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Quantum Gravity: the Canonical, Ashtekar and Loop formalisms

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Chapter 1

Review of Quantum Gravity ideas

1.1 Introduction

Most conceptual problems in Quantum Gravity (QG) concern the status of our conventional ideas on space, time and matter. Such problems are usually linked to the technical problem of the compatibility of the standard geometrical ideas of General Relativity (GR) and Quantum theory (QT). In fact, the analysis of this compatibility may be the only irrefutable motivation for doing QG [Son90]. Thus, as remarked by C. Isham, the discussion of geometrical and conceptual issues in the same framework is justified. The details of the ideas presented here are given in [Ish91].

The significance of the conceptual problems that stem from QG has no consensus. In constructing the theory we have in our grasp only minimal requirements for it: to reproduce i) classical General Relativity and ii) normal Quantum Theory. These should hold in the appropriate domains. Namely, for distances and times much bigger than the *Planck length* l_P *and time* t_P. Here $l_P = \sqrt{\frac{G\hbar}{c^3}} \sim 10^{-33}cm \sim 10^{28}eV$ and $t_P = \frac{l_P}{c} \sim 10^{-42}sec$, where G is the Newton's constant, $\hbar$ is Planck's constant and c the speed of light in vacuum. On the other hand, near the Planck length scale itself the views vary according to the extent to which the conceptual and structural frameworks of GR and QT are still applicable. There is the conservative view that nothing changes at such a scale and the revolutionary one that suggests a reassessment of the traditional ideas of the spacetime and quantum matter e.g. i) Continuum concepts (Differential Geometry) are inapplicable in this domain and ii) Penrose's proposal [Pen86] that QT becomes non-linear on the Planck length scale in the way needed to explain the problem of the reduction of the state vector of Quantum Mechanics.

The above minimal requirements on a QG theory are however not strong enough to single it out. By using a *covariant quantization* method, particle physicists have shown [Gup54, Fey63, Wei64, Wei65, BD75] that any Lorentz invariant theory of a spin-2 massless quantum field coupled to a conserved energy-momentum tensor will necesarily yield the same low energy scattering results as those obtained from the tree graphs of a weak field perturbative expansion of the Einstein Lagrangian[1].

The existence of this set of "equivalent" theories comes from the fact that no precise a priori information is known about the requirements on the theory. Probably the main consequence is that no axiomatic formulation, Wightman-type or C^*-algebras, exists as opposite to Quantum Field Theory.

The above ideas already show that the aims of a QG theory are not well defined. This observation is supported by the many proposals that have appeared. However, it is through such proposals that a "feeling" can be got about what physicists understand as QG. A rough description of them is given below.

The different proposals can be divided into two groups relying on the feature of gravity that is emphasized: the *field properties* of the gravitational interaction is the viewpoint of particle physicists and the *structure of space-time* as linked to gravity that is the approach of general relativists.

More precisely, the former group uses techniques drawn from conventional, Poincare group based Quantum Field Theory. Here, the key concepts are special relativity and gravitons propagating in a fixed Minkowski spacetime. The minimal expectation is to produce scattering amplitudes for gravitons and other particles that are free of irremovable divergences, i.e. to have a renormalizable theory or, may be, a genuinely finite theory. The great goal is to have this theory as a part of a general Grand Unified Theory (GUT) in which the presence of the gravitational sector is essential.

In the early stages of this approach the expansion $g_{\mu\nu}(x) = \eta_{\mu\nu} + \kappa h_{\mu\nu}(x)$, with $\kappa = \sqrt{16\pi G/c^2}$ and $\eta_{\mu\nu}$ the metric in Minkowski spacetime, was used; then the field $h_{\mu\nu}(x)$ was quantised using the standard techniques drawn from relativistic quantum field theory. The conception of gravitons as the quanta of the gravitational field came about. Background field methods were introduced afterwards, taking instead of the Minkowski spacetime another solution of Einstein equations: $\eta_{\mu\nu} \to g^{(0)}_{\mu\nu}(x)$ as the background and then quantising[2] $h_{\mu\nu}(x)$. More recently, attention has been paid to theo-

[1]It has been shown [Wal86] that there exists a consistent theory of a massless interacting spin-2 field that is not generally covariant, that is, it is not possible to change the dynamical field variable in such a way that the background flat metric disappears from the theory. This implies that the equivalence of the theories mentioned above is not strictly valid.

[2]Note that now we have a Quantum field theory for $h_{\mu\nu}(x)$ in a non-dynamical curved

ries of superstrings and other extended objects where massless spin-2 fields (gravitons) also appear.

Among the most important problems in this approach is the lack of a meaningful causal structure of the theory: there is no reason why it should be the same as that of the background space. Also, it is difficult to handle relevant cosmological issues, for example when studying the influence of QG on the initial spacetime singularity the use of a classical cosmological model (e.g. Robertson-Walker) as the background is not sufficient. This is obvious if one thinks, as is done in this approach, that the spacetime structure is given by the background and gravitons – as defined on this background – are the entities that scatter between them and with the rest of the matter particles.

The general relativists group emphasise the geometrical structure of the theory and the role played by the spacetime structure. This can be considered closer to the essence of GR in which the gravitational field is replaced by the geometry of the spacetime. Quantisation adopting any special background spacetime is not accepted. If such a spacetime has a special role it should emerge naturally as part of the structure of the theory itself and not just put in by hand.

The minimal expectation here is to improve the understanding of the problems posed by spacetime singularities like those associated with black holes and similar situations in the classical theory. In particular, one expects to recover Hawking's results on the quantum production of particles by classical black holes and to extend them to tackle the problems of the back reaction of the created particles on the background spacetime as well as the final state of the evaporating hole. A more ambitious idea is to apply the theory to cosmological issues, especially to study the universe as a quantum entity.

The philosophy used here is one of "back to the basics", by relying on QT rather than on Quantum Field Theory. The main problems are related to i) the quantum status of the spacetime concepts of classical GR and ii) the extent to which conventional QT ideas can be applied. Thus in this approach one is more concerned with the conceptual issues that arises in QG.

As pointed out in [Ish91], it is guaranteed that the uncertainty will be maintained about what one is trying to do in QG until the following question is unswered satisfactorily: *is the central problem of Quantum Gravity one of i) physics, ii) mathematics or iii) philosophy?* Moreover, how severe are the conceptual difficulties? and is it possible that one needs to get to grips with them before any serious technical development can be made? A brief account

background $g^{(0)}_{\mu\nu}(x)$. This is because the back reaction is not taken into account.

of some the problems in QG is given below.

1.2 Basic problems in Quantum Gravity

Here we want to stress only the main difficulties one deals with when QG is investigated. One of the broadest of all the problems is the extent to which a quantum theory of gravity maintains: i) the picture of spacetime as afforded by GR and ii) the interpretative and structural frameworks of conventional QT. It is evident that this is a highly non trivial question and in consequence it is worth mentioning it even if no satisfactory answer has been given so far. Instead, more specific problems involved in the construction of a QG theory are next touched.

1.2.1 Spacetime Diffeomorphism group and the definition of the observables

General relativity equations are covariant with respect to the group Diff($\mathcal{M}$) of diffeomorphisms of the spacetime manifold $\mathcal{M}$. In a sense the role of the diffeomorphisms group in both classical and quantum GR is analogous to that of the gauge group in Yang-Mills theory. For instance, in both cases the "gauge fields" are non dynamical[3]. On the other hand, however, the two groups are quite different. Diffeomorphism group moves spacetime points around whereas the transformations involved in Yang-Mills theory are made at a fixed spacetime point. A conclusion can be arrived at that invariance under Diff($\mathcal{M}$) means that individual mathematical points in $\mathcal{M}$ have no intrinsic physical significance. Certainly this is related to the question of what is an "observable" in GR [Sta87, Rov91c]. As an example let us consider the Riemann scalar curvature $R(x) \equiv g^{\mu\nu}(x)R_{\mu\nu}(x)$. It is a scalar function on $\mathcal{M}$ and its value at any $x \in \mathcal{M}$, hence, can not be regarded as an observable. At the quantum level the same result follows by considering a unitary representation of Diff($\mathcal{M}$). For instance, take $f \in$ Diff($\mathcal{M}$) and $U(f)$ in the chosen unitary representation. The action on the quantised metric of spacetime would lead to the transformation law:

$$U(f)R(x)[U(f)]^{-1} = R(f^{-1}(x)), \qquad (1.1)$$

[3]Roughly, when one extendes the normal derivatives to covariant derivatives in a theory of a free matter field say, in order to get the correspondent invariance, one arrives at a coupling between the matter fields and the gauge fields but does not get a kinetic term for the latter.

provided $R(x)$ can be defined as a proper operator function of the metric operator and its derivatives. Since a physical observable is defined as one that commutes with the action of the gauge group, $R(x)$ turns out not to be one. Alternatives can be tried for generating observables. One is to construct genuine invariants by integrating scalar functions of the metric of spacetime over the entire spacetime, e.g. $\int_{\mathcal{M}} R(x)(g(x))^{\frac{1}{2}} d^4x$; $\int_{\mathcal{M}} R_{\mu\nu}(x) R^{\mu\nu}(x)(g(x))^{\frac{1}{2}} d^4x$. It is worth noting that these are highly non-local and the corresponding quantisation will be very different from any conventional quantum field theory. Another possibility is an old idea about observables in GR. The basic point is that although $R(x)$ is not an observable, $R(X)$ is whenever X is a point on the spacetime manifold occupied by an actual physical particle. That is, we locate ourselves on the spacetime manifold with the aid of a material reference system. This implies to some extent, that simple GR is an incomplete theory since the equations of motion do not involve the reference system, for example, its energy-momentum tensor. Adoption of this approach is related to the so called "physical" coordinates which are used sometimes in the canonical quantisation of gravity, that we will deal with, as well as in the treatment of the problem of time in QG.

The diffeomorphism invariance problem can be seen as arising through the insistence that QG should reflect the diffeomorphism group invariance of classical GR [RS90]. Three related examples make this assertion clear. First, in normal quantum field theory the two-point function for a scalar field ϕ shows the behaviour:

$$W(x,y) \equiv \langle 0|\hat{\phi}(x)\hat{\phi}(y)|0\rangle \sim \frac{const.}{(x-y)^2} \tag{1.2}$$

with x, y approaching each other (the distance is measured with the Minkowski metric of the background). The dependence of $W(x,y)$ on its arguments is a direct consequence of the invariance of the vacuum state $|0\rangle$ under the action of the Poincare group. If we require $|0\rangle$ to be Diff$(\mathcal{M})$-invariant, and if $\hat{\phi}(x)$ transforms as $R(x)$ in (1.1) then

$$W(x,y) \equiv W(f(x), f(y)) \quad \forall\, f \in \text{Diff}(\mathcal{M}). \tag{1.3}$$

However, for any two pairs of points (x,y) and (x',y') which are sufficiently close to each other –that lie in a single coordinate chart e.g.– there exists a diffeomorphism f such that $x' = f(x)$ and $y' = f(y)$. It follows that $W(x,y)$ is a constant for any y in a sufficiently small neighbourhood of x. When interpreting this result one has to note that the value of $\phi(x)$ of the scalar field at $x \in \mathcal{M}$ is not an observable in a Diff$(\mathcal{M})$ invariant theory. The conclusion here is that the short-distance behaviour and ultraviolet divergences are likely to be different in quantum gravity than in quantum field theory. In addition, the regularisation method for the operators will have to change since now there is not a background metric affording the measure of nearness of spacetime points.

Second, Diff($\mathcal{M}$) invariance also affects functional integral quantisation methods. One might try to construct a theory of QG by using functional integrals, in analogy to standard quantum field theory, to produce vacuum expectation values of a time oredered product of a set of fields say

$$G(x_1, x_2, ..., x_n) = \int \mathcal{D}[g] R(x_1) R(x_2) ... R(x_n) e^{i \int_{\mathcal{M}} R(g) \sqrt{g} d^4 x}. \qquad (1.4)$$

Here the difficulty is to recognise what "time-ordered" product means in the absence of any background metric providing the preferred notion of spacelike and timelike[4]. Even if such a background is provided there seems to be still inconsistency since the attribute of spacelike or timelike of any pair of points can be changed one into the other by the action of the Diff($\mathcal{M}$) group.

Third, we have the spacetime operator version of quantisation. In quantum field theory, a scalar field $\hat{\phi}$ obeys the microcausality condition

$$[\hat{\phi}(x), \hat{\phi}(y)] = 0, \quad \forall \ x, y \ \textit{spacelike separated.} \qquad (1.5)$$

In the case of QG it has been shown [FH87] that for most pairs of points $x, y \ \in \mathcal{M}$ there will exist at least one Lorentzian metric with respect to which they are not spacelike separated, and hence, as far as all metrics are summed over in a funtional integral (e.g. (1.4)), the r.h.s of (1.5) will not vanish!

1.2.2 Background structure and the problem of time

The background structure is a key feature of any approach to QG that can take different forms. It can consists of choosing a particular mathematical element of the theory or it can refers to the conceptual or interpretative framework assumed a priori. In the former case we have the examples of theories that take a fixed manifold representing spacetime or in which a particular spacetime metric is considered central. Concerning the latter case a remark is in order. It can be argued that the conventional Copenhagen interpretation of QT presupposes as part of its background a fixed spacetime (in both topological and metric sense) and it is therefore intrinsically incompatible with the idea of QG. Also, the very existence of such a background is usually associated with a division of the universe into "system" and "observer".

[4]Apparently this problem can be circumvented by using an Euclidean formalism in which the functional integral is over metrics possessing a Riemannian rather than Lorentzian signature. The problem is transformed into one of interpreting physical amplitudes from Riemann n-point functions.

This split is in fact one of the current problems in Quantum Cosmology. The importance of the understanding of the precise background structure that is assumed should be clear by now. The issue on the compatibility of the ideas of QT and GR mentioned at the beginning of this introduction acquires a defined form in the background structure here discussed.

Time is not an observable in conventional quantum physics since there is not an operator associated to it. Instead, it is treated as a background parameter, as in classical physics, to express the evolution of a system. This applies to non-relativistic quantum theory, relativistic particle dynamics as well as to quantum field theory. Hence, time can be regarded as an element of the classical background that is essential to the Copenhagen interpretation of the theory. We can see now that, in any particular approach to QG, the nature of the problem of time is strongly related to the background structure assumed. As used by particle physicists, the background Minkowski metric provides the usual notion of time of special relativity and quantum field theory. However, whether or not a measure of time, as given in such an approach, is physically correct is not clear. This is an aspect of the question about to the extent to which the spacetime concepts of GR can be accounted adequately by a weak-field perturbation around a Minkowski background. For instance, the behavior of the lightcones at the event horizon of a black hole cannot be readily reproduced in a graviton-based picture. This criticism holds also for the genaralizations in which, instead of a Minkowskian background, another curved background is used.

In the relativists approach the issue of time is different. The background is now the "three-manifold of space". If it is non-compact the asymptotic structure might be used to define an absolute time. In dealing with cosmological models, however, this three-manifold is taken to be compact and the notion of time has to be extracted from the variables involved in the description: canonical variables of gravity, matter fields or particles added to the system.

1.2.3 A minimum length in QG

A problem encountered when dealing with local quantum fields may be the existence of a minimum lenth (or time) related to the Planck units. It has been argued recently [Pad85b, Pad85a, Pad87] that: i)geodesic distance is intrisically bounded from below in QG and ii) the uncertainty relations and the existence of the Schwarzchild radius, impose lower bounds on measurements of both space and time. The latter being a result coming from an analysis of "quantum clocks" as related to time in the canonical quantization of gravity. Another possibility is that the minimum length may arise in the context of a lattice approach to QG [Gre90]. Furthermore, there are indications that string theory may lead to a natural minimum length.

