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Gravitational and Electromagnetic Fields Associated with Shear-free Congruences of Null Geodesics and Cauchy-Riemann Structures

Thesis submitted for the degree of

"Magister Philosophiæ"

CANDIDATE

Paweł Nurowski

SUPERVISOR

Prof. Andrzej Trautman

October 1991

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Most of the ideas behind the original results of this thesis arose during discussions with my supervisor Professor Andrzej Trautman. He introduced me into the subject, attracted my attention to the crucial papers of Cartan, and suggested me that I use Cartan's invariants of CR-structures to study gravitational fields.

Professor Jacek Tafel, who also studied relations between CR-structures and General Relativity, guided me through the theory of symmetric CR-structures. The collaboration with him in 1987 encouraged me not to give up the subject. I include the results of this collaboration in the present work. Doctor Jerzy Lewandowski helped me in understanding the mathematics of CR-structures. The results of our collaboration are also presented here.

I wish to express my cordial thanks to all of them.

Special thanks are also due to all members of the Mathematical Physics Sector in SISSA, and in particular to Professor Cesare Reina, for creating the warm atmosphere during my studies.

I dedicate this work to my wife, Ewa. Without her love this work would never have been completed.

CONTENTS.

I. INTRODUCTION.

Most of the main difficulties one encounters in General Relativity have their origin in the nonlinearity of Einstein's equations. In particular, because of this nonlinearity perturbative solutions to Einstein's equations can not give a full insight into Einstein theory. Knowledge of an exact solution to Einstein's equations is a precondition for study such phenomena like black holes, wormholes etc. Over years a few techniques were introduced to provide examples of such solutions [see e.g. Kramer at al 1980 for a review]. One of such techniques imposes some symmetry conditions on the space-time. Another assumes that there exists some congruence of lines, with some special properties, in the space-time. As a particular example of the latter technique one assumes that there exists a congruence of shear-free and null geodesics in the space-time [Robinson 1961] (for definitions see e.g. (5.2.2)-(5.2.5) of this work). This assumption, being general enough, considerably simplifies vacuum, pure radiation and Einstein-Maxwell equations. This work is mainly devoted to study consequences of this assumption.

Some parts of the work presented here have the character of a review of known results. However, there are some parts which seem to be new.

Following this Introduction we present Chapter II devoted to our notations.

Chapter III shows assumptions which satisfy space-times considered in this work. In particular, assumptions A1) and A2) of Chapter III turn out to be equivalent to the existence of a congruence of shear-free and null geodesics in the space-time. This fact has not been mentioned in the literature so far. It is included in Corollary 5.3.1 and Remark 5.3.1, which summarize different facts equivalent to assumptions A1) and A2).

Chapter IV presents corollaries one can deduce from assumptions A1) and A2). In particular, it is shown that if space-time admits a congruence preserving two-form $\tilde{F}=F+i{*}F$, built out of a real two- form F, then the congruence and F are null. Moreover, congruence defined above satisfies so called Robinson-Trautman condition [Robinson, Trautman 1983].

Chapter V gives the physical meaning to the constructions given in Chapter IV. It presents null Maxwell fields and such well known results as Robinson theorem 5.2.1 [Robinson 1961], Goldberg-Sachs theorem 5.3.3 [Goldberg, Sachs 1962], Cartan-Petrov-Penrose classification [Cartan 1922, Petrov 1954, Penrose 1960](Section V.3) and peeling-off theorem 5.3.4 [Trautman 1958, Sachs 1962]. Also Trautman theorem 5.4.1 is included there [Trautman 1984]. This specifies what are possible metrics which admit <u>the same</u> congruence of shear-free and null geodesics.

Theorem 5.4.1 introduces to the so called optical geometry without shear - i.e. geometry associated with a given congruence of shear-free and null geodesics (Section VI.2). Chapter VI is devoted to the wider class of geometries, namely to optical geometries. These geometries, introduced by A. Trautman [Trautman 1984], are the weakest structures one needs to write vacuum Maxwell equations for a null electromagnetic field. Almost all Chapter VI is based on [Robinson, Trautman 1985].

The crucial chapter for the rest of the work is Chapter VII, which gives a one to one correspondence between a congruence of shear-free null geodesics and a 3-dimensional CR structure. This is called in Section VII.2 as Robinson-Trautman correspondence [Robinson, Trautman 1985, 1989]. This correspondence has been in the air for a long time. It is already apparent in the occurrence of the Cauchy-Riemann operator in the process of solving Einstein's equations in the case of twist-free congruences of null and shear-free geodesics [Robinson, Trautman 1962]. P. Sommers [Sommers 1976, 1977] and J. Tafel [Tafel 1985] pointed out the appearance of the tangential CR operator in connection with twisting congruences.

Chapter VIII is devoted to different definitions of CR structures. Definition 8.1.1 is given to define in the abstract manner an odd-dimensional CR structure. Also other, more technical definitions are given in the case of 3 dimensions. Equivalence of all definitions in this case is proved in Section VIII.1. Section VIII.2 defines realizable CR structures. An important, 5-dimensional example of such structures is given in Section VIII.3. This describes the twistor space of R. Penrose [Penrose 1967].

Two next chapters - IX and X are mainly original.

Chapter IX deals with geometry of 3-dimensional CR structures. Such structures were for the first time studied by H. Poincare [Poincare 1907] in the context of real hypersurfaces of codimension 1 in $\mathbb{C}^2$. He observed that, in general, it is impossible even locally to find biholomorphic transformation which transforms one real hypersurface of codimension 1 in $\mathbb{C}^2$ into another. This observation gave rise to the theory of 3 dimensional CR structures, and in particular to the local equivalence problem for them. In section IX.1 after a definition of equivalence of such structures an argument is given that there exist nonequivalent CR structures. This is firstly pointed out by considering so called symmetric CR structures. These are CR structures which admit symmetries (definition (9.3.1)). Such CR structures were for the first time considered by H. Poincare [Poincare 1907] and then studied by B. Segre [Segre 1931] and E. Cartan [Cartan 1932]. In particular E. Cartan gave a full classification of nonequivalent CR structures admitting at least 3 symmetries, showing possible realizations of such structures in $\mathbb{C}^2$. In Sections IX.3-IX.6 we generalize this classification to all symmetric CR structures. Our method is slightly different from this of Cartan. We extend Cartan classification to CR structures admitting 1 or 2 symmetries. We also give canonical representation for 1-forms defining symmetric CR structures (this was not given by Cartan). Results of Sections IX.3-IX.6 were obtained in collaboration with J. Tafel [Nurowski, Tafel 1988].

Section IX.7 combines results of E. Cartan [1932], N. Tanaka [1962], S. S. Chern and J. Moser [Chern, Moser 1974] and D. Burns Jr., K.Diederich and S. Schnider [Burns at al 1977], and gives classification of <u>all</u> 3-dimensional CR structures. The so called Cartan invariants of these structures are introduced there.[1] The way of presentation of this section is partially taken from J. Lewandowski's Ph.D thesis [Lewandowski 1989]. It is worth noting that Sections IX.3-IX.7 can be regarded as a new presentation of all results of two Cartan's papers [Cartan 1932]. This modern description of these, still not very well known, papers was achieved in collaboration with J. Tafel and J. Lewandowski [Nurowski, Tafel 1988, Lewandowski 1988, Lewandowski 1989, Lewandowski, Nurowski 1990a, 1990b].

In Chapter X we study Lorentz geometries associated with a given CR structure. Two approaches are distinguished. The

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[1] Following [Cartan 1932], we give an effective algorithm of computing them in the Appendix (Chapter XIII).

nonstandard one [Lewandowski, Nurowski 1990b] uses Cartan invariants of the CR structure to construct a preferred by the CR geometry cobasis in space-time. In Section X.2 we use this approach to study Weyl tensor of metrics admitting twisting shear-free congruences of null geodesics. All results of this section are new. In particular, Theorem 10.2.1 describes all such space-times of type N. The theorem is not very powerful, since we failed in finding integrability conditions of equations obtained in terms of Cartan invariants only. However, applying this theorem we can define Fefferman metrics [Fefferman 1976], and prove Theorem 10.3.2 characterizing these metrics in terms of Cartan-Petrov-Penrose type and conformal symmetries. This last theorem was proved in very general setting for the first time by G. A. J. Sparling [Sparling 1985], and then rediscovered by J. Lewandowski [Lewandowski 1989]. However, the proof we present in Section X.3 seems to be new.

Sections X.4-X.6 present known results on solutions to vacuum and pure radiation Einstein and Einstein-Maxwell equations in the case of space-times admitting congruences of shear-free and null geodesics. These results were obtained in the standard approach [Robinson, Trautman 1962, Kerr 1963, Robinson at al 1969], which did not recognize that there was a CR structure behind such space-times. Therefore in Section X.6 we present carefully CR structures associated with Taub-NUT and Kerr solutions. As it was pointed out by I. Robinson and A. Trautman [Robinson, Trautman 1985] in the first case CR structure is equivalent to 3-dimensional sphere imbedded in $\mathbb{C}^2$. This has the highest possible number of symmetries. The CR structure associated with the Kerr metric was for the first time studied in [Nurowski 1987]. In Section X.6 we present results of [Nurowski 1987] using another arguments related to Cartan invariants of this CR structure.

Section X.7 gives the first example of pure radiation Einstein-Maxwell solution with twisting light rays. The results of this section were obtained in collaboration with J. Tafel [Nurowski, Tafel 1991].

Section X.8, preceding Conclusions, discusses Kerr theorem [Kerr UNPUBLISHED, Tafel 1985] to show, that its reformulation in terms of Cartan invariants of CR structures is needed.

II. NOTATIONS

Notations used in this work are intended to be as standard as possible. The following list of notations is an adaptation of notations of [Abraham, Marsden 1978, Kobayashi, Nomizu 1963, 1969, Kramer at al 1980]. These books can be used to find all differential geometric definitions that we do not quote here.

Real and complex numbers.

Real line is denoted by $\mathbb{R}$; complex plane by $\mathbb{C}$.

$i^2=-1$

We denote complex conjugation by a bar over the symbol.

Equivalence relations.

In this work in certain places we use equivalence relations R defined in some sets A (vector spaces, manifolds, bundles etc.) to define quotient sets A/R of classes [a] defined by R in A.

Vectors and tensors.

If $\{h_i\}_{i=1}^{i=n}$ is a basis in a real n-dimensional vector space V then $\{h^i\}_{i=1}^{i=n}$ is a dual basis in a dual vector space V^*.

If W is a vector subspace of V then by W^o we denote these elements of W^* which produce zero acting on all elements of W. W^o is a vector space called annihilator of W.

In terms of a basis $\{h_i\}_{i=1}^{i=n}$ a general element of a tensorial vector space

$$V^r_s = \underbrace{V\otimes V\otimes\ldots\otimes V}_{r}\otimes\underbrace{V^*\otimes V^*\otimes\ldots\otimes V^*}_{s}$$

is given by

$$T = T^{i_1 i_2 \ldots i_s}{}_{j_1 j_2 \ldots j_r}\, h_{i_1}\otimes\ldots\otimes h_{i_s}\otimes h^{j_1}\otimes\ldots\otimes h^{j_r} \qquad (2.1)$$

where i_k, j_l = 1, 2, ...,n = dimV, and $T^{i_1 i_2 \ldots i_s}{}_{j_1 j_2 \ldots j_r}$ are number valued coefficients. Here we used the Einstein summation convention. This will be also always used in the following.

Symmetrization of indices is denoted by a round bracket; antisymmetrization by a square bracket e.g. $v_{(ab)}=\frac{1}{2}(v_{ab}+v_{ba})$, $v_{[ab]}=\frac{1}{2}(v_{ab}-v_{ba})$

Tensor product is denoted by $\otimes$, wedge product by $\wedge$.

Vectors of a 3-dimensional vector space are denoted by an arrow over the symbol, e. g. $\vec{A}$, $\vec{r}$. Scalar product of $\vec{A}$ and $\vec{r}$ is denoted by $\vec{A}\cdot\vec{r}$; their vector product by $\vec{A}\times\vec{r}$.

Manifolds.

Manifolds are denoted by Latin capital letters. We reserve letter M to denote four-dimensional space-time; N - is reserved to denote three dimensional submanifolds of M.

TP and T^*P denote tangent and cotangent bundles to P, respectively; T_pP and T_p^*P denote tangent and cotangent space at the point $p\in P$, respectively.

Set of all vector fields over P is denoted by $\mathfrak{X}(P)$; by $\mathfrak{X}_s^r(P)$ we denote set of all r-covariant and s-contravariant tensor fields over P.

Λ^rP denotes set of all fields of r-forms on P. ΛP denotes the Grassman algebra over P.

If ϕ is a diffeomorphism between P and P' then ϕ_* denotes transport of vector fields from TP to TP'; ϕ^* denotes pull-back of r-forms from T^*P' to T^*P. Transport of a general tensor field is denoted by $\tilde{\phi}^*$.

If k is a vector field then $\phi_t(k)$ denotes one-parameter group of diffeomorphisms generated by k; we also call $\phi_t(k)$ as a flow of k.

Vector fields on P can be regarded as differential operators. If k is a vector field on P then k(f) denotes value of a differential operator k evaluated on a function f. Sometimes, when it is obvious that differential operator interpretation of a vector field ∂ is relevant, we omit parenthesis and write ∂f.

In ΛP three kinds of derivatives are of particular interest. These are:

1) the exterior derivative - denoted by d - which is a derivation of degree +1,

2) inner derivative associated with a given vector field k. This is a derivation of degree -1, denoted by $k\lrcorner$,

3) Lie derivative with respect to a given vector field k, which is a derivation of degree 0. We denote this by $\underset{k}{\mathcal{L}}$.

These three derivatives are connected by the Cartan identity:

$$\underset{k}{\mathcal{L}} = k\lrcorner d + dk\lrcorner \qquad (2.2)$$

$\underset{k}{\mathcal{L}}$ extends to the Lie derivative with respect to k of any tensor field T by the formula

$$\underset{k}{\mathcal{L}}T = \frac{d}{dt}\left[\ \phi_t\tilde{(}k)^{*}\ (T)\ \right]\Big|_{t=0} \qquad (2.3)$$

If $\omega\in\Lambda^r P$ and $X_1, X_2, \dots X_k \in \mathcal{X}(P)$ then

$$\omega\ (X_1, \dots, X_{k-1}, X_k) = X_k\lrcorner(X_{k-1}\lrcorner(\dots(X_1\lrcorner\omega))\dots) \qquad (2.4)$$

Metric and tetrads.

If $\{h_i\}_{i=1}^{i=4}$ is a tetrad on M then in terms of its cotetrad $\{h^i\}_{i=1}^{i=4}$ the metric tensor is denoted by

$$g = g_{ab}\ h^a \otimes h^b \qquad (2.5)$$

where g_{ab} are functional coefficients. Inverse matrix to the matrix (g_{ab}) of coefficients of g is denoted by (g^{ab}) i.e.

$$g_{ab}g^{bc} = \delta_a{}^c \qquad (2.6)$$

Signature of the metric tensor is (+ + + -).

If X and Y are vector fields then g(X) is a one-form defined by

$$g(X) = g_{ab}(X\lrcorner h^a)h^b \qquad (2.7a)$$

and g(X,Y) is a function - scalar product of X and Y - defined by

$$g(X,Y) = g_{ab}(X\lrcorner h^a)(Y\lrcorner h^b) \qquad (2.7b)$$

This last equality is also denoted by

$$g(X,Y) = XY = g_{ab}X^aX^b = X_aX^a \qquad (2.7c)$$

where

$$X^a = X\lrcorner h^a \qquad (2.7d)$$

and lowering and raising of indices is possible by means of g_{ab} and g^{ab} e.g. $X_a=g_{ab}X^b$, $X^a=g^{ab}X_b$.

From now on we assume that M is orientable, hence we can endow M with the volume form η. Given η we define a Hodge dualization $*\omega$ of $\omega\in\Lambda^rM$ as a (4-r)-form satisfying

$$*\omega\ (X_{r+1},X_{r+2},\dots,X_4)\ \eta = \omega\wedge g(X_{r+1})\wedge g(X_{r+2})\wedge\dots\wedge g(X_4) \qquad (2.8)$$

[Trautman 1984].

Very often we use complex null tetrad $\{e_i\}_{i=1}^{i=4}$ i.e. such tetrad that

$$e_2 = \bar{e}_1,\ e_3 = \bar{e}_3,\ e_4 = \bar{e}_4 \qquad (2.9a)$$

and

$$g = 2\ (\ e^1e^2 - e^3e^4) \qquad (2.9b)$$

Here we used an abbreviation

$$e^ie^j = \frac{1}{2}\ (e^i\otimes e^j + e^j\otimes e^i) \qquad (2.10)$$

Convention for the volume form η is such that in the null tetrad (2.9) this reads

$$\eta = i\ e^1\wedge e^2\wedge e^3\wedge e^4 \qquad (2.11)$$

Connection and curvature.

Connection one forms are denoted by

$$\Gamma^a{}_b = \Gamma^a{}_{bc}h^c \qquad (2.12)$$

where $\Gamma^a{}_{bc}$ are connection coefficients.

Operator D denotes covariant exterior differential. On coefficients $T^{i_1 i_2 \dots i_s}{}_{j_1 j_2 \dots j_r}$ of a tensor T this reduces to

$$D\,(T^{i_1 i_2 \dots i_s}{}_{j_1 j_2 \dots j_r}) = T^{i_1 i_2 \dots i_s}{}_{j_1 j_2 \dots j_r ; k}\, h^k =$$

$$= [h_k (T^{i_1 i_2 \dots i_s}{}_{j_1 j_2 \dots j_r}) + \Gamma^{i_1}{}_{lk} T^{l i_2 \dots i_s}{}_{j_1 j_2 \dots j_r} + \dots \qquad (2.13)$$
$$- \Gamma^{l}{}_{j_1 k} T^{i_1 i_2 \dots i_s}{}_{l j_2 \dots j_r} + \dots]\, h^k$$

As it can be seen from (2.13) we denote covariant derivative with respect to the vector h_k by semicolon. Derivatives of a function f with respect to this vector field are denoted by

$$h_k(f) = f_k. \qquad (2.14)$$

In the special case of a vector field $\dfrac{\partial}{\partial x^k}$, where $(x^k)_{k=1}^{k=4}$ are coordinates on considered neighbourhood U, this is also denoted by

$$\frac{\partial}{\partial x^k}(f) = \frac{\partial f}{\partial x^k} = \partial_k f = f_k = f,_k \qquad (2.15)$$

Levi-Civita connection of a metric (2.5) is defined by

$$dh^a + \Gamma^a{}_b \wedge h^b = 0 \qquad (2.16a)$$

$$\Gamma_{(ab)} = 0 \qquad (2.16b)$$

If $(\Gamma^a{}_b)$ is Levi-Civita connection then we denote by $\nabla_k T$ covariant derivative with respect to a vector field k of a tensor T.

Curvature $(\Omega^a{}_b)$ of a connection $(\Gamma^a{}_b)$ is defined by

$$\Omega^a{}_b = \frac{1}{2} R^a{}_{bcd} h^c \wedge h^d = d\Gamma^a{}_b + \Gamma^a{}_c \wedge \Gamma^c{}_b. \qquad (2.17)$$

Ricci tensor is defined by

$$R_{ab} = R^c{}_{acb} \qquad (2.18a)$$

and Ricci scalar by

$$R = g^{ab} R_{ab} \qquad (2.18b)$$

Components of the Weyl tensor are defined by

$$g_{ae}C^{e}{}_{bcd} = R_{abcd} + \frac{R}{3} g_{a[c}g_{db]} - g_{a[c}R_{d]b} + g_{b[c}R_{d]a} \qquad (2.19)$$

In terms of the null tetrad $\{e_i\}_{i=1}^{i=4}$ Weyl tensor scalars are defined by:

$$\begin{aligned} \Psi_0 &= C_{1414} \\ \Psi_1 &= C_{3414} \\ \Psi_2 &= \frac{1}{2} (C_{3434} + C_{3412}) \\ \Psi_3 &= C_{3432} \\ \Psi_4 &= C_{2323} \end{aligned} \qquad (2.20)$$

Other mathematical symbols.

Symbol $A \sim B$ means that A is proportional to B.

Symbol $A \times B$ denotes a Cartesian product of A and B.

In the following we often use the symbol $\neq$ to denote that a considered object (function, vector field, r-form etc.) is nonvanishing in the considered neighbourhood.

As we said N denotes 3-dimensional submanifold of space-time M. Symbol $\omega_{|N}$ denotes restriction of r-form ω to N.

III. ASSUMPTIONS. TWO-FORM $\tilde{F} = F + i{*}F$ INVARIANT UNDER A CONGRUENCE OF LINES k.

Let us consider space-times modeled on a 4-dimensional, oriented manifold M equipped with the metric tensor g. The metric tensor is supposed to be Lorentzian of signature (+ + + −).

The space-times we consider posses some additional structure. This structure is entirely defined by the following assumptions on (M, g):

A1) there exists a real congruence of lines on M,

A2) there exists a real two-form $F \neq 0$ on M such that the complex two-form $\tilde{F} = F + i{*}F$ is preserved by the congruence defined in A1)

The assumptions A1) and A2) can be formalized as follows.

Let k be a nowhere vanishing vector field tangent to the congruence defined in A1). Define a family of vector fields $k<f>$ by

$$k<f> = f \cdot k \tag{3.1}$$

where f is any real function on M. The assumption A2) means that for any a

$$\mathcal{L}_{k<f>} \tilde{F} = 0 \tag{3.2}$$

where $\mathcal{L}_{k<f>}$ denotes Lie derivative of $\tilde{F}$ with respect to $k<f>$.

From now on, all our considerations have LOCAL character. We assume that we deal with neighbourhoods of nonsingular points of our constructions. We also assume that all vector fields considered in these neighbourhoods are REAL ANALYTIC. Therefore we do not encounter problems discussed by J. Tafel in [Tafel 1985].

IV. PROPERTIES OF $\tilde{F}$ AND k.

1. Two form $\tilde{F}$ is null.

Equation (3.2) is satisfied for any real function f. Taking this function as being equal to 1 everywhere we have

$$\mathcal{L}_{k\langle 1\rangle} \tilde{F} = \mathcal{L}_{k} \tilde{F} = 0 \tag{4.1.1}$$

hence

$$0 = \mathcal{L}_{k\langle f\rangle} \tilde{F} = df \wedge (k \lrcorner \tilde{F}) \tag{4.1.2}$$

The last equality, being true for any f, implies that

$$k \lrcorner \tilde{F} = 0 \tag{4.1.3}$$

i.e.

$$k \lrcorner F = k \lrcorner *F = 0 \tag{4.1.4}$$

The fact expressed by the equation (4.1.4) means by definition that two-form F is <u>null</u>. It is also called <u>simple</u> [Trautman 1986].

2. Congruence generated by k is null.

The definition (2.8) of the Hodge star implies in particular that for any p-form ω and any vector field X we have:

$$X \lrcorner *\omega = *(\omega \wedge g(X)) \tag{4.2.1}$$

The other simple consequence of (2.8) is

$$*\tilde{F} = -i\tilde{F} \tag{4.2.2}$$

Applying (4.2.1) for k, $\tilde{F}$ and using (4.1.3) we have:

$$0 = -ik \lrcorner \tilde{F} = k \lrcorner *\tilde{F} = *(\tilde{F} \wedge g(k)) \tag{4.2.3}$$

Since * is an isomorphism between $\Lambda^{p}M$ and $\Lambda^{4-p}M$ then

$$\tilde{F} \wedge g(k) = 0 \tag{4.2.4}$$

Differentiating (4.2.4) using $k^{\lrcorner}$ we have

$$\tilde{F}g(k,k) = 0 \tag{4.2.5}$$

i.e.

$$g(k,k) = 0 \tag{4.2.6}$$

Clearly equation (4.2.6) states that vector field k as well as the congruence generated by k is null.

3. One-forms κ and α defined by $\tilde{F}$.

The equation (4.2.4) implies that there exists a complex one-form s.t.

$$\tilde{F} = \kappa \wedge \alpha \tag{4.3.1}$$

where

$$\kappa = g(k) \tag{4.3.2}$$

The important fact is that

$$\kappa \wedge \alpha \wedge \bar{\alpha} \neq 0 \tag{4.3.3}$$

This can be seen by considering $\bar{\tilde{F}}$ and assuming that

$$\kappa \wedge \alpha \wedge \bar{\alpha} = 0 \tag{4.3.4}$$

In this case (4.3.4) and $\bar{\tilde{F}} \neq 0$ implies that

$$\bar{\alpha} = f\alpha + h\kappa, \tag{4.3.5}$$

where f is a nonvanishing complex function and h is some complex function. Now (4.3.5) and

$$*\bar{\tilde{F}} = i\bar{\tilde{F}} \tag{4.3.6}$$

show that

$$f*(\kappa \wedge \alpha) = if(\kappa \wedge \alpha), \tag{4.3.7}$$

hence

$$*\tilde{F} = i\tilde{F} \tag{4.3.8}$$

The last equality contradicts (4.2.2) and condition $F \neq 0$. This proves (4.3.3) i.e. linear independence of forms κ, α, $\bar{\alpha}$.

