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SCHRODINGER OPERATOR WITH POINT INTERACTIONS DEFINED AS QUADRATIC FORMS

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TABLE OF CONTENTS

INTRODUCTION

In this thesis we discuss some mathematical aspects of the Schrödinger operator with zero range or point interactions, namely its connection with the theory of quadratic forms and boundary value problems.

The Hamiltonian with a zero range potential have been introduced since 1935 by Bethe and Peierls ([1]) in the study of the system consisting of a neutron and a proton. Then this model was employed by Fermi ([2]) to analyse scattering of neutrons in hydrogenous substances and by Thomas ([3]) who showed how to obtain point interactions as a limit of suitable short range interactions. A further application in solid state physics was Krönig and Penney's model of one dimensional crystal ([4]).

The first rigorous analysis goes back to 1961 with Berezin and Faddeev ([5]) whose technique was based on the selfadjoint extension of the laplacian suitably restricted. By now there are several ways to define rigorously a Schrödinger operator with finitely or infinitely many point interactions located at a discrete set in $\mathbb{R}^n$, $n<4$.

Here we simply list some of the techniques one can employ to get such a rigorous definition:

Def. 1. The theory of selfadjoint extensions

Def. 2. Resolvent limits of approximating Schrödinger operators

Def. 3. The theory of Dirichelet forms

Def. 4. Non-standard analysis

The Hamiltonian with point interactions is a highly idealized model, nevertheless it is interesting because it is one of the few solvable models in Quantum Mechanics, in the sense that all the physical relevant quantities, i.e. spectrum, eigenfunctions, scattering data, etc., can be explicitely computed. So this model can be used to deduce properties of more general models which are close to the solvable one in an appropriate sense (see discussion in section 1.3.).

For a comprehensive treatment of the whole subject we refer to [29],[6] and references therein.

In this thesis we provide a new definition of the Schrodinger operator with point interactions by means of a suitable quadratic form, which in turn can be obtained as limit of a sequence of natural approximating forms. This procedure follows the same line of reasoning of Def. 2, if resolvent limits of operators are replaced by Γ-convergence of quadratic forms, which is a notion of convergence largely used in the context of variational problems ([7] and references therein).

Moreover we will show that point interactions are releted to a boundary value problem, as outlined in [8]. In particular we consider the laplace operator in the domain exterior to a sphere with mixed boundary conditions on the surface of the sphere and then we prove convergence to point interaction when the radius is going to zero and the scattering length is kept constant.

The exposition is organized as follows:

in chapter 1 we recall some basic facts about the Schrodinger operator with one or finitely many point interactions.

In chapter 2 we construct the quadratic form associated to such operator and then this form is obtained as Γ-limit of approximating forms; moreover some extensions are presented.

Chapter 3 is devoted to the study of the connection between a mixed boundary value problem and the Schrodinger operator with point interactions; finally some possible developments of the material exposed in chapters 2,3 are outlined.

CHAPTER 1
SCHRODINGER OPERATOR WITH POINT INTERACTIONS

We introduce the rigorous definition of the Schrödinger operator with one or finitely many point interactions and we recall some of its foundamental properties. Then we show how point interactions can considered as the limit of suitable short range potentials. There is no attempt of completeness and the proofs are only outlined or omitted. All the material exposed in this chapter is extensively treated in [6].

1.1 BASIC PROPERTIES

We want to give a rigorous mathematical meaning to the formal expression

$$H = -\Delta + \mu\, \delta_y \qquad (1.1.1.)$$

where $y \in \mathbb{R}^3$, δ_y is the Dirac measure concentrated in y and μ is a coupling constant, as a selfadjoint operator in $L^2(\mathbb{R}^3)$.
Here we sketch Def. 1 (see introduction) which is very clear and direct from a mathematical point of view; in the next section we will describe Def. 2 which clarifies in a more transparent way how the defined operator is the rigorous counterpart of (1.1.1.) .
Let us consider the positive and symmetric operator in $L^2(\mathbb{R}^3)$

$$H_y = -\Delta \quad , \quad D(H_y) = C_0^\infty(\mathbb{R}^3 - \{y\}) \qquad (1.1.2.)$$