Concerning the meaning of the existence of such a minimum length one can interpret it as that length can be "measured" only to an accuracy of the Planck value (in principle w.r.t. some background metric) and that the underlying model of a continuum spacetime still holds. On the other hand, it can be interpreted as a signal of the breakdown of the continuum picture itself. Anyway, both views make more obscure the idea of quantising gravity by quantising the point fields of classical GR. For instance, if time cannot be measured to an accuracy greater than the Planck time, one needs to recast the equal-time commutation relations to make them meaningful as well as the general quantum mechanical idea of a complete commuting algebra of "simultaneously" measurable observables.

A more difficult matter would be the breakdown of the continuum picture. This amounts to think of the Diff($\mathcal{M}$) invariance as only a coarse-grained feature [5]. This also causes problems to the Regge calculus approach to QG in contrast to the case of the gauge group in Yang-Mills theories.

1.2.4 Quantum Topology

The framework of GR involves a pair $(\mathcal{M}, g)$; the spacetime manifold and metrics on it. However, once the classical continuum picture of spacetime has been entertained a number of possibilities may occur: in particular if geometries are to be quantised (g or related objects) it may be possible to consider the quantisation of $\mathcal{M}$. This idea goes back to J.A. Wheeler [Whe64]. It is far from clear what this would mean. The mathematical framework of GR has the following symbolic hierarchical form

$$\text{set} \longrightarrow \text{topology} \longrightarrow \text{differential structure} \longrightarrow \text{Lorentzian metric}$$

in which each step represents a structure superimposed on the previous in the chain. A "priori" quantisation might be applied at any of these stages. To keep $\mathcal{M}$ (the manifold) fixed and just quantise the metric is not an adventurous approach. There are other possibilities, for instance, to keep the differential structure but let the manifold become part of the quantum structure. S. Hawking [Haw78, Haw79] and collaborators developed this idea through the so called "Euclidean" quantum gravity program. This is based on the use of path integrals over Riemannian, rather than Lorentzian, metrics. A typical quantity is:

$$Z = \sum_{\mathcal{M}} \int_{\text{Riem}(\mathcal{M})} \mathcal{D}[g] e^{\int_{\mathcal{M}} R(g)\sqrt{g} d^4x} \tag{1.6}$$

[5] This idea is difficult to implement in practice, partly because of the absence of any finite-dimensional approximation to the Diff($\mathcal{M}$) group.

where the integral is over the set Riem($\mathcal{M}$) of all Riemannian metrics on $\mathcal{M}$ and the sum is over all four-manifolds[6] $\mathcal{M}$. The current theory of wormholes with their possible consequences in determining the constants of nature is an application of this idea [Col88, Haw88, HL88]. The use of certain complicated manifolds $\mathcal{M}$ raises intriguing possibilities, e.g. there may exists lost of information: when particles fall into the event horizon of a (virtual) black hole. An observer external to the black hole would interpret this as a transition from a pure state to one that is mixed, leading to what Hawking calls the "$-matrix": the pure→mixed analogue of a normal S-matrix [Haw82a, Haw82b, Haw84].

There are other alternatives not less interesting based on the quantisation of sets, topologies and manifolds that have been developed in a number of directions [Ish91]. However, as remarked by C. Isham himself , they are more speculative and difficult to relate to the conventional approaches to QG.

Finally, we want to mention other important questions appearing in the context of QG. They are, for example, the interpretation of the role played by complex metrics (e.g. in Asthekar's formalism –Chapter 3), or those that are degenerated [Hor91]. Both have arisen in recent work on QG posing non-trivial problems to the interpretation of the theory. The Quantum Cosmology issue contains several points which deserve discussion. Here we just quote them: essentially, shadows are cast when the interpretative framework of the quantum theory is applied to the entire universe. i)The conventional Copenhagen interpretation of quantum theory emphasizes the role of measurement and probability (often considered in a relative frequency sense). However, an observer can not be out of the universe to measure it and, also, we do not know what an ensemble of universes is. ii) Theories of the Quantum Creation of the Universe (QCU) aspire to have a unique quantum state based on some quantum boundary conditions "near the big-bang". It is not certainly kown if this is compatible with the standard notions of quantum theory. iii)The world around us is remarkable classical. It is a main question how to get this feature from a totally quantum mechanical description. iv) QCU theories involve the idea of a beginning of time. It must be checked the compatibility of such an idea with both GR and conventional QT.

1.3 Approaches to Quantum Gravity

In this section we present a brief description of several of the different approaches to tackle the quantisation of gravity. We include only the particle

[6]It is not quite clear what is the meaning of this since it is not possible to classify four-manifolds in any algorithmic way.

physicists schemes since a more complete discussion of the canonical framework, used by the relativists group, will be given in chapter 2. We make a rough description of these schemes followed by a series of remarks concerning their conceptuals and geometrical aspects.

1.3.1 Quantisation of GR

Detailed reviews in this respect are [Ish85, Ish87, Alv89]. The analysis involves several points to be discussed.

Gravitons

Gravitons are the quanta of the gravitational field. The particle is conceived of as propagating on a background Minkowski spacetime and it is associated (like the other elementary particles) with a specific representation of the Poincaré group labelled by its mass m and spin s [SW64]. The especification of m and s for the case of the graviton is got as follows:

i) t-channel exchange of a particle of mass m can give rise to a static force of the form e^{-mr}/r^2 where r is the distance between two particles. Thus the usual gravitational inverse-square law can be secured only if the graviton is massless.

ii) s cannot be half-integral because the Pauli exclusion principle makes it impossible to construct a classical-sized field from a coherent superposition of fermions.

iii)S. Weinberg showed [Wei64, Wei65] that a particle whose spin is greater than two will not produce a static force. Furthermore, $s = 1$ gives a repulsive force between like particles (e.g. spin-1 photons serves this function in Electrodynamics). We arrived at the only two possibilities that $s = 0$ or $s = 2$.

Scalar fields, $\phi(x)$ are associated with zero spin while symmetric Lorentz tensor fields $h_{\mu\nu}(x)$ are with spin-2. They can be interpreted as corresponding to Newtonian gravity and General Relativity respectively. According to quantum field theory (based on special relativity) a free massless spin-2 field must satisfy the field equation:

$$h_{\mu\nu,\alpha}{}^{\alpha} - h_{\mu}{}^{\alpha}{}_{,\alpha\nu} - h_{\nu}{}^{\alpha}{}_{,\alpha\mu} + h^{\rho}{}_{\rho,\mu\nu} + \eta_{\mu\nu}(h^{\alpha\beta}{}_{,\alpha\beta} - h^{\alpha}{}_{\alpha,}{}^{\beta}{}_{\beta}) = 0. \quad (1.7)$$

Two properties are notable of this equation: it is invariant under a) a redefinition of the field, $h_{\mu\nu} \to h_{\mu\nu} + \lambda\eta_{\mu\nu}h^{\alpha}{}_{\alpha}$, with $\lambda \neq \frac{1}{4}$ to avoid the new fields being traceless, and b) the gauge transformations $h_{\mu\nu} \to h_{\mu\nu} + \xi_{\mu,\nu} + \xi_{\nu,\mu}$, where $\xi_{\nu}(x)$ is an arbitrary Lorentz tensor field[7].

[7] This invariance is a consequence of the masslessness of the graviton and it turns out to be necessary to project out the lower spin ghosts which are otherwise associated with the tensor field $h_{\mu\nu}(x)$.

Gravitons from GR

The derivation of the graviton field from GR is through the lowest order approximation of the Einstein-Hilbert action

$$S[g] = \frac{1}{\kappa^2} \int R(g(x))[|detg|]^{\frac{1}{2}} d^4x \tag{1.8}$$

in the expansion: $g_{\mu\nu}(x) + \kappa h_{\mu\nu}(x)$. The field eq. for $h_{\mu\nu}(x)$ is precisely (1.7) when the lowest order in κ is considered. The two properties mentioned above concerning the field equation have an interpretation here. The first corresponds to using $g_{\mu\nu}(|detg|)^{\lambda}$ as the field variable instead of $g_{\mu\nu}$. The second is just the effect induced by an infinitesimal diffeomorphism of Minkowski space generated by the vector field ξ.

Advantages

The advantages of adopting such a scheme can be summarised in:

1) Short-distance behaviour, operator-product expansions, regularisation and related topics are faced conventionally due to the existence of the background metric.

2) A fixed causal structure is afforded by the background metric that allows to define a microcausal spacetime commutation relations for spacetime fields, equal-time commutation relations for canonical fields and a good notion of time ordering for use in a functional integral or other formalisms of coventional quantum field theories.

3) Some of the difficulties with the Diff($\mathcal{M}$) invariance are translated into into standard problems for gauge-invariant quantum field systems and then they can be tackled with such methods.

4) The definition of observables can be approached by using the underlying Minkowski structure in analysing the asymptotic behaviour of the fields. The key point is that the gauge group generators ξ have compact support and therefore do not affect the fields asymptotically[8].

Objections

Concerning the geometrical and conceptual objections we have:

1) There is no reason to adopt the causal structure of the Minkowski metric as the physically correct one. In fact, it has been suggested that a non-perturbative treatment could lead to light cones that do not coincide

[8]Asymptotic observables played an important role in the seminal investigations of B.S. DeWitt [DeW65, DeW67a, DeW67b] on the spacetime covariant approach to QG.

with those provided by the Minkowski structure. The status of the initial microcausal structure is uncertain.

2) The Minkowski background fixes also the topology of spacetime to coincide with that of a trivial vector space. In this way any feature of classical GR involving non-trivial topological structure is made difficult to discuss, e.g. cosmological problems, spacetimes singularities, black holes and event horizons.

3) The expansion which the graviton field come from is a poor one in the geometrical perspective of classical GR. For example, $g_{\mu\nu}(x)$ will be a genuine metric tensor (an invertible matrix with signature (-1,1,1,1)) only for small values of $h_{\mu\nu}(x)$. However, in some quantisation methods, one integrates over all values of $h_{\mu\nu}$. Indeed, rather than quantising on the space of pseudo-Riemannian metrics, we are quantising on the tangent space to the specific $\eta_{\mu\nu}$.

Non-Renormalizability

When the corresponding expansion, for the metric in terms of the graviton field, is inserted in the Einstein-Hilbert action (1.8) the ensuing Lagrangian for $h_{\mu\nu}(x)$ contains terms that are non-polinomial, derivatively-coupled and with a dimensional coupling constant κ. Each one of the last features is an indication of the non-renormalisability of a quantum field theory in four spacetime dimensions. This can be considered the major disease of the approach to QG[9] we are disscussing now. And that is the reason why several schemes have been proposed to averte it. This is the case, for instance, of the *"$R+R^2$" theories* in which to the Einstein-Hilbert action is added the square of the Riemann curvature [Ste77, FT81, FT82] but which have not succeded because of problems of non-unitarity and ghosts [Tom84]. The *supergravity* [WB83] theory was another major program aimed at removing ultraviolet divergences with the hope that the additional fermi loops would cancel the infinities produced by the bosonic graviton loops. The appealing feature of this approach is that it yields a definite prediction for the fundamental matter Lagrangian which must be attached to GR. Unfortunately, it was found the idea does not work for more than 2 loops in the case of $N = 1$ supersymmetry and for more than 7 loops in the $N = 8$ case.

[9]In the early stages, only power-counting estimates indicated the non-renormalisability of the theory. Due to kinematical reasons the pure gravity one-loop graphs are finite on-shell. This is not the case when matter is included. Finally, two-loop calculations [GS85, GS86] explicitely showed this failure.

1.3.2 Quantisation of a theory that gives GR as its low energy limit

The key step here is to find a system possessing a well-defined quantum theory and which yields classical GR as a low-energy limit, even though that is not the starting point.

Induced gravity

Here, the Einstein-Hilbert action is not fundamental, but rather an effective action induced by the quantum structure (for a detailed account see [Adl82]). In constructing the theory one starts with an action including only the usual coupling of matter fields to the metric tensor and, the pure gravity term will arise as a counterterm from the quantised matter fields. Such term coincides with the weak-field expansion of the Einstein-Hilbert action. Hence, the criticisms to the weak-field scheme apply to the present case.

String theory

This is a more sophisticated scheme in which the graviton occurs as just one of an infinite number of particles associated with the quantised string.

The idea that a quantum theory of gravity can be constructed starting from closed strings comes from studies of zero-slope limit of the dual resonance model for non-hadrons [SS74, Yon74] (an extensive review on this link is [Hor89]).

The main idea is to quantise certain fields appearing in the Polyakov action

$$S[q,X] \equiv \frac{1}{4\pi\alpha'}\int_W q^{ij}(\sigma)\partial_i X^\mu(\sigma)\partial_j X^\nu(\sigma) g_{\mu\nu}(X(\sigma))[|detg|]^{\frac{1}{2}} d^2\sigma \qquad (1.9)$$

where q_{ij} is a metric on the two-manifold W (the world-sheet), $X : W \rightarrow \mathcal{M}$ are the string fields which map W into the spacetime manifold $\mathcal{M}$, and $g_{\mu\nu}$ is a background metric on $\mathcal{M}$. The constant α' is related to the string tension and is assumed to be of the order of the Planck length.

The classical system is invariant under the conformal transformations $q_{ij}(\sigma) \mapsto F(\sigma)q_{ij}(\sigma)$, $F(\sigma) > 0$ which can hold at the quantum level only in the case in which the metric $g_{\mu\nu}$ satisfies an equation that is effectively the vanishing of the trace of the energy momentum tensor of the two-dimensional quantum field theory. The above equations have the form

$$0 = R_{\mu\nu} + \frac{1}{2}\alpha' R_{\mu\alpha\beta\gamma} R_\nu{}^{\alpha\beta\gamma} + \cdots \qquad (1.10)$$

where higher-order powers and derivatives of the Riemann curvature are not written down explicitely. Other background spacetime fields may also be introduced (e.g. massless dilatons and 2-form fields) producing similar equations.

Conformal invariance also constraint the dimension of $\mathcal{M}$ to be equal to some critical value depending on the background fields that are present or any two-dimensional spinor fields added to the system (like in superconformal field theories).

The above field equations are the string theory substitute for the classical equations of GR. The metric $g_{\mu\nu}$ is not considered as background structure since it comes about as a solution of the dynamical equations. However, it works like that once calculations of quantum fluctuations around it are performed.

Several exact solutions to (1.10) have been found but probably among the most interesting ones are those which contain spacetime singularities [HS90, Hor90, Hor91]. The existence of these singular solutions may be taken as disappointing if one thinks that quantum gravity is supposed to remove the singularities coming from the classical framework of GR. Also, it is often said that the existence of a minimum length in the theory imply that quantum amplitudes should be free of ultraviolet divergences that plague conventional quantum field theory. However, this idea does not allow one to conclude whether the strings can only probe to a minimum length (keeping the spacetime continuum) or this fact signals the breakdown of the entire continuum picture. It cannot be decided adopting the Polyakov approach since the presence of a continuum manifold $\mathcal{M}$ is part of the background structure.

The main problem arises when an assessment of the singular solutions to the effective field equations is made. The next step following this line of work involves the computation of the quantum fluctuations around the classical background solutions; we are back to the weak-field scenario of the old approaches to QG. Even when many of these high-energy calculations involve non-perturbative methods for summing the various contributions, any complete, non-perturbative alternative to the Polyakov approach is lacking. This could help to investigate the main issues in QG: spacetime singularities, quantum topology, quantum cosmology, etc.

1.3.3 General-Relativise Quantum Theory

Rather than starting with classical GR which is then quantised, one begins instead with normal quantum theory and studies the extent to which it can be made compatible with the ideas of GR. This is very intriguing but it has not been developed much as the quantum field theory has. Remarkable in this approach is the work of K. Fredenhagen and R. Haag [FH87] who study the problem of making a quantum theory invariant under spacetime diffeomorphisms (see also [Ban88]). This seems to be the reason why C.J. Isham used the term "General-Relativise".

