!\!\!/^\dagger/M^2} \tag{2-110}$$

to regularize the sum above and using the plane wave basis, the anomaly calculation reduces to calculating the following expression

$$-i\lim_{M\to\infty} tr\int\frac{d^4k}{(2\pi)^4} e^{-ik\cdot x}\gamma_5 T_a[e^{-D\!\!\!/^\dagger D\!\!\!//M^2} + e^{-D\!\!\!/ D\!\!\!/^\dagger/M^2}] \tag{2-111}$$

$$\times e^{ik\cdot x} .$$

Now we follow a series of tricks as done in

Fujikava's work. The first step consists in rewriting .

as

$$\not{D} = \gamma^{\mu}(\partial_{\mu} - i L_{\mu})\frac{1-\gamma_5}{2} + \gamma^{\mu}(\partial_{\mu} - i R_{\mu}) \times$$

$$\times \frac{1+\gamma_5}{2} = \not{D}(L)\frac{1-\gamma_5}{2} + \not{D}(R)\frac{1+\gamma_5}{2}, \tag{2-112}$$

where

$$L_{\mu} \equiv V_{\mu} + A_{\mu}, \quad R_{\mu} \equiv V_{\mu} - A_{\mu}. \tag{2-113}$$

From this new expression, one can show that

$$\not{D}^{+}\not{D} = \not{D}(L)^2\frac{1-\gamma_5}{2} + \not{D}(R)^2\frac{1+\gamma_5}{2},$$

$$\not{D}\not{D}^{+} = \not{D}(L)^2\frac{1+\gamma_5}{2} + \not{D}(R)^2\frac{1-\gamma_5}{2}. \tag{2-114}$$

The second step consists in showing that the sum of the convergence factors is

$$e^{-\not{D}\not{D}^{+}/M^2} + e^{-\not{D}^{+}\not{D}/M^2} = e^{-\not{D}(L)^2/M^2} + e^{-\not{D}(R)^2/M^2}. \tag{2-115}$$

In possess of this last result, the anomaly calculation turns to calculating the following expression

$$(-i)\lim_{M\to\infty} tr\int\frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot x}\gamma_5 T_a \times$$
$$\times\left[e^{-\not D(L)^2/M^2} + e^{-\not D(R)^2/M^2}\right]e^{ik\cdot x}. \qquad (2\text{-}116)$$

Then comparing with our calculation for the chiral U(1) anomaly, our problem now is just the same, up to the group generators. The result is

$$\mathcal{A} = -\frac{i}{32\pi^2}\,\epsilon^{\mu\nu\rho\sigma}\, tr[T_a(F_{\mu\nu}(R)\times$$
$$\times F_{\alpha\beta}(R) + F_{\mu\nu}(L)F_{\alpha\beta}(L)]. \qquad (2\text{-}117)$$

which is the expression for the covariant anomaly.

Just to conclude, we would like to point out that the covariant anomaly is more suitable for applications in the presence of both the local gauge and global chiral symmetries, as for example unified theories. However, the condition for the anomaly cancellation, namely tr($T_a\{T_b, T_c\}$) is such that it cancels simultaneously the consistent and the covariant anomalies.

2.3 Non-renormalization of anomalies

In this section, we are going to discuss the Adler-Bardeen theorem, i.e. the non-renormalization theorem[12]. In the following, we shall see that this theorem is very powerful in discussion of the anomaly because it restricts the chiral anomalies only at one-loop level. Due to this theorem, we need not consider the higher loop effects.

The regularization method based on the introduction of higher derivatives is not very practical but it is very good for formal proofs and manipulations. It consists in the introduction of higher derivative terms in the Lagrangian, which give rise to propagators which are better behaved, of the type K^{-4}.

Ordinary gauge theories, this method of regulating the theory has been developed in detail by Slavnov and he proves that the divergences, after the higher derivatives have been introduced, appear only at one loop level. From two loops on, the theory is perfectly finite. Now since the anomalies appear in association with divergent graphs, and the Slavnov method respects gauge invariance, we can conclude that the anomalies appear only at one-loop level. Going to another regularization scheme, an anomaly power counting indicates that this result remains true. For example, in four-dimensions, anomaly power counting gives the following degree of divergence

$$\delta = 8 - E_A - \frac{7}{2} E_\psi - 4 L,$$

where

E_A = number of external vector lines,

E_ψ = number of external fermion lines, (2-118)

L = number of loops.

In the triangle case, $E =3$ and $E =0$. Then

$$\delta = 8 - 3 - 4 L = 5 - 4 L. \qquad (2\text{-}119)$$

Therefore, for $L \geqslant 2$, we have that δ is negative and the triangle graph converges. For $L=1$, it diverges and gives the anomaly.

So concluding, we can say that the anomalies induced by the chiral fermions appear only at one-loop level and do not receive contributions from two loops on.

Chapter 3 A general overview of the axial anomaly problem in globally supersymmetric theories

In the early days of supersymmetry, Ferrara and Zumino showed that the supercurrent (a spinor current), if it is suitably defined, share a super-multiplet with the energy-momentum tensor and the axial vector current of susy. This then suggests that, at the quantum level, there is also an anomaly super-multiplet which has the components of the divergence of the axial current, the anomalous trace of the improved energy-momentum tensor and the γ -trace of the supersymmetry spinor current. All of these anomalies are renormalized to all orders and proportional to the renormalization group β -function.

This fact brings some questions about the validity of the Adler-Bardeen theorem[12] in susy theories. This theorem states that the divergence of the axial current receives only one-loop corrections, while the superconformal anomaly is proportional to the β -function and thus generally receives corrections to all loop orders. This apparently conflicting result and the solution to this puzzle will be the main topic of this chapter. We shall define here the super-current multiplet both in a component field and in a superfield analysis.

Once this is done, we proceed to the anomaly calculation and finally we discuss the way to reconcile the Adler-Bardeen theorem and supersymmetry.

3.1 The superconformal current

The starting point is the Lagrangian

$$
\begin{aligned}
\mathcal{L} &= \mathcal{L}_0 + \mathcal{L}_m + \mathcal{L}_g \\
&= -\tfrac{1}{2}(\partial A)^2 + \tfrac{1}{2}(F^2 + G^2) - \tfrac{1}{2}(\partial B)^2 - \tfrac{1}{2} i \bar{\psi} \gamma \cdot \partial \psi \\
&\quad + m(FA + GB - \tfrac{1}{2} i \bar{\psi}\psi) + g(FA^2 - FB^2 + 2GAB \\
&\quad - i\bar{\psi}\psi A + i\bar{\psi}\gamma_5 \psi B),
\end{aligned}
\tag{3-1}
$$

which is expressed by means of component fields. From this Lagrangian, the equations of motion can be got as follows

$$\Box A + mF + 2gFA + 2gGB - ig\bar{\psi}\psi = 0 \quad ,$$

$$\Box B + mG - 2gFB + 2gGA + ig\bar{\psi}\gamma_5\psi = 0 \, .$$

$$\gamma\cdot\partial\psi + mF + 2g(A - \gamma_5 B)\psi = 0 \, ,$$

$$F + mA + g(A^2 - B^2) = 0 \, ,$$

$$G + mB + 2gAB = 0 \quad .$$

Since the Lagrangian is invariant under the restricted susy transformations (or a larger algebra for m=0), space-time translations and chiral transformations, the Noether theorem gives the spinor current associated with the susy transformations

$$J^\mu = [\gamma\cdot\partial(A - \gamma_5 B)]\gamma^\mu\psi + m\gamma^\mu(A - \gamma_5 B)\psi$$

$$+ g\gamma^\mu(A - \gamma_5 B)^2\psi \qquad (3\text{-}2)$$

$$= [\gamma\cdot\partial(A - \gamma_5 B)]\gamma^\mu\psi - (F + \gamma_5 G)\gamma^\mu\psi$$

the second spinor current associated with the remaining generators of the special supersymmetry transformations

$$I^\mu = -\gamma^\lambda x_\lambda J^\mu - 2(A - \gamma_5 B)\gamma^\mu \psi , \tag{3-3}$$

the canonical energy-momentum tensor

$$T_{\mu\nu} = (\partial_\mu A)(\partial_\nu A) + (\partial_\mu B)(\partial_\nu B) + \tag{3-4}$$
$$+ \frac{1}{4} i (\bar{\psi}\gamma_\mu \partial_\nu \psi + \bar{\psi}\gamma_\nu \partial_\mu \psi) + \eta_{\mu\nu}\mathcal{L}$$

and the axial current

$$J_\mu^{(5)} = A(\partial_\mu B) - B(\partial_\mu A) - \frac{1}{4} i \bar{\psi}\gamma_5 \gamma_\mu \psi , \tag{3-5}$$

which satisfy

$$\partial_\mu J^\mu = 0 ,$$
$$\partial_\mu I^\mu = -2m(A - \gamma_5 B)\psi , \tag{3-6}$$
$$\partial^\mu T_{\mu\nu} = 0 .$$

$$\partial^\mu J^{(5)}_\mu = m(GA - FB + \frac{1}{2} i\bar{\psi}\gamma_5\psi).$$

Here, it should be noted that the second spinor current cannot be expressed only by the first one. To improve this situation, the improved quantities are defined as

$$J^\mu_{imp} = J^\mu + \frac{1}{3}(\gamma^\mu \gamma\cdot\partial - \gamma\cdot\partial\gamma^\mu)\times$$

$$\times(A + \gamma_5 B)\psi \quad , \qquad (3\text{-}7)$$

$$I^\mu_{imp} = -\gamma\cdot x J^\mu_{imp} \quad ,$$

here the improved second spinor current is expressed only by the first one. And the improved energy-momentum tensor is

$$\Theta_{\mu\nu} = T_{\mu\nu} - \frac{1}{6}(\partial_\mu\partial_\nu - \eta_{\mu\nu}\Box)(A^2+B^2) \qquad (3\text{-}8)$$

