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Action-Angle Variables in Integrable Systems

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ACTION-ANGLE VARIABLES IN INTEGRABLE SYSTEMS

We consider(1 + 1) Hamiltonian systems

$$u^i_t = \{H, u^i(x)\}. \tag{0.1}$$

there are two of the simplest construction of invariant finite dimensional manifold for such systems:
a) for any commuting flow $u^i_s = G^i(x,t,u,u')$ such that $u_{st} = u_{ts}$ the set $M = M_G$ of the stationary points of the r-flow is invariant under t-dynamics;
b) for any conservation law $F = \int f(u,u')dx$ the set N_F of its extrema is invariant.
We note that in general if the commuting flow is a Hamiltonian system with Hamiltonian F then $N_F \subset M_G$ because in general there exist the Poisson bracket annihilator.
Let us suppose that the system $u^i_t = \{H, u^i(x)\}$ has a sufficiently large family of commuting conservation law $F_0.....F_M$ $(F_0 = H)$ than :

$$N_{C_0.....C_M} = \{u(x)| \ \frac{\delta(\sum c_\alpha F_\alpha)}{\delta u_j} = 0\}$$

is an invariant finite dimensional manifold.
We can consider the manifold $N = \bigcup_{C_0.....C_M} N_{C_0...C_M}$ this manifold also is time invariant under the t-dynamic but while on $\bigcup_{C_0....C_M} N_{C_0...C_M}$ it is possible to introduce a symplectic structure by P. B., on N it is not possible because on N the P. B. have annihilators, they are generate of the constant $C_0.....C_M$. So N results foliated with symplectic leaves.
Le us suppose that the dimension of the generic symplectic leave $N_{C_0.....C_M}$ is equal to $2m$. If the number of the commuting conservation law F_α of the system (0.1) are not less than $m-1$ such that the Hamiltonian P_C of the

x-flow and $F_2 = H, ...F_m$ are independent for generic $C = (C_0.....C_M)$ and if their common level surface is compact (and connected) for given C then it is a torus and the manifold N is foliated in a family of completely integrable systems.
The corresponding solution of the evolutionary system are periodic or almost periodic function of x, t of the form:

$$u(x,t) = U(kx + wt + \phi^0; C, f)$$

where $U(\phi; C, f)$ is a periodic function in each variable $\phi = (\phi_1......\phi_m)$ depending on the parameter $C = (C_0.....C_M)$ and $f = (f_1.....f_m)$ the last one fix the common level surface of the conservation law. The vectors k and w are the wave-number and frequencies respectively, depends on f and C .

The action-angles variable J_α, ϕ_α $\alpha = (1...m)$ can be introduced on some neighbourhood in N of the torus such that

$$P_C = P_C(J_1...J_m) \quad F_\alpha = F_\alpha(J_1....J_m)$$

$$K_\alpha = \frac{\partial P_C}{\partial J_\alpha} \quad w_\alpha = \frac{\partial H_C}{\partial J_\alpha} \quad \alpha = 1...m.$$

Respect to the problem of the integration of the evolutionary systems it is convenient to find those with rich families of "non obvious" conservation law.
Probably the evolutionary systems possessing commutative representation are those to look for. In this class fall the KdV system. Such a system can be represented with one equation: $L_t = [B, L]$ where: $L = -\partial_x^2 + u; B = 4\partial_x^3 - 6u\partial_x - 3u_x$; $[B, L] = BL - LB$ or in a more convenient form:

$$[\partial_x - U(\lambda), \partial_t - V(\lambda)] = 0 \qquad (0.2)$$

that is equivalent to:

$$U_t(\lambda) - V_x(\lambda) + [U, V] = 0$$

Here, the operator $\mathcal{L} = \partial_x - U(\lambda)$ is the matrix form of the Schrödinger operator L - λ, (λ begin the spectral parameter)

$$\mathcal{L} = \partial_x - U(\lambda) = \partial_x - \begin{pmatrix} 0 & 1 \\ u-\lambda & 0 \end{pmatrix}$$

$$(\partial_x - U(\lambda)) \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0 \Longleftrightarrow \quad \psi_1 = \psi \; \psi_2 = \psi' \tag{0.3}$$

$$L\psi = \lambda\psi$$

And the matrix V(λ) is matrix form of B-operator in the space of solutions (3):

$$V(\lambda) \begin{pmatrix} \psi \\ \psi' \end{pmatrix} = \begin{pmatrix} B\psi \\ (B\psi)' \end{pmatrix} \quad (modulo\ \partial_x - U(\lambda) \begin{pmatrix} \psi \\ \psi' \end{pmatrix} = 0)$$

$$V(\lambda) = \begin{pmatrix} u' & -2(u+2\lambda) \\ u'' - 2(u-\lambda)(u+2\lambda) & -u' \end{pmatrix}$$

the spectral parameter is a new variable, so the (2) is an identity in λ.