Because of the spherical symmetry of the problem, it is convenient to decompose the Hilbert space $L^2(\mathbb{R}^3)$ with respect to angular momenta ([9] pag. 160)

$$L^2(\mathbb{R}^3) = L^2\big((0,+\infty);\, r^2 dr\big) \otimes L^2(S^2) \qquad (1.1.3.)$$

where S^2 is the unit sphere in $\mathbb{R}^3$.
If we introduce the spherical harmonics $Y_{\ell,m}$, $\ell \in \mathbb{N}$ and $m = 0, \pm 1, \dots \pm \ell$, as a basis for $L^2(S^2)$ and define the unitary operator

$$U : L^2((0;+\infty); r^2 dr) \longrightarrow L^2((0;+\infty); dr)$$

$$(Uf)(r) = r f(r) \tag{1.1.4.}$$

the right hand side of $(1.1.3.)$ can be written as

$$L^2(\mathbb{R}^3) = \bigoplus_{\ell=0}^{\infty} U^{-1} L^2((0;+\infty); dr) \otimes [Y_{\ell,-\ell}, \dots Y_{\ell,\ell}] \tag{1.1.5.}$$

where $[Y_{\ell,-\ell}, \dots Y_{\ell,\ell}]$ denotes the subspace of $L^2(S^2)$ spanned by $Y_{\ell,-\ell} \dots$ $\dots Y_{\ell,\ell}$. It is not difficult to see that, with respect to this decomposition, the operator given by $(1.1.2.)$ becomes

$$H_y = T_y^{-1} \left\{ \bigoplus_{\ell=0}^{\infty} \left[U^{-1} h_\ell U \otimes 1 \right] \right\} T_y \tag{1.1.6.}$$

where T_y is the translation operator in $L^2(\mathbb{R}^3)$

$$(T_y g)(x) = g(x+y) \tag{1.1.7.}$$

and

$$h_\ell = -\frac{d^2}{dr^2} + \frac{\ell(\ell+1)}{r^2} \quad , \quad D(h_\ell) = C_0^\infty((0;+\infty)) \tag{1.1.8.}$$

Now we have to distinguish the case $\ell = 0$ from $\ell > 0$; in fact h_0 has deficiency indices $(1,1)$ and its selfadjoint extensions are given by ([9] pag.144)

$$h_{0,\alpha} = \frac{d^2}{dr^2} \quad , \quad D(h_{0,\alpha}) = \Big\{ \phi \in L^2((0,+\infty)) \,\Big|\, \phi, \phi' \in AC_{loc}((0,+\infty)) \, ;$$

$$\phi'(0^+) = 4\pi\alpha\, \phi(0^+) \; ; \; \phi'' \in L^2((0;+\infty)) \Big\} \quad , \quad -\infty < \alpha \le +\infty \tag{1.1.9.}$$

where $AC_{loc}((0;+\infty))$ denotes the set of locally absolutely continuous functions on $(0;+\infty)$. If $\ell > 0$ then h_ℓ is essentially selfadjoint ([9] pag. 161), i.e. the only selfadjoint extension is its closure

$$\dot{h}_\ell = -\frac{d^2}{dr^2} + \frac{\ell(\ell+1)}{r^2} , \quad D(\dot{h}_\ell) = \left\{ \phi \in L^2((0,+\infty)) \,\middle|\, \phi, \phi' \in AC_{loc}((0,+\infty)); \right.$$

$$\left. -\phi'' + \frac{\ell(\ell+1)}{r^2}\phi \in L^2((0,+\infty)) \right\} \qquad (1.1.10)$$

From (1.1.6.) , (1.1.9.) , (1.1.10.) we have that all selfadjoint extensions of H_y are given by

$$-\Delta_{\alpha,y} = T_y^{-1}\left\{ \left[U^{-1} h_{0,\alpha} U \oplus \left(\bigoplus_{\ell=1}^{\infty} U^{-1} \dot{h}_\ell U \right) \right] \otimes 1 \right\} T_y \qquad (1.1.11)$$

Now we define $-\Delta_{\alpha,y}$ as the Schrödinger operator with point or delta interaction in y and parametrized by α .