These improved quantities satisfy similar equations,

$$\partial_\mu J^\mu_{imp} = 0 \quad ,$$

$$\partial_\mu I^\mu_{imp} = -\gamma_\mu J^\mu_{imp} = \partial_\mu I^\mu$$

$$(3\text{-}9)$$

$$= -2m(A - \gamma_5 B)\psi ,$$

$$\partial^\mu \theta_{\mu\nu} = 0 .$$

For the case of presence of interaction, the modified currents and tensor are introduced as follows

$$J^\mu_{mod} = J^\mu_{imp} + \frac{1}{6g}(\gamma^\mu \gamma\cdot\partial - \gamma\cdot\partial\gamma^\mu)\psi ,$$

$$I^\mu_{mod} = -\gamma\cdot x\, J^\mu_{mod} ,$$

(3-10)

$$\theta_{\mu\nu\, mod} = \theta_{\mu\nu} - \frac{m}{6g}(\partial_\mu\partial_\nu - \eta_{\mu\nu}\Box)A ,$$

$$J^\mu_{5\, mod} = J^{(5)}_\mu + \frac{m}{2g}\partial_\mu B ,$$

which satisfy

$$\partial_\mu J^\mu_{mod} = 0 ,$$

$$\partial_\mu I^\mu_{mod} = -\gamma_\mu J^\mu_{mod} = \frac{m^2}{g}\psi ,$$

$$\partial_\mu \theta_{\mu\nu\, mod} = 0 , \qquad (3\text{-}11)$$

$$\partial_\mu \theta^\mu_{\ \mu\,\mathrm{mod}} = -\frac{m^2}{2g} F ,$$

$$\partial_\mu J^{(5)}_\mu{}_{\mathrm{mod}} = -\frac{m^2}{2g} G .$$

After all of these definitions, the charge of the restricted supersymmetry transformations, i.e., the 3-dim integral of the time component of J^μ, has not been changed,

$$\int d^3x\, J^{(0)}_{\ \mathrm{mod}} = \int d^3x\, J^0_{\mathrm{imp}} = \int d^3x\, J^0 = Q . \qquad (3\text{-}12)$$

If m=0, the remaining charge has not been changed, either. Since the larger algebra is preserved only for m=0, this will not bring any difficulty.

Now that the component field expressions for the current have been known, we would like to derive the classical supercurrents for both the chiral and vector multiplet. These are the superfields that contain the superconformal component currents. They can be obtained in principle from the classical action via the Noether theorem, or they can be calculated by considering supergravity covariantized action and then picking up the linear term in the supergravity potential, since we know that the latter couples to the superconformal current (this is the superspace analogue of stating that the energy-momentum tensor couples to the metric tensor of general relativity). In general, the two procedures mentioned above do not lead to the same result, unless we perform field redefinitions. These redefinitions have no physical effect since they only change the currents by terms of proportional to the field equations.

To start, let us consider the action for a scalar multiplet in the presence of background supergravity

$$\mathcal{S} = \int d^4x\, d^4\theta\, E^{-1} \bar{\Phi}\, e^{-H}\, \Phi^{-1}, \qquad (3\text{-}13)$$

where E is the superdeterminant of the Vierbein superfield , and H is the supergravity pre-potential. If one now uses the linearized equation

$$E^{-1} e^{-H} = 1 - \frac{1}{3} \bar{D}_{\dot\alpha} D_\alpha H^{\alpha\dot\alpha} - \frac{1}{3} i\, \partial_{\alpha\dot\alpha} H^{\alpha\dot\alpha} \qquad (3\text{-}14)$$

we obtain the super-current

$$J_{\alpha\dot\alpha} = \frac{\delta \mathcal{S}}{\delta H^{\alpha\dot\alpha}} = -\frac{1}{3} (\bar{D}_{\dot\alpha} \bar{\Phi})(D_\alpha \Phi) + \frac{1}{3} \bar{\Phi}\, i \overleftrightarrow{\partial}_{\alpha\dot\alpha} \Phi \qquad (3\text{-}15)$$

The θ -independent component of $J_{\alpha\dot\alpha}$ is the R-symmetry axial current given by

$$j_{\alpha\dot\alpha} = \frac{1}{3} A^* i \overleftrightarrow{\partial}_{\alpha\dot\alpha} A - \frac{1}{3} \bar\Psi_{\dot\alpha} \Psi_\alpha \qquad (3\text{-}16)$$

the component linear in θ is the supersymmetric current

$$j_{\beta.\alpha\dot\alpha} = -\frac{1}{3} (\partial_{\beta\dot\alpha} A^*) \Psi_\alpha + \frac{1}{3} (\bar\Psi_{\dot\alpha}) F \delta_{\alpha\beta}$$

$$+ \frac{1}{3} A^* i \overleftrightarrow{\partial}_{\alpha\dot\alpha} \Psi_\beta \, , \qquad (3\text{-}17)$$

and at the $\theta\bar\theta$ -level, we find the improved energy-momentum tensor

$$\theta_{\beta\dot\beta,\alpha\dot\alpha} = -\frac{1}{3} \{ 2 F^* F \, \delta_{\alpha\beta} \delta_{\dot\alpha\dot\beta} - 2 (\partial_{\beta\dot\alpha} A^*)(\partial_{\alpha\dot\beta} A)$$

$$+ 2 \bar\Psi_{\dot\beta} i \partial_{\alpha\dot\alpha} \Psi_\beta + \bar\Psi_{\dot\alpha} (\partial_{\alpha\dot\beta} \Psi_\beta) \qquad (3\text{-}18)$$

$$- (\partial_{\beta\dot\alpha} \bar\Psi_{\dot\beta}) \Psi_\alpha + A^* i \overleftrightarrow{\partial}_{\beta\dot\beta} \overleftrightarrow{\partial}_{\alpha\dot\alpha} A \} \, .$$

By using the superfield equations of motion, one can show that

$$\bar D^{\dot\alpha} J_{\alpha\dot\alpha} = 0 \, , \qquad (3\text{-}19)$$

which is the equation expressing the conservation of the R-symmetry axial current, the vanishing of the supersymmetry current γ -trace and the vanishing of the improved energy-momentum tensor trace.

3.2 An investigation of the super-conformal current anomaly

(a) The self-interacting scalar multiplet

Considering the scalar multiplet with components A, B, ψ_α , F and G, where A, F are real and B, G are pseudo-scalar fields, and ψ_α is a Majorana spinor field, the component field Lagrangian for Wess-Zumino model reads

$$\mathcal{L} = -\frac{1}{2}(\partial A)^2 - \frac{1}{2}(\partial B)^2 - \frac{1}{2} i \bar{\chi} \gamma \cdot \partial \chi + \frac{1}{2}(F^2 + G^2)$$

$$+ g[FA^2 - FB^2 + 2GAB - i\bar{\chi}(A - \gamma_5 B)\chi] . \tag{3-20}$$

in the four-component notation, the R-symmetry axial current is given by

$$j^{(s)}_\mu = A\overleftrightarrow{\partial}_\mu B - \frac{i}{4}\bar{\chi}\gamma_5\gamma_\mu\chi . \qquad (3\text{-}21)$$

The one-loop graphs contributing to the divergence of the current are given in fig. 3-1.

This example is interesting because, according to what we generally say, external vector fields are needed to get an anomaly, while here we are taking external spin-zero or spin-half particles, however, using a susy regularization procedures, we find that, at one-loop level,

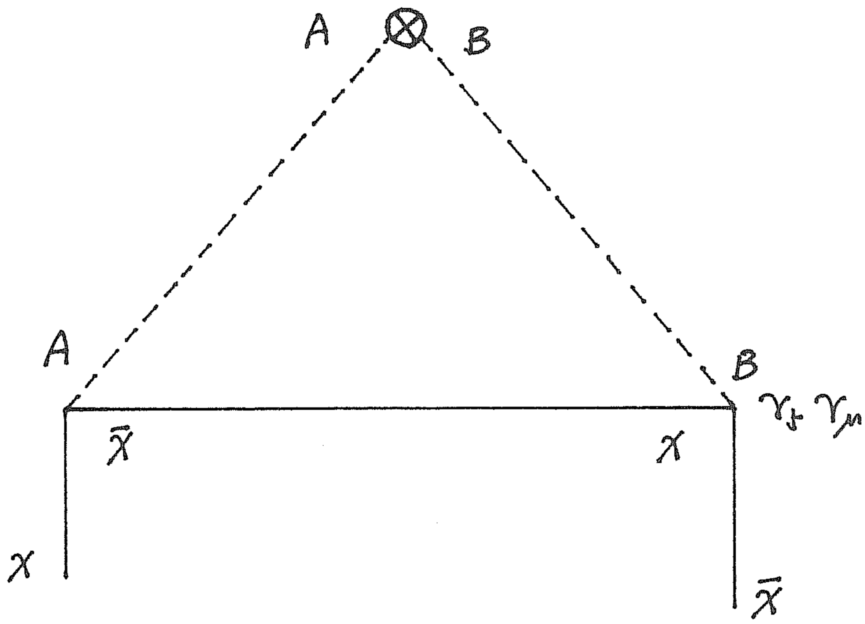
$$\partial^\mu j^{(s)}_\mu = -\frac{g^2}{\pi^2}\partial_\mu [A\overleftrightarrow{\partial}^\mu B + \frac{1}{2}i\bar{\chi}\gamma_5\gamma_\mu\chi] \qquad (3\text{-}22)$$

Therefore, unlike the Adler-Bell-Jackiw anomaly case, one could redefine $j^{(s)}_\mu$ by adding the quantity

$$\frac{g^2}{\pi^2}(A\overleftrightarrow{\partial}_\mu B + \frac{i}{2}\bar{\chi}\gamma_5\gamma_\mu\chi) \qquad (3\text{-}23)$$

so that the new current is anomaly-free. Let us now consider the case of the coupling of the scalar multiplet to a background vector multiplet.

