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THESIS

INVARIANT SETS AND PERIODIC SOLUTIONS FOR DIFFERENTIAL SYSTEMS

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INTRODUCTION

In this work , we study invariance and existence of periodic solutions for the first order differential system $\dot{x} = f(t,x)$, with f continuous .

By a flow-invariant (or positively invariant) set , for such a system ,we mean a set for which every solution of the system , with initial value on the set , remains on it for all future time .

Sometimes , when there is not uniqueness for Cauchy problems of the system , we can guarantee the existence of one solution , for each initial value on the set, remainig on it ,for all the future time . This is the concept of weak flow-invariance. Nagumo's classical theorem (1942) is , essentially , a caracterization of such a set . Nevertheless,for the applications, these concepts are not enough . For instance , if our system means a dynamic of populations for which one wants persistence of the species . So, we'll also give other concepts for invariance .

Another application of flow-invariance is on the research of periodic solutions for $\dot{x} = f(t,x)$ when f depends in a periodic manner on the time t . If one finds a flow-invariant set , with the fixed point property (for instance if it is homeomorphic to a retract of a closed ball in $\mathbb{R}^m$) , one can expect to find a periodic solution on this set.

From the point of view of the applications , it's of particular interest the research of positive periodic solutions . This can be done requiring hypothesis giving positive invariance for the positive cone of $\mathbb{R}^m$, $\mathbb{R}^m_+ = \{(x_1,..,x_m)\in\mathbb{R}^m \mid x_i \geqslant 0$ for $i=1,..,m\}$, and applying a fixed point theorem on the cone.

This thesis is divided intofive sections . We begin section 1 with Nagumo's theorem, for relatively closed sets , and some original geometric interpretations based on several kinds of tangent cones (the contingent cone, the Dubovickii-Miljutin cone, the Clarke's cone and the Bony's cone) . In section 2 , we study the concepts of weak flow-invariance and of flow-invariance , with several abstract theorems , giving special attention to the flow-invariance for open sets. In section 3, we analyse concepts of strong flow-invariance and persistence, giving several examples and quoting recent results in [18] , which solves a question raised by Gard in [24] , about the possibility to obtain an anlogue of Bony's theorem , for compact strongly flow-invariant sets. In the last section , we use the theory of flow-invariant sets and combine it with the fixed-point index theory to get existence theorems,for periodic solutions in the positive cone of $\mathbb{R}^m$, which generalize some results contained in [23] and [47] .

For all this thesis we'll consider several concepts of invariance and their relationships, for a differential equation of the type

(0.1) $\dot{x}=f(t,x)$

where $f:J\times\Omega\to \mathbb{R}^m$ is a continuous function, Ω is an open set in $\mathbb{R}^m$ and J is a non degenerate real interval, with a=inf J and b=sup J (eventually not real numbers).

By a solution of (0.1) in a non degenerate subinterval I, of J, we mean a continuously differentiable function $x:I\to\mathbb{R}^m$ such that $\dot{x}(t)=f(t,x(t))$ for every $t\in I$.

A subset M of $\mathbb{R}$ is said to be flow-invariant (or positively invariant) for (0.1), if each solution of (0.1), with initial value in M, remains in M in the future. In an analogous way, one can define negative invariance, if we deal with the past time, instead of the future time. We note that results for negative invariance, respect to equation (0.1), can be obtained by studying positive invariance respect to equation $\dot{x}=-f(-t,x)$, because if x(t) is a solution for one of the two equations, x(-t) is a solution for the other one.

By an invariant set we mean a set which is both positively and negatively invariant.

So, from now on, we just study flow-invariance.

For a subset $M\subset\Omega$, we denote $\mathrm{int}_\Omega M$, $\mathrm{cl}_\Omega M$ and $\mathrm{fr}_\Omega M$, respectively the interior, the the closure and the boundary of M respect to Ω. If $\Omega=\mathbb{R}^m$, we'll simply write int M, cl M and fr M.

1. Weak flow-invariance and tangent cones.

Let $\mathbb{R}_-=\{x\in\mathbb{R} \mid x\leq 0\}$. For each $(t_0,x_0)\in\mathbb{R}\times\mathbb{R}_-$, there exists a solution of $\dot{x}=\sqrt{|x|}$, with initial condition $x(t_0)=x_0$, remaining in $\mathbb{R}_-$ for all future time. In fact, let y(.) be a solution of $\dot{x}=\sqrt{|x|}$, with $y(t_0)=x_0$, and for which there exists $t^*\geq t_0$ such that $y(t^*)=0$. Taking $t_1=\min\{t\in[t_0,t^*] \mid y(t)=0\}$,

$$w(t)=\begin{cases} y(t) & ,t_0\leq t\leq t_1 \\ 0 & ,t>t_1\end{cases}$$

is a solution of $\dot{x}=\sqrt{|x|}$ in $[t_0,+\infty)$, with $w(t_0)=x_0$ and $w(t)\in\mathbb{R}_-$, for every $t\geq t_0$.

However, for some initial conditions there can exist solutions of $\dot{x}=\sqrt{|x|}$ that don't remain in $\mathbb{R}_-$. For instance, if $t_0=0=x_0$ and $x(t)=t^2/4$, for $t\geq 0$.

This example shows a weak concept of flow-invariance.

Definition 1.1 $M\subset\Omega$ is said to be a weakly flow-invariant (or positively weakly invariant) set for (0.1) if for each $(t_o,x_o)\in J\times M$, there is a solution $x(.)$, of (0.1), with initial condition $x(t_o)=x_o$ and such that $x(t)\in M$, for every t in the right maximal interval of existence of $x(.)$.

From the definition, it is clear that it is sufficient to check the weak flow-invariance for $t_o<b=\sup J$.

With the preceding example, we showed that $\mathbb{R}_-$ is weakly flow-invariant for $\dot{x}=\sqrt{|x|}$.

For some properties of weakly flow-invariant sets we recall the following theorem in [29]:

Theorem 1.2 Suppose J is open and let f_n converge uniformly to f, on compact subsets of $J\times\Omega$, where $f_n:J\times\Omega\to\mathbb{R}^m$ are continuous functions. Let, for each $n\in\mathbb{N}$, $x_n(.)$ be a solution of the Cauchy problem $\dot{x}=f_n(t,x)$ with $x(t_n)=x_{no}$, where $(t_{no},x_{no})\in J\times\Omega$ converges to $(t_o,x_o)\in J\times\Omega$. Then, there exists $x(.)$, a solution of (0.1), with $x(t_o)=x_o$. Moreover, being $(\omega_{n-},\omega_{n+})$ the maximal interval of existence of $x_n(.)$, for each $n\in\mathbb{N}$, and (ω_-,ω_+) the maximal interval of existence of $x(.)$, there exists a sequence of positive integers $n_1<n_2<\dots$ with the property that, if $\omega_-<t^1<t^2<\omega_+$, then $\omega_{n_k-}<t^1<t^2<\omega_{n_k+}$, for k large, and $x_{n_k}(.)$ converges to $x(.)$ uniformly on $[t^1,t^2]$. In particular, $\limsup\omega_{n-}\leqslant\omega_-<\omega_+\leqslant\liminf\omega_{n+}$.

The following theorem shows that, if a set is weakly flow-invariant for approximating equations, then its closure is weakly flow-invariant.

Theorem 1.3 Let f_n converge uniformly to f on compact subsets of $J\times\Omega$, where $f_n:J\times\Omega\to\mathbb{R}^m$ are continuous functions. Then, if for each $n\in\mathbb{N}$, M is weakly flow-invariant for $\dot{x}=f_n(t,x)$, $cl_\Omega M$ is weakly flow-invariant.

Proof: Assume first that $J=[a,b)$. In order to apply theorem 1.2, define $\tilde{f}:(-\infty,b)\times\Omega\to\mathbb{R}^m$ by

$$\tilde{f}(t,x)=\begin{cases} f(t,x), & a\leqslant t<b,\ x\in\Omega\\ f(a,x), & t<a,\ x\in\Omega\end{cases}$$

and for each $n\in\mathbb{N}$, $\tilde{f}_n:(-\infty,b)\times\Omega\to\mathbb{R}^m$ by

$$\tilde{f}_n(t,x)=\begin{cases} f_n(t,x), & a\leqslant t<b,\ x\in\Omega\\ f_n(a,x), & t<a,\ x\in\Omega\end{cases}$$

. As f_n converges uniformly to f on

compact subsets of $J\times\Omega$, $\tilde{f}_n$ converges uniformly to $\tilde{f}$ on compact subsets of $(-\infty,b)\times\Omega$.

Take $(t_0,x_0)\in J\times cl_\Omega M$. Let $x_{n0}\in M$ be such that $x_{n0}\to x_0$. By the weak flow-invariance of M, respect to $\dot{x}=f_n(t,x)$, for each $n\in\mathbb{N}$ there exists $x_n(.)$, solution of $\dot{x}=f_n(t,x)$ with $x_n(t_0)=x_{n0}$ and such that $x_n(t)\in M$, for all $t\geq t_0$ in its right maximal interval of existence. As $x_n(.)$ is also a solution of $\dot{x}=\tilde{f}_n(t,x)$, let $\tilde{x}_n(.)$ be an extension of $x_n(.)$, as a solution of $\dot{x}=\tilde{f}_n(t,x)$, over a maximal interval of existence, $(\omega_{n-},\omega_{n+})$. We observe that $[t_0,\omega_{n+})\subset J$, so that $[t_0,\omega_{n+})$ is the right maximal interval of existence of $x_n(.)$.

Applying Theorem 1.2 to $\tilde{f}_n$, $\tilde{f}$ and $(-\infty,b)$, there exists $\tilde{x}(.)$ a solution of $\dot{x}=\tilde{f}(t,x)$ with $\tilde{x}(t_0)=x_0$. Moreover, being (ω_-,ω_+) its maximal interval of existence, put $x(.)=\tilde{x}\,|\,[t_0,\omega_+)$. $x(.)$ is a solution of $\dot{x}=f(t,x)$ with $x(t_0)=x_0$ and right maximal interval of existence $[t_0,\omega_+)$. Being $t\in[t_0,\omega_+)$, choose $t^1\in(\omega_-,t_0)$. Then, by theorem 1.2, there exists $\tilde{x}_{n_k}$ subsequence of $\tilde{x}_n$, converging uniformly to $\tilde{x}$ in $[t^1,t]$ so that $M\ni x_{n_k}(t)=\tilde{x}_{n_k}(t)\to\tilde{x}(t)=x(t)$, and $x(t)\,(cl\,M)\cap\Omega=cl_\Omega M$.

So, $x(.)$ is the desired solution.

If $J=(a,b]$ we extend f to $(a,+\infty)$ by $\tilde{f}(t,x):=f(b,x)$ for $t>b$ and $x\in\Omega$; if $J=[a,b]$, we take $\tilde{f}(t,x):=f(a,x)$ for $t<a$, and $\tilde{f}(t,x)=f(b,x)$ for $t>b$, proceeding in a similar way. If $J=(a,b)$, we just apply theorem 1.2.

□

As a consequence, we have that the closure of a weakly flow-invariant set is also a weakly flow-invariant set. That is:

<u>Corollary 1.4</u> If $M\subset\mathbb{R}^m$ is weakly flow-invariant for (0.1), so it is $cl_\Omega M$.

Proof: Take, in theorem 1.3, $f_n=f$, for all $n\in\mathbb{N}$.

□

For relatively closed sets, the next theorem gives a caracterization of weak flow-invariance.

<u>Theorem 1.5</u> Let $M\subset\Omega$ be a closed set relative to Ω. Then, M is a weakly flow-invariant set for (0.1) if, and only if,

(1.1) for each $(t_0,x_0)\in(J\setminus\{b\})\times M$, there exist $T>0$ and $x(.)$, a solution of (0.1) in $[t_0,t_0+T]$, with $x(t_0)=x_0$ and such that $x(t)\in M$, for $t\in[t_0,t_0+T]$.

Proof: The necessary condition is obvious. For the sufficient one, fix (t_0, x_0) in $J\setminus\{b\})\times M$, and define

$\mathcal{F}:=\{(x;I) \mid$ I is a non degenerate real subinterval of J, with $t_0 \in I$, and x is a solution of (0.1) in I, with $x(t_0)=x_0$ and such that $x(t)\in M$, $\forall t\in I\}$

By condition (1.1), $\mathcal{F}\neq\emptyset$. Define, on $\mathcal{F}$, "$\leqslant$" by

$$(x;I)\leqslant(x^*;I^*) \Leftrightarrow I\subset I^* \text{ and } x^*_{\mid I}=x$$

"$\leqslant$" is a reflexive and transitive relation.

Let $\emptyset\neq C=\{(x_\alpha;I_\alpha) \mid \alpha\in A\}$ be a chain in $\mathcal{F}$ (that is, if $(x_\alpha;I_\alpha),(x_\beta;I_\beta)\in C$, then $(x_\alpha;I_\alpha)\leqslant(x_\beta;I_\beta)$ or $(x_\beta;I_\beta)\leqslant(x_\alpha;I_\alpha)$). Take $I=\bigcup_{\alpha\in A} I_\alpha$ and $x:I\to\mathbb{R}^m$ defined by $x(t):=x_\alpha(t)$, if $t\in I$. As C is a chain, x is a map and $(x;I)$ is really a supremum of C in $\mathcal{F}$.

Applying Zorn's lemma, let $(\tilde{x};\tilde{I})$ be a maximal element in $\mathcal{F}$. Call $\tilde{T}=\sup\tilde{I}$. If $\tilde{T}\in\tilde{I}$, $\tilde{x}(\tilde{T})\in M$. By condition (1.1) there would be $T>0$ and $z(.)$, solution of (0.1) in $[\tilde{T},\tilde{T}+T]$ such that $z(\tilde{T})=\tilde{x}(\tilde{T})$ and $z(t)\in M$, $\forall t\in[\tilde{T},\tilde{T}+T]$. Define $w:\tilde{I}\cup[\tilde{T},\tilde{T}+T]\to\mathbb{R}^m$ by

$$w(t)=\begin{cases}\tilde{x}(t) & , t\in\tilde{I}\\ z(t) & , t\in[\tilde{T},\tilde{T}+T]\end{cases}$$

. Then, $(w;\tilde{I}\cup[\tilde{T},\tilde{T}+T])\in\mathcal{F}$ and $(w;\tilde{I}\cup[\tilde{T},\tilde{T}+T])\geqslant(\tilde{x};\tilde{I})$, which is a contradiction with the fact that $(\tilde{x};\tilde{I})$ is a maximal element in $\mathcal{F}$. So, $\tilde{T}\notin I$.

If $\tilde{I}\cap[t_0,+\infty)$ would not be the rigth maximal interval of existence of $\tilde{x}$, $\tilde{T}$ would belong to this right interval. So, $\tilde{x}(\tilde{T})\in\Omega$ and, as M is a closed set in Ω and $\tilde{T}=\sup\tilde{I}$, $\tilde{x}(\tilde{T})=\lim\tilde{x}(t)\in(\text{cl } M)\cap\Omega=M$. But, because $(\tilde{x};\tilde{I})$ is a maximal element in $\mathcal{F}$, $\tilde{T}\in\tilde{I}$, which is impossible, as observed before.

So, $\tilde{x}$ is the desired solution.

□

The next result is essentially the classical theorem of Nagumo. See also [51] and [16] . An analogous result, for multifunctions, can be found in [5].

By a locally closed set in $\mathbb{R}^m$ we mean a subset F of $\mathbb{R}^m$, such that for each $x_0\in F$ there exists $r>0$ for which $F\cap B[x_0,r]$ is a closed set in $\mathbb{R}^m$, being $B[x_0,r]$ the closed ball in $\mathbb{R}^m$, with center x_0 and radius. We denote by $B(x_0,r)$ the corresponding open ball.

Observe that a closed set relative to Ω, $F\subset\Omega$, is a locally closed set in $\mathbb{R}^m$. In fact, if $x_0\in F\subset\Omega$, as Ω is an open set, choose $r>0$ small enough so that $B[x_0,r]\subset\Omega$. Then, $F\cap B[x_0,r]=(\mathrm{cl}\ F)\cap\Omega\cap B[x_0,r]$, which is a closed set in $\mathbb{R}^m$.

For $\emptyset\neq A\subset\mathbb{R}^m$, we denote $d(x,A)=\inf_{x\in A}|x-y|$, where $|\cdot|$ is the euclidean norm of $\mathbb{R}^m$.

Theorem 1.6 Let $\emptyset\neq M\subset\Omega$ be a locally closed set in $\mathbb{R}^m$. Then, condition (1.1) is equivalent to

(1.2) for each $t_0\in J\setminus\{b\}$ and $x_0\in M\cap \mathrm{fr}_\Omega M$, $\liminf_{h\to0^+} \dfrac{d(x_0+hf(t_0,x_0),M)}{h}=0$.

Remark: We observe that condition (1.2) is also equivalent to condition

(1.3) for each $t_0\in J\setminus\{b\}$ and $x_0\in M$, $\liminf_{h\to0^+} \dfrac{d(x_0+hf(t_0,x_0),M)}{h}=0$.

In fact, if $x_0\in M\setminus \mathrm{fr}_\Omega M$, for all h small enought, $x_0+hf(t_0,x_0)\in M$.

Proof: It's enought to prove the result in the autonomous case, that is, when f doesn't depend on t. In fact, $M^*=J\times M$ is locally closed in $\mathbb{R}^{m+1}$. If J is open, take $\Omega^*=J\times\Omega$ and, if $J=[a,b)$ (resp. $(a,b]$ or $[a,b]$), take $\Omega^*:=(-\infty,b)\times\Omega$ (resp. $(a,+\infty)\times\Omega$ or $\mathbb{R}\times\Omega$). Put $g:\Omega^*\to\mathbb{R}^{m+1}$ with $g(y):=(1,\tilde f(y))$, where $\tilde f:\Omega^*\to\mathbb{R}^m$ is defined as in theorem 1.3. Then apply the theorem to the autonomous case, as for $(t_0,x_0)\in M^*$ and h small enough $d((t_0,x_0)+hg(t_0,x_0),M^*)=d(x_0+hf(t_0,x_0),M)$.

So, consider the autonomous case and take $f(x)=f(t,x)$ for all $x\in\Omega$ and $t\in J$.

For (1.1) implies (1.2), we observe that, if $t_0\in J\setminus\{b\}$ and $x_0\in M\cap \mathrm{fr}_\Omega M$, being $x(\cdot)$ a solution of $\dot x=f(x)$, with $x(t_0)=x_0$ and $x(t)\in M$, for all t in some interval $[t_0,t_0+T]$, with $T>0$, we have $x(t_0+h)=x_0+h\dot x(t_0)+o(h)=x_0+hf(x_0)+o(h)$, for $h\in[0,T]$. So that $\dfrac{d(x_0+hf(x_0),M)}{h}\leqslant\dfrac{|x_0+hf(x_0)-x(t_0+h)|}{h}=\dfrac{|o(h)|}{h}$. And we have (1.2).

Let us see that (1.2) implies (1.1).

Fix $(t_0,x_0)\in J\times M$. As M is a locally closed set, let $r>0$ be such that $k_0:=M\cap B[x_0,r]$ is a closed set in $\mathbb{R}^m$.

For each $k\in\mathbb{N}$ and $y\in M$, define $N(y):=\left\{x\in\mathbb{R}^m \mid d(x+h_yf(y),M)<\dfrac{h_y}{4k}\right\}$ where, by condition (1.2), h_y is chosen in such a way that $0 < h_y < 1/k$ and $\frac{d(y+h_y f(y),M)}{h_y} < \frac{1}{4k}$.

As for every $y \in M$, $y \in N(y)$, which is an open set in $\mathbb{R}^m$, there exists $\eta_y \in (0,1/k)$ small enough, such that $B(y,\eta_y) \subset N(y)$ and, if $z,w \in k_0$ are such that $|z-w| \leqslant \eta_y$, $|f(z)-f(w)| \leqslant \frac{1}{2k}$, attending to the uniform continuity of f in the compact set k_0.