Since

$$k \lrcorner \kappa = g(k,k) = 0 \tag{4.3.9}$$

then

$$k \lrcorner \tilde{F} = 0$$

implies that

$$\begin{aligned} k \lrcorner \alpha = k \lrcorner \bar{\alpha} = 0 \\ k \lrcorner \alpha = 0 \end{aligned} \tag{4.3.10}$$

Introducing complex vector field a s.t.

$$\alpha = g(a) \tag{4.3.11}$$

and considering

$$0 = i^*(\tilde{F} \wedge \alpha) = a \lrcorner \tilde{F} = (a \lrcorner \kappa)\alpha - g(a,a)\kappa \tag{4.3.12}$$

we conclude that

$$\begin{aligned} a \lrcorner \kappa = a \lrcorner \alpha = 0 \\ \bar{a} \lrcorner \kappa = \bar{a} \lrcorner \bar{\alpha} = 0 \end{aligned} \tag{4.3.13}$$

This in particular means that

$$g(a,a) = g(\bar{a},\bar{a}) = 0 \tag{4.3.14}$$

i.e. that complex vector fields a and $\bar{a}$ are null.

It is worth noting that as a consequence of (4.3.3) and because of the signature of the metric

$$g(a,\bar{a}) \neq 0 \quad \text{and} \quad g(a,\bar{a}) > 0 \tag{4.3.15}$$

4. Relations between forms κ and α and the metric.

Since κ, α and $\bar{\alpha}$ are linearly independent then they can be supplemented by one additional real form λ s.t. κ, α. $\bar{\alpha}$ and λ form a cobasis in M. Moreover conditions (4.3.10), (4.3.13), (4.3.15) ensures that the real 1-form λ can be chosen in such a way that there exist a real vector field l and a real function $\mathcal{P}$ on M s.t.[2]

$$\lambda = g(l) \tag{4.4.1}$$

$$\mathcal{P}\kappa \wedge \alpha \wedge \bar{\alpha} \wedge \lambda \neq 0 \tag{4.4.2}$$

$$g = \mathcal{P}^2(\alpha \otimes \bar{\alpha} + \bar{\alpha} \otimes \alpha) - \kappa \otimes \lambda - \lambda \otimes \kappa \tag{4.4.3}$$

and

$$a \lrcorner \lambda = \bar{a} \lrcorner \lambda = l \lrcorner \lambda = 0 \tag{4.4.4}$$

$$k \lrcorner \lambda \neq 0, \qquad k \lrcorner \lambda < 0 \tag{4.4.5}$$

This in particular shows that vector fields k, l, a, $\bar{a}$ form a complex null tetrad on our manifold.

5. Robinson-Trautman conditions

Condition (4.1.1) ($\tilde{F}$ is preserved by k) and (4.1.4) (F is simple) imply that

$$0 = k \lrcorner d\tilde{F} \tag{4.5.1}$$

Using expression (4.3.1) for $\tilde{F}$ in terms of κ and α we obtain from (4.5.1):

$$(k \lrcorner d\kappa) \wedge \alpha + \kappa \wedge (k \lrcorner d\alpha) = 0 \tag{4.5.2}$$

This implies that

$$k \lrcorner d\kappa = f\kappa \tag{4.5.3}$$

and

$$k \lrcorner d\alpha = h\kappa + p\alpha \tag{4.5.4}$$

for some functions f (real) and h,p (complex).

Equations (4.5.3) and (4.5.4) compared with (4.3.10) show that

$$\mathcal{L}_k\kappa \wedge \kappa = 0 \tag{4.5.5}$$

$$\mathcal{L}_k\alpha \wedge \kappa \wedge \alpha = 0 \tag{4.5.6}$$

---

[2]By an appropriate rescaling of k and α we can always achieve $\mathcal{P}=1$. However, we prefer to keep a general function $\mathcal{P}$ in (4.4.3).

Using equations (4.5.5) and (4.5.6) and the form of the metric (4.4.3) one sees that

$$\mathcal{L}_{k} g = \rho g + \kappa \otimes \omega + \omega \otimes \kappa \qquad (4.5.7)$$

where ρ is an appropriate real function and ω an appropriate real one-form on M.

The condition (4.5.7) is called the Robinson-Trautman condition [Robinson, Trautman 1983]. I will also call conditions (4.5.5)-(4.5.6) as Robinson-Trautman conditions.

6. The freedom in the choice of k, $\tilde{F}$, κ and α

In section III we have chosen a vector field k as any nonvanishing vector field which was tangent to a congruence of lines defined by assumption A1). Such vector field is defined up to a transformation

$$k \rightarrow f \cdot k, \qquad (4.6.1)$$

where f is any nonvanishing function on M. The assumption A2) requires only the existence of two-form $\tilde{F}$ s.t. condition (3.2) of invariance of $\tilde{F}$ with respect to the congruence is satisfied. Let us notice that if we take new two form

$$\tilde{F}' = h\tilde{F} \qquad (4.6.2)$$

s.t. $h \neq 0$ is a complex function and

$$\mathcal{L}_{k} h = 0 \qquad (4.6.3)$$

then

$$F' = \mathrm{Re}(h\tilde{F}') \qquad (4.6.4)$$

also satisfies assumption A2).

Starting from this new F' we obtain new forms κ' and α' by formula (4.3.1). They are related to κ and α by

$$\kappa' = b\kappa \qquad (4.6.5a)$$

$$\alpha' = \frac{h}{b}\alpha + c\kappa, \qquad (4.6.5b)$$

where $b \neq 0$ is real function and c is a complex function.

V. INTERPRETATION

Last section was devoted to the structure which is connected with invariant under the congruence two-forms (assumptions A1) and A2)). Here we give some examples to understand why such structures are relevant in physics. We present some physical problems in which different structures from the preceding Chapter arise.

1. Null Maxwell fields

In the Maxwell theory one can consider electromagnetic fields with electric field $\vec{E}$ and magnetic field $\vec{B}$. The ratio

$$\vec{v} = \frac{\vec{E} \times \vec{B}}{\frac{1}{2}(\vec{E}^2 + \vec{B}^2)} \tag{5.1.1}$$

of the Poynting vector to the energy density characterizes the velocity of the propagation of the field. Its magnitude is always less or equal to unity. The equality occurs if and only if

$$\vec{E}\vec{B} = 0 = \vec{E}^2 - \vec{B}^2 \tag{5.1.2}$$

In this case field propagates with speed of light. In terms of the electromagnetic 2-form

$$F = dt \wedge \vec{E}d\vec{r} + \frac{1}{2}\, d\vec{r} \stackrel{\cdot}{\wedge} (\vec{B} \times d\vec{r}) \tag{5.1.3}$$

this fact can be described by

$$F \wedge F = F \wedge * F = 0 \tag{5.1.4}$$

where

$$*F = dt \wedge \vec{B}d\vec{r} - \frac{1}{2}\, d\vec{r} \stackrel{\cdot}{\wedge} (\vec{E} \times d\vec{r}) \tag{5.1.5}$$

It is matter of checking that conditions (5.1.2) or (5.1.4) imply that the vector field

$$k = \frac{\partial}{\partial t} + \vec{v}\,\frac{\partial}{\partial \vec{r}} \tag{5.1.6}$$

with $\vec{v}$ given by (5.1.1) is such that

$$k \lrcorner F = k \lrcorner * F = 0 \tag{5.1.7}$$

This shows that electromagnetic fields for which $|\vec{v}| = 1$ or (what is equivalent) $\vec{E}\vec{B} = 0 = \vec{E}^2 - \vec{B}^2$ are null fields in sense of the definition (4.1.4). This fields are pure radiation fields, i.e. describe the situation in which all electromagnetic energy propagates in one direction k with the speed of light.

Since we assumed that fields $\vec{E}$ and $\vec{B}$ were electromagnetic i.e. satisfied Maxwell equations

$$dF = d * F = 0 \tag{5.1.8}$$

then, as a consequence of (5.1.7) and (5.1.8), we see that

$$\tilde{F} = F + i{*}F \tag{5.1.9}$$

satisfies

$$\underset{k}{\mathcal{L}}\tilde{F} = 0 \tag{5.1.10}$$

Hence we find the first example of manifold (M,g) (Minkowski space-time) on which there exists an invariant under the congruence (generated by k from (5.1.6)) two-form $\tilde{F}$ (two-form (5.1.9) constructed out of pure radiation electromagnetic field) [Robinson, Trautman 1989]. All other examples will be generalization of this in such sense that they will describe situations of pure radiation classical fields.

2. Robinson theorem

In this Section we study a question when a given space-time (M,g) can admit a pure radiation electromagnetic field i.e. field described by a two-form F s.t.

$$dF = d * F = 0 \tag{5.2.1a}$$

$$F \wedge F = F \wedge {*F} = 0 \tag{5.2.1b}$$

Before answering this question we need some definitions.

A congruence of lines in the space-time consists of geodesics if for any vector field k generating it the following condition is satisfied

$$\nabla_k k = fk. \tag{5.2.2}$$

Here f is any real function on (M,g) and ∇_k denotes the covariant derivation of Levi-Civita connection associated with g. To define the so called shear-free property of a congruence of null geodesics we choose an affine parameterization for k. Then we have

$$k_{a;b}k^b = 0. \tag{5.2.3}$$

Now we can define shear σ by

$$\sigma\bar{\sigma} = \frac{1}{2} k_{(a;b)}k^{a;b} - \frac{1}{4}(k^a{}_{;a})^2 \tag{5.2.4}$$

One says that congruence generated by k is shear-free iff

$$\sigma = 0 \tag{5.2.5}$$

Now, we can give the answer to the question we addressed at the beginning of this section.

Theorem 5.2.1 [Robinson 1961]

A space-time (M,g) admits an electromagnetic pure radiation field if and only if (M,g) admits congruence of shear-free and null geodesics.

To illustrate how a shear-free condition (5.2.5) appears in the Robinson theorem one considers an energy momentum tensor of a pure radiation electromagnetic field. This has the form

$$T_{ab} = \Phi\bar{\Phi}k_a k_b \tag{5.2.6}$$

where $\Phi \neq 0$. The algebraic fact

$$T_a{}^a = 0, \tag{5.2.7}$$

which is true for any energy momentum tensor originating from electromagnetic field, shows that vector field k_a is null. A covariant conservation of T_{ab} i.e.

$$T_{ab}{}^{;b} = 0 \tag{5.2.8}$$

implies geodesic condition (5.2.2) for k_a. The fact that T_{ab} is constructed out of an electromagnetic field

$$F = \frac{1}{2} F_{ab} e^a \wedge e^b \qquad (5.2.9)$$

satisfying Maxwell equations (5.2.1a) shows that shear of k_a defined by (5.2.4) vanishes.

The physical interpretation of shear is as follows. Suppose that we put a 2-dimensional obstacle perpendicularly to the rays of congruence. We observe the shape of this obstacle on the screen which also is put perpendicularly to the congruence. The property of $\sigma = 0$ is reflected in the transformation φ between obstacle and its shape on the screen by the condition that φ is a conformal transformation.

This in particular means that the shape can be rotated and expanded in comparison with the obstacle, but never can be sheared. For instance a circle will never has a shape of an ellipse on the screen. The proof of this fact can be found for example in [Sachs 1962].

Here we will see that if the congruence generated by k is null and satisfies the Robinson-Trautman condition (4.5.7) for $\kappa = g(k)$ then it is geodesic and shear-free [Robinson, Trautman 1983]. The condition (4.5.7) in the language of indices can be written as

$$k_{a;b} + k_{b;a} = \rho g_{ab} + k_a \omega_b + k_b \omega_a \qquad (5.2.10)$$

Contracting (5.2.10) with k^b and using the fact

$$k_a k^a = 0 \quad \text{i.e.} \quad k_{a;b} k^a = 0$$

we obtain

$$k_{a;b} k^b = (\rho + k\omega) k_a \qquad (5.2.11)$$

what is a geodesic condition (5.2.2) for k, where $f = \rho + k\omega$.

Now we can choose an affine parameterization of k. In this way (5.2.10) will change into

$$k_{a;b} + k_{b;a} = \tilde{\rho} g_{ab} + k_a \tilde{\omega}_b + k_b \tilde{\omega}_a \qquad (5.2.12)$$

with $\tilde{\rho}$ and $\tilde{\omega}$ s.t.

$$\tilde{\rho} + k\tilde{\omega} = 0 \tag{5.2.13}$$

Computing σ using (5.2.4) and (5.2.12) we obtain

$$\sigma\bar{\sigma} = 0 \tag{5.2.14}$$

Formulas (5.2.11) and (5.2.14) show that Robinson-Trautman condition for null congruence implies its geodesic and shear-free properties. The converse is also true. All shear-free geodesic null congruences have to satisfy Robinson-Trautman condition [Robinson, Trautman 1983]. In view of the above the Robinson theorem can be formulated as follows.

Theorem 5.2.2

A space-time (M,g) admits an electromagnetic pure radiation field if and only if it admits a congruence of null lines generated by the vector field k which satisfy the Robinson-Trautman condition (4.5.7) with $\kappa = g(k)$ and ρ, ω being arbitrary real function and 1-form, respectively.

Now let us consider technical part of the Robinson theorem 5.2.2. This implies that congruence generated by k, which is null and satisfies Robinson-Trautman condition, defines two-form $F \neq 0$ s.t.

$$F \wedge F = F \wedge {}^*F = 0 \tag{5.2.15a}$$

$$dF = d{}^*F = 0 \tag{5.2.15b}$$

It can be shown that the two-form

$$\tilde{F} = F + i{}^*F \tag{5.2.16}$$

is related to k by

$$k \lrcorner \tilde{F} = 0 \tag{5.2.17}$$

This together with (5.2.15b) show that

$$\underset{k}{\mathcal{L}}\tilde{F} = 0. \tag{5.2.18}$$

Since $\tilde{F}$ satisfies (5.2.17) and (5.2.18) then it is preserved by the congruence k. Hence as a consequence of the Robinson theorem 5.2.2 we see that Robinson-Trautman condition implies that the space-time (M,g) is of the form considered in Chapter IV (i.e. satisfies assumptions A1) and A2)). In IV.5) we also showed that assumptions A1) and A2) imply Robinson-Trautman condition for the congruence of A1). Hence we arrive at the following proposition

Proposition 5.2.1

The following conditions are equivalent:

1) Space-time (M,g) admits a real two-form $F \neq 0$ s.t. the complex two form $\tilde{F} = F + i * F$ is invariant under some congruence.

2) Space-time (M,g) admits null congruence which satisfies Robinson-Trautman condition.

Congruences appearing in 1) and 2) are the same.

3. Goldberg-Sachs theorem

Now let us consider space-time (M,g) which admits a shear-free congruence of null geodesics k. It is interesting to ask what are relations between k and curvature of (M,g). First remark is as follows. Since shear-free geodesic null property of k is conformally invariant it is more reasonable to consider relations between k and conformal curvature (Weyl tensor) of (M,g) rather, than ordinary curvature (Riemann tensor). Proposition 5.2.1 shows that the metric tensor g on our space-time has the form (4.4.3). Computing the Weyl tensor $C^a{}_{bcd}$ for the metric (4.4.3) and using the facts (4.5.5) and (4.5.6) one can see that [Debever 1959]

$$k_{[e}C_{a]bc[d}k_{f]}k^b k^c = 0 \qquad (5.3.1)$$

We will prove this fact more explicitly in the section X.2

The property (5.3.1) tells, by definition, that k is a <u>principal null direction</u> of the Weyl tensor. Principal null directions of the Weyl tensor were considered for the first time by E. Cartan[3] (I thank A. Trautman for this comment). The following theorem is implicit in his note [Cartan 1922].

<u>Theorem 5.3.1</u>

Any space-time (M,g) is either conformally flat or admits at least one, and in general four, principal null directions. The following possibilities can occur:

1) All four principal null directions coincide. This is equivalent to

$$C_{abcd}k^c = 0 \qquad (5.3.2)$$

where k^c is tangent to the common principal null direction.

2) Three principal null directions coincide. This is equivalent to

$$C_{abc[d}k_{f]}k^c = 0 \qquad (5.3.3)$$

where k^c is tangent to the common principal null

---

[3] Since this fact is not very much known we quote here Cartan's considerations from a note "Sur les espaces conformes generalises et l'Universe optique" C.R. Acad. Sc., t 174, p 857 <u>(1922!)</u>: "From a geometric viewpoint, it is worthwhile to note an interesting property. At each point A there exist four privileged null directions (i.e. those for which $ds^2=0$). They can be characterized as follows: Any one of these directions, say AA', is invariant under transport around an infinitesimal parallelogram one of whose sides is AA' and the other of whose sides is along an arbitrary null directions at A. In the case of the ds^2 corresponding to a single attractive mass (ds^2 of Schwarzschild) the four privileged directions reduce to two (degenerate) directions which correspond to null rays pointing to or from the center of attraction." (translation of A. Ashtekar and A. Magnon)

direction.

3) Two principal null directions coincide. This is equivalent to

$$C_{abc[d}k_{f]}k^b k^c = 0 \qquad (5.3.4)$$

where k^c is the common principal null direction.

4) All four principal null directions are distinct. This is equivalent to the existence of four null vector fields k_I, k_{II}, k_{III}, k_{IV} satisfying (5.3.1).

The principal null directions k from the points 1), 2), 3) and 4) of the above theorem are called principal null directions of multiplicity 4, 3, 2 and 1 respectively.

The existence of principal null directions enables algebraical classification of gravitational fields. Given a Weyl tensor of an arbitrary gravitational field associated with a metric g one can ask about the number of principal null directions. The following types of gravitational fields can occur.

1) Type N $\Longleftrightarrow$ there is a principal null direction of multiplicity 4

2) Type III $\Longleftrightarrow$ there is a principal null direction of multiplicity 3

3) Type II $\Longleftrightarrow$ there is one principal null direction of multiplicity 2

3a) Type D $\Longleftrightarrow$ there are two principal null directions of multiplicity 2

4) Type I $\Longleftrightarrow$ all principal null directions have multiplicity 1.

Situations 1), 2), 3) and 3a) are called algebraically special; 4) - algebraically general. The same terminology is applied to the gravitational fields corresponding to particular situations.

The above types are called Petrov types in the literature [see for instance Kramer at al. 1980]. However, in our opinion, their name should be also associated with E. Cartan and R. Penrose. In particular in the note [Cartan 1922] Cartan observed for the first time that for any spacetime there exist in general four principal null directions. He also noticed that in the case of Schwarzschild solution [Schwarzschild 1916] these directions coincide in such a way that the metric is of type D. The Cartan work was overlooked by the relativity community. His principal null directions were then rediscovered by R. Penrose in his works on spinorial approach to gravitation [Penrose 1960]. He was also the first who spelled out all possible degeneracies 1), 2), 3), 3a) and 4) of principal null directions. On the other hand A. Z. Petrov [Petrov 1954] gave a slightly different classification based on a study of eigenbivectors of a Weyl tensor. This classification was insufficient to distinguish some of the particular types given above. Namely types N and D were hidden in types II and I, respectively [Ehlers, Kundt 1962, Pirani 1962]. Therefore we will call types I, II, III, N and D as Cartan-Petrov-Penrose types.

At the beginning of this section (eq.(5.3.1)) we proved the following theorem.

Theorem 5.3.2

Any shear-free geodesic null congruence is a principal null direction of the Weyl tensor [Robinson, Trautman 1989, Lewandowski, Nurowski 1990b].

There exists even stronger theorem for gravitational fields which satisfy Einstein equations with the energy momentum tensor of pure radiation (5.2.6).

Theorem 5.3.3 [Goldberg-Sachs 1962]

Suppose that space-time (M,g) satisfies pure radiation Einstein equations

$$R_{ab} = \kappa_o \Phi\bar{\Phi}\, k_a k_b \qquad (5.3.5)$$

where $\Phi\bar{\Phi} \geq 0$, $k_a k^a = 0$.

If

1) k_a is shear-free and geodesic in the case $\Phi \neq 0$

or

2) g admits a shear-free geodesic and null congruence in the case $\Phi = 0$

then the metric is algebraically special.

For vacuum fields ($\Phi = 0$) the converse is also true i.e. any algebraically special vacuum metric admits shear-free geodesic null congruence.

The importance of algebraically special gravitational fields in gravitation theory is due to the following theorem.

Peeling-off Theorem 3.3.4 [Sachs 1962]

Suppose that (M,g) satisfies vacuum Einstein equations, and that the gravitational field is asymptotically flat.

The Riemann tensor (which equals the Weyl tensor since $R_{ab} = 0$) has the form:

$$C = \frac{N}{r} + \frac{III}{r^2} + \frac{II}{r^3} + \frac{G}{r^4} + \frac{I}{r^5} + \vartheta(r^{-6}) \qquad (5.3.6)$$

Here indices are suppressed, letters in the numerators indicate an algebraic type of the Weyl tensor[4] (G is a subtype of I s.t.

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[4]Here following F. A. E. Pirani [Pirani 1962] we use the name "Weyl tensor" in a wider than usual sense, namely as any tensor which has the same symmetries that usual Weyl tensor possesses.

there exists a geodesic principal null direction) and r is a radial coordinate along null rays which measures distance from the sources of gravitational field.

The above theorem shows that the order (with respect to the powers of r) of terms in (5.3.6) coincides with the order with respect to the algebraical complexity of corresponding to these terms gravitational fields. Namely, the more leading term in r of (5.3.6) corresponds to the more algebraically special field this describes. In particular, very far from the line $r=0$, which can be considered as a world line of a particle being close to the sources of gravitational field, the term

$$\frac{N}{r} \tag{5.3.7}$$

dominates. This means that gravitational radiation observed very far from the sources should behave as being of Cartan-Petrov-Penrose type N. This fact, for the first time observed by A. Trautman [Trautman 1958], was then generalized to the peeling-off theorem [Sachs 1962], and gave a support for looking for gravitational radiation as being of type N [Pirani 1957, Robinson, Trautman 1962].

Concluding what we have so far presented here we can collect facts included in Theorem 5.2.1 and 5.2.2, Proposition 5.2.1 and Theorem 5.3.3 in the following corollary.

Corollary 5.3.1

Suppose that space-time (M,g) satisfies vacuum Einstein equations.

The following conditions are equivalent

1) (M,g) admits shear-free geodesic null congruence
2) (M,g) admits null congruence satisfying Robinson-Trautman condition (4.5.7)
3) (M,g) admits pure radiation electromagnetic field

4) (M,g) is algebraically special

5) (M,g) admits two-form F such that $\tilde{F} = F + i{*}F$ is invariant under some congruence

Remark 5.3.1

1) As far as only equivalence of 1), 2), 3) and 5) is concerned in the above corollary the assumption about Ricci flatness of (M,g) can be abandoned.

2) To prove implications 1), 2), 3), 5) $\Rightarrow$ 4) it is enough to assume that (M,g) satisfies pure radiation Einstein equations (5.3.5).

Since any of the conditions 1), 2), 3) and 5) of corollary 5.3.1 are equivalent, for the characterization of space-time we consider, we choose condition 1). Hence we can say that from now on (as well as from the very beginning of this work) we consider only such space-times which admit a shear-free geodesic null congruence.

4. Metrics associated with shear-free geodesic null congruences.

So far we have studied a given space-time M with a given metric g and the implication of existence on (M,g) of a congruence of shear-free and null geodesics k. Now we can extend our considerations and ask how many different metrics g′ one can associate with (M,g,k) in such a way that a congruence k is still shear-free geodesic and null in the metric g′. The answer is given by the following theorem

Theorem 5.4.1 [Trautman 1984]

If vector field k is shear-free null and geodesic in the metric g then it is also shear-free null and geodesic in the metric g′ s.t.

$$g' = \tau^2 g + \kappa \otimes \omega + \omega \otimes \kappa \qquad (5.4.1)$$

where τ is a nowhere vanishing real function, $\kappa = g(k)$ and ω is any real 1-form on M. The form (5.4.1) of g′ is the most general metric for which k is null geodesic and shear-free.

To prove the first part of theorem 5.4.1 it is enough to check that g′ satisfies Robinson-Trautman condition (4,5.7) for k. This is obvious if one notices that g satisfies Robinson-Trautman condition (4.5.7) and applies (4.5.5). The proof of the second part of the Theorem 5.4.1 can be found in [Robinson, Trautman 1989].