The Lagrangian is


A
B
A
B
$\gamma_5 \gamma_\mu$
$\bar{\chi}$
χ
χ
$\bar{\chi}$

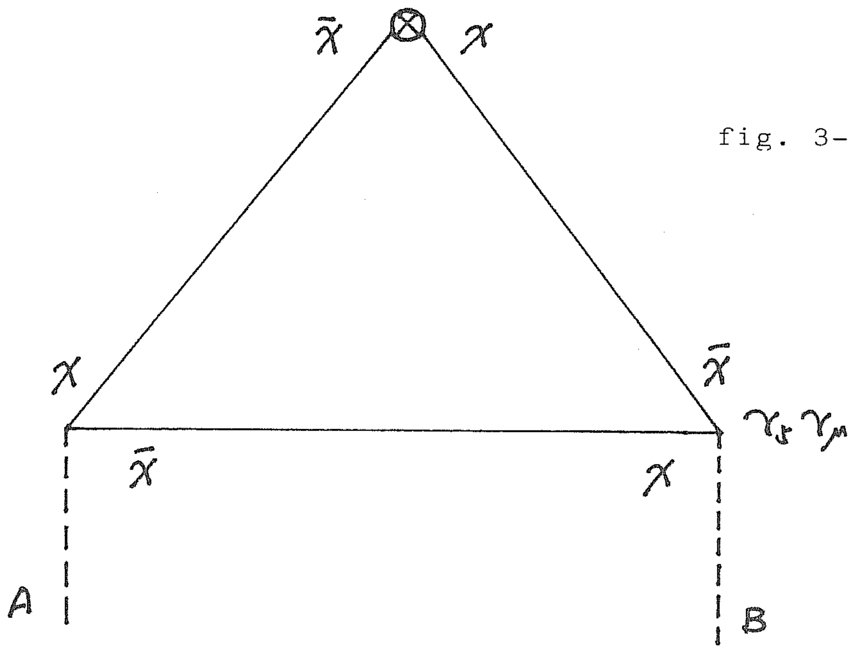

$\bar{\chi}$
χ
χ
$\bar{\chi}$
$\gamma_5 \gamma_\mu$
$\bar{\chi}$
χ
A
B

fig. 3-1

$$\mathcal{L} = -\frac{1}{2}[(\partial A)^2+(\partial B)^2]-\frac{i}{2}\bar{\chi}^a\gamma^\mu(D_\mu\chi)^a-$$

$$-\frac{1}{2}g^2[A^2B^2-(A^aB^a)^2]-gf^{abc}\bar{\lambda}^a(A^b+\gamma_5B^b)\chi^c \quad (3\text{-}24)$$

$$-[\frac{1}{4}v_{\mu\nu}^2+\frac{i}{2}\bar{\lambda}^a\gamma^\mu(D_\mu\lambda)^a]\ ,$$

from which we can get, by using the Noether theorem, the following currents and tensor

$$S=\gamma^\lambda\gamma_\mu[D_\lambda(A+\gamma_5B)]^a\chi^a+$$

$$+\frac{4}{3}\sigma_{\mu\lambda}\partial^\lambda[(A^a+\gamma_5B^a)\chi^a]\ ,$$

$$\theta_{\mu\nu}=(\partial_\mu A)(\partial_\nu A)+(D_\mu B)(D_\nu B)+\frac{i}{4}\bar{\chi}(\gamma_\mu D_\nu+\gamma_\nu D_\mu)\chi \quad (3\text{-}25)$$

$$-\eta_{\mu\nu}\mathcal{L}-\frac{1}{6}(\partial_\mu\partial_\nu-\eta_{\mu\nu}\Box)(A^2+B^2)\ ,$$

$$j_\mu^{(5)}=A\overleftrightarrow{\partial}_\mu B-\frac{i}{4}\bar{\chi}\gamma_5\gamma_\mu\chi\ .$$

We look at the one-loop matrix elements of the above quantities between the vacuum and states of the vector multiplet. The anomaly of $j_{\alpha\dot{\alpha}}^{(5)}$ is the usual chiral anomaly as follows

$$\partial^\mu j_\mu^{(5)} = -C\, v_{\mu\nu} v^{\mu\nu} ,$$

$$C = -\frac{g^2}{32\pi^2} C_V , \qquad (3-26)$$

where C_V is the Casimir operator for the Y-M group. This expression receives contribution only from the spinor part of the current , that is, the graph, in fig. 3-2, formed by spinor fields. This anomaly, together with the γ -trace of $S_{\mu\alpha}$ and the trace of $\theta_{\mu\nu}$, may form a superfield as its components and transform into each other under the superconformal transfor- mation.

(b) A discussion of the supercurrent anomaly in susy gauge theories

In the previous subsection, we have presented different models where the axial current of the R-symmetry receives anomalous contributions. We know that this anomaly lies in the same multiplet as the γ -trace of the susy current and the dilation anomaly. Thus, in a susy theory where the scalar invariance is broken, the axial current has a chiral anomaly and the susy current has an S-susy anomaly, the coefficients of all 3 anomalies being equal. However,

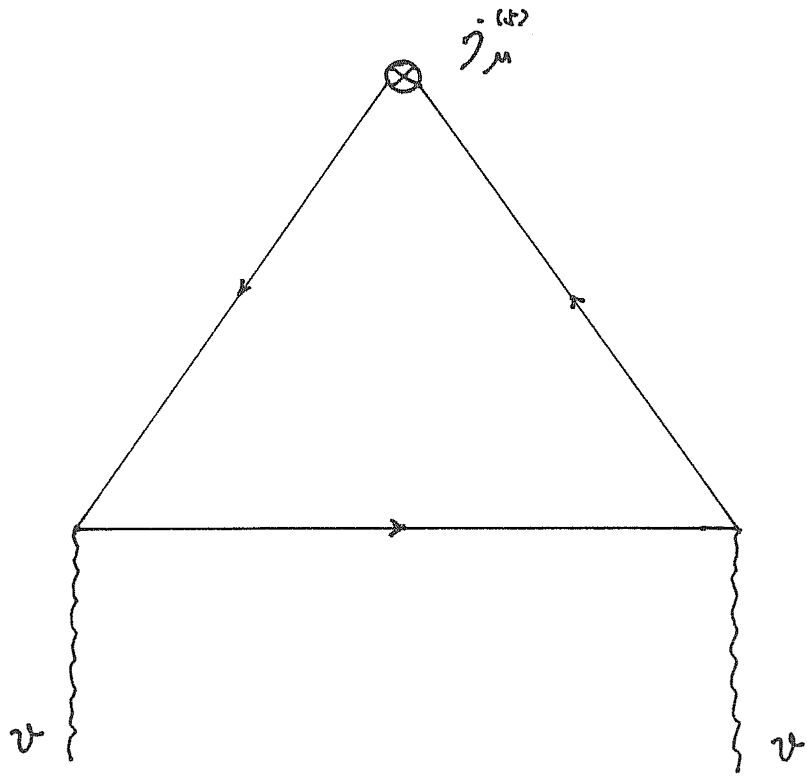


fig. 3-2

just as translational invariance is not violated (the trace of the energy- momentum tensor is anomalous, not its divergence), neither is ordinary Q-susy (the γ -trace of the susy current is anomalous, not its divergence).

In the present subsection, we would like to point out that an anomaly may exist for the divergence of the susy current in the case of a susy non-Abelian gauge theory. It is worthwhile, however, to stress that one cannot derive the anomaly from a susy transformation of the axial vector current anomaly since these two anomalies do not lie in the same supermultiplet.

By considering as SU(2) susy gauge theory, with Lagrangian in the Wess-Zumino gauge , given by

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \frac{1}{2} i \bar{\psi}^a \gamma_\mu (D_\mu \psi)^a \tag{3-27}$$

where the Yang-Mills field strength is

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g \epsilon_{abc} A^b_\mu A^c_\nu \tag{3-28}$$

with a =1,2,3 and ψ^a is an I=1 Majorana spinor. The covariant derivative is

$$(D_\mu \psi)^a = \partial_\mu \psi^a + g\,\epsilon_{abc} A_\mu^b A_\nu^c \; . \qquad (3\text{-}29)$$

It can be shown that this Lagrangian transforms into a total derivative under the following susy transformation laws

$$\delta A_\mu^a = i \bar{\epsilon}\, \gamma_\mu \psi^a \; ,$$

$$(3\text{-}30)$$

$$\delta \psi^a = \sigma^{\mu\nu} \epsilon\, F_{\mu\nu}^a \; .$$

By using the Noether theorem, one can show that the Noether's current generating the above transformations is

$$S_{\mu\alpha} = -i\,(\sigma^{kl} \gamma_\mu)_{\alpha\beta}\, \psi^{a\beta} F_{kl}^a \; . \qquad (3\text{-}31)$$

The contributions to the anomaly of the susy current, $J_{\alpha\dot{\alpha}}$, come from the 4 graphs drawn in fig.3-3. Their respective calculations with a suitable regularization procedure can be found in reference[13], where the opera- tor form of the anomaly was shown to be

$$\partial^\mu S_{\mu\alpha} = -i \frac{3g^2}{8\pi^2} (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)(\gamma^\nu \partial_\mu \psi_a)_\alpha \qquad (3\text{-}32)$$

