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Thesis submitted for the degree of Magister Philosophiae

Stochastic differential equations with boundary conditions

Candidate:
Marco R. Ferrante

Supervisor:
Prof. D. Nualart

Academic Year 1990/91

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Contents

Chapter 1

Introduction

The stochastic calculus with anticipating integrands has been recently developed by several authors (see for example [7]). This theory allows to define $\int \varphi_s dW_s$, when the integrand φ_s is not adapted to the filtration generated to the Brownian motion $\{W_t : t \in [0,1]\}$. Moreover it allows us to study different types of stochastic differential equations driven by $\{W_t : t \in [0,1]\}$, where the solution turns out to be non necessarily adapted to the filtration generated by W_t.

In the present thesis we are concerned with the stochastic differential equations of the type:

$$\frac{dX_t}{dt} + f(X_t) = B\frac{dW_t}{dt} \tag{1.1}$$

where $t \in [0,1]$ and instead of the usual initial condition, where we fix the value of X_0, we impose a boundary condition

$$h(X_0, X_1) = \bar{h} \tag{1.2}$$

which involves both X_0 and X_1. We assume that $\{W_t : t \in [0,1]\}$ is a k-dimensional Brownian motion and $\{X_t\}$ takes values in $\mathbb{R}^d$ (h being a function from $\mathbb{R}^{2d}$ into $\mathbb{R}^d$, $\bar{h} \in \mathbb{R}^d$).

The goal of this thesis is to develop, where possible, the results of the jointly paper of Nualart-Pardoux [8] about the Markov property of the solution of (1.1), from the one-dimensional case to the general d dimensional case. More precisely, in [8] is proved in dimension one (i.e. $d = k = 1$), that the solution of (1.1) (if exists and is unique) is a Markov field if and only if $f'' \equiv 0$ and is proved via a counterexample that in dimension larger than one the solution can be a Markov process, even with non linear f's. In the present thesis we provide a necessary measurability condition for the solution X_t of equation (1.1) to be a Markov field. Using this new condition, we can prove in an easier way than in [8] the dichotomy result in dimension one (section 6.2) and state a necessary result, in dimension larger than one, in the "triangular" situation.

The thesis is organized as follows: in chapter 2 we prove some existence and uniqueness theorems (following [8]) for the stochastic differential equation of type (1.1) with a boundary value problem of type (1.2). In chapter 3 we present a short introduction about the anticipative calculus (referring to [7] for a comprehensive exposition) and state two Lemmas that we need in the following sections. In chapter 4 we state an extended version of the Girsanov theorem for non necessarily adapted processes which is due to Kusuoka(in [6]). Moreover we prove that we can apply it to our problem and compute a Radon-Nikodym derivative. In chapter 5 we study the Markov property in the linear case and find out a measurability condition that the solution of a general non linear stochastic differential equation of type (1.1) (under boundary condition (1.2)) has to satisfy if we assume that it is a Markov field. In chapter 6 we prove another existence and uniqueness theorem for a particular class of problems with linear boundary condition and apply the previous measurability condition to this class. In the second part of this chapter we give a new proof of Theorem 4.4 of [8]. In the last chapter, we show, via a counterexample, that the result holding for $d = 1$(i.e. that the solution of (1.1) (if exists and is unique) is a Markov field if and only if $f'' \equiv 0$) , can not be extended to the case $d > 1$. We utilize again the existence and uniqueness result proved in chapter 6 for the particular "triangular" class.

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Chapter 2

Existence and Uniqueness

Let $\{W_t; t \in [0,1]\}$ be a standard k-dimensional Wiener process defined on a probability space $(\Omega, \mathcal{F}, P)$. We are looking for a solution $\{X_t; t \in [0,1]\}$ of equation (1.1),(1.2) as an $\mathbb{R}^d$-valued process. We shall assume without loss of generality that $k \leq d$ and that the kernel of the $d \times k$ matrix B reduce to $\{0\}$. We suppose moreover that the mapping $f : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ takes the form:

$$f(x) = Ax + B\bar{f}(x)$$

where A is a $d \times d$ matrix, and $\bar{f} : \mathbb{R}^d \longrightarrow \mathbb{R}^k$ is measurable and locally bounded. We are finally given a mapping $h : \mathbb{R}^{2d} \longrightarrow \mathbb{R}^d$ and a vector $\bar{h} \in \mathbb{R}^d$, and we consider the equation:

$$\begin{cases} \frac{dX_t}{dt} + f(X_t) &= B\frac{dW_t}{dt} \\ \\ h(X_0, X_1) &= \bar{h} \end{cases} \tag{2.1}$$

In other words, a solution is thought of as an element $X \in C([0,1]; \mathbb{R}^d)$ which is s.t.:

$$\begin{cases} X_t + \int_0^t f(X_s)ds = X_0 + BW_t & 0 \leq t \leq 1 \\ \\ h(X_0, X_1) = \bar{h} \end{cases} \tag{2.2}$$

Remark 2.1 *Notice that in this section we shall consider the solution $\{X_t\}$ as a function of the input $\{W_t\}$ defined pointwise on the space $C(\mathbb{R}_+; \mathbb{R}^k)$, so that the fact that $\{W_t\}$ is a Wiener process is in fact irrelevant.*

Following [8], we first associate to (2.1) the equation with $\bar{f} \equiv 0$:

$$\begin{cases} \frac{dY_t}{dt} + AY_t = B\frac{dW_t}{dt} \\ \\ h(Y_0, Y_1) = \bar{h} \end{cases} \tag{2.3}$$

A solution of (2.2), if any , is of the form:

$$Y_t = e^{-At}\left[Y_0 + \int_0^t e^{As} B dW_s\right]$$

where the last expression makes sense for any continuous function $\{W_t\}$ by integration by parts. Therefore a solution of (2.2) must to satisfy:

$$h\left(Y_0, e^{-A}\left(Y_0 + \int_0^1 e^{As} B dW_s\right)\right) = \bar{h}$$

Let us define :

$$F := \left\{\int_0^1 e^{As} B d\varphi(t); \varphi \in C([0,1]; \mathbb{R}^k)\right\} \subset \mathbb{R}^d.$$

We now formulate our first assumption:

$$(H.1)\begin{cases} \forall z \in F, \text{the equation } h(y, e^{-A}(y+z)) = \bar{(h)} \\ \text{has a unique solution } y = g(z) \end{cases}$$

Under (H.1) the equation (2.3) has the unique solution:

$$Y_t = e^{-At}\left[g\left(\int_0^1 e^{As}BdW_s\right) + \int_0^t e^{As}BdW_s\right]$$

We now define the sets of functions:

$$C_0([0,1];\mathbb{R}^d) = \left\{\eta \in C([0,1];\mathbb{R}^d); \eta(0) = 0\right\}$$

$$\begin{aligned}\Sigma \;\; = \{\xi \in C([0,1];\mathbb{R}^d);\;\; & \xi_t - \xi_0 + \int_0^t A\xi_s ds \in ImB \\ & 0 \le t \le 1\;;\; h(\xi_0,\xi_1) = \bar{h}\;\}\end{aligned}$$

It is easily seen that there exists a bijection ψ from $C_0([0,1];\mathbb{R}^d)$ into Σ s.t.:

$$Y_t = \psi_t(W)$$

We define the mapping T from $C_0([0,1];\mathbb{R}^d)$ into itself by:

$$T(\eta) = \eta + \int_0^{\cdot} \bar{f}(\psi_s(\eta))ds$$

We can state the following theorem (for the proofs in this chapter we refer always to [8])

Theorem 2.1 *T is a bijection if an only if equation (2.1) has the unique solution* $X = \psi \circ t^{-1}(W)$

In the following we shall give some sufficient conditions for T to be one-to-one and onto. Let us start with:

Proposition 2.1 *Each of the following conditions implies that T is one-to-one:*

$$(H.2i)\begin{cases} \exists\lambda \in I\!R s.t. f + \lambda I \, is \ monotone \ and \\ e^{\lambda}|g(z) - g(z')| \geq |e^{-A}(z - z' + g(z) - g(z')|; z, z' \in F \\ \Rightarrow g(z) = g(z') \end{cases}$$

$$(H.2ii)\begin{cases} \exists\lambda \in I\!R s.t. f + \lambda I \, is \ strictly \ monotone \ and \\ e^{\lambda}|g(z) - g(z')| \leq |e^{-A}(z - z' + g(z) - g(z')|; z, z' \in F \end{cases}$$

Remark 2.2 *It is possible to prove that the following condition also implies that T is one-to-one:*

$$(H.2iii) \quad g \ and \ \ e^{At}B\bar{f}(e^{-At}) \ are \ monotone, \forall 0 \leq t \leq 1$$

We now give a sufficient condition for T to be onto:

Proposition 2.2 *The following three conditions imply that T is onto:*

$$(H.3) \qquad \bar{f} \ \ is \ locally \ Lipschitz$$

$$(H.4) \qquad \lim_{a \to \infty} \tfrac{1}{a} \sup_{|x| \leq a} |\bar{f}(x)| = 0$$

$$(H.5) \quad \exists c \ \ s.t. \ \ |g(x)| \leq c(1 + |x|), x \in F$$

In chapter 6 we shall prove directly that our nonlinear equation admits a unique solution (as a consequence it implies that T is a bijection (Proposition 2.1)). It is nevertheless interesting to present some example where the previous general conditions are satisfied (for the proof see again [8]). We shall consider the case where the boundary condition is linear, i.e.

$$h(y, z) = H_0 y + H_1 z$$

where H_0 and H_1 are $d \times d$ matrices. Then a sufficient condition for (H.1) is that the matrix $H_0 + H_1 e^{-A}$ is invertible, and moreover:

$$g(z) = (H_0 + H_1 e^{-A})^{-1}(\bar{h} - H_1 e^{-A} z)$$

(this condition is satisfied in chapter 6 by our triangular model). Clearly (H.5) is satisfied in this case. Assume (H.3) and (H.4), if we suppose that H_0 is invertible, we have that a sufficient condition for (H.2ii) is that:

$$f - \left(\log |H_0^{-1} H_1|\right) I \quad \text{is strictly monotone}$$

(see again [8]).

Two interesting examples of this kind of problems are those of the periodic boundary condition (i.e. $H_0 = -H_1 = I$ and $\bar{h} = 0$) and of the proportional initial and final value (i.e.$H_0 = aI, H_1 = bI$, where $a, b \in I\!R$, and again $\bar{h} = 0$).

Chapter 3

Some remarks about the Anticipative Calculus

In this section we shall recall the notions of derivation on Wiener space and Skorohod integral (see Nualart [7] for a complete exposition of the basic results about the anticipating stochastic calculus). Moreover we shall prove two Lemmas that we shall need in the following sections.

Let $\{W(t),\ 0 \le t \le 1\}$ be a d-dimensional Wiener process defined on the canonical probability space $\Omega = C_0([0,1];\mathbb{R}^d)$. Let us denote by H the Hilbert space $L^2(0,1;\mathbb{R}^d)$. For any $h \in H$, we will denote by $W(h)$ the Wiener integral

$$\int_0^1 \langle h(t), dW_t \rangle .$$

We denote by S the dense subset of $L^2(\Omega)$ consisting of those random variables of the form

$$F = f(W(h_1), ..., W(h_n)) \tag{3.1}$$

where $n \ge 1$, $h_1, \cdots, h_n \in H$ and $f \in C_b^\infty(\mathbb{R}^n)$ (that means, f and all its partial derivatives are bounded).

The random variables of the form (3.1) are called smooth functional. For a smooth functional $F \in S$ of the form (3.1) we define its derivative DF (resp. D_hF) as the d-dimensional stochastic process $\{D_tF, 0 \le t \le n\}$ given by

$$D_tF = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(W(h_1), ..., W(h_n))h_i(t) \tag{3.2}$$

(resp. $D_hF = \langle h, DF\rangle_{L^2(\Omega\times[0,1];\mathbb{R}^d)}$). Then D (resp. D_h) is a closable unbounded operator from $L^2(\Omega)$ into $L^2(\Omega \times [0,1]; \mathbb{R}^d)$. We will denote by $\mathbb{D}^{1,2}$ (resp. $\mathbb{D}^{h,2}$) the completion of S with respect to the norm $\|\cdot\|_{1,2}$ (resp. $\|\cdot\|_{h,2}$) defined by

$$\|F\|_{1,2} = \|F\|_2 + \|DF\|_{L^2(\Omega\times[0,1];\mathbb{R}^d)} \tag{3.3}$$

(resp.$\|F\|_{h,2} = (E[F^2 + (D_hF)^2])^{\frac{1}{2}}$).

We will denote by δ the adjoint of the derivation operator D. That means, δ is a closed and unbounded operator from $L^2(\Omega \times [0,1]; \mathbb{R}^d)$ into $L^2(\Omega)$ defined as follows: the domain of δ, *Dom* δ, is the set of processes $u \in L^2(\Omega \times [0,1]; \mathbb{R}^d)$ such that there exists a positive constant c_u verifying

$$\mid E\int_0^1 \langle D_tF, u_t\rangle dt \mid \le c_u\|F\|_{L^2(0,1;\mathbb{R}^d)} \tag{3.4}$$

for all $F \in S$. If u belongs to the domain of δ then $\delta(u)$ is the square integral random variable determined by the duality relation

$$E(\int_0^1 \langle D_tF, u_t\rangle dt) = E(\delta(u)F) \qquad F \in \mathbb{D}^{1,2}. \tag{3.5}$$

The operator δ is an extension of the Itô integral in the sense that the class L_a^2 of processes u in $L^2(\Omega \times [0,1]; \mathbb{R}^d)$ which are adapted to the Brownian filtration is included into $Dom(\delta)$ and $\delta(u)$ is equal to the Itô integral if $u \in L_a^2$. The operator δ is called the Skorohod stochastic integral.

Define $\mathbb{L}^{1,2} = (L^2([0,1]; \mathbb{D}^{1,2})^d$. Then the space $\mathbb{L}^{1,2}$ is included into the domain of δ. The operator D (resp. D_h) and δ are local in the following sense:

(a) $\mathbf{1}_{\{F=o\}}DF = 0$ for all $F \in \mathbb{D}^{1,2}$;

(b) $\mathbf{1}_{\{\int_0^1 |u|_t^2 dt=0\}}\delta(u) = 0$ for all $u \in \mathbb{L}^{1,2}$.