It's clear from (1.1.9.) that when $\alpha = +\infty$ then $-\Delta_{\alpha,y}$ reduces to the free Hamiltonian $-\Delta$.

The above construction of $-\Delta_{\alpha,y}$ shows that point interaction acts, as perturbation of the laplacian, only in the s-wave since, in the angular momenta decomposition, $-\Delta_{\alpha,y}$ and $-\Delta$ coincide for $\ell > 0$.

Many informations about the properties of the Schrödinger operator $-\Delta_{\alpha,y}$ just defined can be obtained; here we simply collect the main results. First we can explicitely compute the resolvent; let

$$G_k = (-\Delta - k^2)^{-1} , \quad \mathrm{Im}\, k > 0 \qquad (1.1.12)$$

the free resolvent which is ([9] pag. 58) an integral operator with kernel

$$G_k(x,x') = \frac{e^{ik|x-x'|}}{4\pi\,|x-x'|} , \quad \mathrm{Im}\, k > 0 , \quad x, x' \in \mathbb{R}^3 , \quad x \neq x' \qquad (1.1.13)$$

A straightforward application of the Krein's formula ([10]) leads to the resolvent of $-\Delta_{\alpha,y}$

$$\left(-\Delta_{\alpha,y}-k^2\right)^{-1} = G_k + \frac{1}{\alpha - \frac{ik}{4\pi}} \left(\overline{G_k^y}, \cdot\right) G_k^y \quad , \quad k^2 \in \rho\left(-\Delta_{\alpha,y}\right),$$

$$\operatorname{Im} k > 0 \quad , \quad G_k^y(x) = G_k(x,y) \qquad (1.1.14.)$$

with kernel

$$\left(-\Delta_{\alpha,y} - k^2\right)^{-1}(x,x') = G_k(x,x') + \frac{1}{\alpha - \frac{ik}{4\pi}} G_k^y(x)\, G_k^y(x') \quad , \quad k^2 \in \rho\left(-\Delta_{\alpha,y}\right),$$

$$\operatorname{Im} k > 0 \, , \quad x, x' \in \mathbb{R}^3 \quad , \quad x \neq y \, , \; x' \neq y \, , \; x \neq x' \qquad (1.1.15.)$$

Then we can obtain additional informations about the domain and the local character of $-\Delta_{\alpha,y}$ ([6]); in particular the domain $D\left(-\Delta_{\alpha,y}\right)$ consists of all elements Ψ of the type

$$\Psi(x) = \phi_k(x) + \frac{1}{\alpha - \frac{ik}{4\pi}} \phi_k(y)\, G_k^y(x) \qquad (1.1.16.)$$

where $\phi_k \in D(-\Delta)$ and $k^2 \in \rho\left(-\Delta_{\alpha,y}\right)$, $\operatorname{Im} k > 0$.

The decomposition $(1.1.16.)$ is unique and with $\Psi \in D\left(-\Delta_{\alpha,y}\right)$ of this form we obtain

$$\left(-\Delta_{\alpha,y} - k^2\right) \Psi = \left(-\Delta - k^2\right) \phi_k \qquad (1.1.17.)$$

Moreover if $\Psi \in D\left(-\Delta_{\alpha,y}\right)$ and $\Psi = 0$ in an open set $U \subseteq \mathbb{R}^3$ then $-\Delta_{\alpha,y}\Psi = 0$ in U .

REMARK 1.1.1. Suppose a particle, subject to a point interaction centered in y , is in a state $\Psi \in D\left(-\Delta_{\alpha,y}\right)$ of finite energy with $\Psi = 0$ in an arbitrarily small neighbourhood of y, then from $(1.1.16.)$, $(1.1.17.)$, we see that

$$-\Delta_{\alpha,y}\Psi = -\Delta\Psi \, , \qquad (1.1.18.)$$

i.e. the particle behaves as a free particle in agreement with the idea of a zero-range interaction.