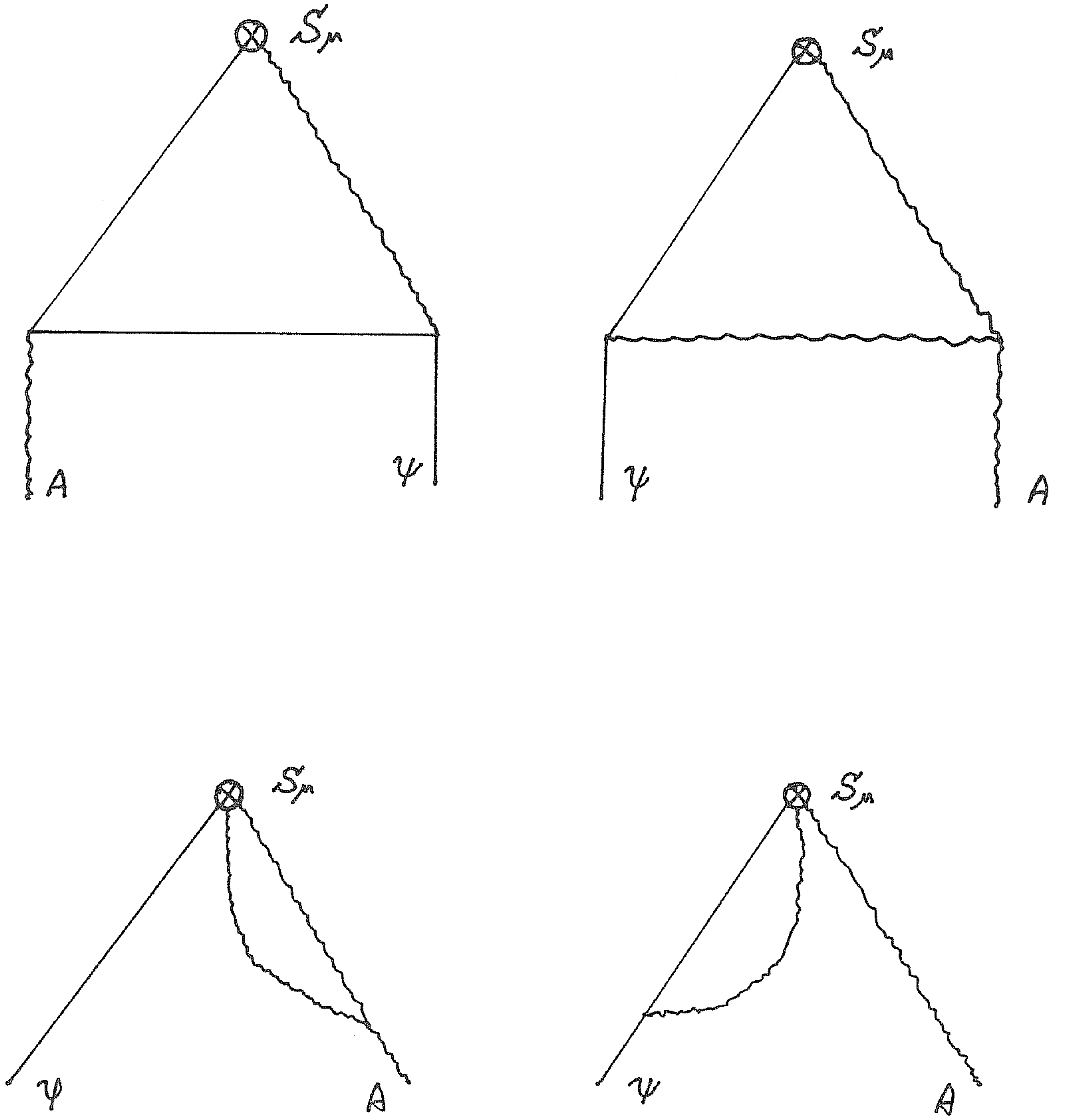


fig. 3-3

where the on-shell conditions for the external legs were used.

The existence of such an anomaly for the divergence of $S_{\mu\alpha}$ is not a disaster for the theory and the renormalizability of the above susy model is not spoiled since no gauge particle is coupled to the susy current (remember that susy is global in this model).

The most serious problem one should worry about is that the appearance of a non-vanishing $\partial^{\mu} S_{\mu\alpha}$ could destroy translational invariance in superspace and then break global susy. However, this is not quite the case since one can define a modified supercurrent $\tilde{S}_{\mu\alpha}$ given by

$$\tilde{S}_{\mu\alpha} = -F^{a}_{\alpha\beta}(\sigma^{\alpha\beta}\gamma_{\mu}\psi^{a}) + i\frac{3g^{2}}{8\pi^{2}}[A^{a}_{\lambda}(\gamma^{\mu}\partial^{\lambda}\psi^{a} - \tag{3-33}$$

$$-\gamma^{\lambda}\partial^{\mu}\psi^{a}) - 3A^{a}_{\mu}(\gamma\cdot\partial\psi^{a}) + 2\epsilon_{\mu\nu\lambda\sigma}A^{a}_{\mu}(\gamma_{5}\gamma^{\lambda}\partial^{\sigma}\psi^{a})] .$$

This modified operator is such that

$$\partial_{\mu}\langle 0|\tilde{S}_{\mu\alpha}|\psi, A\rangle = 0 , \tag{3-34}$$

and

$$\langle 0 | \gamma^{\mu} \tilde{S}_{\mu\alpha} | \psi, A \rangle = 0 \,, \tag{3-35}$$

which means that the one boson-one fermion matrix element is conserved and satisfies the spin 3/2 constraint. The existence of the conserved operator $\tilde{S}_{\mu\alpha}$ means that translational invariance in superspace can be restored and susy is not broken by one-loop effects. This is in agreement with the fact that if susy is not broken at the tree-level approximation, it is not broken by radiative corrections in any finite order of the perturbation theory.

Just conclude this subsection, we would like to remark that one can always find a mechanism to cancel the anomaly of the divergence of the super-current by suitably coupling a scalar multiplet in some representation of the gauge group to the susy Yang-Mills multiplet.

(c) A sample calculation: the superconformal current anomaly in superspace for the Wess-Zumino model [14]

We have derived in the subsection (b), the superfield which accommodate R-symmetry axial current, susy current and the energy-momentum tensor. By coupling the Wess-Zumino model to a supergravity background field, we obtained the classical superconformal current to be

$$J^{(0)}_{\alpha\dot{\alpha}} = -\frac{1}{3}\bar{D}_{\dot{\alpha}}\bar{\Phi}D_{\alpha}\Phi + \frac{1}{3}\bar{\Phi}\, i\overleftrightarrow{\partial}_{\alpha\dot{\alpha}}\Phi \,, \qquad (3\text{-}36)$$

which satisfies the conservation equation

$$\bar{D}^{\dot{\alpha}} J^{(0)}_{\alpha\dot{\alpha}} = 0 \qquad (3\text{-}37)$$

by virtue of the equation of motion. This conservation equation expresses the invariance of the classical theory under superconformal transformations.

We wish now to make use of supergraph techniques to compute the expression for the anomaly of $J_{\alpha\dot{\alpha}}$ at one-loop level for the Wess-Zumino model. We then begin by computing the one-loop graph of the graph, in fig. 3-4, with two external lines and one insertion of the operator $J^{(0)}_{\alpha\dot{\alpha}}$. By using standard supergraph techniques, we computed the graph drawn above, which we denote by $\Gamma(\Phi, \bar{\Phi}; J)$. Our final answer is

$$\Gamma = g^2\int\frac{d^4q}{(2\pi)^4}[q^2(q+p)^2(q-p')^2]^{-1}\{[\tfrac{1}{3}(q+p)_{\beta\dot{\alpha}}(q-p')_{\alpha\dot{\beta}} +$$

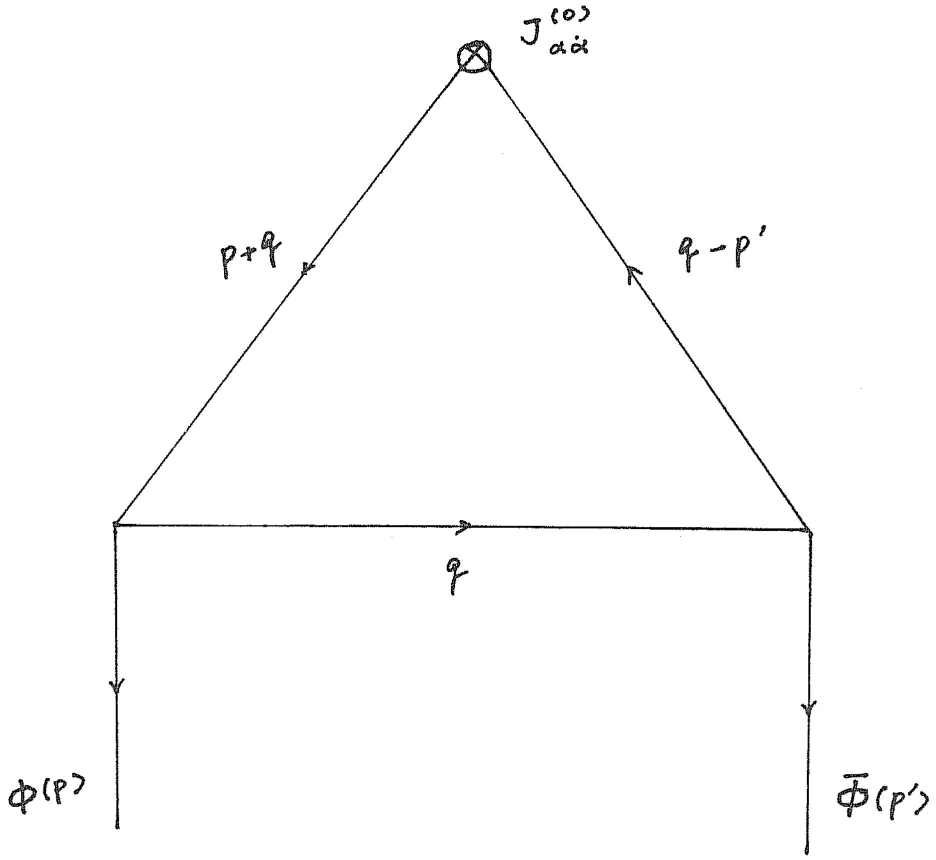


fig. 3-4

$$+\frac{1}{3}(2q+p-p')_{\alpha\dot{\alpha}}\, q_{\rho\dot{\rho}}]\, D^{\rho}\phi\bar{D}^{\dot{\rho}}\bar{\phi}+[\frac{1}{3}(q+$$

(3-38)

$$+p)_{\rho\dot{\alpha}}(q-p')_{\alpha\dot{\rho}}\, q^{\rho\dot{\rho}}+\frac{1}{3}(2q+p-p')_{\alpha\dot{\alpha}}\, q^{2}]\,\bar{\phi}\phi-$$

$$-\frac{1}{3}(2q+p-p')_{\alpha\dot{\alpha}}\bar{D}^{2}\bar{\phi}D^{2}\phi\}.$$

One should note that the divergent part of Γ is given by

$$\Gamma_{div}=\frac{1}{2}g^{2}\int\frac{d^{4}q}{(2\pi)^{4}}[(q+p)^{2}(q-p')^{2}]^{-1}\times$$

(3-39)

$$\times[\bar{\phi}\, i\overleftrightarrow{\partial}_{\alpha\dot{\alpha}}\phi-\bar{D}_{\dot{\alpha}}\bar{\phi}D_{\alpha}\phi].$$

On the other hand, if one now uses again the standard algebra, it's possible to show that the wave-function renormalization of the ϕ -superfield, which is calculated from the graph, drawn in fig. 3-5, whose answer

$$\frac{1}{2}g^{2}\int\frac{d^{4}q}{(2\pi)^{4}}[q^{2}(q+p)^{2}]^{-1}\int d^{4}\theta\,\bar{\phi}(p)\phi(p), \qquad (3-40)$$

is sufficient to renormalize the Green function Γ with

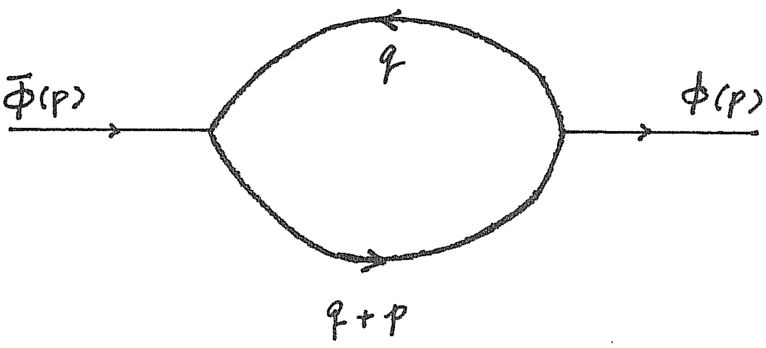


fig. 3-5

current insertion. This simply shows that Γ is made finite by the renormalization of the external lines, so that its anomalous dimension vanishes.