VI. OPTICAL GEOMETRY

The theory called now optical geometry evolved from the studies of the objects we have so far presented here. The following abstractions which lead to the definition of an optical geometry have obvious representatives in the examples we have considered in the preceding sections.

1. Flag geometry and adapted p-forms

The flag geometry on a 4-dimensional oriented manifold M is a pair (K,L) of two line bundles s.t. $K \subset L \subset TM$ and fibers of K,L are 1- and 3- dimensional respectively.

The Lorentzian metric tensor g on M is adapted to (K,L) if the bundles K and L are perpendicular with respect to g. It is worth noticing that since $K \subset L$ then K is perpendicular to itself i.e. the bundle K is null with respect to any adapted metric tensor.

The property of being adapted to (K,L) for g can be equivalently expressed in terms of sections of K and $L^{o} \subset T^{*}M$ - the bundle which consists of all 1-forms on M annihilating L. Bundles K and L^{o} have fibers of dimensions 1 as a consequence of their definitions. Therefore they can be described by any of their nonvanishing sections i.e. a vector field k which is defined up to the transformation

$$k \longrightarrow k' = \rho k \qquad 6.1.1a)$$

and a field of a 1-form κ defined up to the transformation

$$\kappa \longrightarrow \kappa' = \tau\kappa \qquad (6.1.1b)$$

It is easy to see that the metric tensor g is adapted to (K,L) iff

$$k \lrcorner\, g(k) = 0 \qquad (6.1.2a)$$

$$\kappa \wedge g(k) = 0 \qquad (6.1.2b)$$

Similarly, we define an adapted to (K,L) p-form (p = 1,2,3) F on M as such a p-form on M which satisfies

$$k \lrcorner\, F = 0 \qquad (6.1.3a)$$

$$\kappa \wedge F = 0 \tag{6.1.3b}$$

Note, that as one would expect the definitions (6.1.2) and (6.1.3) are invariant under the transformations (6.11). This shows that property of being adapted for p-form depends only on (K,L).

Considering k and κ as above one sees that property

$$\mathcal{L}_k g(k) \wedge g(k) = 0 \tag{6.1.4a}$$

or equivalently

$$\mathcal{L}_k \kappa \wedge \kappa = 0 \tag{6.1.4b}$$

is also invariant under (6.1.1). This expresses the fact that the congruence generated by k is a congruence of null geodesic with respect to any adapted metric g. The flag geometry (K,L) satisfying (6.1.4) is called <u>geodesic</u>. Since from now on we will consider only geodesic flag geometries we give proposition which characterizes them from different points of view.

<u>Proposition</u> <u>6.1.1</u> [Robinson-Trautman 1985]

The following properties of flag geometry (K,L) (represented by k and κ as above) are equivalent:

i) $\mathcal{L}_k \kappa \wedge \kappa = 0$

ii) $\kappa \wedge d\kappa$ is adapted

iii) the lines of the flow $\varphi_t(k)$ generated by the vector field k define a congruence of null geodesics with respect to any metric tensor adapted to (K,L)

iv) if F is an adapted 2-form, then $\kappa \wedge dF = 0$.

A simple consequence of Proposition 6.1.1 describes the following corollary

<u>Corollary</u> <u>6.1.1</u>

If the bundle L of the flag geometry (K,L) is integrable then (K,L) is geodesic.

Proof follows directly from ii) in Proposition 6.1.1 and the

Fröbenius theorem which states that L is integrable if and only if

$$\kappa \wedge d\kappa = 0 \tag{6.1.5}$$

2. Optical geometry, isomorphism of optical geometries, shear-free property

The flag geometry is enough to define a congruence of null geodesics on M and the notion of an adapted (simple, null) p-form. In particular, this is enough to define a null 2-form F and the first part of the Maxwell equations

$$dF = 0 \tag{6.2.1a}$$

for the Maxwell field F defined on the Lorentzian manifold M with any adapted Lorentzian metric tensor g. However, if the second Maxwell equation

$$d \underset{g}{*} F = 0 \tag{6.2.1b}$$

is satisfied in one of the adapted metric tensors g, it is not necessarily satisfied in any other adopted metric tensor. Therefore a given adapted Maxwell field (i.e. satisfying (6.2.1)) is not an object of the flag geometry (K,L). The weakest geometry needed on a 4-dimensional manifold M to write the full set of Maxwell's equations for null electromagnetic field is called <u>an optical geometry</u>. This is a subclass of flag geometries defined as follows.

Let (K,L) be a flag geometry on M and let A be the set of all adapted metric tensors. Suppose that F is an adapted 2-form. It is easy to see that if $g \in A$ then $\underset{g}{*} F$ is also adapted 2-form. We introduce an equivalence relation R in A by

$$gRg' \Leftrightarrow \underset{g}{*} F = \underset{g'}{*} F \tag{6.2.2}$$

An <u>optical geometry</u> on M consists of the pair (K,L) together with an element $[g] \in A/R$ and an orientation of the vector bundle L/K with fibers of dimension 2. The last condition that L/K is oriented endows L/K with the structure of a complex line bundle over M. This occurs because one can define a linear bundle morphism

$$\mathfrak{I} : L/K \longrightarrow L/K \tag{6.2.3a}$$

s.t.

$$\mathfrak{I}^2 = -\mathrm{id} \tag{6.2.3b}$$

This is defined by the demand that on any fiber $(L/K)_x$ $\mathfrak{I}$ transforms any vector $v \in (L/K)_x$ to the vector v' rotated by $\frac{\pi}{2}$ accordingly with the orientation in $(L/K)_x$. (This is possible because fibers of $(L/K)_x$ are 2-dimensional). Saying that $v \in (L/K)_x$ is rotated by $\frac{\pi}{2}$ requires specification of the metric tensor on $L/K \otimes L/K$, or, at least, conformal class of such metrics. This class can be defined by using a field of a complex 1-form $\alpha \neq 0$ defined on M by the relation

$$*(\kappa \wedge \alpha) = -i\kappa \wedge \alpha \tag{6.2.4}$$

where κ is associated with (K,L) by (6.1.1b).

The relation (6.2.4) defines α up to the transformation

$$\alpha \longrightarrow \alpha' = h\alpha + p\kappa \tag{6.2.5}$$

where $h \neq 0$ and p are arbitrary complex functions on M. The arguments similar to those in IV.3) show that

$$\kappa \wedge \alpha \wedge \bar{\alpha} \neq 0 \tag{6.2.6}$$

Taking k as any nonvanishing section of K and applying the identity (4.2.1) to $X = k$, $\omega = \kappa \wedge \alpha$ we have

$$k \lrcorner \alpha = 0 \tag{6.2.7}$$

where we also used the property (6.1.3a). This allows us to define metric tensor

$$g_2 = \alpha \otimes \bar{\alpha} + \bar{\alpha} \otimes \alpha \tag{6.2.8}$$

which defines conformal class $[g_2]$ in L/K because of (6.2.7) and (6.2.5).

It is worth noting here that since fibers of an annihilator K^o of K are three dimensional and since (6.2.6) is valid, then K^o is spanned by κ, $Re\alpha$, $Im\alpha$.

The fact that L/K has the structure of a complex line bundle over M will be of great importance in next sections.

The following proposition is due to Robinson and Trautman:

Proposition 6.2.1

i) Relation R defined by (6.2.2) does not depend on F.

ii) Two metric tensors g and g′ are in the relation R iff

$$g' = \tau^2 g + \kappa \otimes \omega + \omega \otimes \kappa \qquad (6.2.9)$$

where $\tau \neq 0$ is any real function on M, ω is any real 1-form on M and κ is any nonvanishing section of the bundle annihilating L in (K,L). κ can also be characterized as

$$\kappa = fg(k) \qquad (6.2.10)$$

for some real nonvanishing function f and k being any nonvanishing section of K.

The convenient way of defining an optical geometry on an oriented manifold M can be described as follows. Suppose we have a pair (g,k) where g is a Lorentzian metric tensor on M and k is a null vector field on M. We can define an optical geometry on M by saying that:

i) $K = \{k' \in \mathcal{X}(M) : k' = fk \,|\, f \neq 0 \text{ is a real function on } M\}$

ii) $L = \{l \in \mathcal{X}(M) : g(l,k) = 0\}$

iii) an element $[g] \in A/R$ is given by g or any other g′ related to g by (6.2.9).

iv) an orientation in L/K is clock wise.

Hence, to define an optical geometry (K,L) on a 4- dimensional oriented manifold M it is enough to specify a pair (g,k), where g is a Lorentzian metric on M and k is a null vector field on M, and then to apply procedure i), ii), iii) as above. We will prefer this way of defining optical geometries. Therefore we will speak about optical geometry generated by the pair (g,k) or simply optical geometry (g,k). This last name does not introduce any confusion if one realizes that different pairs (g,k) can give rise to the same optical geometry as it is clear from the above presented procedure i), ii), iii) and iv). Suppose now that we have two 4-dimensional oriented manifolds M and M′ with optical geometries generated by pairs (g,k) and (g′,k′) respectively. We say that <u>optical geometries</u> (M,g,k) and (M′,g′,k′) <u>are (optically) isomorphic</u> iff there exists a diffeomorphism

$$\Phi : M \longrightarrow M'$$

s.t.

$$\Phi_* k = f' k' \tag{6.2.11a}$$

$$\Phi^* g' = \tau^2 g + g(k) \otimes \omega + \omega \otimes g(k) \tag{6.2.11b}$$

for some real function $f' \neq 0$ on M′ and some real function $\tau \neq 0$ and 1-form ω on M.

An optical geometry (g,k) was introduced as the weakest structure which is needed to <u>write</u> full set of Maxwell's equations for the null two-form F. So far, however, we have only exploited the fact that

$$\underset{g}{*} F = \underset{g'}{*} F \tag{6.2.12}$$

for any g′ adopted to an optical geometry (g,k) (i.e. related to g by (6.2.9)). If we additionally assume that it is possible not only <u>to write</u> Maxwell's equations for F but also <u>to find</u> such null two-form F that Maxwell's equations

$$dF = d \underset{g}{*} F = 0 \tag{6.2.13}$$

are satisfied, then we will restrict the class of optical geometries. To see this let us note that in virtue of iv) of proposition 6.1.1 optical geometry which satisfies (6.2.13) is geodesic. Moreover, the Robinson theorem 5.2.2 shows that

$$\mathcal{L}_k g = \rho g + \kappa \otimes \omega + \omega \otimes \kappa \qquad (6.2.14)$$

where ρ, κ, ω are as those in (4.5.7). It is obvious that the Robinson-Trautman condition (6.2.14) is also satisfied for any other g' related to g by (6.2.9) and any other k' related to k by (6.1.1a). Therefore this is universal condition for an optical geometry (g,k). Since k is null the condition (6.2.14) means that k is shear free. Hence, the optical geometry (g,k), in which one can find a null two-form F satisfying Maxwell's equations (6.2.13) (or equivalently optical geometry satisfying Robinson-Trautman condition (6.2.14)) is called <u>shear-free</u> <u>optical</u> <u>geometry</u>.

Before passing to the next section we present one more point of view for an optical geometry. We know that this is generated by the pair (g,k) where g is Lorentzian metric tensor on M and k is null vector field. It is obviously also generated by any other pair (g',k') where g' is related to g by (6.2.9) and k' is related to k by (6.1.1a). The pair (g,k) defines a real 1-form

$$\kappa = g(k) \qquad (6.2.15)$$

The pair (g',k') defines a real 1-form κ' which relates to κ by

$$\kappa' = f\kappa \qquad (6.2.16)$$

for some real function $f \neq 0$. The definition (6.2.4) of a complex 1-form α can be applied to κ from (6.2.15). Such α is defined up to the freedom (6.2.5). If one defines α starting from κ' of (6.2.16) one sees that such α is in the class (6.2.5). Therefore we see that the optical geometry (g,k) defines a class of pairs of 1-forms $[(\kappa,\alpha)]$ s.t. κ is real, α is complex and $(\kappa',\alpha') \in [(\kappa,\alpha)]$ iff

$$\kappa' = f\kappa \qquad (6.2.17a)$$

$$\alpha' = h\alpha + p\kappa. \qquad (6.2.17b)$$

Conversely if $[(\kappa,\alpha)]$ constitutes a class of pairs of one-forms (real and complex respectively) defined up to the freedom (6.2.11) on the oriented manifold M then one can define a class of metric tensors [g] given by any tensor of the form

$$g = \alpha \otimes \bar{\alpha} + \bar{\alpha} \otimes \alpha - \kappa \otimes \varphi - \varphi \otimes \kappa \tag{6.2.18}$$

where κ and α are any forms from the class $[(\kappa,\alpha)]$ and φ is a real 1-form s.t.

$$\kappa \wedge \alpha \wedge \bar{\alpha} \wedge \varphi \neq 0 \tag{6.2.19}$$

It is the matter of checking that if g is a metric tensor corresponding to the pair (κ,α) then the metric tensor g′ for any other pair (κ',α') from the class $[(\kappa,\alpha)]$ is related to g by (6.2.9) with appropriate τ and ω. Introducing real vector field k $\neq 0$ by the relation

$$k \rfloor \alpha = k \rfloor \kappa = 0 \tag{6.2.20}$$

we see that it is defined by the pair $[(\kappa, \alpha)]$ up to

$$k \longrightarrow k' = \rho k \tag{6.2.21}$$

Moreover k is null in any metric of the form (6.2.18). The pair (g,k) defined by $[(\kappa,\alpha)]$ through (6.2.18), (6.2.20) generates an optical geometry.

Therefore we have <u>another definition of an optical geometry</u> as a structure on a 4-dimensional oriented manifold M consisting of a class of pairs of one-form κ (real), α (complex) s.t. the pair (κ,α) is in the same class what (κ',α') if and only if (κ',α') is related to (κ,α) by (6.2.17).

The following proposition follows from the Robinson theorem 5.2.2 and considerations of the section III and IV (compare (4.5.5), (4.5.6) with (4.5.7))

<u>Proposition 6.2.2</u>

The optical geometry $[(\kappa,\alpha)]$ is shear-free if and only if

$$\mathcal{L}_k \alpha \wedge \kappa = 0 \tag{6.2.22a}$$

$$\mathcal{L}_{\mathbf{k}}\, \alpha \wedge \kappa \wedge \alpha = 0 \qquad (6.2.22b)$$

where k is related to κ, α by (6.2.20).

VII. SHEAR-FREE OPTICAL GEOMETRIES AND CAUCHY-RIEMANN STRUCTURES

1. Quotient of space-time and the congruence of shear-free and null geodesics.

From now on we consider only shear-free optical geometries. We know that they can be represented by the class of one forms $[(\kappa, \alpha)]$ defined on a 4-dimensional oriented manifold M up to the transformations

$$\kappa \longrightarrow \kappa' = f\kappa \tag{7.1.1a}$$

$$\alpha \longrightarrow \alpha' = h\alpha + p\kappa \tag{7.1.1b}$$

where κ is real, α is complex

$$\kappa \wedge \alpha \wedge \bar{\alpha} \neq 0 \tag{7.1.1c}$$

and $f \neq 0$ is a real function on M, $h \neq 0$, p are complex functions on M. The shear-free property is expressed by relations

$$\mathcal{L}_k \kappa \wedge \kappa = 0 \tag{7.1.2a}$$

$$\mathcal{L}_k \alpha \wedge \alpha \wedge \kappa = 0 \tag{7.1.2b}$$

where $k \neq 0$ is a real vector field on M defined by

$$k \lrcorner \alpha = k \lrcorner \kappa = 0. \tag{7.1.3}$$

The most general form of metric tensor g adapted to the optical geometry in question is

$$g = P^2(\alpha \otimes \bar{\alpha} + \bar{\alpha} \otimes \alpha) - \varphi \otimes \kappa - \kappa \otimes \varphi \tag{7.1.4a}$$

where $P \neq 0$ is any real function on M and φ is any real 1-form on M s.t.

$$\kappa \wedge \alpha \wedge \bar{\alpha} \wedge \varphi \neq 0. \tag{7.1.4b}$$

The most general form of two-form F adapted to our optical geometry is given by

$$F = \mathrm{Re}(A\kappa \wedge \alpha) \tag{7.1.5}$$

where $A \neq 0$ is a complex function on M. The complex two-form

$$\tilde{F} = F + i{*}F \tag{7.1.6}$$

is given by

$$\tilde{F} = A\kappa \wedge \alpha \tag{7.1.7}$$

As we know equations (7.1.3) define in fact congruence of lines which are shear-free geodesic and null in the metric (7.1.4a). Consider now a flow $\varphi_t(k)$ generated by the field k. The conditions (7.1.2) are infinitesimal versions of the expressions:

$$\phi_t^*(k)\kappa = f\kappa \tag{7.1.8a}$$

$$\varphi_t^*(k)\alpha = h\alpha + p\kappa \tag{7.1.8b}$$

for some functions $f \neq 0$ (real) and $h \neq 0$, p (complex). Now let us pass to the quotient manifold

$$N = M/S \tag{7.1.9}$$

where S is the equivalence relation identifying points on the same line of the congruence generated by k [Robinson, Trautman 1985]. Let

$$\pi : M \longrightarrow N \tag{7.1.10}$$

be a canonical projection.

Since for any t a pair $(\varphi_t^*(k)\kappa,\ \varphi_t^*(k)\alpha)$ is in the same class as the pair (κ,α) we see that a class $[(\kappa,\alpha)]$ has its counterpart $[(\kappa_N,\alpha_N)]$ in the quotient manifold N, and $\pi^*(\kappa_N,\alpha_N)$ is in the same class as $[(\kappa,\alpha)]$. Therefore a class $[(\kappa,\alpha)]$ in M defines a class $[(\kappa_N,\alpha_N)]$ in N. It follows from the construction that two pairs (κ'_N,α'_N) and (κ_N,α_N) are in the same class in N if and only if

$$\kappa'_N = f_N\kappa_N \tag{7.1.11a}$$

$$\alpha'_N = h_N\alpha_N + p_N\kappa_N \tag{7.1.11b}$$

where $f_N \neq 0$ is a real function and $h_N \neq 0$, p_N are complex functions on N = M/S. It is also easy to see that

$$\kappa_N \wedge \alpha_N \wedge \bar{\alpha}_N \neq 0 \tag{7.1.11c}$$

From now on we will omit subscripts N in the expressions for κ_N and α_N.

2. Robinson-Trautman correspondence

The construction derived here shows that with any shear-free optical geometry we can associate a 3-dimensional real manifold N with a class of pairs of one-forms $[(\kappa,\alpha)]$ (κ - real, α - complex) on N satisfying (7.1.11c) and defined up to the transformations (7.1.11a-b).

Definition 7.2.1

A three-dimensional real manifold N with a class of pairs of one forms $[(\kappa,\alpha)]$ (κ - real, α - complex) satisfying

$$\kappa \wedge \alpha \wedge \bar{\alpha} \neq 0 \tag{7.2.1}$$

and defined up to the transformations

$$\kappa \longrightarrow \kappa' = f\kappa \tag{7.2.2a}$$

$$\alpha \longrightarrow \alpha' = h\alpha + p\kappa \tag{7.2.2b}$$

where $f \neq 0$ is real function and $h \neq 0$, p are complex functions is called a 3-dimensional Cauchy-Riemann (CR) structure [see for example Nurowski, Tafel 1988].

Thus we have the following corollary

Corollary 7.2.1 [Robinson-Trautman 1985]

Any shear-free optical geometry defines a 3-dimensional CR structure.

Conversely, given a CR structure N with $[(\kappa,\alpha)]$ one can built a manifold

$$M = \mathbb{R} \times N \tag{7.2.3}$$

and a real vector field k on M s.t.

$$k = \frac{\partial}{\partial r} \tag{7.2.4}$$

where r is a coordinate along $\mathbb{R}$ in (7.2.3). One can also extend κ, α and $\bar{\alpha}$ to M by the condition

$$\mathcal{L}_k \kappa = \mathcal{L}_k \alpha = 0 \tag{7.2.5}$$

Forms κ and α are defined up to the transformations (7.2.2). One can now supplement κ, α and $\bar{\alpha}$ to the cobasis in M by any real 1-form φ on M s.t.

$$\kappa \wedge \alpha \wedge \bar{\alpha} \wedge \varphi \neq 0 \tag{7.2.6}$$

The optical geometry generated by (g,k) where k is given by (7.2.4) and g is given by (7.1.4a) with κ, α, $\bar{\alpha}$ and φ as above is shear-free due to (7.2.5). Thus we arrive at the following corollary

Corollary 7.2.2

Any 3-dimensional CR structure can be supplemented to the structure of optical geometry over some 4-dimensional manifold.

Combining the facts given by corollaries 7.2.1 and 7.2.2 with informations about optical geometries yields the following theorem

Theorem 7.2.1 [Robinson, Trautman 1985]

Any point of a manifold with a shear-free optical geometry has a neighbourhood optically isomorphic to the Cartesian product of $\mathbb{R}$ by a CR structure.

VIII. CR STRUCTURES

1. Equivalence of different definitions

In the preceding section the definition 7.2.1 was used to define a 3-dimensional CR-structure. This definition is convenient for our purposes but very technical. Here we present an abstract definition of a CR structure. It defines a (2n-1)-dimensional ($n = 2,3,4,\ldots$) CR structure.

Definition 8.1.1[Wells 1982]

A real (2n-1) manifold N with a subbundle $\mathcal{H} \subset TN$, where $\mathcal{H}$ is a complex vector bundle with fibers of complex dimension (n-1) is called (2n-1)-dimensional CR manifold. CR manifold is called integrable if

$$[\Gamma(\mathcal{H}),\Gamma(\mathcal{H})] \subset \Gamma(\mathcal{H}) \qquad (8.1.1)$$

where $\Gamma(\mathcal{H})$ denotes set of all sections of $\mathcal{H}$. Manifold N is also said to possess CR structure (or to be a CR structure).

From now on we consider only integrable CR structures. Therefore we will omit the word "integrable" in the following.

Let us show that definition 7.2.1 is equivalent to the definition 8.1.1 for n = 2.

First, let us notice that forms κ, α and $\bar{\alpha}$, which define a CR structure on N according to the definition 7.2.1, define a complex vector field $\partial \neq 0$ s.t.

$$\partial \lrcorner \kappa = \partial \lrcorner \bar{\alpha} = 0 \qquad (8.1.2)$$

This field is defined by (8.1.2) up to the transformations

$$\partial \longrightarrow \partial' = c\partial \qquad (8.1.3)$$

where $c \neq 0$ is some complex function. Since

$$\kappa \wedge \alpha \wedge \bar{\alpha} \neq 0 \qquad (8.1.4)$$

then the complex conjugate vector field $\bar{\partial}$ is linearly independent (in the complex sense) of ∂. The real bundle $\mathcal{H} \subset TN$ s.t.

$\mathcal{H} = \{v \in \mathcal{X}(N) \,|\, v = a\, \mathrm{Re}\partial + b\mathrm{Im}\partial$ where a, b are real functions on N$\}$

has a complex structure. To see this define a real-linear operator J in $\mathcal{H}$ which in any point p of N acts on a basis $(\mathrm{Re}\partial)_p$, $(\mathrm{Im}\partial_p)$ of $\mathcal{H}_p$ by:

$$J[(\mathrm{Re}\partial)_p] = -(\mathrm{Im}\partial)_p \qquad (8.1.5a)$$

$$J[(\mathrm{Im}\partial)_p] = (\mathrm{Re}\partial)_p \qquad (8.1.5b)$$

Obviously

$$J^2 = -\mathrm{id} \qquad (8.1.6)$$

what shows that $\mathcal{H}$ is equipped with the complex structure. (The integrability conditions (8.1.1) are automatically satisfied).