The spectral properties of $-\Delta_{\alpha,y}$ can be completely characterized; let σ_{ess}, σ_{ac}, σ_{sc}, σ_p denote the essential, absolutely continuous, singular continuous and point spectrum respectively, then we have ([6])

$$\sigma_{ess}(-\Delta_{\alpha,y}) = \sigma_{ac}(-\Delta_{\alpha,y}) = [0,+\infty) \quad , \quad \sigma_{sc}(-\Delta_{\alpha,y}) = \phi , \tag{1.1.19}$$

if $\alpha \geq 0$ then $\sigma_p(-\Delta_{\alpha,y}) = \phi$ (1.1.20)

if $\alpha < 0$ then $\sigma_p(-\Delta_{\alpha,y}) = \{-(4\pi\alpha)^2\}$ (1.1.21)

with normalized eigenfunction

$$\Psi_0(x) = \sqrt{-\alpha}\; \frac{e^{4\pi\alpha|x-y|}}{|x-y|} \quad , \quad x \neq y \tag{1.1.22}$$

Now it remains to compute the scattering quantities associated to $-\Delta_{\alpha,y}$. We confine ourself to the stationary theory but it is not difficult to extend the analysis to the time depenedent scattering theory.
The starting point is to determine the generalized eigenfunctions of $-\Delta_{\alpha,y}$. We define

$$\Psi_{\alpha,y}(k\omega,x) = e^{ik\omega x} + \frac{e^{ik\omega y}}{4\pi\alpha - ik}\,\frac{e^{ik|x-y|}}{|x-y|} \quad , \quad k \geq 0 \;,\; \omega \in S^2,$$
$$x,y \in \mathbb{R}^3 \quad , \quad x \neq y \tag{1.1.23}$$

A direct calculation shows that $\Psi_{\alpha,y}$ satisfies

$$-4\pi\alpha|x-y|\,\Psi_{\alpha,y}(k\omega,x) + \frac{(x-y)}{|x-y|}\,\nabla_x \Psi_{\alpha,y}(k\omega,x)\Big|_{x=y} = 0 \tag{1.1.24}$$

$$\left(-\Delta_{\alpha,y}\Psi_{\alpha,y}\right)(k\omega,x) = k^2\,\Psi_{\alpha,y}(k\omega,x) \tag{1.1.25}$$

$$\lim_{\varepsilon\to 0} \lim_{\substack{|x'|\to\infty \\ \omega=-\frac{x'}{|x'|}}} 4\pi|x'| \, e^{-i(k+i\varepsilon)|x'|}\left(-\Delta_{\alpha,y}-(k+i\varepsilon)^2\right)^{-1}(x,x') = \Psi_{\alpha,y}(k\omega,x) \tag{1.1.26.}$$

Then ([11] ch.VI) $\Psi_{\alpha,y}$ is a set of generalized eigenfunctions of $-\Delta_{\alpha,y}$.
From $\Psi_{\alpha,y}$ we can compute the relevant scattering quantities associated to $-\Delta_{\alpha,y}$ such as the on-shell scattering amplitude

$$f_{\alpha,y}(k,\omega,\omega') = \lim_{\substack{|x|\to\infty \\ \omega=\frac{x}{|x|}}} |x| \, e^{-ik|x|}\left[\Psi_{\alpha,y}(k\omega',x) - e^{ik\omega' x}\right] =$$

$$= \frac{e^{ik(\omega-\omega')y}}{4\pi\alpha - ik} \quad , \quad k\geq 0 \quad , \quad \omega,\omega' \in S^2 \tag{1.1.27.}$$

and the on-shell scattering operator in $L^2(S^2)$

$$S_{\alpha,y}(k) = 1 - \frac{k}{2\pi i}\,\frac{\left(e^{-ik(\cdot)y}\,,\,\cdot\right)e^{-ik(\cdot)y}}{4\pi\alpha - ik} \quad , \quad k\geq 0 \tag{1.1.28.}$$

$S_{\alpha,y}(k)$ can be continued in k to the whole complex plane and the poles coincides with the bound state (if $\mathrm{Im}\, k > 0$) or resonance (if $\mathrm{Im}\, k \leq 0$) of $-\Delta_{\alpha,y}$.