After removing the infinity, we are left with a well-defined one-loop renormalized current $J^{(1)}_{\alpha\dot{\alpha}}$. This renormalized current has a super-conformal anomaly, whose value can be calculated by using Pauli-Villar's regularization Its value turns out to be

$$\bar{D}^{\dot{\alpha}} J^{(1)}_{\alpha\dot{\alpha}} = -\frac{1}{6}\left(\frac{g^2}{4\pi}\right)^2 D_\alpha(\phi \bar{D}^2 \bar{\phi}). \qquad (3\text{-}41)$$

Now, we are going to mention the chiral anomaly problem in global supersymmetry and the Adler-Bardeen theorem.

As pointed out in the previous sections, the R-symmetry axial current, the supercurrent and the energy-momentum tensor are the physical components of the superconformal current multiplet. Quantum corrections break superconformal invariance in an explicit way and then give rise to anomalies for the associated currents. However, since the quantum corrections can be computed in a supersymmetric manner, one can also deduce that the anomalies will be members of some multiplet, that is the divergence of the R-symmetry current, the γ-trace of the supercurrent and the trace of the energy-momentum tensor belonging to the same superfield and can be transformed into each other by a supersymmetry transformation.

The fact that the trace of the energy-momentum tensor is proportional to the β-function means that the dilatation anomaly receives contribution to all orders of perturbation theory. On the other hand, the divergence of the R-current should, by invoking the Adler-Bardeen theorem for global chiral currents, receive only a one-loop contribution. However, these two quantities share the same multiplet and should then be transformed into one another. This appears to be a clash between the Adler-Bardeen theorem and supersymmetry. This is the apparent paradox that we are going to try to explain in the course of this section.

(a) To start, let us consider the massless Wess-Zumino model and its superconformal current given by

$$J^{(0)}_{\alpha\dot\alpha} = -\frac{1}{3}\bar D_{\dot\alpha}\bar\Phi D_\alpha \Phi + \frac{1}{3}\bar\Phi\, i\overleftrightarrow{\partial}_{\alpha\dot\alpha}\Phi \,. \qquad (3\text{-}42)$$

According to the superfield calculation carried out in the previous section for the two-point function with a current-insertion, one obtains a one-loop anomaly given by

$$\bar{D}^{\dot{\alpha}} J_{\alpha\dot{\alpha}} = -\frac{1}{6}\left(\frac{g}{4\pi}\right)^2 D_\alpha[\phi \bar{D}^2 \bar{\phi}] . \tag{3-43}$$

Let us extract the scalar component of the above equation, that is, the divergence of the R-symmetry current:

$$-i\{D^\alpha, \bar{D}^{\dot{\alpha}}\} J_{\alpha\dot{\alpha}}\Big|_{\theta=\bar{\theta}=0} = \partial^{\alpha\dot{\alpha}} j^{(5)}_{\alpha\dot{\alpha}} = \left(\frac{g}{4\pi}\right)^2 \partial^{\alpha\dot{\alpha}} K^{(0)}_{\alpha\dot{\alpha}} \tag{3-44}$$

where

$$K^{(0)}_{\alpha\dot{\alpha}} = -\frac{1}{3}[\bar{\psi}_{\dot{\alpha}} \psi_\alpha + \frac{1}{2} A^* i \overleftrightarrow{\partial}_{\alpha\dot{\alpha}} A] . \tag{3-45}$$

This shows that one can redefine a renormalized component chiral current by removing the anomaly of $j^{(5)}_{\alpha\dot{\alpha}}$. This means that one can work with a new subtracted current given by

$$\tilde{j}^{(5)}_{\alpha\dot{\alpha}} = j^{(5)}_{\alpha\dot{\alpha}} - \left(\frac{g}{4\pi}\right)^2 K^{(0)}_{\alpha\dot{\alpha}} . \tag{3-46}$$

In general, a finite subtraction on the chiral current can not be extended to the whole current supermultiplet without destroying the conservation or symmetry properties of the renormalized energy-momentum tensor. However, the massless Wess-Zumino model is an exception to this general rule. Indeed, the new chiral current defined above is the first component of a superfield given by

$$\tilde{J}_{\alpha\dot{\alpha}} = J^{(0)}_{\alpha\dot{\alpha}} - \left(\frac{g}{4\pi}\right)^2 K^{(0)}_{\alpha\dot{\alpha}} \tag{3-47}$$

where

$$\begin{aligned} K^{(0)}_{\alpha\dot{\alpha}} &= -\frac{1}{6}[\bar{D}_{\dot{\alpha}}, D_\alpha](\bar{\phi}\phi) \\ &= -\frac{1}{3}[\bar{D}_{\dot{\alpha}}\bar{\phi} D_\alpha \phi + \frac{1}{2}\bar{\phi}\, i\overleftrightarrow{\partial}_{\alpha\dot{\alpha}}\phi]. \end{aligned} \tag{3-48}$$

For this redefined current, we have now the following super-trace

$$\bar{D}^{\dot{\alpha}}\tilde{J}_{\alpha\dot{\alpha}} = \frac{1}{2}\left(\frac{g}{4\pi}\right)^2 \bar{D}^2 D_\alpha(\bar{\phi}\phi) . \tag{3-49}$$

By extracting components one gets a conserved chiral current but a γ-trace of the supercurrent and a trace for the energy-momentum tensor, both of which are proportional to the β-function given by

$$\beta(g)=\frac{3}{2}\left(\frac{g}{4\pi}\right)^2 \tag{3-50}$$

The final conclusion is that we can arrive at a situation of a conserved chiral current and a non-vanishing β -function. The next step is to extend this analysis to the case of a N=1 gauge theory.

(b) Pure N=1 Yang-Mills theory [14]

We consider the N=1 super-Yang-Mills theory whose superfield action is given by

$$S=\frac{1}{4g^2}\,Tr\int d^4x\, d^2\theta\, W^{\alpha}W_{\alpha}+h.c. \tag{3-51}$$

where

$$W_{\alpha}=i\bar{D}^2\left(e^{-gV}D_{\alpha}\,e^{gV}\right). \tag{3-52}$$

As in the case of the Wess-Zumino model, one can couple the above action to supergravity and, at the linearized level, one can read off the following superconformal current

$$J^{(0)}_{\alpha\dot\alpha} = \mathrm{Tr}\, \overline{W}_{\dot\alpha} W_\alpha \ , \tag{3-53}$$

which satisfies the classical superconformal law

$$\overline{D}^{\dot\alpha} J^{(0)}_{\alpha\dot\alpha} = 0 \ . \tag{3-54}$$

By reading components, one can extract the classical expression for the R-symmetry axial current

$$j^{(5)}_{\alpha\dot\alpha} = \mathrm{Tr}\, \overline{\lambda}_{\dot\alpha} \lambda_\alpha \quad , \tag{3-55}$$

which is conserved,

$$\partial^{\alpha\dot\alpha} j^{(5)}_{\alpha\dot\alpha} = 0. \tag{3-56}$$

The one-loop graphs contributing to the renormalization of the current $J^{(0)}_{\alpha\dot\alpha}$ are given in fig. 3-6.

By evaluating these supergraphs, one can find that their overall divergence is of the type

$$Z_V \overline{W}_{\dot\alpha} W_\alpha \tag{3-57}$$

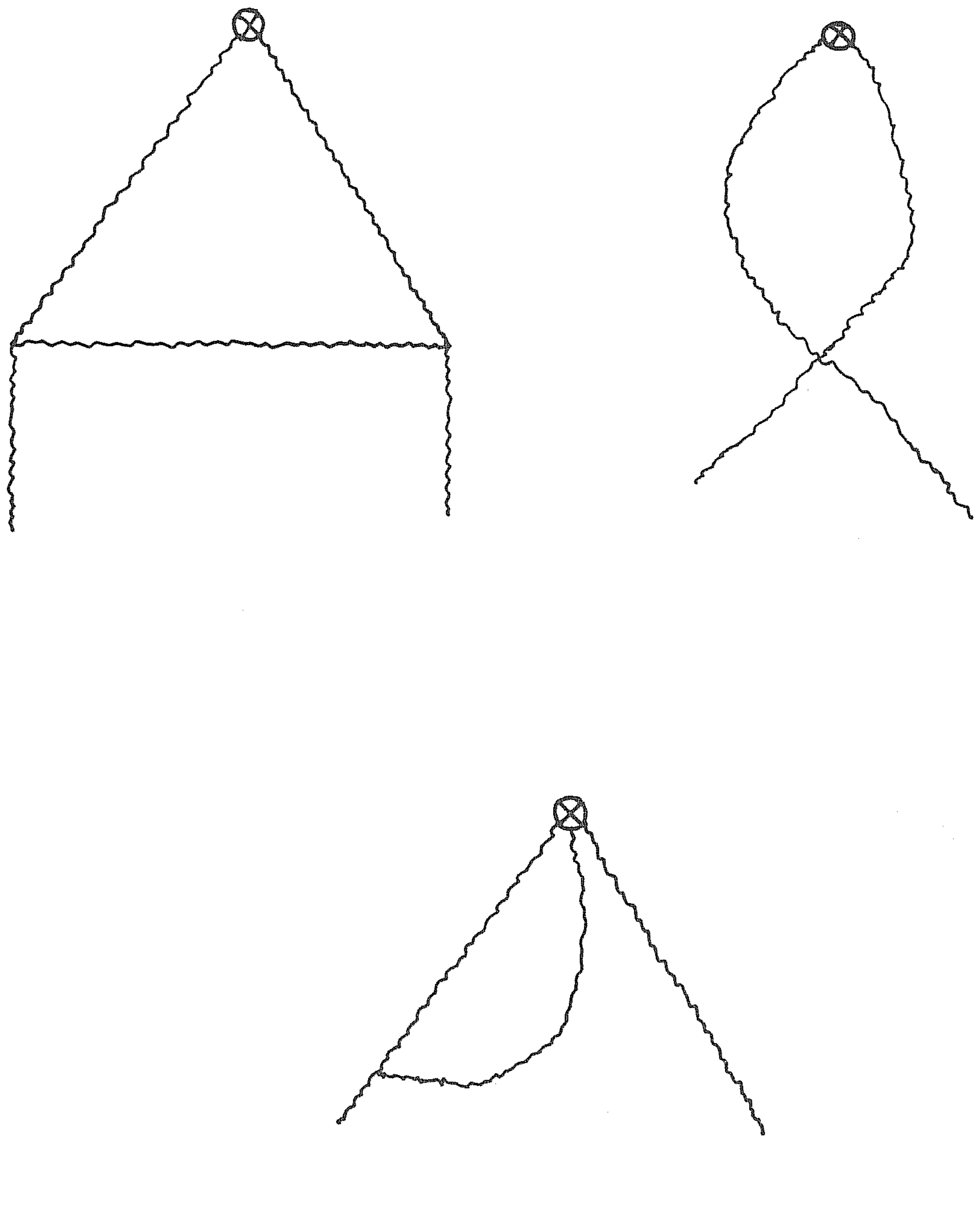


fig. 3-6

where Z_V is the wave-function renormalization factor of the vector superfield. This means that a renormalization of the external vector lines suffices to renormalize the super-conformal current. Then, by subtracting the infinity of the one-loop graphs drawn in the figure above, one gets that the renormalized current satisfies