Conversely, given a complex bundle $\mathcal{H} \subset TN$, which defines a CR structure on N according to the def. 8.1.1, one takes any nonvanishing section X of $\mathcal{H}$. Then one can locally define a vector field Y in $\mathcal{H}$ s.t.

$$Y = -J(X) \qquad (8.1.7a)$$

Since $J^2 = -\mathrm{id}$ then

$$J(Y) = X \qquad (8.1.7b)$$

Defining a vector field ∂ by

$$\partial = X + iY \qquad (8.1.8)$$

it is easy to see that ∂ is defined by X, Y satisfying (8.1.7) up to the transformation

$$\partial \longrightarrow \partial' = c\partial \qquad (8.1.9)$$

where c is any nonvanishing complex function on N. Supplementing operators ∂ and its complex conjugate $\bar{\partial}$ to a basis in TN by adding one real operator ∂_o we see that this basis is defined up to the transformations:

$$(\partial, \bar{\partial}, \partial_o) \longrightarrow \left(\frac{1}{h}\,\partial,\ \frac{1}{\bar{h}}\,\bar{\partial},\ \frac{1}{f}\,\partial_o - \frac{p}{h}\,\partial - \frac{\bar{p}}{\bar{h}}\,\bar{\partial}\right) \qquad (8.1.10)$$

where $h \neq 0$ and p are arbitrary complex functions, and $f \neq 0$ is an arbitrary real function on N. The dual basis $(\alpha, \bar{\alpha}, \kappa)$ to $(\partial, \bar{\partial}, \partial_o)$ is defined up to the transformations

$$(\alpha,\ \bar{\alpha},\ \kappa) \longrightarrow (h\alpha + p\kappa,\ \overline{h\alpha} + \bar{p}\kappa,\ f\kappa) \tag{8.1.11}$$

The forms α and κ given up to the gauge (8.1.11) define a CR structure on N as in the definition 7.2.1. It follows from the above that the third definition of 3-dimensional CR structure is possible.

Definition 8.1.2 [Chern, Moser 1974]

A three dimensional real manifold N is called a three dimensional CR structure if and only if N is equipped with a complex vector field ∂ defined up to the transformation

$$\partial \longrightarrow \partial' = c\partial. \tag{8.1.12}$$

∂ must be such that a complex conjugated vector field $\bar{\partial}$ is linearly independent of ∂.

This definition can be also generalized for (2n-1) dimensional CR-structures.

2. Realizability

The most important examples of CR structures are given by real hypersurfaces of real dimension (2n-1) imbedded in $\mathbb{C}^n$ [Cartan 1932, Tanaka 1962, Chern, Moser 1974].

Let us see how such a hypersurface can be endowed with the CR structure considering an example of a 3-dimensional real hypersurface N in $\mathbb{C}^2$. Let $(z_i = x_i + iy_i)_{i=1}^{i=2}$ be holomorphic coordinates in $\mathbb{C}^2$. In any point $q \in \mathbb{C}^2$ we can write any vector $v_q \in T_q\ (\mathbb{C}^2)$ as a complex linear combination of a basis

$$\left(\frac{\partial}{\partial x_1},\ \frac{\partial}{\partial y_1},\ \frac{\partial}{\partial x_2},\ \frac{\partial}{\partial y_2}\right). \tag{8.2.1}$$

Let us introduce a linear operator J_q s.t.

$$J_q\ :\ T_q(\mathbb{C}_2) \longrightarrow T_q(\mathbb{C}_2) \tag{8.2.2a}$$

$$J_q\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial y_i} \tag{8.2.2b}$$

$$J_q\left(\frac{\partial}{\partial y_i}\right) = -\frac{\partial}{\partial x_i} \tag{8.2.2c}$$

Obviously

$$J_q^2 = -\mathrm{id}_q \tag{8.2.3}$$

Suppose now that the point $q \in N \subset \mathbb{C}^2$. It is clear that any vector $w_q \in T_q(N)$ is in general a real linear combination of a basis (8.2.1). Define a vector space

$$\mathcal{H}_q = T_q(N) \cap J_q(T_qN) \tag{8.2.4}$$

Since $\mathcal{H}_q \subset J_q(T_qN)$, then it is equipped with the complex structure J_q. This also means that $\mathcal{H}_q$ has an even real dimension. The dimension of $\mathcal{H}_q$ cannot be zero because in this case J_q should send all 3 real vectors spanning T_qN into 1-dimensional vector space V_q transversal to T_qN. This would contradict the fact that J_q is nondegenerate. Therefore the only possibility is that $\mathcal{H}_q$ has real dimension 2 (note that $\mathcal{H}_q \subset T_qN$ and $\dim T_qN = 3$). The vector line bundle

$$\mathcal{H} = \bigcup_{q\in N} \mathcal{H}_q \tag{8.2.5}$$

is a subbundle of TN and constitutes a complex vector bundle of complex rank 1. The integrability conditions (8.1,1) are automatically satisfied. According to the definition 8.1.1 N is equipped with the CR-structure.

Similar considerations can show that any (2n-1) real dimensional manifold in $\mathbb{C}^n$ ($n \geq 2$) is equipped with the CR-structure as in the definition 8.1.1. CR structures which can be considered as real hypersurfaces of real dimension (2n-1) in a (2n+2) real dimensional manifold $\mathcal{M}$ with a complex structure J are called <u>realizable CR structures.</u> Hence we can have CR structures which are realizable in $\mathbb{C}^n$, $\mathbb{C}P^n$ etc. However there exist examples of nonrealizable CR structures [Nirenberg 1973, 1974, Jacobowitz, Treves 1982]. Moreover, R. Penrose predicts that nonrealizable CR-structure should have important connections with physics [Penrose 1983].

3. Important example of a 5-dimensional CR-structure - null twistors

Before passing to the 3-dimensional CR-structures which will be objects of the main interest in the following, we give an example of a 5-dimensional CR-structure. This structure was introduced by R. Penrose as a basic object of the twistor theory [Penrose 1967, 1977]. This will be useful for us in next sections when we comment on twistor description of the Kerr theorem. The rest of this section is a summary of constructions given by R. Penrose in [Penrose 1983].

Let us consider the Minkowski space-time M with the coordinates

$$\varkappa = (\tau,\ \xi,\ \eta,\ \zeta) \in \mathbb{R}^4 \tag{8.3.1}$$

and interval between any two points κ and κ' given by

$$S(\varkappa,\varkappa') = -(\tau - \tau')^2 + (\xi - \xi)^2 + (\eta - \eta')^2 + (\zeta-\zeta')^2 \tag{8.3.2}$$

It is known that M can be identified with the space $\mathbb{H}$ of Hermitian 2×2 matrices A of the form

$$A = \begin{pmatrix} \tau + \zeta & \xi + i\eta \\ \\ \xi - i\eta & \tau - \zeta \end{pmatrix} \tag{8.3.3}$$

This gives a one-to-one correspondence between κ in (8.3.1) and A in (8.3.3). The interval (8.3.2) is represented in this formalism as

$$S(\varkappa,\varkappa') = S(A,A') := -\det(A - A') \tag{8.3.4}$$

where matrices A and A′ correspond to $\varkappa$ and $\varkappa'$, respectively. It is clear from (8.3.4) that two points $\varkappa$ and $\varkappa'$ constitute a null (optical) vector $\varkappa - \varkappa'$ if and only if the difference between corresponding matrices $A - A'$ is a matrix of zero determinant.

Let us consider now a 4-complex-dimensional vector space $\mathbb{T} = \mathbb{C}^4$, called dual twistor space, whose points are

$$z = (z_1, z_2, z_3, z_4). \tag{8.3.5}$$

The point $z \in \mathbb{T}$ is called incident with $x \in M$ iff

$$(z_1 - z_3,\ z_2 - z_4) = \frac{1}{i\sqrt{2}}(z_1 + z_3,\ z_2 + z_4)\begin{pmatrix} \tau + \zeta & \xi + i\eta \\ \xi - i\eta & \tau - \zeta \end{pmatrix} \tag{8.3.6}$$

The important fact is that not all points $z \in \mathbb{T}$ can satisfy equality (8.3.6). The necessary condition for $z \in \mathbb{T}$ to satisfy (8.3.6) is that

$$\Sigma(z) = 0 \tag{8.3.7}$$

where Σ is Hermitian form of signature $(+ + - -)$ defined by

$$\Sigma(z) = |z_1|^2 + |z_2|^2 - |z_3|^2 - |z_4|^2 \tag{8.3.8}$$

To see this, consider scalar product of the form

$$(z_1 - z_3,\ z_2 - z_4)\begin{pmatrix} \bar{z}_1 + \bar{z}_3 \\ \bar{z}_2 + \bar{z}_4 \end{pmatrix} \tag{8.3.9}$$

by multiplying (8.3.6) by the vector $\begin{pmatrix} \bar{z}_1 + \bar{z}_3 \\ \bar{z}_2 + \bar{z}_4 \end{pmatrix}$. Since the matrix A in (8.3.6) is Hermitian and $\begin{pmatrix} \bar{z}_1 + \bar{z}_3 \\ \bar{z}_2 + \bar{z}_4 \end{pmatrix}$ is Hermitian conjugation of $(z_1 + z_3,\ z_2 + z_4)$ then after this multiplication right hand side of (8.3.6) is purely imaginary. To have also left hand side purely imaginary one has to impose condition (8.3.7) on $z \in \mathbb{T}$.

Since equation (8.3.6) is invariant under the transformations

$$\mathbb{T} \ni z \longrightarrow \lambda z \in \mathbb{T}, \qquad \lambda \in \mathbb{C}\backslash\{0\} \tag{8.3.10}$$

it is reasonable to consider 3-complex dimensional projective space $\mathbb{PT}$, rather then $\mathbb{T}$. This space is a complex manifold whose points are labelled by three complex numbers given by the three ratios

$$(z_1 : z_3 : z_2 : z_4). \tag{8.3.11}$$

$\mathbb{PT}$ can be split into three manifolds. Two of them $\mathbb{PT}_-$, $\mathbb{PT}_+$ are complex open manifolds given by the relations $\Sigma(Z)<0$ and $\Sigma(Z)>0$ respectively (here we used capital letter Z to denote the direction in $\mathbb{PT}$ associated with the point z in $\mathbb{T}$). The third manifold given by the relation

$$\Sigma(Z) = 0 \tag{8.3.12}$$

is a 5-dimensional <u>real</u> manifold. This 5-dimensional real manifold $\mathbb{PT}_0$ is by its construction imbedded in 3-complex-dimensional complex manifold $\mathbb{PT}$. Hence $\mathbb{PT}_o$ is an example of 5-dimensional realizable CR-structure.

The space $\mathbb{PT}$ is called <u>projective</u> <u>twistor</u> <u>space</u> and the space $\mathbb{PT}_o$ <u>projective</u> <u>null</u> <u>twistor</u> <u>space</u>. The space $\mathbb{PT}$ is important because it gives a good description of the Minkowski space-time M. To see this consider points of $\mathbb{PT}_o$. First, note that not all points of $\mathbb{PT}_o$ are those which satisfy (8.3.6). However, after excluding one complex direction Z_o in $\mathbb{PT}_o$ represented by a point

$$z_o = (z_1,\ z_2,\ -z_1,\ -z_2) \tag{8.3.13}$$

it is easy to see that all points $Z \in (\mathbb{PT}_o - \{Z_0\})$ can satisfy (8.3.6). Now suppose that we have fixed a point

$$Z \in \mathbb{PT}_o - \{Z_o\} \tag{8.3.14}$$

We are looking for points in M incident with Z i.e. for points $x \in$ M of the form (8.3.1) satisfying (8.3.6), where $z = (z_1, z_2, z_3, z_4)$ is any point in $\mathbb{T}$ which multiplied by an arbitrary $\lambda \in \mathbb{C}$ span the direction Z. The remarkable fact is that the solution for this problem is not unique. If

$$A = \begin{pmatrix} \tau + \zeta & \xi + i\eta \\ \xi - i\eta & \tau - \zeta \end{pmatrix} \tag{8.3.15}$$

satisfies (8.3.6) for z as above, then also

$$A' = A + k \begin{pmatrix} (z_2 + z_4)(\bar{z}_2 + \bar{z}_4) & -(z_2 + z_4)(\bar{z}_1 + \bar{z}_3) \\ -(z_1 + z_3)(\bar{z}_2 + \bar{z}_4) & (z_1 + z_3)(\bar{z}_1 + \bar{z}_3) \end{pmatrix}, \quad (8.3.16)$$

where k is an arbitrary real constant, does. Since the matrix

$$\begin{pmatrix} (z_2 + z_4)(\bar{z}_2 + \bar{z}_4) & -(z_2 + z_4)(\bar{z}_1 + \bar{z}_3) \\ -(z_1 + z_3)(\bar{z}_2 + \bar{z}_4) & (z_1 + z_3)(\bar{z}_1 + \bar{z}_3) \end{pmatrix} \quad (8.3.17)$$

has vanishing determinant then the solution for the problem represents a null straight line (light ray) in the Minkowski space-time. This light ray passes through $x \in M$ which is associated with A given by (8.3.15).

Conversely, the point $x \in M$ can be interpreted in terms of points of $\mathbb{PT}$. Fixing x in (8.3.6) we see that z is determined by two complex numbers $\omega_{13} = z_1 + z_3$ and $\omega_{24} = z_2 + z_4$. Therefore z which is a solution of (8.3.6) for a fixed point $x \in M$ give us a 2-complex dimensional linear subspace in $\mathbb{T}$, hence a complex projective line in $\mathbb{PT}$. Clearly this line lies in $\mathbb{PT}_o - \{Z_o\}$ and, moreover, all lines lying entirely in $\mathbb{PT}_o - \{Z_o\}$ arise in this way from points in M. The above considerations can be concluded in the following theorem.

Theorem 8.3.1 (Penrose's correspondence)

Points of $\mathbb{PT}_o - \{Z_o\}$ represent light rays in the Minkowski space-time M.

The projective lines lying entirely in $\mathbb{PT}_o - \{Z_o\}$ represent points of the Minkowski space-time M.

Remark 8.3.1

If we replace $\mathbb{PT}_o - \{Z_o\}$ by $\mathbb{PT}_o$ and the Minkowski space-time M by compactified Minkowski space-time $\tilde{M}$ in the above theorem we also have a true statement, which also is called Penrose's correspondence.

The interpretation of the CR-structure of $\mathbb{PT}_o - \{Z_o\}$ and its realizability in terms of physical objects in the Minkowski space-time can be found in [Penrose 1983]. We will mention only that it follows from the construction presented here that the space of all light rays in the Minkowski space-time is 5-dimensional and is equivalent to $\mathbb{PT}_o - \{Z_o\}$. This space can be also obtained by the following "physical" considerations:

We want to find a space of all light rays in the Minkowski space-time. In order not to count the light rays twice we choose a spacelike hypersurface U_{τ_o} given by $\tau=\tau_0=$ const. Any light ray intersects U_{τ_o} in precisely one point. This point is characterized by three coordinates $\vec{x}(\tau_o) = (\xi(\tau_o), \eta(\tau_o), \zeta(\tau_o))$. All light rays which pass through the $\vec{x}(\tau_o)$ are all light rays which comes from the heaven sphere of an observer situated at $\vec{x}(\tau_o)$. Hence any ray passing through $\vec{x}(\tau_o)$ can be parameterized by two polar angles $(\vartheta_{\vec{x}(\tau_o)}, \varphi_{\vec{x}(\tau_o)})$. The five parameters$(\xi(\tau_o), \eta(\tau_o), \zeta(\tau_o), \vartheta_{\vec{x}(\tau_o)}, \varphi_{\vec{x}(\tau_o)})$ constitute a coordinate system of the space of all light rays in the Minkowski space-time. This space coincides in a certain sense with the realizable CR structure $\mathbb{PT}_o - \{Z_o\}$ and does not depend on the choice of τ_o.

Finally, we comment on the possible generalization of the above presented construction to the nonflat space-time $\mathcal{M}$. It turns out that in this case with any spacelike hypersurface U in $\mathcal{M}$ one can associate a 5-dimensional CR-structure - a space of all null geodesics intersecting U. However now, CR-structures CR_{u} and $CR_{u'}$ associated with different hypersurfaces U and U' will be, in general, intrinsically different one from the other. Moreover, taking $\mathcal{M}$ and U included in it as being suitably nonanalitic one can obtain CR-structure CR_u which can be nonrealizable [Penrose 1983].

IX. THREE DIMENSIONAL CR STRUCTURES

Because of the role which 3-dimensional CR-structures play in the theory of shear-free optical geometry and in the theory of exact solutions of Einstein equations this chapter is entirely devoted to their mathematics. In the following we mainly use definition 7.2.1 of CR-structures. Sometimes the other definitions are used.

1. Nonequivalence of CR-structures

As we know in order to impose a CR structure on a real manifold N one has to specify two one-forms κ (real) and α (complex) s.t.

$$\kappa \wedge \alpha \wedge \bar{\alpha} \neq 0 \tag{9.1.1}$$

The CR structure is then given by a class of pairs of one-forms $[(\kappa',\alpha')]$ which are related to κ, α by

$$\kappa' = f\kappa \tag{9.1.2a}$$
$$\alpha' = h\alpha + p\kappa \tag{9.1.2b}$$

where $f \neq 0$ is any real function and $h \neq 0$, p are some complex functions on N.

Given two different manifold N and N′ and CR structures $[(\kappa,\alpha)]$ on N and $[(\kappa',\alpha')]$ on N′ we say that this structures are <u>equivalent</u> if and only if there exists a (local) diffeomorphism

$$\varphi : N \longrightarrow N' \tag{9.1.3a}$$

s.t.

$$\varphi^*(\kappa') = f\kappa \tag{9.1.3b}$$

$$\varphi^*(\alpha') = h\alpha + p\kappa \tag{9.1.3c}$$

for some functions f (real) and h,p (complex) on N. Two structures $[(\kappa,\alpha])$ and $[(\kappa',\alpha')]$ on the same manifold N are <u>isomorphic</u> if

(κ,α) is related to (κ',α') by (9.1.2).

The first observation is that there do exist nonequivalent CR structures [Poincare 1907]. This can be illustrated by the following very simple example.

Example 9.1.1

Consider three-dimensional manifold N and impose on it CR-structure CR_1 by forms (κ_1,α_1) s.t. they satisfy (9.1.1) and

$$\kappa_1 \wedge d\kappa_1 = 0 \qquad (9.1.4)$$

The second CR-structure CR_2 may be imposed on N by forms (κ_2,α_2) which also satisfy (9.1.1) but

$$\kappa_2 \wedge d\kappa_2 \neq 0 \qquad (9.1.5)$$

Since equation (9.1.4) is invariant under transformations (9.1.2) then it is obvious that CR_1 and CR_2 are nonequivalent.

The CR-structures satisfying (9.1.4) are called degenerate. Nondegenerate structures of the kind (9.1.5) are known to physicists since they correspond via Robinson-Trautman theorem 7.2.1 to so called twisting shear-free congruences in space-time.

The proof of the facts that

1) all degenerate CR structures are locally equivalent,

2) there exist nonequivalent nondegenerate CR structures

will be given in Section IX.3.

2. Tangential CR equation.

Before passing to the notion of symmetry of a three dimensional CR-structure we reformulate a definition of realizability in $\mathbb{C}^2$.

Let $(N,[(\kappa,\alpha)])$ be a 3-dimensional CR-structure. As we know this can be also defined by complex operator $\partial \neq 0$ (called CR operator) which is given up to the complex factor

$$\partial \longrightarrow \partial' = f\partial \qquad (9.2.1)$$

where $f \neq 0$ is any complex function on N. Forms (κ,α) are related to ∂ by

$$\partial \lrcorner \kappa = \partial \lrcorner \bar{\alpha} = 0 \qquad (9.2.2)$$

$$\kappa\wedge\alpha\wedge\bar{\alpha} \neq 0$$

It can be easily seen, that in the language of CR-operator ∂, realizability of an associated with it CR structure N is equivalent to the existence of an embedding

$$\iota : N \longrightarrow \mathbb{C}^2 \qquad (9.2.3a)$$

such that

$$\iota_*\partial = A(z,\omega) \frac{\partial}{\partial\omega} + B(z,\omega) \frac{\partial}{\partial z} \qquad (9.2.3b)$$

at each point of the image $\iota(N) \subset \mathbb{C}^2$, where (z,ω) are holomorphic coordinates on $\mathbb{C}^2$ (A,B are not required to be holomorphic). Moreover, a <u>sufficient</u> condition for realizability consists in the existence of two independent (complex) solutions of the so called <u>(tangential) CR equation</u>

$$\bar{\partial}f = 0 \qquad (9.2.4)$$

Independence here means that

$$df_1 \wedge df_2 \neq 0 \qquad (9.2.5)$$

Given two independent solutions $f_1 = \xi(x^i)$, $f_2 = \eta(x^i)$ of (9.2.4) ($(x^i)_{i=1,2,3}$ are coordinates on N) one defines an embedding by

$$\iota((x^1,x^2,x^3)) = (\xi(x^i),\eta(x^i)) \in \mathbb{C}^2 \qquad (9.2.6)$$

It is also true that <u>all</u> realizable 3-dimensional CR structures have to admit two independent solutions of CR equation (9.2.4). Therefore the necessary and sufficient condition of realizability of 3-dimensional CR-structures consists in the existence of such two independent solutions of the CR equation.

3. Groups of automorphisms of dimension D.

A diffeomorphism $\phi: N\to N$ is called a symmetry (automorphism) of a Cauchy-Riemann space iff the pullbacks $\phi^*\kappa$, $\phi^*\alpha$ are related to κ,α by the transformation (9.1.2) [Nurowski, Tafel 1988]. An equivalent condition is $\phi_*\partial \sim \partial$. We say that a vector field X is an infinitesimal symmetry iff

$$\mathcal{L}_X\kappa = a\kappa \ , \quad \mathcal{L}_X\alpha = b\alpha + c\kappa \ , \tag{9.3.1}$$

or, equivalently,

$$[X,\partial_0] = -b\partial_0 \ , \tag{9.3.2}$$

where $\mathcal{L}_X$ denotes the Lie derivative along X, a is a real function and b, c are complex functions. Note that the definition of an infinitesimal symmetry is invariant under transformations (9.1.2). Therefore CR structures admitting different groups of symmetries are nonequivalent.

If a Cauchy-Riemann structure is degenerate then $\kappa \wedge d\kappa = 0$ and it follows from the Frobenius theorem that $\kappa \sim du$, where u is a real function. Then equation (9.2.4) admits a complex solution ξ, hence $\alpha = Bd\xi + C\kappa$. In virtue of transformations (9.1.2) we can assume

$$\kappa = du, \qquad \alpha = d\xi \ . \tag{9.3.3}$$

without loss of generality. If η=u then the hypersurface in $\mathbb{C}^2$ given by (9.2.6) is the plane

$$\mathrm{Im}\eta = 0.$$

In this case the local symmetry group is infinitely dimensional. This consists of transformations

$$u \to f(u), \qquad \xi \to h(\xi,u). \tag{9.3.4}$$

If a Cauchy-Riemann structure is nondegenerate then it follows from a remark of Segre that its symmetry group is a finite dimensional Lie transformation group [Segre 1931]. In virtue of the Palais theorem [Kobayashi 1972] this fact can be considered a direct consequence of equations (9.3.1) and their integrability conditions. They show that the exterior derivatives of $\partial_0 a$, b, c and components of X must be linear functions of these variables with coefficients defined by κ and α. Hence, given a CR structure, a general solution X of the symmetry conditions (9.3.1) depends at most on 8 real parameters (values of $\partial_0 a$, b, c and components of X at a fixed point, provided no further constraints follow). Thus the Lie algebra generated by infinitesimal symmetries has dimension $D\le 8$. The Palais theorem says that the corresponding group of transformations is a Lie transformation group of dimension $D\le 8$.

In the following we investigate separately the cases when D is equal to 1,2 or $D\ge 3$. It will turn out that there do exist CR structures with different symmetry groups. This proves that there are nonequivalent CR structures among nondegenerate ones.

4. D = 1

If only one infinitesimal symmetry is present then we can easily adjust a transformation (9.1.2) in order to get

$$\mathcal{L}_X \kappa = 0, \qquad \mathcal{L}_X \alpha = 0. \tag{9.4.1}$$

It follows from (9.4.1) that $\kappa = A_i(x,y)dx^i$ and $\alpha = B_i(x,y)dx^i$ in coordinates $x^i=(u,x,y)$ such that

$$X = \partial_u .$$

Condition (9.1.5) requires $A_1\neq 0$. A residual freedom of transformations (9.1.2) allows us to assume $A_1=1$ and $B_1=0$. Then α becomes proportional to an exact form $d\xi$, where $\xi=\xi(x,y)$. The functions $\mathrm{Re}\xi, \mathrm{Im}\xi$ can be chosen as new coordinates x,y. A suitable transformation of α and u leads to the following canonical expressions for κ and α

$$\kappa = du + f(x,y)dx, \qquad \alpha = dx + idy \ , \tag{9.4.2}$$

where $\partial_y f \neq 0$. Particular solutions ξ and η of (9.2.4) are given by

$$\xi = x + iy \ , \qquad \eta = u + h(x,y), \tag{9.4.3}$$

where $2h_{\bar\xi} = f$. It follows from (9.4.3) that

$$\mathrm{Im}\eta = F(\mathrm{Re}\xi, \mathrm{Im}\xi) \ , \tag{9.4.4}$$

where $F = \mathrm{Im}h$ and $F_{\xi\bar\xi} \neq 0$.