REMARK 1.1.2
If we perform the low energy limit $k \to 0$ in (1.1.27.) we get the scattering lenght

$$a_\alpha = -\lim_{k\to 0} f_{\alpha,y}(k,\omega,\omega') = -\frac{1}{4\pi\alpha} \tag{1.1.29.}$$

(1.1.29.) provides the physical meaning of the parameter α ([21]).

Finally we briefly mention the definition of the Schrödinger operator with point interaction in $L^2(\mathbb{R}^n)$, with $n \neq 3$. First we note that for $n \geq 4$ there is a "no-go theorem"; the reason is that the minimal operator

$$H_y = -\Delta \quad , \quad D(H_y) = C_0^\infty(\mathbb{R}^n - \{y\}) \quad , \quad n \geq 4 \tag{1.1.30.}$$

is already essentially selfadjoint ([9] Th. X.11) and so the only selfadjoint extension of H_y is the laplacian in $\mathbb{R}^m$, $m \geq 4$.

The definition of point interaction in dimension two is identical to the three dimensional case ([6]). Again one starts with

$$H_y = -\Delta \quad , \quad D(H_y) = C_0^\infty(\mathbb{R}^2 - \{y\}) \tag{1.1.31.}$$

then, performing the decomposition of $L^2(\mathbb{R}^2)$ with respect to angular momenta and extending the radial part of the operator, one gets a family of selfadjoint operators $-\Delta_{\alpha,y}$, with $-\infty < \alpha \leq +\infty$.

By definition $-\Delta_{\alpha,y}$ is the Schrödinger operator with point interaction in y parametrized by α in dimension two. The physical meaning of α is again related to the scattering lenght and the case $\alpha = +\infty$ leads to the free Hamiltonian.

The one dimensional case is simpler and in a sense peculiar. Following the usual procedure based on extension of the laplacian restricted to $\mathbb{R} - \{y\}$ ([6]) one defines a family of selfadjoint operators parametrized by α, $-\infty < \alpha \leq +\infty$

$$-\Delta_{\alpha,y} = -\frac{d^2}{dx^2} \quad , \quad D(-\Delta_{\alpha,y}) = \{ g \in H^1(\mathbb{R}) \cap H^2(\mathbb{R} - \{y\}) \; | \\ g'(y^+) - g'(y^-) = \alpha \, g(y) \} \tag{1.1.32.}$$

where $H^m(\Omega)$, Ω open subset of $\mathbb{R}^n$ and $m, n \in \mathbb{N}$, is the standard Sobolev space of the functions in $L^2(\Omega)$ whose weak derivatives up to order m are in $L^2(\Omega)$ ([12]).

Clearly for $\alpha = 0$ $-\Delta_{\alpha,y}$ reduces to the free Hamiltonian in $L^2(\mathbb{R})$, while for $\alpha = +\infty$ the two half-lines $(-\infty, y)$ and $(y, +\infty)$ are decoupled by a Dirichelet boundary condition in y and so

$$-\Delta_{+\infty,y} = (-\Delta_{D^-}) \oplus (-\Delta_{D^+}) \tag{1.1.33.}$$

where $-\Delta_{D^\pm}$ is the Dirichelet laplacian on $(y, \pm\infty)$.

We define $-\Delta_{\alpha,y}$ as the Schrödinger operator with point interaction centered in y of strenght α.
While in the three and two dimensional cases α is related to the scattering lenght, now α is just the coupling constant of the point interaction. Another peculiarity of the one-dimensional case, as we will see in the next chapter, is that δ_y is a small perturbation of $-\Delta$ in the sense of the forms, therefore $-\Delta_{\alpha,y}$ can be easily defined as quadratic form.
Exploiting carefully this fact we will obtain a new definition of $-\Delta_{\alpha,y}$ in dimension three and two by means of quadratic forms.