Using these local properties one can define the spaces $\mathbb{D}^{1,2}_{loc}$ (resp. $\mathbb{D}^{l,2}_{loc}$) and $\mathbb{L}^{1,2}_{loc}$ by a standard localization procedure. For instance $\mathbb{D}^{1,2}_{loc}$ (resp. $\mathbb{D}^{h,2}_{loc}$) is the space of the random variables F such that there exists a sequence $\{(\Omega_n, F_n), n \geq 1\}$ such that $\Omega_n \in \mathcal{F}$, $\Omega_n \uparrow \Omega$ a.s. , $F_n \in \mathbb{D}^{1,2}$ and $F_n = F$ on Ω_n for each $n \geq 1$. By property (a) the derivation operator D can be extended to random variables of the space $\mathbb{D}^{1,2}_{loc}$.

We shall now state two Lemmas, that we shall need in the sequel. We shall present here only the proof of the second one, referring for the proof of the first one to [1]:

Lemma 3.1 *Let $F : \Omega \to \mathbb{R}$ and $u : \mathbb{R}^k \times \Omega \to \mathbb{R}$ be measurable functions such that:*

(i) $u(\cdot, \omega) \in C^1(\mathbb{R}^k)$, for all $\omega \in \Omega$;

(ii) $u(x, \cdot) \in \mathbb{D}^{1,2}$, for all $x \in \mathbb{R}^k$ and there exists a version of $Du(x)$ such that the mapping $x \to Du(x)$ is continuous from $\mathbb{R}^k$ into H;

(iii) for all $a > 0$ we have :

$(iii)_1$ $E[\sup_{|x|\leq a} \mid u(x) \mid^2] < \infty$;

$(iii)_2$ $\| \sup_{|x|\leq a} \mid \nabla u(x) \| < \infty$;

$(iii)_3$ $E(sup_{|x|\leq a} \|Du(x)\|^2) < \infty$;

(iv) $F_j \in \mathbb{D}^{1,2}_{loc}$ $, 1 \leq j \leq k$.

Define $G(\omega) = u(F(\omega), \omega)$. Then $G \in \mathbb{D}^{1,2}_{loc}$ and

$$DG = (\nabla u)(F) \cdot DF + (Du)(F) \tag{3.6}$$

Let $\mathcal{H}$ be a real separable Hilbert space. Consider a stochastic process $\{W(h); h \in \mathcal{H}\}$ defined in some probability space $(\Omega, \mathcal{F}, P)$, such that:

(i) { W(h) } is a Gaussian process ;

(ii) E(W(h))=0 for all h in $\mathcal{H}$;

(iii) E(W(h)W(g)) $=\langle h,g\rangle_{\mathcal{H}}$.

In this context one can introduce as before the class S of smooth functionals of the process W(h), as the set of random variables of the form (3.1) where the elements h_i are in $\mathcal{H}$. Then for any smooth functional F of the form (3.1), its derivative DF is the $\mathcal{H}$-valued random variable defined by

$$DF=\sum_{i=1}^{n}\frac{\partial f}{\partial x_i}(W(h_1),\cdots,W(h_i))h_i.$$

Then, as before, D is a closable unbounded operator from $L^2(\Omega)$ into $L^2(\Omega,\mathcal{H})$. All the preceding notions can be extended to this more abstract framework. The following example will play a basic role in the sequel:

let us take the Hilbert space $L^2([0,1];\mathbb{R}^d)$ with the scalar product:

$$\begin{aligned}\langle h,g\rangle &= \langle e^{Au}h(u),e^{Au}g(u)\rangle_{L^2([0,1];\mathbb{R}^d)}\\ &= \sum_{i,j}\int_0^1(e^{A+A^T})_{ij}h^i(u)g^j(u)du\end{aligned}$$

(where A is a $d\times d$ matrix) We can define the derivation operator D with respect to the d-dimensional Gaussian process

$$\xi_s=\int_0^s e^{Au}dW_u.$$

Let $\mathcal{G}_1=\sigma(\xi(h_s);t\leq s\leq 1)$ and $\mathcal{G}_2=\sigma(\xi(h_1),\xi(h_2))$ where $\xi(h)=\int_0^1 h(u)d\xi_u$ and $h_s=\mathbf{1}_{[t,s]},h_1=\mathbf{1}_{[0,t]},h_2=\mathbf{1}_{[t,1]}$. Let $K_1=span\{h_s;t\leq s\leq 1\}$ and $K_2=span\{h_1,h_2\}$; we can prove the following:

Lemma 3.2

(i) Let $F\in\mathbb{D}^{1,2}_{loc}$, $A\in\mathcal{G}_1$ and $\mathbf{1}_AF$ be $\mathcal{G}_1$-measurable. Then $DF\in K_1$ a.s.

(ii) Let $F\in\mathbb{D}^{1,2}_{loc}$, $A\in\mathcal{G}_2$ and $\mathbf{1}_AF$ be $\mathcal{G}_2$-measurable. Then $DF\in K_2$ a.s.

Proof We give the proof just of the second part of the Lemma (the first part can be proved in a similar way). Since we can approximate F by $\varphi_n(F)$ with $\varphi_n \in C_b^\infty(\mathbb{R})$, $\varphi_n(x) = x$ for $|x| \leq n$, it is sufficient to prove the result for $F \in \mathbb{D}_{loc}^{1,2} \cap L^2(\Omega)$. Let $h \in \bar{H}$ with $h \perp K_2$. Then since $E^{\mathcal{G}}(F)$ is a function of $\xi(h_1)$ and $\xi(h_2)$, it is easily seen that $E^{\mathcal{G}}(F) \in \mathbb{D}^{h,2}$, and that $D_h[E^{\mathcal{G}}(F)] = 0$. However, $F \in \mathbb{D}_{loc}^{h,2}$ and $F = E^{\mathcal{G}}(F)$ a.s on A. It then follows from the local property of D_h that:

$$D_h F = \int_0^1 h(t) D_t F dt = 0 \quad a.s \quad on \quad A$$

and it holds for any $h \perp K_2$. It remains to choose a countable set $\{h_n : n \in \mathbb{N}\} \subseteq \bar{H}$ s.t.

$$\bar{h} \in K_2 \Leftrightarrow \langle \bar{h}, h_n \rangle_{\bar{H}} = 0 \ \ \forall n$$

and to remark that

$$D_{\bar{h}_n} F = 0 \ \ \forall n, \qquad \text{a.s on A}$$

Q.E.D.

Chapter 4

Computation of a Radon-Nikodym derivative

From now on we shall assume that $k = d$ and $B = I$. In this section we recall some results proved in [8]: first we state the extended Girsanov theorem of Kusuoka (Theorem 6.4 of [6]) and then apply it to our situation. As before we assume that $\Omega = C_0([0,1];\mathbb{R}^d)$ equipped with the topology of uniform convergence, $\mathcal{F}$ is the Borel field over Ω, P is standard Wiener measure and $W_t(\omega) = \omega(t)$ is the canonical process.

Theorem 4.1 *Let $T : \Omega \longrightarrow \Omega$ be a mapping of the form*

$$T(\omega) = \omega + \int_0^\cdot K_s(\omega)ds$$

where K is a measurable mapping from Ω into $H = L^2(0,1 : \mathbb{R}^d)$, and suppose that the following conditions are satisfied:

(i) T is bijective;

(ii) For all $\omega \in \Omega$, there exists a Hilbert-Schmidt operator $DK(\omega)$ from H into itself such that:

(1) $\|K(\omega + \int_0^\cdot h_s ds) - K(\omega) - DK(\omega)h\|_H = o(\|h\|)$ as $\|h\|_H$ tends to zero.

(2) $h \to DK(\omega + \int_0^\cdot h_s ds)$ is continuous from H into $\mathcal{L}^2(H)$,

the space of Hilbert-Schmidt operators.

(3) $I + DK(\omega)$ is invertible.

Then if Q is the measure on $(\Omega, \mathcal{F})$ s.t. $P = Q\,T^{-1}$, Q is absolutely continuous with respect to P and

$$\frac{dQ}{dP} = |d_c(-DK)| \exp\left(-\delta(K) - \frac{1}{2}\int_0^1 |K_t|^2 dt\right),$$

where $d_c(-DK)$ denotes the Carleman-Fredholm determinant of the Hilbert-Schmidt operator $-DK$, and $\delta(K)$ is the Skorohod integral of K.

To define the Carleman-Fredholm determinant (see e.g. [12]), it is sufficient to say that:

(i) If A is a linear operator from $I\!R^N$ into itself,

$$d_c(A) = \prod_j (1 - \lambda_j) \exp(\lambda_j)$$

where the λ_j's are the nonzero eigenvalues of A counted with their multiplicities.

(ii) $A \longrightarrow d_c(A)$ is continuous from $\mathcal{L}^2(H)$ into $I\!R$.

We want to apply Theorem 4.1 to the mapping T defined in chapter 2. Moreover we shall compute the Radon-Nikodym derivative $\frac{dQ}{dP}$ in this particular case.

We have $K_t(\omega) = \bar{f}(\psi_t(\omega))$. Assume that $\bar{f}, g \in C^1(I\!R^d; I\!R^d)$.

$$\begin{aligned} D_s K_t(\omega) &= \bar{f}(\psi_t(\omega)) D_s \psi_t(\omega) \\ &= \bar{f}(\psi_t(\omega)) e^{-At} [g'(\xi_1) e^{As} + e^{As} \mathbf{1}_{[0,1]}(s)] \end{aligned}$$

where $\xi_t = \int_0^t e^{As} dW_s$. The operator $DK(\omega) \in \mathcal{L}^2(H)$ is given as

$$(DK(\omega)(h))^i(t) = \sum_{j=1}^{d} \int_0^1 D_s^j K_t^i(\omega) h_s^j ds$$

Conditions (ii.1) and (ii.2) are here satisfied (as a consequence of the fact that $h \longrightarrow (\bar{f}(\psi_t(\omega + \int_0^\cdot h_s ds), \xi_1(\omega + \int_0^\cdot h_s ds))$ is continuous from H into $I\!R^d$, for any $\omega \in \Omega$).

Remark 4.1 *Notice that everything here is defined for any $\omega \in \Omega$ and not just a.s., by integrating by parts the Wiener integrals.*

Let $\{\phi_t; 0 \leq t \leq 1\}$ denote the $d \times d$ matrix value solution of

$$\begin{cases} \frac{d\phi_t}{dt} & = & -f'(\psi_t)\phi_t \\ \phi(0) & = & I \end{cases}$$

We have (for the proof see [8]):

Proposition 4.1 *Suppose that $\bar{f} \in C^1(I\!R^d, I\!R^d)$, (H.1) holds and $g \in C^1(I\!R^d, I\!R^d)$. Assume moreover that T defined in chapter 2 is bijective, and furthermore that*

$$det(I - e^A \phi_1 g'(\xi_1) + g'(\xi_1)) \neq 1 \tag{4.1}$$

Then the conditions of Theorem 4.1 are satisfied.

Remark 4.2 *From the computation of the Carleman-Fredholm determinant in this particular case, we obtain that condition (4.1) is equivalent to the fact that $d_c(-DK) \neq 0$ and form the theory of the Hilbert-Schmidt operators (see e.g [12]) that it holds if and only if condition (ii.3) is satisfied.*

We shall now give two sufficient conditions for (4.1) (that we shall use in section 2 of chapter 6):

Proposition 4.2 *Suppose that $\bar{f}, g \in C^1(\mathbb{R}^d, \mathbb{R}^d)$. Then (4.1) follows from each of the following hypotheses:*

$$(H.2.i)' \begin{cases} \exists \lambda \in \mathbb{R} s.t. f + \lambda I \text{ is monotone and } \forall x, y \in \mathbb{R}^d \\ e^{\lambda}|g'(y)x| \geq |e^{-A}(I + g'(y))x| \quad \Rightarrow \quad x = 0 \end{cases}$$

$$(H.2.ii)' \begin{cases} \exists \lambda \in \mathbb{R} s.t. f + \lambda I > 0 \forall y \in \mathbb{R}^d \text{ and} \\ \forall x, y \in \mathbb{R}^d \; e^{\lambda}|g'(y)x| \leq |e^{-A}(I + g'(y))x| \end{cases}$$

It remains to compute the Radon-Nicodym derivative $J = \frac{dQ}{dP}$ of Theorem 4.1. The main step is the computation of the Carleman-Fredholm determinant $d_c(-DK)$. In Appendix A we shall compute the Carleman-Fredholm determinant in a more general case (itself a generalization of a similar result present in [2]) and we obtain that under assumption of Proposition 4.1, if $K_t = \bar{f}(\psi_t)$,

$$d_c(-DK) = det(I - e^{A}\phi_1 g'(\xi_1) + g'(\xi_1))$$

$$\exp\left(-\int_0^1 Tr\left[\bar{f}(\psi_t)e^{-At}g'(\xi_s)e^{At}\right]dt\right)$$

At the end we obtain that:

Theorem 4.2 *Under the assumptions of Proposition 4.1, if Q is defined as in Theorem 4.1 with $K_t = \bar{f}(\psi_t)$, then*

$$J = |det(I - e^{A}\phi_1 g'(\xi_1) + g'(\xi_1))|$$

$$\exp\left[-\frac{1}{2}\int_0^1 Tr\bar{f}(\psi_t)dt - \int_0^1 \bar{f}(\psi_t) \circ dW_t - \frac{1}{2}\int_0^1 |\bar{f}(\psi_t)|^2 dt\right]$$

where $\int_0^1 \bar{f}(\psi_t) \circ dW_t$ is the generalized Stratonovich integral (see [7]).

Chapter 5

The Markov Property

In this section we shall study the Markov property of the solution $\{X_t\}$ of equation (1.1). Following Rozanov [10], we shall first define the splitting σ-algebras and state some basic results about these σ-algebras. In this first part we shall assume that $(\Omega, \mathcal{A}, P)$ is a general probability space and all the σ-algebras considered are sub σ-algebras of $\mathcal{A}$.

Definition 5.1 *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ two σ-algebras; we say that the σ-algebra $\mathcal{B}$ splits $\mathcal{A}_1$ and $\mathcal{A}_2$ (or is splitting) if:*

$$P(A_1 \cap A_2|\mathcal{B}) = P(A_1|\mathcal{B}) \cdot P(A_2|\mathcal{B}) \tag{5.1}$$

for any $A_1 \in \mathcal{A}_1$ and $A_2 \in \mathcal{A}_2$.