Theorem 9.4.1 [Nurowski, Tafel 1988]

If a CR structure admits a 1-dimensional group of symmetries then it is equivalent to the CR structure defined by (9.4.2) with some f. The corresponding hypersurface in $\mathbb{C}^2$ is given by equation (9.4.4).

5. D = 2

It is easy to show that the orbits of a 2-dimensional symmetry group G must be 2-dimensional. Indeed, let us assume that it is not so. Then the vector fields X_1 and X_2 representing any two independent infinitesimal symmetries have to be proportional at each point, i.e. $X_1 = qX_2$, where $q \neq const$ (otherwise X_1 and X_2 represent the same symmetry). It follows from (9.3.2) that either $\partial \sim X_2$ or $\partial q = 0$. The first possibility is in contradiction with (9.1.5) which is equivalent to

$$\partial,\ \bar{\partial},\ [\partial,\bar{\partial}] \text{ are independent at each point.} \qquad (9.5.1)$$

The second possibility violates (9.1.5) or the assumption $q \neq const$ (since $\partial q = 0$ implies $\bar{\partial} q = 0$ and $[\partial,\bar{\partial}]q = 0$).

Since X_r, $r=1,2$, are infinitesimal symmetries we know that

$$\mathcal{L}_r \kappa = a_r \kappa, \qquad \mathcal{L}_r \alpha = b_r \alpha + c_r \kappa \ , \qquad (9.5.2)$$

where $\mathcal{L}_r$ denotes the Lie derivative along X_r. We now want to transform κ and α to forms κ', α' strictly invariant under the action of G,

$$\mathcal{L}_r \kappa' = 0, \qquad \mathcal{L}_r \alpha' = 0 \ . \qquad (9.5.3)$$

Comparing (9.5.2) and (9.5.3) yields the following equations for the parameters A,B,C of the required transformation (9.1.2)

$$\mathcal{L}_r f = -fa_r \ , \quad \mathcal{L}_r p = -pb_r \ , \quad \mathcal{L}_r h = -ha_r - pc_r \ . \qquad (9.5.4)$$

Solutions to these equations exist since the integrability conditions of (9.5.4) follow from equations obtained by the Lie differentiation of (9.5.2). Hence we can assume (9.5.3) without loss of generality.

There are two nonisomorphic 2-dimensional Lie algebras. We can choose the fields X_r in such a way that

$$[X_1, X_2] = \varepsilon X_1, \qquad \varepsilon = 0,1 . \tag{9.5.5}$$

Since the orbits of G are 2-dimensional it follows from (9.5.5) that

$$X_1 = \partial_u , \qquad X_2 = \varepsilon u \partial_u + \partial_x \tag{9.5.6}$$

in some coordinates u,x,y. Now it is easy to find general forms κ' and α' satisfying equations (9.5.3). In virtue of residual transformations (9.1.2) and a freedom in the choice of the coordinates they can be reduced to the following expressions

$$\kappa = \exp(-\varepsilon x)\, du + f(y)dx , \qquad \alpha = dx + idy, \tag{9.5.7}$$

where $\partial_y f \neq 0$ and we have dropped the primes. Particular solutions ξ and η of equation (9.2.4) are given by

$$\xi = x + iy , \qquad \eta = u + h(y)\exp(\varepsilon x) , \tag{9.5.8}$$

where

$$ih_y + \varepsilon h = f .$$

It follows from (9.5.8) that

$$\mathrm{Im}\eta = \exp(\varepsilon \mathrm{Re}\xi)\, F(\mathrm{Im}\xi), \tag{9.5.9}$$

where F = Imh. By an appropriate choice of f we can obtain any real function F satisfying the condition

$$F_{yy} + \varepsilon F \neq 0 .$$

Theorem 9.5.1. [Nurowski, Tafel 1988]

If a CR structure admits a 2-dimensional group of symmetries then it is equivalent to the structure defined by (9.5.7). The corresponding hypersurface in $\mathbb{C}^2$ is given by equation (9.5.9).

6. $D \geq 3$[5]

[5] This case was considered by E. Cartan [Cartan 1932]. Our approach

Our approach to this case can be described as follows. First we observe that any Lie group of dimension $D\geq3$ contains locally a 3-dimensional subgroup. Hence in order to find all symmetrical CRstructures with $D\geq3$ it is sufficient to consider the case $D=3$. We prove that the action of a 3-dimensional symmetry group G on N is (locally) simply transitive and the forms κ and α can be transformed to left invariant forms on G. In virtue of the Bianchi classification of 3-dimensional Lie groups the whole problem reduces to the problem of finding all nonequivalent left invariant frames on these groups.

Lemma 9.6.1

Any Lie algebra $\mathcal{G}'$ of dimension greater than 2 contains a 3-dimensional subalgebra $\mathcal{G}$.

Proof. [Nurowski, Tafel 1988]

According to the Levi-Malcev theorem [Barut, Raczka 1977] $\mathcal{G}$ can be decomposed into the semidirect sum of a solvable subalgebra $\mathcal{R}$ (radical) and a semisimple subalgebra $\mathcal{S}$. If $\mathcal{S}$ is nontrivial then it contains either the subalgebra su(2) (if $\mathcal{S}$ is compact [Helgason 1962, p.219]) or su(1,1) (if $\mathcal{S}$ is noncompact [Helgason 1962, p.245]). If $\mathcal{S}$ is trivial then dim $\mathcal{R}\geq3$ and $\mathcal{R}$ must contain a 3-dimensional subalgebra since any solvable algebra $\mathcal{R}$ contains subalgebras of all dimensions between 1 and dim$\mathcal{R}$ [Helgason 1962, p.133].

q.e.d.

The lemma has its local equivalent at the group level. Hence it follows that highly symmetrical ($D\geq4$) CR structures are particular cases of CR structures with a 3-dimensional symmetry group G.

It is easy to prove that orbits of G are 3-dimensional. It

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simplifies his results.

follows already from Section 4 that they cannot be 1-dimensional. Let us assume for a moment that they are 2-dimensional, i.e.

$$X_3 = pX_1 + rX_2, \qquad X_2 \neq qX_1 , \tag{9.6.1}$$

where X_r (r=1,2,3) are independent infinitesimal symmetries and either p≠const or r≠const. It follows from (9.3.2) that

$$(\partial p\, X_1 + \partial r\, X_2) \sim \partial . \tag{9.6.2}$$

Equation (9.6.2) and its conjugate yield

$$X_r = s_r \partial + \bar{s}_r \bar{\partial} , \tag{9.6.3}$$

where s_r are complex nonvanishing functions. Substituting (9.6.3) into (9.3.2) yields contradiction with (9.1.5).

Since the orbits are 3-dimensional the manifold N can be locally identified with the group manifold G and the action of G on N can be identified with the natural left action of G on itself.

An analysis analogous to that for D=2 shows that the forms κ and α satisfying the symmetry conditions (9.5.2) (now r=1,2,3) can be replaced by new forms, which satisfy the equations

$$\mathcal{L}_r \kappa = 0, \qquad \mathcal{L}_r \alpha = 0 . \tag{9.6.4}$$

It follows from (9.6.4), (9.1.1), (9.1.5) and the identification of N with G that (κ, Reα, Imα) is a left invariant frame on G satisfying (9.1.5).

<u>Theorem</u> <u>9.6.1</u> [Nurowski, Tafel 1988]

If a CR structure admits a symmetry group G' of dimension D≥3 then it is equivalent to the CR structure defined by a left invariant basis (κ, Reα, Imα), $\kappa \wedge d\kappa \neq 0$, on a 3-dimensional local subgroup G of G'.

The local properties of G are determined by its Lie algebra $\mathcal{G}$. All the 3-dimensional Lie algebras are explicitly known

[Bianchi 1897]. Let (ϑ^r) be a particular left invariant basis related to $\mathcal{G}$. Then κ and α are general linear combinations of ϑ^r with constant coefficients such that (9.1.5) is satisfied. A number of free parameters can be reduced by means of transformations (9.1.2) (with constant A,B,C) and transformations of ϑ^r preserving the Maurer-Cartan equations. Moreover we can always obtain

$$\alpha \wedge d\alpha = 0 ,$$

hence $\alpha \sim d\xi$ and ξ satisfies equation (9.2.4). If $\mathcal{G}$ = su(2) or su(1,1) (Bianchi types IX and VIII) then one real parameter remains. It means that 1-parameter families of CR structures are related to these algebras. For other algebras (except Bianchi types I and V, which are excluded by (9.1.5)) there is only one corresponding CR structure. For all Bianchi types (except I and V) we list below the following data [Nurowski, Tafel 1988]

i) reduced forms κ and α in terms of coordinates u, ξ, $\bar{\xi}$ or u, x, y where ξ=x+iy and u, x, y are real

ii) symmetry transformations generated by G (p,q,r being constant parameters)

iii) a second solution η of (9.2.4) and an equation of the hypersurface in $\mathbb{C}^2$ defined by (9.2.6)

We also give the corresponding Cartan type [Cartan 1932] in square brackets.

Type II [A] (9.6.5)

i) $\kappa = du - i\bar{\xi}\, d\xi + i\xi\, d\bar{\xi}$, $\alpha = d\xi$

ii) $u' = u + i\bar{q}\xi - iq\bar{\xi} + p$, $\xi' = \xi + q$, $q \in \mathbb{C}$, $p \in R$

iii) $\eta = u + i\xi\bar{\xi}$, $\mathrm{Im}\eta = \xi\bar{\xi}$

Type IV [F] (9.6.6)

i) $\kappa = y^{-1}(du - \ln y\, dx)$, $\alpha = y^{-1}(dx + idy)$

ii) $u' = ru + r\ln r\, x + p$, $x' = rx$, $y' = ry + q$

$p, q, r \in R$, $r > 0$

iii) $\eta = u + x + iy\ln y$, $\mathrm{Im}\eta = \mathrm{Im}\xi \ln(\mathrm{Im}\xi)$

Type VI_h (including VI_0 and III) [E,B] (9.6.7)

i) $\kappa = y^{b}\, du - y^{-1}dx$, $\alpha = y^{-1}(dx + idy)$, where $b = \dfrac{1-\sqrt{-h}}{1+\sqrt{-h}}$

ii) $u' = r^{-b}\, u + p$, $x' = rx$, $y' = ry + q$, $p, q, r \in R$, $r>0$

iii) $\eta = -bu + iy^{-b}$, $\mathrm{Im}\eta = (\mathrm{Im}\xi)^{-b}$ for $h \neq -1$

$\eta = u + i\ln y$, $\mathrm{Im}\eta = \ln(\mathrm{Im}\xi)$ for $h = -1$ (type III)

Type VII_h (including VII_0) [H] (9.6.8)

i) $\kappa = du + e^{(A+i)u}\, d\xi + e^{(A-i)u}\, d\bar{\xi}$, $\alpha = e^{(A+i)u}\, d\xi$,

where $A = \sqrt{h}$

ii) $u' = u - p$, $\xi' = e^{(A+i)p}\, \xi + q$, $p \in R$, $q \in \mathbb{C}$

iii) $\eta = (i-A)\bar{\xi} + e^{(i-A)u}$, $\mathrm{Im}\, [\eta + (A-i)\bar{\xi}\,]^{A+i} = 0$

Type IX [D,L] (upper signs) and VIII [C,K] (lower signs) (9.6.9)

i) $\kappa = du + \dfrac{ke^{iu} - i\bar{\xi}}{\xi\bar{\xi} \pm 1}\, d\xi + \dfrac{ke^{-iu} + i\xi}{\xi\bar{\xi} \pm 1}\, d\bar{\xi}$, $\alpha = \dfrac{2e^{iu}}{\xi\bar{\xi} \pm 1}\, d\xi$,

where $0 \leq k \in R$, $k^2 \pm 1 \neq 0$

ii) $u' = u - i\ln(\bar{q}\xi + \bar{p}) + i\ln(q\bar{\xi} + p)$, $\xi' = \dfrac{p\xi \mp q}{\bar{q}\xi + \bar{p}}$,

where $p\bar{p} \pm q\bar{q} = 1$ and $p, q \in \mathbb{C}$

iii) $\eta = \dfrac{\xi e^{iu} - ik}{e^{iu} \pm ik\bar{\xi}}$, $|\xi - \eta| = k|1 \pm \xi\bar{\eta}|$ for $k > 0$

$$\eta = \left[(\xi\bar{\xi} \pm 1)e^{-iu}\right]^{\frac{1}{2}}, \qquad \eta\bar{\eta} = |\xi\bar{\xi} \pm 1| \qquad \text{for } k = 0$$

As we have seen in Section IX 3 all degenerate Cauchy-Riemann structures are isomorphic and admit locally an infinitely dimensional group of automorphisms. A nondegenerate structure may have no symmetries. If it does then they form a Lie transformation group of dimension $D\le 8$. The action of this group is transitive if $D\ge 3$. CR structures with continuous symmetries ($D\ge 1$) are locally equivalent to the structures defined by (9.4.2),(9.5.7) or the forms κ,α listed in Section IX 6. We have found the corresponding hypersurfaces in $\mathbb{C}^2$ Those for $D\ge 3$ are locally equivalent to the hypersurfaces obtained by Cartan [Cartan 1932].

Finally let us make a comment on CR structures admitting groups of symmetries of dimension $D\ge 4$. It was proved by Cartan [Cartan 1932] that from a local point of view they are all isomorphic and admit the group SU(1,2). They can be characterized by the vanishing of the Cartan relative invariant R [Cartan 1932, we will define R by (9.7.18)]. They are known to physicists since they are related to the Robinson congruence [Penrose 1967], which exists e.g in Minkowski space. On our list from Section IX 6 these structures occur for the types II, III, VI_{-9} , VIII ($k=0$ or $k=\sqrt{2}$), IX ($k=0$). The other structures from this list are nonisomorphic and admit precisely 3-dimensional groups of automorphisms.

7. Cartan invariants of 3-dimensional CR structures

In last few sections we gave a classification of symmetric CR structures. It follows from this classification that all degenerate CR structures are equivalent and that there are plenty of nonequivalent nondegenerate CR structures. In this section we address a question whether there exists an algorithm, which enables us in finite number of steps, to distinguish between nonequivalent CR structures. We will give a full set of invariants which pairwise identity is necessary and sufficient for two nondegenerate CR structures to be equivalent.

Consider a 3-dimensional CR-structure $(N,[(\kappa,\alpha)])$ which is nondegenerate i.e.

$$\kappa \wedge d\kappa \neq 0 \tag{9.7.1}$$

Since $(\kappa,\alpha,\bar{\alpha})$ constitute a cobasis on N then (9.7.1) implies that there exists a real function $\rho \neq 0$ on N s.t.

$$\kappa \wedge d\kappa = i\rho\kappa \wedge \alpha \wedge \bar{\alpha} \tag{9.7.2}$$

Suppose now that we have another forms (κ_1,α_1) on N which also satisfy relations of the form (9.7.1) and (9.7.2). We ask whether forms (κ_1,α_1) can be related to (κ_2,α_2) by transformations:

$$\kappa_1 = f\kappa \tag{9.7.3a}$$

$$\alpha_1 = h\alpha + p\kappa \tag{9.7.3b}$$

where $f \neq 0$ is any real function on N and $h \neq 0$, p are some complex functions on N. If it is so, then the CR structures related to (κ,α) and (κ_1,α_1) are equivalent; if no - they are nonequivalent. The strategy for the answer to this question is as follows. We need to find some preferred forms in the class $[(\kappa,\alpha)]$, as well as an universal way of choosing them. By universality here we mean that whatever (κ, α) from the class $[(\kappa, \alpha)]$ we chose, after our procedure, we would always obtain the same pair (Ω, Ω_1) belonging to the class $[(\kappa, \alpha)]$. We start with (κ,α) satisfying (9.7.1) and (9.7.2). We will single out Ω and Ω_1 by reducing the allowed freedom

$$\kappa \longrightarrow \kappa' = f\kappa \tag{9.7.4a}$$

$$\alpha \longrightarrow \alpha' = h\alpha + p\kappa \tag{9.7.4b}$$

By a suitable choice of κ' and α' (see (9.7.4)) we always can achieve:

$$\kappa' \wedge d\kappa' = i\alpha' \wedge \bar{\alpha}' \wedge \kappa'. \tag{9.7.5}$$

Without loss of generality we assume that (9.7.5) holds.

Suppose that κ' and α' belonging to the class $[(\kappa,\alpha)]$ satisfy (9.7.5). Let

$$\Omega = f\bar{f}\kappa' \tag{9.7.6a}$$

$$\Omega_1 = f(\alpha' + h\kappa') \tag{9.7.6b}$$

where f, h are arbitrary complex functions ($f \neq 0$). It is obvious that Ω and Ω_1 also belong to the class $[(\kappa,\alpha)]$ and preserve (9.7.5) with κ' and α' being replaced by Ω and Ω_1 respectively.

Our aim now is to single out f and h in (9.7.6). We wish to do it in such a way that the resulting Ω, Ω_1 will not depend on the choice of an initial pair (κ,α) representing a fixed CR structure. First, let us quote the Cartan-Chern-Moser theorem.

Theorem 9.7.1 [Cartan 1932, Chern, Moser 1974].

Let 1-forms Ω, Ω_1 be given by (9.7,6). There exist complex-valued 1-forms Ω_2, Ω_3, a real-valued 1-form Ω_4 and complex function $\mathcal{R}$, S which satisfy the following equations

$$d\Omega = i\Omega_1 \wedge d\bar{\Omega}_1 + (\Omega_2 + \bar{\Omega}_2) \wedge \Omega \tag{9.7.7a}$$

$$d\Omega_1 = \Omega_2 \wedge \Omega_1 + \Omega_3 \wedge \Omega \tag{9.7.7b}$$

$$d\Omega_2 = 2i\Omega_1 \wedge \bar{\Omega}_3 + i\bar{\Omega}_1 \wedge \Omega_3 + \Omega_4 \wedge \Omega \tag{9.7.7c}$$

$$d\Omega_3 = \Omega_4 \wedge \Omega_1 + \Omega_3 \wedge \bar{\Omega}_2 + \mathcal{R}\bar{\Omega}_1 \wedge \Omega. \tag{9.7.7d}$$

$$d\Omega_4 = i\Omega_3 \wedge \bar{\Omega}_3 - (\Omega_2 + \bar{\Omega}_2) \wedge \Omega_4 - S\,\Omega \wedge \Omega_1 - \bar{S}\,\Omega \wedge \bar{\Omega}_1 \tag{9.7.7e}$$

The forms Ω_2, Ω_3, Ω_4 and the function $\mathcal{R}$, S are related to Ω, Ω_1 and, as a consequence, they depend on f and h of (9.7.6).

They are given up to the freedom:

$$\begin{aligned} \Omega'_2 &= \Omega_2 + \rho\Omega \\ \Omega'_3 &= \Omega_3 + \rho\Omega_1 \\ \Omega'_4 &= \Omega_4 + d\rho + \rho(\Omega_2 + \bar{\Omega}_2) + \rho^2\Omega \end{aligned} \tag{9.7.8}$$

where ρ is any real function on N.[6]

Forms Ω, Ω_1, Ω_2, Ω and Ω_4 can be collected in such a way that they constitute a matrix of one-forms ω defined by [Chern, Moser 1974]:

$$\omega = \begin{bmatrix} \frac{1}{3}(2\Omega_2+\bar{\Omega}_2) & i\bar{\Omega}_3 & -\frac{1}{2}\,\Omega_4 \\ \Omega_1 & \frac{1}{3}(\bar{\Omega}_2 - \Omega_2) & -\frac{1}{2}\,\Omega_3 \\ 2\Omega & 2i\bar{\Omega}_1 & -\frac{1}{3}(2\bar{\Omega}_2 + \Omega_2) \end{bmatrix} . \qquad (9.7.9)$$

Suppose now that we have computed ω for f = 1, h = 0 in (9.7.6), then ω' which is computed for any other choice of f and h is given by

$$\omega' = A^{-1}\,\omega A + A^{-1}dA, \qquad (9.7.10)$$

where A is a matrix belonging to the group SU(2,1) and is related to f and h by

$$A = \begin{bmatrix} |f|e^{i\vartheta} & i\bar{h}e^{-2i\vartheta} & -\frac{e^{i\vartheta}}{4|f|}\,(\frac{1}{2}\,\rho + i|h|^2) \\ 0 & e^{-2i\vartheta} & -\frac{e^{i\vartheta}}{2|f|}\,h \\ 0 & 0 & \frac{e^{i\vartheta}}{|f|} \end{bmatrix} . \qquad (9.7.11a)$$

Here

$$f = |f|e^{3i\vartheta}, \qquad (9.7.11b)$$

ϑ is a real function, and ρ is an arbitrary function which is connected with the freedom (9.7,8) in the choice of Ω_2,Ω_3 and Ω_4.

According to the transformation rule (9.7.10) it is natural to associate a name connection to the form ω defined in (9.7.9). It is called Cartan-Chern-Moser (CCM) connection of nondegenerate CR structure (N,[(κ,α)]).

The group G of matrices A of the form (9.7.11) defines a gauge

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[6] See an Appendix at the end of this work.

group G of Cartan-Chern-Moser connection

$$\mathcal{G} = \frac{G}{J} \tag{9.7.12}$$

where

$$J = \{A \in G : A = \sigma \cdot \mathrm{id},\ \sigma^3 = 1,\ \sigma = \mathrm{const},\ \sigma \in \mathbb{C}\} \tag{9.7.13}$$

Moreover

$$\mathcal{G} \subset SU(2,1) \tag{9.7.14}$$

and

$$\omega \in su(2,1) \tag{9.7.15}$$

where su(2,1) is a Lie algebra of the group SU(2,1). It can be seen that applying transformations (9.7.10)-(9.7.11) for ω one obtains all the algebra su(2,1).

The curvature of CCM connection (9.7.9)

$$\tilde{\Omega} = d\omega + \omega \wedge \omega \tag{9.7.16}$$

has usual transformation properties

$$\tilde{\Omega}' = A^{-1}\, \tilde{\Omega}\, A \tag{9.7.17}$$

under the action of the group $\mathcal{G}$. According to the relations (9.7.7) it has a relatively simple form

$$\tilde{\Omega} = \begin{bmatrix} 0 & i\bar{\mathcal{R}}\Omega_1 \wedge \Omega & -\frac{1}{2}(S\Omega_1 + \overline{S\Omega}_1)\wedge \Omega \\ 0 & 0 & -\frac{1}{2}\mathcal{R}\bar{\Omega}_1 \wedge \Omega \\ 0 & 0 & 0 \end{bmatrix} \tag{9.7.18}$$

where functions $\mathcal{R}$, S are defined by (9.7.7).

To complete the discussion let us note that the function R is transformed by (9.7.17) to

$$\mathcal{R}' = \frac{1}{|f|^4}\, e^{6i\vartheta}\, \mathcal{R} \tag{9.7.19}$$

and unless $\mathcal{R} = 0$, $\mathcal{R}$ may be normalized to get

$$\mathcal{R} = 1 \tag{9.7.20}$$

Then the second condition

$$\mathrm{Re}\ \omega^{\circ}_{\circ} := \mathrm{Re}\ \Omega_2 = 0 \qquad (9.7.21)$$

may be imposed on A (i.e. on f,h in (9.7.6)).