1.3.4 The semi-classical option

An idea of Moller [Mol62] in the opposite extreme to the one here exposed is that perhaps it is not necessary to quantise the gravitational field but only the matter to which it couples. This is in general what is meant by semi-classical approach. The induced gravity approach mentioned above is an example.

Originally, the idea was to study the system

$$G_{\mu\nu}(g) = {}_t\langle\psi|T_{\mu\nu}(\widehat{matter}, g)|\psi\rangle_t, \tag{1.11}$$

$$i\hbar\frac{d}{dt}|\psi\rangle_t = H(\widehat{matter}, g)|\psi\rangle_t \tag{1.12}$$

where the source of the gravitational field is the expectation value of the energy-momentum tensor in some especial state $|\psi\rangle_t$.

Several remarks are in order,

1) higher powers of the Riemann curvature appear when regularisation and renormalisation of the energy-momentum tensor are made. These are needed because of the quantum matter fields considered as source in the first equation [SK80].

2) the system seems to be intrisically unstable [HW78, Hor80, Sue89a, Sue89b]; the two equations are strongly coupled and the effective equations for the metric tensor are far more non-linear than those of normal GR . The calculations used were, however, mainly of perturbative nature and have been recently challenged [Sim90, Ful90].

3) the effective equation for $|\psi\rangle_t$ is non-linear, hence, the superposition principle is lost. Whether or not this is a problem is related to one's attitude to conventional quantum theory.

It is not at all clear that quantising everything but the gravitational field is something inconsistent. For instance, the Bohr-Rosenfeld argument [BR33] showing that the electromagnetic field should be quantised if it couples consistenly to the current generated by the quantised matter does not apply to gravitational case. This can be seen as follows. The proof for electromagnetism involves taking to infinity the ratio e/m of the electrical charge e to the inertial mass m of a test particle. This is precluded in the gravitational case by the equivalence principle since the analogue of e is the gravitational mass [Ros63]. Several attempts have appeared [EH77, PG81, Unr84] but no one has succeded in clarifying the situation completely (see also [Son90]).

The semiclassical approach has been recently reappeared in the form of a Born-Oppenheimer/WKB approximation to QG. Again, quantum matter effectively couples to a classical gravitational field. Now, however, this is considered as an approximation to the unknown full theory of Quantum Gravity (see [SP89] for a clear explanation).

Chapter 2

Canonical Quantisation of Gravity

Now we discuss the general relativists group approach known as the canonical Quantisation of Gravity. In order to arrive at the current description of this approach to QG several steps have to be done. We outline them and give the features that make this canonical framework appealing as well as its main problems.

As summarized by K. Kuchař[Kuc81] the canonical quantisation program applied to any classical theory can be set as follows: 1) Translate the classical theory into the Hamiltonian formalism and identify the correspondent canonically conjugate variables. 2) Turn such variables into operators satisfying the Dirac commutation relations, and substitute these operators into the Hamiltonian in order to get the Schrödinger equation. 3) An inner product should be defined that is conserved by this equation along the dynamical evolution of the state. The existence of this product renders the space of solutions a structure of Hilbert space where the probabilistic interpretation of the theory comes from.

In the case of gravity, however, there is no Hamiltonian in the usual sense but Hamiltonians constraints. The implementation of the canonical quantisation approach is not straightforward and, in particular, a completely satisfactory Hilbert space has not been constructed so far. Among the most important consequences is that a clear probabilistic interpretation of the theory is lacking. Nevertheless, an understanding of the sources of the difficulties deserves attention for, eventually, circumventing them.

2.1 The Canonical Structure of Classical GR

This section can be considered as the first step, mentioned above, of the canonical quantisation program. Early studies concerning with the casting of classical GR in a canonical form, that is, adopting it to a Hamiltonian structure, were developed by using a specific coordinate system for the spacetime and the involved global aspects were not emphasized [Ber56, Dir58, RAM62]. This aspects of global character turn out to be of vital importance for the analysis of topological structure in QG, e.g. changes of topology, wormholes, but also a clear conception of them is needed to account for the "convenient" 3+1 foliation of spacetime. This geometrical, global view is given in [Kuc72, Kuc76b, Kuc76c, Kuc76a, Kuc77] (see also [Ish91]). We briefly describe it before going into the canonical approach to classical GR.

2.1.1 The foliation of spacetime and related definitions

Lets consider spacetime as a four-manifold $\mathcal{M}$ and a three-manifold Σ, assumed to be compact[1], playing the role of physical three-space. Assume that the topology of $\mathcal{M}$ admits a foliation by a one-parameter family of copies of Σ. By this we mean that there exists a set of embeddings $X_t : \Sigma \rightarrow \mathcal{M}$, with $t \in I\!R$ and such that the induced map

$$X : \Sigma \times I\!R \rightarrow \mathcal{M}; \quad X(x,t) = X_t(x) \tag{2.1}$$

is a diffeomorphism of $\Sigma \times I\!R$ with $\mathcal{M}$. The set of all embeddings of Σ in $\mathcal{M}$, $Emb(\Sigma, \mathcal{M})$, is a subset of $C^\infty(\Sigma, \mathcal{M})$, the set of all smooth maps from Σ to $\mathcal{M}$. To the latter it can be give the structure of a C^∞-manifold and $Emb(\Sigma, \mathcal{M})$ becomes an open subset of it. It inherits in this way a differential structure we will implicitly use here.

$X^{-1} : \mathcal{M} \rightarrow \Sigma \times I\!R$ is also a diffeomorphism since X is. It can be written as:

$$X^{-1}(y) = (S(y), T(y)) \in \Sigma \times I\!R, \quad y \in \mathcal{M} \tag{2.2}$$

where $S : \mathcal{M} \rightarrow \Sigma$ and $T : \mathcal{M} \rightarrow I\!R$. The map T is the so called "global time function" and provides the natural time coordinate associated to the foliation: $T(X_t(x)) = t, \quad \forall x \in \Sigma$. Of course, one cannot talk about "time" until a Lorentzian metric on $\mathcal{M}$ is given such that each $X_t(\Sigma)$ is spacelike. We have to keep this in mind.

At this point three remarks are in order:

First, the requirement on $\mathcal{M}$ to admit foliations restricts its possible topologies. Also the topology of Σ must be compatible with a given topology

[1] If it is not but it is asymtotically flat, modifications of the corresponding expressions are needed by adding surface terms.

of $\mathcal{M}$, hence the "quantum topology fluctuations" of Σ, that would amount to consider all the topologies of Σ, cannot be discussed.

Second, the selected foliation is not unique, there exists many other diffeomorphisms of $\Sigma \times I\!R$ with $\mathcal{M}$. Thus we have a choice of time function for each of them.

Third, mathematically we should take in to account only those foliations whose leaves $(T(X_t(\Sigma)) = t = const.$ can be deformed smoothly into the leaves of the foliations X we started with. Furthermore, the set of Lorentzian metrics g with which we want to eqquip $\mathcal{M}$ should be restricted to those for which there exists at least one foliation whose leaves are all spacelike with respect to g.

The deformation vector $\dot{X}_t$, of a given foliation is the tangent vector to the curve in $Emb(\Sigma, \mathcal{M})$ representing the map $t \to X_t$. Thus this vector belongs to the tangent space $T_{X_t} Emb(\Sigma, \mathcal{M})$ which is the set of all tangent vectors to the infinite dimensional manifold $Emb(\Sigma, \mathcal{M})$ at the particular embedding X_t.

On the other hand, the map $t \to X_t(x)$, for any $x \in \Sigma$, represents a curve in $\mathcal{M}$. This curve has tangent vector belonging to the tangent spaces of $\mathcal{M}$. For instance, given any t, the tangent vector associated with $x \in \Sigma$ lies in $T_{X_t(x)}\mathcal{M}$ at the point $X_t(x) \in \mathcal{M}$.

Whenever one deals with tangent spaces to function spaces, as above, it can be established that

$$T_f C^\infty(\Sigma, \mathcal{M}) = \{v : \Sigma \to T\mathcal{M} | v(x) \in T_{f(x)}\mathcal{M}\}, f : \Sigma \to \mathcal{M}, \qquad (2.3)$$

through which one relates the tangent spaces in the different manifolds $C^\infty(\Sigma, \mathcal{M})$ and $\mathcal{M}$. Any $v \in T_f C^\infty(\Sigma, \mathcal{M})$ is "hybrid" in the sense that to each $x \in \Sigma$ it associates a vector $v(x)$ in the tangent space to $\mathcal{M}$ at $f(x) \in \mathcal{M}$. Given a coordinate system on $\mathcal{M}$ this vector is written as $v^\mu(x)$ where $\mu = 0, 1, 2, 3$.

Similarly, cotangent vectors to a function space can be defined by

$$T_f^* C^\infty(\Sigma, \mathcal{M}) = \{\omega : \Sigma \to T\mathcal{M} | \omega(x) \in T_{f(x)}^*\mathcal{M}\} \qquad (2.4)$$

Again, by introducing a coordinate system on $\mathcal{M}$, any element $\omega \in T_f^* C^\infty(\Sigma, \mathcal{M})$ associates to each $x \in \Sigma$ a covariant vector $\omega_\mu(x)$ at the point $f(x) \in \mathcal{M}$.

Apart the deformation vector the normal vector, lapse and shift functions are useful definitions in accomplishing our aim of casting classical GR in canonical form.

The normal vector to the foliation at the embedding $X_t \in Emb(\Sigma, \mathcal{M})$ is an element of $T^*_{X_t}Emb(\Sigma, \mathcal{M})$ which is unique, and denoted n_t, that satisfies:

$$\begin{aligned} X_t^*(n_t(x)) = 0 \Leftrightarrow n_\mu(x,t)X^\mu{}_{,a}(x,t) &= 0 \\ g^{\mu\nu}(X(x,t))n_\mu(x,t)n_\nu(x,t) &= -1 \end{aligned} \tag{2.5}$$

where $X_t^* : T^*_{X_t(x)}\mathcal{M} \rightarrow T^*_x\Sigma$ is the pull-back on cotangent spaces and $a = 1,2,3$ refers to a coordinate system on Σ. Furthermore, the change of notation $(n_t)_\mu(x) \rightarrow n_\mu(x,t)$ has been performed. The first equation defines the normality aspect of n_t w.r.t. Σ_t at the point $X_t(x) \in \mathcal{M}$, in a way that is independent of the coordinates. The latter supposes the existence of a Lorentzian metric on $\mathcal{M}$ to define the timelike character of n as well as its normalization.

As clearly seen from the relation between n the metric and the global time function

$$n_\mu(x,t) = \left.\frac{T_{,\mu}}{(g^{\alpha\beta}T_{,\alpha}T_{,\beta})^{1/2}}\right|_{y=X(x,t)} \tag{2.6}$$

the normal vector is a functional of both the Lorentzian metric and the folation X.

An element of $T_{X_t}Emb(\Sigma, \mathcal{M})$ can be always decomposed into two components, one lying along Σ_t, and the other parallel to n_t. In the case of the deformation vector, we have:

$$\begin{aligned} \dot{X}^\mu(x,t) &= N(x,t)g^{\mu\nu}(X(x,t))n_\nu(x,t) \\ &+ N^a(x,t)X^\mu{}_{,a}(x,t) \end{aligned} \tag{2.7}$$

$N(x,t)$ and $N^a(x,t)$ are called the lapse function and the shift vector, respectively. It is evident from their definition (2.7) that they are functionals of both the spacetime metric and the foliation.

From this global point of view one can arrive at the specific coordinate approach of ADM (See [RAM62]). The key point is to identify the lapse and shift with some components of the metric. We start by taking a fixed foliation and looking at the pull-back of the metric g, supposed to be given also, by $X : \Sigma \times I\!R \rightarrow \mathcal{M}$. This provides us with a Lorentzian metric X^*g on $\Sigma \times I\!R$ whose components are:

$$(X^*g)_{\alpha\beta}(x,t) \equiv g_{\mu\nu}(X(x,t))X^\mu{}_{,\alpha}(x,t)X^\nu{}_{,\beta}(x,t) \tag{2.8}$$

where $\alpha, \beta = 0,1,2,3$ correspond to a coordinate system in $\Sigma \times \mathbb{R}$. Next, we choose coordinates "adapted" to the foliations, that is, $x^{\alpha=0}(x,t) = t$ and $\alpha = 1,2,3$ referring to a coordinate system on Σ. Then (2.8) becomes[2]

$$
\begin{aligned}
(X^*g)_{ab}(x,t) &= {}_t\gamma_{ab}(x) \equiv g_{\mu\nu}(X(x,t))X^{\mu}{}_{,a}(x,t)X^{\nu}{}_{,b}(x,t) \\
(X^*g)_{oo}(x,t) &= N^a(x,t)N^b(x,t)\, {}_t\gamma_{ab}(x) - [N(x,t)]^2 \\
(X^*g)_{0a}(x,t) &= N^b(x,t)\, {}_t\gamma_{ab}(x)
\end{aligned}
\qquad (2.9)
$$

The first of these equations can be seen as the pull-back of $X_t : \Sigma \rightarrow \mathcal{M}$ of the Lorentzian metric g on $\mathcal{M}$ to give a Riemannian metric ${}_t\gamma$ on Σ. We can recognise the shift and lapse functions as the g_{00} and g_{0a} parts of the metric.

2.1.2 Canonical variables for GR

Classical GR is cast in a canonical form as follows. A foliation $X : \Sigma \times \mathbb{R} \rightarrow \mathcal{M}$ is fixed and the case in which $\mathcal{M}$ carries a Lorentzian metric g satisfying the vacuum Einstein equations, $G_{\mu\nu} = 0$, is considered and is such that $\Sigma_t \equiv X_t(\Sigma)$ is a submanifold of $\mathcal{M}$ spacelike w.r.t. g. Next, a set of canonical variables for this system and the correspondent 1st. order differential equation describing their evolution from one leaf to another have to be found.

All the treatment so far has been given to adopt the Riemannian metric ${}_t\gamma$ on Σ as the canonical configuration variable. The meaning of ${}_t\gamma$ is that, geometrically, it is related to the intrinsic curvature of Σ_t. Thus it is said that ${}_t\gamma$ measures the intrinsic curvature of Σ_t.

The canonical variable conjugated to ${}_t\gamma$ can be better understood in terms of the extrinsic curvature of Σ_t. This extrinsic curvature can be interpreted as the "look" of Σ_t as embedded in the four manifold $\mathcal{M}$, *i.e.*, using the Lorentzian metric g on $\mathcal{M}$. By using a coordinate system it is defined as:

$$K_{ab}(x,t) \equiv -n_{\nu;\mu}(x,t)X^{\nu}{}_{,a}(x,t)X^{\mu}{}_{,b}(x,t) \qquad (2.10)$$

where $n_{\nu;\mu}$ is the covariant derivative[3] w.r.t. g on $\mathcal{M}$. It can be transformed further into

[2]The adopted coordinates allows us to identify $X^{\mu}{}_{,0}$ with $\dot{X}^{\mu}$. Hence $(X^*g)_{00} = g_{\mu\nu}[Ng^{\mu\lambda}n_\lambda + N^a X^{\mu}{}_{,a}] \cdot [Ng^{\nu\tau}n_\tau + N^b X^{\nu}{}_{,b}]$ and $(X^*g)_{0a} = g_{\mu\nu}[Ng^{\mu\lambda}n_\lambda + N^b X^{\mu}{}_{,b}]X^{\nu}{}_{,a}$ give us the above results once the normality condition of n w.r.t. the foliation is introduced.