There is a family of equation (KdV hierarchy) that have a commutative representation of the form (2) with the operator $\mathcal{L} = \partial_x - U(\lambda)$ being fixed. Let us try to find all the equations of the form (2) with fixed U(λ), under the assumption of the polynomiality in λ of the matrix V(λ):

$$U_S(\lambda) - V_x(\lambda) + [U, V] = 0$$

So the "n-th analogue KdV equation" has the form:

$$u_S = 2\partial_x \sum_{i=0}^{n} c_{n-i} v_{i+1}(u, u', ..., u^{2i}) \tag{0.4}$$

where $v_k(u, u', ..., u^{2i})$ are polynomials in $u, u', ..., u^{2i}$and $c_0, ..., c_n$ are constants.

The "basis" in the family of equations of the form (4) consist of the following equations:

$$u_{t_k} = -2^{2k+1} \partial_x v_{k+1}(u, u', ..., u^{2k}) \quad (k = 0, ..., n) \tag{0.5}$$

this equation for any k admits a zero curvature representation:

$$U_{t_k}(\lambda) - V_k(\lambda) + [U(\lambda), V(\lambda)] = 0$$

For any k the equations (5) give a flow which commute pairwise: $(u_{t_k})_{t_l} = (u_{t_l})_{t_k}$, moreover these flows are Hamiltonian systems, having a common infinite family of conservation laws. More precisely the system (5) for all k is a Hamiltonian system with respect to Gardner-Zakharov-Faddev Poisson bracket (P.B.):

$$\{u(x), u(y)\} = \delta'(x-y)$$

with the Hamiltonian $2I_{k+1}$:

$$u_{t_k} = 2\partial_x \tfrac{\delta I_{k+1}}{\delta u(x)}$$

where:

$$I_m = \textstyle\int \chi_{2m+1}(u, u', ..., u^{2m+1})dx$$

and χ_{2m+1} are polynomials in $u, u', ...$ and Hamiltonian system with respect to Magri P.B.:

$$\{u(x), u(y)\} = \frac{1}{2}\delta''(x-y) + [u(x) + u(y)]\delta'(x-y)$$

with Hamiltonian I_k.

Now we should like to construct the invariant tori for the evolutionary systems and the corresponding action-variables.
At this end, let us consider the set of stationary points of any flow of the KdV hierarchy:

$$\partial_x \sum_{i=0}^{n} 2^{2(i+1)} c_{n-i} v_{i+1}(u, u', ...u^{2i}) = 0$$

this equation is equivalent to:

$$W(\lambda) = -\sum_{i=0}^{n} c_{n-i} V_i(\lambda)$$

$$\partial_x W(\lambda) = [U(\lambda), W(\lambda)]$$

moreover $W(\lambda)$ satisfy also the equation in t_k (for any k):

$$\partial_t W = [V_k, W]$$

so the invariant manifold for the KdV system is given as solution of the system:

$$\begin{cases} \partial_{t_k} W(\lambda) = [V_k, W](\lambda) \\ \partial_x W(\lambda) = [U(\lambda), W(\lambda)] \end{cases}$$

This solution can be investigated with the help of the algebraic geometry. We consider the case k=1.

In general we shall mean for "finite gap" solution of equations admitting a commutational representation those solutions for which a matrix-valued function $W(x,t,\lambda)$ exists depending meromorphically of the parameter λ, such that:

$$\begin{cases} [\partial_x - U(x,t,\lambda), W(x,t,\lambda)] = 0 \\ (\partial_t - V(x,t,\lambda), W(x,t,\lambda)) = 0 \end{cases} \tag{0.6}$$

It is possible to demonstrate that the common eigenvectors $\psi(x,t,\lambda,\mu)$ of the operators $\partial_x - U$, $\partial_t - V$ and W is meromorphic on the algebraic curve Γ defined by the characteristic equation:

$$Q(\lambda,\mu) = det(W(x,t,\lambda) - \mu I) = 0$$

outside the points P_α, the preimages of the poles λ_i of the matrices U and V.

The poles of $\psi(x,t,\lambda,\mu)$ are independent on x and t, the number of poles of ψ is equal to g+l-1 where g is the genus of the curve Γ and l is the dimension of the matrix W.
In a neighbourhood of the points P_α ψ has the form:

$$\psi(x,t,\lambda,\mu) = (\sum_{S=0}^{\infty} \xi_{S_\alpha}(x,t) K_\alpha^{-S}) exp(q_\alpha(x,t,K_\alpha))$$

where the factor in the bracket is the expansion with respect to a local parameter K_α^{-1} ($K_\alpha^{-1}(P_\alpha) = 0$) of a holomorphic vector, and q_α is a polynomial in K_α.