It is worth noting that conditions (9.7.21) and (9.7.20) are independent of each other. Therefore they can be imposed also in the reverse order. If it is the case (9.7.21) subordinates h to f in (9.7.6) and (9.7.20) determines f up to <u>the change of the sign</u>:

$$f \longrightarrow -f \qquad (9.7.22)$$

The forms Ω and Ω_1 satisfying (9.7.20) and (9.7.21) have, by their construction, property of being invariant under the transformation (9.7.4) of initial forms κ, α. The same applies to the forms $\Omega_2,\Omega_3,\Omega_4$ restricted to the conditions (9.7.20)-(9.7.21). Apart from the case

$$\mathcal{R} = 0 \qquad (7.6.23)$$

the full set of forms $(\Omega,\Omega_1,\Omega_2,\Omega_3,\Omega_4)$ constitutes set of all invariants of a nondegenerate 3-dimensional CR structure. According to the transformation (9.7.22) two CR structures (κ,α) and (κ',α') on a manifold N are equivalent if and only if we have

$$\begin{aligned} \Omega &= \Omega' \\ \Omega_1 &= \pm\ \Omega'_1 \\ \Omega_2 &= \Omega'_2 \\ \Omega_3 &= \pm\ \Omega'_3 \\ \Omega_4 &= \Omega'_4 \end{aligned} \qquad (9.7.24)$$

Note, that since Ω, Ω_1 and $\bar{\Omega}_1$ constitute a basis on N′ then forms Ω_2, Ω_3 and Ω_4 can be decomposed onto them. The functional coefficients of this decomposition are called <u>Cartan's invariants</u> of CR manifold [Cartan 1932]. These are α, β, γ, ϑ, η, ξ where

$$\begin{aligned} \Omega_2 &= -\ \bar{\Omega}_2 = \alpha\Omega_1 - \overline{\alpha\Omega}_1 + i\beta\Omega \\ \Omega_3 &= i\gamma\Omega_1 + \vartheta\bar{\Omega}_1 + \eta\Omega \\ \bar{\Omega}_3 &= \bar{\vartheta}\Omega_1 - i\gamma\bar{\Omega}_1 + \bar{\eta}\Omega \\ \Omega_4 &= \bar{\Omega}_4 = -\ \frac{1}{2}\ i\bar{\eta}\Omega_1 + \frac{1}{2}\ i\eta\bar{\Omega}_1 + \xi\Omega \end{aligned} \qquad (9.7.25)$$

According to the rules (9.7.7), (9.7.20), (9.7.21) forms Ω and Ω_1 together with Cartan invariants satisfy

$$\begin{aligned} d\Omega &= i\Omega_1 \wedge \bar{\Omega}_1 \\ d\Omega_1 &= \bar{\alpha}\Omega_1 \wedge \bar{\Omega}_1 + i(\beta - \gamma)\Omega \wedge \Omega_1 - \vartheta\Omega\,\bar{\Omega}_1 \\ d\bar{\Omega}_1 &= -\alpha\Omega_1 \wedge \bar{\Omega}_1 - \bar{\vartheta}\Omega \wedge \Omega_1 - i(\beta - \gamma)\Omega \wedge \bar{\Omega}_1 \end{aligned} \tag{9.7.26}$$

Combining (9.7.7), (9.7.20), (9.7.21), (9.7.25) and (9.7.26) we obtain also relations

$$\begin{aligned} &\bar{\partial}\alpha + \partial\bar{\alpha} = 2\alpha\bar{\alpha} - \beta - 3\gamma \\ &\bar{\partial}_o\alpha - i\partial\beta = -i\alpha(\beta - \gamma) - \overline{\alpha\vartheta} - \frac{3}{2}\, i\bar{\eta} \\ &\partial_o\bar{\alpha} + i\bar{\partial}\beta = i\bar{\alpha}(\beta - \gamma) - \alpha\vartheta + \frac{3}{2}\, i\eta \\ &\partial\vartheta - i\bar{\partial}\gamma = 2\alpha\vartheta - \frac{3}{2}\, i\eta \\ &\overline{\partial\vartheta} + i\partial\gamma = 2\overline{\alpha\vartheta} + \frac{3}{2}\, i\bar{\eta} \\ &\partial_o\vartheta - \bar{\partial}\eta = 2i\beta\vartheta + \bar{\alpha}\eta - 1 \\ &\partial_o\bar{\vartheta} - \partial\bar{\eta} = -2i\beta\bar{\vartheta} + \alpha\bar{\eta} - 1 \\ &\partial\eta - i\partial_o\gamma = \alpha\eta + \gamma^2 - \vartheta\bar{\vartheta} - \xi \\ &\overline{\partial\eta} + i\partial_o\gamma = \overline{\alpha\eta} + \gamma^2 - \vartheta\bar{\vartheta} - \xi \end{aligned} \tag{9.7.27}$$

where operators $(\partial_o, \partial, \bar{\partial})$ constitute a dual basis to $(\Omega, \Omega_1, \bar{\Omega}_1)$. To complete the discussion of the case $\mathcal{R} \neq 0$ we also give commutators of ∂_o, ∂ and $\bar{\partial}$ which follows from (9.7.26)

$$\begin{aligned} [\bar{\partial}, \partial] &= i\partial_o + \bar{\alpha}\partial - \alpha\bar{\partial} \\ [\partial, \partial_o] &= i(\beta - \gamma)\partial - \overline{\vartheta\partial} \\ [\bar{\partial}, \partial_o] &= -i(\beta - \gamma)\bar{\partial} - \vartheta\partial \end{aligned} \tag{9.7.28}$$

So far we did not consider the case

$$\mathcal{R} = 0 \tag{9.7.29}$$

for which we can not perform transformation (9.7.20). In this case, however, it follows from exterior differentiation of (9.7.7d) and from definition (9.7.7e) that also function S vanishes. Therefore all curvature $\tilde{\Omega}$ of the associated CR structure is zero. This can happen exactly in one case [Cartan 1932] when the CR structure is equivalent to

$$\Omega_1 = dz$$
$$\Omega = du - \frac{i}{2}\bar{z}dz + \frac{i}{2}zd\bar{z} \tag{9.7.30}$$

As we know this structure corresponds to the structure with the highest possible symmetry group SU(1,2) (compare (9.6.)5 and the comment at the end of Section IX.6). We will call this structure hyperquadric.

Remark 9.7.1

The forms (9.7.30) in the case of $\mathcal{R} = 0$ and forms Ω, Ω_1, Ω_2, Ω_3, Ω_4 in the case (9.7.20)-(9.7.21) are called Cartan's invariant forms [Cartan 1932].

The meaning of this term becomes more apparent if we assume that the CR structure admits a symmetry. It follows from the construction of the forms Ω, Ω_1, Ω_2, Ω_3, Ω_4 that if X is a symmetry of Cauchy-Riemann space then

$$\mathcal{L}_X\Omega = \mathcal{L}_X\Omega_1 = \mathcal{L}_X\Omega_2 = \mathcal{L}_X\Omega_3 = \mathcal{L}_X\Omega_4 = 0 \tag{9.7.31a}$$

$$X(\alpha) = X(\beta) = X(\gamma) = X(\vartheta) = X(\eta) = X(\xi) = 0 \tag{9.7.31b}$$

X. 3-DIMENSIONAL CR STRUCTURES AND LORENTZ GEOMETRIES

1. Standard and nonstandard approaches to the problem

Let us turn to the theory of shear-free congruences of null geodesics in space-time i.e. to the theory of shear-free optical geometry.

We know (Theorem 7.2.1) that any space-time M which admits a congruence of null and shear-free geodesics defines a 3-dimensional CR structure. We also know that the converse of the above mentioned statement is also true. Therefore taking all nonequivalent 3-dimensional CR structures $(N,[(\kappa,\alpha)])$ and constructing a Lorentzian manifolds

$$M = \mathbb{R} \times N \tag{10.1.1}$$

with the metric

$$g = 2p^2(\alpha \otimes \bar{\alpha} + \bar{\alpha} \otimes \alpha - \kappa \otimes \varphi - \varphi \otimes \kappa) \tag{10.1.2}$$

we obtain all space-times admitting such congruences. Forms κ and α, which stand in (10.1.2), are any representatives of the class $[(\kappa,\alpha)]$ and are understood as being pull-backed to M by π^*, where π is a natural projection

$$\pi : M \longrightarrow N. \tag{10.1.3}$$

The real function p and a real 1-form φ which stay in (10.1.2) are arbitrary provided that

$$p\kappa \wedge \alpha \wedge \bar{\alpha} \wedge \varphi \neq 0 \tag{10.1.4}$$

The shear-free congruence of null geodesics is represented by a real vector field $k \neq 0$ s.t.

$$k \lrcorner \alpha = k \lrcorner \bar{\alpha} = k \lrcorner \kappa = 0 \tag{10.1.5}$$

The congruence defined by k can be in general <u>twisting</u> and <u>expanding</u>. This means that if a 2-dimensional obstacle is placed perpendicularly to the rays of a congruence, then the shape of this obstacle on the screen which is also placed perpendicularly to the rays, can be rotated and increased when compared to the position and size of an obstacle. In the language of CR geometry associated with M the vanishing of a twist is equivalent to [Robinson, Trautman 198.]:

$$\kappa \wedge d\kappa = 0 \qquad (10.1.6)$$

This means that the presence of twist of the congruence is equivalent to nondegeneracy of an associated CR structure.

The vanishing of an expansion of a congruence associated with a vector field k given by (8.1.5) is equivalent to

$$\mathcal{L}_k \eta = 0 \qquad (10.1.7)$$

where η is a volume form

$$\eta = ip^4 \kappa \wedge \alpha \wedge \bar{\alpha} \wedge \varphi \qquad (10.1.8)$$

on M given by (10.1.1).

In this chapter we study space-times (10.1.1)-(10.1.2) for different CR structures.

Such space-times were studied in the context of gravitation theory [Robinson, Trautman 1962, Kerr 1963, and many others. See Kramer at al 1980 for a review]. There the metrics of the form (10.1.2) were subjected to the Einstein equations. In that case one of the Einstein equations ensures that the tangential CR equation (9.2.4) for the associated CR structure is solvable [Tafel 1985]. Therefore there exists a complex function ξ on N s.t.

$$\alpha = d\xi \qquad (10.1.9)$$

and

$$d\xi \wedge d\bar{\xi} \neq 0 \qquad (10.1.10)$$

The functions ξ and $\bar{\xi}$ can be supplemented by a one real function u on N in such a way that

$$du \wedge d\xi \wedge d\bar{\xi} \neq 0 \qquad (10.1.11)$$

The set of functions $(u, \zeta, \bar{\xi})$ constitute a coordinate system on N.

By means of transformations (9.7.4a) we can always choose an appropriate factor to achieve

$$\kappa = du + Ld\zeta + Ld\overline{\zeta}, \tag{10.1.12}$$

where L is an appropriate complex function

$$L = L(u, \xi, \overline{\xi}) \tag{10.1.13}$$

on N. The metric tensor (10.1.2) takes the form

$$g = 2(e^1e^2 - e^3e^4) \tag{10.1.14}$$

where

$$e^1 = -\frac{d\xi}{\Pi} = \overline{e^2}$$

$$e^3 = du + Ld\zeta + \overline{L}d\overline{\zeta} \tag{10.1.15}$$

$$e^4 = dr + Wd\xi + \overline{W}d\overline{\zeta} + He^3$$

where $\Pi \neq 0$, W are any complex functions on $M = \mathbb{R}xN$ and H, r are any real functions on M provided that

$$e^1 \wedge e^2 \wedge e^3 \wedge e^4 \neq 0 \tag{10.1.16}$$

Here we used a historical convention in which

$$e^ie^j := \frac{1}{2}(e^i \otimes e^j + e^j \otimes e^i) \tag{10.1.17}$$

As we said in Geenral Relativity metrics (10.1.14)-(10.1.15) are subjected to the Einstein equations. A number of questions may be asked at this stage. Here we list three of them:

1) Do Einstein equations imposed on metrics (10.1.14) - (10.1.15) imply realizability of an associated CR structures (10.1.9), (10.1.12)?

2) Are there any (and if so, any) physically interesting, solutions to the Einstein equations among the metrics (10.1.14) - (10.1.15)?

3) Given a CR structure (10.1.9), (10.1.12) is it possible to find such Π, W and H that the corresponding metric (10.1.15) describes, for example, Minkowski space-time?

In order to answer the above questions, it is enough to work in the formalism described between equations (10.1.9) - (10.1.17). However, in the context of nondegenerate CR structures, we prefer another approach, which is coordinate free and invariant under the allowed transformations for κ and α. This approach is related to the invariants of CR structures defined in IX. Within this approach, which is suitable only for the nondegenerate CR structures, we have a preferred forms Ω and Ω_1 among the class of forms $[(\kappa,\alpha)]$. These are Cartan invariant forms defined in the section IX 7 by equations (9.7.6), (9.7.7), (9.7.20), (9.7.21) or by (9.7.30). Therefore we have a representation of the metric (10.1.2) which suits to the associated CR geometry - namely:

$$g = 2\mathcal{P}^2[\Omega_1\bar{\Omega}_1 - \Omega(dr + \mathcal{W}\Omega_1 + \overline{\mathcal{W}}\bar{\Omega}_1 + \mathcal{H}\Omega)] \qquad (10.1.18)$$

where $\mathcal{P}$, $\mathcal{H}$, r are any real functions and $\mathcal{W}$ is any complex function on

$$M = \mathbb{R} \times N \qquad (10.1.19)$$

provided that

$$\mathcal{P}\Omega_1 \wedge \bar{\Omega}_1 \wedge \Omega \wedge dr \neq 0 \qquad (10.1.20)$$

[Lewandowski, Nurowski 1990b].

The approaches (10.1.9), (10.1.12)-(10.1.17) and (10.1.18)-(10.1.20) to the problem of Lorentzian geometries admitting shear-free congruences of null geodesics we call standard and nonstandard, respectively. Here, we first give examples of the utility of nonstandard approach and then examples of solvable problems in the framework of the standard one.

2. Weyl tensor of metrics admitting twisting shear-free congruences of null geodesics

As an application of a nonstandard approach to the space-times admitting a twisting shear-free congruences of null geodesics described in X.1 we compute a Weyl tensor for such metrics.

As we know such space-times can be considered locally a product

$$M = \mathbb{R} \times N \tag{10.2.1}$$

where N is any nondegenerate CR structure described by the Cartan invariant forms Ω and Ω_1. In the case when the CR structure $(N, (\Omega, \Omega_1)])$ is nonequivalent to the hyperquadric the forms Ω and Ω_1 satisfy equations (9.7.26) with Cartan invariants α, $\beta-\gamma$, ϑ subjected to the conditions (9.7.27). In the case of hyperquadric (9.7.30) one can also use equations (9.7.26) with

$$\alpha = \beta - \gamma = \vartheta = 0 \tag{10.2.2}$$

The metric tensor has the form

$$g = 2e^1e^2 - 2e^3e^4 \tag{10.2.3}$$

where $(e^1,\ e^2,\ e^3,\ e^4)$ denote the null cobasis on M defined by

$$\begin{aligned} e^1 &= \mathcal{P}\Omega_1 = \bar{e}^2 \\ e^3 &= \Omega \\ e^4 &= \mathcal{P}^2(dr + \mathcal{W}\Omega_1 + \overline{\mathcal{W}}\Omega_1 + \mathcal{H}\Omega) \end{aligned} \tag{10.2.4}$$

The congruence of twisting shear-free and null geodesics is tangent to the vector field

$$k \sim e_4 = \frac{1}{\mathcal{P}^2}\,\partial_r \tag{10.2.5}$$

Since we are interested in the Weyl tensor of metrics (10.2.3)-(10.2.4), and this is invariant under conformal transformations then we can put

$$\mathcal{P} = 1 \tag{10.2.6}$$

in (10.2.4).

Simple but lengthy calculations show that in this case and in the dual basis $(e_1e_2e_3e_4)$ to (e^1, e^2, e^3, e^4) the following relations hold:

a) 1-forms of the Levi-Civita connection associated with the metric (10.2.3), (10.2.4), (10.2.6) are given by

$$\Gamma_{12} = \alpha e^1 - \bar{\alpha} e^2 + i(B + \beta - \gamma)e^3 + \frac{i}{2}\, e^4 \tag{10.2.7a}$$

$$\Gamma_{13} = -\bar{\vartheta} e^1 + iBe^2 + Ae^3 + Ce^4 \tag{10.2.7b}$$

$$\Gamma_{14} = \frac{i}{2} e^2 + Ce^3 \tag{10.2.7c}$$

$$\Gamma_{23} = -iBe^1 - \vartheta e^2 + \bar{A}e^3 + \bar{C}e^4 \tag{10.2.7d}$$

$$\Gamma_{24} = -\frac{i}{2} e^1 + \bar{C}e^3 \tag{10.2.7e}$$

$$\Gamma_{34} = Ce^1 + \bar{C}e^2 - De^3 \tag{10.2.7g}$$

where functions A, B, C, D are defined as follows:

$$A = \mathcal{H}_1 - \mathcal{W}_3 - i(\beta - \gamma)\mathcal{W} + \overline{\mathcal{W}\vartheta} \tag{10.2.8a}$$

$$2iB = \overline{\mathcal{W}}_1 - \mathcal{W}_2 + \mathcal{W}\bar{\alpha} - \overline{\mathcal{W}}\alpha + i\mathcal{H} \tag{10.2.8b}$$

$$2C = -\mathcal{W}_4 \tag{10.2.8c}$$

$$D = \mathcal{H}_4 \tag{10.2.8d}$$

The subscripts 1, 2, 3, 4 in the above expressions denote derivatives with respect to vector fields

$$e_1 = \partial - \mathcal{W}\partial_r \tag{10.2.9a}$$

$$e_2 = \bar{\partial} - \overline{\mathcal{W}}\partial_r \tag{10.2.9b}$$

$$e_3 = \partial_o - \mathcal{H}\partial_r \tag{10.2.9c}$$

$$\partial_4 = \partial_r \tag{10.2.9d}$$

where operators ∂_o, ∂ and $\bar{\partial}$ constitute a dual basis to Ω, Ω_1 and $\bar{\Omega}_1$ on N and are extended to M by the demand that

$$\partial(r) = \bar{\partial}(r) = \partial_o(r) = 0 \tag{10.2.10}$$

b) The components of the Riemann and Ricci tensors can be computed by using (10.2.7) and the definition of the curvature

$$\Omega^a_b = \frac{1}{2} R^a_{bcd} e^c \wedge e^d = d\Gamma^a_b + \Gamma^a_c \wedge \Gamma^c_b \tag{10.2.11}$$

Since they are not invariant under the conformal transformations they are important only for computing the Weyl tensor. However, for some other reasons we quote here the Ricci tensor components, and Ricci scalar.

These are:

$$R_{11} = -i\bar{\vartheta} - 2C\alpha - 2C^2 - 2C_1 = \overline{R_{22}} \quad (10.2.12a)$$

$$R_{12} = 2B + \bar{\alpha}_1 + \bar{\alpha}_2 - 2\alpha\bar{\alpha} + \beta - \gamma + C\bar{\alpha} + \bar{C}\alpha - 2C\bar{C} - \bar{C}_1 - C_2 \quad (10.2.12b)$$

$$R_{13} = 2\overline{\alpha\vartheta} - \bar{\vartheta}_2 + \overline{\vartheta C} - iC(\beta-\gamma) - \frac{i}{2}A - 2iBC - iB_1 - C_3 + A_4 = \overline{R_{23}} \quad (10.2.12c)$$

$$R_{14} = -C_4 - 2iC = \overline{R_{24}} \quad (10.2.12d)$$

$$R_{33} = -2\vartheta\bar{\vartheta} - 2A\bar{\alpha} - 2\bar{A}C - 2iBD + 2B^2 + 2A_2 - 2iB_3 \quad (10.2.12e)$$

$$R_{34} = 2C_2 + D_4 + 4C\bar{C} - 2C\bar{\alpha} + B + \frac{i}{2}D - iB_4 \quad (10.2.12f)$$

$$R_{44} = \frac{1}{2} \quad (10.2.12g)$$

$$R = 2B + 2\alpha_2 + 2\bar{\alpha}_1 - 4\alpha\bar{\alpha} + 6C\bar{\alpha} + 2\bar{C}\alpha - 12C\bar{C} - 2\bar{C}_1 - 6C_2 - 2D_4 + $$
$$+ 2iB_4 - iD \quad (10.2.13)$$

Such quantities as R_{33}, R_{34} and R are real due to the identities which follow from

$$d^2e^4 = 0 \quad (10.2.14)$$

These are:

$$\bar{A}_1 - A_2 + 2iB_3 + A\bar{\alpha} - \bar{A}\alpha + 2(A\bar{C} - C\bar{A}) + 2iBD = 0 \quad (10.2.15a)$$

$$\bar{C}_1 - C_2 + iB_4 + C\bar{\alpha} - \bar{C}\alpha - \frac{i}{2}D = 0 \quad (10.2.15b)$$

$$A_4 - 2C_3 - D_1 - 2\overline{C\vartheta} - 2iC(\beta - \gamma) = 0 \quad (10.2.15c)$$

c) The relevant Weyl tensor components are:

$$C^1_{\ 424} = 0 \quad (10.2.16a)$$

$$C^1_{\ 434} = -\frac{1}{2}(\bar{C}_4 + i\bar{C}) \quad (10.2.16b)$$

$$C^3_{\ 334} = \frac{1}{3}(-4B - \bar{\alpha}_1 - \alpha_2 + 2\alpha\bar{\alpha} + 3C\bar{\alpha} - \bar{C}\alpha +$$
$$+ \bar{C}_1 - 3C_2 + D_4 + 2iB_4 - iD) \quad (10.2.16c)$$

$$C^1_{\ 134} = i(B_4 + \frac{1}{2}D) \quad (10.2.16d)$$

$$C^1_{\ 334} = \frac{1}{4}(4\alpha\vartheta - 2\vartheta_1 + 2\vartheta C - 2o\bar{C}(\beta - \gamma) +$$
$$+ 3i\bar{A} + 2iB_2 + 2\bar{C}_3 - 2\bar{A}_4) \quad (10.2.16e)$$

$$C^1_{\ 323} = \vartheta_3 - 2i\beta\vartheta + 2i\vartheta\gamma + 2iB\vartheta + \overline{A\alpha} + D\vartheta + \bar{A}_2 \quad (10.2.16f)$$

All other components can be obtained from the above by applying either symmetries of the Weyl tensor or by the fact that $C^a_{\ bcd}$ is traceless in any pair of indices.

d) The first important fact we can observe in (10.2.16) is that

$$C^{1}_{424} = 0 \tag{10.2.17}$$

This is only a consequence of the assumed form of the metric [Robinson, Trautman 1989, Lewandowski, Nurowski 1990b]. Comparing this with the definition of principal null direction (5.3.1) applied to the vector field

$$k \sim e_4 = \partial_r \tag{10.2.18}$$

we can easily see that our congruence of shear-free null geodesics is tangent to a principal null direction of the Weyl tensor. This is a direct proof of the statement quoted in the section V.3 in particular case of twisting congruences.

e) Applying Theorem 5.3.1 we see that our space-times are of Cartan-Petrov-Penrose type II iff

$$C^{1}_{434} = 0 \tag{10.2.19}$$

i.e. iff

$$C = ae^{ir} \tag{10.2.20}$$

$$\mathcal{W} = 2iae^{ir} + b \tag{10.2.21}$$

and a, b are arbitrary complex functions s.t.

$$e_4(a) = e_4(b) = 0 \tag{10.2.22}$$

f) The important case of type N space-times with twisting shear-free congruences is given by all space-times conformally equivalent to (10.2.4), (10.2.6) with all Weyl tensor components (10.2.16), except (10.2.16f) equal to zero. By simple integrations of (10.2.16b)-(10.2.16e) we arrive at the following theorem:

Theorem 10.2.1 [Lewandowski, Nurowski 1990b].

All space-times which are of Cartan-Petrov-Penrose type N and admit congruences of shear-free null geodesics with twist are locally given by:

a) $M = \mathbb{R} \times N$ (10.2.23)

b) N is any nondegenerate CR structure with Cartan invariant forms Ω and Ω_1 satisfying either (9.7.26)-(9.7.27) or (9.7.30)

c) the metric tensor has the form

$$g = 2\mathcal{P}^2[\Omega_1\bar{\Omega}_1 - \Omega(dr + \mathcal{W}\Omega_1 + \overline{\mathcal{W}}\bar{\Omega}_1 + \mathcal{H}\Omega)] \tag{10.2.24}$$

$$\mathcal{W} = 2aie^{ir} + b \tag{10.2.25}$$

$$\mathcal{H} = (\bar{\Delta} - i\bar{b})ae^{ir} + (\Delta + ib)\bar{a}e^{-ir} + h \tag{10.2.26}$$

$$h = -\frac{1}{2}(\Delta\bar{\alpha} + \bar{\Delta}\alpha + \beta - \gamma) - 6a\bar{a} + i(\Delta\bar{b} - \bar{\Delta}b) \tag{10.2.27}$$

where $\mathcal{P} \neq 0$ is an arbitrary real function on M, the complex functions a and b are subjected to the conditions

$$\frac{\partial a}{\partial r} = \frac{\partial b}{\partial r} = 0 \tag{10.2.28}$$

and

$$2iha - 2\delta a - i\partial\bar{\Delta}a - \partial(\bar{b}a) - b\bar{\Delta}a + ib\bar{b}a = 0 \tag{10.2.29}$$

$$\begin{aligned} &4\alpha\bar{\vartheta} - 2\partial\bar{\vartheta} + 3i(\delta b - \bar{\vartheta}\bar{b}) + \partial(\bar{\Delta}b - \Delta\bar{b} - 4ih) + \\ &+ 8i(a\Delta\bar{a} - \partial(a\bar{a}) + iba\bar{a}) = 0 \end{aligned} \tag{10.2.30}$$

In the equations (10.2.26)-(10.2.30) we used an abbreviation Δ and δ to denote operators

$$\Delta = \partial - \alpha \tag{10.2.31}$$

$$\delta = \partial_o + i(\beta - \gamma) \tag{10.2.32}$$

d) the twisting congruence of null shear-free geodesics is tangent to the vector field

$$k = \frac{\partial}{\partial r} \tag{10.2.33}$$

If we demand that in addition to the vanishing of (10.2.16b)-(10.2.16e) also (10.2.16f) vanishes, then we obtain all possible space-times which are conformally flat and admit twisting shear-free congruence of null geodesics. Hence imposing condition

$$C^1_{\ 323} = 0$$

we arrive at the following corollary.