[3]The explicit appearance of the covariant derivative of n allows us to interpret the extrinsic curvature as a measure of the bending of Σ_t in the spacetime in which it is embedded.

$$K_{ab}(x,t) = \frac{1}{2N}(-\frac{\partial\gamma_{ab}}{\partial t} + N_{a/b} + N_{b/a}) = \frac{1}{2N}(-\frac{\partial\gamma_{ab}}{\partial t} + (\mathcal{L}_{\vec{N}}\, \gamma)_{ab}) \quad (2.11)$$

when use is made of the definition of γ and $N_{a/b}$ is taken as the covariant derivative w.r.t. γ on Σ. $(\mathcal{L}_{vecNt}\gamma)_{ab})$ means the Lie derivative of the metric γ along the vector field N^a. We can see that K_{ab} is an explicit functional of γ_{ab}, $\dot{\gamma}_{ab}$ and $\vec{N}$ (see(2.11)). The utility of K_{ab} just defined can be appreciated once the pull-back of the Einstein lagrangian density by the foliation X : $\Sigma \times I\!R \rightarrow \mathcal{M}$ is expressed as a function of the extrinsic curvature, the metric γ and the lapse and shift functions:

$$\begin{aligned} X^*[R(g)(-\det g)^{1/2} &= N(\det\gamma)^{1/2}[K_{ab}K^{ab} - (K_a{}^a)^2 + R^{(3)}(\gamma)] \quad (2.12) \\ &- 2\frac{d}{dt}[(\det\gamma)^{1/2}K_a{}^a] - [(\det\gamma)^{1/2}(K_a{}^a N^b - \gamma^{ab}N_{,a})]_{,b} \end{aligned}$$

where $R^{(3)}(\gamma)$ is the curvature scalar of the metric γ on Σ and the label t on γ is understood. Since we assume Σ to be compact, the spatial divergence term vanishes. On the other hand, in order to have a genuine action principle, the total time derivative must be dropped [4]. One arrives at the Einstein lagrangian for matter-free gravity.

$$\begin{aligned} L(t) &= \frac{1}{\kappa^2}\int_\Sigma N(\det\gamma)^{1/2}[K_{ab}K^{ab} - (K_a{}^a)^2 + R^{(3)}(r)]d^3x, \\ &\kappa^2 \equiv \frac{16\pi G}{c^2} \quad (2.13) \end{aligned}$$

The canonical conjugate variable to γ, as in the standard canonical analysis, is got by using the time parameter associated with the foliation as follows:

$$\pi^{ab} \equiv \frac{\delta L}{\delta\dot{\gamma}_{ab}} = -\frac{(\det\gamma)^{1/2}}{\kappa^2}(K^{ab} - \gamma^{ab}K_c{}^c) \quad (2.14)$$

or equivalently: $K^{ab} = -\kappa^2 \det\gamma^{-1/2}(\pi^{ab} - 1/2\gamma^{ab}\pi_c{}^c)$.

However, in the case of N and N^a

$$\pi_N \equiv \frac{\delta L}{\delta\dot{N}} = 0; \quad \pi_{\vec{N}} = \frac{\delta L}{\delta\dot{\vec{N}}} = 0 \quad (2.15)$$

we get the so called primary constraints of the theory. That is, the evolution of N and $\vec{N}$ is not determined by the Einstein eqs.

[4] In fact, we know that the Hilbert-Einstein action contains 2nd order time derivatives and hence the equations of motion could be third order in time derivatives. However, it is well established this is not the case because the correspondent term can be identified as boundary term that vanishes when the variation of the action is performed.

2.1.3 The constraint equations

Already from the Einstein's equations themselves the existence of constraint equations is noted. From the Einstein equations the $G^{\mu 0} = 0$ contains only first order time derivatives [5]. Thus only six of the original equations are time dynamical equations. In fact it can be shown [Wal84] that the $G_{ab} = 0$ equations are equivalent to:

$$
\begin{aligned}
\dot{\gamma}_{ab} &= \frac{2N}{(\det\gamma)^{1/2}}(\pi_{ab} - 1/2\gamma_{ab}\pi^{a}{}_{a}) + 2N_{(b/a)} \\
\dot{\pi}^{ab} &= -N(\det\gamma)^{1/2}({}^{(3)}R^{ab} - \frac{1}{2}{}^{(3)}R\gamma^{ab}) \\
&+ \frac{1}{2}N\frac{\gamma^{ab}}{(\det\gamma)^{1/2}}(\pi_{cd}\pi^{cd} - \frac{1}{2}(\pi^{a}_{a})^2) \\
&- \frac{2N}{(\det\gamma)^{1/2}}(\pi^{ac}\pi^{b}_{c} - 1/2\pi\pi^{ab}) \\
&+ (\det\gamma)^{1/2}(N_{|ab} - \gamma^{ab}N^{|c}_{|c} \\
&+ (\det\gamma)^{\frac{1}{2}}\left(\frac{N^c\pi^{ab}}{(\det\gamma)^{1/2}}\right)_{|c} - 2\pi^{c(a}N^{b)}{}_{|c}
\end{aligned}
\tag{2.16}
$$

while the $G_{0\mu}(g) = 0$ amount to:

$$
\begin{aligned}
C_a(\gamma,\pi)(x) &= -\pi_{a}{}^{b}{}_{1b}(x) = 0 \\
C_\perp(\gamma,\pi)(x) &= \frac{\kappa^2}{(\det\gamma(x))^{1/2}}[\gamma_{ab}(x)\gamma_{cd}(x) - 1/2\gamma_{ac}(x)\gamma_{bd}(x)]\pi^{ac}(x)\pi^{bd}(x) \\
&- \frac{(\det\gamma(x))^{1/2}}{\kappa^2}{}^{(3)}R(\gamma(x)) = 0
\end{aligned}
\tag{2.17}
$$

which are constraint equations between the canonical variables $\gamma_{ab}(x)$ and $\pi^{cd}(x)$. This leads us to the possibility of writing the first order action:

$$
S(\gamma,\pi,N,\vec{N}) \equiv \int dt \int_\Sigma d^3x[\pi^{ab}\dot{\gamma}_{ab} - NC_\perp(\gamma,\pi) - N^aC_a(\gamma,\pi)] \tag{2.18}
$$

in which γ_{ab}, π^{cd},N, and $\vec{N}$ are to be varied independently to produce the above constraint and dynamical equations. Here it is evident that N and $\vec{N}$ can be considered Lagrange multipliers and hence without any dynamics.

[5] This is seen as follows. The Bianchi identities $R^{\mu}{}_{\nu(\sigma\tau;\lambda)} = 0$ imply $R_{\nu\tau;\lambda} - R_{\nu\lambda;\tau} + R^{\mu}{}_{\nu\tau\lambda;\mu} = 0$ and $R_{;\lambda} = 2R^{\nu}{}_{\lambda;\nu}$. Thus we get $G^{\mu\nu}{}_{;\nu} = 0 \Leftrightarrow G^{\mu 0}{}_{;0} = -G^{\mu\kappa}{}_{;\kappa}$. The r.h.s. of the last equation contains at most 2nd order time derivatives (eqs. of motion) and we arrive at the conclusion that $G^{\mu 0}$ are first order in time derivatives.

At this stage a Hamiltonian can be extracted for our system,

$$H[N,\vec{N}] = \int_\Sigma (N(x)C_\perp(x) + N^a(x)C_a(x))d^3x \tag{2.19}$$

which vanishes by virtue of (2.17).

Concerning the constraints several points deserve a brief explanation. Among the most important ones are:

1) Initial value problem. In the case of GR the Cauchy data on an initial hypersurface must satisfy the constraints. It has been found that given a Lorentzian metric g on $\mathcal{M} \simeq \Sigma \times I\!R$ satisfying the vacuum Einstein field equations then the above constraints equations are satisfied on any hypersurface in $\mathcal{M}$ and conversely, that given a Lorentzian metric g on $\mathcal{M} \simeq \Sigma \times I\!R$ satisfying the above constraint equations on all spacelike hypersurfaces then it will necessarily satisfy the full field equations $G_{\mu\nu}(g) = 0$. Thus the relevance of the role of the constraint equation is appreciated.

2) The algebra of the constraints. Classically the canonical algebra of the system is expressed by the Poisson brackets:

$$\begin{aligned}
\{\gamma_{ab}(x), \gamma_{cd}(x')\} &= 0 \\
\{\pi^{ab}(x), \pi^{cd}(x')\} &= 0 \\
\{\gamma_{ab}(x), \pi^{cd}(x')\} &= \delta^c{}_{(a}\delta^d{}_{b)}\delta(x,x').
\end{aligned} \tag{2.20}$$

When he Poisson brackets among the constraints is computed, taking in to account the above canonical algebra, one gets the Dirac algebra ([Dir65])

$$\begin{aligned}
\{C_a(x), C_b(x')\} &= -C_b(x)\partial_a^{x'}\delta(x,x') + C_a(x')\partial_b^{x}\delta(x,x') \\
\{C_a(x), C_\perp(x')\} &= C_\perp(x)\partial_a^{x}\delta(x,x') \\
\{C_\perp(x), C_\perp(x')\} &= \gamma^{ab}(x)C_a(x)\partial_b^{x'}\delta(x,x') - \gamma^{ab}(x')C_a(x')\partial_b^{x'}\delta(x,x')
\end{aligned} \tag{2.21}$$

The meaning of these relations can be better understood by introducing their smeared versions:

$$\begin{aligned}
C(\eta) &= \int_\Sigma C_\perp(x)\eta(x)d^3x, \ \eta \text{ a scalar function on } \Sigma \text{ and} \\
C(\vec{\xi}) &= \int_\Sigma C_a(x)\xi^a(x)d^3x, \ \vec{\xi} \text{ a vector field on } \Sigma
\end{aligned} \tag{2.22}$$

Which substituted in the original Poisson brackets between he constraints, produce:

$$\begin{aligned}
\{C(\vec{\xi}_1), C(\vec{\xi}_2)\} &= C([\vec{\xi}_1, \vec{\xi}_2]) \\
\{C(\vec{\xi}), C(\eta)\} &= C(L_{\vec{\xi}}\eta) \\
\{C(\eta_1), C(\eta_2)\} &= C(\vec{\xi}(\eta_1, \eta_2))
\end{aligned} \tag{2.23}$$

where in the last relation, $\vec{\xi}(\eta_1, \eta_2) = \gamma^{ab}(\eta_1 \eta_{2,b} - \eta_2 \eta_{1,b})$ and $[\vec{\xi_1}, \vec{\xi_2}]$ is the commutator of vector fields $\vec{\xi}$ on Σ that generate infinitesimal coordinate transformations on Σ (see below)

Geometrically they are interpreted as follows:

a) It is well established that the infinitesimal coordinate transformations $x^a \longrightarrow x^a + \xi^a$ on Σ, provide us with a representation of the Lie algebras of the diffeomorphism group Diff(Σ). The Lie bracket being determined on Diff(Σ) the negative of the commutator:

$$[\xi_1, \xi_2]^a \equiv \xi^b{}_1 \xi_{2,b}^{\ a} - \xi^b{}_2 \xi_{1,b}^{\ a} \tag{2.24}$$

The first of the smeared Poisson brackets (2.23) tells us that the map $\vec{\xi} \longrightarrow -C(\vec{\xi})$ is a homomorphism of the Lie algebra of Diff(Σ) into the Poisson bracket algebra of the constraints. Furthermore,

$$\{\gamma_{ab}(x), -C(\vec{\xi})\} = -(\gamma_{ab,c}\xi^c + \gamma_{ac}\xi^c{}_{,b} + \gamma_{cb}\xi^c{}_{,a}) = \delta_{\vec{\xi}}\gamma_{ab}(x) \tag{2.25}$$

where $\delta_{\vec{\xi}}\gamma_{ab}(x)$ is the change in $\gamma_{ab}(x)$ induced by a coordinate transformation generated by $\vec{\xi}$. Hence, $-C(\vec{\xi})$ can be interpreted as the generator of spatial diffeomorphisms. The same holds in the case of $\pi^{cd}(x)$. The $C_a(x)$ constraints are normally called the "transverse" constraints or even "momentum" constraints.

b) $C_\perp(x)$ can be interpreted by using the result:

$\{\gamma_{ab}(x), C(\eta)\} = \eta(\det\gamma)^{-1/2}(2\pi_{ab}(x) - \pi_c{}^c(x)\gamma_{ab}(x))$ as responsible for the dynamics. It is thus called the "Hamiltonian" constraint. It can be shown also to generate deformations of a hypersurface normal to itself as it is embedded in $\Sigma \times I\!\!R \simeq \mathcal{M}$ and for this reason it is also known as "longitudinal" constraints.

Finally, a couple of observations have to be done with respect to the smeared version of the algebra of the constraints. (2.23). The first is that it is not the underlying Lie algebra of Diff$(\mathcal{M})$ of the theory. The second is related to the appearance of γ^{ab} in the r.h.s. of the third eq. in (2.23); it implies that what we have is not a genuine Lie algebra. Essentially, the Dirac algebra turns out to be the Lie algebra of Diff$(\mathcal{M})$ projected along and normal to a spacelike hypersurface and hence the above two troubles can be accounted.

2.2 Canonical Quantisation

From here on we consider the first-order action (2.18) where the canonical variables $\gamma_{ab}(x)$ and $\pi^{cd}(x)$ are related by the constraints (2.17). Whenever constraints are involved in a canonical theory the quantisation becomes difficult. Several possibilities may be tried. The most natural perhaps is to reduce the theory to a true canonical form by eliminating the constranits and the correspondent Lagrange multipliers before the quantisation is carried out. We have already faced the constraints equation of GR and, in fact, they cannot be solved in a closed form the only alternative being the Ashtekar formalism (where the correspondent constraints have a simpler structure, see chapter 3) and the perturbative weak-field methods we critiziced in chapter 1. Further unappealing reasons, given in [Ish91, Kuc91], make this reduction to the true canonical form a program not easy to implement.

The possibility that has received more attention is the one in which the complete set of variables $(\gamma_{ab}(x), \pi^{cd}(x))$ are given a quantum states and only at the quantum level the problem of the constraints and other (like gauge fixing) are tackled. The first step consists setting the form of the canonical commutation relations:

$$\begin{aligned} [\hat{\gamma}_{ab}(x), \hat{\gamma}_{cd}(x')] &= 0 \\ \left[\hat{\pi}^{ab}(x), \hat{\pi}^{cd}(x')\right] &= 0 \\ \left[\hat{\gamma}_{ab}(x), \hat{\pi}^{cd}(x')\right] &= i\hbar \delta_{(a}{}^{c}\delta_{b)}{}^{d}\delta(x, x') \end{aligned} \tag{2.26}$$

Remarks in order here are:

i) The Schrodinger representation is adopted since the canonical variables do not carry any time dependence (however, see the analysis of the quantum constraints below).

ii) Smeared operators should be introduced in order to avoid eventual use of the components of γ and π in a specific coordinate system. That is:

$$\begin{aligned} [\hat{\gamma}(h), \hat{\gamma}(h')] &= 0 \\ [\hat{\pi}(k), \hat{\pi}(k')] &= 0 \\ [\hat{\gamma}(h), \hat{\pi}(k)] &= i\hbar \int_{\Sigma} h^{ab}(x) k_{ab}(x) d^3x \end{aligned} \tag{2.27}$$

where h and k are, respectively, a tensor density and a tensor [6].

[6] The third of these equations is coordinate independent since the r.h.s. is integrated over all Σ

iii) Microcausality. The first equation in (2.26) might be interpreted as a mean to guarantee that the points on Σ can be taken as spacelike separated, independently tha spacetime structure that could be adopted.

iv) Affine commutation relations. By insisting in the metric character of $\gamma_{ab}(x)$ it should be investigated whether there exists the correspondent quantum operator and if it is compatible with the above canonical commutation relations. It has been argued that a positive definite smeared operator may correspond to the notion of the classical metric γ_{ab} and even more there are suggestions that certain degeneracy should be allowed [IK84, Ash87, Wit88, Hor91], *i.e.* the action of the smeared operator on non-zero vectors can give zero. Furthermore, it turns out to be case that there is no compatibility of the semi-definite smeared operator and the canonical commutation relation. Affine commutation relations should substitute the canonical ones [7] [Ish84].