As $k_0 \subset M$ and $k_0 \subset \bigcup_{y \in K_0} B(y,\eta_y)$, from the compactness of k_0, there exist $y_1,\dots,y_q \in K_0$ such that $K_0 \subset \bigcup_{j=1}^{q} B(y_j,\eta_{y_j})$.

Let us call $h_j = h_{y_j}$ and $\eta_j = \eta_{y_j}$, for $j \in \{1,\dots,q\}$.

So, if $x \in k_0$, there exists $j \in \{1,\dots,q\}$ such that $x \in B(y_j,\eta_j) \subset N(y_j)$ and then $d(x+h_j f(y_j),M) < \frac{h_j}{4k}$. Therefore, there is $x_j \in M$ such that $|x+h_j f(y_j)-x_j| \leqslant$

$d(x+h_j f(y_j),M) + \frac{h_j}{4k} < \frac{h_j}{2k}$. Take $u_j := \frac{x_j - x}{h_j}$. Then, $|f(y_j)-u_j| = |f(y_j) - \frac{x_j - x}{h_j}| \leqslant$

$\leqslant \frac{1}{2k}$. But, as $x \in B(y_j,\eta_j)$, by construction of η_j, $|f(y_j)-f(x)| \leqslant \frac{1}{2k}$. Therefore, $u_j \in B[f(x),1/k]$. We have also $x+h_j u_j = x_j \in M$.

Let $h_0(k) := \min_{1 \leqslant j \leqslant q} h_j > 0$.

We proved that, for every $x \in k_0$, there exist $h \in [h_0(k),1/k]$ and $u \in B[f(x),1/k]$ such that $x+hu \in M$.

Put $M := \max_{x \in K_0} |f(x)|$ and $T := \frac{r}{1+M}$.

As $x_{0,k} := x_0 \in K_0$, there exist $h_{0,k} \in [h_0(k),1/k]$ and $u_{0,k} \in B[f(x_0),1/k]$ such that $x_{1,k} := x_{0,k} + h_{0,k} u_{0,k} \in M$.

If $h_{0,k} \leqslant T$, $|x_{1,k}-x_0| \leqslant h_{0,k}(|u_{0,k}-f(x_0)| + |f(x_0)|) \leqslant h_{0,k}(M+1/k) \leqslant$ $\leqslant T(1+M) = r$. And then $x_{1,k} \in M \cap B[x_0,r] = K_0$.

In this case $(x_{1,k} \in K_0)$, there exist $h_{1,k} \in [h_0(k),1/k]$ and $u_{1,k} \in B[f(x_{1,k}),1/k]$ such that $x_{2,k} := x_{1,k} + h_{1,k} u_{1,k} \in M$.

If $h_{0,k} + h_{1,k} \leqslant T$, we have $x_{2,k} \in K_0$. And so on.

We observe that, for $1/k \leqslant T$, as $h_{n,k} \geqslant h_0(k) > 0$, there exists $m(k) \in \mathbb{N}_0$ such that

$h_{o,k}+h_{1,k}+\ldots+h_{m(k),k}\leqslant T<h_{o,k}+h_{1,k}+\ldots+h_{m(k)+1,k}$.

Then, $x_{m(k)+1,k}\in K_o$ and $x_{m(k)+2,k}\in M$.

Put $\tau_k^o:=t_o$ and $\tau_k^p:=t_o+h_{o,k}+\ldots+h_{p-1,k}$, for $1\leqslant p\leqslant m(k)+2$.

Define for $k\in\mathbb{N}$, with $1/k\leqslant T$,and $t\in[t_o,t_o+T]$,

$$x_k(t):=x_{p-1,k}+(t-\tau_k^{p-1})\,u_{p-1,k}\quad ,\text{ if } t\in[\tau_k^{p-1},\tau_k^p] .$$

x_k is a continuous function on $[t_o,t_o+T]$.

One can easly check that $|x_k(s)-x_k(t)|\leqslant(s-t)(1+M)$,for every $s,t\in[t_o,t_o+T]$ such that $s>t$. So that, $\{x_k\}_k$ is an equicontinuous set in $C([t_o,t_o+T],\mathbb{R}^m)$, the space of continuous functions from $[t_o,t_o+T]$ into $\mathbb{R}^m$. If $t\in[\tau_k^{p-1},\tau_k^p]$, $|x_k(t)|\leqslant|x_{p-1,k}|+h_{p-1,k}|u_{p-1,k}|\leqslant M_1+1/k(1+M)\leqslant 1+M+M_1$, where $M_1:=$ $=\max\limits_{x\in K_o}|x|$. So that, $\{x_k\}_k$ is also equibounded .

By Ascoli-Arzelā, if necessary passing to a subsequence, we may assume that x_k converges uniformly to some continuous function, $x(.)$, in $[t_o,t_o+T]$.

Let us see that $x(t)\in M$, for every $t\in[t_o,t_o+T]$, and that $x(.)$ is a solution of $\dot{x}=f(x)$, satisfying $x(t_o)=x_o$.

Fix $t\in[t_o,t_o+T]$. For each $k\in\mathbb{N}$,with $1/k\leqslant T$, $[t_o,t_o+T]\subset\bigcup\limits_{p=1}^{m(k)+2}[\tau_k^{p-1},\tau_k^p]$

So, exists $p_k\in\mathbb{N}$, with $1\leqslant p_k\leqslant m(k)+2$, such that $t\in[\tau_k^{p_k-1},\tau_k^{p_k}]$, and $|x_{p_k,k}-x(t)|\leqslant$

$$\leqslant|x_{p_k,k}-x_k(t)|+|x_k(t)-x(t)|\leqslant(\tau_k^{p_k}-t)\,|u_{p_k-1,k}|+|x_k-x|_\infty\leqslant$$

$\leqslant 1/k(1+M)+|x_k-x|_\infty$, where $|\,.\,|_\infty$ denotes the uniform convergence norm on $[t_o,t_o+T$

So, $x_{p_k,k}\to x(t)$. As $x_{p_k,k}\in K_o$, for $k\in\mathbb{N}$ such that $1/k\leqslant T$, and K_o is closed in $\mathbb{R}^m$, $x(t)\in K_o\subset M$.

To see that $x(.)$ is a solution of $\dot{x}=f(x)$, define, for every $k\in\mathbb{N}$, with $1/k\leqslant T$, and $t\in[t_o,t_o+T]$, $g^*(t):=f(x(t))$ and $g_k(t):=f(x_{p-1,k})$ if $t\in[\tau_k^{p-1},\tau_k^p]$.

g_k converges uniformly to g^* on $[t_0, t_0+T]$ and $\int_{t_0}^{\tau_k^{p_k}} g_k \to \int_{t_0}^{t} g^*$, where $t \in [\tau_k^{p_k-1}, \tau_k^{p_k}] \cap [t_0, t_0+T]$ for every $k \in \mathbb{N}$, with $1/k \leq T$.

As $x_{p_k,k} \to x(t)$, as $t \in [\tau_k^{p_k-1}, \tau_k^{p_k}] \cap [t_0, t_0+T]$, for every $k \in \mathbb{N}$ with $1/k \leq T$, proving that $|x_{p_k,k} - x_0 - \int_{t_0}^{\tau_k^{p_k}} g_k| \leq \frac{T+1}{k}$ and passing to the limit as $k \to +\infty$, we'll have $x(t) - x_0 - \int_{t_0}^{t} g^* = 0$. That is $x(t) = x_0 + \int_{t_0}^{t} f(x(\tau))\, d\tau$, for all $t \in [t_0, t_0+T]$. So $x(.)$ is a solution of $\dot{x} = f(x)$ with $x(t_0) = x_0$.

□

Therefore, theorems 1.5 and 1.6, imply the following criterion on weak flow-invariance, for relatively closed sets ([40] and [51]).

Theorem 1.7 Let $\phi \neq M \subset \Omega$ be a closed set relative to Ω. Then, M is weakly flow-invariant for (0.1) if, and only if,

$$\text{for each } t_0 \in J \setminus \{b\} \text{ and } x_0 \in \mathrm{fr}_\Omega M, \quad \liminf_{h \to 0^+} \frac{d(x_0 + h\, f(t_0, x_0), M)}{h} = 0 .$$

We give a geometrical meaning to the tangential condition (1.2), using the Bouligand's contingent cone, which derives from a concept of a contingent set introduced by Bouligand in the 1930's.

In order to give equivalent formulations to theorem 1.7, we introduce other tangent cones, like those of Bony ([40]), Dubovickii-Miljutin ([26]) and Clarke ([15], [46]). For a discussion of the main properties of all these cones, see, for instance, [43].

Let F be a subset of $\mathbb{R}^m$ and $x \in \mathbb{R}^m$.

We define $T(F,x)$, the contingent cone to F at x (or tangent cone, in the sense of Bouligand to F at x), by

$$T(F,x) := \left\{ v \in \mathbb{R}^m \mid \liminf_{h \to 0^+} \frac{d(x+h\,v, F)}{h} = 0 \right\} .$$

Condition (1.2) can , then, be reformulated as

(1.2′) for each $t_0 \in J\setminus\{b\}$ and $x_0 \in M \cap fr_\Omega M$, $f(t_0,x_0) \in T(M,x_0)$.

Remark that, if $x \notin cl\, F$, $T(F,x)=\phi$. And, if $x \in int\, F$, $T(F,x)=\mathbb{R}^m$.

$T_D(F,x)$, the <u>tangent cone</u>, in the sense of <u>Dubovickii-Miljutin</u>, to F at x, is defined by

$$T_D(F,x)=\left\{ v\in\mathbb{R}^m \mid \lim_{h\to 0^+} \frac{d(x+hv,F)}{h} = 0 \right\} .$$

By $T_C(F,x)$, the <u>tangent cone</u>, in the sense <u>of Clarke,</u> to F at x, we mean

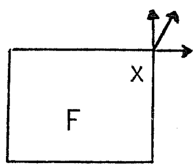
$$T_C(F,x)=\left\{ v\in\mathbb{R}^m \mid \limsup_{\substack{h\to 0^+\\ y\to x}} \frac{d(y+hv,F)-d(y,F)}{h} \leqslant 0 \right\} .$$

To define the Bony's tangent cone, we introduce the concept of outer normal in the sense of Bony ([7] , [43] , [45]).

<u>Definition 1.8</u> If $x \in cl\, F$, $\nu \in \mathbb{R}^m$ is said to be an <u>outer normal</u>,in the sense <u>of Bony</u> , to F at x , if $\nu \neq 0$ and $F \cap B(x+\nu, |\nu|) = \phi$.

Observe that, if ν is an outer normal, so is $\lambda\nu$ for $\lambda \in (0,1]$. And, if F is a convex set, the same is true, for all $\lambda > 0$.

<u>Remark</u>: we point out that outer normals in the sense of Bony at a point x are not necessarily unique. And, in the case of a convex set F, outward normals are Bony's outer normals (see page 23).



For $x \in cl\, F$, we define $T_B(F,x)$, the tangent cone, in the sense of Bony, to F at ,by

$$T_B(F,x) = \left\{ v \in \mathbb{R}^m \mid (v \mid \nu_x) \leqslant 0 \right\}, \text{ for every } \nu_x \text{ outer normal in the sense of Bony, to F at } x \},$$

here $(.\mid .)$ means the canonical inner product of $\mathbb{R}^m$.

We have the following relations among the cones:

roposition 1.9 Let $x \in cl\, F$. Then, $T_C(F,x) \subset T_D(F,x) \subset T(F,x) \subset T_B(F,x)$.

roof: Let $v \in T_C(F,x)$. $\limsup_{h \to 0^+} \frac{d(x+hv,F)}{h} = \limsup_{h \to 0^+} \frac{d(x+hv,F)-d(x,F)}{h} \leqslant$

$\limsup_{\substack{h \to 0^+ \\ y \to x}} \frac{d(y+hv,F)-d(y,F)}{h} \leqslant 0$. And, as $0 \leqslant \liminf_{h \to 0^+} \frac{d(x+hv,F)}{h}$,

$\lim_{h \to 0^+} \frac{d(x+hv,F)}{h} = 0$, that is , $v \in T_D(F,x)$.

By the definitions, $T_D(F,x) \subset T(F,x)$.

Let $v \in T(F,x)$. If ν is a Bony's outer normal to F at x and $h \geqslant 0$, $|\nu| \leqslant$ $d(x+\nu,F)$ $|\nu - hv| + d(x+hv,F)$. Therefore, $|\nu| \leqslant (|\nu-hv| + d(x+hv,F))^2 =$ $|\nu|^2 + h^2 |v|^2 - 2h(v|\nu) + d(x+hv,F)^2 + 2|\nu-hv|\, d(x+hv,F)$. So ,

$\leqslant h\, |v|^2 - 2(v|\nu)_+ \frac{d(x+hv,F)}{h}\, d(x+hv,F) + 2\, |\nu-hv| \frac{d(x+hv,F)}{h}$. Passing to the

im inf as $h \to 0^+$, $(v|\nu) \leqslant 0$. As ν was arbitrary, $v \in T_B(F,x)$.

Then , $T(F,x) \subset T_B(F, x)$.

□

emarks: We observe that all these cones are closed cones (that is, they are closed , n $\mathbb{R}^m$, and for every $\lambda \geqslant 0$ and v in the tangent cone, λv belongs to this cone).

$T_C(F,x)$ and $T_B(F,x)$ are convex cones, but $T(F,x)$ and $T_D(F,x)$ are not ne- essarily convex, as shown in the example below. The same example shows that all these ones can be different among them. However, if F is a closed and convex set, they coin- ide ([46]).

There are some caracterizations of these cones. For instance, for the contingent one (see [5] , [46]),

$$T(F,x)=\left\{v\in\mathbb{R}^m \mid \exists h_n\to 0^+ \ \exists (y_n)_n\subset F : \frac{y_n-x}{h_n}\to v\right\}=$$

$$=\bigcap_{\varepsilon>0}\ \bigcap_{\lambda>0}\ \bigcup_{0<h<\lambda}\left(\frac{F-x}{h}+B(0,\varepsilon)\right),$$

and, for the Clarke's cone (see [46] , [43])

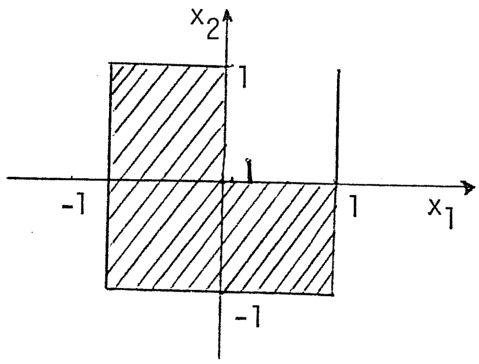
$$T_C(F,x)=\left\{v\in\mathbb{R}^m \mid \forall h_n\to 0^+ \ \forall (x_n)_n\subset \text{cl}\, F,\ x_n\to x \ \ \exists (y_n)_n\subset F : \frac{y_n-x_n}{h_n}\to v\right\}=$$

$$=\bigcap_{\varepsilon>0}\ \bigcup_{\substack{\lambda>0\\ \delta>0}}\ \bigcap_{\substack{y\in F\cap B(x,\delta)\\ 0<h<\lambda}}\left(\frac{F-y}{h}+B(0,\varepsilon)\right).$$

Example: Take in $\mathbb{R}^2$, $F=([-1,0]\times[-1,1])\cup([0,1]\times[-1,0])\cup$

$$\cup\left(\bigcup_{n\in\mathbb{N}_0}\left\{\frac{1}{2^{2n}}\right\}\times\left[0,\frac{1}{2^{2n}}\right]\right)$$

and $x=(0,0)$.



We have: $T_B(F,0)=\mathbb{R}^2$

$T(F,0)=\{(x_1,x_2)\in\mathbb{R}^2 \mid x_1\leqslant 0 \text{ or } x_2\leqslant|x_1|\}$

$T_D(F,0)=\{(x_1,x_2)\in\mathbb{R}^2 \mid x_1\leqslant 0 \text{ or } x_2\leqslant 0\}$

$T_C(F,0)=\{0\}\times(-\infty,0]$.

The next theorem allows the equivalent formulations for Nagumo's theorem, with the different cones.

We give the following lemma ([16]), for the sake of completness:

Lemma 1.10 Let F be a locally closed set in $\mathbb{R}^m$ and $g:F\to\mathbb{R}^m$ a continuous function. If, for each $x\in F$, $g(x)\in T_B(F,x)$, then for each $K\subset F$ compact in $\mathbb{R}^m$,

$$\lim_{h\to 0^+} \left(\sup_{x\in K} \frac{d(x+hg(x),F)}{h} \right) = 0$$

Proof: Fix $K\subset F$ compact. As F is locally closed, there exists a compact set $K_0\supset K$ such that $K_0\cap F$ is a compact set. Therefore, there is $h_0>0$ such that, for $0\leqslant h\leqslant h_0$ and $x\in K$, $d(x+hg(x),F) = d(x+hg(x),K_0\cap F)$.

For $r>0$, put $\gamma(r) = \sup\{ \, |g(z)-g(y)| \;\; | \;\; z,y\in K_0\cap F$ and $|z-y|\leqslant r\}$.

Set $L = \max_{x\in K_0\cap F} |g(x)|$.

Fix $x\in K$ and, for $0\leqslant \tau\leqslant h_0$, let $y_\tau := x+\tau g(x)$. As $K_0\cap F$ is closed in $\mathbb{R}^m$, there exists $x\;\in K_0\cap F$ such that $|y_\tau - x_\tau| = d(y_\tau, F\cap K_0) = d(y_\tau, F)$.

Define f , in $[0,h_0]$, by $f(\tau) = d(y_\tau, F)^2$. Then, for $0\leqslant s<\tau\leqslant h_0$,

$$f(\tau)-f(s)\leqslant |y_\tau - x_\tau|^2 - |y_s-x_s|^2 \quad |y_\tau - x_s|^2 - |y_s-x_s|^2 =$$

$$= |y_\tau - y_s|^2 + 2\,(y_\tau - y_s)\,|\,(y_s-x_s) =$$

$$=(\tau-s)^2\,|g(x)|^2 + 2\,(\tau-s)(g(x_s)\,|\,(y_s-x_s)) + 2(\tau-s)((g(x)-g(x_s))\,|\,(y_s-x_s))\leqslant$$

$$\leqslant (\tau-s)^2 L^2 + 2(\tau-s)(g(x_s)\,|\,(y_s-x_s)) + 2(\tau-s)\,\gamma(\,|x-x_s|\,)\sqrt{f(s)}\;.$$

As y_s-x_s is a Bony's outer normal to F at x_s and $g(x_s)\in T_B(F,x_s)$,

$(x_s)|\,(y_s-x_s))\leqslant 0$.

$|x-x_s| \leqslant |x-y_s| + |y_s-x_s| \leqslant 2\,|x-y_s| = 2s\,|g(x)| \leqslant 2sL$. So, as γ is increasing, $\gamma(\,|x-x_s|\,)\leqslant \gamma(2sL)$. Therefore,

.4) $$\frac{f(\tau)-f(s)}{\tau-s}\leqslant (\tau-s)\,L^2 + 2\;\gamma(2sL)\,\sqrt{f(s)}\;.$$

Let $h\in(0,h_0]$. As $\sqrt{f(\tau)} = d(x+\tau g(x),F)$, $\sqrt{f}$ is lipschitz on $[0,h]$. Therefore, $\sqrt{f}$ and f are absolutely continuous on $[0,h]$. So, there exist $f'(\tau)$ and $(\sqrt{f})'(\tau)$ almost everywhere in $[0,h]$.