An important result is that the enunciated properties are sufficient to reconstruct the vector ψ and hence the operator U, V, W which satisfy the commutation conditions (6).

Precisely the following theorem holds:

let $P_1, ..., P_n$ be points on a Riemann surface Γ of genus g; let K_α^{-1} a local parameter in a neighbourhood of these points, $K_\alpha^{-1}(P_\alpha) = 0 \quad \alpha = 1, ..., n$; let $q_1(K), ..., q_n(K)$ be a set of polynomials, and D a divisor on Γ of degree N; than there are functions (n-points Baker-Akhiezer functions) meromorphic on Γ outside the points P_α, with the divisor (ψ) satisfying the condition $(\psi) + D \geq 0$, and such that, for $P \to P_\alpha \quad \psi(P)exp(-q_\alpha(K_\alpha(P))$ is analytic. The dimension of the space of such functions is $N - g + 1$ if D is non special.

Summarizing there exists a one to one correspondences between the system (6) and a Riemann surface Γ, a set of polynomial $q_\alpha(x, t, K)$, a non special divisor of degree $g + l - 1$ and $P_1, ..., P_n$ points on Γ.

Moreover the set of finite gap solutions corresponding to a nonsingular curve Γ is isomorphic to a torus $J(\Gamma)$: the Jacobi variety of this curve.

In the KdV case, the algebraic curve is hyperelliptic of genus n, if n is the order of the hierarchy. So, the invariant variety, under the KdV flow, after complexification, is isomorphic to the torus $J(\Gamma)$.

Now consider the variety M^n that describes the set of algebraic curves of genus g. For any $K \geq 1$ there exists a natural fibering:

$$N^{k+n} \longrightarrow M^n$$

(n is the complex dimension of M) where the fiber in any point is $S^k\Gamma$, i.e. the k-th symmetric power of the curve Γ.

Let us assume that the family of algebraic curves is given in the form of m-sheeted coverings of the λ-plane: $\lambda : \Gamma \to C$, and suppose that locally the image on C of the branch points are good coordinates on M^n: hence a point of N^{k+n} locally is written as:

$$(\lambda_1, ..., \lambda_N, \lambda(P_1) \equiv \gamma_1, ..., \lambda(P_k) \equiv \gamma_k)$$

where $\lambda_1, ..., \lambda_N$ represent the λ-coordinates of the branch points.

a) Let a set A of complex meromorphic functions on M^n be given.

b) Let a meromorphic 1-form $Q(\Gamma)$ on Γ be given.

It is required that the derivatives of $Q(\Gamma)$ along all the direction of the base space tangent to the manifold M^n be a globally defined meromorphic differential form on algebraic curve Γ itself.

If the closed 2-form

$$\Omega_Q = \sum_{i=1}^{k} dQ(\Gamma, \gamma_i) \wedge d\gamma_i$$

is non degenerate in the points of a region of the manifold

$$N_A = \{x \in N^{n+k} : \forall f \in A \quad f(x) = const.\}$$

then we say that an analytic Poisson bracket is given by the properties:

$$\begin{aligned} \{\gamma_i, \gamma_j\} &= 0 \qquad & i,j = 1, ..., k \\ \{Q(\gamma_i), Q(\gamma_j)\} &= 0 \qquad & i,j = 1, ..., k \\ \{Q(\gamma_j), \gamma_i\} &= \delta_{ij} \qquad & i,j = 1, ..., k \\ \{f, \gamma_i\} = \{f, Q(\gamma_i)\} &= 0 \quad & i,j = 1, ..., k \quad \forall f \in A \end{aligned}$$

For any Hamiltonian of the form $H = H(\Gamma)$ one proves that, if $\gamma_1, ..., \gamma_k$ are in general position, the complex variables

$$\psi_j = \sum_{i=1}^{k} \int_{p_0}^{\gamma_i} \nabla_{\tau_j} Q$$

are independent and have linear dynamics with respect to the time, where τ_j are the tangent directions to M^n.

If k=g and all forms $\nabla_{\tau_j} Q$ are holomorphic, the Abel map linearizes the dynamics of all the Hamiltonians $H(\Gamma)$.

If we want define the action variables we have to define a real structure on M^n, and if Q and A are compatible with such structure than:

$$J_i = \oint_{a_i} Q(\Gamma, \lambda) d\lambda$$

where a_i are appropriate cycles,are the variables conjugate to ψ_j, i.e. the action variables.

To construct the explicit form of the action-angle variables we have to study the geometry of the moduli-space of algebraic curves.