As usually, condition (5.1) is equivalent to the following condition in terms of random variables:

$$E(\xi_1 \cdot \xi_2|\mathcal{B}) = E(\xi_1|\mathcal{B}) \cdot E(\xi_2|\mathcal{B}) \tag{5.2}$$

for any $\xi_1 \in L^2(\mathcal{A}_1)$ and $\xi_2 \in L^2(\mathcal{A}_2)$.

We shall now state, without proving (for the proof see [3], pagg. 56-58), two basic results about the splitting σ-algebras :

Proposition 5.1 *If the σ-algebra $\mathcal{B}$ splits $\mathcal{A}_1$ and $\mathcal{A}_2$, then it also splits the σ-algebras $\mathcal{A}_1' = \mathcal{A}_1 \vee \mathcal{B}$ and $\mathcal{A}_2' = \mathcal{A}_1 \vee \mathcal{B}$*

(with a more compact notation:

$$\mathcal{A}_1 \underset{\mathcal{B}}{\overset{||}{-}} \mathcal{A}_2 \quad \Rightarrow \quad \mathcal{A}_1 \vee \mathcal{B} \underset{\mathcal{B}}{\overset{||}{-}} \mathcal{A}_2 \vee \mathcal{B} \quad)$$

Proposition 5.2 *If the σ-algebra $\mathcal{B}_I$ is splitting for $\mathcal{A}_1$ and $\mathcal{A}_2$ then so is every σ -algebra $\mathcal{B} \supseteq \mathcal{B}_0$ of the form*

$$\mathcal{B} = \mathcal{B}_1 \vee \mathcal{B}_2$$

where $\mathcal{B}_1 \subseteq \mathcal{A}_1 \vee \mathcal{B}_0$ and $\mathcal{B}_2 \subseteq \mathcal{A}_2 \vee \mathcal{B}_0$.

Remark 5.1 *Let us note that trivially holds:*

$$\mathcal{A}_1 \underset{\mathcal{B}}{\overset{||}{-}} \mathcal{A}_2 \ \text{and} \ \mathcal{A}_1' \subseteq \mathcal{A}_1 \ , \ \mathcal{A}_2' \subseteq \mathcal{A}_2 \quad \Rightarrow \quad \mathcal{A}_1' \underset{\mathcal{B}}{\overset{||}{-}} \mathcal{A}_2'$$

We shall use these two propositions in the last section of the present thesis to prove, in a very direct way, a general result about the Markov property of the solution of a Stochastic Differential Equation depending on another independent Markov process.

We can now define the two type of Markov properties which are of interest in our framework:

Definition 5.2 *A process $\{X_t; t \in [0,1]\}$ is said to be Markov if for any $t \in [0,1]$, the σ-algebra $\sigma\{X_t\}$ splits the two σ-algebras $\sigma\{X_s; 0 \leq s \leq r, t \leq s \leq 1\}$ and $\sigma\{X_s; r \leq s \leq t\}$*

Definition 5.3 *A process $\sigma\{X_t; t \in [0,1]\}$ is said to be a Markov field if for any $0 \leq r < t \leq 1$, the σ-algebra $\sigma\{X_r, X_s\}$ splits $\sigma\{X_s; 0 \leq s \leq r, t \leq s \leq 1\}$ and $\sigma\{X_s; r \leq s \leq t\}$*

It is possible to prove (see [4]), that any Markov process is a Markov field, but the converse is not true in general. In the case of periodic boundary condition $X_0 = X_1$, we can not clearly expect $\{X_t\}$ to be a Markov process, but it could be a Markov field.

It has been proved (see [9]) that in the Gaussian case (f affine and h linear) the solution is always a Markov field and is moreover a Markov process if $h(x,y) = H_0x + H_1y$ is such that $ImH_0 \cap ImH_1 = \{0\}$.

It is possible to extend the previous result (see Nualart-Pardoux [8]) to the case in which the function h is not linear. Let us consider the equation

$$\begin{aligned} \frac{dX_t}{dt} + AX_t + c &= B\frac{dW_t}{dt} \\ h(X_0, X_1) &= \bar{h} \end{aligned} \tag{5.3}$$

where A, B, h and $\bar{h}$ are as in chapter 2, and $c \in I\!R^k$. We assume that (H.1) holds with $F = I\!R^d$, which implies that (5.3) has the unique solution:

$$X_t = e^{-At}g(\int_0^1 e^{As}(BdW_s - cds)) + e^{-At}\int_0^t e^{As}(BdW_s - Cds) \tag{5.4}$$

Theorem 5.1 *The process $\{X_t; t \in [0,1]\}$ given by (5.4) is a Markov field.*

Proof Let us define

$$\xi_t := \int_0^t e^{As}(BdW_s - cds)$$

For $\varphi \in C_b(I\!R^d)$, and $0 \le s < r < t \le 1$, we have

$$\begin{aligned} E\left[\varphi(X_r)|X_u; u \in [0,1]\backslash(s,t)\right] &= E\left[\varphi(e^{-rA}(g(\xi_1) + \xi_r))|\xi_u; u \in [0,1]\backslash(s,t)\right] \\ &= E\left[\varphi(e^{-rA}(y + \xi_r))|\xi_u; u \in [0,1]\backslash(s,t)\right]_{|y=g(\xi_1)} \end{aligned}$$

$\{\xi_t\}$ is a Gauss-Markov process, hence also a Markov field. Therefore the conditional law of ξ_r given $\sigma(\xi_u : u \in [0,1]\backslash(s,t))$ is Gaussian with mean $c_0 + C_1\xi_s + C_2\xi_t$ and constant covariance, where $c_0 \in I\!R^d$ and C_1, C_2 are $d \times d$ matrices satisfying $C_1 + C_2 = I$. Therefore the quantity

$$E\left[\varphi(e^{-rA}(y+\xi_r))|\xi_u; u \in [0,1]\backslash(s,t)\right]$$

is a function of $y + C_1\xi_s + C_2\xi_t = C_1(y+\xi_s) + C_2(y+\xi_2)$.

We have

$$E\left[\varphi(X_r)|\xi_u; u \in [0,1]\backslash(s,t)\right]$$

is a function of X_s,X_t, hence equals

$$E\left[\varphi(X_r)|X_s, X_t)\right]$$

Q.E.D.

Let us now come back to the general equation. We shall assume that k = d and B = I and we shall consider

$$\begin{aligned} \frac{dX_t}{dt} + f(X_t) &= \frac{dW_t}{dt} \\ h(X_0, X_1) &= \bar{h} \end{aligned} \tag{5.5}$$

Recall that $\Omega = C_0([0,1]; I\!R^d)$, $\mathcal{F}$ is its Borel field, P Wiener measure, $P = QT^{-1}$, $J = \frac{dQ}{dP}$ and $W_t(\omega) = \omega(t)$ the canonical process.

We shall prove in the following Proposition a general measurability condition that has to be satisfied whether $\{X_t; t \in [0,1]\}$, solution of (3.3), is a Markov field. In the following section we shall use this Proposition to prove, in a different way than in the paper of Nualart-Pardoux [8], that the Markov field property of the solution $\{X_t; t \in [0,1]\}$ implies in the scalar case [d = 1] that $f(\cdot)$ is affine. Moreover we shall characterize in dimension $d > 1$ the Markov field property, in the particular case where f has a triangular form (i.e. $\forall i = 1, ..., d$ $f^i(x)$ depends only on $(x_1, ..., x_i)$) and the boundary condition is linear and has a similar triangular form.

Proposition 5.3 *Suppose that (H.1) holds, equation(2.1) has a unique solution, f and g are of class C^2 and (H.2)" holds. Then, if $\{X_t; t \in [0,1]\}$ is a Markov field, we have*

$$Tr((I - e^A\phi(1)g'(\xi_1) + g'(\xi_1))^{-1}(e^A\phi(1)\phi^{-1}(s)\nabla_l\partial f(Y_s)\phi(s)g'(\xi_1)) \in \mathcal{F}_t^e$$

$$t \in [0,1]\ ,\ t \leq s \leq 1$$

where $(\nabla_l\partial f(Y_s))_{ij} := \frac{\partial^2 f^i}{\partial x_l \partial x_j}(Y_s)$, $\mathcal{F}_t^e := \sigma\{Y_0, Y_s; t \leq s \leq 1\}$

Proof : recall that $Y_t = g(\xi_1) + \xi_t$ and $\xi_t = \int_0^t e^{Au} dW_u$. Let $t \in [0,1]$ and define the following three σ -algebras :

$$\begin{aligned}
\mathcal{F}_t &:= \sigma\{Y_0, Y_t\} = \sigma\{g(\xi_1), \xi_t\} \\
\mathcal{F}_t^i &:= \sigma\{Y_s; 0 \leq s \leq t\} = \sigma\{g(\xi_1), \xi_s; 0 \leq s \leq t\} \\
\mathcal{F}_t^e &:= \sigma\{Y_0, Y_s; t \leq s \leq 1\} = \sigma\{\xi_s; t \leq s \leq 1\}
\end{aligned}$$

(where e stand for "exterior" and i for "interior").

Since

$$\begin{aligned}
X(\omega) &= \psi \circ T^{-1}(\omega) \\
Y(\omega) &= \psi(\omega)
\end{aligned}$$

we have, for every non negative measurable function f on Ω, that:

$$\int_\Omega f(X(\omega))dP(\omega) = \int_\Omega f(Y(\omega))dQ(\omega)$$

i.e. the law of $\{X_t; t \in [0,1]\}$ under P is the same as the law of $\{Y_t; t \in [0,1]\}$ under Q. This implies that for any non negative (or equivalently Q integrable) random variable χ which is $\mathcal{F}_t^i$ measurable,

$$\Lambda_x = E_Q(\chi|\mathcal{F}_t^e) = \frac{E_P(\chi J|\mathcal{F}_t^e)}{E_P(J|\mathcal{F}_t^e)}$$

is $\mathcal{F}_t^i$ - measurable.

Recall that :

$$
\begin{aligned}
J &= |det(I - e^{A}\phi(1)g'(\xi_1) + g'(\xi_1))| \times \\
&\times \exp\left[\frac{1}{2}\int_0^1 Tr\bar{f}'(Y_s)ds - \int_0^1 \bar{f}'(Y_s)\circ dW_s - \frac{1}{2}\int_0^1 |\bar{f}'(Y_s)|^2 ds\right] \\
&= Z \cdot J_t^i \cdot J_t^e
\end{aligned}
$$

where

$$\xi_1 = \int_0^1 e^{At}dW_t$$

$$Z = |det(I - e^{A}\phi(1)g'(\xi_1) + g'(\xi_1))|$$

$$J_t^i = \exp\left[\frac{1}{2}\int_0^t Tr\bar{f}'(Y_s)ds - \int_0^t \bar{f}'(Y_s)\circ dW_s - \frac{1}{2}\int_0^t |\bar{f}'(Y_s)|^2 ds\right]$$

$$J_t^e = \exp\left[\frac{1}{2}\int_t^1 Tr\bar{f}'(Y_s)ds - \int_t^1 \bar{f}'(Y_s)\circ dW_s - \frac{1}{2}\int_t^1 |\bar{f}\prime(Y_s)|^2 ds\right]$$

Since the increments of $\{W_t\}$ in any interval I are $\sigma\{Y_t : t \in I\}$ measurable, we conclude that

$$J_t^i \text{ is } \mathcal{F}_t^i\text{-measurable and } J_t^e \text{ is } \mathcal{F}_t^e\text{-measurable .}$$

Then we have

$$\frac{E_P(\chi J|\mathcal{F}_t^e)}{E_P(J|\mathcal{F}_t^e)} = \frac{E_P(\chi Z \cdot J_t^i \cdot J_t^e|\mathcal{F}_t^e)}{E_P(Z \cdot J_t^i \cdot J_t^e|\mathcal{F}_t^e)}$$

$$= \frac{J_t^e}{J_t^e}\frac{E_P(\chi Z \cdot J_t^i|\mathcal{F}_t^e)}{E_P(Z \cdot J_t^i|\mathcal{F}_t^e)}$$

$$= \frac{E_P(\chi Z \cdot J_t^i|\mathcal{F}_t^e)}{E_P(Z \cdot J_t^i|\mathcal{F}_t^e)}$$

Let us choose $\chi_1 = \eta(J_t^i)^{-1}$, for any η non negative , $\mathcal{F}_t^i$ - measurable random variable, and $\chi_2 = (J_t^i)^{-1}$. Clearly χ_1 and χ_2 are both non negative, $\mathcal{F}_t^i$ - measurable random variables: therefore

$$\frac{E_P(\chi_1 Z \cdot J_t^i|\mathcal{F}_t^e)}{E_P(Z \cdot J_t^i|\mathcal{F}_t^e)} = \frac{E_P(\eta Z \cdot J_t^i|\mathcal{F}_t^e)}{E_P(Z \cdot J_t^i|\mathcal{F}_t^e)} \quad \text{is } \mathcal{F}_t - \text{measurable}$$

and

$$\frac{E_P(\chi_2 Z \cdot J_t^i|\mathcal{F}_t^e)}{E_P(Z \cdot J_t^i|\mathcal{F}_t^e)} = \frac{E_P(Z|\mathcal{F}_t^e)}{E_P(Z \cdot J_t^i|\mathcal{F}_t^e)} \quad \text{is } \mathcal{F}_t - \text{measurable.}$$

Moreover their quotient will be $\mathcal{F}_t$ - measurable, i.e.

$$\frac{E_P(\eta Z|\mathcal{F}_t^e)}{E_P(Z|\mathcal{F}_t^e)} \quad \text{is } \mathcal{F}_t - \text{measurable}$$

for any non negative $\mathcal{F}_t^i$ - measurable random variable η. We are therefore interested just in the part of the Carleman-Fredholm determinant $d_c(-DK)$ given by

$$Z = |det(I - e^A\phi(1)g'(\xi_1) + g'(\xi_1))|.$$

Recall that

$$\phi(1) = \phi(1)(\phi(t))^{-1}\phi(t) = \phi(t,1)\phi(t)$$

where $\phi(t)$ is $\mathcal{F}_t^i$ - measurable and $\phi(t,1)$, as the solution of the differential equation

$$\begin{aligned} \frac{d\phi(t,1)}{dt} &= -\phi(t,1)f'(Y_t) \\ \phi(t,i)_{|t=1} &= I \end{aligned} \tag{5.6}$$

is $\mathcal{F}_t^e$ - measurable.