If $0<\tau<s\leqslant h$, by (1.4), as γ is increasing, we have

$$\frac{f(\tau)-f(s)}{\tau-s} \leqslant (\tau-s)L^2 + 2\gamma(2hL)\sqrt{f(s)} .$$

Passing to the limit as $s\uparrow\tau$, where $f'(\tau)$ exists, it satisfies

$f'(\tau)\leqslant 2\gamma(2hL)\sqrt{f(\tau)}$. Therefore, for $\tau\in[0,h)$, where it exists, $(\sqrt{f})'(\tau)\leqslant\gamma(2hL)$. As $\sqrt{f}$ is absolutely continuous on $[0,h]$,it follows that $\sqrt{f(h)}\leqslant\gamma(2hL)\,h$. So, for every $h\in(0,h_0]$, $\dfrac{d(x+hg(x),F)}{h}\leqslant\gamma(2hL)$.

As x was arbitrarly chosen on K, we have also, for every $h\in(0,h_0]$, $0\leqslant\sup\limits_{x\in K}\dfrac{d(x+hg(x),F)}{h}\leqslant\gamma(2hL)$. But, by the uniform continuity of g on the compact set $k_0\cap F$, $\lim\limits_{r\to 0}\gamma(r)=0$. So, $\lim\limits_{h\to 0^+}\left(\sup\limits_{x\in K}\dfrac{d(x+hg(x),F)}{h}\right)=0$.

□

Theorem 1.11 Let F be a locally closed set in $\mathbb{R}^m$ and $g:F\to\mathbb{R}^m$ be a continuous function. Then,the following conditions are equivalent:

(a) $\forall x\in F\cap \mathrm{fr}\,F$ $\quad g(x)\in T_C(F,x)$

(b) $\forall x\in F\cap \mathrm{fr}\,F$ $\quad g(x)\in T_D(F,x)$

(c) $\forall x\in F\cap \mathrm{fr}\,F$ $\quad g(x)\in T(F,x)$

(d) $\forall x\in F\cap \mathrm{fr}\,F$ $\quad g(x)\in T_B(F,x)$.

Remarks: For $x\in\mathrm{int}\,F$, $\mathbb{R}^m = T_C(F,x) = T_D(F,x) = T(F,x) = T_B(F,x)$ .So, we could also substitute $F\cap\mathrm{fr}\,F$ by F .

In [43] , theorem 3.9 , Penot proved that the same theorem holds for $f:F\to E$ continuous,where F is a subset of a strongly smooth Banach space E,and F is locally closed and proximinal at each point $x\in F$. We note that Penot assumptions on F and E are always satisfied, whenever $E=\mathbb{R}^m$ (with the euclidean norm) and $F\subset\Omega$, with Ω an open set in $\mathbb{R}^m$, is a closed set relative to Ω .

roof: By proposition 1.9, (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) . So, we have only to see (d) $\Rightarrow$ (a).

Fix $x_0 \in F \cap \mathrm{fr}\, F$ and $\varepsilon > 0$. As F is locally closed in $\mathbb{R}^m$ and g is continuous at $_0$, there exists $\delta > 0$ such that $F \cap B[x_0, \delta]$ is closed and, for $z \in F \cap B[x_0, \delta]$, $g(z)-g(x_0) | < \varepsilon/2$.

For $y \in B[x_0, \delta/2]$, let $z_y \in F$ be such that $|y-z_y| = d(y, F \cap B[x_0, \delta]) = d(y,F)$.

As (d) implies that $\forall x \in F \quad g(x) \in T(F,x)$, by lemma 1.10 , applied to the compact et $F \cap B[x_0, \delta]$, there exists $h_0 > 0$ such that, for $h \in (0,h_0)$,

$$\sup_{\in F \cap B[x_0, \delta]} \frac{d(x+hg(x),F)}{h} < \frac{\varepsilon}{2} .$$

So, if $y \in B[x_0, \delta/2]$ and $h \in (0,h_0)$,

$$\frac{(y+hg(x_0),F)-d(y,F)}{h} \leqslant \frac{d(z_y+hg(x_0),F) + |y-z_y| - d(y,F)}{h} = \frac{d(z_y+hg(x_0),F)}{h} \leqslant$$

$$\leqslant \frac{d(z_y+hg(z_y),F)}{h} + |g(z_y)-g(x_0)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon ,$$

s $|z_y - x_0| \leqslant |z_y - y| + |y-x_0| \leqslant 2|y-x_0| \leqslant \delta$.

As $\varepsilon > 0$ was arbitrary, $\limsup_{\substack{h\to 0^+ \\ y \to x_0}} \frac{d(y+hg(x_0),F) - d(y,F)}{h} \leqslant 0$. That is, $(x_0) \in T_C(F,x)$.

□

Combining theorem 1.7 with theorem 1.11 ,we get:

orollary 1.12 Let $\emptyset \neq M \subset \Omega$ be a closed set relative to Ω .Then, the following con-itions are equivalent:

(1) M is weakly flow-invariant for (0.1) .

(2) $\forall t \in J \quad \forall x \in \mathrm{fr}_\Omega M \qquad f(t,x) \in T_C(F,x)$

(3) $\forall t \in J \quad \forall x \in \mathrm{fr}_\Omega M \qquad f(t,x) \in T_D(F,x)$

(4) $\forall t \in J \quad \forall x \in \mathrm{fr}_\Omega M \qquad f(t,x) \in T(F,x)$

(5) $\forall t \in J \quad \forall x \in \mathrm{fr}_\Omega M \qquad f(t,x) \in T_B(F,x)$.

Proof: Observe that as $M \subset \Omega$ is closed relative to Ω , M is locally closed in $\mathbb{R}^m$ and $\mathrm{fr}_\Omega M = M \cap \mathrm{fr}\, M$. Apply theorem 1.7 for $(1) \Leftrightarrow (4)$ and, for each $t \in J$ apply theorem 1.11 to $g = f(t,\cdot)_{|M}$.

□

This corollary also shows that Crandall (1972), Hartman (1972), Martin (1973) and Yorke (1967) rediscovered Nagumo's theorem (1942). Also Brézis (1970), Bony (1969) and Redheffer (1972) formulated it , in the particular case of f lipschitz.

2. weak flow-invariance and flow-invariance.

We have already seen in section 1, that $\mathbb{R}_-$ is not flow-invariant for $\dot{x} = \sqrt{|x|}$, but it is weakly flow-invariant . However, for special differential equations, both concepts coincide. In fact,

Definition 2.1 $M \subset \mathbb{R}^m$ is said to be a flow-invariant (or positively invariant) set for (0.1) if, for each $(t_0, x_0) \in J \times M$, each solution $x(.)$ of (0.1), with initial condition $x(t_0) = x_0$, is such that $x(t) \in M$, for every t in the right maximal interval of existence of $x(.)$.

Obviously, if f satisfies uniqueness for solutions of Cauchy problems, for (0.1) a set is flow-invariant if and only if it is weakly flow-invariant.

From the above definition, some easy properties can be derived at once: the intersection and the union of flow-invariant sets is flow-invarant, a set is flow-invariant if and only if its complementary is negatively invariant (see [6] , [49] and [51]).

Remark that the closure of a flow-invariant set need not be flow-invariant as well as its interior .

In the previous example, $\mathbb{R}_- \setminus \{0\}$ the interior of $\mathbb{R}_-$ is not flow-invariant for $\dot{x} = \sqrt{|x|}$ In fact, for the initial condition $x(0) = -1/4$,

$$x(t) = \begin{cases} \dfrac{-(t-1)^2}{4} & , \quad t < 1 \\ 0 & , \quad t \geqslant 1 \end{cases}$$ is a solution of $\dot{x} = \sqrt{|x|}$.

We are particulary interested in results of flow-invariance for open sets. Indeed, the concepts of strong flow-invariance, persistence and uniform persistence, that we shall present in the next sections, require that the interior of a set will be flow-invariant.

If $V: \Omega \to \mathbb{R}$ is a scalar function, we denote by ∇V the gradient of V (wherever it is defined) and, for $c \in \mathbb{R}$ $[V \leqslant c] = \{x \in \Omega \mid V(x) \leqslant c\}$. The sets $[V=c]$ and $[V \geqslant c]$ are defined in an analogous way.

We observe that, for V continuous, $fr_\Omega [V \leqslant c] \subset [V=c]$, but the inclusion cannot be reversed. For instance, if $V: \mathbb{R} \to \mathbb{R}$ is defined by $V(x) = x^2(x+1)(x-1)$, $[V \leqslant 0] = [-1,1]$ and $fr_\Omega [V \leqslant 0] = \{-1,1\} \neq \{-1,0,1\} = [V=0]$.

It is easly seen that $fr_\Omega [V \leqslant c] = fr_\Omega [V < c] = [V=c]$, if V is of class C^1 and $\nabla V(x) \neq 0$ on $[V=c]$ (that is, if c is a regular value for V).

For open sets, we have the following theorem:

Theorem 2.2 Let G be an open set in $\mathbb{R}^m$ such that, for each $u \in fr_\Omega G$, there exists a continuous function $V_u: \Omega \to \mathbb{R}$ such that $V_u(u) = 0$ and $G \subset \bigcap_{x \in fr_\Omega G} [V_x < 0]$.

Assume

(2.1) for each $u \in fr_\Omega G$ and $t \in J$, there exists $\varepsilon > 0$ such that V_u is C^1 on $G \cap B(u, \varepsilon)$ and $(f(s,y) \mid \nabla V_u(y)) \leqslant 0$, for every $s \in J$ with $t - \varepsilon \leqslant s < t$ and $y \in G \cap B(u, \varepsilon)$.

Then, G is a flow-invariant set and $cl_\Omega G$ is a weakly flow-invariant set, for (0.1).

Proof: By contradiction, suppose that there is $(t_0, x_0) \in (J \setminus \{b\}) \times G \subset J \times \Omega$ such that there exist a solution $x(.)$ of (0.1), with $x(t_0) = x_0$, and $b > \tilde{t} > t_0$ verifying $x(\tilde{t}) \notin G$.

Let $t_1 = \min \{t \in [t_0, \tilde{t}] \mid x(t) \notin G\}$. Then, $t_1 > t_0$, $x(t_1) \notin G$ and $x(t) \in G$, for every $t \in [t_0, t_1)$. Put $\bar{u} = x(t_1)$. So, $\bar{u} \in fr_\Omega G$. Defining $v(t) = V_{\bar{u}}(x(t))$, we have $v(t_1) = V_{\bar{u}}(x(t_1)) = V_{\bar{u}}(\bar{u}) = 0$. Moreover, v is continuous on $[t_0, t_1]$ with $v(t) = V_{\bar{u}}(x(t)) < 0$, for $t \in [t_0, t_1)$, as $x(t) \in G \subset \bigcap_{u \in fr_\Omega G} [V_u < 0]$.

By condition (2.1) , let $\varepsilon > 0$ be such that $(f(s,y) \mid \nabla V_u(y)) \leqslant 0$,for every $s \in J$, with $t_1 - \varepsilon < s < t_1$, and $y \in G \cap B(\bar{u}, \varepsilon)$. As $x(.)$ is continuous at t_1 ,there exists $\delta > 0$ such that $\delta < \min(t_1 - t_0, \varepsilon)$ and, for $t \in (t_1 - \delta, t_1)$, $x(t) \in B(\bar{u}, \varepsilon)$.

Then, $v(.)$ is continuously differentiable on $(t_1 - \delta, t_1) \subset [t_0, t_1)$ and $v'(t) = (\nabla V_{\bar{u}}(x(t)) \mid \dot{x}(t)) = (f(t,x(t)) \mid \nabla V_{\bar{u}}(x(t))) \leqslant 0$, for $t_1 - \delta < t < t_1$. Since $v(t_1 - \delta) = V_{\bar{u}}(x(t_1 - \delta)) < 0 = v(t_1)$, we have a contradiction .

So, G is a flow-invariant set for (0.1).

To conclude, as G is, in particular, a weakly flow-invariant set for (0.1), apply corollary 1.4 .

□

In particular, for sets of the type $[V<c]$, we have:

<u>Corollary 2.3</u> Let $V:\Omega \to \mathbb{R}$ be a C^1 function and let $c \in \mathbb{R}$ .Assume

(2.2) for each $x \in [V=c]$, there exists an $\varepsilon > 0$ such that $(f(t,y) \mid \nabla V(y)) \leqslant 0$, for every $t \in J$ and $y \in [V<c] \cap B(x, \varepsilon)$.

Then, $[V<c]$ is flow-invariant for (0.1) .

Proof: Take $W := -c+V$ and let $G := [V<c] = [W<0]$. Putting $V_u := W$,for all $u \in fr_\Omega\, G = fr_\Omega\, [V<c] \subseteq [V=c]$, condition (2.2) implies (2.1). Apply, then, theorem 2.2 .

□

Observe that $cl_\Omega\, G \subset [V \leqslant c]$, not necessarily equal.

We point out that with the above result we cannot guarantee the flow-invariance of $cl_\Omega\, [V<c]$. So, not even that of $cl_\Omega\, G$, in theorem 2.2 .

In fact, $\mathbb{R}_-$ is not flow-invariant for $\dot{x} = \sqrt{|x|}\,\text{sign}\, x$, where

$$\text{sign}\, x = \begin{cases} -1 & ,x<0 \\ 0 & ,x=0 \\ 1 & ,x>0 \end{cases} \quad , \text{as} \quad x(t) = \begin{cases} 0 & ,t \leqslant 0 \\ \frac{t^2}{4} & ,t>0 \end{cases} \quad \text{is a solution with } x(0)=0 \in \mathbb{R}_- .$$

Nevertheless , $f(t,x) = \sqrt{|x|}\,\text{sign}\, x$ satisfies (2.2), for $\Omega = \mathbb{R} = J$, $V(x) = x$ and $c=0$.

This example, also shows that the closure of a flow-invariant set is not necessarily flow-invariant, as, by corollary 2.3, $\mathbb{R}\setminus\{0\} = [V<0]$ is flow-invariant for $\dot{x} = \sqrt{|x|}\,\text{sign}\, x$.

A slight modification of theorem 2.2, reads as follows:

Theorem 2.4 Let G be an open set in $\mathbb{R}^m$ such that, for each $u\in fr_\Omega G$, there exists a continuous function $V_u:\Omega \to \mathbb{R}$ such that $V_u(u) = 0$ and $G \subset \bigcap_{x\in fr_\Omega G} [V_x \leqslant 0]$. Assume

(2.3) for each $u\in fr_\Omega G$ and $t\in J$, there exists $\varepsilon>0$ such that V_u is C^1 in $G\cap B(u,\varepsilon)$ and $(f(s,y)\,|\,\nabla V_u(y))<0$, for every $s\in J$, with $t-\varepsilon<s<t$, and $y\in G\cap B(u,\varepsilon)$.

Then, G is flow-invariant and $cl_\Omega G$ is weakly flow-invariant, for (0.1).

Proof: Similar to the one of theorem 2.2. Using the same notations, we point out that $v(t) = V_{\bar{u}}(x(t))\leqslant 0$, as $x(t)\in G \subset \bigcap_{u\in fr_\Omega G}[V_u\leqslant 0]$, for $t\in [t_0,t_1)$. So, that the contradiction is obtained with condition (2.3), as $0\leqslant \dfrac{v(t_1)-v(t_1-\delta)}{\delta} = v'(s) =$

$=(\nabla V(x(s))\,|\,f(s,x(s)))$, for some $s\in J$ with $t_1-\varepsilon<t_1-\delta<s<t_1$, being $x(s)\in G\cap B(\bar{u},\varepsilon)$.

□

An obvious consequence is now the following ([18] lemma 1, [38] theorem 7.4):

Corollary 2.5 Let G be an open set in $\mathbb{R}^m$ such that, for each $u\in fr_\Omega G$, there exists $V_u:\Omega \to \mathbb{R}$ a C^1 function such that $V_u(u) = 0$ and $G\subset \bigcap_{x\in fr_\Omega G} [V_x \leqslant 0]$. Assume

(2.4) for each $u\in fr_\Omega G$ and $t\in J$, $(f(t,u)\,|\,\nabla V_u(u))<0$.

Then, G is flow-invariant and $cl_\Omega G$ is weakly flow-invariant, for (0.1).

We remark that the conclusion for $cl_\Omega G$ cannot be improved. In fact, let $f(t,x,y) = (1,-x^2+y+2|y|^{1/2})$, for $t\in\mathbb{R} = J$ and $(x,y)\in\mathbb{R}^2 = \Omega$. Set

$$G:=\{(x,y)\in\mathbb{R}^2 \mid y<0<x \text{ or } (-1/4<x\leqslant 0 \text{ and } y<x)\}.$$

G is an open set . Moreover, defining for each $u = (u_1,u_2) \in fr\ G$, $V_u : \mathbb{R}^2 \to \mathbb{R}$ by

$$V_u(x,y) := \begin{cases} y-(x-u_1)^2 & , u_1 > 0 \ , u_2 = 0 \\ y-x & , -1/4 < u_1 \leqslant 0 \ , u_2 = u_1 \\ -x-1/4 & , u_1 = -1/4 \ , u_2 \leqslant -1/4 \ . \end{cases}$$

$G \subset \bigcap_{u \in fr\ G} [\ V_u \leqslant 0\ ]$ and $V_u(u) = 0$. And condition (2.4) is satisfied .

However, cl G is not flow-invariant for $\begin{cases} \dot{x} = 1 \\ \dot{y} = -x^2+y+2|y|^{1/2} \end{cases}$, as $z(t) = (t,t^2)$, for $t \geqslant 0$, is a solution of this system and $z(0) = (0,0) \in cl\ G$.

While in theorems 2.2—2.4 ,conditions (2.1)—(2.3) must be satisfied for y in the open set G , the next proposition is for y outside cl G , and concerns the closed sets of the type $M = [\ V \leqslant c\ ]$.

<u>Proposition 2.6</u> Let $V : \Omega \to \mathbb{R}$ be a continuous function and $c \in \mathbb{R}$. Set $M = [\ V \leqslant c]$. Assume

(2.5) for each $u \in fr_\Omega\ M$ and $t \in J$ there exists $\varepsilon > 0$ such that V is C^1 in $[V > c] \cap B(u,\varepsilon\ )$ and $(f(s,y) \mid \nabla V(y)) \leqslant 0$,for every $s \in J$ with $t < s < t+\varepsilon$ and $y \in [V > c] \cap B(u,\varepsilon\ )$.

Then, M is flow-invariant for (0.1) .

We omit the proof , since it is similar to the previous ones .

We can't say about the flow-invariance of $[V < c\ ]$. Not even if we have the strict inequality in (2.5) .

In fact , $\mathbb{R} \setminus \{0\}$ is not flow invariant for $\dot{x} = -\sqrt{|x|}\,\text{sign}\ x$, as

$$x(t) = \begin{cases} \dfrac{-(t-2)^2}{4} & ; t \leqslant 2 \\ 0 & , t > 2 \end{cases}$$

is a solution with $x(0) = -1$. And taking $V(x) = x$, $\mathbb{R} \setminus \{0\} = [V < 0\ ]$ and (2.5) is satisfied for $c=0$ and $\Omega = \mathbb{R} = J$, with the strict inequality .

Proposition 2.6 gives the following consequence:

Corollary 2.7 Let $V:\Omega \to \mathbb{R}$ be of C^1 and let $c\in\mathbb{R}$ be a regular value of V. Assume

(2.6) $\quad (f(t,u) \mid \nabla V(u)) \leqslant 0$, for every $t\in J$ and $u\in [V=c]$.

Then $[V\leqslant c]$ is weakly flow-invariant for (0.1) .

Proof: Define on $J\times\Omega$, $f_n(t,x):= f(t,x)-\frac{1}{n}\nabla V(x)$. Then , f_n converges uniformly to f on compact subsets of $J\times\Omega$.

As , for each $n\in\mathbb{N}$, $(f_n(t,u) \mid \nabla V(u))<0$, for $t\in J$ and $u\in fr_\Omega [V\leqslant c] \subseteq [V=c]$, it is enough to apply proposition 2.6 and theorem 1.3 .