Corollary 10.2.1

All space-times admitting shear-free twisting congruence of null geodesic and being conformally related to the Minkowski space-time are given by Theorem 10.2.1 provided that

$$C^{1}_{323} = 0 \tag{10.2.34}$$

This condition is equivalent to

$$(\partial + \alpha)(\partial h - \delta b + \overline{b\vartheta} - 4a(\Delta + ib)\bar{a}) - 2a(\partial + ib)(\Delta + ib)\bar{a} + \\ + \bar{\vartheta}(2i\beta + 6ia\bar{a}) + \partial_0\bar{\vartheta} = 0 \tag{10.2.35}$$

3. Important example - Fefferman metric

Let us notice that if we put

$$\begin{aligned} a &= 0 \\ b &= \frac{2}{3}\, i\alpha \\ h &= -\frac{2}{3}\,\beta \end{aligned} \tag{10.3.1}$$

in Theorem 10.2.1 then in all cases different from the hyperquadric (9.7.30) equations (10.2.28)-(10.2.30) are satisfied due to the equations (9.7.27). In the case of hyperquadric, which may be characterized by $\alpha = \beta = \gamma = \vartheta = 0$ situation is even better - even equation (10.2.35) is satisfied. Using equations (9.7.27) we can see that in all cases except hyperquadric (9.7.30) the left hand side of the equation (10.2.35) is equal to -1. Therefore in all these cases the metrics given by Theorem 10.2.1 with a, b, h defined by (10.3.1) can not be conformally flat. Therefore we arrive at the following theorem.

Theorem 10.3.1 [Lewandowski 1988]

For any nondegenerate CR structure the metric tensor

$$g = 2\mathcal{P}^2[\Omega_1\bar{\Omega}_1 - \Omega(dr + \frac{2}{3}\, i\alpha\Omega_1 - \frac{2}{3}\, i\overline{\alpha\Omega} - \frac{2}{3}\,\beta\Omega)] \tag{10.3.2}$$

is of the Cartan-Petrov-Penrose type N. Here forms Ω and Ω_1 and functions α, β are Cartan invariants defined in section IX.7. The only case when this metric tensor is conformally flat occurs when $(N,[(\Omega,\Omega_1)])$ is equivalent to hyperquadric (9.7.30) (in this case

$\alpha=\beta=0$).

The metrics (10.3.2) are called Fefferman metrics [Fefferman 1976]. They can be considered as a recipe for associating with any nondegenerate CR structure a conformal Lorentzian geometry of Cartan-Petrov-Penrose type N.

The following theorem admits another characterization of Fefferman metrics.

Theorem 10.3.2 [Sparling 1985, Lewandowski 1989].

Given any nondegenerate CR structure $(N,[(\kappa,\alpha)])$ the Fefferman metric is the only class among metrics (10.2.3)-(10.2.4) on

$$M = \mathbb{R} \times N, \tag{10.3.3}$$

which is of Cartan-Petrov-Penrose type N, possessing a conformal symmetry generated by the vector field k tangent to the shear-free congruence of null geodesic associated with M by (10.2.5).

Proof

If a metric (10.2.3)-(10.2.4) is of Petrov type N then it is described by Theorem 10.2.1. The fact that $k = \partial_r$ generates a conformal symmetry, by definition, means that

$$\mathcal{L}_k g = \varphi\cdot g \tag{10.3.4}$$

where g is given by Theorem 10.2.1 and φ is any real function on $M = \mathbb{R} \times N$. This is the matter of checking that (10.3.4) imposed on the metric from Theorem 10.2.1 is equivalent to

$$a = 0. \tag{10.3.5}$$

To complete the proof it is enough to show that if b and h satisfy (10.2.29)-(10.2.30) then

$$e^4 = dr + b\Omega_1 + \overline{b\Omega}_1 + h\Omega \tag{10.3.6}$$

differs from

$$e^4_F = dr + \frac{2}{3}\, i\alpha\, \Omega_1 - \frac{2}{3}\, i\overline{\alpha\Omega}_1 - \frac{2}{3}\, \beta\Omega \tag{10.3.7}$$

which appears in Fefferman metrics by a differential of a real function. This occurs (locally) if and only if

$$de^4 = de^4_F \tag{10.3.8}$$

However, this last equation is satisfied due to the fact that b, h satisfy equations (10.2.29)-(10.2.30), as can be seen by writing the expression (10.3.8) in the basis Ω, Ω_1 and $\bar{\Omega}_1$. This completes the proof.

□

The Fefferman metrics were introduced since they could be used to define an important class of curves on CR manifolds. These curves are called <u>chains</u> [Cartan 1932]. They are obtained from null geodesics in Fefferman metrics $[g_F]$ by projecting them on N along lines of a congruence of shear-free null geodesics associated with $[g_F]$ [Fefferman 1976]. The careful study of chains in geometries (10.2.3)-(10.2.4)can be found in [Koch 1988, Lewandowski, Nurowski 1990a]

4. Einstein equations in the framework of standard approach

Let us turn to the study of Einstein equations on space-times admitting congruences of shear-free and null geodesics in the standard approach mentioned in section X.1. This approach is due to I. Robinson and A.Trautman [Robinson, Trautman 1962]. They succeded in reducing vacuum Einstein equations for space-times admitting <u>non-twisting</u> congruences (of shear-free and null geodesics) to one equation for one real function of three variables. They solved this equation in certain situations obtaining first examples of gravitational waves with spherical wave-fronts. Their approach was then applied to the twisting case. The first full set of sufficiently reduced vacuum equations in this case was given by Kerr [Kerr 1963]. We present these equations in a form given in [Kramer at al 1980]. They can be described by the following theorem:

Theorem 10.4.1

A space-time M admits a geodesic, shear-free null congruence k exactly if the metric can be written as

$$g = 2(e^1e^2 - e^3e^4) \tag{10.4.1}$$

$$e^1 = - \frac{d\zeta}{P\bar{\rho}} = \overline{e^2} \tag{10.4.2}$$

$$e^3 = du + Ld\zeta + \bar{L}d\bar{\zeta} \tag{10.4.3}$$

$$e^4 = dr + Wd\zeta + \bar{W}d\bar{\zeta} + He^3 \tag{10.4.4}$$

If, in addition, the Ricci tensor components satisfy

$$R_{44} = R_{14} = R_{11} = R_{12} = R_{34} = 0 \tag{10.4.5}$$

(according to the basis $(e_1e_2e_3e_4)$ dual to $(e^1e^2e^3e^4)$) the metric functions appearing in (10.4.2)-(10.4.3) are given by

$$\frac{1}{\rho} = -(r + i\Sigma) \tag{10.4.6}$$

$$2i\Sigma = P^2(\bar{\partial}L - \partial\bar{L}) \tag{10.4.7}$$

$$\partial = \partial_\zeta - L\partial_u \tag{10.4.8}$$

$$W = \frac{L_{,u}}{\rho} + i\partial\Sigma \tag{10.4.9}$$

$$H = -r\ (\ln P)_u - \frac{m\ r + M\Sigma}{r^2 + \Sigma^2} + \frac{K}{2} \tag{10.4.10}$$

$$K = 2P^2\ \mathrm{Re}[\partial(\bar{\partial}\ \ln P - \bar{L}_u)] \tag{10.4.11}$$

$$M = \Sigma K + P^2\mathrm{Re}[\partial\bar{\partial}\Sigma - 2\bar{L}_u\partial\Sigma - \Sigma\partial_u\partial\bar{L}] \tag{10.4.12}$$

The unknown functions here, are: L (complex); P and m (real). All of them are independent of the coordinate r (thus they depend only on u, ξ, $\bar{\zeta}$ which together with r constitute a coordinate system on M).

The congruence of null geodesics without shear is tangent to the vector field

$$k = e_4 \tag{10.4.13}$$

The metric tensor (10.1) satisfies pure radiation Einstein equations

$$R_{ab} = \Phi\bar{\Phi}\ k_a k_b \tag{10.4.14}$$

if and only if the functions P, L, m and M are additionally subjected to the following conditions:

$$(3L_{,u} - \partial)(m + iM) = 0 \qquad (10.4.15)$$

$$P^4(\partial - 2L_{,u} + 2\partial \ln P)\partial[\bar{\partial}(\bar{\partial} \ln P - \bar{L}_{,u}) + (\bar{\partial} \ln P - \bar{L}_u)^2] + \overline{}$$
$$- P^3[P^{-3}(m + iM)]_{,u} = \kappa_o \, \Phi^o_2 \bar{\Phi}^o_2 \qquad (10.4.16)$$

The complex function

$$\Phi^o_2 = \Phi^o_2(u, \zeta, \bar{\zeta})$$

is absolutely arbitrary. Imposing some other conditions on this function one can interpret what kind of pure radiation is described by the corresponding metric. The following two cases are of particular interest.

a) $\Phi^o_2 \equiv 0.$ (10.4.17)

In this case the metric tensor (10.4.1) with functions P,L,M and m satisfying (10.4.7), (10.4.8), (10.4.11), (10.4.12), (10.4.15), (10.4.16) describes vacuum space-time [Robinson, Trautman 1962, Kerr 1963, Debney, Kerr, Schid 1969, Robinson at al 1969].

b) $$(\partial - L_u)\left(\frac{\Phi_2}{P}\right) = 0 \qquad (10.4.18a)$$

It can be shown that if the metric (10.4.1) with functions P,L,M and m satisfying (10.4.7), (10.4.8), (10.4.11), (10.4.12), (10.4.15) and (10.4.16) also satisfies condition (10.4.18a) then it corresponds to electromagnetic pure radiation i.e. to Einstein-Maxwell null field with all the energy propagated along the congruence k.

The electromagnetic field $F = F_{ab} e^a \wedge e^b$ has the form

$$F = \rho \Phi^o_2 e^2 \wedge e^3 + \overline{\rho \Phi^o_2}\, e^1 \wedge e^3 \qquad (10.4.18b)$$

It can be further shown that if the metric (10.4.1) satisfies pure radiation Einstein equations (10.4.14) (or its specializations i.e. $\Phi = 0$ or Einstein-Maxwell equations) then the Weyl scalars are given in terms of functions P, L, m, M and ρ through:

$$\psi_o = \psi_1 = 0 \tag{10.4.19}$$

$$\psi_2 = (m + iM)\rho^3 \tag{10.4.20}$$

$$\psi_3 = -P^3\rho^2\partial I + \vartheta(\rho^2) \tag{10.4.21}$$

$$\psi_4 = P^2\rho\partial_u I + \vartheta(\rho^2) \tag{10.4.22}$$

where

$$I = \bar{\partial}(\bar{\partial}\ln P - \bar{L}_u) + (\bar{\partial}\ln P - \bar{L}_u)^2 \tag{10.4.23}$$

The terms of higher order in ρ occurring in ψ_3 and ψ_4 vanish identically if $\psi_2 = 0$ or $\psi_2 = \psi_3 = 0$, respectively [Trim, Wainwright 1974].

The CR structure N associated with the space-time M of the Theorem 10.4.1 can be described as follows.

Take any hypersurface N given by the equation

$$r = r_o(u,\zeta,\bar{\zeta}) \tag{10.4.24}$$

The CR structure on N is defined by forms

$$\kappa' = e^3|_N \tag{10.4.25}$$

$$\alpha' = e^1|_N \tag{10.4.26}$$

It is obvious that forms κ and α given by

$$\kappa = du + Ld\zeta + \bar{L}d\bar{\zeta} \tag{10.4.27}$$

$$\alpha = d\zeta \tag{10.4.28}$$

are in the class (κ', α') given by (10.4.25)-(10.4.26). The condition of no twist of the congruence k given by (10.4.13) means that

$$\kappa \wedge d\kappa = 0 \tag{10.4.29}$$

This is equivalent to

$$\bar{\partial}L - \partial\bar{L} = 0 \tag{10.4.30}$$

where the operator ∂ is given by (10.4.8).

The most general coordinate transformation which leaves the metric tensor of the theorem 10.4.1 invariant is

$$\zeta' = f(\zeta) \tag{10.4.31}$$

$$u' = F(\zeta,\bar{\zeta},u) \tag{10.4.32}$$

$$r' = r\, F^{-1}_{,u} \tag{10.4.33}$$

Such transformation change L, m + iM and P into

$$L' = -\frac{\eth F}{f'} \tag{10.4.34}$$

$$(m + iM)' = \frac{m + iM}{F^{3}_{,u}} \tag{10.4.35}$$

$$P' = \frac{|f'|}{F_{,u}}\, P \tag{10.4.36}$$

The transformation (10.4.31)-(10.4.33) [Robinson at al 1969] can be used to simplify functions L or P in the integration procedure of the field equations given above. In particular one can ask when L can be transformed to

$$L' = 0 \tag{10.4.37}$$

The necessary and sufficient condition for this is the existence of a real function $F = F(u,\zeta,\bar{\zeta})$ s.t.

$$\eth F = 0 \tag{10.4.38}$$

(compare (10.4.34)). The integrability condition for this equation follows from the identity

$$(\eth\bar{\eth} - \bar{\eth}\eth)F = (\bar{\eth}L - \eth\bar{L})\partial_u F \tag{10.4.39}$$

Applying (10.4.39) to (10.4.38) we see that:

$$(\bar{\eth}L - \eth\bar{L})\partial_u F = 0 \tag{10.4.40}$$

Since $\partial_u F \neq 0$ (because otherwise transformation (10.4.31)-(10.4.33) would be degenerate) then the necessary and sufficient condition for L to be transformed to $L' = 0$ is

$$\bar{\eth}L - \eth\bar{L} = 0 \tag{10.4.41}$$

A comparison with (10.4.29)-(10.4.30) shows the following corollary

Corollary 10.4.1. [Robinson, Trautman 1962]

Space-time M of the theorem 10.4.1 admits twistfree congruence of shearfree null geodesics if and only if the function L can be transformed to zero value by coordinate transformation (10.4.31)-(10.4.33).

5. Robinson-Trautman solutions

Space-times described by Corollary 10.4.1 are called Robinson-Trautman space-times. In the process of integration of Einstein equations (10.4.7), (10.4.8), (10.4.11), (10.4.12), (10.4.15), (10.4.16) we can put

$$L = 0 \tag{10.5.1}$$

without loss of generality. Then equations (10.4.7), (10.4.11) and (10.4.12) show that

$$M = 0 \tag{10.5.2}$$

The equation (10.4.15) gives m as a function of u only, i.e.

$$m = m(u) \tag{10.5.3}$$

The remaining equation (10.4.16) has the form

$$\Delta\Delta \ln P + 12\, m(\ln P)_{u} - 4m_{,u} = \kappa_{o}\Phi^{o}_{2}\overline{\Phi^{o}_{2}} \tag{10.5.4}$$

where

$$\Delta = 2P^{2}\partial_{\zeta}\partial_{\overline{\zeta}} \tag{10.5.5}$$

This is the only equation to be solved. Moreover, if we are interested in pure radiation solutions only, it is enough to choose such m and P that left-hand-side of (10.5.4) is nonnegative.

If in addition function Φ^{o}_{2} defined by left-hand-side of (10.5.4) is such that $\frac{\Phi^{o}_{2}}{P}$ is holomorphic in $\overline{\xi}$ then this pure radiation is an Einstein-Maxwell pure radiation. If Φ^{o}_{2} vanishes we obtain vacuum solution [Robinson, Trautman 1962].

Examples

a) Historically the first example of a solution belonging to the Robinson-Trautman class was given (using another methods) by Schwarzschild [Schwarzschild 1916]. In the above formalism this solution is determined by

$$m = \text{const}, \qquad P = 1 + \frac{1}{2}\zeta\bar{\zeta} \tag{10.5.6}$$

The solution describes gravitational field outside a spherically symmetric mass. The parameter m is interpreted as a value of this mass. Explicitly the solution reads:

$$g = \frac{2r^2 d\zeta d\bar{\zeta}}{(1 + \frac{1}{2}\zeta\bar{\zeta})^2} - 2dudr - \left(1 - \frac{2m}{r}\right)du^2 \tag{10.5.7}$$

This is of Cartan-Petrov-Penrose type D [Cartan 1922].

After a coordinate transformation:

$$\xi = \sqrt{2}\, e^{i\varphi} \operatorname{ctg}\frac{\vartheta}{2}$$

$$u = t - \int \frac{dr}{1 - \frac{2m}{r}}$$

this is transformable to a standard form of the Schwarzschild metric:

$$g = r^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2) + \frac{dr^2}{1 - \frac{2m}{r}} - \left(1 - \frac{2m}{r}\right)dt^2 \tag{10.5.8}$$

The shear-free congruence of null geodesic is tangent to a vector field

$$k = \frac{\partial}{\partial r} \tag{10.5.9}$$

The forms κ and α defining CR structure on Schwarzschild space-time are given by

$$\kappa = du = dt - \frac{dr}{1 - \frac{2m}{r}} \tag{10.5.10}$$

$$\alpha = d\xi = \sqrt{2}\, e^{i\varphi}\left(i\operatorname{ctg}\frac{\vartheta}{2}\, d\varphi - \frac{d\vartheta}{2\sin^2\frac{\vartheta}{2}}\right) \tag{10.5.11}$$

Since $\kappa \wedge d\kappa = 0$ we see that the congruence is twistfree.

b) An example of Einstein-Maxwell, pure radiation solution is given by

$$m = 0$$

$$P = l(u)k(\zeta)\bar{k}(\bar{\zeta})(1 + \frac{1}{2}\zeta\bar{\zeta}) \tag{10.5.12}$$

$$\sqrt{\kappa_o}\,\frac{\Phi_2^0}{P} = \sqrt{2}\; l(u)\bar{k}(\bar{\zeta})\bar{k}'(\bar{\zeta})$$

where $l = l(u)$ is any real function, and $k = k(\zeta)$ is any holomorphic function [Bartrum 1967].

The explicit forms of Maxwell field and the metric are:

$$F = \frac{\sqrt{2}}{\sqrt{\kappa_o}}\; l\; du \wedge (kk'd\zeta + \bar{k}\bar{k}'d\bar{\zeta})$$

$$g = \frac{2r^2 d\zeta d\bar{\zeta}}{l^2 k^2 \bar{k}^2 (1 + \frac{1}{2}\zeta\bar{\zeta})^2} - 2dudr - l^2 k^2 \bar{k}^2 du^2 \qquad (10.5.13).$$

The solution (10.5.12)-(10.5.13) will be used to construct examples of <u>twisting</u> pure radiation Einsten-Maxwell fields.

6. Twisting solutions

The first examples of twisting vacuum solutions to equations from the theorem 10.4.1 were given by A. H. Taub (using different methods)[Taub 1951] and then rediscovered and generalized by Newman, Unti and Tamburino (NUT) [Newman at al 1963]. The next very important twisting solution was given by Kerr [Kerr 1963]. This solution describes gravitational field outside a rotating star. Both Taub-NUT and Kerr solutions can jointly be written in the formalism of theorem 10.2.4 [Kramer at al 1980]. The solutions read

$$g = 2\,\frac{r^2 + \Sigma^2}{P^2}\, d\zeta d\bar{\zeta} - 2(du + Ld\zeta + \bar{L}d\bar{\zeta}) \times$$
$$\times\; [dr + Wd\zeta + \bar{W}d\bar{\zeta} + H(du + Ld\zeta + \bar{L}d\bar{\zeta}] \qquad (10.6.1)$$

where

$$P = 1 + \frac{K}{2}\zeta\bar{\zeta} \qquad K = \pm 1$$
$$L = -\frac{2iM}{P^2\zeta} - \frac{i\bar{\zeta}(M + a)}{P^2}$$
$$W + \frac{Ka\zeta}{P^2} \qquad (10.6.2)$$
$$\Sigma = KM - a\,\frac{1 - \frac{k\zeta\bar{\zeta}}{2}}{1 + \frac{k\zeta\bar{\zeta}}{2}}$$

$$H = \frac{K}{2} - \frac{mr + M\Sigma}{r^2 + \Sigma^2} .$$

m and M appearing in (10.6.2) are constant parameters. (These are the same m and M as in (10.4.15)). A constant parameter a is real.

The null congruence tangent to a vector field

$$k = \frac{\partial}{\partial r} \tag{10.6.3}$$

is shear-free and geodesic.

The associated CR structure is given by forms

$$\begin{aligned} \kappa &= du + L d\zeta + \bar{L} d\bar{\zeta} \\ \alpha &= d\zeta \end{aligned} \tag{10.6.4}$$

with L as in (10.6.2).

The congruence tangent to k is twistfree if and only if

$$\Sigma = 0 \Leftrightarrow M = a = 0 \tag{10.6.5}$$

The solution (10.6.1)-(10.6.2) satisfying this condition is the Schwarzschild solution described in section X.5.

All other solutions (10.6.1)-(10.6.2) correspond to twisting congruences.

The Taub-NUT solution is (10.6.1)-(10.6.2) with

$$K = 1, \ a = 0, \ M \neq 0 \tag{10.6.6}$$

The corresponding CR structure is given by

$$\begin{aligned} \kappa &= du - \frac{2iMd\zeta}{(1 + \frac{1}{2}\zeta\bar{\zeta})\bar{\zeta}} + \frac{2iMd\bar{\zeta}}{(1 + \frac{1}{2}\zeta\bar{\zeta})\zeta} \\ \alpha &= d\zeta \end{aligned} \tag{10.6.7}$$

Performing a coordinate transformation

$$\begin{aligned} \xi' &= \frac{\sqrt{2}}{\xi} \\ u' &= -\frac{u}{2M} \end{aligned} \tag{10.6.8}$$

we see that

$$\kappa = -2M\left(du' - \frac{i\bar{\xi}'d\zeta'}{1 + \xi'\bar{\zeta}'} + \frac{i\xi'd\bar{\zeta}'}{1 + \xi\bar{\zeta}'}\right)$$

$$\alpha = -\frac{\sqrt{2}d\xi'}{\xi'^2} \tag{10.6.9}$$

i.e that the CR structure under consideration is equivalent to

$$\kappa' = du - \frac{i\bar{\xi}d\zeta}{1 + \xi\bar{\zeta}} + \frac{i\xi d\bar{\zeta}}{1 + \xi\bar{\zeta}}$$

$$\alpha' = d\zeta \tag{10.6.10}$$

Comparing this with (9.6.9) for k=0 we see, that CR structure associated with Taub-NUT geometry has a three dimensional symmetry group of Bianchi type IX which can be extended to SU(2.1) (see discussion at the end of section IX.6). Therefore this CR structure is a hyperquadric (9.7.30) [Robinson, Trautman 1985].

The Kerr solution is a solution (10.6.1)-(10.6.2) with

$$M = 0, \qquad K = 1 \tag{10.6.11}$$

It is interpreted as a gravitational field outside a rotating star with an angular momentum a and a mass m.

The Kerr solution has two Killing symmetries: one of them is timelike ensuring stationarity of the metric, the other is an axial symmetry. These symmetries are given by vector fields

$$X_1 = \frac{\partial}{\partial u}$$

$$X_2 = i\ (\xi\partial_\xi - \bar{\xi}\partial_{\bar{\xi}}) \tag{10.6.12}$$

A CR structure which corresponds to a Kerr metric can be given by forms

$$\kappa = du - \frac{i\bar{\xi}d\xi}{(1 + \frac{\xi\bar{\zeta}}{2})^2} + \frac{i\xi d\bar{\xi}}{(1 + \frac{\xi\bar{\zeta}}{2})^2}$$

$$\alpha = d\zeta \tag{10.6.13}$$

It is easy to see, that X_1 and X_2 from (10.6.12) restricted to a CR manifold $(N,[(\kappa,\alpha)])$ constitute symmetries of this CR structure [Nurowski 1987].

In order to show that these are all symmetries [Nurowski 1987] one can use the following proposition.

<u>Proposition</u> <u>10.6.1</u>

If a nondegenerate CR structure N admits three (or more) symmetries then its all Cartan's invariants $\alpha,\beta,\gamma,\vartheta,\xi,\eta$ are constants.

Proof

By their construction Cartan's invariants are constant on orbits of a symmetry group. Since 3-dimensional group acts in 3-dimensional N in a simple transitive fashion (see a discussion in IX.6 after the proof of Lemma 9.6.1) then Cartan's invariants have to be constant everywhere on N.