2.2.1 Treatment of the constraints a la Dirac

What Dirac thought us concerning the quantisation of the theories with constraints is that such constraints should be imposed on the physical states. In the present case we have:

$$\begin{aligned} C_a(\hat{\gamma},\hat{\pi})\Psi &= 0 \\ C_\perp(\hat{\gamma},\hat{\pi})\Psi &= 0 \end{aligned} \tag{2.28}$$

They are the quantum analogue of the classical result about the equivalence of the constraints and dynamical equation, i.e., they are the sole technical content of the theory of QG. This can be related also to the following result. The canonical Hamiltonian (2.19) is taken with N and $\vec{N}$ regarded as c-number function in constructing the "Schrodinger equation":

$$i\frac{d}{dt}\Psi_t = \hat{H}[N,\vec{N}]\Psi_t \tag{2.29}$$

which by virtue of (2.28) implies tha Ψ_t is time independent. Also, it seems to be meaningless to talk about "Schrodinger" or "Heisenberg" picture since matrix elements between physical states in both pictures will coincide. These aspects of QG can be traced back to the absence of any intrinsic definition of "time" in GR. We have not gauge-fixed the theory and hence no such coordinate has been selected.

The problem arising when the implementation of Dirac scheme is attempted may be very severe. Probably among the most difficult ones whether

[7] The analogue is a particle restricted to move in $\mathbb{R}^+$. Canonical commutation relation, $[\hat{x},\hat{\pi}] = i\hbar$, imply the spectrum of $\hat{x}$ is $\mathbb{R}$. However, compatibility is got by replacing the conventional commutation relation by the affine algebra: $[\hat{x},\hat{p}] = i\hbar\hat{x}$ [KA70]

and to what extent the classical Poisson algebra (2.22) structure can be translated into the quantum theory. Other important issues are:

1) Regularisation and renormalisation of the operators constraints (2.17). The origin of these problem is the non-linearity of these equations in the field operators expressed also as products evaluated at the same point. This is the analogue of the ultraviolet divergence problem in the quantum field theory.

2) Operator-ordering. Its origin is the appearance of product operators in the constraints and more precisely,

a) *e.g.* it has to be decided where to place $\hat{\gamma}_{ab}(x)$ on the r.h.s. of the third equation in (2.22). A restriction here can be set on that no further constraints on the physical state vectors desired to be generated by the commutators of the given first-class constraints.

b) the hermeticity or non-hermeticity of the constraints [HK90a, HK90b]. The non-hermeticity option comes about because of the absence of a well stablished relation between the Hilbert space structure carrying the representation of the canonical (or affine) algebra and the one that should be imposed on the physical states (i.e. those satisfying tha constraints).

c) singular operator products in the constraints seem to imply that any ordering is likely to be ambiguous [FJ88].

d) an anomaly should, possibly, be present in the theory as a genuine Planck length effect. The mathematical problem here is that, for example, not much is known about central extension of the Dirac algebra.

3) The unexistent clear definition of the inner product on the physical states to be constructed from the Hilbert space structure on the original space $\mathcal{H}$ that carries the representation of the canonical, or affine, commutation relations.

4) The recovering of the Diff($\mathcal{M}$) group. This is the converse problem of the translation of the Diff($\mathcal{M}$) invariance into the Dirac algebra in the canonical decomposition [IK84].

2.2.2 The meaning of the quantum constraints

Our aim here is to extract the information from the quantum constraints since, as we have seen, they are all the dynamical contents of QG. We will see that the quantum version of the vector constraints imposes a structure on the domain of the state vectors of QG leading to the notion of superspace while the classical Hamiltonian constraint become the so called Wheeler-DeWitt equation which provides us with the, more properly said, dynamics of gravity.

To achieve the above goals, however, the introduction of a given representation for the canonical algebra is needed. It is natural to adopt the analogue of the quantum operators associated to the canonical conjugated variables in standard quantum mechanics, i.e.

$$\begin{aligned}(\hat{\gamma}_{ab}(x)\Psi)[\gamma] &\equiv \gamma_{ab}(x)\Psi[\gamma] \\ (\hat{\pi}^{cd}(x)\Psi)[\gamma] &\equiv -i\hbar\frac{\delta\Psi}{\delta\gamma_{cd}(x)}[\gamma]\end{aligned} \tag{2.30}$$

for the quantum operators $\hat{\gamma}$ and $\hat{\pi}$. They are commonly used in spite of the following:

1) the incompatibility of the positiveness (or semi-positiveness) of the classical Riemannian metric with the above canonical algebra implies that the states functionals do not have domain Riem(Σ). This is possible only if the commutation relations are the affine ones.

2) the problem of the measure. There do not exist Lebesgue measures on Riem(Σ) of Riemannian metrics with which an hermitian inner product between state vectors can be defined [8].

3) distributional metrics are expected to be objects around which any measure can be concentrated. However, Riem(Σ) is not a vector space and hence its dual space cannot be defined (at least conventionally). The above distributions could live in such a dual space. On the other hand, affine commutation relations do allow an appropriate distribution of a distributional metric and, furthermore, they admit representations in which state vectors are concentrated on distributional analogues of degenerate metrics as well as some in which the state vector has an internal index analogously to the spin of a relativistic particle [Ish84, IK84].

Now we face the interpretation of the quantum momentum-constraint. We expect as in the classical case, to have the $\hat{C}(\vec{\xi})$ as the generator of Diff(Σ). While insisting in keeping at quantum level the structure of the classical algebra of the $\hat{C}(\vec{\xi})'s$ (i.e. Diff(Σ) algebra) operator-ordering problems come about.

One way to avoid these problems is to force the $\hat{C}(\vec{\xi})'s$ generators to form a hermitian representation of the algebra of Diff(Σ). Thus it is assumed that the quantum theory carry a unitary representation of Diff(Σ) and the quantum momentum constraint is translated into [9]:

$$(D(f)\Psi)[\gamma] = \Psi[\gamma] \tag{2.31}$$

[8] There is the possibility of introducing an infinite dimensional weighted measure which requires $(\hat{\pi}^{ab}(x)\Psi)[\gamma] \equiv -i\hbar\frac{\delta\Psi}{\delta\gamma_{ab}(x)}[\gamma] + i\rho(\gamma)\Psi[\gamma]$ where $\rho(\gamma)$ is a function that compensates for the weight factor in the measure [Ish91]

[9] Roughly, $D(f)$ is an element of the group Diff(Σ) and then it can be obtained by exponentiating the generators $\hat{C}(\vec{\xi})$ belonging to the Lie algebra of Diff(Σ). Since $(\hat{C}(\vec{\xi})\Psi)[\gamma] = 0$ we get $(exp(\hat{C}(\vec{\xi}))\Psi)[\gamma] = \Psi[\gamma]$, the desired result.

where $D(f)$ is a unitary operator representing $f \in \text{Diff}(\Sigma)$. On the other hand, the natural representation of the operators $\hat{\gamma}$ and $\hat{\pi}$ (2.30) suggests the action of $D(f)$ on Ψ as:

$$(D(f)\Psi)[\gamma] \equiv \Psi[f^*\gamma] \tag{2.32}$$

where $f^*\gamma$ is the usual pull-back of γ by diffeomorphism $f : \Sigma \to \Sigma$. From the above two aspects of the action of $D(f)$ on Ψ one is lead to the conclusion that

$$\Psi[f^*\gamma] = \Psi[\gamma] \;\; \forall f \in \text{Diff}(\Sigma),\; \gamma \in \text{Riem}(\Sigma) \tag{2.33}$$

We see the group $\text{Diff}(\Sigma)$ acts on the space $\text{Riem}(\Sigma)$ by sending γ to $f^*\gamma$ both elements of $\text{Riem}(\Sigma)$ through $f \in \text{Diff}(\Sigma)$. Modulo metrics with isometry groups [10], one can think $\text{Riem}(\Sigma)$ as a fiber bundle with base space $\text{Riem}(\Sigma)/\text{Diff}(\Sigma)$ and fiber the orbits of $\text{Diff}(\Sigma)$.

The base space $\text{Riem}(\Sigma)/\text{Diff}(\Sigma)$ of "inequivalent Riemannian metrics" under diffeomorphisms was called superspace by J.A.Wheeler [Whe64, Whe68]. The quantum vector constraints in its version (2.33) says that the state functional Ψ is constant on the orbits of $\text{Diff}(\Sigma)$ and in consequence it is a function on superspace, namely, it is superspace the true domain space of the QG state vectors.

Among the cautionary remarks once the above view is adopted are:

i) θ-states may be present due to the possible existence of non-trivial transformations under large diffeomorphism which cannot be continuously connected to the identity. We have discussed here only infinitesimal transformations [FS80, Ish81].

ii) (2.32) contains a unitary action only if the Hilbert space measured on the domain space is itself a $\text{Diff}(\Sigma)$ invariant which probably does not exist. The only alternatives are to modify the structure of (2.32) and hence the idea that the state functional is constant on the $\text{Diff}(\Sigma)$ orbits does not hold anywhere.

iii) If the domain space of the state vectors is a space of distributional metrics, the action of $\text{Diff}(\Sigma)$ on it would change. The bundle picture is not longer correct.

The analysis of the quantum hamiltonian constraint is not as straightforward as in the case of the quantum vector constraint. Essentially, the explicit γ factor in the r.h.s. of (2.22) spoils the Lie algebra structure and as a consequence the operator-ordering problem is far more severe. By choosing

[10] This problem can be circumvented by considering only those diffeomorphisms that leave fixed some particular frame at a base point in Σ

the simple ordering in which the π^{cd}'s are always to the right of the $\gamma'_{ab}s$ the quantum hamiltonian constraint becomes:

$$-\hbar^2\kappa^2\mathcal{G}_{abcd}(\gamma)\frac{\delta^2\Psi}{\delta\gamma_{ac}\delta\gamma_{bd}}[\gamma]-\frac{(\det\gamma)^{1/2}}{\kappa^2}R(\gamma)[\gamma]=0 \qquad (2.34)$$

where $\mathcal{G}_{abcd}(\gamma) \equiv (\gamma_{ab}\gamma_{cd} - 1/2\gamma_{ac}\gamma_{bd})(\det\gamma)^{-1/2}$. They are the Wheeler-DeWitt equation (WDW) and the metric on Riem(Σ) respectively [DeW67a].

We now briefly account the main problems concerning WDW equation. Their relevance coming from the fact that all the canonical quantisation program has been reduced to this equation.

First, factor ordering. The WDW equation was got by using a specific ordering inspired on simplicity. Other orderings are possible, for example one commonly used is to express the "kinetic term" (the one containing functional derivatives) as a covariant functional Laplacian taking the DeWitt metric as the underlying structure. The appealing of this ordering is the invariance under "coordinate transformations" on Riem(Σ). account when deciding the operator ordering, is whether the $\hat{C}_\perp(x)$ are expected to be hermitian.

Second, regularization. The second order functional derivative taken at the same spatial point when acting on some state functionals is likely to produce $\delta^3(0)$ singularities. A regularization procedure will be eventually required.

Third, time and time evolution. These concepts should be introduced in some way. The key idea is to consider time as an internal property of the gravitational system, probably including matter, instead of taking it as an external parameter of the universe.

Fourth, solutions of WDW. The immediate plan of attack may be to handle it by using the notions on functional differential equations. The validity of this view depends on the interpretation of the constraint equation $\hat{C}_\perp(x)\Psi = 0$. If it is considered to be self adjoint functional differential equation with eigenvector Ψ corresponding to the eigenvalue zero, some sort of boundary conditions should be imposed on Ψ, as in conventional eigenfunction problems. The theory, however, does not give much information about it. In addition, problems arising due to the apparent zero eigenvalue lead to the suggestion that a renormalisation of the Wheeler-DeWitt operator is required [Ish91]. Other ways to look for solutions of WDW are the expansion in $1/G$ corresponding to a perturbation theory where the coupling constant is the coefficient in front of $R(\gamma)$ in WDW and the WKB approximation, which has been used in tackling the problem of time. There exists other possibility that has become the most popular way of studying the WDW. This is the minisuperspace technique, it involves freezing all but a finite number of the infinite degrees of freedom in Riem(Σ) and quantising the small number that remain. WDW becomes a second order partial differential equation that can be studied using the conventional methods of differential equations. Such an approach is mainly used in the studies of quantum cosmology since the finite degree of freedom can be chosen in a way that is adopted to the classical models of cosmology. It must be noted that there is no way of estimating the effect of the infinite degrees of freedom that are dropped and thus any conclusion coming from this approach should be handled carefully. Remarkable, however, is the utility of the minisuperspace models in the discussion of the problem of time, interpretation of the state vector and related issues.

We end this review of methods for solving WDW by taking about the use of functional integrals. The motivation is traced back to the result of ordinary quantum mechanics that a solution of the time dependent Schrödinger equation $i\hbar\partial\Psi/\partial t = \hat{H}\Psi$ can be given as

$$\Psi(x,t) = \int \mathcal{D}x[s]e^{\frac{i}{\hbar}\int_{t_0}^{t} L(s)ds} \tag{2.35}$$

where the path integral is over paths that end at the point x of the path at the initial time t_0. The case of gravity was first studied in detail by C. Teitelboim [Tei82, Tei83, CTe83], and has been recently used in the Hartle-Hawking approach to quantum cosmology [HH83] in which the euclidianised version of the Hilbert-Einstein action is adopted. It was formally shown that the state function constructed in this form is in fact a solution of WDW [Hal88, Hal90]. It is worth mentioning that all the above proposal provide approximate solutions in the sense that the full WDW is not solved. The situation seem to be improved by the introduction of the Ashtekar variables (see chapter 3).

2.2.3 Minisuperspace Model: an example

Some of the features of the WDW equation can be seen more easily in terms of a minisuperspace model of canonical QG. We present here the homogeneous and isotropic Robertson-Walker cosmological model whose metric is given by

$$ds^2 \equiv -[N(t)]^2dt^2 + [a(t)]^2h_{ab}dx^a dx^b \tag{2.36}$$

where h_{ab} is the metric for a three-space of constant curvature κ. The three possibilities $\kappa = +1, 0, -1$ correspond to take Σ as a three sphere, flat and hyperbolic space respectively. We can see that N^a does not appear. This amounts to work in the Diff(Σ) gauge in which $N^a = 0$. This is a feature of minisuperspace models which does not produce important limitations on the information that can be extracted from them.

The Einstein tensor $G_{\mu\nu}$ computed with the above metric is non-zero and, by consistency with the Einstein equations, a matter component can be introduced. In the case a scalar field is taken to be the matter contents of the theory, it can be shown that the dynamical equation can be produced by starting with the first order action

$$S[a, \pi_a, \phi, \pi_\phi, N] \equiv \int (\pi_a \dot{a} + \pi_\phi \dot{\phi} - NC) dt \tag{2.37}$$

where units in which $16\pi G/c^4 = 1$ have been used, and

$$C \equiv -\frac{\pi_a^2}{24a} - 6\kappa a + \frac{p_\phi^2}{2a^3} + a^3 V(\phi). \tag{2.38}$$

The variables $\pi_a, a, \pi_\phi, \phi, N$ are to be varied independently.

The evident problems now are two. The first concerns the possible values for tha classical variable $a : a \geq 0$. The second is related to the operator ordering problems in the term $\pi a^2/24a$.

The first problem can be tackled in several ways, for instance:

1) Impose the standard commutation relation: $[\hat{a}, \hat{\pi}] = i\hbar$. They imply the Hilbert will be $L^2(I\!R, da)$, where $I\!R$ means that the spectrum of $\hat{a}$ is all $I\!R$ and da is the correspondent measure, when the operators are defined as

$$\begin{aligned} (\hat{a}\Psi)(a) &\equiv a\Psi(a) \\ (\hat{\pi}_a\Psi)(a) &\equiv -i\hbar \frac{d\Psi}{da}(a) \end{aligned} \tag{2.39}$$

the problem is reduced to the interpretation of the negative values of a.