1.2 THE MANY CENTERS CASE

Here we introduce the Schrödinger operator with finitely many point interactions in dimension three, i.e. the rigorous counterpart of the formal expression

$$H = -\Delta - \sum_{J=1}^{N} \mu_J \delta_{y_J} \tag{1.2.1.}$$

where $y_1, \ldots, y_N \in \mathbb{R}^3$ and $\mu_1, \ldots, \mu_N$ are coupling constants.
To this aim one can employ the same procedure of the last section, based on selfadjoint extension, but it is more instructive to outline another method (Def. 2 in the introduction) which provides a definition of (1.2.1.) as resolvent limit of approximating Hamiltonians.
We will use a suitable momentum cut-off and a normalization of the coupling constants μ_J which will make clear the singular character of the point interactions. Let us start with the formal operator (1.2.1.) and consider the unitary equivalent operator in the space of Fourier transforms

$$\hat{H} = F H F^{-1} \tag{1.2.2.}$$

where F is the Plancherel operator in $L^2(\mathbb{R}^3)$ defined by

$$(Ff)(p) = \hat{f}(p) = s\text{-}\lim_{R \to \infty} \int_{|x|<R} e^{-ixp} f(x)\, dx \tag{1.2.3.}$$

Some formal manipulations lead us to the explicit expression of $\hat{H}$

$$(\hat{H} \hat{f})(p) = p^2 \hat{f}(p) - \sum_{J=1}^{N} \mu_J (\phi_{y_J}, \hat{f})\, \phi_{y_J}(p) \tag{1.2.4.}$$

where $\phi_{y_J}(p) = \dfrac{e^{-ipy_J}}{(2\pi)^{3/2}}$, $J = 1, \ldots, N$

Clearly $\hat{H}$ is not an operator in $L^2(\mathbb{R}^3)$ because the plane waves ϕ_{y_J} doesn't belong to $L^2(\mathbb{R}^3)$ .To avoid this difficulty we impose a momentum cut-off defining

$$\phi^{\omega}_{Y_J}(P) = 1_{B_\omega(0)}(P)\,\phi_{Y_J}(P) = \begin{cases} \phi_{Y_J}(P) & \text{if } |P| \leq \omega \\ 0 & \text{if } |P| > \omega \end{cases} \qquad (1.2.5.)$$

where $B_\omega(0)$ is the sphere centered in 0 of radius ω and $1_{B_\omega(0)}$ is its characteristic function, so that we get a well defined operator

$$\hat{H}^{\omega} = P^2 - \sum_{J=1}^{N} \mu_J(\omega)\left(\phi^{\omega}_{Y_J}, \cdot\right)\phi^{\omega}_{Y_J} \qquad (1.2.6.)$$

the coupling constant μ_J being considered explicitely dependent on the cut-off. The reason is that we have to perform the limit $\omega \to +\infty$ and so $\mu_J(\omega)$ must compensate the divergences arising from $\phi^{\omega}_{Y_J}$.

We note that the operator $-\sum_{J=1}^{N} \mu_J(\omega)\left(\phi^{\omega}_{Y_J}, \cdot\right)\phi^{\omega}_{Y_J}$ is infinitesimally small with respect to P^2 ([9] pag. 162) so $\hat{H}^{\omega}$ is selfadjoint on

$$D(\hat{H}^{\omega}) = D(P^2) = \left\{\hat{f} \in L^2(\mathbb{R}^3) \;\middle|\; P^2\hat{f}(P) \in L^2(\mathbb{R}^3)\right\} \qquad (1.2.7.)$$

and it can be used as the starting point to define our Schrödinger operator.