In section IX.7 we showed how to compute Cartan's invariants for any nondegenerate CR structure. Applying this for a CR structure of Kerr geometry (10.6.13) we can see that its Cartan's invariants are nonconstant. Therefore CR structure of Kerr geometry has two-dimensional symmetry group. This group is generated by X_1 and X_2 given by (10.6.12). We see that since

$$[X_1,X_2] = 0 \qquad (10.6.14)$$

then this structure has to be equivalent to the one of (9.5.7) with $\varepsilon=0$. To see that this is the case let us consider transformation:

$$\begin{aligned} u &\to u \\ \xi &\to \exp\,(ix-y) \\ \bar{\xi} &\to \exp(-ix-y) \end{aligned} \qquad (10.6.15)$$

After this transformation forms (10.6.13) take the form

$$\begin{aligned} \kappa &= du + \frac{4\exp(-2y)}{(2+\exp(-2y))^2}\,dx\,, \\ \alpha &\sim dx + idy \end{aligned} \qquad (10.6.16)$$

Another approach to the determining of a symmetry group of a CR structure (10.6.13) consists on explicit solving of symmetry equations

$$\mathop{\mathcal{L}}_{X}\kappa = f\kappa$$

$$\mathop{\mathcal{L}}_{X}\alpha = h\alpha + p\kappa$$

for an unknown vector field X, and functions f, h and p [Nurowski 1987]. This approach is very general and can be used to find symmetries of any CR structure. However, in some particular cases it is better to use methods similar to those presented for the Kerr metric. In particular, the fact that Killing symmetries generate symmetries of an associated CR structure is almost always (i.e. apart from some pathological cases) true (for more detailed discussion see [Lewandowski, Nurowski 1990b]).

To close this section we remark that nowadays many other twisting solutions of vacuum Einstein equations are known [see Kramer at al 1980 for an overview]. However, it is interesting that only few of them have applications in physics. Moreover CR structures corresponding to all known physically interesting solutions are equivalent either to hyperquadric or to a CR structure of the Kerr metric [Trautman 1991].

7. Example of pure radiation Einstein-Maxwell solution with a twisting congruence

Many vacuum solutions of Einstein equations with a metric tensor admitting a congruence of <u>twisting</u> shear-free and null geodesics can be found by putting

$$L_u = P_u = (m + iM)_u = 0 \qquad (10.7.1)$$

in the equations (10.4.12), (10.4.15) and (10.4.16) [Kramer at al 1980]. As it was shown by I. Robinson and J. Robinson [Robinson, Robinson 1969] the vacuum Einstein equations can be reduced to the field equation (10.5.4) for non-twisting vacuum solutions with m = 0 and $\Phi^o_2 = 0$ and for an additional equation for L. This additional equation can be solved for certain cases.

The similar method can be applied in order to find an example of a pure radiation solution of Einstein-Maxwell equations admitting a congruence of twisting, shear-free and null geodesics.

Indeed, if in equations from the Theorem 10.4.1 we put

$$L_u = P_u = (m + iM)_u = 0 \tag{10.7.2}$$

then the Einstein-Maxwell equations (10.4.7), (10.4.8), (10.4.11), (10.4.12), (10.4.15), (10.4.16) (10.4.18) reduce to

$$m + iM = A(\bar{\xi}) \tag{10.7.3}$$

$$\Delta\Delta \ln P = 4\kappa_o \Phi^o_2 \overline{\Phi^o_2} \tag{10.7.4}$$

$$\partial\left(\frac{\phi^o_2}{P}\right) = 0 \tag{10.7.5}$$

$$M = \Sigma\Delta \ln P + \frac{1}{2}\Delta\Sigma \tag{10.7.6}$$

where $A = A(\bar{\xi})$ is any holomorphic function in $\bar{\xi}$

$$\Delta = 2P^2\partial_\zeta\partial_{\bar{\zeta}} \tag{10.7.7}$$

$$\Sigma = P^2\,\frac{\bar{\partial}L - \partial\bar{L}}{2i} \tag{10.7.8}$$

$$\partial = \partial_\zeta - L\partial_u \tag{10.7.9}$$

Here (10.7.6) corresponds to (10.4.12); (10.7.3) corresponds to (10.4.15); (10.7.4) corresponds to (10.4.16); and (10.7.5) corresponds to (10.4.18). The important fact is that the equations (10.7.4) and (10.7.5) can be solved independently of (10.7.3) and (10.7.6). Moreover, these are the same equations as in the case of pure radiation Einstein-Maxwell solutions admitting nontwisting congruences of shear-free, null geodesics with m = 0 (see section X.5b). In section X.5b we gave an explicit example of solution of the equations (10.7.4) and (10.7.5). This is given by equations (10.5.12) with

$$l(u) = \text{const} \tag{10.7.10}$$

The restriction (10.7.10) is a consequence of an assumption (10.7.2). This solution of equations (10.7.4) and (10.7.5) will be used in the following. However, we can perform more general considerations.

Suppose that we have solutions of equations (10.7.4) and (10.7.5). Then, the only condition we have to impose on functions L and P is a condition of antiholomorphicity of the function $A = A(\bar{\zeta})$ appearing in (10.7.3). This is equivalent to the condition of harmonicity of the function M given by (10.7.6) i.e. to the equation

$$\Delta(\Sigma\Delta \ln P + \frac{1}{2}\Delta\Sigma) = 0 \tag{10.7.11}$$

This equation is a linear equation for Σ, hence a linear equation for L (see (10.4.7) for $L_u = 0$).

Given a solution to (10.7.4) and (10.7.5) this is the only equation to solve, in order to find a solution of pure radiation Einstein-Maxwell equations with twisting and shear-free light rays. Many solutions can be found assuming the solution (10.5.12) for (10.7.4) and (10.7.5) [Nurowski, Tafel 1991] and harmonicity of Σ i.e.

$$\Sigma = f(\xi) + \bar{f}(\bar{\xi}) \tag{10.7.12}$$

A particular one is given by taking

$$k(z) = \frac{1}{z} \text{ and } l(u) = \text{const} \tag{10.7.13}$$

in the solution (10.5.12). Then the explicit expressions for a metric tensor g and electromagnetic field F read (see Theorem 10.4.1 and (10.4.18))

$$\begin{aligned} g = 2\,\frac{r^2+\Sigma^2}{P^2}\,d\xi d\bar{\zeta} - 2(du + Ld\zeta + \bar{L}d\bar{\zeta}) \times \\ \times\,[dr + Wd\zeta + \bar{W}d\bar{\zeta} + H(du + Ld\zeta + \bar{L}d\bar{\zeta})] \end{aligned} \tag{10.7.14a}$$

$$F = (du + Ld\zeta + \bar{L}d\bar{\zeta}) \wedge \left(\frac{\bar{\Phi}^{o}_{2}}{P}\,d\xi + \frac{\Phi^{0}_{2}}{P}\,d\bar{\xi}\right) \tag{10.7.14b}$$

where:

$$P = \frac{1}{z\bar{z}}\,(z\bar{z} + b^2) \qquad \mathbb{R} \ni b = \text{const} \tag{10.7.14c}$$

$$\sqrt{\kappa_o}\,\frac{\Phi^{o}_{2}}{P} = -\,2\,\frac{b}{\bar{z}^3} \tag{10.7.14d}$$

$$\Sigma = a\left(z^2 + \bar{z}^2\right) \qquad R \ni a = \text{const} \tag{10.7.14e}$$

$$L = 2ai\left[z^2\bar{z} - \frac{b^4 z}{z\bar{z} + b^2} - 2b^2 z\ln(z\bar{z} + b^2)\right] \tag{10.7.14f}$$

$$W = 2aiz \tag{10.7.14g}$$

$$M = 2b^2 a\left(\frac{1}{z^2} + \frac{1}{\bar{z}^2}\right) \qquad m = 2b^2 ai\left(\frac{1}{\bar{z}^2} - \frac{1}{z^2}\right) \tag{10.7.14h}$$

$$H = -\frac{mr + m\Sigma}{r^2 + \Sigma^2} + \frac{b^2}{z^2\bar{z}^2} \tag{10.7.14i}$$

The solution presented above seems to be the first solution of Einstein-Maxwell equations with pure radiation electromagnetic field propagating along <u>twisting</u> congruence of shear-free and null geodesics. Further solutions can be found in [Nurowski, Tafel 1991].

8. Kerr theorem

In section X.1 we addressed few questions about relation between CR structures and gravitational fields. The formalism we described in section X.4 is enough to answer for the question whether vacuum Einstein equations imposed on a metric tensor admitting a congruence of shear-free and null geodesic imply reliability of an associated CR structure. This answer is given by the following theorem

<u>Theorem 10.8.1</u>

If a metric tensor admits shear-free congruence of null geodesics and satisfies vacuum Einstein equations then a CR structure associated with g is realizable.

Proof of this theorem can be found in [Lewandowski at al 1990]. It uses functions and equations of Theorem 10.4.1 to construct solutions to the tangential CR equation (9.2.4). Because of the very technical character of the proof we do not quote it here.

Instead, we give a more careful answer to another question posed in Section X.1, namely to the question of lifting of a given CR structure to the Minkowski space-time.

As we know from section VII.2 given a CR structure $(N,[(\kappa,\alpha)])$ such a lifting is defined by constructing a spacetime

$$M = \mathbb{R} \times N \tag{10.8.1a}$$

and a metric

$$g = 2(\mathcal{P}^2\alpha\bar{\alpha} - \kappa\varphi) \tag{10.8.1b}$$

where

$$\mathcal{P}\kappa \wedge \alpha \wedge \bar{\alpha} \wedge \varphi \neq 0, \tag{10.8.1c}$$

$\mathcal{P}$ is any real function, and φ is any real one form on M. Here we require a lifting to a Minkowski space-time i.e. we want to know when for a given $(N,[(\kappa,\alpha)])$ we can find such $\mathcal{P}$ and φ that g given by (10.8.1b) is the Minkowski metric.

Only a partial answer to this question is known.

A first remark follows from Theorem 10.8.1. This theorem shows that only realizable CR structures can be lifted to space-times satisfying vacuum Einstein equations. Therefore a CR structure which can be lifted to the Minkowski space-time has to be realizable.

Theorem 10.4.1 shows that any spacetimes being of the form (10.8.1) and satisfying Einstein equations can be transformed to a form (10.4.1)-(10.4.12). It also shows that forms κ and α defining a CR structure on N can be transformed to

$$\kappa = du + Ld\xi + \bar{L}d\bar{\zeta} \tag{10.8.2a}$$

$$\alpha = d\xi \tag{10.8.2b}$$

where the function $L = L(u,\zeta,\bar{\zeta})$ is subjected to the vacuum Einstein equations (10.4.12), (10.4.15) and (10.4.16) with $\overset{\circ}{\Phi}_2 = 0$. If we want the metric to be Minkowskian we, in addition, wish all Weyl scalars (10.4.19)-(10.4.22) to vanish. It is easy to see that all this conditions are equivalent to

$$m + iM = 0 \tag{10.8.3a}$$

$$\partial I = 0 \tag{10.8.3b}$$

$$\partial_u I = 0 \tag{10.8.3c}$$

where

$$I = \bar{\partial}(\bar{\partial}\ln P - \bar{L}_u) + (\bar{\partial}\ln P - \bar{L}_u)^2 \tag{10.8.3d}$$

These are equations for m, M, P and L. Using the identity

$$-\,\overline{\partial L_u} + \bar{L}_u^2 = -\partial_u\overline{\partial L} \tag{10.8.4}$$

we see that a particular solution of (10.8.3) is

$$m = M = 0 \tag{10.8.5a}$$

$$P = 1 \tag{10.8.5b}$$

$$\partial L = 0 \tag{10.8.5c}$$

The only restriction on the CR structure (10.8.2) is hidden in equation (10.8.5c). Therefore any CR structure $(N,[(\kappa,\alpha)])$ for which forms κ and α can be expressed as

$$\kappa = du + Ld\zeta + \bar{L}d\bar{\zeta} \tag{10.8.6a}$$

$$\alpha = d\zeta \tag{10.8.6b}$$

where L satisfies

$$\partial L = 0 \tag{10.8.6c}$$

with

$$\partial = \partial_\xi - L\partial_u \tag{10.8.6d}$$

lifts to Minkowski space-time. If (10.8.6c) is satisfied then the lifting is given by a metric

$$g = 2(r^2 + \Sigma^2)d\zeta d\bar{\zeta} - 2(du + Ld\zeta + \bar{L}d\bar{\zeta}) \times$$
$$\times\ [dr + Wd\zeta + \bar{W}d\bar{\zeta} + H(du + Ld\zeta + \bar{L}d\bar{\zeta})] \tag{10.8.7a}$$

where

$$2i\Sigma = (\bar{\partial}L - \partial\bar{L}) \tag{10.8.7b}$$

$$W = -L_u(r + i\Sigma) + i\partial\Sigma \tag{10.8.7c}$$

$$H = -\mathrm{Re}\,\partial\bar{L}_u \tag{10.8.7d}$$

The Kerr theorem [Kerr UNPUBLISHED, Tafel 1985] states that CR structures (10.8.6) are the only ones which can be lifted to the Minkowski space-time. More formally this reads:

Theorem 10.8.2 [Kerr UNPUBLISHED]

Any CR structure $(N,[(\kappa,\alpha)])$ which lifts to the Minkowski space-time is equivalent to one of CR structures given by (10.8.6).

The theorem requires a few comments.

Suppose that we have a CR structure $(N,[(\kappa,\alpha)])$ defined by forms

$$\kappa = du + Ld\zeta + Ld\bar{\zeta} \tag{10.8.8a}$$

$$\alpha = d\zeta \tag{10.8.8b}$$

s.t.

$$\partial L \neq 0 \tag{10.8.8c}$$

Does this mean that such a CR structure can not be lifted to Minkowski space-time? The negative answer to this question is obtained if we notice, that Kerr Theorem 10.8.2 says, that CR structure (10.8.8) has only to be equivalent to (10.8.6) in order to be liftable to Minkowski space-time. Therefore (10.8.8) will not be liftable to Minkowski space-time if we show that neither possible transformations of forms κ and α

$$\kappa \longrightarrow f\kappa$$

$$\alpha \longrightarrow h\alpha + p\kappa$$

nor possible coordinate transformations transform (10.8.8) to the desired form (10.8.6). This is usually almost impossible to be shown in practice. Therefore Kerr theorem does not constitute an effective algorithm for checking whether a given CR structure lifts to Minkowski space-time [Lewandowski 1986].

Such an effective algorithm does not seem to have been established so far. We believe that in order to obtain it, it is necessary to apply Cartan invariants of CR structures and find integrability conditions for Einstein equations in terms of Cartan invariants. This was the main motivation of introducing nonstandard approach to the problem described in sections X.1 - X.3. However, we have not been able to realize this programme, so far. It seems to be quite difficult one.

A geometrical interpretation of Kerr theorem is worth quoting [Penrose, Rindler 1986, Tafel 1985]. This can be achieved by reinterpretation of the equation (10.8.6c) which is equivalent to

$$dL \wedge (du + Ld\zeta + \bar{L}d\bar{\zeta}) \wedge d\bar{\xi} = 0 \qquad (10.8.9a)$$

or

$$dL \wedge d(u + L\xi) \wedge d\bar{\xi} = 0 \qquad (10.8.9b)$$

This last equation shows that if L is analytic then L satisfies an equation

$$F(L, u + L\xi, \bar{\xi}) = 0, \qquad (10.8.10)$$

where F is an arbitrary function analytic in the three complex variables $L, u + L\xi, \bar{\xi}$.

This allows for a reformulation of Kerr theorem 10.8.2 to the following form.

<u>Theorem 10.8.3</u>

Any analytic CR structure which lifts to Minkowski space-time is either degenerate or equivalent to a CR structure with forms

$$\begin{aligned} \text{[illegible]} &= du + Ld\zeta + Ld\bar{\zeta} \\ \alpha &= d\zeta \end{aligned} \qquad (10.8.11a)$$

where L satisfies an equation

$$F(L, u + L\xi, \bar{\xi}) = 0 \qquad (10.8.11b)$$

for any function F analytic in the variables L, $u + L\zeta$, $\bar{\xi}$.

Equation (10.8.11b) can also be written as

$$u + L\zeta = f(L, \bar{\zeta}) \qquad (10.8.12a)$$

where f is an analytic function of the variable L, $\bar{\zeta}$. This shows that CR structure (10.8.11) is realizable as a hypersurface

$$\mathrm{Im}[L\xi - f(L, \bar{\zeta})] = 0 \qquad (10.8.12b)$$

in $\mathbb{C}^2$ with coordinates $(\xi, \bar{\zeta}, L, \bar{L})$. The twistor theory enables to show that any such hypersurface can be obtained by the following construction. Let us take a projective twistor space $\mathbb{PT}$ described in section VIII.3 and its five dimensional real submanifold $\mathbb{PT}_o$ given by equation

$$\Sigma(Z) = 0 \tag{10.8.13}$$

(see (8.3.12)). The twistor form of the Kerr theorem says that any analytic shear-free congruence of null geodesics in compactified Minkowski space corresponds to the intersection N of $\mathbb{PT}_o$ with a complex surface of equation

$$h(z_1, z_2, z_3, z_4) = 0,$$

where h is holomorphic and homogeneous function of its arguments [Penrose and Rindler 1986], which are homogeneous coordinates in $\mathbb{PT}$. The submanifold N of $\mathbb{PT}_o$ is in one-to-one correspondence with the hypersurface given by (10.8.12b). This is a 3-dimensional CR structure. Penrose points out that the freedom in defining N involves one complex holomorphic function of two variables whereas a general, realizable CR structure may be defined by an analytic function of three variables. Therefore <u>most</u> of realizable CR structures do not lift to Minkowski spacetime.

As we have already said no effective procedure is known to distinguish this majority of realizable CR structures which do not lift to Minkowski space-time.

XI. CONCLUSIONS AND OPEN PROBLEMS.

Gravitational fields admitting a congruence of shear-free and null geodesics (CSNG) have been studied for more than thirty years. In this work we tried to give an overview of known results in this matter up to the present date.

We payed a particular attention on still not very well known relations between space-times with CSNGs and 3-dimensional CR-structures. In the case of space-times admitting twisting CSNGs we proposed a new approach. It uses the fact that CR structures associated with such space-times are nondegenerate and their invariants distinguish a certain tetrad. One of possible applications of this fact is a method of comparing whether two metrics admitting twisting CSNGs are equivalent.

The distinguished by CR geometry tetrad was used to express the Weyl tensor of an associated metric as being invariant under the gauge transformations of the CR structure. In the future we are going to extend these considerations. We want to express Einstein (and Einstein-Maxwell) equations for metrics with twisting CSNGs to be invariant under gauge transformations of their CR-structures. The solution to this problem seems to be easy to achieve. [Lewandowski, Nurowski 1990b, Lewandowski at al 1991a]. Difficulties arise when integrability conditions for Einstein equations in terms of Cartan invariants of the CR-structures only have to be found. Even in the simplest case of finding such conditions for the metric to be Minkowskian we have not been able to give a satisfactory solution, yet. Such solution would constitute an invariant characterization of the Kerr theorem and would definitely solve the problem of lifting of a given CR-structure to the Minkowski space.

An application of the distinguished by the CR geometry tetrad to construct new Einstein and Einstein-Maxwell fields admitting twisting CSNGs seems to be promising. In particular this new method can be used to study vacuum Einstein equations for the twisting type N. The class of known solutions in this case consists only of one-parameter Hauser's family of metrics [Hauser 1974]. Only type D vacuum metrics with twisting CSNGs are known completely [Kinnersley 1969]. Most of known vacuums of other types either possess high symmetries (if the dimension of a (conformal)group of motion is grater than 2 all solutions are known [Kerr, Debney 1970, Lewandowski, Nurowski 1990b]) or are generated from non-twisting ones by a procedure given in [Robinson, Robinson 1969; see also a new application of this method in Section 10.7 of this work]. We believe that in order to extend known vacuum classes of solutions the standard approach of Section 10.4 is too weak. Therefore it is worth checking whether nonstandard approach in which the metric has the form (10.1.8) gives some new solutions. In [Lewandowski, Nurowski 1990b, Lewandowski at al 1991a, 1991b] we applied this method obtaining many new pure radiation solutions. Application to the vacuum case has not been performed so far.

XII. APPENDIX. ALGORITHM FOR COMPUTING CARTAN INVARIANTS.

Here we present an algorithm for computing Cartan invariants for a nondegenerate CR structure $(N, [(\kappa, \alpha)])$ in the most important case when the CR structure admits at least one complex solution to the tangential CR equation (9.2.4) [Cartan 1932]. Suppose that a complex function z is a such solution i. e.:

$$dz\wedge\kappa\wedge\alpha = 0 \tag{12.1.1}$$

In this case we can always achieve

$$\alpha' = dz \tag{12.1.2a}$$

$$d\kappa' = i\alpha'\wedge\bar{\alpha}' + b\,\kappa'\wedge\alpha' + \bar{b}\kappa'\wedge\bar{\alpha}' \tag{12.1.2b}$$

by means of transformations (9.7.4). Here (12.1.2b) can be considered a definition of a complex function b.

After this preparations the algorithm is as follows:

1. Given any function f let us define symbols f_1, $f_{\bar{1}}$, f_0 by

$$df = f_0\kappa' + f_1\alpha' + f_{\bar{1}}\bar{\alpha}' \tag{12.1.3}$$

2. Let real functions c and g be defined by

$$c = b_{\bar{1}} \tag{12.1.4a}$$

$$g = c_{1\bar{1}} - \frac{i}{2}c_0 \tag{12.1.4b}$$

and complex functions l, r be defined by

$$l = c_1 - bc - 2ib_0 \tag{12.1.4c}$$

$$r = \frac{1}{6}(\bar{l}_1 - 2\bar{b}\bar{l}) \tag{12.1.4d}$$

3. Let us compute forms

$$\omega_2 = -b\alpha' + \frac{i}{4}c\kappa'$$

$$\omega_3 = \frac{i}{4}c\alpha' + \frac{1}{6}\bar{l}\kappa' \tag{12.1.5}$$

$$\omega_4 = -i\frac{l+4ib_0}{12}\alpha' + i\frac{\bar{l}-4i\bar{b}_0}{12}\bar{\alpha}' + [\frac{11}{48}c^2 + \frac{1}{6}(b\bar{b}c+bl+bl-g)]\kappa'$$

4. Then forms defined by (9.7.7) read

$$\Omega = f\bar{f}\kappa'$$

$$\Omega_1 = f(\kappa' + h\alpha')$$

$$\Omega_2 = d\log f + \omega_2 - 2i\bar{h}\alpha' - ih\bar{\alpha}' + (\rho - \frac{3}{2}ih\bar{h})\kappa'$$

$$\Omega_3 = \frac{1}{\bar{f}}\ [dh + \omega_3 + h\bar{\omega}_2 + (\rho+\tfrac{i}{2}h\bar{h})\alpha' + ih^2\bar{\alpha}' + h(\rho+\tfrac{i}{2}h\bar{h})\kappa'\,]$$

$$\Omega_4 = \frac{1}{f\bar{f}}\ [\ d\rho + \tfrac{i}{2}(hd\bar{h}-\bar{h}dh) + \omega_4 - i\bar{h}\omega_3 + ih\bar{\omega}_3 + (\rho+\tfrac{i}{2}h\bar{h})\omega_2 +$$

$$+ (\rho-\tfrac{i}{2}h\bar{h})\bar{\omega}_2 - i\bar{h}(\rho+\tfrac{i}{2}h\bar{h})\alpha' + ih(\rho-\tfrac{i}{2}h\bar{h})\bar{\alpha}' + (\rho^2+\tfrac{1}{4}h^2\bar{h}^2)\kappa'\,]$$

and function $\mathcal{R}$ of (9.7.7d) is given by

$$\mathcal{R} = \frac{r}{f\bar{f}^3}$$

5. If

$$r = 0$$

then forms Ω and Ω_1 satisfying (9.7.21) are given by (9.7.30).

6. If

$$r \neq 0$$

then Cartan invariant forms Ω, Ω_1, Ω_2, Ω_3, Ω_4 (satisfying (9.7.20), (9.7.21)) are given by the forms of Point 4. of this algorithm with functions f, h, ρ defined as follows:

$$f = \varepsilon \sqrt{\frac{\bar{r}}{\sqrt[4]{r\bar{r}}}}\ , \quad \varepsilon = \pm 1$$

$$h = \frac{i}{4}\ (\log r\bar{r})_{\bar{1}} - i\bar{b}$$

$$\rho = -\ \frac{1}{8}\ (\log r\bar{r})_0$$

where r and b are defined by (12.1.4).

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