2) Use the Hilbert space $L^2(I\!R^+, da)$ of function with support on $I\!R_+$ keeping the usual commutation relations. $\hat{\pi}^a$ is not longer a self adjoint operator, but it is possible to handle $\hat{C}_\perp$ in a way for it to be self adjoint.

3) Perform a canonical transformation at classical level, say $a = e^\Omega$. Ω ranges over all $I\!R$ and hence the canonical commutation relations can be used with the Hilbert space $L^2(I\!R, da)$. The associated conjugated variable is $\pi_\Omega \equiv e^{-\Omega}\pi_a$; the constraint becomes

$$C \equiv e^{-3\Omega}\left(-\frac{\pi_\Omega^2}{24} + \frac{\pi_\phi^2}{2}\right) - 6\kappa e^\Omega + e^{3\Omega}V(\phi) \tag{2.40}$$

4) Use affine commutation relations instead of canonical ones. The affine momentum is related classically to the canonical momentum by: $p_a \equiv a\pi_a$. The constraint now is:

$$C \equiv \frac{1}{a^3}\left(-\frac{p_a^2}{24} + \frac{\pi_\phi^2}{2}\right) - 6\kappa a + a^3 V(\phi). \tag{2.41}$$

It is possible to define a self-adjoint representation of the affine commutation relations on the Hilbert space $L^2(\mathbb{R}_+, da/a)$ with:

$$
\begin{aligned}
(\hat{a}\Psi)(a) &\equiv a\Psi(a) \\
(\hat{p}_a\Psi)(a) &\equiv -i\hbar a\frac{d\Psi}{da}(a)
\end{aligned}
\tag{2.42}
$$

NB. The transformation $a \longrightarrow e^{\Omega}$ is the connection between this approach and number 3.

A WDW equation can be written now. Let us take method 3 with Ω satisfying canonical commutation relations, apart of the operator ordering, we get

$$
\left[\hbar^2 e^{-3\Omega}\left(\frac{1}{24}\frac{\partial^2}{\partial\Omega^2} - \frac{1}{2}\frac{\partial^2}{\partial\Psi^2}\right) - 6\kappa e^{\Omega} + e^{3\Omega}V(r)\right]\Psi(\Omega,\phi) = 0 \tag{2.43}
$$

As simple as it may appear this model illustrates several features of the full canonical QG.

2.3 Final comments

We end this chapter by noting the advantages of a canonical approach to QG. Since canonical QG is discussed in an operator-based framework, the involved problem appear more explicitely than in the conventional methods of quantisation of gravity mentioned in chapter 1. In the canonical framework several techniques are background metric- independent. This leads to the possibility of developing non-perturbative analysis of QG and therefore the problems of quantum cosmology, spacetime singularities and related issues can be faced in a more suitable way. Also, as we have seen, in this approach a strong emphasis is made on the geometrical structure of the spacetime as viewed in GR, and thus, the extent to which it holds in the quantum theory can be adressed as well as other deep conceptual problems like the "time" one.

Chapter 3

Ashtekar and Loop formalisms of Quantum Gravity

Apart from the severe conceptual problems mentioned in chapter 1, progress in the canonical approach to gravity has been far slowed down by the highly non trivial structure of the field equations when expressed in the canonical variables (γ_{ab}, π^{cd}). Recently, A. Ashtekar introduced a set of new variables which improved the situation [Ash86b, Ash86a, Ash87]. In terms of the Ashtekar variables all the equations of the theory become polynomial (at most quartic). Also, when these variables are used, a relation between Yang-Mills theory and GR is revealed. This relation allows an exchange of techniques between them. These two features, polynomiality and Yang-Mills like structure, hold even in the cases where one adds to the gravity system a non-zero cosmological constant, matter fields (scalars, spinors or Yang-Mills fields) or considers the supersymmetric extension of the theory [AAT88, Jac88b, Jac88a].

Shortly later, C. Rovelli and L. Smolin built a Loop formalism which, when adapted to the Ashtekar one (without matter fields), produced the first non trivial solutions to the dynamical equations of QG [RS88, RS90] (see also [JS88b, Ble90]) in terms of loop-supported objects. As a consequence, based on these two approaches, nonperturbative canonical QG is resurging.

We present here a brief description of the above formalisms, the details of which can be found in the literature quoted in turn. Our aim is to remark their appealing features but also to stress their problems w.r.t to the conventional approach given in chapter 2.

3.1 Classical Canonical Gravity with Ashtekar variables

As in chapter 2 we discuss, for the sake of simplicity, the case of source free GR. (For inclusion of the sources and supergravity extensions see [AAT88, Jac88b, Jac88a]). There exists several approaches to arrive at the Ashtekar variables, all equivalent, each one stressing different aspects of the construction (see e.g. [Ash90] and references there). Here, we follow Rovelli's construction [Rov91a] in which the interpretation of the new variables, as related to the conventional entities in GR, is more clearly established. This is so, partly, because an explicit coordinate system is chosen. The global character, as provided by the introduction of a foliation analogue to that used in the standard canonical formalism, is discussed in [Ash90].

We consider Ricci flat ($R_{\mu\nu} = 0$) metrics $g_{\mu\nu}$ on a real four manifold $\mathcal{M}$ and fix a compact three-manifold Σ s.t. $\mathcal{M}$ will have the topology $\Sigma \times \mathbb{R}$. The non compact case, which we will not consider here, would introduce boundary terms in the correspondent expressions as well as asymptotic conditions on the dynamical variables. For the analysis of such a case see [Ash90].

The basic canonical variable in this program is not longer a three-metric but a densitized triad on Σ. A triad $e^i(x)$, $i = 1, 2, 3$, is a set of three linearly independent cotangent vectors at the point $x \in \Sigma$ with components $e^i{}_a(x)$, $a = 1, 2, 3$. The metric tensor γ_{ab} is related to the triad fields by

$$\gamma_{ab}(x) = \sum_{i,j=1}^{3} e^i{}_a(x) e^j{}_b(x) \delta_{ij} = \mathrm{Tr}[e_a(x) e_b(x)]. \tag{3.1}$$

In the last equality the "i" index is taken to belong to the Lie algebra $so(3)$ of the Lie group $SO(3)$. Hence, $e_a(x) = e^i{}_a(x)\lambda_i$, where λ_j, $j = 1, 2, 3$, are the generators of $SO(3)$ normalized so that $\mathrm{Tr}(\lambda_i \lambda_j) = \delta_{ij}$.

Whenever fermions are coupled to gravity one introduces triads and, therefore, an extra gauge group comes about: $C^\infty(\Sigma, SO(3))$, the set of smooth maps from Σ to $SO(3)$. The origin of the gauge symmetry comes from the fact that an element $\Lambda \in C^\infty(\Sigma, SO(3))$ acts on a triad in the way:

$$e^i{}_a(x) \mapsto e^j{}_a(x) \Lambda_j{}^i(x) \tag{3.2}$$

leaving the metric tensor γ_{ab} invariant.

The Ashtekar canonical variables are $(\tilde{E}^a{}_i, A_b{}^j)$, where $\tilde{E}^a{}_i(x) \equiv (\det\gamma(x))^{\frac{1}{2}} e^a{}_i(x)$ is a triad density of weight one[1] and $A_b{}^j$ is its conjugate canonical momentum[2].

[1] $e^a{}_i$ is the inverse of the matrix $e^i{}_a$

[2] modulo an i, see below.

$A_b{}^j$ is a $SO(3)$-connection one-form on Σ, where "b" is a vector index and "j" the $SO(3)$ internal index.

3.1.1 The canonical structure

Now we outline how the Ashtekar variables are defined starting from GR and how they are cast in a canonical structure.

Lets split up the construction in steps as follows. 1) Start from the Palatini first-order formalism of GR and then, adapt it to a tetrad formalism. 2) Define a convenient complex selfdual spin connection s.t. GR is reformulated in terms of it and the real tetrad fields. 3) Split up the Lagrangian variables in an ADM-like way to go to the canonical theory.

The above steps consist in the following.

1) The Palatini first order formalism of GR asserts that the Einstein equations can be obtained from the action[3]

$$S[g,\Gamma] = \int d^4x(\det g)^{\frac{1}{2}} g^{\mu\nu} R_{\mu\nu}[\Gamma] \tag{3.3}$$

which contains, at most, first order derivatives of the affine connections $\Gamma^{\sigma}_{\mu\nu}$, and these affine connections and the metric are to be varied independently. Variation w.r.t. Γ yields the definition of the Christoffel symbols in terms of the metric, while varying w.r.t. the metric produces the Einstein equations in terms of the Γ's. Everything can be expressed using only the metric by substituting the former equation in the latter. The tetrad formalism amounts to introduce tetrad fields s.t.

$$g_{\mu\nu}(x) = e_\mu{}^I(x) e_\nu{}^J(x) \eta_{IJ}, \quad \forall x \in \mathcal{M}, \tag{3.4}$$

where $\eta_{IJ} = \mathrm{diag}\{-1,+1,+1,+1\}$ is the Minkowski metric, which acts on uppercase latin indices as usually. An internal $SO(3,1)$ symmetry, corresponding to the Lorentz group, is introduced in this way [DI76]. The analogue of the above Γ's are now the spin connections defined via the second Cartan structure equation

$$\partial_{[\mu} e^I_{\nu]} + \omega^I_{[\mu\ J}[e] e^J_{\nu]} = 0. \tag{3.5}$$

In terms of the tetrads and spin connections the Riemannian curvature can be written as

[3] From here on we will use units in which $c = 1$ *and* $16\pi G = 1$.

$$R^{\mu}{}_{\nu\tau\sigma}[g[e]] = e^{\mu}{}_{J} e_{\nu J} R_{\tau\sigma}{}^{IJ}[\omega[e]]. \tag{3.6}$$

When it is introduced in the Palatini action (3.3), and the rest of this action is expressed in tetrad language, one gets

$$S[e,\omega] = \int d^4x (\det e) e^{\mu}{}_{I} e^{\nu}{}_{J} R_{\mu\nu}{}^{IJ}[\omega], \tag{3.7}$$

where $\det e$ is the determinant and $e^{\mu}{}_{I}$ is the inverse of the matrix $e^{I}{}_{\mu}$. This action yields as dynamical equations the second Cartan structure equation (3.5), defining $\omega[e]$, and the Einstein equations for the tetrad fields, when the correspondent independent variations are performed.

2) Next, we define the selfdual spin connection ${}^4A_{\mu}{}^{IJ}$ by the properties

$$\begin{aligned} {}^4A_{\mu}{}^{IJ} &= -\frac{i}{2}\epsilon^{IJ}{}_{MN}\,{}^4A_{\mu}{}^{MN} &(3.8)\\ {}^4A_{\mu}{}^{IJ} &= [\delta^{I}{}_{M}\delta^{J}{}_{N} - \frac{i}{2}\epsilon^{IJ}{}_{MN}]\omega_{\mu}{}^{MN} = \omega_{\mu}{}^{IJ} - \frac{i}{2}\epsilon^{IJ}{}_{MN}\omega_{\mu}{}^{MN}, &(3.9) \end{aligned}$$

where ϵ is the totally antisymmetric tensor normalized to the unity and δ is the Kronecker delta, both in four dimensions, with indices acted upon by η_{IJ}.

The curvature of the selfdual spin connection[4] ${}^4A_{\mu}{}^{PQ}$, ${}^4F_{\mu\nu}{}^{PQ}$, and that of $\omega_{\nu}{}^{MN}$, $R_{\mu\nu}{}^{MN}$, turn out to be related by

$${}^4F_{\mu\nu}{}^{IJ}[A[\omega]] = R_{\mu\nu}{}^{IJ}[\omega] - \frac{i}{2}\epsilon^{IJ}{}_{MN}R_{\mu\nu}{}^{MN}[\omega] = [\delta^{I}{}_{M}\delta^{J}{}_{N} - \frac{i}{2}\epsilon^{IJ}{}_{MN}]R_{\mu\nu}{}^{MN}[\omega]. \tag{3.10}$$

We conclude that ${}^4F_{\mu\nu}{}^{IJ}$ is the selfdual part of $R_{\mu\nu}{}^{IJ}$ and, in fact, from here, we see that it is composed by a real and an imaginary part[5]. It can be shown that the integral

$$T[e,\omega] = \int d^4x (\det e) e^{\mu}{}_{I} e^{\nu}{}_{J} \epsilon^{IJ}{}_{MN} R_{\mu\nu}{}^{MN}[\omega] \tag{3.11}$$

is a topological quantity in the sense that

$$\delta_e T[e,\omega[e]] = 0, \tag{3.12}$$

where δ_e means local variations w.r.t. $e^{\lambda}{}_{K}$, and therefore the imaginary component of ${}^4F_{\mu\nu}{}^{IJ}$ (see(3.10)) will not yield any dynamical equation. This

[4]The selfdual connection defines a covariant derivative and hence a curvature given by ${}^4F_{\mu\nu}{}^{IJ}[A] = \partial_{\mu}^4A_{\nu}{}^{IJ} - \partial_{\nu}^4A_{\mu}{}^{IJ} + {}^4A_{\mu}{}^{IM}\,{}^4A_{\nu M}{}^{J} - {}^4A_{\nu}{}^{IN}\,{}^4A_{\mu N}{}^{J}$.

[5]$R_{\mu\nu}{}^{MN}[\omega]$ calculated from real ω's is a real quantity.

allows us to substitute in (3.7) $R_{\mu\nu}{}^{IJ}$ by ${}^4F_{\mu\nu}{}^{IJ}$ *without altering the contents of the theory.* Finally, using

$$\begin{aligned} S[e,\omega] &= \int d^4x(\det e)e^{\mu}{}_{I}e^{\nu}{}_{J}{}^4F_{\mu\nu}{}^{IJ}[A[\omega]] \\ &= \int d^4x(\det e)e_{\mu I}e_{\nu J}\ {}^4F_{\sigma\tau}{}^{IJ}[A[\omega]]\epsilon^{\mu\nu\sigma\tau}, \end{aligned} \tag{3.13}$$

and by going from the variables (e,ω) to (e,A), we arrive at the underlying action

$$S[e,A] = \int d^4x(\det e)e_{\mu I}e_{\nu J}\ {}^4F_{\sigma\tau}{}^{IJ}[A]\epsilon^{\mu\nu\sigma\tau}. \tag{3.14}$$

In turn, this action generates as equation of motion (3.9), with $\omega_{\mu}{}^{MN}[e]$ defined by (3.5), and the vanishing of the Ricci tensor. Note that ${}^4A$ is not an arbitrary complex Lagrangian variable but it has to be related to a real ω by (3.9) (see e.g. [Rov91a]). This is the Ashtekar theory in the Lagrangian formalism.

Originally, A. Ashtekar introduced the new variables at the canonical framework level. The Lagrangian versions of the formalism came later [Sam87, JS87, JS88a], among which the given by Capovilla et.al. [RCD89, RCJ90] is notable since the only variable appearing in the action is a connection, without any tetrad field at all.