This corollary could also be obtained using Nagumo's theorem . See also [3], theorem 16.9 , for a different proof .

We point out that corollary 2.7 cannot be extended , as well as theorem 2.4 and proposition 2.6 ,as shows the last example , to closed sets .

We present,now, some consequences of the main theorems of this section, using in a more specific way , geometrical conditions on the boundary of the considered set .

Dealing with outer normals, in the sense of Bony, corollary 2.5 has the interesting consequence:

Corollary 2.8 Let $G\subset\Omega$ be an open set . Assume

(2.7) for each $u\in fr_\Omega G$, there exists an outer normal η_u ,in the sense of Bony , to F at u for which $(f(t,u) \mid \eta_u)<0$, for every $t\in J$.

Then, G is flow-invariant, and $cl_\Omega G$ is weakly flow-invariant, for (0.1) .

Proof: For each $u\in fr_\Omega G$, define $V_u:\Omega\to\mathbb{R}$ by $V_u(x):=\frac{1}{2}(|\eta_u|^2-|u+\eta_u-x|^2)$, as in [41] . V_u is a C^1 function with $V_u(u) = 0$ and $\nabla V_u(x) = u+\eta_u-x$.

As for each $u\in fr_\Omega G$, η_u is an outer normal to G at u , we have $G\cap B(u+\eta_u, |\eta_u|) =\emptyset$. So, $G\subset \{x\in\Omega \mid |u+\eta_u-x| \geqslant |\eta_u|\} = [V_u\leqslant 0]$, for each

$u \in fr_\Omega G$. Then, as condition (2.7) implies (2.4) because $\nabla V_u(u) = n_u$, applying corollary 2.5 , we have the desired conclusion .

□

A significant application of Bony outer normals, is in the case of convex sets. More precisely, let $F \subset \Omega$ be a convex set . If F is a closed set, relative to Ω , then, for each $u \in fr_\Omega F$ there is a non zero vector n_u (outward normal) such that

(A) $\quad F \subset \{ x \in \Omega \mid ((x-u) \mid n_u) \leqslant 0 \}$

(use separation theorems in exercise 2 of section 3.6 in [50]). Moreover, each n_u verifying (A) is also a Bony outer normal . And,conversely, Bony outer normals satisfy (A) .

We also remark that,if such an F has non empty interior , then $cl_\Omega int F = F$ and $fr_\Omega int F = fr_\Omega F$.

Accordingly , we get:

Corollary 2.9 Let $F \subset \Omega$ be a convex set, with non empty interior, closed in Ω . Assume

(2.8) for each $u \in fr_\Omega F$, there exists n_u an outward normal to F at u , for which $(f(t,u) \mid n_u) < 0$, for every $t \in J$.

Then , int F is flow-invariant and F is weakly flow-invariant , for (0.1) .

If we are only interested in the weak flow-invariance of convex bodies (that is,closed and convex sets with non empty interior) in Ω , we can use a condition weaker than (2.8) . For it, let us see the next lemma:

Lemma 2.10 If F is a closed convex set, in Ω , then, for each $x \in int F$, there exists $\varepsilon > 0$ such that $((u-x) \mid n_u) \geqslant \varepsilon$, for every $u \in fr_\Omega F$ and n_u outward normal to F at u , with $|n_u| = 1$.

Proof: Let $x \in int F$. Then, there exists $\varepsilon > 0$ with $B[x,\varepsilon] \subset F$. Fix $u \in fr_\Omega F$ and let n_u be an outward normal to F at u, with $|n_u| = 1$.

By the definition of outward normal to F at u, we have $((y-u) \mid n_u) \leqslant 0$ for each $y \in F$. Hence, taking $y = x + \varepsilon n_u \in B[x,\varepsilon] \subset F$, as $|n_u| = 1$, we have $(n_u \mid (x + \varepsilon n_u - u)) \leqslant 0$,that is $((u-x) \mid n_u) \geqslant \varepsilon |n_u|^2 = \varepsilon$.

□

Theorem 2.11 Let $F \subset \Omega$ be a closed convex set, in Ω, with non empty interior. Assume

(2.9) for each $u \in fr_{\Omega} F$, there is an outward normal, η_u, to F at u, for which $(f(t,u) \mid \eta_u) \leqslant 0$, for every $t \in J$.

Then, F is weakly flow-invariant for (0.1).

Proof: As $\text{int}\, F \neq \phi$, fix $x_0 \in \text{int}\, F$. By the preceeding lemma, let $\varepsilon > 0$ be such that $((u-x_0) \mid \overset{*}{\eta_u}) \geqslant \varepsilon$, for every $u \in fr_{\Omega} F$ and $\overset{*}{\eta_u}$ outward normal to F at u, with $|\overset{*}{\eta_u}| = 1$. We observe that, as F is convex, $\frac{\eta_u}{|\eta_u|}$ satisfies this condition.

For each $n \in \mathbb{N}$, define $g_n : J \times \Omega \to \mathbb{R}^m$ by $g_n := f(t,x) + \frac{1}{n}(x_0 - x)$. So, g_n is a continuous function and, for $u \in fr_{\Omega} F$ and $t \in J$, $(g_n(t,u) \mid \eta_u) = (f(t,u) \mid \eta_u) + \frac{1}{n}((x_0-u) \mid \eta_u) \leqslant -\frac{1}{n}\, \varepsilon\, |\eta_u| < 0$.

By corollary 2.9, F is, then, weakly flow-invariant for $\dot{x} = g_n(t,x)$. As g_n converges uniformly on compact subsets of $J \times \Omega$, by theorem 1.3, as F is a closed set in Ω, F is weakly flow-invariant for (0.1).

□

An application of theorem 2.11 is the following corollary, similar to the one obtained by Pavel and Turicini in [42] :

Corollary 2.12 Let $h : J \times \prod_{i=1}^{m} [a_i, b_i] \to \mathbb{R}^m$ be a continuous function and $a_i < b_i$, for all $i \in \{1,..,m\}$.

A necessary and sufficient condition that, for each $(t_0, x_0) \in J \times \prod_{i=1}^{m} [a_i, b_i]$, there exists $x(.)$ solution of $\dot{x} = h(t,x)$ with $x(t_0) = x_0$ and $a_i \leqslant x_i(t) \leqslant b_i$, for every $i \in \{1,..,m\}$ and $t \in J \cap [t_0, +\infty)$, is that

(2.10) for each $i \in \{1,..,m\}$, $t \in J$ and $x_j \in [a_j, b_j]$, with $j \in \{1,..,i-1,i+1,..,m\}$,

$$h_i(t, x_1, .., x_{i-1}, a_i, x_{i+1}, .., x_m) \geqslant 0 \geqslant h_i(t, x_1, .., x_{i-1}, b_i, x_{i+1}, .., x_m)$$

holds, where h_i is the i^{th} component of h.

Proof: Let us see the necessary condition . Fix $i\in\{1,..,m\}$, $t_0\in J$ and $x_j\in [a_j,b_j]$, with $j\in\{1,..,i-1,i+1,..,m\}$.

Let $x(.)$ be a solution of $\dot{y} = h(t,y)$, with initial condition $y(t_0) = x_0$, where $x_{oi} = a_i$ and $x_{oj} = x_j$ for $j\neq i$, and $a_k \leqslant x_k(t) \leqslant b_k$, for every $k\in\{1,..,m\}$, and $t\in J\cap [t_0,+\infty)$.

Then, $h_i(t_0,x_1,..,x_{i-1},a_i,x_{i+1},..,x_m) = h_i(t_0,x_0) = \dot{y}_i(t_0) \geqslant 0$.

In an analogous way , taking $x_{oi} = b_i$, we obtain $h_i(t_0,x_1,..,x_{i-1},b_i,x_{i+1},..,x_m) \leqslant 0$.

For the sufficient condition , we note that $f:J\times\mathbb{R}^m \rightarrow \mathbb{R}^m$ defined by $f_i(t,y):= h_i(t,x)$, with $x_j = a_j$ if $y_j < a_j$, $x_j = y_j$ if $a_j \leqslant y_j \leqslant b_j$ and $x_j = b_j$ if $y_j > b_j$, for all $i\in\{1,..,m\}$, is a continuous extension of h. So, if $\prod_{i=1}^{m} [a_i,b_i]$ is weakly flow-invariant for $\dot{x} = f(t,x)$, our result is proved.

Putting $F = \prod_{i=1}^{m} [a_i,b_i]$ and $\Omega = \mathbb{R}^m$, one can easly check condition (2.9) , in theorem 2.11 , with condition (2.10) ,using outward normals.

□

A sufficient condition for the flow-invariance of $\prod_{i=1}^{m} (a_i,b_i)$ can be easily deduced from corollary 2.9 , replacing the weak inequalities (2.10) by the strict ones .

3. Strong flow-invariance and persistence.

In this section, we'll consider J an interval open on the right, so that for each $t_0\in J$ and $x_0\in \Omega$, there exist a solution, $x(.)$, of (0.1) defined on a maximal interval to the right of t_0, which is open on the right, and satisfying $x(t_0) = x_0$ (the initial condition).

In many applications, the existence of flow-invariant sets is very important. For instance, if (0.1) represents a model of dynamic of populations, with $x_i(t)$

(the i^{th} component of x(t)) the amount of the i^{th} population at the time t , we are interested in solutions x(.) of (0.1) with $x_i(t) \geqslant 0$. Then $\mathbb{R}^m_+$ must be a flow-invariant set. The most of the times, we are also interested in the fact that none of the populations come into extinction(that is $x_i(t) > 0$ for every i), so that we must require that the interior of $\mathbb{R}^m_+$ is flow-invariant.

In some applications, however, this requirement is not sufficiently significant. We have to impose further conditions in order to prevent that some positive $x_i(t)$ come arbitrarly close to zero, as time evolves. In fact, such possibility would represent a practical extinction of the considered population in a long period.

For a more general point of view, we can formulate the following problem: given a set $M \subset \Omega$,with non empty and flow-invariant interior, we want that the solutions of (0.1) , with initial value in int M, do not approach the boundary fr_Ω M as t grows. The manner in which the solutions have to remain far from the boundary has been described by various authors in different ways. For instance, if each solution x(.) of (0.1), with initial value in int M, is such that, for all future time t , the distance from x(t) to the boundary of M is bigger than a certain positive value , depending on x(.), we have persistence ([22]). And, if that positive value doesn't depend on the solution x(.), we have uniform persistence([11] , [12]).

See [20] and [33] ,for an exhaustive list of references concerning this problem and for more details from the point of view of the applications.

Freedman and Waltman , in [21] , considered systems of the type

(3.1) $\dot{x}_i = x_i \; g_i(t,x) \qquad i=1,..,m$

where $g_i: J\times\Omega \rightarrow \mathbb{R}^m$ is a continuous function .

They were interested in solutions x(.) of (3.1) , with $x_i(t) > 0$ in the future, and such that $\limsup_{t\to b_0^-} x_i(t) > 0$, for i=1,..,m , where b_0 is the supremum of the right maximal interval of existence of x(.) .

See also [24] and [25] .

If this happens for all initial values in int $\mathbb{R}^m_+$, (3.1) is said to be weakly persistent for $\mathbb{R}^m_+$.

A stronger concept in a more general setting, is that of strong flow-invariance, introduced by Gard ([24]):

Definition 3.1 $M \subset \Omega$ is said to be strongly flow-invariant for (0.1) if ,

(3.2) for each $(t_0,x_0) \in J\times \text{int } M$, any solution $x(.)$ of (0.1) , with $x(t_0) = x_0$, is such that

$$\limsup_{t\to \tau^-} d(x(t), fr_\Omega M) > 0 ,$$

for any $\tau > t_0$ in the closure of the right maximal interval of existence of $x(.)$.

Remark that condition (3.2) can be equivalently written as:

(3.2') $x(t) \in \text{int } M$, for every $t > t_0$ in the maximal interval of existence of $x(.)$ and

$$\limsup_{t\to b_0^-} d(x(t), fr_\Omega M) > 0 ,$$ where b_0 is the supremum of the right maximal interval of existence of $x(.)$.

If $\Omega = \mathbb{R}^m$, $M = \mathbb{R}^m_+$ and $f_i(t,x) = x_i\, g_i(t,x)$, for $i=1,..,m$,strong flow-invariance implies weak persistence for $\mathbb{R}^m_+$, with respect to (3.1).

The next example , of May and Leonard ([39]) , shows that , for $\mathbb{R}^m_+$ and (3.1), these concepts do not always coincide.

Example 3.1 Consider , in $\mathbb{R}^3$, the following system of the Gauss-Lotka-Volterra type, modeling competition between three species , with densities x_1 ,x_2 and x_3 :

(3.3)
$$\begin{cases} \dot{x}_1 = x_1\,(1-x_1-\alpha x_2-\beta x_3) \\ \dot{x}_2 = x_2\,(1-\beta x_1-x_2-\alpha x_3) \\ \dot{x}_3 = x_3\,(1-\alpha x_1-\beta x_2-x_3) \end{cases}$$

with α and β real constants such that $0 < \alpha < 1 < \beta$ and $\alpha+\beta > 2$.

Because of the particular form of (3.3) , for each initial value x_0 in the boundary of $\mathbb{R}^3_+$, there is a solution in a coordinated plane containing x_0 . Then , by the uniqueness of solutions for Cauchy problems of (3.3) , each solution with initial value in the interior of $\mathbb{R}^3_+$ cannot touch its boundary. So, int $\mathbb{R}^3_+$ is flow-invariant for (3.3) . And we have $x_i(t) > 0$, for $i=1,2,3$ and $t \geq t_0$ in the right maximal interval of existence of any solution $x(.)$, with initial condition $x(t_0) = x_0 \in \text{int } \mathbb{R}^3_+$.

We point out that positive solutions of (3.3) are defined for $t \in [t_o, +\infty)$, for t_o an initial time . In fact , we prove that , for all $k \geqslant 3$, the compact set $F_k = \{ (x_1,x_2,x_3) \in \mathbb{R}^3_+ \mid x_1+x_2+x_3 \leqslant k \}$ is flow-invariant for (3.3) . The conclusion follows as every $x_o \in \text{int}\, \mathbb{R}^3_+$ belongs to some F_k .

As F_k are closed and convex sets , we apply theorem 2.11 (with $\Omega = \mathbb{R}^3$) , as we have uniqueness for solutions of Cauchy problems of (3.3) .

For $u = (u_1,u_2,u_3) \in \text{fr}\, F_k$,we choose one outward normal that satisfies :

$$n_u = \begin{cases} (-1,0,0) & , u_1=0 \\ (0,-1,0) & , u_2=0 \\ (0,0,-1) & , u_3=0 \\ (1,1,1) & , u_1+u_2+u_3=k \text{ and } u_1,u_2,u_3 \neq 0 \ . \end{cases}$$

Call f the vectorial field associated to (3.3) .

So, if for some $i \in \{1,2,3\}$ $u_i=0$, $(f(t,u) \mid n_u) = -f_i(t,u) = 0$.

Otherwise , $(f(t,u) \mid n_u) = f_1(t,u)+f_2(t,u)+f_3(t,u) \leqslant u_1+u_2+u_3-(u_1{}^2+u_2{}^2+u_3{}^2) \leqslant k-\frac{k^2}{3}$ $=k\,(1-\frac{k}{3}) \leqslant 0$, as $k \geqslant 3$ and $k=u_1+u_2+u_3 \leqslant \sqrt{3}\ \sqrt{u_1{}^2+u_2{}^2+u_3{}^2}$.

By theorem 2.11 , F_k is,then , flow-invariant for (3.3) .

May and Leonard (see also theorem 1 in [48]) proved that all positive solutions of (3.3),with exception for the equilibrium point $(1,1,1)/(1+\alpha+\beta)$,have $F_o := \{(x_1,x_2,x_3) \in \mathbb{R}^3_+ \mid x_1+x_2+x_3=1$ and there exists $i \in \{1,2,3\}$ such that $x_i=0\}$ as their ω-limit set . So that,for positive solutions $x(.)$ of (3.3) , we have $\limsup_{t \to +\infty} x_i(t) > 0$, for all $i \in \{1,2,3\}$.

Then , (3.3) is weakly persistent for $\mathbb{R}^3_+$.

However, $\mathbb{R}^3_+$ is not strongly flow-invariant . In fact , being $x(.)$ a positive solution of (3.3) , with initial condition $x(0) = x_o \in \text{int}\, \mathbb{R}^3_+$ and $x_o \neq \frac{(1,1,1)}{1+\alpha+\beta}$, $d(x(t),F_o) \to 0$,as F_o is the ω-limit set of $x(.)$. But , as $F_o \subset \text{fr}\, \mathbb{R}^3_+$, we have also $d(x(t), \text{fr}\, \mathbb{R}^3_+) \to 0$.

An easy example of a flow-invariant set, that is not strongly flow-invariant , is $\mathbb{R}_+$, for $\dot{x} = \alpha x$, with $\alpha < 0$.

A stronger concept than strong flow-invariance is that of persistence.

For systems of the type (3.1) , considered by Freedman and Waltman, (3.1) is said to be persistent for $\mathbb{R}_+^m$, if int $\mathbb{R}_+^m$ is flow-invariant for (3.1) and, for each initial condition $(t_o,x_o)\in J\times \text{int}\, \mathbb{R}_+^m$, $\liminf_{t\to b_o^-} x_i(t)>0$, for $i=1,..,m$, where b_o is the supremum of the right maximal interval of existence of the solution $x(.)$ of (3.1) , with initial condition $x(t_o) = x_o$.

Of course, persistence implies weak persitence . But they do not necessarily coincide, as shows example 3.1 . In fact, this example also shows that each solution with initial value $x_o\in \text{int}\, \mathbb{R}_+^3$, with $x_o \neq (1,1,1)/(1+\alpha + \beta)$, is such that $\liminf_{t\to+\infty} x_i(t) = 0$. So , (3.3) is not persistent for $\mathbb{R}_+^3$.

A simple example of a persistent set is the following one:

Example 3.2 ([8]) Consider, in $\mathbb{R}^2$, the Volterra system :

$$(3.4)\qquad \begin{cases} \dot{x}_1 = x_1\,(a-bx_2) \\ \dot{x}_2 = x_2\,(-c+dx_1) \end{cases}$$

with a,b,c and d positive constants.

Let $x(.) = (x_1(.),x_2(.))$ be a solution of (3.4) , with initial condition $x(t_o) = x_o\in \text{int}\, \mathbb{R}_+^2$. Then , $\frac{d}{dt}\left(\frac{e^{dx_1}}{x_1^{\,c}}\cdot\frac{e^{bx_2}}{x_2^{\,a}}\right) = 0$. So that , $x(.)$ is contained in some $E_k := \left\{(x_1,x_2)\in \text{int}\, \mathbb{R}_+^2 \mid \frac{e^{dx_1}}{x_1^{\,c}}\cdot\frac{e^{bx_2}}{x_2^{\,a}} = k\right\}$, with k a positive constant .

We point out that E_k is a closed and bounded set in $\mathbb{R}^2$. Then , $x(t)$ exists for all $t\geq t_o$, $x_i(t)>0$ and $\liminf_{t\to+\infty} x_i(t)>0$, for $i=1,2$.

So, (3.4) is persistent for $\mathbb{R}_+^2$.

Remark: As for $x\in\mathbb{R}_+^m$, $d(x,\text{fr}\, \mathbb{R}_+^m) = \min_{1\leq i\leq m} x_i$, where $x = (x_1,..,x_m)$, the following definition generalizes the concept of persistence already given for $\mathbb{R}_+^m$ (with $\Omega = \mathbb{R}^m$) .

Definition 3.2 (0.1) is said to be persistent for $M \subset \Omega$ if , for each (t_o, x_o) in $J \times \text{int } M$, any solution $x(.)$ of (0.1) , with $x(t_o) = x_o$, is such that

(3.5) $\liminf_{t \to \tau^-} d(x(t), fr_\Omega M) > 0$, for $\tau > t_o$ in the closure of the right maximal interval of existence of $x(.)$ (see [11]) .