Let us consider, for every deterministic $d \times d$ matrix y, the function

$$y \longrightarrow I - e^A y\phi(1)g'(\xi_1) + g'(\xi_1) \quad ;$$

clearly $det|I - e^A y\phi(1)g'(\xi_1) + g'(\xi_1)|$ is $\mathcal{F}_t^i$ - measurable and it holds that

$$\frac{E_P(\eta Z|\mathcal{F}_t^e)}{E_P(Z|\mathcal{F}_t^e)} = \frac{E_P(\eta det|I - e^A y\phi(1)g'(\xi_1) + g'(\xi_1)||\mathcal{F}_t^e)_{y=\phi(t,1)}}{E_P(det|I - e^A y\phi(1)g'(\xi_1) + g'(\xi_1)||\mathcal{F}_t^e)_{y=\phi(t,1)}}$$

(we use the wellknown result that $E(f(Y,Z)|\mathcal{G}) = E(f(y,Z)|\mathcal{G})_{|y=Y}$ if Y is $\mathcal{G}$ - measurable)

Calling

$$\psi_\omega(y) = \frac{E_P(\eta det|I - e^A y\phi(1)g'(\xi_1) + g'(\xi_1)||\mathcal{F}_t^e)}{E_P(det|I - e^A y\phi(1)g'(\xi_1) + g'(\xi_1)||\mathcal{F}_t^e)} \cdot 1_\Gamma \tag{5.7}$$

where

$$\Gamma := \{(\omega, y) : \ E_P(det|I - e^A y\phi(1)g'(\xi_1) + g'(\xi_1)||\mathcal{F}_t^e) \neq 0\}$$

we have that

$$\frac{E_P(\eta Z|\mathcal{F}_t^e)}{E_P(Z|\mathcal{F}_t^e)} = \psi_\omega(\phi(t,1))$$

is $\mathcal{F}_t$ - measurable.

The process $\{\xi_s = \int_0^s e^{Au} dW_u \ ; \ 0 \leq s \leq 1 \}$ is a Gaussian process and we shall denote by D the derivation with respect to this Gaussian process (see chapter 3). Note that $\mathcal{F}_t^e \subseteq \mathcal{G} = \sigma\{\xi(h_s); t \leq s \leq 1\}$ with $\xi(h) = \int_0^1 h(u) d\xi_u$ and $h_s = 1_{[t,s]}(u)$. Let $K = span\{h_s; t \leq s \leq 1\}$. To evaluate $D[\psi_\omega(\phi(t,1))]$ we shall first construct a sequence $\{(\Omega_k, u_k)\} \subseteq \mathcal{F} \times I\!D^{1,2}_{loc}$ with :

(a) $\Omega_k \uparrow \Omega$

(b) $u_k = \psi$ a.s on Ω_k.

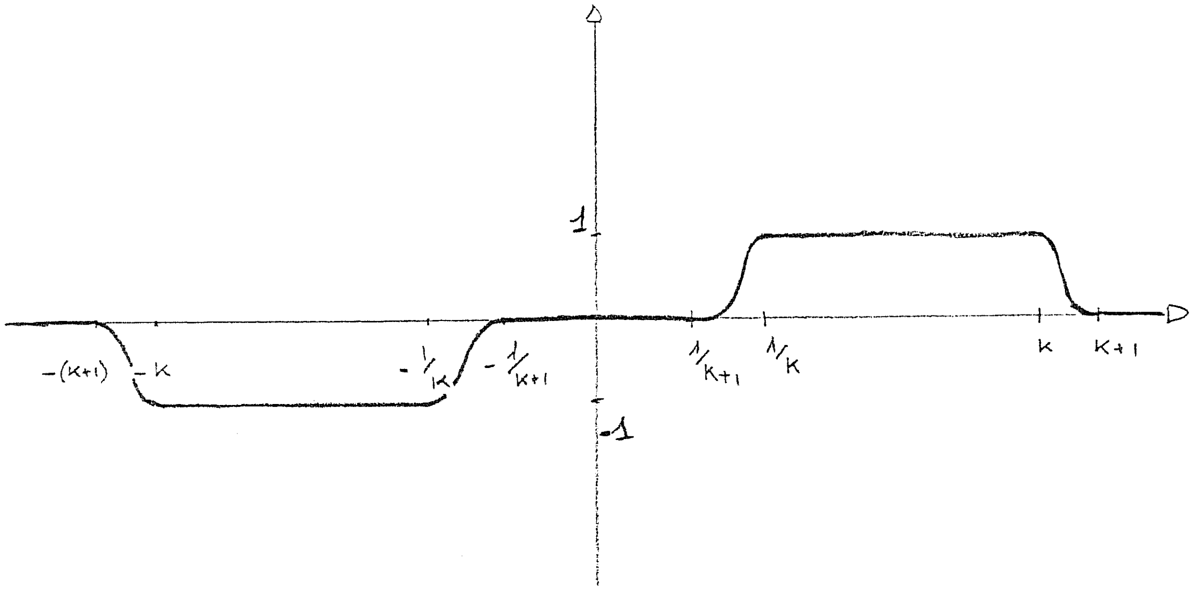


Figure 5-1: $\varphi_k(x)$

Let us consider the deterministic function:

$$\begin{array}{ccc} (y,z,x) & \longrightarrow & det(I - e^{A}yzg'(x) + g'(x)) = \Delta(y,z,x) \\ M_d \times M_d \times I\!R^d & \longrightarrow & I\!R. \end{array} \tag{5.8}$$

Clearly this function is continuous. Consider the increasing sequence of open sets

$$A_k := \{(y,z,x) : \frac{1}{k} < |\Delta(y,z,x)| < k\}$$

We have

$$A_k \uparrow \{(y,z,x) : \Delta(y,z,x) \neq 0\}.$$

Let now φ_k be a $C_b^\infty(I\!R)$ (all the partial derivatives are bounded) such that

$$\varphi_k(x) = \begin{cases} 1 & \text{if } \frac{1}{k} < x < k \\ \\ -1 & \text{if } -k < x < -\frac{1}{k} \\ \\ 0 & \text{if } x \neq [\frac{1}{k+1}, k] \cup [-(k+1), -\frac{1}{k+1}] \end{cases}$$

Moreover, let us $\zeta_k(y) \in C_b^\infty(M_d)$:

$$\zeta_k(y) = \begin{cases} y & \text{if } \|y\| \leq k \\ \\ 0 & \text{if } \|y\| > k \end{cases}$$

and in a same way $\chi_k(x) \in C_b^\infty(\mathbb{R}^d)$. Defining, for η smooth random variable (i.e. $\eta = \varphi(g(\xi_1), \xi_{s_1}, \cdots, \xi_{s_d})$ where $s_1, \cdots, s_d \in [t, 1]$ and $\varphi \in C_b^\infty(\mathbb{R}^{k-1})$) :

$$u_k(y, \omega) := \frac{E[\eta\zeta_k(y)\chi_k(\xi_1)\chi_k(\phi(t))\varphi_k(\Delta(y, \phi(t), \xi_1)) \cdot \Delta(y, \phi(t), \xi_1)|\mathcal{F}_t^e}{E[\zeta_k(y)\chi_k(\xi_1)\chi_k(\phi(t))\varphi_k(\Delta(y, \phi(t), \xi_1)) \cdot \Delta(y, \phi(t), \xi_1)|\mathcal{F}_t^e}$$

it is now possible to prove that $u_k(y, \omega)$ satisfies all the conditions of Lemma 3.1 in chapter 3 . Therefore we obtain that

$$D_s(u_k(\phi(t, 1)), \omega)) = (D_s u_k)(\phi(t, 1)) + \sum_{i,j=1}^{d} \frac{\partial u_k}{\partial y_{ij}(\phi(t, 1))} D_s\phi_{ij}(t, 1))$$

Consider the set

$$\Omega_k = \{\omega : |\phi(t, 1)| \leq k, |\phi(t)| \leq k, |\xi_1| \leq k, (\phi(t, 1), \phi(t), \xi_1) \in A_k\}.$$

Clearly

$$\psi_\omega(\phi(t, 1)) = u_k(\phi(t, 1), \omega) \quad a.s. \;\; on \;\; \Omega_k$$

and $\Omega_k \uparrow \Omega$ as $k \to +\infty$. Therefore we obtain that $\psi_\omega(\phi(t, 1)) \in \mathbb{D}_{loc}^{1,2}$ and

$$D_s(\psi_\omega(\phi(t, 1), \omega)) = (D_s\psi_\omega)(\phi(t, 1)) + \sum_{i,j=1}^{d} \frac{\partial \psi_\omega}{\partial y_{ij}(\phi(t, 1))} D_s\phi_{ij}(t, 1) \qquad (5.9)$$

Since $\psi(\phi(t, 1))$ and $\psi(y)$ are $\mathcal{F}_t^e$ - measurable, from Lemma 3.2,(ii) and condition (5.7) we obtain

$$D_s(\psi_\omega(\phi(t, 1), \omega)) = A_t^1 + A_t^2 1_{\{s \leq t\}}$$

$$(D_s\psi_\omega)(\phi(t, 1), \omega) = \bar{A}_t^1 + \bar{A}_t^2 1_{\{s \leq t\}}.$$

Therefore

$$\frac{d}{ds}(D_s(\psi_\omega(\phi(t,1),\omega))) = 0 \quad t \le s \le 1$$

$$\frac{d}{ds}(D_s(\psi_\omega)(\phi(t,1),\omega)) = 0 \quad t \le s \le 1.$$

From (5.9) we obtain that

$$\frac{d}{ds}\left(\sum_{i,j=1}^{d} \frac{\partial \psi_\omega}{\partial y_{ij}(\phi(t,1))} D_s \phi_{ij}(t,1)\right) = 0 \quad t \le s \le 1. \tag{5.10}$$

Let us compute $\sum_{i,j=1}^{d} \frac{\partial \psi_\omega}{\partial y_{ij}(\phi(t,1))} D_s \phi_{ij}(t,1)$ for $0 \le s \le 1$ and $0 \le t \le 1$. Since D_s^k commute with the integration, we have, for any $k = 1, ..., d$

$$\begin{aligned} D_s^k \phi_{ij}(t,1) &= -\int_t^1 D_s^k[\phi_{i\alpha}(u,1)\frac{\partial f^\alpha}{\partial x_j}(Y_u)]du \\ &= -\int_t^1 [D_s^k \phi_{i\alpha}(u,1)]\frac{\partial f^\alpha}{\partial x_j}(Y_u)du + \\ &\quad -\int_t^1 \phi_{i\alpha}(t,1)\sum_{l=1}^{d} \frac{\partial}{\partial x_l}\frac{\partial f^\alpha}{\partial x_j}(Y_u) D_s^k(Y_u^l)du \end{aligned}$$

$$D_s^k \phi_{ij}(t,1)_{|t=1} = 0$$

Therefore, we obtain that

$$\begin{aligned} D_s^k \phi_{ij}(t,1) &= -\int_t^1 \phi_{i\alpha}(u,1)\sum_{l=1}^{d} \frac{\partial}{\partial x_l}\frac{\partial f^\alpha}{\partial x_m}(Y_u) D_s^k(Y_u^l)\phi_{mj}(t,u)du \\ &= -\sum_{l=1}^{d}\int_t^1 [\phi_{i\alpha}(u,1)]\frac{\partial}{\partial x_l}\frac{\partial f^\alpha}{\partial x_m}(Y_u)\phi_{mj}(t,u) D_s^k(Y_u^l)du \end{aligned}$$

(notice that for every l and k fixed, $D_s^k(Y_u^l)$ is a scalar that commutes with $\phi_{mj}(t,u)$). Since

$$D_s^k(Y_u^l) = (e^{-At})_{l\beta}[\frac{\partial g^\beta}{\partial x_k}(\int_0^1 e^{Ar} dW_r) + (1_{\{s \le t\}})\delta_{\beta k}]$$

(recall that D_s^k is the derivative with respect to ξ_s)

$$D_s^k \phi_{ij}(t,1) = (B_1^k(s,t))_{ij} + (B_1^k(s,t))_{ij}$$

where

$$(B_1^k(s,t))_{ij} = -\sum_{l=1}^{d} \int_t^1 [\phi_{i\alpha}(u,1)] \frac{\partial}{\partial x_l} \frac{\partial f^\alpha}{\partial x_m}(Y_u) \phi_{mj}(t,u)(e^{-At})_{l\beta} \frac{\partial g^\beta}{\partial x_k}(\xi_1) du$$

$$(B_2^k(s,t))_{ij} = -\sum_{l=1}^{d} \int_t^1 [\phi_{i\alpha}(u,1)] \frac{\partial}{\partial x_l} \frac{\partial f^\alpha}{\partial x_m}(Y_u) \phi_{mj}(t,u)(e^{-At})_{l\beta} \mathbf{1}_{\{s \leq t\}} \delta_{\beta k} du$$

When $t \leq s \leq 1$ we obtain that

$$(B_2^k(s,t))_{ij} = -\sum_{l=1}^{d} \int_s^1 [\phi_{i\alpha}(u,1)] \frac{\partial}{\partial x_l} \frac{\partial f^\alpha}{\partial x_m}(Y_u) \phi_{mj}(t,u)(e^{-At})_{lk} du,$$

and condition (5.10) implies

$$\frac{d}{ds} \Big(\sum_{i,j=1}^{d} \frac{\partial \psi_\omega}{\partial y_{ij}(\phi(t,1))} \{ B_1^k(s,t) + B_2^k(s,t) \} \Big) = 0$$

for $t \leq s \leq 1$ and $k = 1, \ldots d$.

i.e.

$$\sum_{i,j=1}^{d} \frac{\partial \psi_\omega}{\partial y_{ij}(\phi(t,1))} [\phi_{i\alpha}(u,1)] \frac{\partial}{\partial x_l} \frac{\partial f^\alpha}{\partial x_m}(Y_u) \phi_{mj}(t,u)(e^{-At})_{l\delta} = 0$$

for $\delta = 1, \ldots d$.

Multiplying by e^{As}, we obtain:

$$\sum_{i,j=1}^{d} \frac{\partial \psi_\omega}{\partial y_{ij}(\phi(t,1))} [\phi_{i\alpha}(u,1)] \frac{\partial}{\partial x_l} \frac{\partial f^\alpha}{\partial x_m}(Y_u) \phi_{mj}(t,u) = 0$$

for $l = 1, \ldots d$.

We have now to compute $\frac{\partial\psi_\omega}{\partial y_{ij}}$; let us recall that

$$\psi_\omega(y) = \frac{E_P(\eta|det(\Delta(y))||\mathcal{F}^e_t)}{E_P(|det(\Delta(y))||\mathcal{F}^e_t)} \cdot \mathbf{1}_\Gamma$$

where η is a non negative smooth $\mathcal{F}^i_t$ - measurable random variable and

$$\Delta(y) = I - e^A y\phi(1)g'(\xi_1) + g'(\xi_1)$$

$(y = (y_{ij}))$.