$$\bar{D}^{\dot\alpha} J^{(1)}_{\alpha\dot\alpha} = -\frac{1}{3}\frac{\beta^{(1)}(g)}{g} D_\alpha (W^2) \quad , \tag{3-58}$$

where $\beta^{(1)}(g)$ is the one-loop renormalization group β - function for the N=1 super-Yang-Mills model.

Going over into components, one can find by computing the quantity

$$\{D^\alpha, \bar{D}^{\dot\alpha}\} J^{(1)}_{\alpha\dot\alpha} \Big|_{\theta=\bar\theta=0} \tag{3-59}$$

that

$$\partial^{\alpha\dot\alpha} j^{(5)}_{\alpha\dot\alpha} = -\frac{4}{3}\frac{\beta^{(1)}(g)}{g}\left[\frac{1}{4}F\tilde{F} + \frac{1}{2}\partial^{\alpha\dot\alpha}(\bar\lambda_{\dot\alpha}\lambda_\alpha)\right] . \tag{3-60}$$

This shows that, due to the second term of the right-hand side of the above equation, the R-current does not satisfy the Adler-Bardeen theorem. The problem now would be the search for a different renormalized chiral current which satisfies the Adler-Bardeen theorem. In other words, this means that one has to search for another supermultiplet which accommodates a chiral current that satisfies the Adler-Bardeen theorem and this new superfield cannot contain the energy-momentum tensor. The panorama now would be the following:

(i) the renormalized supercurrent $J^{(1)}_{\alpha\dot{\alpha}}$ contains an axial current, the energy-momentum tensor and the supersymmetry current, however the axial current here present is not the Adler-Bardeen current.

(ii) the modified supercurrent $\tilde{J}^{(1)}_{\alpha\dot{\alpha}}$ contains the axial current that satisfies the Adler-Bardeen theorem as its first component. It cannot contain a symmetric, conserved energy-momentum tensor nor the supersymmetry current.

In reference[14], the modified current has been formed and its divergence turns out to be

$$\bar{D}^{\dot{\alpha}}\hat{J}^{(1)}_{\alpha\dot{\alpha}} = \frac{\beta^{(1)}}{3g}\left[\frac{1}{2}W^{\beta}D_{(\alpha}W_{\beta)} - \frac{3}{2}W_{\alpha}(D^{\beta}W_{\beta})\right]. \qquad (3\text{-}61)$$

This now gives for the first component

$$\partial^{\alpha\dot{\alpha}} \tilde{J}^{(5)}_{\alpha\dot{\alpha}} = -\frac{4}{3}\frac{\beta^{(1)}}{g}\left(\frac{1}{4}F\tilde{F}\right). \qquad (3\text{-}62)$$

which then shows that there is in the theory an appropriate chiral current which satisfies the Adler-Bardeen theorem.

As a summary, one can say that there is no clash, as it was thought for quite a time, between the Adler-Bardeen theorem and supersymmetry. What one has to understand is that the renormalized superconformal current contains a chiral current which does not satisfy the Adler-Bardeen theorem along with the energy-momentum tensor and the super-current. However, a chiral current which does satisfy the Adler-Bardeen theorem can always be found, but it is located in another supermultiplet. Therefore, one can say that, though at the classical level the axial current and energy-momentum tensor are members of the same multiplet, quantum corrections split them to different superfields.

All these results have been established up to now to one-loop. Following again Grisaru and West, it is shown that at two-loops the modified superconformal current is such that

$$\bar{D}^{\dot{\alpha}} J^{(2)}_{\alpha\dot{\alpha}} = \left[-\frac{\beta^{(2)}}{3g} - 2\frac{\beta^{(1)}}{3g}\frac{\beta^{(1)}}{3g}\right] D_{\alpha}(W^2). \qquad (3\text{-}63)$$

It should be said that the first term on the right hand side (proportional to $\beta^{(2)}$) comes from two-loop graphs with the insertion of the classical current $J^{(0)}_{\alpha\dot{\alpha}}$, according to the graph in fig.3-7. The second term, which is proportional to $\beta^{(1)}\cdot\beta^{(1)}$, originates from the one-loop superfield which subtracted from $J^{(1)}_{\alpha\dot{\alpha}}$, to give $\tilde{J}_{\alpha\dot{\alpha}}$ ($\tilde{J}^{(1)}_{\alpha\dot{\alpha}} = J^{(1)}_{\alpha\dot{\alpha}} - K^{(1)}_{\alpha\dot{\alpha}}$), as in the graph in fig.3-8.

Since from the equation above the coefficient of the term, $F\tilde{F}$, is

$$-\frac{\beta^{(2)}}{3g} - 2\,\frac{\beta^{(1)}}{3g}\,\frac{\beta^{(1)}}{3g} \,, \qquad (3\text{-}64)$$

and the Adler-Bardeen current can be defined by subtracting the $\bar{\lambda}_{\dot{\alpha}}\lambda_{\alpha}$ term, this coefficient must be zero, since the chiral current in this multiplet must satisfy the Adler-Bardeen theorem and cannot receive any two-loop contribution. This then implies that

$$\frac{\beta^{(2)}}{g} = -\frac{2}{3}\,\frac{\beta^{(1)}}{g}\,\frac{\beta^{(1)}}{g} \,. \qquad (3\text{-}65)$$

This process illustrates how the β -function at higher orders can be fixed from the one-loop value for β .