It is clear , by the definitions , that persistence implies strong flow-invariance. But the reverse is not true in general, as shows the following example :

Example 3.3 Modifying slightly Volterra system , we can get an example of strong flow-invariance which doesn't give persistence . In fact , take

$$(3.6) \quad \begin{cases} \dot{x} = x \left[a-by+ \frac{\varepsilon}{1+x^2}(-c+dx)\right] \\ \dot{y} = y \left[-c+dx- \frac{\varepsilon}{1+y^2}(a-by)\right] \end{cases}$$

with a,b,c and d positive constants and $\varepsilon = \min (\frac{a}{2c} , \frac{c}{2b})$.

As we have uniqueness of solutions for Cauchy problems of (3.6), according to the particular form of (3.6) , the x and y axes are invariant sets . So, int $\mathbb{R}^2_+$ is flow-invariant for (3.6) .

The determinant of $\begin{bmatrix} 1 & \frac{\varepsilon}{1+x^2} \\ -\frac{\varepsilon}{1+y^2} & 1 \end{bmatrix}$ is different from zero, so that, for

$$\begin{cases} (a-by)+\frac{\varepsilon}{1+x^2}(-c+dx) = 0 \\ -\frac{\varepsilon}{1+y^2}(a-by)+(-c+dx) = 0 \end{cases}$$

we have the only solution

$$\begin{cases} x = c/d \\ y = a/b \end{cases}$$

For $x,y > 0$, as we choosed ε, we have

$$a + \frac{\varepsilon}{1+x^2}(-c+dx) \geqslant a - \frac{c\,\varepsilon}{1+x^2} \geqslant a-c\,\varepsilon \geqslant \frac{a}{2}$$

and

$$-c - \frac{\varepsilon}{1+y^2}(a-by) \leqslant -c + \frac{by}{1+y^2}\varepsilon \leqslant -c + b\varepsilon \leqslant -\frac{c}{2} .$$

So, in $\mathbb{R}_+^2$, the equilibrium points , of (3.6) , are (0,0) and $(\frac{c}{d} , \frac{a}{b})$.

Let H , defined in int $\mathbb{R}_+^2$, be the energy function considered in the example 3.2,

$$H(x,y) = \frac{e^{dx}}{x^c} \cdot \frac{e^{by}}{y^a} . \text{ So } \frac{\partial H}{\partial x}(x,y) = \frac{H}{x}(-c + dx) \text{ and } \frac{\partial H}{\partial y}(x,y) = -\frac{H}{y}(a-by) ,$$

for $x,y > 0$.

The minimum , in int $\mathbb{R}_+$, of $\phi(s) = \frac{e^{fs}}{s^g}$, where f and g are positive constants,

is attained at g/f , so that , H attains its minimum at $(\frac{c}{d} , \frac{a}{b})$.

As $\lim_{s\to 0^+} \phi(s) = +\infty = \lim_{s\to+\infty} \phi(s)$,we have that $[H \leqslant \alpha]$ is a compact set contained in int $\mathbb{R}_+^2$, for every $\alpha \in \mathbb{R}$.

If $\alpha < H(\frac{c}{d} , \frac{a}{b})$, $[H \geqslant \alpha] = \text{int } \mathbb{R}_+^2 = [H > \alpha]$.

If $\alpha \geqslant H(\frac{c}{d} , \frac{a}{b})$, applying corollary 2.3 with $V = \alpha - H$, as

$$\left((x\,[a-by + \frac{\varepsilon}{1+x^2}(-c+dx)] ,y\,[(-c+dx)- \frac{\varepsilon}{1+y^2}(a-by)]\,)\,|\, \nabla V(x,y)\right) =$$

$$= -((x(a-by),y(-c+dx))\,|\, \nabla H(x,y)\,) - ((\frac{\varepsilon}{1+x^2}(-c+dx)- \frac{\varepsilon}{1+y^2}(a-by))\,|\, \nabla H(x,y)) =$$

$$= -\frac{\partial H}{\partial x}(x,y)\frac{\varepsilon}{1+x^2}(-c+dx) + \frac{\partial H}{\partial y}(x,y)\frac{\varepsilon}{1+y^2}(a-by) =$$

$$= -\frac{H(x,y)}{x}\frac{\varepsilon}{1+x^2}(-c+dx)^2 + \frac{H(x,y)}{y}\frac{\varepsilon}{1+y^2}(a-by) < 0$$, for $(x,y) \in [H > \alpha]$, we have the flow-invariance of $[H > \alpha]$ and $[H \geqslant \alpha]$, for (3.6) .

Then, for each $\alpha \in \mathbb{R}$,$[H \leqslant \alpha]$ is compact and negatively invariant .So that , every solution z(.) , of (3.6), with initial condition $z(t_0) = z_0 \in \text{int } \mathbb{R}_+^2$, exists for all $t \leqslant t_0$ and , by LaSalle' s theorem ([37]) , $\lim_{t\to-\infty} z(t) = (\frac{c}{d},\frac{a}{b})$ .Therefore, system (3.6) cannot have non trivial periodic orbits and , in particular, limit cycles.

Let us see that the positive solutions of (3.6), different of the equilibrium point $(\frac{c}{d},\frac{a}{b})$, turn around it,for all future time.

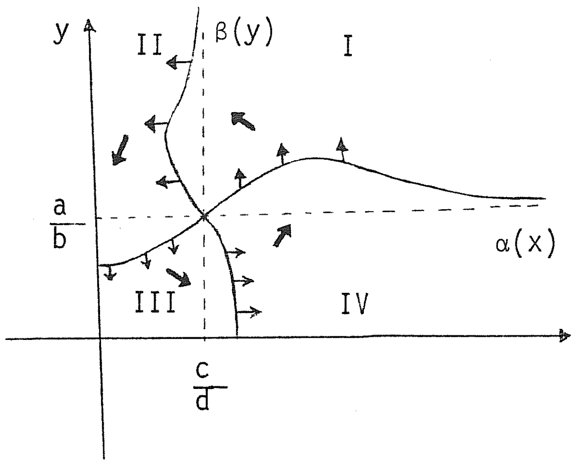
For such a conclusion we divide int $\mathbb{R}^2_+$ in four open regions , according to the behaviour of $\dot{x}$ and $\dot{y}$,bounded by the curves $y = \alpha(x)$ and $x = \beta(y)$, for $x,y > 0$, where

$$\alpha(x) = \frac{a}{b} + \frac{\varepsilon}{b(1+x^2)}(-c+dx) \qquad (\geqslant \frac{a}{2b})$$

and

$$\beta(y) = \frac{c}{d} + \frac{\varepsilon}{d(1+y^2)}(a-by) \qquad (\geqslant \frac{c}{2d}) \quad .$$

We have the following phase portrait:



Suppose $z(.)=(x(.),y(.))$ a solution of (3.6), with initial condition $z(t_0) = (x_0,y_0)\in \text{int}\ \mathbb{R}^2_+ \setminus \{(\frac{c}{d},\frac{a}{b})\}$.

If $x_0 \in I$, while $z(.)$ remains in I,say for $t\in [t_0,t^*)$, we have $\dot{x} < 0$ and $\dot{y} > 0$ and for some positive constant k , $0 < -\frac{dy}{dx} \leqslant ky$. So that, $y(t) \leqslant y_0\ e^{k(x_0-x(t))}$, as x(.) is decreasing on $[t_0,t^*)$. So, z(.) is bounded on $[t_0,t^*)$. Being $(\frac{c}{d},\frac{a}{b})$ repulsive (as $[H \geqslant \alpha]$ are flow-invariant) and as there are no more equilibrium points in int $\mathbb{R}^2_+$ and no limit cycles, by Poincaré- Bendixon theorem, z(.) cannot remain in I, for all time t in the right maximal interval of existence. So, $x(t^*) = \beta(y(t^*))$. And , as $\dot{x}(t^*) < 0$, there exists $t_1 > t^*$ with $z(t_1)\in II$.

If x_0 II , as $\dot{y} < 0$ and x(.) is bounded , as long as z(t) belongs to II , say for $t\in [t_0,t^*)$, z(.) is bounded on $[t_0,t^*)$. By the repulsivity of $(\frac{c}{d},\frac{a}{b})$ and, as there are no more equilibrium points in the closure of II, nor limit cycles, $y(t^*) = \alpha(x(t^*))$. Being $\dot{y}(t^*) < 0$, there exists $t_1 > t^*$ with $z(t_1)\in III$.

If $x_0 \in III$, as $\dot{y} < 0$ and $x(.)$ is bounded, as long as $z(t)$ belongs to III, say for $t \in [t_0, t^*)$, $z(.)$ is bounded on $[t_0, t^*)$. As $\dot{x}(t) > 0$ for $t \in [t_0, t^*)$, $x(t) > x_0$ on $[t_0, t^*)$. So, $z(t)$ cannot approach $(0,0)$ as t approaches t^* . As $(\frac{c}{d}, \frac{a}{b})$ is repulsive and there are no limit cycles in cl III, $x(t^*) = \beta(y(t^*))$. Being $\dot{x}(t^*) > 0$, there exists $t_1 > t^*$ with $z(t_1) \in IV$.

Finally , if $x_0 \in IV$, while $z(.)$ remains in IV , say for $t \in [t_0, t^*)$, we have $\dot{x} > 0$ and $\dot{y} > 0$, and, for some constant $k > 0$, $0 < \frac{dx}{dy} \leqslant kx$. So, $x(t) \leqslant x_0 e^{k(y(t)-y_0)}$, as $x(.)$ increases in $[t_0, t^*)$. Then, $z(.)$ is bounded on $[t_0, t^*)$. By the repulsivity of $(\frac{c}{d}, \frac{a}{b})$, the only equilibrium point in cl IV , which contains no limit cycles, $y(t^*) = \alpha(x(t^*))$. As $\dot{y}(t^*) > 0$, there exists $t_1 > t^*$ with $z(t_1) \in I$.

So, for all future time, positive solutions of (3.6) go around $(\frac{c}{d}, \frac{a}{b})$.

(3.6) is not persistent for $\mathbb{R}^2_+$. In fact , let $x_0 = \frac{c}{d}$ and $0 < y_0 < \frac{a}{b}$. If $z(.) = (x(.), y(.))$ is the solution of (3.6) , with initial condition $z(t_0) = (x_0, y_0)$, as $z(.)$ goes around $(\frac{c}{d}, \frac{a}{b})$ in its right maximal interval of existence, say $[t_0, t^0)$, there exists t_n strictly increasing converging to t^0 , with $x(t_n) = \frac{c}{d}$ and $y(t_n)$ decreasing . Then, $y(t_n)$ converges to some y_1 .

Suppose , $y_1 > 0$. By Poincaré-Bendixon theorem, $z(.)$ cannot remain, for the future, in $[H \leqslant H(\frac{c}{d}, y_1)]$, as this is a compact set and $(\frac{c}{d}, \frac{a}{b})$ is repulsive . As $[H \leqslant H(\frac{c}{d}, y_1)]$ is also a negatively invariant set for (3.6) , there exists $t_* > t_0$ such that $H(x(t), y(t)) > H(\frac{c}{d}, y_1)$, for $t_* < t < t^0$. But then , for n large enough , as $y_1 \leqslant y(t_n) < a/b$ and $\phi(s) = e^{bs}/s^a$ is decreasing for $s < a/b$,

$$\frac{e^{b\,y(t_n)}}{(y(t_n))^a} \leqslant \frac{e^{b\,y_1}}{y_1^{\,a}} \quad , \text{ so that } H(x(t_n), y(t_n)) = H(\tfrac{c}{d}, y(t_n)) \leqslant H(\tfrac{c}{d}, y_1) \text{ , which}$$

gives a contradiction .

So, $y_1 = 0$ and we have $\liminf_{t \to t^0} d(z(t), \text{fr}\, \mathbb{R}^2_+) = 0$, which shows that (3.6) is not persistent for $\mathbb{R}^2_+$.

$\mathbb{R}^2_+$ is strong flow-invariant for (3.6) . As a matter of fact , every positive solution z(.) of (3.6) , not coinciding with $(\frac{c}{d}, \frac{a}{b})$, is such that

$\limsup_{t\to b_0^-} d(z(t), \mathrm{fr}\, \mathbb{R}^2_+) = +\infty$, where b_0 is the right hand side of the maximal interval of existence of z(.).

In fact, being $(x_0, y_0) = z(t_0)$ an initial condition for $z(.) = (x(.), y(.))$, as z(.) turns around $(\frac{c}{d}, \frac{a}{b})$, as $t\to b_0$ there exists t_n converging to b_0 and strictly increasing , with $y(t_n) = \frac{a}{b} + \frac{ad}{bc}(x(t_n) - \frac{c}{d})$ and $x(t_n) > \frac{c}{d}$. So, $y(t_n)$ and $x(t_n)$ are increasing functions . If one of them would be bounded (so would be the other one), there would be $x_1 > \frac{a}{b}$ with $(x(t_n), y(t_n)) \to (x_1, y_1)$, for $y_1 = \frac{a}{b} + \frac{ad}{bc}(x_1 - \frac{c}{d})$. Reasoning in a similar way , as for the non persistence, with the energy function , for n big enough we should have $H(x(t_n), y(t_n)) > H(x_1, y_1)$, but this contradicts the fact that $\frac{e^{d\,x(t_n)}}{(x(t_n))^c} \leqslant \frac{e^{d\,x_1}}{x_1{}^c}$ and $\frac{e^{b\,y(t_n)}}{(y(t_n))^a} \leqslant \frac{e^{b\,y_1}}{y_1{}^a}$, as $x(t_n) < x_1$ and $y(t_n) < y_1$.

So, both $x(t_n)$ and $y(t_n)$ converge to $+\infty$, which means ,

$d((x(t_n), y(t_n)), \mathrm{fr}\, \mathbb{R}^2_+) = \min(x(t_n), y(t_n)) \to +\infty$.

□

For properties on strong flow-invariance and persistence,see , for instance, [24] and [18] . We just quote some results in the last one.

The main result, in [18] ,is the following theorem, for compact sets:

<u>Theorem 3.3</u> Let $M \subset \Omega$ be a compact set in $\mathbb{R}^m$, with non empty interior, such that, for each $u \in \mathrm{fr}\, M$, $V_u : \Omega \to \mathbb{R}$ is a C^1 function , with $V_u(u) = 0$, and $M \subset \bigcap_{u\in \mathrm{fr}\, M} [V_u \leqslant 0]$. Assume (with $J = [a,b)$)

(3.7) for each $u \in \mathrm{fr}\, M$ and $(t,x) \in (J \cup \{b\}) \times (M \cap [V_u = 0])$

$$\limsup_{\substack{s\to t^- \\ \mathrm{int}\, M \ni y \to x}} (f(s,y) \mid \nabla V_u(y)) < 0 .$$

Then, (0.1) is persistent for M and, hence, M is strongly flow-invariant for (0.1) .

As observed in [18] , condition (3.7) is satisfied whenever both

(3.8) $(f(t,x) \mid \nabla V_u(x)) < 0$, for $t \in J$, $u \in fr\ M$ and $x \in M \cap [V_u=0]$

and

(3.9) $\limsup_{\substack{s \to b^- \\ int\ M \ni y \to x}} (f(s,y) \mid \nabla V_u(y)) < 0$, for $t \in J$, $u \in fr\ M$ and $x \in M \cap [V_u=0]$

hold.

Being (3.7) equivalent to (3.8) and (3.9), if (3.7) is supposed to be satisfied for $x \in [V_u=0] \cap cl\ int\ M$.

We remark that , in the autonomous case ($f(x) = f(t,x)$) condition (3.7) is satisfied if

(3.10) $(f(x) \mid \nabla V_u(x)) < 0$, for $x \in M \cap [V_u=0]$ and $u \in fr\ M$

holds.

For the autonomous case , theorem 3.3 gives theorem 3 of Gard's paper [24] .

We also note that condition (3.8) is not enough to guarantee persistence and , not even , strong flow-invariance, as the next example shows.

Example 3.4 Take $f: [0,+\infty) \times \mathbb{R} \to \mathbb{R}$ defined by $f(t,x) = -x + \arctan t$ and set $M = [-\pi/2, \pi/2]$. For $u \in \{-\pi/2 , \pi/2\} = fr\ M$, let $V_u : \mathbb{R} \to \mathbb{R}$ be such that $V_u(x) = x^2 - (\pi/2)^2$. Then , $M = \bigcap_{u \in fr\ M} [V_u \leq 0]$ is a compact set and condition (3.8) is satisfied, because $(f(t,x) \mid \nabla V_u(x)) = -2x(x + \arctan t) < 0$, $u \in fr\ M$ and $x \in [V_u=0] \cap M = fr\ M$. Nevertheless, $x(t) = \arctan t - e^{-t} \int_0^t \frac{e^s}{1+s^2}\, ds$ is a solution of $\dot{x} = -x + \arctan t$ in $[0,+\infty)$, with $x(0)=0$ and $\lim_{t \to +\infty} x(t) = \pi/2 \in fr\ M$. So that, M is not strongly flow-invariant for this equation .

Also compactness cannot be dropped in theorem 3.3, neither in the autonomous case. In fact, Example 3.5 Take $f:\mathbb{R}^2\to\mathbb{R}^2$ defined by $f(x_1,x_2) = (1, -x_2 + \arctan x_1)$ and $M = \{(x_1,x_2)\in\mathbb{R}^2 \mid x_2 \leqslant \pi/2\}$. For $u\in \mathrm{fr}\, M = \mathbb{R}\times\{\pi/2\}$, let $V_u:\mathbb{R}^2\to\mathbb{R}$ with $V_u(x) = x_2 - \pi/2$. Then $M = \bigcap_{u\in \mathrm{fr}\, M} [V_u \leqslant 0]$, which is not a compact set in $\mathbb{R}^2$.

$x(t) = (t, \arctan t - e^{-t}\int_0^t \frac{e^s}{1+s^2}\, ds)$ is a solution of $\dot{x}=f(x)$ in $[0,+\infty)$, with $x(0) = (0,0)\in \mathrm{int}\, M$. But $d(x(t),\mathrm{fr}\, M) = \left|\arctan t - e^{-t}\int_0^t \frac{e^s}{1+s^2}\, ds - \frac{\pi}{2}\right| \to 0$ as $t\to+\infty$. So that, M is not strongly flow-invariant for $\dot{x}=f(x)$. Nevertheless, condition (3.10) is satisfied as, for $x,u\in \mathrm{fr}\, M = M\cap[V_u=0]$, $(f(x)\mid \nabla V_u(x)) = ((1, -\frac{\pi}{2} + \arctan x_1)\mid (0,1)) = -\frac{\pi}{2} + \arctan x_1 < 0$.

Theorem 2, in [18], shows that, in the non autonomous case, no condition on $f(t,.)\big|_{\mathrm{fr}\, M}$ is sufficient to ensure the strong flow-invariance of M, for (0.1), so that, condition (3.7) in theorem 3.3 cannot be substituted by the more natural one

$$\sup_{t\geqslant a} (f(t,x) \mid \nabla V_u(x)) < 0 , \quad \text{for } u\in \mathrm{fr}\, M \text{ and } x\in M\cap[V_u=0] .$$

In the autonomous case, and for compact sets, we have, with the same condition (2.6) of corollary 2.7, a result for persistence, with outer normals in the sense of Bony (corollary 2 in [18]). This corollary answers a question raised by Gard's paper [24] (page 289) and is the following one:

Corollary 3.4 Let $M\subset\Omega$ be a compact set in $\mathbb{R}^m$, with non empty interior. Suppose $f(t,x) = f(x)$, for all $t\in J$ and $x\in\Omega$, and assume

(3.11) for each $u\in \mathrm{fr} M$ there is an outer normal n_u, in the sense of Bony, to M at u, such that $(f(u) \mid n_u) < 0$.

Then, $\dot{x} = f(x)$ is persistent for M and, hence, M is srongly flow-invariant for $\dot{x} = f(x)$.