3) We proceed now with the spliting *a' la ADM* of the tetrad variables. Lets make the following definitions, using the spatial indices $a = 1,2,3$,

$$N^a = g^{a0} = e^a{}_I e^{0I}, \tag{3.15}$$

$$\underline{N} = \frac{(\det g)^{\frac{1}{2}}}{\det\gamma} = \frac{\det e}{\det\gamma}, \tag{3.16}$$

$$\tilde{E}^a{}_I = (\det\gamma)^{\frac{1}{2}}[e^a{}_I - N^a e^0{}_I]. \tag{3.17}$$

We can see from the these definitions that $\underline{N}$ and $\tilde{E}^a{}_I$ are densities of weight -1 and $+1$, respectively. Furthermore, $\underline{N}$ differs from the ADM lapse function given in chapter 2 precisely in this aspect. The action (3.14), in terms of the tetrads only, contains the sixteen components of the tetrads. The same number of variables remains when going to the N's, $\tilde{E}$ variables, as it should be. Next, we go on by introducing the antisymmetric three indices tensor

$$\epsilon^{IJK} = e^0{}_L\epsilon^{LIJK} \tag{3.18}$$

and the related one index connection

$$A_\mu{}^I[e] = \epsilon^I{}_{JK}{}^4A_\mu{}^{JK}[e]. \tag{3.19}$$

Once all these variables are introduced in the action (3.14), and making use of the selfdual character of the curvature ${}^4F_{\mu\nu}{}^{IJ}$, they translate it into

$$S[\tilde{E}\ ,\underline{N}, N^a, e^0{}_I] = \int d^4x \ \Big\{ iA_a{}^I[e]\dot{\tilde{E}}{}^a{}_I + iA_0{}^I[e]D_b\tilde{E}^b{}_I + iN^a\tilde{E}^b{}_JF_{ab}{}^J + \underline{N}\tilde{E}^a{}_I\tilde{E}^b{}_J{}^4F_{ab}{}^{IJ}\Big\}, \tag{3.20}$$

where D_a and $F_{ab}{}^I$ have been identified with the covariant derivative and curvature, respectively, defined by the one index connection (3.19). We should remark here that in the above action, expressed in terms of ADM-like variables, the only dynamical variable is $\tilde{E}^a{}_I$. This is because its time derivative, $\dot{\tilde{E}}{}^a{}_I$, is the only time derivative that appears in the action. Of course, since we adopted an specific coordinate system, the one in which the "global time" is defined by the direction of $e^0{}_L$ (orthogonal to Σ), the time derivative refers to this frame. As expected, $\underline{N}$ and N^a, like in the conventional canonical approach, are not dynamical variables. However, now there exists extra non dynamical variables. They are the $e^0{}_I$ components of the tetrads coming from the internal gauge freedom. The three non dynamical objects can be considered as Lagrange multipliers and freely fixed.

We use in the following the internal gauge freedom to render the theory into the canonical structure. By choosing $e^0{}_i = 0,\ i = 1,2,3,$ and[6] $e^0{}_{I=0} = 1$, we are reducing the internal symmetry from the $SO(3,1)$ to the $SO(3)$ group associated to a triad formalism [DI76]. We get two basic results. One concerns the three indices antisymmetric tensor (3.18); its only non zero components are those for which I, J, K are 1,2 or 3 and thus $\epsilon^{IJK} \mapsto \epsilon^{ijk}$. The second is that the only non vanishing components of the one index connection $A_a{}^I[e]$ become $A_a{}^i[e]$. The important consequence of the latter is that the dynamical variables will be $\tilde{E}^a{}_i,\ \ a,i = 1,2,3$ ($\tilde{E}^a{}_0$ desappears) and they have the form $\tilde{E}^a{}_i = (\det\gamma)^{\frac{1}{2}} e^a{}_i$. In this gauge the action turns out to be

$$S[\tilde{E}\ ,\underline{N}, N^a, e^0{}_I] = \int dt \int_\Sigma d^3x \left\{ iA_a{}^i[e]\dot{\tilde{E}}{}^a{}_i + iA_0{}^j[e]C_j + iN^aC_a + \underline{N}C \right\}, \tag{3.21}$$

where

$$C_i[A,E] \ = \ D_b\tilde{E}^b{}_i, \tag{3.22}$$

[6]This is the normalization to the unit of the normal to the foliations as in the case studied in chapter 2

$$C_a[A,E] = \tilde{E}^b{}_i F_{ab}{}^i, \quad (3.23)$$
$$C[A,E] = \tilde{E}^a{}_i \tilde{E}^b{}_j {}^4F_{ab}{}^{ij}. \quad (3.24)$$

From here on, the arguments of every quantity will refer to space only as we have considered the decomposition of $\mathcal{M}$ in $\Sigma \times I\!R$ in the specific coordinate system.

From the above action the conjugate canonical momenta are defined as

$$p_a{}^i = \frac{\delta L}{\delta \dot{\tilde{E}}^a{}_i} = iA_a{}^i[e], \quad (3.25)$$

and the equations of motion are [Ash88b]

$$\dot{p}_b{}^i = -\underline{N}\tilde{E}^a{}_j F_{abk}\epsilon^{ijk} + iN^a F_{ab}{}^i, \quad (3.26)$$
$$\dot{\tilde{E}}^b{}_i = -iD_a(\underline{N}\tilde{E}^a{}_j \tilde{E}^b{}_k)\epsilon_{ijk} + 2D_a(N^{[a}\tilde{E}^{b]}{}_i). \quad (3.27)$$

$A_a{}^i$ can be interpreted as the three dimensional projection of ${}^4A_\mu{}^{IJ}$. It is usually referred as the *Ashtekar connection.* The complex phase space of the theory has coordinates $(p_a{}^i, \tilde{E}^k{}_j)$ with the Poisson bracket structure

$$\{p_a{}^i(x), p_b{}^j(y)\} = 0$$
$$\{\tilde{E}^a{}_i(x), \tilde{E}^b{}_j(y)\} = 0$$
$$\{p_a{}^i(x), \tilde{E}^b{}_j(y)\} = -\delta_a{}^b \delta^i{}_j \delta^3(x,y). \quad (3.28)$$

From (3.21) we can read off the Hamiltonian of the theory, since it has the structure $\int dt[p\dot{q} - H]$. It vanishes weakly, as expected according to the discussion in chapter 2 concerning the strcuture of GR. Note, however, that here we have introduced complex variables that could modify such a structure but they do not. This is a remarkable feature of the Ashtekar formalism.

The canonical structure of gravity with Ashtekar variables can be seen also from the perspective of complex canonical transformation by going from the standard canonical triad variables to the Ashtekar ones. This is given, e.g., in the contribution of J.L. Friedman and I. Jack to [Ash88b].

We shall follow the standard convention of taking $iA_a{}^i$ as the canonical conjugate variable to $\tilde{E}^b{}_j$ instead of the above momenta.

3.1.2 The constraint equations

By adopting the Ashtekar approach in GR, two kind of constraints emerge in the theory. The first kind are the primary constraints that follow from the definiton of the canonical momenta and that turn out to be a sort of reality condition on the connection $A_a^{\ i}$. The second kind are the secondary constraints, analogue of those in the conventional canonical treatment of GR given in chapter 2, that follow either from interpreting $\underline{N}$, N^a, $e^0{}_I$ as Lagrange multipliers (as we did) or by giving them the status of dynamical variables and then eliminating the redundant sector of phase space by a gauge fixing.

The first kind of constraints can be studied by analysing the momenta. From the definition (3.25) we have that their real part can be expressed as

$$Re\{p_a^{\ i}\} = Re\{i\epsilon^i{}_{jk}{}^4A_a^{\ jk}[e]\} = \epsilon^i_{jk}\epsilon^{jk}{}_{MN}\omega_a^{\ MN}[e] = \omega_a^{\ 0i}[e]. \tag{3.29}$$

They can be readily related to the standard canonical momenta in the triad formalism of gravity coming from a gauge fixing of a tetrad formalism [DI76]. On the other hand, we know that the real part contribution to the action (3.14), associated with the real part of (3.10), is just the standard tetrad action. Then we can interpret the real part of the momenta as just the momenta of the standard canonical triad formalism and, as in that case, they can be related to the extrinsic curvature. In the present case we should write

$$K_{ab} = Re\{p^{\ i}_{(a}e_{b)i}\}. \tag{3.30}$$

What about the imaginary part? It can be written as

$$Im\{p_a^{\ i}\} = Im\{i\epsilon^i{}_{jk}{}^4A_a^{\ jk}[e]\} = \epsilon^i{}_{jk}\omega_a^{\ jk}[e] \tag{3.31}$$

The second Cartan structure equation tells us that $\omega_a^{\ jk}[e]$ does not contain time derivatives of $e^a{}_j$; indeed, it is the three dimensional spin connection of the densitized triad $\tilde{E}$. Namely, the imaginary part of the momenta is completely constrained. This is equivalent to say that the following primary constraint holds

$$Im\{p_a^{\ i}\} - \epsilon^i{}_{jk}\omega_a^{\ jk}[\tilde{E}\ ] = 0. \tag{3.32}$$

This result can be traced back to the non dynamical character of the imaginary part of the action (3.14), associated with the imaginary part of (3.10), which is a topological term. It is worth noting that (3.32) can be expressed in a way that shows the character of "reality" of the constraint:

$$\bar{A}_a^{\ i} = A_a^{\ i} + 2i\epsilon^i{}_{jk}\omega_a^{\ jk}[\tilde{E}\ ]. \tag{3.33}$$

The bar here means complex conjugated. This is why it is known as the *reality constraint.* To show the polynomial character of such a constraint, as the other constraints do, there is the alternative way of expressing it [Ash90]

$$Re\{D_c(\tilde{E}^{[c}_{i}\tilde{E}^{(a]}_{j})\tilde{E}^{b)}_{k}\epsilon^{ijk}\} = 0. \tag{3.34}$$

(As usual, (ab) means symmetrization while $[ca]$ is antisymmetrization.) Even if the structure of all the constraint equations is now similar, we will see that at the quantum level an essential difference between the *reality* one and the others comes about.

In the Ashtekar formalism then, the constraint contents of the theory altogether is the above *reality* constraint and the second type, we were talking about above:

$$C_i[A,E] = 0, \quad C_a[A,E] = 0, \quad C[A,E] = 0. \tag{3.35}$$

It is convenient, as in the case of the conventional cononical formalism, to understand the geometrical meaning of the Ashtekar constraints.

It is the Poisson bracket structure, we defined above, that plays a basic role in the interpretation of the constraint equations. From the Poisson brackets between the constraints, and with some functions, we should decode the information they carry, as it was done in chapter 2.

1) *Gauss law* constraints C_i. The Poisson brackets between them,

$$\{C_i(x), C_j(y)\} = \epsilon_{ij}{}^{k} C_k(x)\delta^3(x,y), \tag{3.36}$$

is the $so(3)$ current algebra. In general it can be associated to the invariance under the rotations in the internal space to which the subindices belong. In the present case, where the system is the spacetime geometry, it can be interpreted as the freedom in choosing the triads in terms of which, for instance, the three-metric can be expressed. Obviusly, this also occurs in the conventional canonical triad formulation for gravity. The Ashtekar scheme, however, gives further structure through the C_i's. This is seen by introducing the smeared version of C_i's, $C(\lambda)$, defined by

$$C(\lambda) = \int d^3x \ \lambda^i(x) C_i(x). \tag{3.37}$$

Its Poisson bracket with the Ashtekar connection yields

$$\{A_a{}^i, C(\lambda)\} = D_a\lambda^i = \delta_\lambda A_a{}^i, \tag{3.38}$$

which is just the transformation law of a connection in any Yang-Mills theory. The conclusion is, therefore, that the object we called Ashtekar connection, is a true connection defined on the hypersurface Σ of the spacetime manifold $\mathcal{M}$. This is the origen of the "Gauss law" name for the C_i's. It is in this sense that the phase space of GR has the same structure of a non-abelian Yang-Mills theory.

Note that the real part of A , being a spin connection, transforms as it should. On the other hand, its imaginary part transforms homogeneously since, taking the components $\lambda^i \in I\!R$, the non homogeneous term $\partial_a \lambda^i$ does not affect it. This imaginary part of A is related to the extrinsic curvature which then is well behaved.

2) *Spatial diffeomorphism constraint.* It has been shown, e.g. in [Ash88a], that the smeared version of the *vector* and *Gauss* constraints combination given by

$$C(\vec{N}) = \int d^3x N^a(x)[C_a(x) - A_a^{\ i}(x)C_i(x)] \tag{3.39}$$

yields the Poisson bracket with a function $f[A\ ,E]$

$$\{f[A\ ,E], C(\vec{N})\} = \mathcal{L}_{\vec{N}} f[A,E] = \delta_{\vec{N}} f[A,E], \tag{3.40}$$

where $\mathcal{L}_{\vec{N}}$ is the Lie derivative along $\vec{N}$ and $\delta_{\vec{N}}$ is the change of $f[A,E]$ under the infinitesimal coordinate transformation $\vec{x} \mapsto \vec{x} + \vec{N}(x)$. Thus $C(\vec{N})$ generates three-dimensional diffeomorphism transformations and so it is called the diffeomorphism constraint.

3) *Hamiltonian constraint.* It can be shown also that given the smeared scalar constraint

$$C(\underline{N}) = \int d^3x \underline{N}(x)C(x), \tag{3.41}$$

the Poisson bracket with $f[A,E]$ turns out to be

$$\{f[A,E], C(\underline{N})\} = \underline{N}\partial_t f[A,E]. \tag{3.42}$$

Here t means the "global time function" associated with the chosen foliation s.t. the time direction is that of $e^0{}_J$. This $C(\underline{N})$ is then called the Hamiltonian constraint.

3.2 Ashtekar Quantisation Program

We outline the quantisation program given by A. Ashtekar [Ash88b]. The scheme will be given as insisting in the canonical algebra and the connection representation even when there exists alternatives. For instance, one of them is the $\mathcal{T}$-algebra and its loop representation that will be considered in the next section.

Lets set the program in the following steps.

1) Introduce operator-valued distributions, $\tilde{E}^a{}_i(x)$ and $A_b{}^j(x)$, subject to the canonical commutation relations

$$\begin{aligned}
\left[\hat{A}_a{}^i(x), \hat{A}_b{}^j(y)\right] &= 0, \\
\left[\hat{\tilde{E}}^a{}_i(x), \hat{\tilde{E}}^b{}_j(y)\right] &= 0, \\
\left[\hat{\tilde{E}}^a{}_i(x), \hat{A}_b{}^j(y)\right] &= \hbar \delta^a{}_b \delta_i{}^j \delta^3(x,y).
\end{aligned} \tag{3.43}$$

2) On the algebra generated by these operators, introduce a $*$-operation by requiring that $\tilde{E}^a{}_i$ be its own $*$-adjoint as well as its "time derivative", yielded by the commutator with the Hamiltonian, be its own $*$-adjoint. Thus the reality conditions are incorporated at the algebraic level.

3) Choose a representation for the algebra. The most convenient choice is to use for states holomorphic[7] functions of the complex connection $A_b{}^j$, represent $\hat{A}_b{}^j$ as a multiplication operator and $\hat{\tilde{E}}^a{}_i$ as a differential operator, $\hbar \frac{\delta}{\delta A_a{}^i}$. At this stage the $*$-relations are ignored. This is so because to incorporate such relations requires the availability of a Hermitian inner product. An unambiguous inner product is expected to exist only on *physical states* and thus it is appropriate to postpone the incorporation of these " quantum reality" conditions until after the *physical states* have been extracted.

4) Solve the quantum constraints. Since at the classical level the constraints involve only $A_b{}^j$ and $\tilde{E}^a{}_i$ and **not** their complex conjugates, we can continue avoiding to use the $*$-relations in the algebra. The space of solutions is the complex vector space of physical states. The operators of interest will be those in our algebra that map this space to itself.

5) On this space of physical states, introduce a Hermitian inner product that now incorporates the $*$-relations. The operators $\tilde{E}^a{}_i$ and its "time derivative" themselves will not be observables. Nevertheless, the $*$-relations of the initial algebra induce $*$-relations on the space of observables, which maps the space of physical states onto itself, and these relations are to be faithfully reflected in the Hermitian adjointness relations by the appropriate choice of the inner product. Thus, in the quantum theory, the secondary constraints (Gauss,vector and scalar) and the reality conditions are not on

[7]In simple examples as the harmonic oscillator this requirement is equivalent to choosing the Hilbert space $L^2(\mathbb{R}, dx)$. In more general cases the situation is not clear.

the same footing; the former determine the space of physical states (step 4) while the latter constrains the inner product on this space. In practice, the introduction of an inner product may require that we isolate "time" from among the various components of $A_b{}^j$ and interpret the scalar constraint as a Schrödinger equation.