The line of the construction is now the following ([6]): one writes explicitely the resolvent of $\hat{H}^{\omega}$ which contains an integral divergent for $\omega \to +\infty$; to compensate this infinity one has to fix

$$\mu_J(\omega) = \frac{1}{\alpha_J + \frac{\omega}{2\pi}}, \qquad \alpha_J \in \mathbb{R}, \quad J = 1, \dots, N \qquad (1.2.8.)$$

Then it is easy to compute the limit and to prove that it is the resolvent of a selfadjoint operator denoted by

$$-\hat{\Delta}_{\alpha^{(N)}, Y^{(N)}}, \quad \alpha^{(N)} = (\alpha_1, \dots, \alpha_N), \quad Y^{(N)} = (Y_1, \dots, Y_N) \qquad (1.2.9.)$$

Finally we come back to the x-space and define

$$-\Delta_{\alpha^{(N)}, Y^{(N)}} = F^{-1} - \hat{\Delta}_{\alpha^{(N)}, Y^{(N)}} F \qquad (1.2.10.)$$

as the Schrödinger operator with point interactions centered in $Y^{(N)} = (y_1, \dots, y_N)$ parametrized by $\alpha^{(N)} = (\alpha_1, \dots, \alpha_N)$.

It is remarkable that the above construction provides the explicit expression of the resolvent of $-\Delta_{\alpha^{(N)}, Y^{(N)}}$

$$\left(-\Delta_{\alpha^{(N)}, Y^{(N)}} - k^2\right)^{-1} = G_k + \sum_{J,J'=1}^{N} \left[\underline{\underline{\Gamma}}_{\alpha^{(N)}, Y^{(N)}}(k)\right]^{-1}_{JJ'} \left(\overline{G_k^{y_{J'}}}, \cdot\right) G_k^{y_J}$$

$$k^2 \in \rho\left(-\Delta_{\alpha^{(N)}, Y^{(N)}}\right) \quad , \quad \mathrm{Im}\, k > 0 \qquad (1.2.11.)$$

where

$$\left[\underline{\underline{\Gamma}}_{\alpha^{(N)}, Y^{(N)}}(k)\right]_{JJ'} = \left(\alpha_J - \frac{ik}{4\pi}\right) \delta_{JJ'} - \left[\underline{\underline{G}}_k\right]_{JJ'} , \qquad (1.2.12.)$$

$$\left[\underline{\underline{G}}_k\right]_{JJ'} = \begin{cases} G_k(y_J, y_{J'}) & \text{if} \quad J \neq J' \\ 0 & \text{if} \quad J = J' \end{cases} \qquad (1.2.13.)$$

Making use of the same methods as in the one-center case with only some technical complications one can study the properties of $-\Delta_{\alpha^{(N)}, Y^{(N)}}$ ([6]).

First one can prove locality and characterize the domain $D\left(-\Delta_{\alpha^{(N)}, Y^{(N)}}\right)$ as the set of the functions

$$\Psi(x) = \phi_k(x) + \sum_{J=1}^{N} a_J G_k^{y_J}(x) \quad , \quad x \neq y_1, \dots, y_N \qquad (1.2.14.)$$

where

$$a_J = \sum_{J'=1}^{N} \left[\underline{\underline{\Gamma}}_{\alpha^{(N)}, Y^{(N)}}(k)\right]^{-1}_{JJ'} \phi_k(y_{J'}) \quad , \quad J = 1, \dots, N \qquad (1.2.15.)$$

$$\phi_k \in D(-\Delta) = H^2(\mathbb{R}^3) \quad , \quad k^2 \in \rho\left(-\Delta_{\alpha^{(N)}, Y^{(N)}}\right) \quad , \quad \mathrm{Im}\, k > 0 \qquad (1.2.16.)$$

Moreover

$$\left(-\Delta_{\alpha^{(N)}, Y^{(N)}} - k^2\right) \Psi = \left(-\Delta - k^2\right) \phi_k \qquad (1.2.17.)$$