Since $\Delta(y)$ is linear in y, and holds that

$$d(det(A)) = (detA)Tr(A^{-1}dA)$$

it holds :

$$\sum_{i,j=1}^{d} \frac{\partial\psi_\omega}{\partial y_{ij}(\phi(t,1))}[\phi_{i\alpha}(u,1)]\frac{\partial}{\partial x_l}\frac{\partial f^\alpha}{\partial x_m}(Y_u)\phi_{mj}(t,u) =$$

$$= \frac{1}{[E_P(|det(\Delta(\phi(t,1)))||\mathcal{F}^e_t)]^2} \cdot [E_P(\eta|det(\Delta(\phi(t,1)))|Tr(\Theta)|\mathcal{F}^e_t)\times$$

$$\times E_P(|det(\Delta(\phi(t,1)))||\mathcal{F}^e_t) - E_P(\eta|det(\Delta(\phi(t,1)))||\mathcal{F}^e_t) \times E_P(|det(\Delta(\phi(t,1)))|Tr(\Theta)|\mathcal{F}^e_t)]$$

where

$$\Theta = \Delta^{-1}(-(e^A)_{\beta i}\phi_{i\tau}(1)\phi^{-1}_{\tau\alpha}(s)\frac{\partial}{\partial x_l}\frac{\partial f^\alpha}{\partial x_m}(Y_s)\phi_{m\gamma}(s)\phi^{-1}_{\gamma j}(t)\phi_{j\delta}(t)\frac{\partial g^\delta}{\partial x_p}(\xi_1).$$

Since

$$\sum_{i,j=1}^{d} \frac{\partial\psi_\omega}{\partial y_{ij}(\phi(t,1))}[\phi_{i\alpha}(u,1)]\frac{\partial}{\partial x_l}\frac{\partial f^\alpha}{\partial x_m}(Y_u)\phi_{mj}(t,u) = 0$$

for $t \leq s \leq 1$ and $l = 1, ...d.$ and the previous equation holds for every non negative smooth random variable η that is $\mathcal{F}^i_t$ -measurable (and trivially for every non negative smooth random variable η that is $\mathcal{F}^e_t$ - measurable), then it holds for every non negative smooth random variable. We obtain:

$$Tr(\Theta) \cdot E_P(|det(\Delta(\phi(t,1)))||\mathcal{F}_t^e) = E_P(Tr(\Theta)|det(\Delta(\phi(t,1)))||\mathcal{F}_t^e)$$

It implies that (in a more compact notation):

$$Tr((I - e^A\phi(1)g'(\xi_1) + g'(\xi_1))^{-1}(e^A\phi(1)\phi^{-1}(s)\nabla_l\partial f(Y_s)\phi(s)g'(\xi_1))$$

is $\mathcal{F}_t^e$-measurable for any $l = 1, ..., d$ and $t \leq s \leq 1$.

Chapter 6

Necessary conditions

6.1 A particular class for $d \geq 1$

In the present section we shall consider a particular class of nonlinear stochastic differential equations with linear boundary conditions. We are interested here in finding a class of problems for which the previous measurability condition, necessary if the solution $\{X_t; t \in [0,1]\}$ is a Markov field, would be more tractable.

Since the set of triangular matrices $d \times d$ (i.e. the matrices $A = (a_{ij})$ for which $a_{ij} = 0 \quad \forall i < j$) is closed under the sum and the multiplication of matrices, and the inverse of a non-singular triangular matrix is itself a triangular matrix, we shall prove that, if f is a triangular function, and the boundary conditions are also triangular, the measurability condition of Proposition 5.1 is relatively simpler.

We shall consider the following stochastic differential equation:

$$\frac{dX_t}{dt} + f(X_t) = \frac{dW_t}{dt} \tag{6.1}$$

where $t \in [0,1]$, $\{W_t : t \in [0,1]\}$ is a standard d-dimensional Brownian motion ($d \geq 1$) and $f : I\!R^d \longrightarrow I\!R^d$ is a measurable function ($A = 0, B = I$).

We shall assume that $f(x) = (f^1(x), ..., f^d(x))$ has a triangular form, i.e. $\forall i = 1, ..., d$,

$f^i(x)$ is a function of $x_1, ..., x_i$ only and that satisfies

$$(H.6) \qquad |f(x_1, ..., x_{k-1}, x)| \leq L_k(x_1, ..., x_{k-1})(1 + |x|)$$

$$(H.7) \qquad |f(x_1, ..., x_{k-1}, x) - f(x_1, ..., x_{k-1}, y)| \leq L_k(x_1, ..., x_{k-1})|x - y|$$

where $L_k : \mathbb{R}^{k-1} \longrightarrow \mathbb{R}_+$ is measurable and locally bounded.

The boundary condition, that we consider, will be linear, i.e.

$$h(X_0, X_1) = H_0 X_0 + H_1 X_1 = \bar{h} \in \mathbb{R}^d$$

and we shall assume, in addition:

$(B.1)$ $\quad H_0$ and H_1 are triangular $d \times d$ matrices;

(i.e., putting $H_0 = (h^0_{ij})$ and $H_1 = (h^1_{ij})$, $h^\alpha_{ij} = o \ \forall i < j, \alpha = 1, 2$)

$(B.2)$ $\quad H_0 + H_1$ is invertible and $h^0_{ii} \cdot h^1_{ii} > 0 \ \forall i = 1, ..., d$

From (B.2) we obtain that condition $(H.1)$ of chapter 2 is satisfied with

$$g(z) = (H_0 + H_1)^{-1}(\bar{h} - H_1 z)$$

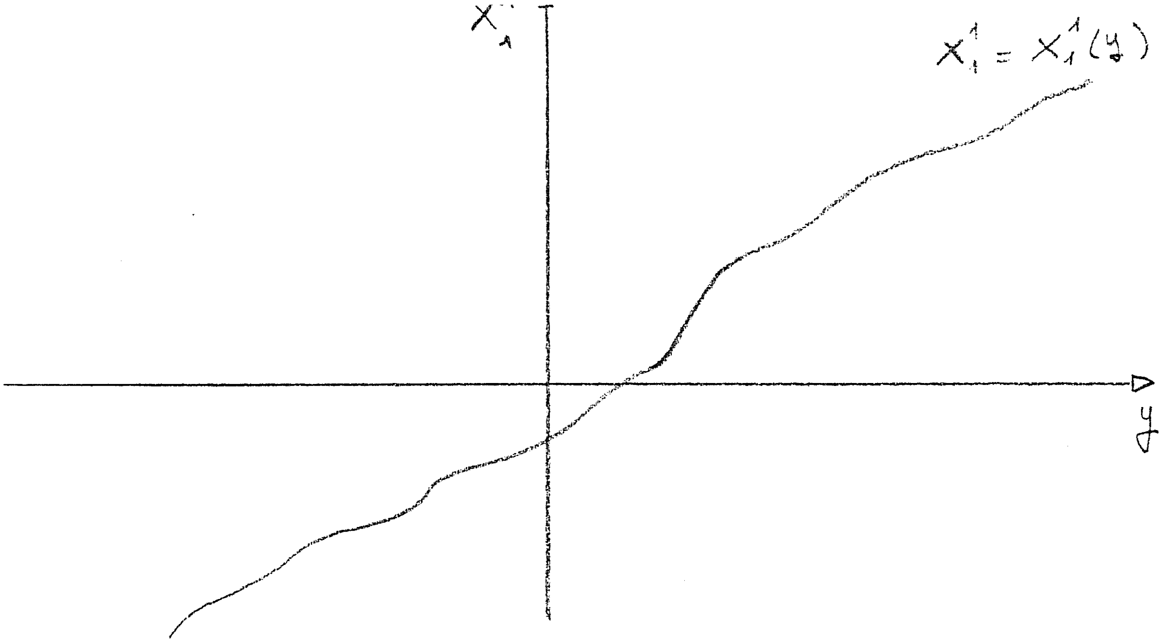
and that

$$g'(z) = -(H_0 + H_1)^{-1} H_1$$

is a triangular matrix, with $-1 < g'_{ii}(z) < 0 \ \forall i = 1, ..., d$. Let us prove the following:

Proposition 6.1 *Under assumptions $(H.6), (H.7), (B.1)$ and $(B.2)$ our problem admits an unique solution.*

Proof : Let us consider the first equation



$$\frac{dX_t^1}{dt} + f^1(X_t^1) = \frac{dW_t^1}{dt}$$

$$h_{11}^0 X_0^1 + h_{11}^1 X_1^1 = \bar{h}_1 \tag{6.2}$$

$t \in [0,1]$, $\bar{h}_1 \in I\!R$.

Notice that this is a closed equation. By assumptions $(H.6)$ and $(H.7)$ we have that the previous equation admits, for every fixed initial value y a unique solution $X_t^1(y)$. By the uniqueness theorem the function

$$y \longrightarrow X_1^1(y)$$

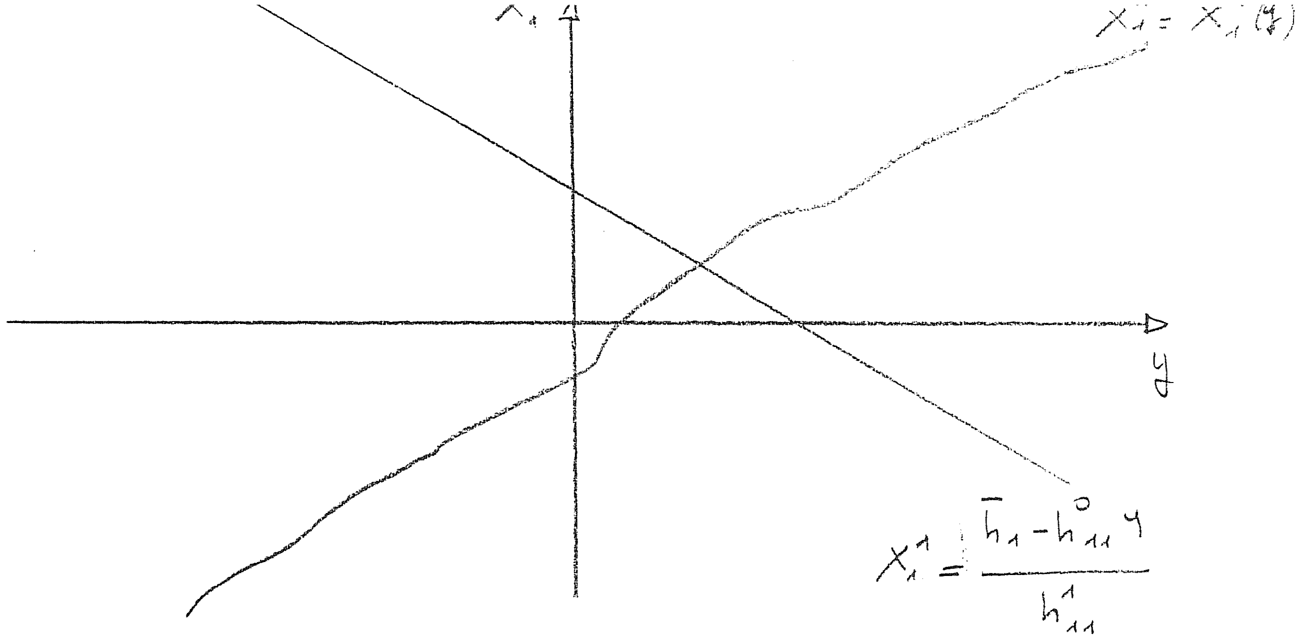
is strictly increasing. By assumption (B.2), $h_{11}^0 \cdot h_{11}^1 > 0$ and therefore the linear map

$$y \longrightarrow X_1^1(\omega) = \frac{\bar{h}_1 - h_{11}^0 y}{h_{11}^1}$$

is strictly decreasing. Consequently there exists a unique point $y = X_0^1(\omega)$ such that

$$X_1^1(X_0^1(\omega)) = \frac{\bar{h}_1 - h_{11}^0 X_0^1(\omega)}{h_{11}^1}$$

In this way we obtain that equation (6.1) admits a unique solution $X_t^1 = X_t^1(X_0^1)$.



Consider now the second equation

$$
\begin{aligned}
\frac{dX_t^2}{dt} + f^2(X_t^1, X_t^2) &= \frac{dW_t^2}{dt} \\
h_{21}^0 X_0^1 + h_{22}^0 X_0^2 + h_{21}^1 X_1^1 + h_{22}^1 X_1^2 &= \bar{h}_2
\end{aligned}
\qquad (6.3)
$$

$t \in [0,1]$, $\bar{h}_2 \in I\!R$.

Since $\{X_t^1 : t \in [0,1]\}$ is now the unique solution of equation (6.2) and therefore is fixed, our boundary condition is, for every $\omega \in \Omega$:

$$h_{22}^0 X_0^2 + h_{22}^1 X_1^2 = \bar{h}_2 - h_{21}^0 X_0^1 - h_{21}^1 X_0^1$$

For every initial condition X_0^2, equation (6.3) has a unique solution ($\{X_t^1\}$ is now a fixed process), again using $(H.9)$ and $(H.10)$. We are in the same situation as before and again we obtain that equation (6.3) has a unique solution. It is straigthforward that the same proof can be applied to each equation of our system and therefore the Proposition is proved. Q.E.D.

We want now apply Propositio 5.1 of previous section.