Remark that, if h is autonomous and satisfies strict inequalities in(2.10) $\dot{x} = h(x)$ is persistent for $\prod_{i=1}^{m} [a_i, b_i]$.

We point out that compactness is, also, essential in corollary 3.4, as shows example 3.5 . In fact, in this example, for each $u \in fr\ M = \mathbb{R} \times \{\pi/2\}$, $n_u = (0,1)$ is an outer normal, to M at u, and $(f(u) \mid n_u) = ((1, -\pi/2 + \arctan u_1) \mid (0,1)) =$ $= -\pi/2 + \arctan u_1 < 0$.

In practice, the strict inequality in condition (3.7) of theorem 3.3, is not good to apply because in many applications fr M is a piece of some invariant set and therefore, the field is tangent to fr M. To avoid this difficulty, we give the next theorem (which generalizes theorem 1' in [18]) that combines arguments of theorem 3.3 and similar ones presented in papers [30] and [32] .

<u>Theorem 3.5</u> Let $M \subset \Omega$ be a compact set in $\mathbb{R}^m$, with non empty interior, such that, for each $u \in fr\ M$, $V_u : \Omega \to \mathbb{R}$ is a C^1 function with $V_u(u)=0$ and $\nabla V_u(x) \neq 0$, for all $x \in M \cap [V_u = 0]$. Moreover, $M \subset \bigcap_{u \in fr\ M} [V_u \leq 0]$. Assume that, for each $u \in fr\ M$, there are $\Psi_u : J \times M \to \mathbb{R}$ and $\phi_u : \mathbb{R}^+ \to \mathbb{R}^+$ continuous functions, with $\phi_u(s) > 0$ for $s > 0$, such that

(3.12) $\quad (f(t,x) \mid \nabla V_u(x)) \leq \phi_u(|V_u(x)|)\Psi_u(t,x)$, for every $u \in fr\ M$, $t \in J$ and $x \in M$

and

(3.13) $\quad \limsup_{\substack{s \to t^- \\ int\ M \ni y \to x}} \Psi_u(s,y) < 0$, for every $u \in fr\ M$, $t \in J \cup \{b\}$ and $x \in M \cap [V_u = 0]$.

Then, (0.1) is persistent for M and, hence, M is strongly flow-invariant for (0.1) .

Proof: The proof is essentially the same as the one presented in theorem 1 of [18]. We only prove the different parts.

To see that int M is flow-invariant, we apply theorem 2.6 . In fact, as assumed in the proof of theorem 1 in [18] , M = cl int M . So, fr int M = fr M . By (3.13), with $x = u \in fr\ M$ ($u \in [V_u = 0]$ M, M compact) , and $t \in J$, exists $\varepsilon > 0$ such that, if $y \in B(u, \varepsilon) \cap int\ M$ and $|s-t| < \varepsilon$, $\Psi_u(s,y) < 0$. So, according to (3.12), as $y \in int\ M \subset$

$\subset [V_u < 0]$ (see Remark 1 in [18]) implies $\phi_u(|V_u(y)|) > 0$, $(f(s,y) | \nabla V_u(y)) \leq \phi_u(|V_u(y)|) \Psi_u(s,y) < 0$. Then, condition (2.3) in theorem 2.4 is satisfied and therefore, int M is flow-invariant for (0.1) .

For the proof of the claim

$$\forall u \in \text{fr } M \quad \exists \varepsilon_u > 0 \quad : (f(t,y) | \nabla V_u(y)) < 0, \quad \forall_{t \in J} \quad \forall_{y \in \text{int } M \, [V_u \geq -\varepsilon_u]}$$

we suppose , by contradiction , that it is not true . So, there exists $u \in \text{fr } M$ such that , for every $n \in \mathbb{N}$, exists $t_n \in J$ and $y_n \in \text{int } M \cap [V_u \geq -1/n]$, with $(f(t_n,y_n) | \nabla V_u(y_n)) \geq 0$. As $y_n \in \text{int } M \subset M$, M compact , there are subsequences t_{n_k} and y_{n_k} , of t_n and y_n , converging to some $\zeta \in J \cup \{b\}$ and $z \in M$.

$y_n \in \text{int } M \subset M \subset \bigcap_{x \in \text{fr } M} [V_x \leq 0]$, so $-1/n \leq V_u(y_n) \leq 0$. And, by continuity of V_u , $V_u(z)=0$. Then $z \in M \cap [V_u=0]$. As $y_n \in \text{int } M = M \cap (\bigcap_{x \in \text{fr } M} [V_x < 0])$ (by Remark 1 in [18]) , we have $V_u(y_n) < 0$ and , therefore, $\phi_u(| V_u(y_n)|) > 0$. Applying (3.12) ,

$$\limsup_{\substack{s \to \zeta^- \\ \text{int } M \ni y \to z}} \Psi_u(s,y) \geq \lim_{n \to +\infty} \Psi_u(t_n,y_n) \geq \frac{(f(t_n,y_n) | \nabla V_u(y_n))}{\phi_u(|V_u(y_n)|)} \geq 0 ,$$

a contradiction with (3.13) .

□

Remark that , for $\phi_u(s)=1$, for every $s \geq 0$ and $u \in \text{fr } M$, and $\Psi_u(t,x) = (f(t,x) | \nabla V_u(x))$, for every $t \in J$, $u \in \text{fr } M$ and $x \in M$, theorem 3.5 becomes theorem 3.3 .

The choice $\phi_u(s) = s$ may be useful for systems of the type (3.1) . In fact,

<u>Corollary 3.6</u> Let $b_1,..,b_m$ be positive real constants and $g: \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$ continuous and p- periodic in the first variable. Assume that

(3.14) for each $i \in \{1,..,m\}$, $t \in \mathbb{R}$ and $x_j \in [0,b_j]$, with $j=1,..,i-1,i+1,..,m$,

$$g_i(t,x_1,..,x_{i-1},b_i,x_{i+1},..,x_m) < 0 < g_i(t,x_1,..,x_{i-1},0,x_{i+1},..,x_m) \; .$$

Then, system $\dot{x}_i = x_i g_i(t,x)$, with $i=1,..,m$,is persistent for $\prod_{i=1}^{m} [0,b_i]$.

Proof: Let $M = \prod_{i=1}^{m} [0,b_i]$, $J = \mathbb{R}$ and $\Omega = \mathbb{R}^m$. To apply theorem 3.5 , let us define, for each $u \in fr\, M$, V_u, ϕ_u and ψ_u.

If $u \in fr\, M$ is such that there exists $i \in \{1,..,m\}$ with $u_i=0$, let $i_u = \min\{i \in \{1,..,m\} \mid u_i=0\}$. If $u_i > 0$ for every $i \in \{1,..,m\}$, take $i_u := \min\{i \in \{1,..,m\} \mid u_i = b_i\}$.

Define for $u \in fr\, M$, $V_u : \mathbb{R}^m \to \mathbb{R}$, $\phi_u : \mathbb{R}_+ \to \mathbb{R}_+$ $\psi_u : \mathbb{R} \times M \to \mathbb{R}$ by

$$V_u(x) := \begin{cases} -x_{i_u} & \text{, if there exists } i \in \{1,..,m\} \text{ such that } u_i=0 \\ x_{i_u} - b_{i_u} & \text{, otherwise .} \end{cases}$$

$$\phi_u(s) := \begin{cases} s & \text{, if there exists } i \in \{1,..,m\} \text{ such that } u_i=0 \\ 1 & \text{, otherwise} \end{cases}$$

$$\psi_u(t,x) := \begin{cases} -g_{i_u}(t,x) & \text{, if there exists } i \in \{1,..,m\} \text{ such that } u_i=0 \\ x_{i_u} g_{i_u}(t,x), & \text{otherwise} \end{cases}$$

Let $x \in M$. If $u \in fr\, M$ is such that exists $i \in \{1,..,m\}$ with $u_i=0$, $V_u(x) = -x_{i_u} \leq 0$, as $x \in M$. If $u \in fr\, M$ is such that $u_i > 0$ for every $i \in \{1,..,m\}$, $V_u(x) = x_{i_u} - b_{i_u} \leq 0$, as $x \in M$. So, $M \subset \bigcap_{u \in fr\, M} [V_u \leq 0]$. By construction of V_u , $\nabla V_u(x) \neq 0$, for every $u \in fr\, M$ and $x \in \mathbb{R}^m$.

Call, for each $i \in \{1,..,m\}$, $f_i(t,x) = x_i\, g_i(t,x)$, where $t \in \mathbb{R}$ and $x \in \mathbb{R}^m$. Condition (3.13) is verified . In fact, let $u \in fr\, M$, $t \in \mathbb{R}$ and $x \in M$. If there exists $i \in \{1,..,m\}$, with $u_i=0$, $(f(t,x) \mid \nabla V_u(x)) = -f_{i_u}(t,x) = -x_{i_u} g_{i_u}(t,x) = -\mid V_u(x) \mid g_{i_u}(t,x) = \phi_u(\mid V_u(x) \mid)\, \psi_u(t,x)$. If $u_i > 0$ for all $i \in \{1,..,m\}$, $(f(t,x) \mid \nabla V_u(x)) = f_{i_u}(t,x) = x_{i_u} g_{i_u}(t,x) = \phi_u(\mid V_u(x) \mid)\, \psi_u(t,x)$.

suppose, by contradiction, that condition (3.13) is not verified . Then, there are $u \in fr\, M$, $x \in [V_u=0]$ and $t_0 \in \mathbb{R} \cup \{+\infty\}$, for which there exist, for $n \in \mathbb{N}$, $s_n \in \mathbb{R}$ and $y_n \in int\, M$, with $s_n \to t_0^-$ and $y_n \to x$,and $\lim_{n \to +\infty} \psi_u(s_n, y_n) = h > 0$. Taking

$t_n = s_n - r_n p \in [0,p]$ with $r_n \in \mathbb{Z}$, there exists a subsequence of t_n , t_{n_k}, converging to some $t^* \in [0,p]$. Then, as Ψ_u is p-periodic in the first variable, $\Psi_u(t_{n_k}, y_{n_k}) = \Psi_u(s_{n_k}, y_{n_k}) \to h$. And , as Ψ_u is continuous , $\Psi_u(t^*,x)=h \geq 0$.

If there exists $i \in \{1,..,m\}$ such that $u_i=0$, then, according to (3.14), as $x_{i_u}=0$, $0 \leq h = \Psi_u(t^*,x) = -g_{i_u}(t^*,x) < 0$, a contradiction .

If $u_i > 0$, for every $i \in \{1 ..,m\}$, again by (3.14) , as $x_{i_u} = b_{i_u}$, $0 \leq h = \Psi_u(t^*,x) = b_{i_u} g_{i_u}(t^*,x) < 0$, a contradiction.

4. Some remarks on uniform persistence .

Recently, it has been considered a stronger concept . That of uniform persistence. In the literature, there are several versions involving this concept .

Let J be as in section 3 .

Considering a system of type (3.1) , one can say that (3.1) is uniformly persistent for $\mathbb{R}^m_+$ if int $\mathbb{R}^m_+$ is flow-invariant and there exists $\delta > 0$ such that $\liminf_{t\to b_0^-} x_i(t) > \delta$, for every i=1,..,m and x(.) solution of (3.1) , with initial condition $x(t_0) = x_0 \in \text{int}\, \mathbb{R}^m_+$, having right maximal interval of existence $[t_0, b_0)$.

As, for $x=(x_1,..,x_m) \in \mathbb{R}^m_+$, $d(x, \text{fr}\, \mathbb{R}^m_+) = \min_{1 \leq i \leq m} x_i$, one can generalize this notion, for more general sets and equations , in the following one :

<u>Definition 4.1</u> (0.1) is <u>uniformly persistent for</u> M if int M is flow-invariant and there exists $\delta > 0$ such that $\liminf_{t\to b_0^-} d(x(t), \text{fr}\, M) > \delta$, for every x(.) solution of (0.1) , with initial value $x(t_0) \in \text{int}\, M$ and right maximal interval of existence $[t_0, b_0)$.

This definition has been considered in very recent papers by Butler and Waltman

([12]) and by Butler, Freedman and Waltman ([11]) . In the last one , conditions are given under which strong flow-invariance implies uniform persistence.

Another version , which is also called cooperativeness ([30] , [19]) or permanent coexistence ([32]) asserts that:

<u>Definition 4.2</u> (0.1) is <u>cooperative</u> on M if int M is flow-invariant and there exists a compact set in $\mathbb{R}^m$, $K \subset \text{int } M$ such that , for every solution x(.) of (0.1), with initial value $x(t_0) \in \text{int } M$, there exists $t_1 \in J$ with $t_1 \geqslant t_0$, for which $x(t) \in K$, for all $t \geqslant t_1$ in the right maximal interval of existence of x(.) .

We point out that , if M is a compact set , both definitions coincide .

5. Application to the existence of periodic solutions .

In this section , we are interested in finding solutions of equation

(5.1) $\dot{x} = f(t,x)$

satisfying the boundary condition

(5.2) $x(0) = x(p)$

where $p > 0$ and $f: [0,p] \times \Omega \longrightarrow \mathbb{R}^m$ is a continuous function, with Ω a nonempty open subset of $\mathbb{R}^m$.

We observe that a solution of (5.1)-(5.2) ,in [0,p] ,is not necessarily the restriction of a p- periodic (that is $x(t+p) = x(t)$, for all t) C^1 function, because $\dot{x}(0) = \dot{x}(p)$ is satisfied only if $f(0,x(0)) = f(p,x(p))$. However, for brevity , solutions of (5.1)-(5.2) will be called p- periodic .

There are several methods to prove existence of periodic solutions. One of them is proving the existence of fixed points for the translation operator, that is , for T defined by

$$T(z) = \{ x(p) \mid x \text{ is a solution of (5.1) , in } [0,p] \text{ , with } x(0) = z \}$$

where $z \in M \subset \Omega$, being M such that $T(z) \neq \phi$.

If f has no uniqueness for solutions of Cauchy problems for (5.1), T is a multivalued map ([13]) . And, a fixed point for T will be a point $z \in T(z)$. Then , the existence of a p-periodic solution is equivalent to the existence of a fixed point for T.

A fixed point, z_0, for T, is such that $z_0 \in M \cap T(z_0)$. For M compact and weakly flow-invariant for (5.1) , we'll have, for every $z \in M$, $T(z) \cap M \neq \phi$. In fact, by the weak flow-invariance of M, for each $z \in M$, there exists a solution, x(.), of (5.1) defined on a maximal interval of existence, with initial condition $x(0)=z$ and such that $x(t) \in M$, for every t in the interval of existence of x(.), which is $[0,p]$, as M is a compact set . Then , $x(p) \in M \cap T(z)$. So that, $T(z) \cap M \neq \phi$.

Our problem was reduced to the application of a fixed point theorem to the translation operator, which can be found , for instance in [5] .

For our applications, however, we will approximate the continuous function f by C^1 functions,with the uniform convergence over compact subsets of $[0,p] \times \Omega$. In fact, for C^1 functions, we have uniqueness for Cauchy problems , so that the translation operator will be a continuous and one valued map. So, we'll apply a fixed point theorem to each approximating problem to find an approximating solution.

We begin with the continuity of the translation operator:

Proposition 5.1 Let M be a subset of Ω and $g: [0,p] \times \Omega \to \mathbb{R}^m$ be a continuous function . Assume that, for each $z \in M$, there exists one,and only one , solution in $[0,p]$ for the Cauchy problem

$$\begin{cases} \dot{x} = g(t,x) & (5.3) \\ x(0) = z & (5.4) \end{cases}$$

which will be called $x(.;z)$.

Then the translation operator $T: M \to \Omega$, $z \to x(p;z)$ is a continuous map on M.

Proof: Suppose , by contradiction , that T is not continuous on M . Then,there exist a sequence $(z_n)_n \subseteq M$ and $z_0 \in M$ such that z_n converges to z_0 ,but $x(p;z_n)$ does not converge to $x(p;z_0)$. Therefore , there exist $\delta > 0$ and z_{n_k} , a subsequence of z_n , such that $|x(p;z_{n_k})-x(p;z_0)| \geqslant \delta$, for every $k \in \mathbb{N}$.

Define $\tilde{f}: \mathbb{R} \times \Omega \to \mathbb{R}^m$ by

$$\tilde{f}(t,x) = \begin{cases} g(0,x) & , t<0 , x\in\Omega \\ g(t,x) & , 0\leq t\leq p , x\in\Omega \\ g(p,x) & , t>p , x\in\Omega \end{cases}$$

and take $f_{n_k} = \tilde{f}$ for every $k\in\mathbb{N}$. Applying theorem 1.2, as $x(.;z_{n_k})$ are solutions of $\dot{x} = f_{n_k}(t,x)$ on $[0,p]$, with $x(0;z_{n_k}) = z_{n_k} \to z_0 \in M\subset\Omega$, there exists $y(.)$, a solution of $\dot{x} = \tilde{f}(t,x)$, with $y(0) = z_0$ and defined on a maximal interval of existence (ω_-,ω_+) . But according to the definition of $\tilde{f}$, $y(.)$ is also a solution of $\dot{x} = g(t,x)$ on $[0,p] \cap (\omega_-,\omega_+)$ with $y(0) = z_0\in M$. By hypothesis, it follows that $(\omega_-,\omega_+) \supset [0,p]$ and $y(t) = x(t;z_0)$ for all $t\in[0,p]$. Being $[0,p]$ a compact subinterval of (ω_-,ω_+) , by theorem 1.2 , there exists a subsequence $x(.;z_{n_{k'}})$ converging uniformly on $[0,p]$ to $x(.;z_0)$. So, in particular, $x(p;z_{n_{k'}})$ converge to $x(p;z_0)$, which is a contradiction as $|x(p;z_{n_{k'}})-x(p;z_0)| \geq \delta > 0$, for every $k'\in\mathbb{N}$.

□

A subset M of $\mathbb{R}^m$ has the fixed point property if for every continuous map $h:M\to M$, there is a fixed point for h (that is , there exists $x\in M$ such that $h(x) = x$). Every subset homeomorphic to a retract of a compact and convex set of $\mathbb{R}^m$, has the fixed point property.

For such sets , we have the following existence theorem on periodic solutions:

<u>Theorem 5.2</u> Let $M\subset\Omega$ be a compact set with the fixed point property and flow-invariant for (5.3) , where $g:[0,p]\times\Omega \to \mathbb{R}^m$ is continuous and such that, for each $z\in M$, there exists one and only one solution , $x(.;z)$, of (5.3)-(5.4), on $[0,p]$.

Then, there is a p-periodic solution , $y(.)$ of (5.3) , with $y(t)\in M$, for every $t\in[0,p]$.

Proof: We observe that , as M is a compact set which is flow-invariant, solutions, $x(.)$, of (5.3) with initial value $x(0)\in M$, exist in all $[0,p]$. Therefore,

:he translation operator $T: M \to \Omega$, $z \to x(p;z)$ applies M into M . By proposition 5.1, T is
ı continuous map . Then, as M has the fixed point property, there exists a fixed
›oint $z_0 \in M$, for T. Therefore, $x(.;z_0)$ is a solution of (5.3) with $x(0;z_0) = z_0 =$
$= T(z_0) = x(p;z_0)$. And this solution remains in M for all $t \in [0,p]$, as M is flow-
-invariant .

Take, then, $y(.) = x(.;z_0)$.

□

An application of theorem 5.2 is the following existence theorem, for compact
nd convex sets:

heorem 5.3 Let $M \subset \Omega$ be a compact and convex set, with non empty interior .
ssume

5.5) for each $u \in fr\, M$, there exists n_u an outward normal to M at u,
such that $(f(t,u) \mid n_u) \leqslant 0$ (resp. $\geqslant 0$) for all $t \in [0,p]$.