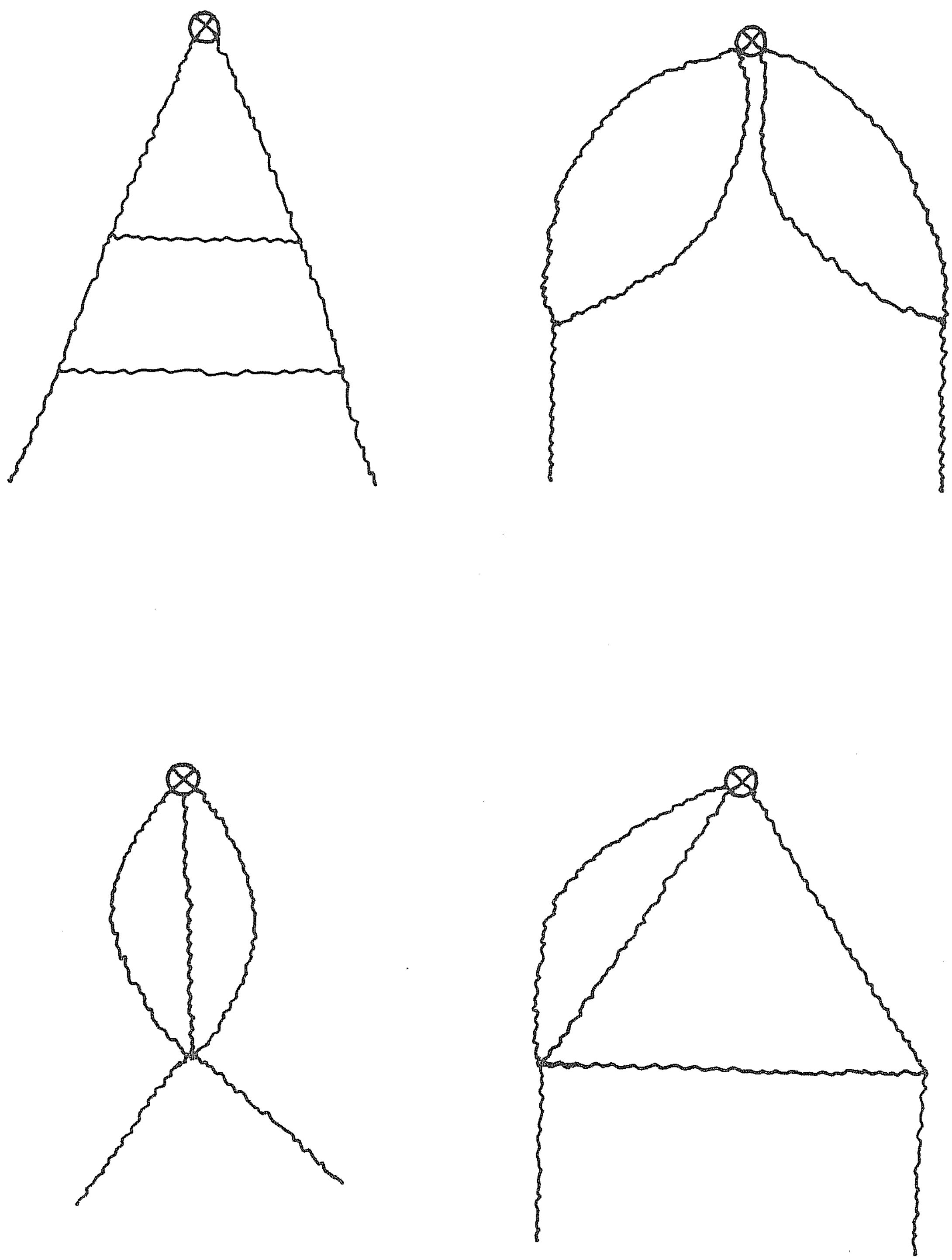


fig. 3-7

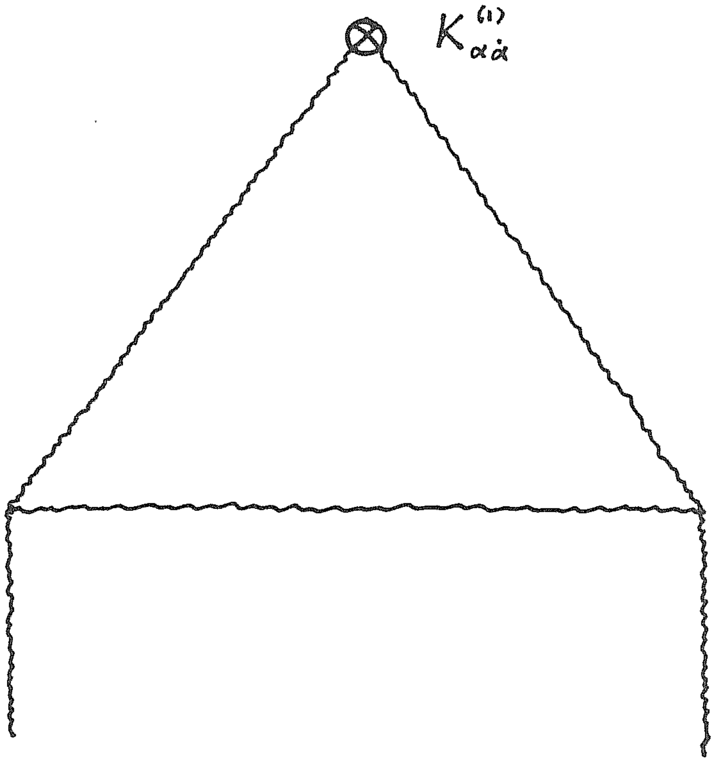


fig. 3-8

Chapter 4 Two-loop investigation of the chiral anomaly problem in N=1 supersymmetric gauge theories

The Adler-Bardeen non renormalization theorem for the anomaly of an axial current is a very outstanding result in both perturbative and non-perturbative field theories. The fact that the divergence of pseudo-vector currents receives just a one-loop and no contributions from higher orders should be well understood in the framework of supersymmetric theories, where one expects different currents affected by higher order corrections to be members of a same supermultiplet. Actually, as pointed out by Ferrara and Zumino[4], the R-symmetry axial current, $j_\mu^{(5)}$, the improved energy-momentum tensor, $\theta_{\mu\nu}$, and the (improved) supersymmetry current, $S_{\mu\alpha}$, are identified with the components of a vector superfield. This already anticipates constraints existing among the divergences of the currents sharing the same supermultiplet.

The apparent puzzle arising in this context is the fact that the β-function, which in principle may receive loop contributions to all orders, can be transformed under supersymmetry into the chiral current anomaly which, as already stated, is not affected by corrections above one loop. This situation has been clarified in the papers by Novikov et al. [15], Grisaru and West[14], Tonin et al. [16], and Piguet et al. [17].

In references[18], supergraph methods were employed in association with background field calculations in superspace, and this allows for a remarkable simplification. In the work we are going to report here, we propose to pursue an investigation through two loops of the divergence of the superconformal current by using the supersymmetric extension of the heat kernel method in superspace. We shall here perform explicit supergraph evaluations with a quantum background splitting and then use the results of reference[19] for the Seeley coefficients. We shall then verify that the two loop contributions to the superconformal anomaly is actually governed by the two loop β-function of the supersymmetric theory under consideration, which is in our case, a general pure N=1 super-Yang-Mills theory, whose superfield action is given by

$$\mathcal{S} = \frac{1}{4g^2} \mathrm{Tr} \int d^4x \, d^2\theta \, W^\alpha W_\alpha + h.c. \qquad (4-1)$$

where

$$W_\alpha = i\bar{D}^2(e^{-gV} D_\alpha e^{gV}). \tag{4-2}$$

By coupling the gauge field to the N=1 supergravity multiplet and linearizing in the vierbein superfield, one can show that the superconformal current turns out to be

$$J_{\alpha\dot{\alpha}} = \mathrm{Tr}\, W_\alpha \bar{W}_{\dot{\alpha}}. \tag{4-3}$$

It satisfies the classical conservation law

$$\bar{D}^{\dot{\alpha}} J_{\alpha\dot{\alpha}} = 0. \tag{4-4}$$

The authors of reference [19] have considered the one-loop correction to equation (4-4) in terms of the heat kernel method. In such a case, there is only one graph to be considered and the whole contribution comes from the Seeley coefficient $a_2(z, z') = (W^2 \mathcal{D}^2 + \bar{W}^2 \bar{\mathcal{D}}^2)\, \delta^4(\theta - \theta')$. We propose here to extend the results of reference[19] going one order ahead in perturbation theory. The whole set of two loop graphs relevant for our calculations is depicted in fig. 4-1.

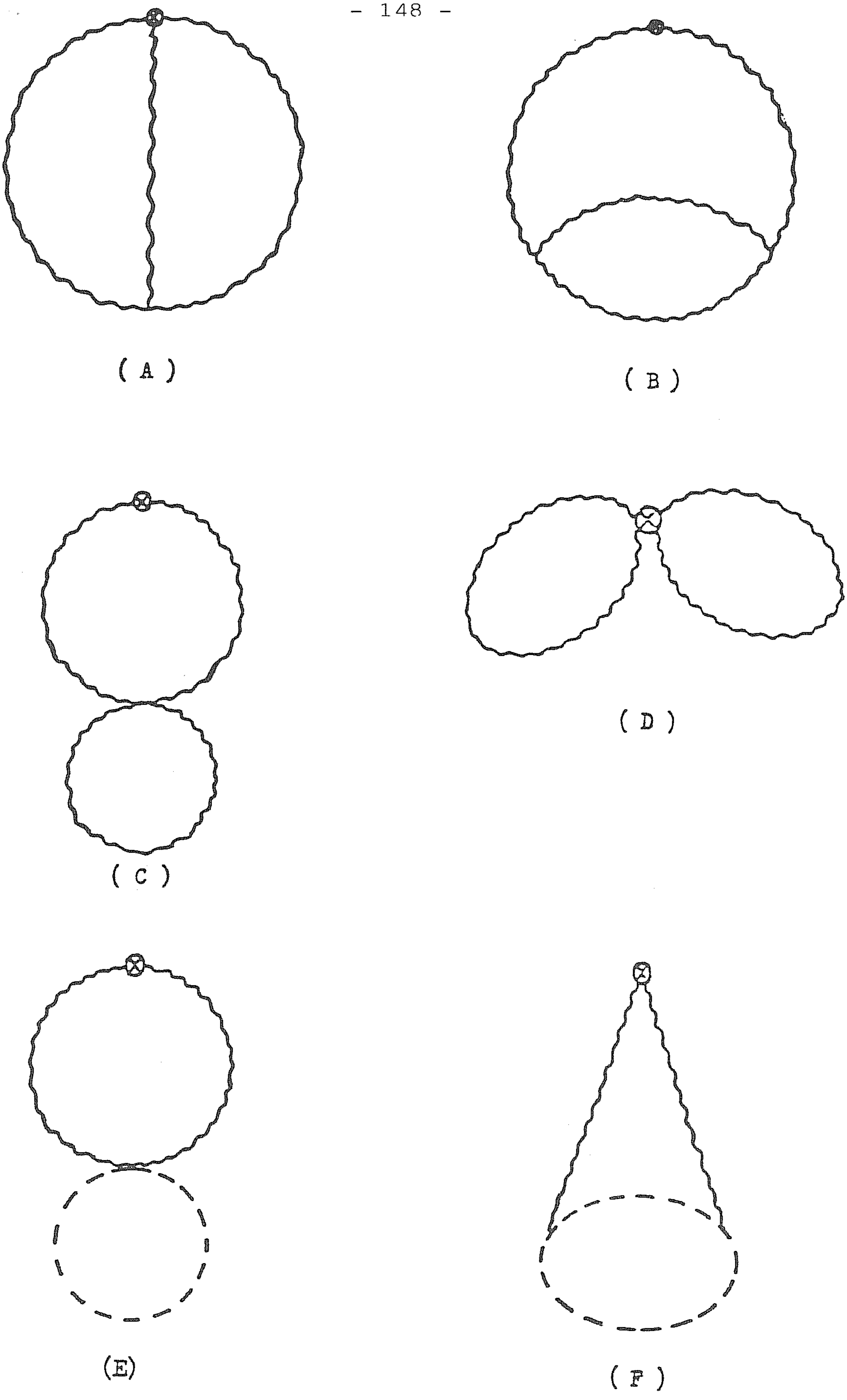


fig. 4-1

Before discussing and quoting the answers we obtained by calculating the above supergraphs, we would like to mention that , apart from the current vertex insertions, the Feynman rules for the vertices coming from the Lagrangian are the ones given in reference[20]. To read off the current insertions, the relevant pieces of , to the order we wish to contemplate, are given below

$$J_{\alpha\dot\alpha} = \mathrm{Tr}\{\bar D^2 (g D_\alpha V - \tfrac{1}{2} g^2 [V, D_\alpha V] + \\ + \tfrac{1}{6} g^3 [V, [V, D_\alpha V]]) \cdot D^2 (g \bar D_{\dot\alpha} V - \\ - \tfrac{1}{2} g^2 [V, \bar D_{\dot\alpha} V] + \tfrac{1}{6} g^3 [V, [V, \bar D_{\dot\alpha} V]])\} \qquad (4-5)$$