Theorem 6.1 *Under assumption $(H.6), (H.7), (B.1)$ and $(B.2)$, f is of class C^3 while g of class C^2, if $\{X_t : t \in [0,1]\}$ is a Markov field, then $\forall i = 1, ..., d$,*

$\exists u_i(x_1,...,x_{i-1}) \in C^1(I\!R^{i-1})$ *and* $a_i, b_i \in I\!R$ *such that*

$$f^i(x_1,...,x_i) = a_i x_i + b_i + u_i(x_1,...,x_{i-1})$$

Proof : Proceeding as in the previous section, let us fix $t \in (0,1)$ and set $\mathcal{F}^i_t = \sigma\{Y_s : 0 \le s \le t\}$, $\mathcal{F}^e_t = \sigma\{Y_0, Y_s : t \le s \le 1\}$. From Proposition 6.1 and Proposition 2.1 we have that T is a bijection, and since $h^0_{ii} \cdot h^1_{ii} > 0$,

$$det(I - \phi(1)g'(\xi_1) + g'(\xi_1)) = \prod_{i=1}^{d}(1 - \phi_{ii}(1)g'_{ii} + g'_{ii}) > 0$$

(we shall prove that $\phi(1)$ is itself triangular). We can therefore apply Proposition 5.1 and we obtain that

$$Tr((I - \phi(1)g'(\xi_1) + g'(\xi_1))^{-1}(\phi(1)\phi^{-1}(s)\nabla_l \partial f(Y_s)\phi(s)g'(\xi_1)) \in \mathcal{F}^e_t$$

for $t \le s \le 1$

Since

$$\partial f = (\frac{\partial f^i}{\partial x_j})_{i,j=1,...,d}$$

is a triangular matrix, we obtain that the solution of the differential equation:

$$\begin{aligned} \frac{d}{dt}\phi(t) &= -f'(Y_t)\phi(t) \\ \phi(0) &= I \end{aligned} \tag{6.4}$$

is itself a triangular matrix and furthermore that

$$\phi_{ii}(t) = \exp(-\int_0^t \frac{\partial f^i}{\partial x_i}(Y_r)dr) \qquad \forall i = 1,...,d$$

To compute the previous trace, let us recall that the product of triangular matrices and the inverse of a triangular matrix (non-singular) are triangular matrices. Therefore it will be enough to compute the value of the diagonal elements of the matrices involved. We have

$$(I - \phi(1)g'(\xi_1) + g'(\xi_1))_{ii}^{-1} = [1 - (\phi_{ii}(1) - 1)g'_{ii}]^{-1}$$

$$(\phi(1)\phi^{-1}(s)\nabla_l \partial f(Y_s)\phi(s)g'(\xi_1))_{ii} = \phi_{ii}[\frac{\partial^2}{\partial x_l \partial x_i} f_i(Y_s)]g'_{ii}$$

$i = 1, ..., d$, $l \leq i$ and $t \leq s \leq 1$.

Then our general condition will in the present case:

$$\sum_{i=1}^{d} \frac{\phi_{ii}\frac{\partial^2}{\partial x_l \partial x_i} f_i(Y_s)g'_{ii}}{1 - (\phi_{ii}(1) - 1)g'_{ii}} \in \mathcal{F}_t^e$$

$\forall t \in (0,1)$, $t \leq s \leq 1$ and $l = 1, ..., d$.

Putting

$$\alpha_i(x) = \frac{x g'_{ii}}{1 - [\phi_{ii}(1) - 1]g'_{ii}}$$

we can write

$$\begin{pmatrix} \alpha_1(\phi_{11}(1)), & \cdots & \alpha_d(\phi_{dd}(1)) \end{pmatrix} \begin{pmatrix} \frac{\partial^2 f_1}{\partial x_1^2}(Y_s) & 0 & \cdots & 0 \\ \frac{\partial^2 f_2}{\partial x_1 \partial x_2}(Y_s) & \frac{\partial^2 f_2}{\partial x_2^2}(Y_s) & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ \frac{\partial^2 f_d}{\partial x_1 \partial x_d}(Y_s) & \frac{\partial^2 f_d}{\partial x_2 \partial x_d}(Y_s) & \cdots & \frac{\partial^2 f_d}{\partial x_d^2}(Y_s) \end{pmatrix} \qquad (6.5)$$

is $\mathcal{F}_t^e$ -measurable.

From the last column of the matrix, we obtain that:

$$\alpha_d(\phi_{dd})(1)\frac{\partial^2 f_d}{\partial x_d^2}(Y_s(\omega)) \in \mathcal{F}_t^e \qquad 0 \leq t \leq 1, \quad t \leq s \leq 1$$

Let us suppose that there exists $\bar{x} \in \mathbb{R}^d$ such that

$$\frac{\partial^2 f_d}{\partial x_d^2}(\bar{x}) \neq 0$$

From the continuity of $\frac{\partial^2 f_d}{\partial x_d^2}$, we obtain that there exists U open set in $\mathbb{R}^d$ such that

$$\frac{\partial^2 f_d}{\partial x_d^2}(x) \neq 0 \qquad \forall x \in U$$

Define

$$G^d := \{\omega : \frac{\partial^2 f_d}{\partial x_d^2}(Y_t(\omega)) \neq 0\} \in \mathcal{F}_t^e.$$

Since in the present case the law of the random vector

$$Y_t = g(W_1) + W_t$$

has $\mathbb{R}^d$ as its support, we have:

$$P(G^d) > 0.$$

We have therefore:

$$1_{G^d}\alpha_d(\phi_{dd}(1)) \in \mathcal{F}_t^e$$

and, since $\phi_{dd}(1) = \exp[-\int_0^1 \frac{\partial f_d}{\partial x_d}(Y_r)dr]$, then

$$1_{G^d}\int_0^1 \frac{\partial f_d}{\partial x_d}(Y_r)dr \in \mathcal{F}_t^e$$

In the present case

$$\mathcal{F}_t^e = \sigma(\eta(h_s); t \leq\leq s \leq 1)$$

where $\eta_i(h) = \int_0^1 h(t)dW_t$ and $h_s(t) = 1_{[t,1]}(s)$ and so

$$1_{G^d}(\frac{d}{dt}(D_u^k \int_0^1 \frac{\partial f_d}{\partial x_d}(Y_r)dr)) = 0 \qquad a.s.$$

for $0 \leq u \leq t$, $1 \leq k \leq d$. We have

$$
\begin{aligned}
D_u^k[\int_0^1 \frac{\partial f_d}{\partial x_d}(Y_r)dr)] &= -\int_0^1 D_u^k(\frac{\partial f_d}{\partial x_d}(Y_r))dr \\
&= \int_0^1 \sum_{l=1}^d \frac{\partial^2 f_d}{\partial x_l \partial x_d}(Y_r) D_u^d Y_r^l dr \\
&= \int_0^1 \sum_{l=1}^d \frac{\partial^2 f_d}{\partial x_l \partial x_d}(Y_r) g'_{lk} dr + \int_u^1 \sum_{l=1}^d \frac{\partial^2 f_d}{\partial x_l \partial x_d}(Y_r) \delta_{lk} dr
\end{aligned}
$$

For k = d, we obtain

$$
\begin{aligned}
\frac{d}{dt}[D_u^d[\int_0^1 \frac{\partial f_d}{\partial x_d}(Y_r)dr)]] &= \frac{d}{du}\left(\int_u^1 \frac{\partial^2 f_d}{\partial x_d^2}(Y_r)dr\right) \\
&= -\frac{\partial^2 f_d}{\partial x_d^2}(Y_u) = 0 \qquad 0 \leq u \leq t
\end{aligned}
$$

almost surely on G^d. But this is possible only if $P(G^d) = 0$ and this leads to a contradiction. So we have that

$$
\frac{\partial^2 f_d}{\partial x_d^2}(x) = 0 \qquad \forall x \in I\!R^d.
$$

and so , $\exists \alpha(x_1, \cdots, x_{d-1})$ and $\beta(x_1, \cdots, x_{d-1})$ such that:

$$
f_d(x) = \alpha(x_1, \cdots, x_{d-1})x_d + \beta(x_1, \cdots, x_{d-1}).
$$

We have therefore that it holds:

$$
\begin{pmatrix} \alpha_1(\phi_{11}(1)), & \cdots & \alpha_d(\phi_{dd}(1)) \end{pmatrix}
\begin{pmatrix}
\frac{\partial^2 f_1}{\partial x_1^2}(Y_s) & \cdots & 0 & 0 \\
. & \cdots & . & . \\
\frac{\partial^2 f_{d-2}}{\partial x_1 \partial x_{d-2}}(Y_s) & \cdots & 0 & 0 \\
\frac{\partial^2 f_{d-1}}{\partial x_1 \partial x_{d-1}}(Y_s) & \cdots & \frac{\partial^2 f_{d-1}}{\partial x_{d-1}^2}(Y_s) & 0 \\
\frac{\partial^2 f_d}{\partial x_1 \partial x_d}(Y_s) & \cdots & \frac{\partial^2 f_d}{\partial x_{d-1} \partial x_d}(Y_s) & 0
\end{pmatrix}
\qquad (6.6)
$$

Consider now the $d-1$ column: we have

$$\alpha_{d-1}(\phi_{d-1d-1})\frac{\partial^2 f_{d-1}}{\partial x_{d-1}^2}(Y_s) + \alpha_d(\phi_{dd})\frac{\partial^2 f_d}{\partial x_{d-1}\partial x_d}(Y_s) \in \mathcal{F}_t^e$$

$t \leq s \leq 1$.

From the previous condition on f_d, we have that $\frac{\partial^2 f_d}{\partial x_{d-1}\partial x_d}(x)$ depends only on the first $d-1$ variables. Let us suppose that $\exists\ \bar{x} \in I\!R$ such that

$$\frac{\partial^2 f_{d-1}}{\partial x_{d-1}^2}(\bar{x}) \neq 0 \qquad or \qquad \frac{\partial^2 f_d}{\partial x_{d-1}\partial x_d}(\bar{x}) \neq 0.$$

Defining:

$$G^{d-1} := \{\omega : \left(\frac{\partial^2 f_{d-1}}{\partial x_{d-1}^2}(Y_t(\omega))\right)^2 + \left(\frac{\partial^2 f_d}{\partial x_{d-1}\partial x_d}(Y_t(\omega))\right)^2 \neq 0\}$$

again from the fact that the law of Y_t has $I\!R^d$ as its support, we obtain

$$P(G^{d-1}) > 0$$

We have

$$\mathbf{1}_{G^d}\{\alpha_{d-1}(\phi_{d-1d-1}(1))\frac{\partial^2 f_{d-1}}{\partial x_{d-1}^2}(Y_t) + \alpha_d(\phi_{dd}(1))\frac{\partial^2 f_d}{\partial x_{d-1}\partial x_d}(Y_t)\} \ \in \mathcal{F}_t^e$$

Proceeding as before we have:

$$\mathbf{1}_{G^d}\left(\frac{d}{dt}D_u^k[\alpha_{d-1}(\phi_{d-1d-1}(1))\frac{\partial^2 f_{d-1}}{\partial x_{d-1}^2}(Y_t) + \alpha_d(\phi_{dd}(1))\frac{\partial^2 f_d}{\partial x_{d-1}\partial x_d}(Y_t)]\right) = 0$$

$0 \leq u \leq t$ and $1 \leq k \leq d-1$.

The first term , for $k = d-1$, is:

$$\frac{d}{dt}D_u^k\left[\alpha_{d-1}(\phi_{d-1d-1}(1))\frac{\partial^2 f_{d-1}}{\partial x_{d-1}^2}(Y_t)\right] = \frac{d}{dt}\Big[\alpha'_{d-1}(\phi_{d-1d-1}(1))D_u^{d-1}\phi_{d-1d-1}(1)\frac{\partial^2 f_{d-1}}{\partial x_{d-1}^2}(Y_t) +$$

$$+\alpha_{d-1}(\phi_{d-1d-1}(1))\sum_{l=1}^{d-1}\frac{\partial^3 f_{d-1}}{\partial x_l \partial x_{d-1}^2}(Y_t)D_u^{d-1}Y_t^l\Bigg]$$

$$= \alpha'_{d-1}(\phi_{d-1d-1}(1))\phi_{d-1d-1}(1)\left(\frac{\partial^2 f_{d-1}}{\partial x_{d-1}^2}(Y_u)\right)\left(\frac{\partial^2 f_{d-1}}{\partial x_{d-1}^2}(Y_t)\right)$$

where

$$\alpha'_{d-1}(x) = \frac{g'_{d-1d-1} + (g'_{d-1d-1})^2}{[1-[x-1]g'_{d-1d-1}]^2}$$

$\forall 0 < x < 1$.

Since

$$g'_{d-1d-1} = -\frac{h^1_{d-1d-1}}{h^0_{d-1d-1} + h^1_{d-1d-1}}$$

we obtain that

$$g'_{d-1d-1} + (g'_{d-1d-1})^2 = -\frac{h^0_{d-1d-1}h^1_{d-1d-1}}{(h^0_{d-1d-1} + h^1_{d-1d-1})^2}$$

Therefore, by assumption (B.2),

$$\alpha'_{d-1}(x) < 0 \qquad\qquad \forall 0 < x < 1.$$

The same calculation on the second term gives, for $u = t$:

$$\alpha'_{d-1}(\phi_{d-1d-1}(1))\phi_{d-1d-1}(1)\left[\frac{\partial^2 f_{d-1}}{\partial x_{d-1}^2}(Y_u)\right]^2 +$$

$$+\alpha'_d(\phi_{dd}(1))\phi_{dd}(1)\left[(\frac{\partial^2 f_d}{\partial x_d^2}(Y_u))\right]^2 = 0$$

Since

$$\phi_{ii}(1)(\omega) > 0 \qquad a.e.\ \ on \quad \Omega \qquad \text{for} i = d-1, d$$

and

$$\alpha_i(\phi_{ii}(1)(\omega)) < 0 \qquad a.e.\ \ on \quad \Omega \qquad \text{for} i = d-1, d$$

we have that

$$P\{\omega : \left(\frac{\partial^2 f_{d-1}}{\partial x_{d-1}^2}(Y_t(\omega))\right)^2 + \left(\frac{\partial^2 f_d}{\partial x_{d-1}\partial x_d}(Y_t(\omega))\right)^2 = 0\} = 1.$$

This is possible if and only if $P(G^{d-1}) = 0$ and therefore both $\frac{\partial^2 f_{d-1}}{\partial x_{d-1}^2}(x)$ and $\frac{\partial^2 f_d}{\partial x_{d-1}\partial x_d}(x)$ are identically zero.

It is clear that the same computation can be now done for the $d-2$-th column of the previous matrix. At the end we shall have:

$$\frac{\partial^2 f_i}{\partial x_l \partial x_i}(x) \equiv 0 \qquad \forall i = 1, \cdots, d; l \leq i; \forall x \in I\!R^i$$

and immediately that there exist $a_i, b_i \in I\!R$ and $u_i(x_1, \cdots, x_{i-1}) \in C^1(I\!R^{i-1})$ such that

$$f_i(x) = a_i x_i + b_i + u_i(x_1, \cdots, x_{i-1}) \qquad \forall i = 1, \cdots, d \; ; \; x \in I\!R^d$$

Q.E.D.