Then, (5.1) has at least one p-periodic solution, x(.), with $x(t) \in M$, for eve-
y $t \in [0,p]$.

roof: As we deal with a convex set , we can assume $|n_u| = 1$.
et us consider first that $(f(t,u) \mid n_u) \leqslant 0$.
) Suppose f is C^1 . Then, as we have uniqueness for Cauchy problems, applying theo-
em 2.11, M is flow-invariant for (5.1). As M has the fixed point property, apply,
ıen, theorem 5.2.
) For the general case, with f only continuous, let for $k \in \mathbb{N}$, $f_k: [0,p] \times \Omega \to \mathbb{R}^m$
e such that f_k converges uniformly to f on compact subsets of $[0,p]$ and f_k is C^1 .

For each $k \in \mathbb{N}$, let $\tilde{f}, \tilde{f}_k: \mathbb{R} \times \Omega \to \mathbb{R}^m$ be defined by

$$\tilde{f}_k(t,x) = \begin{cases} f_k(0,x) & , t < 0 \\ f_k(t,x) & , 0 \leqslant t \leqslant p \\ f_k(p,x) & , t > p \end{cases} \quad \text{and} \quad \tilde{f}(t,x) = \begin{cases} f(0,x) & , t < 0 \\ f(t,x) & , 0 \leqslant t \leqslant p \\ f(p,x) & , t > p . \end{cases}$$

Choose $x_0 \in int\, M$. By lemma 2.10 , let $\varepsilon > 0$ be such that $((u-x_0) \mid n_u) \geqslant \varepsilon$,for
/ery $u \in fr\, M$.

Take , for each $k,n\in\mathbb{N}$, $g_{k,n}:\mathbb{R}\times\Omega \to \mathbb{R}^m$ defined by $g_{k,n}(t,x):=$
$= \widetilde{f}_k(t,x)+\frac{1}{n}(x_0-x)$. $g_{k,n}$ is a C^1 function and, for every $t\in\mathbb{R}$ and $u\in \mathrm{fr}\, M$,
$(g_{k,n}(t,u)|n_u) = (\widetilde{f}_k(t,u)|n_u)+\frac{1}{n}((x_0-u)|n_u) \leqslant (\widetilde{f}_k(t,u)|n_u)-\frac{\varepsilon}{n}$.

For each $n\in\mathbb{N}$, choose $j_n\in\mathbb{N}$ large enough , so that j_n is strictly increasing with n and $\frac{1}{j_n}<\frac{\varepsilon}{n}$. And, as f_k converges uniformly to f on the compact set $[0,p]\times \mathrm{fr}\, M$, choose $k_n\in\mathbb{N}$ strictly increasing with n and such that
$\sup_{\substack{t\in[0,p]\\ u\in \mathrm{fr}\, M}} |f_{k_n}(t,u)-f(t,u)| < \frac{1}{j_n}$.

Then, for $t\in[0,p]$ and $u\in \mathrm{fr}\, M$, using (5.5)
$(g_{k,n}(t,u)|n_u) \leqslant ((f_{k_n}(t,u)-f(t,u))|n_u) + (f(t,u)|n_u) - \frac{\varepsilon}{n} \leqslant \frac{1}{j_n} - \frac{\varepsilon}{n} < 0$.

By case 1) , there exists $x_n(.)$, a p-periodic solution of $\dot{x} = g_{k_n,n}(t,x)$ in $[0,p]$, with $x_n(t)\in M$, for every $t\in[0,p]$.

As M is a compact set and $x_n(0)\in M$, for every $n\in\mathbb{N}$, suppose that $x_n(0)$ converges to $x_0\in M$. As $g_{k_n,n}$ converges uniformly to $\widetilde{f}$, on compact subsets of $\mathbb{R}\times\Omega$, applying theorem 1.2 , there exists a non trivial solution $x(.)$ of $\dot{x}=\widetilde{f}(t,x)$ with $x(0) = x_0$, and , if $t\in[0,p]$ belongs to the maximal interval of existence of $x(.)$, there exists $x_{n_r}(.)$, a subsequence of of $x_n(.)$, such that $x_{n_r}(t)$ converges to $x(t)$. So that, $x(t)\in M$, as $x_{n_r}(t)\in M$, for every $r\in\mathbb{N}$ .As , for every $t\in[0,p]$ in the maximal interval of existence of $x(.)$, $x(t)\in M$, which is a compact set , $x(.)$ is defined in all $[0,p]$. Being $[0,p]$ a compact subinterval of the maximal interval of existence of $x(.)$, there exists a subsequence $x_{m_s}(.)$, of $x_n(.)$, converging uniformly , on $[0,p]$, to $x(.)$. So that $x_{m_s}(0) =$
$x_{m_s}(p) \to x(0) = x(p)$.

Then, $x(.)$ is a p-periodic solution of $\dot{x} = f(t,x)$, on $[0,p]$, with values in M .

Consider,now,that $(f(t,u)|n_u) \geqslant 0$, for $t\in[0,p]$ and $u\in \mathrm{fr}\, M$.

Define $g: [0,p]\times\Omega \to \mathbb{R}^m$ by $g(t,x) = -f(p-t,x)$. g satisfies (5.5) , as $(g(t,u)|n_u)\leqslant 0$ for all $t\in[0,p]$ and $u\in \mathrm{fr}\, M$.

So that , applying what just proved , there exists $y(.)$, a p-periodic solution of $\dot{x} = g(t,x)$ with $y(t) \in M$, for every $t \in [0,p]$. But then , $x(t) = y(p-t)$ is a p-periodic solution of (5.1) with $x(t) \in M$, for $t \in [0,p]$.

□

orollary 5.4 Let $f: [0,p] \times \prod_{i=1}^{m} [a_i,b_i] \to \mathbb{R}^m$ be a continuous function , where or $i \in \{1,..,m\}$ $a_i < b_i$. Assume

5.6) for each $i \in \{1,..,m\}$, $t \in [0,p]$ and $x_j \in [a_j,b_j]$ for $j \in \{1,..,i-1,i+1,..,m\}$

$$f_i(t,x_1,..,x_{i-1},a_i,x_{i+1},..,x_m) \geqslant 0 \geqslant f_i(t,x_1,..,x_{i-1},b_i,x_{i+1},..,x_m)$$

where f_i is the i^{th} component of f .

Then, there exists,at least , one p-periodic solution , x(.) , for (5.1) ith $a_i \leqslant x_i(t) \leqslant b_i$, for $i \in \{1,..,m\}$ and $t \in [0,p]$.

roof: Take in theorem 5.3 , $M = \prod_{i=1}^{m} [a_i,b_i]$ and $\Omega = \mathbb{R}^m$, as (5.6) implies (5.5)

□

We can apply successfully this corollary to obtain the existence of a positie and p-periodic solution for the competing two species model of Lotka-Volterra:

$$\text{5.7)} \quad \begin{cases} \dot{x} = x(a(t)-b(t)x-c(t)y) \\ \dot{y} = y(d(t)-e(t)x-f(t)y) \end{cases}$$

ith continuous, positive and p-periodic coeficients a,b,c,d,e and f , defined n $\mathbb{R}$, provided that $b_L > e_M$, $f_L > c_M$, $a_L > c_M d_M / f_L$ and $d_L > a_M e_M / b_L$, where or a continuous function $g: [0,p] \to \mathbb{R}$ we denote

$$g_L = \min_{t \in [0,p]} g(t) \qquad \text{and} \qquad g_M = \max_{t \in [0,p]} g(t) .$$

To prove this result with corollary 5.4 , we must find a rectangle $[A_1,B_1] \times [A_2,B_2]$, for which (5.6) is verified , with $A_1,A_2 > 0$. To obtain it, e proceed like in [1] and [17] , getting first estimations for the solution . n fact , we prove that , if (x(.),y(.)) is a p-periodic and positive soluion of (5.7) , then

$$\frac{a_L f_L - c_M d_M}{b_M f_L - c_M e_L} \leqslant x(t) \leqslant \frac{a_M f_M - c_L d_L}{b_L f_M - c_L e_M}$$

(5.8) for $t \in [0,p]$

$$\frac{b_L d_L - a_M e_M}{b_L f_M - c_L e_M} \leqslant y(t) \leqslant \frac{b_M d_M - a_L e_L}{b_M f_L - c_M e_L}$$

Observe that by hypothesis $a_L f_L > c_M d_M$, $b_L d_L > a_M e_M$, $d_L > a_M e_M / b_L \geqslant a_L e_M / b_M >$ $> c_M d_M e_M / b_M f_L \geqslant d_L c_M e_L / b_M f_L$, so that $b_M f_L > c_M e_L$, and , in an analogous way , $b_L f_M > c_L e_M$. So, the lower bounds in (5.8) are really positive.

Let $t_1, t_2 \in [0,p]$ be such that $x(t_1) = x_M$ and $y(t_2) = y_L$.

We have $0 = \dot{x}(t_1) = x(t_1)\ [a(t_1) - b(t_1)x_M - c(t_1)y(t_1)]$. As $x(t_1) > 0$, $b_L x_M \leqslant b(t_1)x_M = a(t_1) - c(t_1)y(t_1) \leqslant a_M - c_L y_L$. In an analogous way , using that $0 = \dot{y}(t_2)$, $f_M y_L \geqslant d_L - e_M x_M$. Combining these two results , one has $b_L x_M \leqslant a_M - c_L y_L \leqslant a_M - c_L \dfrac{d_L - e_M x_M}{f_M}$ which implies $x_M \leqslant \dfrac{a_M f_M - c_L d_L}{b_L f_M - c_L e_M}$ and, similarly

$$y_L \geqslant \frac{b_L d_L - a_M e_M}{b_L f_M - c_L e_M} .$$

In a similar way , for x_L and y_M we obtain

$$x_L \geqslant \frac{a_L f_L - c_M d_M}{b_M f_L - c_M e_L} \qquad \text{and} \qquad y_M \leqslant \frac{b_M d_M - a_L e_L}{b_M f_L - c_M e_L} .$$

So , we have (5.8),as $x_L \leqslant x(t) \leqslant x_M$ and $y_L \leqslant y(t) \leqslant y_M$, for $t \in [0,p]$.

Call

$$A_1 := \frac{a_L f_L - c_M d_M}{b_M f_L - c_M e_L} \qquad B_1 := \frac{a_M f_M - c_L d_L}{b_L f_M - c_L e_M}$$

$$A_2 := \frac{b_L d_L - a_M e_M}{b_L f_M - c_L e_M} \qquad B_2 := \frac{b_M d_M - a_L e_L}{b_M f_L - c_M e_L}$$

Then , for $y \in [A_2, B_2]$ and $t \in [0,p]$, we have

$$f_1(t,A_1,y) = A_1(a(t)-b(t)A_1-c(t)y) \geq A_1(a_L-b_M A_1-c_M B_2) = 0$$

$$f_1(t,B_1,y) = B_1(a(t)-b(t)B_1-c(t)y) \leq B_1(a_M-b_L B_1-c_L A_2) = 0$$

and for $x \in [A_1,B_1]$ and $t \in [0,p]$,

$$f_2(t,x,A_2) = A_2(d(t)-e(t)x-f(t)A_2) \geq A_2(d_L-e_M B_1-f_M A_2) = 0$$

$$f_2(t,x,B_2) = B_2(d(t)-e(t)x-f(t)B_2) \leq B_2(d_M-e_L A_1-f_L B_2) = 0 .$$

So, (5.6) is verified , as we wanted .

We point out that in [1] and [17] , the existence of such a solution is proved with topological degree tools , as those also permit , in this case, to guarantee uniqueness and stability for the solution we want .

Another consequence of theorem 5.2 is the following existence theorem on the cone $\mathbb{R}^m_+$.

For $x = (x_1,\dots,x_m) \in \mathbb{R}^m$, we note $x \geq 0$ if $x_i \geq 0$, for every $i \in \{1,\dots,m\}$.

Theorem 5.5 Let $f:[0,p]\times\mathbb{R}^m \to \mathbb{R}^m$ be a continuous map . Suppose there exist $r,R \in \mathbb{R}$, with $0 < r < R$, and such that , for $x \geq 0$, one has

i) $|x| = r \Rightarrow (f(t,x) \mid x) \geq 0$, for $t \in [0,p]$

ii) $|x| = R \Rightarrow (f(t,x) \mid x) \leq 0$, for $t \in [0,p]$

iii) $r \leq |x| \leq R$ and $x \not> 0 \Rightarrow \exists i \in \{1,\dots,m\}$: $x_i = 0$ and $\forall t \in [0,p]$ $f_i(t,x) \geq 0$.

Then , there exists at least one p-periodic solution $x(.)$ of (5.1) , with $x(t) \geq 0$ and $r \leq |x(t)| \leq R$, for every $t \in [0,p]$.

Moreover , if (i) (resp. (ii)) has a strict inequality , $|x(t)| < R$ (resp. $|x(t)| > r$) , for every $t \in [0,p]$.

Proof: We will apply corollary 2.8 to get invariance for an approximating problem.

Define $q: \mathbb{R}_+ \to \mathbb{R}$ by $q(\rho) := R\sqrt{m} - \dfrac{R\sqrt{m} - (r/\sqrt{m})}{R - r}(\rho - r)$.

q is a decreasing function with $q(r) = R\sqrt{m}$ and $q(R) = r/\sqrt{m}$.

Take $P(x) := q(|x|)\,(1,\dots,1)$, for $x \in \mathbb{R}^m$.

As f is continuous , let for $k \in \mathbb{N}$, $f_k : [0,p]\times\mathbb{R}^m \to \mathbb{R}^m$ be C^1 functions such that f_k converges uniformly on compact subsets of $[0,p]\times\mathbb{R}^m$, to f .

Define $\tilde{f}$ and $\tilde{f}_k$ as in the proof of theorem 5.3 . And take ,for $k,n\in\mathbb{N}$, $g_{k,n}:\mathbb{R}\times\mathbb{R}^m\to\mathbb{R}^m$ defined by $g_{k,n}(t,x):=\tilde{f}_k(t,x)+\frac{1}{n}(P(x)-x)$.

Set $G:=\{x\in\mathbb{R}^m \mid r<|x|<R \text{ and } x>0\}$. As f_k converges uniformly to f on the compact set $[0,p]\times \text{fr } G$, choose , for each $n\in\mathbb{N}$, k_n strictly increasing with n and such that $\sup_{\substack{t\in[0,p]\\ u\in \text{fr } G}} |f_{k_n}(t,x)-f(t,x)| < \min\left(\frac{R-r}{2n}, \frac{r}{n\sqrt{m}}\right)$.

We verify condition (2.7) , in corollary 2.8 , for each $g_{k_n,n}$.

If $u\in\text{fr } G$ is such that $|u|=r$, $n_u=-u/r$ is an outer normal to G at u . Then , using (i) , for $t\in[0,p]$, $(g_{k_n,n}(t,u)|n_u)=(f_{k_n}(t,u)|n_u)+\frac{1}{n}((P(u)-u)|n_u)=$

$$=((f(t,u)-f_{k_n}(t,u))|\frac{u}{r})-(f(t,u)|\frac{u}{r})-\frac{1}{n}\left[(P(u)|\frac{u}{r})-\frac{(u|u)}{r}\right]<\frac{R-r}{2n}-\frac{1}{n}\left(\frac{q(|u|)}{r}\left(\sum_{i=1}^m u_i\right)-r\right)$$

$$\leqslant\frac{R-r}{2n}-\frac{1}{n}\left(\frac{q(|u|)}{r}\frac{r}{\sqrt{m}}-r\right)=-\frac{R-r}{2n}<0 .$$

If $u\in\text{fr } G$ is such that $|u|=R$, $n_u=u/R$ is an outer normal to G at u .

Using (ii) , for $t\in[0,p]$, $(g_{k_n,n}(t,u)|n_u)\leqslant\frac{R-r}{2n}+(f(t,u)|\frac{u}{r})+((P(u)-u)|\frac{u}{r})\leqslant$

$$\leqslant\frac{R-r}{2n}+\frac{1}{n}\left[\frac{q(|u|)}{R}\left(\sum_{i=1}^m u_i\right)-R\right]\leqslant\frac{R-r}{2n}+\frac{1}{n}\left(\frac{q(|u|)}{R}\sqrt{m}\,R-R\right)=-\frac{R-r}{2n}<0 .$$

Finally, if $u\in\text{fr } G$ is such that $r<|u|<R$ and $t\in[0,p]$, by (iii) , let $j\in\{1,..m\}$ be such that $u_j=0$ and $f_j(t,u)\geqslant 0$. $n_u=-e_j$, the unitary vector with component j equal to -1 , is an outer normal to G at u . And , $(g_{k_n,n}(t,u)|n_u)<$

$$<\frac{r}{n\sqrt{m}}-(f(t,u)|e_j)-\frac{1}{n}((P(u)-u)|e_j)\leqslant\frac{r}{n\sqrt{m}}-f_j(t,u)-\frac{1}{n}(q(|u|)-u_j)\leqslant 0 .$$

Applying corollary 2.8 , for $J=[0,p]$ and $\Omega=\mathbb{R}^m$, cl G is weakly flow-invariant for $\dot{x}=g_{k_n,n}(t,x)$. But , as $g_{k_n,n}$ is a C^1 function, cl G is flow-invariant for $\dot{x}=g_{k_n}(t,x)$.

Then , by theorem 5.2 ,there exists a p-periodic solution $x_n(.)$, of $\dot{x}=g_{k_n}(t,x)$, in $[0,p]$, with $x_n(t)\in\text{cl } G$ for every $t\in[0,p]$.

Using theorem 1.2 , as $g_{k_n,n}$ converges uniformly on compact subsets of $\mathbb{R}\times\Omega$, to $\tilde{f}$, as in the final part of the proof of theorem 5.3 , we get the solution we want.

Suppose that (i) satisfies a srict inequality . By contradiction , assume there exists $s \in [0,p]$ such that $|x(s)| = r$. Then , $\frac{d}{dt} |x(t)|^2 \Big|_{t=s} = 2(x(s)|\ f(s,x(s))> 0$, by (i) . But , if $s \in (0,p)$, $\frac{d}{dt} |x(t)|^2 \Big|_{t=s} = 0$. And ,if $s \notin (0,p)$, as $x(0) = x(p)$, we should have $\frac{d}{dt} |x(t)|^2 \Big|_{t=p} \leqslant 0$, as $|x(t)| \geqslant r$ for all $t \in [0,p]$. In any case,we have a contradiction.

In an analogous way , if (ii) has a strict inequality , one can prove that $|x(t)| < R$, for every $t \in [0,p]$.

□

The following corollary contains , for $\alpha=0$, a generalization of a Santanilla's result (theorem 4.1 in [47]):

Corollary 5.6 Let $f: [0,p] \times \mathbb{R}^m \to \mathbb{R}^m$ be a continuous map. Assume that there exist $r, R \in \mathbb{R}$, with $0 < r < R$ and a constant $\alpha \geqslant 0$ such that , for every $t \in [0,p]$ and $x \geqslant 0$, one has

i) $|x| = r \Rightarrow f(t,x) \geqslant 0$

ii) $|x| = R \Rightarrow (f(t,x)|\ x) \leqslant 0$

iii) $r \leqslant |x| \leqslant R \Rightarrow f(t,x) \geqslant -\alpha x$.

Then , there exist at least one p-periodic solution , $x(.)$, of (5.1) ,with $x(t) \geqslant 0$ and $r \leqslant |x(t)| \leqslant R$, for every $t \in [0,p]$.

Moreover, if (i) (resp. (ii))has a strict inequality , $|x(t)| > r$ (resp. $|x(t)| < R$) , for every $t \in [0,p]$.

Proof: As $x \geqslant 0$, (i) implies that $(f(t,x)|\ x) \geqslant 0$, for every $t \in [0,p]$,if $|x| = r$. If $r < |x| < R$ and $x \not\gg 0$, but $x \geqslant 0$, choose by (iii) , $i \in \{1,..,m\}$ such that $x_i = 0$ and $f_i(t,x) - \alpha x_i = 0$, for all $t \in [0,p]$. So, all conditions of theorem 5.5 are satisfied.