We can now start the discussion of the results of our two-loop calculations. First of all the tadpole graphs of figure (4-1,c) and (4-1,e) can be shown to be identically vanishing by using the algebra of covariant derivatives and the complete expression for the quantum superpropagators in terms of the Seeley coeffcients. Due to the complexity of the vertices involving three or four vector superfields, a single supergraph of fig. 4-1 contains indeed a considerably large number of graphs, due to the permutations of derivatives in the vertices. However, many of them can be identified by exploiting the symmetry of the group theoretical factors, the algebra of covariant derivatives and partial intergration in superspace. Once this first step is accomplished, there remain a number of few different structures to be actually evaluated. Just to illustrate what we have said, we can take, for example, the graph of fig. (4-1,b). From the expression for the three vector vertex, one can immediately see that there are, in principle, 36 graphs to be computed. Nevertheless, symmetric properties identify many of them so that we end up with 18 different structures to be calculated. Proceduring similarly for all the other non-vanishing supergraphs, we finally get the following answers:

fig. 4-1 A =

$$= \frac{13}{8} g^4 \int d^8 z_1 (\bar{D}_{1\dot\alpha} D_1^2 G(z-z_1))(\Box_1 D_{1\alpha} \bar{D}_1^2 G(z-z_1)) G(z-z_1) -$$

$$- \frac{1}{8} g^4 \int d^8 z_1 (\bar{D}_1^2 D_{1\beta} \partial_{1\dot\alpha}^{\beta} G(z-z_1))(D_{1\alpha} \bar{D}_1^2 G(z-z_1)) G(z-z_1) +$$

$$+ \frac{3}{4} g^4 \int d^8 z_1 (\partial_{1\alpha\dot\beta} \bar{D}_1^{\dot\beta} D_1^2 G(z-z_1))(\partial_{1\beta\dot\alpha} D_1^{\beta} \bar{D}_1^2 G(z-z_1)) G(z-z_1) -$$

$$- \frac{1}{4} g^4 \int d^8 z_1 (\bar{D}_1^2 D_1^2 G(z-z_1))(\bar{D}_1^2 D_1^2 \partial_{1\alpha\dot\alpha} G(z-z_1)) G(z-z_1) +$$

$$+ \frac{1}{16} g^4 \int d^8 z_1 (D_1^2 \bar{D}_1^2 G(z-z_1))(\bar{D}_1^2 D_1^2 \partial_{1\alpha\dot\alpha} G(z-z_1)) G(z-z_1) -$$

$$- \frac{3}{32} g^4 \int d^8 z_1 (\bar{D}_1^2 D_1^2 G(z-z_1))(D_1^2 \bar{D}_1^2 \partial_{1\alpha\dot\alpha} G(z-z_1)) G(z-z_1) \tag{4-6}$$

fig. 4-1 B =

$$= \Big\{ \frac{1}{32} g^4 \int d^8 z_1 d^8 z_2 [D_{1\alpha} \bar{D}_1^2 G(z-z_1)][\bar{D}_{2\dot\alpha} D_2^2 G(z-z_2)] G(z_1-z_2) \times$$

$$\times [\Box_1 D_1^2 \bar{D}_1^2 G(z_1-z_2)] - \frac{1}{8} g^4 \int d^8 z_1 d^8 z_2 [-\bar{D}_{2\dot\alpha} D_2^2 G(z-z_2)] \times$$

$$\times [\partial_{1\alpha\dot\beta} \bar{D}_1^2 G(z-z_1)] G(z_1-z_2) [\Box_1 \bar{D}_1^{\dot\beta} D_1^2 G(z_1-z_2)] -$$

$$-\frac{1}{8}g^4\int d^8z_1\, d^8z_2 [\bar{D}_{2\dot{\alpha}} D_2^2 G(z-z_2)][\partial_{1\gamma\dot{\beta}} D_1^{\gamma} \bar{D}_1^2 G(z-z_1)] G(z_1-z_2)\times$$

$$\times[\partial_{1\alpha}{}^{\dot{\beta}} \bar{D}_1^2 D_1^2 G(z_1-z_2)] + \frac{1}{4}g^4\int d^8z_1\, d^8z_2 [\bar{D}_{2\dot{\alpha}} D_2^2 G(z-z_2)]\times$$

$$\times[\Box_1 \bar{D}_1^2 G(z-z_1)] G(z_1-z_2)[\partial_{1\alpha\dot{\beta}} \bar{D}_1^{\dot{\beta}} D_1^2 G(z_1-z_2)] - \frac{1}{128}g^4\times$$

$$\times\int d^8z_1\, d^8z_2 [D_1^2 \bar{D}_1^2 G(z-z_1)][\partial_2{}^{\lambda}{}_{\dot{\alpha}} \bar{D}_2^2 D_2^2 G(z-z_2)] G(z_1-z_2)\times$$

$$\times[D_{1\alpha} \bar{D}_1^2 D_{1\lambda} G(z_1-z_2)] + \frac{1}{8}g^4\int d^8z_1\, d^8z_2 [\Box_1 \bar{D}_1^2 G(z-z_1)]\times$$

$$\times[\partial_2{}^{\lambda}{}_{\dot{\alpha}} D_2^2 G(z-z_2)] G(z_1-z_2)[D_{1\lambda} \bar{D}_1^2 D_{1\alpha} G(z_1-z_2)] -$$

$$-\frac{1}{8}g^4\int d^8z_1\, d^8z_2 [\partial_1^{\lambda\dot{\lambda}} D_{1\lambda} \bar{D}_1^2 G(z-z_1)][\partial_2{}^{\kappa}{}_{\dot{\alpha}} D_2^2 G(z-z_2)]\times$$

$$\times G(z_1-z_2)[D_{1\kappa\dot{\lambda}} \bar{D}_1^2 D_{1\alpha} G(z_1-z_2)] - \frac{1}{32}g^4\int d^8z_1\, d^8z_2\times$$

$$\times[\Box_1 D_{1\alpha} \bar{D}_1^2 G(z-z_1)][\bar{D}_{2\dot{\alpha}} D_2^2 G(z-z_2)] G(z_1-z_2)\times$$

$$\times[D_1^2 \bar{D}_1^2 G(z_1-z_2)]\Big\} + h.c. \qquad (4\text{-}7)$$

fig. 4-1 C = 0.

fig. 4-1 D =

$$= \frac{16}{3} i g^4 [\bar{D}_1^2 D_{1\alpha} G(z_1-z_2)]\Big|_{z_1=z_2} [D_1^{\beta} \partial_{1\beta\dot{\alpha}} G(z_1-z_2)]\Big|_{z_1=z_2} +$$

$$+\frac{16}{3}ig^4[D_1^2\bar{D}_{1\dot{\alpha}}G(z_1-z_2)]\Big|_{z_1=z_2}[\bar{D}_1^{\dot{\beta}}\partial_{1\alpha\dot{\beta}}G(z_1-z_2)]\Big|_{z_1=z_2}+$$

$$+2g^4[\bar{D}_1^2D_{1\alpha}G(z_1-z_2)]\Big|_{z_1=z_2}[\bar{D}_{1\dot{\alpha}}D_1^2G(z_1-z_2)]\Big|_{z_1=z_2} \qquad (4\text{-}9)$$

fig. 4-1 E = 0. (4-10)

fig. 4-1 F =

$$=-\frac{1}{32}g^4\int d^8z_1d^8z_2[D_{1\alpha}\bar{D}_1^2G(z-z_1)][\bar{D}_{2\dot{\alpha}}D_2^2G(z-z_2)]\times$$

$$\times G(z_1-z_2)[\Box_1\bar{D}_1^2D_1^2G(z_1-z_2)]+\frac{1}{8}g^4\int d^8z_1d^8z_2\times$$

$$\times[\partial_{1\alpha}{}^{\dot{\beta}}\bar{D}_1^2G(z-z_1)][\bar{D}_{2\dot{\alpha}}D_2^2G(z-z_2)]G(z_1-z_2)\times$$

$$\times[\Box_1\bar{D}_{1\dot{\beta}}D_1^2G(z_1-z_2)]-\frac{1}{8}g^4\int d^8z_1d^8z_2[\Box_1\bar{D}_1^2G(z-z_1)]\times$$

$$\times[\bar{D}_{2\dot{\alpha}}D_2^2G(z-z_2)]G(z_1-z_2)[\partial_{1\alpha\dot{\lambda}}\bar{D}_1^{\dot{\lambda}}D_1^2G(z_1-z_2)]\times+$$

$$+\frac{1}{32}g^4\int d^8z_1d^8z_2[\partial_1{}^{\lambda\dot{\beta}}D_{1\lambda}\bar{D}_1^2G(z-z_1)][\bar{D}_{2\dot{\alpha}}D_2^2G(z-z_2)]\times$$

$$\times G(z_1-z_2)[\partial_{1\alpha\dot{\beta}}\bar{D}_1^2D_1^2G(z_1-z_2)] \qquad (4\text{-}11)$$

where G(z,z) stands for the full Green function for the vector or ghost quantum fields, whose background dependence is entirely contained in the Seeley coefficients and covariant derivatives appearing as vertex factors in the graphs.

Now, we should notice that all graphs which survive the superspace manipulations correspond effectivelly to nothing but the gauge-coupling renormalization. To understand this statement, it suffices to remember that the supergraphs of fig. (4-1,b), (4-1,d), (4-1,f) give contributions to wave-function renormalization. However, since we are working in the background field formalism, we should recall the well-known relation[21]

$$Z_g \cdot Z_A^{1/2} = 1 \qquad (4\text{-}12)$$

between the wave-function (Z_A) and coupling constant (Z_g) renormalization factors. According to reference [22], this useful relation can be extended to the case of supersymmetric Yang-Mills theories. In view of such a result, one can actually conclude that the overall two-loop correction to the divergence of the superconformal current , is basically given by the two-loop β -function, $\beta^{(2)}(g)$, of the theory (the graphs which contribute to $\bar{D}^{\dot\alpha} J_{\alpha\dot\alpha}$ are basically the ones which renormalize the gauge coupling constant).

At this point, there still remains the problem of determinating the operatoral form of the superconformal anomaly. For this purpose, the explicit form of the Green functions in terms of the Seeley expansion is crucial. The relevant part for us is [19]

$$G(z,z') = G_0(x-x')a_0(z,z') + \\ + G_1(x-x')a_1(z,z') + G_2(x-x')a_2(z,z') \tag{4-13}$$

Inserting (4-13) into the supergraph results (4-6) to (4-11), one actually obtains, after lengthy manipulations with the covariant derivatives and partial integration, that the operatoral form of the superconformal anomaly is

$$D_\alpha (W^2) \tag{4-14}$$

in agreement with the well-known results of reference [14, 15].

So, we can finally conclude by writing down the net result of our supergraph calculations:

$$\bar{D}^{\dot{\alpha}} J_{\alpha\dot{\alpha}}\Big|_{2\text{-loop}} = -\frac{\beta^{(2)}(g)}{3g} D_\alpha (W^2) \qquad (4\text{-}15)$$

We do not claim that ours is a new result but, as already stressed at the beginning of this letter, we have used the supersymmetric extension of the heat kernel expansion in connection with the background field method to calculate the two-loop contribution to the anomaly. Using standard super-graph techniques and useful results of the background field methods, we could check a result (equation(4-14)) previously obtained and discussed in several references quoted here and verify the validity of the heat kernel expansion for a higher loop computation.

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