6.2 The scalar case

In the present section we shall study the equation

$$\begin{aligned} \frac{dX_t}{dt} + f(X_t) &= \frac{dW_t}{dt} \\ h(X_0, X_1) &= \bar{h} \end{aligned} \tag{6.7}$$

in the case $k = d = 1$, $B = I$.

We have already proved that $\{X_t : t \in [0,1]\}$ solution of (6.5) is a Markov field if $f(\cdot)$

is affine, $(H.1)$ is satisfied and T is a bijection. As an application of our general result of Proposition 5.1 we can prove, in a different way than in Nualart-Pardoux [8], that the converse to that property is true.

Theorem 6.2 *Suppose that $d = 1$, $(H.1)$ holds, f and g are of class C^2, T is bijective and $(H.2)''$ holds. Assume moreover that $g' > 1 \ \forall x \in \mathbf{R}$ and $g' \neq 0$. Then if $\{X_t : t \in [0,1]\}$ is a Markov field, f is affine.*

Proof : Since $g' > 1$, from $(H.2)$" we obtain that

$$det(I - e^A\phi(1)g'(\xi_1) + g'(\xi_1)) = (1 - e^A\phi(1)g'(\xi_1) + g'(\xi_1)) > 0 \quad a.s.$$

Putting $\mathcal{F}_t^e := \sigma\{Y_0, Y_s : t \leq s \leq 1\}$, from Proposition 5.1 we have

$$\frac{e^A\phi(1)\phi^{-1}(s)f''(Y_s)\phi(s)g'(\xi_1)}{1 - e^A\phi(1)g'(\xi_1) + g'(\xi_1)} \in \mathcal{F}_t^e$$

$$\forall t \leq s \leq 1.$$

Since $g' \neq 0$, we can assume that $\exists a \neq 0$ such that: $a \in Im(g')$. From the fact that the law of ξ_1 has $I\!R$ as its support we obtain that

$$P\{\omega : g'(\xi_1(\omega)) \in]a - \varepsilon, a + \varepsilon[\} > 0.$$

and for ε sufficientily small that $0 \in]a - \varepsilon, a + \varepsilon[$.

Let us assume that $f'' \neq o$: again we can assume that $\exists b \neq 0$ such that: $b \in Im(f'')$ and

$$P\{\omega : f''(e^{-A}(g(\xi_1(\omega)) + \xi_s(\omega))) \in]b - \varepsilon, b + \varepsilon[\} > 0.$$

and for ε sufficientily small that $0 \in]b - \varepsilon, b + \varepsilon[$. The function

$$(x, y) \longrightarrow (f''(e^{-A}(g(x) + y)), g'(x))$$

is clearly continuous. Therefore the inverse image of $]a - \varepsilon, a + \varepsilon[\times]b - \varepsilon, b + \varepsilon[$ is an open sunset of $I\!R^2$, which does not contain (0,0). Since (ξ_1, ξ_s) is a 2-dimensional Gaussian random vector with non-singular covariance matrix, we obtain that

$$P\{\omega : (f''(e^{-A}(g(\xi_1(\omega)) + \xi_s(\omega))), g'(\xi_1(\omega))) \in]a-\varepsilon, a+\varepsilon[\times]b-\varepsilon, b+\varepsilon[\} > 0$$

Let

$$G^s := \{\omega : f''(Y_s(\omega)) \neq 0\} \cap \{\omega : g'(\xi_1(\omega)) \neq 0\};$$

since G^s contains the previous set, we have that

$$P(G^s) > 0$$

and furthermore $G^s \in \mathcal{F}_t^e, t \leq s \leq 1$.

Since

$$\phi(1)\phi^{-1}(s)f''(Y_s)\phi(s)\phi^{-1}(s) \in \mathcal{F}_t^e \qquad t \leq s \leq 1$$

we have

$$\frac{1+g'(\xi_1)}{1-e^A\phi(1)g'(\xi_1)+g'(\xi_1)} \cdot \phi(1)\phi^{-1}(s)f''(Y_s)\phi(s)\phi^{-1}(s) \in \mathcal{F}_t^e$$

Recalling that

$$\phi(1)\phi^{-1}(s) = \exp[-\int_s^1 f'(Y_u)du] > 0 \qquad t \leq s \leq 1.$$

we have

$$1_{G^S}\frac{1+g'(\xi_1)}{1-e^A\phi(1)g'(\xi_1)+g'(\xi_1)} \in \mathcal{F}_t^e \qquad t \leq s \leq 1.$$

and more

$$1_{G^S}\phi(1) \in \mathcal{F}_t^e$$

(since $1+g'(\xi_1)$ and $g'(\xi_1)$ are $\mathcal{F}_t^e$ - measurable). We obtain, therefore

$$1_{G^S}\int_0^1 f'(Y_u)du \in \mathcal{F}_t \qquad t \leq s \leq 1.$$

We can apply the derivation operator (with respect to ξ_u) D_u and applying again Lemma 3.1 of section 3, we obtain

$$\mathbf{1}_{G^s}\frac{d}{dt}[D_u(\int_0^1 f'(Y_r)dr)] = 0 \qquad \forall 0 \le u \le t \le s \le 1.$$

where

$$D_u(\int_0^1 f'(Y_r)dr) = \int_0^1 f''(Y_r)[e^{-Ar}(g'(\xi_1) + \mathbf{1}_{[0,r]}(u))]dr.$$

Therefore, for $0 \le u \le t \le s \le 1$

$$\begin{aligned}\frac{d}{dt}\left(D_u(\int_0^1 f'(Y_r)dr)\right) &= \frac{d}{dt}\left[\int_0^1 f''(Y_r)e^{-Ar}(g'(\xi_1) + \right.\\ &\left. + \int_u^1 f''(Y_r)e^{-Ar}dr\right] = -f''(Y_u) = 0\end{aligned}$$

We obtain that

$$f''(Y_u) = 0 \qquad a.e. \quad on \quad G^s \quad , \quad 0 \le u \le t \le s \le 1$$

and for $u = t = s$ that

$$f''(Y_t) = 0 \qquad a.e. \quad on \quad G^t.$$

This implies that

$$P(G^t) = 0$$

and this leads to a contradiction, since $P(G^t) > 0$. Therefore it has to be $f''(x) \equiv 0$ and the theorem is proved. Q.E.D.

Chapter 7

A counterexample (for $d > 1$)

We want to show, via a counterexample, that in dimension higher then one the solution to a nonlinear stochastic differential equation of type (1.1) with boundary condition (1.2) may be a Markov process although f is nonlinear. The example which we are going to consider is not covered by the existence and uniqueness results of Chapter 2 (it would be impossible to satisfy condition (H.2i) or (H.2ii)). Indeed, we want to consider linear boundary conditions of the type:

$$\begin{cases} X_0^{i_k} & = & a_k \\ & & \\ X_1^{j_k} & = & b_k \end{cases} \tag{7.1}$$

(here $1 \leq l < d$) where $a_1, \cdots, a_l, b_1, \cdots, b_{d-l}$ are arbitrary real numbers and $i_k \neq j_{k'} \quad \forall k = 1, \cdots, l$ and $\forall k' = 1, \cdots, d-l$. Again we shall consider the equation

$$\frac{dX_t}{dt} + f(X_t) = \frac{dW_t}{dt} \tag{7.2}$$

with $f(\cdot)$ triangular and satisfying (H.6) and (H.7). We have already proved that equation (7.2), under (H.6) and (H.7), has a unique solution, completely determined by (7.1). We have to prove that this solution is a Markov process.

Let us start with

$$\begin{cases} \frac{dX_t^1}{dt} + f_1(X_t) = \frac{dW_t^1}{dt} \\ \\ X_0^1 = a_1 \qquad (or X_1^1 = b_1) \end{cases} \tag{7.3}$$

Clearly $\{X_t^1; t \in [0,1]\}$, in both cases, is a Markov process and proceeding by induction it will be sufficient to prove the following:

Lemma 7.1 *Let $\{Y_t; t \in [0,1]\}$ be an m-dimensional Markov process and let $\{V_t; t \in [0,1]\}$ be an n-dimensional standard Wiener process independent of $\{Y_t\}$. Suppose we are given measurable and locally bounded mapping $g : I\!R^{m\times n} \longrightarrow I\!R^n$, $L : I\!R^m \longrightarrow I\!R_+$ such that:*

$$|g(y,x)| \leq L(y)(1+|x|)$$

$$|g(y,x) - g(y,x')| \leq L(y)|x-x'|$$

Let $\{X_t; t \in [0,1]\}$ denote the unique solution of the Stochastic differential equation

$$\frac{dX_t}{dt} = g(Y_t, X_t) + \frac{dV_t}{dt} \tag{7.4}$$

Then the $m+n$ dimensional process $\{(Y_t, X_t) : t \in [0,1]\}$ is a Markov process.

Proof: It is easy to prove, using the standard results about the existence and uniqueness of Stochastic differential equation (see [3]) that equation (7.3) has a unique solution $\{X_t; t \in [0,1]\}$. To prove that the $m+n$ process $\{(Y_t, X_t) : t \in [0,1]\}$ is Markov we shall apply the two general Propositions about the splitting σ-algebras, that we stated in Chapter 5. In fact, since Y_t is a Markov process, it holds that

$$\sigma(Y_s; s \leq t) \underset{\sigma(Y_t)}{\|} \sigma(Y_s; s \geq t)$$

Clearly, from the independence of $\{Y_t\}$ and $\{V_t\}$, we have that

$$\sigma(Y_s, V_s; s \leq t) \underset{\sigma(Y_t)}{\|} \sigma(Y_s, V_s - V_t; s \geq t)$$

Since $\sigma(X_t) \subseteq \sigma(Y_t, V_t)$ it holds, by Proposition 5.2 that

$$\sigma(Y_s, V_s; s \leq t) \, \frac{||}{\sigma(Y_t, X_t)} \, \sigma(Y_s, V_s - V_t; s \geq t).$$

Now, by Proposition 5.1,

$$\sigma(Y_s, V_s; s \leq t) \, \frac{||}{\sigma(Y_t, X_t)} \, \sigma(Y_s, V_s - V_t, X_t; s \geq t)$$

and, since

$$\sigma(Y_s, X_s; s \leq t) \subseteq \sigma(Y_s, V_s; s \leq t)$$

$$\sigma(Y_s, X_s; s \geq t) \subseteq \sigma(Y_t, V_t; s \geq t)$$

and, by Remark 5.1 , if $\mathcal{A}_1' \subseteq \mathcal{A}_1$, $\mathcal{A}_2' \subseteq \mathcal{A}_2$

$$\mathcal{A}_1 \, \frac{||}{\mathcal{B}} \, \mathcal{A}_2 \Rightarrow \mathcal{A}_1' \, \frac{||}{\mathcal{B}} \, \mathcal{A}_2'$$

that

$$\sigma(Y_s, X_s; s \leq t) \, \frac{||}{\sigma(Y_t, X_t)} \, \sigma(Y_s, X_s; s \geq t)$$

Therefore the $m+n$ process $\{(Y_t, X_t) : t \in [0,1]\}$ is Markov and the Lemma is proved.

Q.E.D.

Applying this lemma to our problem, we see that at each step k, $1 \leq k \leq d$, we have that $\{(X_t^1, \cdots, X_t^{k-1}) : t \in [0,1]\}$ is a $k-1$ dimensional Markov process and conditions (H.9) and (H.10) for $f^k(\cdot)$ are the same of Lemma 7.1 with $g = -f^k$. Therefore the process $\{(X_t^1, \cdots, X_t^{k-1}, X_t^k) : t \in [0,1]\}$ is a Markov process and we can conclude that the solution of equation (7.1) under the boundary condition (7.2) is a Markov process.

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Appendix A

The computation of the Carleman-Fredholm determinant

Let f be an element of $L^2([0,1];M_d)$ and g , $h \in L^4([0,1];M_d)$ (such that $g \cdot h \in L^2([0,1];M_d)$) where M_d is the vector space of the $d \times d$ real matrices. For every s and t belonging to $I\!R_+$, let us define the L^2-kernel

$$K(s,t) = f(t)\left(g(s) + \mathbf{1}_{[0,1]}(s)\right) h(s).$$

Let us denote by K the operator of $L^2([0,1];I\!R^d)$ into itself defined by:

$$(K\varphi)(t) = \int_0^1 K(s,t)\varphi(s)ds$$

$\varphi \in L^2([0,1];I\!R^d)$.

Let ψ_t be the solution of the following differential equation:

$$\begin{aligned} \frac{d\psi_t}{dt} &= -h(t)f(t)\psi_t \\ \psi_0 &= I \end{aligned}$$

Then the Carleman-Fredholm determinant of the Hilbert-Schmidt operator $-K$ is given by:

Lemma A.1

$$d_c(-K) = det\{I + \int_0^1 g(t)h(t)f(t)dt\} \times$$

$$\times \exp(-\int_0^1 Tr(f(t)g(t)h(t))dt$$

Proof First of all let us note that it will be sufficient to prove the result for f, g and h continuous (since d_c is continuous with respect to the $L^2([0,1]; \mathbb{R}^d)$-norm).

For f and g continuous, the idea is to approximate $K(s,t)$ by a sequence of finite-dimensional operators $K^n(s,t)$, for which the computation of the Carleman-Fredholm (C-F) determinant reduces to a classical computation of a finite-dimensional determinant. The continuity in L^2 of the C-F determinant shall give us the result.