□

We can get an analogous , of theorem 5.5 , reversing inequalities in conditions (i) and (ii) . For such a proof we use , not only invariance theorems , but also the fixed-point index theory in $\mathbb{R}^m$.

To present the fixed-point index ,as given in [27] , let us see some definitinitions:

Definition 5.7 Let $f:X\to Y$ be a map between the topological spaces X and Y . f is called a compact map if f is continuous and cl f(X) is a compact subset of Y .

Definition 5.8 If U is an open set of a topological space X and $f:U\to X$ is a continuous map , we'll say that f is admissible if $\{x\in U \mid x=f(x)\}$, the set of fixed points of f , is a compact set .

Definition 5.9 If U is an open subset of the topological space X and $H: [0,p]\times U\subset \mathbb{R}\times U \to X$ is a continuous map , H is said to be an admissible homotopy if $\{x\in U \mid \exists t\in[0,1] : H(t,x) = x\}$ is a compact set .

Definition 5.10 A metric space X is said to be an ANR (absolute neighbourhood retract) if for each metric space Y , each closed subset C of Y and each continuous map $f:C\to X$, there exist an open subset A of Y , with $A\supset C$, and $f:A\to X$ a continuous extension of f to A .

We observe that a closed convex subset of a Banach space is an ANR ([31]) .

With an existence theorem,we give an axiomatic definition for the fixed-point index:

Theorem 5.11 To each (X,U,f) , with X an ANR , U an open subset of X and $f:U\to X$ a compact and admissible map , we can associate an integernumber , $i_X(f,U)$, called the fixed-point index of f respect to U , satisfying the following axioms, where $X_f:=\{x\in U \mid x=f(x)\}$:

1) Excision:

If U' is an open subset of U such that $X_f\subset U'$, then the restriction of f to U' , $f_{|U'}: U'\to X$, is a compact and admissible map , with

$$i_X(f_{|U'},U') = i_X(f,U) \quad .$$

2) Additivity:

If $U = \bigcup_{j=1}^{k} U_j$, with U_j open subset of X , and $X_{f_{|U_j}}$ are mutually disjoint,

$$i_X(f,U) = \sum_{j=1}^{k} i_X(f_{|U_j}, U_j) \quad .$$

3) Existence of fixed-points:

If $i_X(f,U) \neq 0$, then $\chi_f \neq \phi$, that is f has a fixed point in U .

4) Homotopy:

Let $H: [0,1] \times U \to X$ be an admissible homotopy and a compact map . Then ,

$$i_X(H(0,.),U) = i_X(H(1,.),U) \quad .$$

5) Multiplicativity:

If $f_1: U_1 \to X_1$ and $f_2: U_2 \to X_2$ are compact and admissible maps , then so is the product map $f_1 \times f_2: U_1 \times U_2 \to X_1 \times X_2$, $(x_1,x_2) \to (f(x_1),f(x_2))$ and

$$i_{X_1 \times X_2}(f_1 \times f_2, U_1 \times U_2) = i_{X_1}(f_1,U_1) \cdot i_{X_2}(f_2,U_2)$$

as $X_1 \times X_2$ is an ANR for the product topology .

6) Commutativity:

Let $U \subset X$ and $U' \subset X'$ be open subsets of X and X' and $f: U \to X'$ and $g: U' \to X$ be continuous maps . If one of the maps

$$g \circ f: V = f^{-1}(U') \to X$$

$$f \circ g: V' = g^{-1}(U) \to X'$$

is a compact and admissible map , then so is the other one and

$$i_X(g \circ f, V) = i_{X'}(f \circ g, V') \quad .$$

7) Normalization: (see [27])

If U=X and $f: X \to X$ is compact and admissible , then

(*) f is a Lefschetz map

(**) $i_X(f,X) = \Lambda(f)$, the Lefschetz number of f .

An easy consequence of 3) and 7) is the following one:

Proposition 5.12 Let U be an open set of the ANR X and $f: U \to X$ be a constant map , $f(x)=x_0$. Then , f is a compact and admissible map and

$$i_X(f,U) = \begin{cases} 1 & , x_0 \in U \\ 0 & , x_0 \notin U \end{cases} \quad .$$

Remark: We will apply fixed point index with : X a closed and convex set of $\mathbb{R}^m$; U a non empty, open and bounded set in X ; $f: \mathrm{cl}\, U \to X$ continuous and such that $f(x) \neq x$, for every $x \in \mathrm{fr}_X U$; and $H: [0,1] \times \mathrm{cl}\, U \to X$ continuous and such that $H(t,x) \neq x$, for every $t \in [0,1]$ and $x \in \mathrm{fr}_X U$. In fact , in such conditions both f and H are compact maps, as they are continuous on their compact domains . If, for every $n \in \mathbb{N}$, $x_n \in U$ and $x_n = f(x_n)$, being $x_n \in \mathrm{cl}\, U$, which is compact , there are $x_0 \in \mathrm{cl}\, U$ and x_{n_k} , subsequence of x_n , such that $x_{n_k} \to x_0$. So, by continuity of f, on $\mathrm{cl}\, U$, $x_{n_k} = f(x_{n_k}) \to x_0 = f(x_0)$. As, by assumption , $x \neq f(x)$ for $x \in \mathrm{fr}_X U$, it must be $x_0 \in U$. So $f_{|U}$ is a compact and admissible map . In an analogous way , if, for $n \in \mathbb{N}$, $x_n \in U$ is such that there exists $t_n \in [0,1]$ with $x_n = H(t_n, x_n)$, there will be $t_0 \in [0,1]$, $x_0 \in \mathrm{cl}\, U$, t_{n_k} and x_{n_k} subsequences of t_n and x_n , with $t_{n_k} \to t_0$ and $x_{n_k} \to x_0$. So, $x_0 = H(t_0, x_0)$ and it must be $x_0 \in U$. And then , $H_{|[0,1] \times U}$ is a compact and admissible homotopy .

Lemma 5.13 Let $f: [0,p] \times \mathbb{R}^m \to \mathbb{R}^m$ be a C^1 map . Assume that there exist $r, R \in \mathbb{R}$ with $0 < r < R$ and such that , for every $t \in [0,p]$ and $x \geqslant 0$, one has:

(i) $|x| = r \implies (f(t,x) \mid x) < 0$

(ii) $|x| = R \implies (f(t,x) \mid x) > 0$

(iii) $r < |x| < R$ and $x_i = 0 \implies f_i(t,x) > 0$, for all $i \in \{1, \ldots, m\}$.

Then there exists at least one p-periodic solution $x(.)$, of (5.1) , with $x(t) \geqslant 0$ and $r < |x(t)| < R$, for every $t \in [0,p]$.

Proof: Let $f^*: [0,p] \times \mathbb{R}^m \to \mathbb{R}^m$ be defined by

$$f^*(t,x) = \begin{cases} f(t, Rx/|x|) & , \ |x| > R \\ f(t,x) & , \ r \leqslant |x| \leqslant R \\ \dfrac{|x|^4}{r^4} f(t, rx/|x|) & , \ 0 < |x| < r \\ 0 & , \ x = 0 \end{cases}$$

As f is a C^1 function, f^* is C^1 in a small neighbourhood of the origin. For all $x,y \in \mathbb{R}^m \setminus \{0\}$, we have $\left|\frac{x}{|x|} - \frac{y}{|y|}\right| \leq \frac{2}{|x|}|x-y|$. So, one can easily conclude that f^* is locally lipschitzean in all $[0,p] \times \mathbb{R}^m$. Extending f^* to $\mathbb{R} \times \mathbb{R}^m$, with $\tilde{f}(t,x) := f^*(0,x)$, if $t<0$, and $\tilde{f}(t,x) := f^*(p,x)$, if $t>p$, the map $\tilde{f}: \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$ is still locally lipschitzean. So, we have uniqueness for solutions of Cauchy problems of $\dot{x} = \tilde{f}(t,x)$.

If we find a solution, $y(.)$, of $\dot{x} = \tilde{f}(t,x)$, with $y(t) \geq 0$ and $r < |y(t)| < R$, for all $t \in [0,p]$, $y(.)$ is also a solution of (5.1), as $\tilde{f}(t,x) = f(t,x)$ for $t \in [0,p]$ and $r < |x| < R$. And we are done.

Because of the definition of $\tilde{f}$, $\tilde{f}$ also satisfies conditions (i) and (ii), for all $t \in \mathbb{R}$ and $x \geq 0$. And, for $i \in \{1,..m\}$ and $x \in \mathbb{R}^m \setminus \{0\}$, with $x_i = 0$, $\tilde{f}_i(t,x) > 0$, if $t \in \mathbb{R}$. This last property implies with theorem 2.11, taking $\Omega = \mathbb{R}^m$ and $J = \mathbb{R}$, the flow-invariance of $\mathbb{R}^m_+$ for $\dot{x} = \tilde{f}(t,x)$.

As $\tilde{f}$ is bounded, solutions of $\dot{x} = \tilde{f}(t,x)$ exist in all $[0,p]$.

So, the translation operator $T: \begin{matrix} \mathbb{R}^m \to \mathbb{R}^m \\ z \to x(p;z) \end{matrix}$ applies $\mathbb{R}^m_+$ into $\mathbb{R}^m_+$ and is a continuous map.

Take $G_1 := \{x \in \mathbb{R}^m_+ \mid |x| \leq r\}$ and $G_2 := \{x \in \text{int}\, \mathbb{R}^m_+ \mid |x| > R\}$.

By theorem 2.11, G_1 is flow-invariant for $\dot{x} = \tilde{f}(t,x)$. And, by corollary 2.7, cl G_2 is also flow-invariant for $\dot{x} = \tilde{f}(t,x)$.

Take $K := \mathbb{R}^m_+$, which is a closed convex cone in $\mathbb{R}^m$. Then, K is an ANR.

Set $U_1 := K \cap \{x \in \mathbb{R}^m \mid |x| < r\}$ and $U_2 := K \cap \{x \in \mathbb{R}^m \mid |x| < R\}$.

U_1 and U_2 are non empty open and bounded sets in K. If we prove that $i_K(T,U_1) = 1$ and $i_K(T,U_2) = 0$, by additivity property of fixed-point index, $i_K(T, U_2 \setminus \text{cl } U_1) = -1 \neq 0$. Then, there exists $x_0 \in U_2 \setminus \text{cl } U_1$, with $x_0 = T(x_0)$. So, $y(.) = x(.;x_0)$ is a solution of $\dot{x} = \tilde{f}(t,x)$ in $[0,p]$, as $x_0 \in \mathbb{R}^m_+$, and $r < |x_0| < R$. As G_1 and cl G_2 are flow-invariant, for $\dot{x} = \tilde{f}(t,x)$, and $y(0) = y(p)$, it must be $r < |y(t)| < R$, for all $t \in [0,p]$. And we got the solution we wanted.

To see that $i_K(T,U_1) = 1$, define $H: \begin{matrix} [0,1] \times \text{cl } U_1 \to \mathbb{R}^m \\ (\lambda, x) \longrightarrow \lambda T(x) \end{matrix}$. $H_{|U}$ is a compact map and if it is an admissible homotopy, by homotopy property $i_K(H(0,.),U_1) = i_K(H(1,.),U_1)$. By proposition 5.12, $1 = i_K(0,U_1) = i_K(T,U_1)$, as $0 \in U_1$. According to the previous remark, we just have to see that $H(t,x) \neq x$, for all $t \in [0,p]$ and

$x \in fr_K U_1 = (fr\ U_1) \setminus U_1$. By contradiction , suppose there exists $\lambda_0 \in [0,1]$ and $x_* \in (fr\ U_1) \setminus U_1$ such that $x_* = \lambda_0 T(x_*)$. Then , $|x_*| = r$. Therefore , $\lambda_0 \neq 0$, as $x_* = \lambda_0 T(x_*)$. So , $|T(x_*)| = |x_*|/\lambda_0 \geqslant r$, as $\lambda_0\ (0,1]$. But , as G_1 is flow-invariant, $0 \leqslant \frac{d}{dt}|x(t;x_*)|^2\Big|_{t=p} = 2(x_*|\ f(p,x_*)) < 0$, as $|x_*| = r$, which is a contradiction .

To see that $i_K(T,U_2)=0$, define $F: [0,1] \times cl\ U_2 \to \mathbb{R}^m$, $(\lambda,x) \longrightarrow T(x)+\lambda_1 \lambda P$, where $P=(1,..,1)$ and $\lambda_1 > \frac{R+\mu}{|P|}$, with $\mu = \sup_{|x| \leqslant R} |T(x)|$.

$F_{|[0,1] \times U_2}$ is a compact map and , if it is an admissible homotopy , by homotopy property , $i_K(F(0,.),U_2) = i_K(F(1,.),U_2)$, that is $i_K(T,U_2) = i_K(T+\lambda_1 P,U_2)$. If $i_K(T+\lambda_1 P,U_2) \neq 0$, there would be $x^* \in U_2$ such that $x^* = T(x^*)+\lambda_1 P$. And then , $R > |x^*| = |\lambda_1 P+T(x^*)| \geqslant \lambda_1\ |P| - |T(x^*)| \geqslant \lambda_1 |P| - \mu$. So that $\lambda_1 \leqslant \frac{R+\mu}{|P|}$, a contradiction by definition of λ_1 . By the previous remark , to prove that F is an admissible homotopy , it is enough to verify that , if $x \in fr_K U_2$ we cannot have $x = F(\lambda,x)$, for some $\lambda \in [0,1]$. In fact , suppose ,by contradiction , that there exists $y^* \in fr_K U_2 = (fr\ U_2) \setminus U_2$ and $\lambda^* \in [0,1]$ such that $y^* = F(\lambda^*,y^*)$. So , $|y^*| = R$ and $y^* = T(y^*)+\lambda_1 \lambda^* P$. As $cl\ G_2$ is flow-invariant for $\dot{x} = \tilde{f}(t,x)$, $|T(y^*)| \geqslant R = |y^*|$. So, there exists $j \in \{1,..,m\}$ such that $0 \leqslant y_j^* \leqslant (T(y^*))_j$ .But then , $0 \geqslant y_j^* - (T(y^*))_j = \lambda_1 \lambda^* P_j = \lambda_1 \lambda^*$. So that , $\lambda^* = 0$, which implies $y^* = T(y^*)$. As $cl\ G_2$ is flow-invariant for $\dot{x} = \tilde{f}(t,x)$, we have $0 \geqslant \frac{d}{dt}|x(t;y^*)|^2\Big|_{t=p} = 2(T(y^*)|\ f(p,T(y^*))) > 0$, as $|T(y^*)| = |y^*| = R$, a contradiction .

□

Condition (iii) in lemma 5.13 can be weakened . So , the following theorem is a generalization for a result of Gaines and Santanilla (theorem 3.1 in [23]) :

<u>Theorem 5.14</u> Let $f: [0,p] \times \mathbb{R}^m \to \mathbb{R}^m$ be a continuous map . Suppose that there exist $r,R \in \mathbb{R}$, with $0 < r < R$, such that , for every $t \in [0,p]$ and $x \geqslant 0$, one has:

i) $|x| = r \Rightarrow (f(t,x) \mid x) \leqslant 0$

ii) $|x| = R \Rightarrow (f(t,x) \mid x) \geqslant 0$

iii) $r \leqslant |x| \leqslant R$ and $x_i = 0 \Rightarrow f_i(t,x) \geqslant 0$, for all $i \in \{1,..,m\}$.

Then , there exists at least one p-periodic solution of $\dot{x} = f(t,x)$, with $x(t) \geqslant 0$ nd $r \leqslant |x(t)| \leqslant R$, for all $t \in [0,p]$.

Moreover , if (i) (resp. (ii)) has a srict inequality , $r < |x(t)|$ (resp. $|x(t)| > R$, or all $t \in [0,p]$.

roof: Take , for $k \in \mathbb{N}$, $f_k : [0,p] \times \mathbb{R}^m \to \mathbb{R}^m$ C^1 maps such that f_k converges uniformly o f on compact subsets of $[0,p] \times \mathbb{R}^m$. Let $\tilde{f}_k$ and $\tilde{f}$ be as in the proof of theorem 5.3 .

Define $q: \mathbb{R}_+ \to \mathbb{R}$ by $q(\rho) = \dfrac{r}{2\sqrt{m}} + \dfrac{2R\sqrt{m} - r/(2\sqrt{m})}{R - r}(\rho - r)$.

is an increasing function , with $q(r) = r/(2\sqrt{m})$ and $q(R) = 2R\sqrt{m}$.

Put $P(x) := q(|x|)(1,..,1)$ and , for $k,n \in \mathbb{N}$, take $g_{k,n} : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$ defined by

$$g_{k,n}(t,x) = f_k(t,x) + \frac{1}{n}(P(x) - x) .$$

For each $n \in \mathbb{N}$, take $k_n \in \mathbb{N}$ strictly increasing with n and such that

$$\sup_{\substack{|x| \leqslant R \\ t \in [0,p]}} |f_{k_n}(t,x) - f(t,x)| < \min\left(\frac{r}{2n\sqrt{m}} , \frac{R}{n} \right)$$, as f_k converges uniformly o f on compact subsets of $[0,p] \times \mathbb{R}^m$.

For every $n \in \mathbb{N}$, $g_{k_n,n}\big|_{[0,p] \times \mathbb{R}^m}$ is a C^1 function and for $x \geqslant 0$ and $t \in [0,p]$, e have:

$$|x| = r \Rightarrow (g_{k_n,n}(t,x) \mid x) = ((f_{k_n}(t,x) - f(t,x)) \mid x) + (f(t,x) \mid x) + \frac{1}{n}((P(x) - x) \mid x)$$

$$\frac{r}{2n} \cdot r + \frac{1}{n}\left(\frac{r}{2\sqrt{m}}\sqrt{m}\, r - r^2 \right) = 0$$

$$|x| = R \Rightarrow (g_{k_n,n}(t,x) \mid x) > -\frac{R}{n} \cdot R + \frac{1}{n}\left(2R\sqrt{m}\frac{R}{\sqrt{m}} - R^2 \right) = 0$$

$$\leqslant |x| \leqslant R \ , \ x_i = 0 \Rightarrow (g_{k_n,n})_i(t,x) = (f_{k_n} - f)_i(t,x) + f_i(t,x) + \frac{1}{n}(q(|x|) - x_i) >$$

$$> -\frac{r}{2n\sqrt{m}} + \frac{r}{2n\sqrt{m}} = 0 .$$

Applying lemma 5.13 to each $g_{k_n,n}$, we get $x_n(.)$, a p-periodic solution of $= g_{k_n,n}(t,x)$, in $[0,p]$, with $x_n(t) \geqslant 0$ and $r < |x_n(t)| < R$, for all $t \in [0,p]$.

As $g_{k_n,n}$ converges uniformly to $\tilde{f}$ over compact subsets of $\mathbb{R}\times\mathbb{R}^m$, we have , as in the proof of theorem 5.3 , $x(.)$, a solution of (5.1) in $[0,p]$, with $x(t)\geqslant 0$ and $r\leqslant |x(t)|\leqslant R$, for all $t\in[0,p]$.

Suppose , that (i) verifies a strict inequality . By contradiction , let $s\in[0,p]$ be such that $|x(s)| = r$. Then , $\frac{d}{dt}|x(t)|^2\Big|_{t=s} = 2(x(s)\mid f(s,x(s)))<0$, as $|x(s)| = r$. But, if $s\in(0,p)$, $\frac{d}{dt}|x(t)|^2\Big|_{t=s} = 0$. If $s\notin(0,p)$, $\frac{d}{dt}|x(t)|^2\Big|_{t=p}\leqslant 0$ as $|x(t)|\geqslant r$ for $t\in[0,p]$ and $|x(p)| = r$. In any case , we have a contradiction .

In an analogous way , there is no $s\in[0,p]$ such that $|x(s)| = R$.

$\square$

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