In general we can consider a moduli-space $M = N^N$ $N = 2g + n + m - 2$ of sets $(\Gamma, Q_1, ..., Q_m,\ \lambda)$ where Γ is a smooth algebraic curve of genus g, with m marked point $Q_1, ..., Q_m$ and with a meromorphic function λ of degree n such that $\lambda^{-1}(\infty) = Q_1 \cup ... \cup Q_n$ and n_a is the degree of λ in Q_a.
We need that the λ -projections $u^1, ..., u^N$ of the branch points $P_1, ..., P_N$:

$$d\lambda|_{P_j} = 0 \quad u^j = \lambda(P_j) \ \ j = 1, .., N$$

are good local coordinates in an open domain in M.
In case g=0, m=1 the space M is the set of all polynomial of he form:

$$\lambda(p) = p^n + q_{n-2}p^{n-2} + ... + q_0$$

$q_0, ..., q_{n-2} \in C$
If $g > 0$, m=1, n=2 M is the set of all hyperelliptic curves:

$$\mu^2 = \prod_{j=1}^{2g+1} (\lambda - u^j)$$

and $u^1, ..., u^{2g+1}$ are the local coordinates on M.

On this manifold M, or on its coverings $\tilde{M}$ define the subspace af all primary differentials N-dimensional spanned by the differentials:
a) normalized ($\oint_{a_\alpha} \Omega = 0$) Abelian differential $\Omega_\alpha^{(k)}$ of the second kind with a single pole in the point Q_a, and with a principal part:

$$\Omega_a^{(k)}(P) = \frac{dz_a}{z_a^{k+1}} + regular\ term, \quad P \to Q_a \quad with\ z_a = \lambda^{\frac{-1}{n_a}}$$

b) normalized Abelian differentials $\psi_a^{(0)}$ of the third kind with poles in Q_a, Q_m with residue ± 1 respectively a=1,...,m-1
c) holomorphic differentials $\omega_\alpha = \omega_\alpha^{(0)}$ $\alpha = 1, ..., g$ normalized as follows:

$$\oint_{a_\beta} \omega_\alpha = 2\pi i \delta_{\alpha\beta}$$

d) multivalued normalized differentials holomorphic on C σ_α^1 with the increments of the form:

$$\sigma_\alpha^{(1)}(P + b_\alpha) - \sigma_\alpha^{(1)}(P) = -d\lambda; \quad \alpha = 1, ..., g$$

For any of such differentials it is defined a flat Egoroff metric on $\tilde{M}$:

$$dS^2_\Omega = \sum_{i=1}^{N} g^\Omega_{ii}(u)(du^i)^2$$

where:

$$g^\Omega_{ii} = res_{p_i}\frac{\Omega^2}{d\lambda} \quad i = 1, ..., N$$

The corresponding flat coordinates are:

$$\begin{cases} t_{k,a} = -\frac{1}{k} res_{Q_a} z_a^{-k}\Omega & a = 1, ..., m \\ t_{0,a} = v.p. \int_{Q_0}^{Q_a} \Omega + t_0 & a = 1, ..., m \quad k = 1, ..., n_a \\ t'_\alpha = -\frac{1}{2\pi i} \oint_{a_\alpha} \lambda\Omega & \alpha = 1, ..., g \\ t'' = \oint_{a_\alpha} \Omega & \alpha = 1, ..., g \end{cases}$$

with two constraints:

$$\sum_{a=1}^{m} t_{0,a} = \sum_{a=1}^{m} t_{n_a,a} = 0$$

these coordinates are globally independent analytic functions on M.

The flat metric dS^2_Ω on $\tilde{M}$ determines a Poisson structure of on the loop space $\mathcal{L}$ $(\tilde{M})$ of function of $x \in S^1$ having their values in M, via the formula:

$$\{I_1, I_2\} = \int \frac{\delta I}{\delta u^i(x)} g_{ij} \nabla_k (\frac{\delta I_2}{\delta u^j(x)})\, du^k(x)$$

and ∇_k is the Levi-Civita connection for the metric dS^2_Ω.

In the case of the moduli space of hyperellipctic curves the flat coordinates for the metric dS^2_Ω are the annihilators of the G.Z.F. Poisson brackets of the KdV hierarchy, $t'_1, ..., t'_g$ are the action variables, and $t''_1, ..., t''_g$ are the components of the wave number vector.
In fact one can demonstrates that, in this case, $\Omega = dQ$.

Actually one can demonstrate that these results hold, not only for the KdV hierarchy, but also for the generalized KdV hierarchy, that is of the form:

$$\partial_{t_a} L = [L, L_+^{\frac{a}{n}}] \quad a \neq kn$$

with:

$$L = \partial_n + q_{n-2}\partial^{n-2} + ... + q_0$$

There are, instead, cases in which the invariant manifold are not Jacobian but Prym variety, and this is true when the algebraic curves have an involution.
In these cases it isn't known a general procedure to obtain the action variables.

My aim is to check in this direction.