Let $n \geq 1$: define $e_i(t) = \sqrt{n}\ \mathbf{1}_{[t_{i-1},t_i]}(t)$, $1 \leq i \leq n$, an orthonormal family of functions, with $t_i = \frac{i}{n}$. We define the approximating sequence of finite-dimensional operators by:

$$K^n(s,t) = \frac{1}{n} \sum_{i,j=1}^{n} f(t_{i-1}) \left(g(t_{j-1}) + \mathbf{1}_{\{i \leq j\}} I\right) h(t_{j-1}) e_i(t) e_j(t)$$

and we have that

$$d_c(-K^n) \longrightarrow d_c(-K)$$

in L^2, by the continuity of the C-F determinant in L^2. Let us compute $d_c(-K^n)$:

$$d_c(-K^n) = det(I + K^n) \exp(-Tr K^n)$$

The matrix associated to K^n is:

$$\frac{1}{n}\begin{bmatrix} f(t_0)g(t_0)h(t_0) & f(t_0)g(t_1)h(t_1) & \cdots & f(t_0)g(t_{n-1})h(t_{n-1}) \\ f(t_1)(g(t_0)+I)h(t_0) & f(t_1)g(t_1)h(t_1) & \cdots & . \\ f(t_2)(g(t_0)+I)h(t_0) & f(t_0)(g(t_1)+I)h(t_1) & \cdots & . \\ . & . & \cdots & . \\ . & . & \cdots & . \\ . & . & \cdots & . \\ f(t_{n-1})(g(t_0)+I)h(t_0) & f(t_{n-1})(g(t_1)+I)h(t_1) & \cdots & f(t_{n-1})g(t_{n-1})h(t_{n-1}) \end{bmatrix}$$

Therefore

$$TrK^n = \frac{1}{n}\sum_{i=0}^{n-1} Tr(f(t_i)g(t_i)h(t_i))$$

When n tends to $+\infty$ this quantity tends to

$$TrK = \int_0^1 Tr(f(t)g(t)h(t))dt$$

Let us write now the matrix associated to $I + K^n$:

$$\begin{bmatrix} I+\frac{1}{n}f(t_0)g(t_0)h(t_0) & \frac{1}{n}f(t_0)g(t_1)h(t_1) & \cdots & \frac{1}{n}f(t_0)g(t_{n-1})h(t_{n-1}) \\ \frac{1}{n}f(t_1)(g(t_0)+I)h(t_0) & I+\frac{1}{n}f(t_1)g(t_1)h(t_1) & \cdots & . \\ \frac{1}{n}f(t_2)(g(t_0)+I)h(t_0) & \frac{1}{n}f(t_2)(g(t_1)+I)h(t_1) & \cdots & . \\ . & . & \cdots & . \\ . & . & \cdots & . \\ . & . & \cdots & . \\ \frac{1}{n}f(t_{n-1})(g(t_0)+I)h(t_0) & \frac{1}{n}f(t_{n-1})(g(t_1)+I)h(t_1) & \cdots & I+\frac{1}{n}f(t_{n-1})g(t_{n-1})h(t_{n-1}) \end{bmatrix}$$

(note that every element is a $d \times d$ matrix).

We shall now assume that $f(t_i)$ is an invertible matrix for every $i = 1, \cdots, n-1$(we can replace $f(t_i)$ by $f(t_i)+\varepsilon I$ and take at the end the limit for $\varepsilon \to o$). Let $\Delta_0 = det(I+K^n)$

and $I + K^n = D_0$: we have that

$$D_0 = \begin{bmatrix} f(t_0) & 0 & \cdots & 0 \\ 0 & f(t_1) & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & f(t_{n-1}) \end{bmatrix} \cdot D_1$$

where

$$D_1 = \begin{bmatrix} f(t_0)^{-1} + \frac{1}{n}g(t_0)h(t_0) & \frac{1}{n}g(t_1)h(t_1) & \cdots & \frac{1}{n}g(t_{n-1})h(t_{n-1}) \\ \frac{1}{n}(g(t_0)+I)h(t_0) & f(t_1)^{-1} + \frac{1}{n}g(t_1)h(t_1) & \cdots & \cdot \\ \frac{1}{n}(g(t_0)+I)h(t_0) & \frac{1}{n}(g(t_1)+I)h(t_1) & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \frac{1}{n}(g(t_0)+I)h(t_0) & \frac{1}{n}(g(t_1)+I)h(t_1) & \cdots & f(t_{n-1})^{-1} + \frac{1}{n}g(t_{n-1})h(t_{n-1}) \end{bmatrix}$$

Subtracting every row from the next one we have:

$$\Delta_1 = det D_1 = det \begin{bmatrix} f(t_0)^{-1} + \frac{1}{n}g(t_0)h(t_0) & \frac{1}{n}g(t_1)h(t_1) & \cdots & \frac{1}{n}g(t_{n-1})h(t_{n-1}) \\ \frac{1}{n}h(t_0) - f(t_0)^{-1} & f(t_1)^{-1} & \cdots & \cdot \\ 0 & \frac{1}{n}h(t_1) - f(t_1)^{-1} & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & f(t_{n-1})^{-1} \end{bmatrix}$$

Since $det(A \cdot B) = det(B \cdot A)$ and the above result, we obtain that:

$$\Delta_0 = det \begin{bmatrix} I + \frac{1}{n}g(t_0)h(t_0)f(t_0) & \frac{1}{n}g(t_1)h(t_1)f(t_1) & \cdots & \frac{1}{n}g(t_{n-1})h(t_{n-1})f(t_{n-1}) \\ \frac{1}{n}h(t_0)f(t_0) - I & I & \cdots & . \\ 0 & \frac{1}{n}h(t_1)f(t_1) - I & \cdots & . \\ . & . & \cdots & . \\ . & . & \cdots & . \\ . & . & \cdots & . \\ 0 & 0 & \cdots & I \end{bmatrix}$$

(since f, g and h are matrices, $g \cdot h \cdot f \neq f \cdot g \cdot h$ in general).

Let us now define the $(d \times d)$ matrices $B_0, \cdots, B_{n-2}$ by:

$$\begin{aligned} B_0 &= I - \frac{1}{n}h(t_0)f(t_0) \\ B_1 &= \left(I - \frac{1}{n}h(t_1)f(t_1)\right) B_0 \\ &\cdot\ \cdot\ \cdot \\ B_{n-2} &= \left(I - \frac{1}{n}h(t_{n-2})f(t_{n-2})\right) B_{n-3} \end{aligned}$$

For n large enough, all the matrices $B_0, \cdots, B_{n-2}$ are invertible and we have

$$\begin{aligned} I - \frac{1}{n}h(t_0)f(t_0) &= B_0 \\ I - \frac{1}{n}h(t_1)f(t_1) &= B_1 B_0^{-1} \\ &\cdot\ \cdot\ \cdot \\ I - \frac{1}{n}h(t_{n-2})f(t_{n-2}) &= B_{n-2} B_{n-3}^{-1} \end{aligned}$$

Therefore it holds that:

$$\Delta_0 = det \begin{bmatrix} I + \frac{1}{n} g(t_0)h(t_0)f(t_0) & \frac{1}{n} g(t_1)h(t_1)f(t_1) & \cdots & \frac{1}{n} g(t_{n-1})h(t_{n-1})f(t_{n-1}) \\ -B_0 & I & \cdots & . \\ 0 & -B_1 B_0^{-1} & \cdots & . \\ . & . & \cdots & . \\ . & . & \cdots & . \\ . & . & \cdots & . \\ 0 & 0 & \cdots & I \end{bmatrix}$$

Again we can write

$$\Delta_0 = det \begin{bmatrix} I & 0 & \cdots & 0 \\ 0 & B_0 & \cdots & . \\ . & . & \cdots & . \\ 0 & 0 & \cdots & B_{n-2}^{-1} \end{bmatrix} \cdot det D_2$$

where

$$D_2 = \begin{bmatrix} I + \frac{1}{n} g(t_0)h(t_0)f(t_0) & \frac{1}{n} g(t_1)h(t_1)f(t_1) & \cdots & \frac{1}{n} g(t_{n-1})h(t_{n-1})f(t_{n-1}) \\ -I & B_0^{-1} & \cdots & 0 \\ 0 & -B_0^{-1} & \cdots & 0 \\ . & . & \cdots & . \\ . & . & \cdots & . \\ . & . & \cdots & . \\ 0 & 0 & \cdots & -B_{n-2}^{-1} \end{bmatrix}$$

and it holds that

$$\Delta_0 = det \left(D_2 \cdot \begin{bmatrix} I & 0 & \cdots & 0 \\ 0 & B_0 & \cdots & . \\ . & . & \cdots & . \\ 0 & 0 & \cdots & B_{n-2}^{-1} \end{bmatrix} \right) =$$

$$= det \begin{bmatrix} I + \frac{1}{n}g(t_0)h(t_0)f(t_0) & \frac{1}{n}g(t_1)h(t_1)f(t_1)B_0 & \cdots & \frac{1}{n}g(t_{n-1})h(t_{n-1})f(t_{n-1})B_{n-2} \\ -I & I & \cdots & 0 \\ 0 & -I & \cdots & 0 \\ . & . & \cdots & . \\ . & . & \cdots & . \\ . & . & \cdots & . \\ 0 & 0 & I & \end{bmatrix}$$

Replacing the $n-1$ column by the sum of itself and the last column (and the same for the $n-2$ column and so on) we obtain that:

$$\Delta_0 = det \begin{bmatrix} I + \frac{1}{n}g(t_0)h(t_0)f(t_0) + \frac{1}{n}\sum_{i=1}^{n-1} g(t_i)h(t_i)f(t_i)B_{i-1} & 0 & \cdots & 0 \\ 0 & I & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ . & . & \cdots & . \\ . & . & \cdots & . \\ . & . & \cdots & . \\ 0 & 0 & I & \end{bmatrix}$$

and therefore

$$\Delta_0 = det \left(I + \frac{1}{n}g(t_0)h(t_0)f(t_0) + \frac{1}{n}\sum_{i=1}^{n-1} g(t_i)h(t_i)f(t_i)B_{i-1} \right)$$

Recalling that we have denoted by ψ_t the solution of

$$\frac{d\psi_t}{dt} = -h(t)f(t)\psi_t$$

$I - \frac{1}{n}h(t_i)f(t_i) = B_i B_{i-1}^{-1}$ can be approximated (for $n \to \infty$) by $\psi_{t_{i+1}}\psi_{t_i}^{-1}$; in fact

$$\begin{aligned}\psi_{t_{i+1}}\psi_{t_i}^{-1} &= I - \int_{t_i}^{t_{i+1}} h(s)f(s)\psi_s\psi_{t_i}^{1} ds \\ &= I - \frac{1}{n}h(t_i)f(t_i) + \frac{R^i(n)}{n}\end{aligned}$$

where

$$\sup_i |R^i(n)| \longrightarrow 0 \qquad \text{for } n \to \infty$$

Therefore

B_0 can be approximate by $\psi_{t_1}\psi_{t_0}^{-1} = \psi_{t_1}$

B_1 can be approximate by $\psi_{t_2}\psi_{t_1}^{-1}\psi_{t_1}\psi_{t_0}^{-1} = \psi_{t_2}$

and the generic B_i by $\psi_{t_{i+1}}$ for $0 \leq i \leq n-2$.

We obtain that

$$\begin{aligned}\Delta_0 &\sim det\left(I + \frac{1}{n}g(t_0)h(t_0)f(t_0) + \frac{1}{n}\sum_{i=1}^{n-1} g(t_i)h(t_i)f(t_i)\psi_{t_i}\right) \\ &\overset{n\to\infty}{\longrightarrow} det\left(I + \int_0^1 g(t)h(t)f(t)\psi_t dt\right)\end{aligned}$$

Q.E.D.

Corollary A.1 *If $g(\cdot)$ is constant, then :*

$$d_c(-K) = det\{I - \psi_1 g + g\} \exp\left[-\int_0^1 Tr(f(t)h(t)g(t))dt\right]$$

Proof Proceeding as before, we obtain that:

$$\Delta = det \begin{bmatrix} I + \frac{1}{n}gh(t_0)f(t_0) & \frac{1}{n}gh(t_1)f(t_1)B_0 & \cdots & \frac{1}{n}gh(t_{n-1})f(t_{n-1})B_{n-2} \\ -I & I & \cdots & 0 \\ 0 & -I & \cdots & 0 \\ . & . & \cdots & . \\ . & . & \cdots & . \\ . & . & \cdots & . \\ 0 & 0 & I & \end{bmatrix}$$

Assume that g is invertible (if not we shall make the same for $g_\varepsilon = g + \varepsilon I$ and at the end let ε tends to 0). We have

$$\Delta_0 = det(gI \cdot D_3) = det(D_3 \cdot gI) =$$

$$= det \begin{bmatrix} I + \frac{1}{n}h(t_0)f(t_0)g & \frac{1}{n}h(t_1)f(t_1)B_0g & \cdots & \frac{1}{n}h(t_{n-1})f(t_{n-1})B_{n-2}g \\ -I & I & \cdots & 0 \\ 0 & -I & \cdots & 0 \\ . & . & \cdots & . \\ . & . & \cdots & . \\ . & . & \cdots & . \\ 0 & 0 & I & \end{bmatrix}$$

Proceeding now as in the previous proof, at the end we obtain:

$$\Delta_0 \longrightarrow det\left(I + \int_0^1 h(t)f(t)\psi_t g dt\right) = det\left(I - \psi_1 g + g\right)$$

Q.E.D.

Corollary A.2 *Let:*

$f(t) = \bar{f}'(Y_t)e^{As}$

$g(s) = g'(\xi_1)(constant)$

$h(s) = e^{As}$

In this case we have

$$\begin{aligned} d_c(-K) &= det\left(I - e^{A}\phi(1)g'(\xi_1) + g'(\xi_1)\right) \\ &\quad \exp\left(-\int_0^1 Tr\left[\bar{f}'(Y_t)e^{As}e^{-At}g'e^{At}dt\right]\right) \end{aligned}$$

where

$$\begin{aligned} \frac{d}{dt}\phi(t) &= -f'(Y_t)\phi(t) \\ \phi(0) &= I \end{aligned}$$

Proof We have to prove that $e^{At}\phi(t)$ is solution of (A.1) i.e.

$$\frac{d\psi_t}{dt} = -e^{At}\bar{f}'(Y_t)e^{-At}\psi_t$$

$$\psi_0 = I$$

In fact:

$$\begin{aligned} \frac{d}{dt}\left(e^{At}\phi(t)\right) &= A\left(e^{At}\phi(t)\right) + e^{At}\frac{d}{dt}\phi(t) \\ &= A\left(e^{At}\phi(t)\right) + e^{At}\left(f'(Y_t)\phi(t)\right) \\ &= A\left(e^{At}\phi(t)\right) + e^{At}f'(Y_t)e^{-At}\left(e_{At}\phi(t)\right) \end{aligned}$$

Recalling that :

$$f'(Y_t) = A + \bar{f}'(Y_t)$$

we obtain:

$$\frac{d}{dt}\left(e^{At}\phi(t)\right) = -\left(e^{At}\bar{f}'(Y_t)e^{-At}\right)e^{At}\phi(t)$$

and it holds that: $e^{A0}\phi(0) = I$.

Therefore $e^{At}\phi(t)$ is solution of (7.1) and so:

$$\begin{aligned} d_c(-K) &= det\left(I - e^{A}\phi(1)g' + g'\right) \\ &\quad \exp\left(-\int_0^1 Tr\left[\bar{f}'(Y_t)e^{As}e^{-At}g'e^{At}dt\right]\right) \end{aligned}$$

Q.E.D.