6) Select physically interesting observables and make predictions.

Several remarks are in order here.

a) The variable which is diagonalised is $A_a{}^i$, the analogue of the canonical momentum in the conventional theory, whereas the triad variable acts through a functional derivative. The A-representation is analogous to the functional Fourier transform of the representation used in the conventional canonical program, and thus the geometrical interpretation of the functionals on which the above operators act is very different from the familiar functionals $\Psi[\gamma]$.

b) When imposing the constraints a' la Dirac it is necessary first, to choose an ordering for the operator version of the constraints. Jacobson and Smolin [JS88b] chose the expressions

$$\hat{C}_i(x) = D_a \frac{\delta}{\delta A_a{}^i(x)} \tag{3.44}$$

$$\hat{C}_a(x) = F_{ab}{}^i(x) \frac{\delta}{\delta A_b{}^i(x)} \tag{3.45}$$

$$\hat{C}(x) = \epsilon^{ij}{}_k F_{ab}{}^k \frac{\delta}{\delta A_a{}^i(x) \delta A_b{}^j(x)} \tag{3.46}$$

which have the virtue that $\hat{C}_i$ and $\hat{C}_a$ correctly generate the Lie algebras of the gauge group $C^\infty(\Sigma, SO(3))$ and Diff(Σ), respectively. Remarkably, Jacobson and Smolin were able to find a number of exact solutions to the WDW equation in the present case: $\hat{C}\Psi = 0$. Among them there is the functional $\Psi[A] = 1$ that satisfy all the constraints. The lack of knowledge about measures in the space of connections does not allow any physical interpretation of this result. Another, formal, solution to the WDW equation is given by

$$\Psi_{\{\eta_a\}}[A] \equiv \prod_{a \in I} H_{\eta_a}[A] \tag{3.47}$$

where the product extendes over the finite set I of indices s.t. $\{\eta_a | a \in I\}$ is a set of smooth non intersecting loops, and

$$H_\eta[A] = Tr(Pexp \oint_\eta A) \tag{3.48}$$

is an element of a class of holomorphic $C^\infty(\Sigma, SO(3))$ gauge invariant functionals. The P means that the line integral is a path-ordered one and η is supposed to be smooth in Σ. Further solutions exists which include intersecting curves; they involve linear combinations of states corresponding to the different ways in which such curves can be spliced.

3) The main problems raised by following this line of work are: a) The loop based solutions given above comes from exploting the antisymmetry properties of F_{ab}. It is surprising they capture the full content of the WDW equation. b) The operator products in the quantum constraints are ill-defined since they contain factors of $\delta^3(0)$. The *regularisation* method is still subject of debate (a point-spliting regularization necessitates of a backgroun metric and curve, the ultimate effect of this unwanted background is notclear) c) The dependence of the functional solutions on loops in Σ amounts to their non DiffΣ invariant character. The DiffΣ constraints seem to be intractable here whereas in the conventional approach they are considered as innocuous. A way to get round this difficulty is given below in the loop formalism of QG.

The problem of finding physical states, and hence also of implementing the $*$-relations, remains open in the connection representation. Consequently, as far as full quantum gravity is concerned, so far, the connection representation has not led to qualitatively new insight about the dynamics of the gravitational field in the Planck regime. At present, its importance lies mainly in the fact that it provides a suitable general framework to address certain conceptual issues of QG in a concrete way. Among these are the issue of time and the large gauge transformations as related to θ-vacua and CP-violation. They are discussed in [Ash90]. Also using the connection representation some solutions to the quantum minisuperspace models have been found [Kod88, Kod90].

3.3 Loop formalism

The motivations leading to this construction were the Ashtekar reformulation of GR and the Jacobson and Smolin's discovery of a class of solutions to the WDW equation, in terms of the Ashtekar variables, related to loops in three dimensions. No solution was found to the spatial diffeomorphism constraint. The loop representation was invented to solve this problem by introducing a representation space on which the spatial diffeomorphism group acts naturally, whereas the simplicity of the action of the Hamiltonian constraint in the selfdual representation is preserved.

Once the loop representation is introduced, the complete set of solutions to the constraints that generate diffeomorphisms of Σ are readily found. They can be related to a countable basis, whose elements are in one-to-one correspondence with the knot and link classes of Σ (More properly said, the elements are in one-to-one correspondence with the generalized link classes, which allows the loops to intersect and be kinked). The basic tool to handle the structure of the space of physical states of nonperturbative QG will be knot theory [Kau83, Kau87a, Kau87b, Kau87c]. The action of the Hamiltonian constraint on elements of the loop representation gets simplified. Thus, a large class of solutions to the Hamiltonian constraint is got which contains, in turn, a set of states that are also annihilated by the diffeomorphism constraints. It may happen that they are not the most general solutions to the combined set of constraints but they are exact physical states of the gravitational field.

3.3.1 The Dirac quantisation procedure

The strategy followed in the case of the loop approach is the Dirac method whose steps we give now.

1) Choice of a preferred subalgebra $\mathcal{A}$ of the classical observables to be the elementary observables of the quantum theory.

2) Choice of a linear space $\mathcal{S}$, on which there exists a completely regulated algebra of linear operators $\tilde{\mathcal{A}}$ that is a deformation of the classical algebra of the elementary observables, $\mathcal{A}$.

3) Definitions of the constraints and the Hamiltonian of GR in terms of the elements of $\mathcal{A}$.

4) Solution of the quantum constraints by finding the subspace $\mathcal{S}_{\mathrm{Phys}}$ of $\mathcal{S}$ that is in the kernel of the regularized constraints in the limit that the regularization is removed.

5) Definition of the physical observables that constitute the operator algebra on the space $\mathcal{S}_{Phys}$. At this stage, we arerequired to do two things; first, find the algebra, and second, give its elements a physical interpretation.

6) Definition of the physical inner product on $\mathcal{S}_{Phys}$. This choice must implement both the reality conditions of the classical theory and the physical interpretation of the physical observables in the sense that operators that correspond to classical physical observables that are real must be Hermitian w.r.t. the physical inner product.

The first three steps have been completed in the loop representation, whereas 4 and 5 are still under study. We sketch the progress that has been made in this approach.

3.3.2 Classical loop algebra

Quantisation of any classical theory consists of the association of classical observables, defined as functions on the phase space of the theory, with linear operators on some representation space such that the commutator algebra of the latter goes over into the Poisson algebra of the former in the limit $\hbar \to 0$. Due to the operator ordering and regularisation problems in any quantisation of a field theory most of the classical observables will not have an unambiguous representation in terms of the operator algebra of the quantum theory. What we can do is to choose a subalgebra of classical observables that will be represented unambiguously in terms of the operator algebra of the quantum theory. These are called the elementary observables. We can say that the rest of the quantisation procedure is constrained by the choice of these elementary observables in the following sense. The set of elementary observables should form a closed algebra under the Poisson brackets. This set must be small enough so that every element in its algebra can be represented in terms of a well defined linear operator on the representation space (regularisation). Also, the set must be large enough so that the constraints, Hamiltonian, and a large enough set of physical observables must be expressible at the classical level through limits of sequences of elementary observables. When this happens it is said that the algebra of the elementary observables is complete.

In what follows it will be convenient to use the spinorial character of the Ashtekar variables, as originally introduced. The translation is straightforward. Take the Pauli matrices divided by $\sqrt{2}$: $\tau_i{}^A{}_B$, $A, B = 1, 2$ and let

$$A_a{}^A{}_B = A_a{}^i \tau_i{}^A{}_B, \tag{3.49}$$
$$\tilde{E}^{aA}{}_B = \tilde{E}^{ai} \tau_i{}^A{}_B. \tag{3.50}$$

The uppercase indices of the connection $A_a{}^A{}_B$ take values in the spin one-half representation of the $so(3)$ Lie algebra.

To construct the loop representation we choose a set of elementary observables based on loops in the three-manifold Σ. The phase space of GR will be coordinatized by the Ashtekar variables (A, E). The loops are assumed to be piecewise smooth and parametrized, with non vanishing tangent vectors.

Given a loop γ, and two points on it given by the parameter values, s and t, one defines the parallel transport to be

$$[U_\gamma(s,t)]^A{}_B \equiv \left[P e^{\int_s^t du A_a(\gamma(u)) \dot{\gamma}^a(u)} \right]^A{}_B, \tag{3.51}$$

where P means path ordered. The trace of this parallel transport all around the loop is known as the Wilson loop of the Ashtekar connection. In fact another symbol is used for it

$$T^0[\gamma] \equiv \mathrm{Tr} U_\gamma = \mathrm{Tr} P e^{\oint_\gamma A}, \tag{3.52}$$

and it is one of the loop variables we will define that form a closed algebra under Poisson brackets which is called the classical $\mathcal{T}$-algebra. It is found necessary to introduce observables corresponding to unordered sets of loops in Σ. Such a set is called a multiloop, and is denoted $\{\gamma\} \equiv \{\gamma_1, \gamma_2, \ldots\}$. Corresponding to each multiloop $\{\gamma\}$, we have also a T^0 observable

$$T^0[\{\gamma\}] \equiv \prod_i \mathrm{Tr} U_{\gamma_i}. \tag{3.53}$$

Under Poisson bracket, the T^0's form an overcomplete set of commuting $SU(2)$-gauge invariant observables. This result comes from certain relations that hold because of the involved $SL(2, C)$ matrices and their definitions in terms of parallel transport. They include

i) Invariance under reparametrisation of the loop parameter s.

ii) Invariance under inversion

$$T^0[\gamma^{-1}] = T^0[\gamma]. \tag{3.54}$$

iii) The spinor identity:

$$T^0[\alpha]T^0[\beta] = T^0[\alpha\#\beta] + T^0[\alpha\#\beta^{-1}], \tag{3.55}$$

where the loop $\alpha\#\beta$is defined as follows. If α and β intersect in a point p, it is the loop obtained starting from p, going through α, then through β, and finally closing at p. This equation only holds if *alpha* and β intersect.

iv) The "retracing" identity:

$$T^0[\alpha] = T^0[\alpha \cdot l \cdot l^{-1}] \tag{3.56}$$

where l is a line with one end on α and $\alpha \cdot l \cdot l^{-1}$ is the loop obtained by going around α, then along the line, and then back along the line to α.

In order to have a complete algebra of observables we need some observables that also depend on the conjugate E fields. Looking ahead to the problem of regularization, at quantum level, we should require that the elementary observables not include any that involve more than one E at any point of Σ. A T^1 observable is defined by inserting a conjugate E field into the trace of the parallel transport around the loop at some given point s:

$$T^1[\gamma]^a(s) \equiv \mathrm{Tr}[U_\gamma(s)\tilde{E}^a(\gamma(s))]. \tag{3.57}$$

This definition can be extended to the T^n observable, and to multiloops also. Such insertions of the E variable is called a "hand". The important consequences relevant to the quantization are

I) The T^n's form a closed algebra under Poisson brackets.

$$\{T^n[\alpha], T^m[\beta]\} = i \sum_{\text{grasps}} \Delta[\alpha, \beta] T^{n+m-1}[\text{result of the grasp}] \tag{3.58}$$

the grasps are the resulting loops by considering the possible combinations of the initial loops at the "hands".

II) Completeness on the gauge invariant observables. Any gauge invariant and local functions of F_{ab}, E^c may be constructed in terms of limits of sequences of T observables.

III) The loop algebra is closed under the action of the spatial diffeomorphisms. The esasiest example is the T^0. Given $\phi \in \text{Diff}(\Sigma)$

$$\phi \circ T^0[\alpha] = T^0[\phi \circ \alpha]. \tag{3.59}$$

IV) The distributional singularities appearing in the loop algebra may be removed by an appropriate smearing procedure.

3.3.3 Loop representation construction

The key idea in the programme is to use the above algebra as the basic algebra whose representations determine the quantum theory. In particular, it is possible to construct a type of Fock space quantization in which the analog of a "n-particle" state is a function $\psi(\eta_1, \eta_2, \ldots \eta_n)$ of n loops. The $\tilde{T}^0$ acts like a creation operator, for example

$$(\tilde{T}^0_\eta \psi)[\xi] \equiv \psi[\eta, \xi], \tag{3.60}$$

while the $\tilde{T}^1$ operators map each n-loop sector into itself. By this means a deformation of the classical $\mathcal{T}$-algebra is succesfuly constructed:

$$\left[\tilde{T}^n, \tilde{T}^m\right] = \hbar \Delta \tilde{T}^{n+m-1} + \hbar^2 \Delta\Delta \tilde{T}^n + \cdots + \hbar^n \Delta \ldots \Delta \tilde{T}^m. \tag{3.61}$$

The next major step is to construct the quantum constraints as a limit of sequences of these $\tilde{T}$ variables. Rovelli and Smolin showed that the WDW equation, $\hat{C}\Psi = 0$, can be satisfied provided the Fock space functions $\psi(\eta_1, \eta_2, \ldots \eta_n)$ are concentrated on smooth, non intersecting loops.

Evidently, these states are not Diff(Σ) invariant because the diffeomorphism group moves the loops around. Nevertheless, Diff(Σ) invariant states can be found by requiring the n-loop functions $\psi(\eta_1, \eta_2, \ldots \eta_n)$ to be constant on the Diff(Σ) orbits, which are the *link* classes of the manifold Σ.

3.3.4 Advances and perspectives

So far, it is possible to summarize the advances in the loop representation as follows.

1) The loop representation can be considered as a complete quantisation of the phase space of GR. It is completely regulated and diffeomorphism covariant since the operators involved, once regulated, carry arepresentation of the spatial diffeomorphism group.

2) The diffeomorphism and Hamiltonian constraints may be expressed in the loop representation, the former by their natural geometrical action on the loop space, the latter in regulated form.

3) The general solution to the diffeomorphism constraint is found in the loop representation, and expressed in terms of a countable basis. This countable basis is in one-to-one correspondence with the generalized link classes of the manifold.

4) An infinite, but not complete, set of states that are in the kernel of the Hamiltonian constraint is also found. These states consist of all loop functionals with support on loops that are smooth and non intersecting.

5) The Hamiltonian and diffeomorphism constraints are compatible, in the sense that an infinite set of physical states that are in the simultaneous kernel may be constructed. This space has a countable basis, which is in one-to-one correspondence with the ordinary link classes.

6) A functional transform taking states in the selfdual representation to states in the loop representation may be constructed formally.

7) For free field theories this transform may be explicitely constructed, and gives a construction of a loop representation for the Fock space of free photons and free gravitons [AR91, AAL91].

The perspectives, on the otherhand, can be set as follows.

1) Completness of the solution space of the constraints. The set of solutions to the Hamiltonian constraint mentioned above is almost certainly not a complete set. There are two reasons for that. First, a large set of additional solutions has been found in the self dual representation associated with intersecting loops [JS88b, Hus89]. We expect that those solutions will exist in the loop representation as well. The second is that the mentioned solutions are constructed using only the antisymmetry of the indices of the operator. So, it is not impossible that there exists other operators whose continuum limit classically is not the Hamiltonian constraint that also annihilates these states.

2) Physical interpretation of the physical operator algebra. Given a description of the solution space to the constraints in terms of a countable basis

implies that one knows how to construct the general operator acting on that space. Thus given the results about states, we have the general diffeomorphism invariant operator, and a large class of completely physical operators. What we do not have is any correspondence between these operators and difeomorphism invariant or physical observables in classical GR.

3) The Physical inner product. We already mentioned that the choice of an inner product is related to the reality conditions. This means that, if anyone proposes an inner product on the space of physical states, one must be able to check that any operator on the physical states whose classical limit is real when the reality condition are imposed is Hermitian. However, this condition requires that we have a correspondence between classical and quantum physical observers, which, as we have just mentioned, we do not have. Thus, at present, while there are candidates for the physical inner product (e.g. the L^2 norm given the countable basis of link classes) it has not been possible to check whether any of them correctly expresses the reality condition.

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