The informations about spectrum and eigenfunctions of $-\Delta_{\alpha^{(N)},Y^{(N)}}$ are summarized in the following proposition

$$\sigma_{ess}\left(-\Delta_{\alpha^{(N)},Y^{(N)}}\right) = \sigma_{ac}\left(-\Delta_{\alpha^{(N)},Y^{(N)}}\right) = [0,+\infty) \quad , \quad \sigma_{sc}\left(-\Delta_{\alpha^{(N)},Y^{(N)}}\right) = \phi \qquad (1.2.18.)$$

and $-\Delta_{\alpha^{(N)},Y^{(N)}}$ has at most N negative eigenvalues counting multiplicity. If $\operatorname{Im} k > 0$ then

$$k^2 \in \sigma_P\left(-\Delta_{\alpha^{(N)},Y^{(N)}}\right) \quad \text{iff} \quad \det \underline{\underline{\Gamma}}_{\alpha^{(N)},Y^{(N)}}(k) = 0 \qquad (1.2.19.)$$

and the multiplicity of the eigenvalue k^2 equals the multiplicity of the eigenvalue zero of $\underline{\underline{\Gamma}}_{\alpha^{(N)},Y^{(N)}}(k)$.

Moreover if $E_0 = k_0^2 < 0$ is an eigenvalue of $-\Delta_{\alpha^{(N)},Y^{(N)}}$ the corresponding eigenfunctions are of the form

$$\Psi_0(x) = \sum_{J=1}^{N} c_J \, G_{k_0}^{y_J}(x) \quad , \operatorname{Im} k_0 > 0 \qquad (1.2.20.)$$

where $(c_1, \ldots, c_N)$ are eigenvectors with eigenvalue zero of $\underline{\underline{\Gamma}}_{\alpha^{(N)},Y^{(N)}}(k)$.

If $-\Delta_{\alpha^{(N)},Y^{(N)}}$ has a ground state then it is non degenerate and the corresponding eigenfunction is strictly positive.

Finally the generalized eigenfunctions of $-\Delta_{\alpha^{(N)},Y^{(N)}}$ can be obtained from the resolvent and then the scattering data can be computed. The on-shell scattering amplitude is

$$f_{\alpha^{(N)},Y^{(N)}}(k,\omega,\omega') = \frac{1}{4\pi} \sum_{JJ'=1}^{N} \left[\underline{\underline{\Gamma}}_{\alpha^{(N)},Y^{(N)}}(k)\right]^{-1}_{JJ'} e^{ik(y_{J'}\cdot\omega' - y_J\cdot\omega)} ,$$

$$\det \underline{\underline{\Gamma}}_{\alpha^{(N)},Y^{(N)}}(k) \neq 0 \quad , \quad k \geqslant 0 \quad , \quad \omega,\omega' \in S^2 \qquad (1.2.21.)$$

and the unitary on-shell scattering operator in $L^2(S^2)$ is

$$S_{\alpha^{(N)},Y^{(N)}}(k) = \mathbb{1} - \frac{k}{8\pi^2 i} \sum_{JJ'=1}^{N} \left[\underline{\underline{\Gamma}}_{\alpha^{(N)},Y^{(N)}}(k)\right]^{-1}_{JJ'} \left(e^{-iky_{J'}\cdot(\cdot)}, \cdot\right) e^{-iky_J(\cdot)}$$

$$\det \underline{\underline{\Gamma}}_{\alpha^{(N)}, \underline{y}^{(N)}}(k) \neq 0 \quad , \quad k \geq 0 \qquad (1.2.22.)$$

In conclusion we recall that even in dimension two and one the Schrödinger operator with finitely many point interactions can be defined and the main properties characterized (for details see [6]).

1.3 APPROXIMATION BY SHORT RANGE INTERACTIONS

Zero-range interaction is only a useful idealization which enable us to obtain a solvable model. Then it is reasonable to ask in what sense $-\Delta_{\alpha,Y}$ can be considered a limit of a suitable class of Hamiltonians with more realistic short range potentials. Besides clarifying the meaning of $-\Delta_{\alpha,Y}$ from a physical point of view this offers also the possibility of gaining informations about quantities associated with the regular potentials in terms of exactly known quantities associated with the limiting point interaction.

Here we shall limit ourself to discuss this problem only in the one center case. The generalization to finitely many point interactions can be obtained following the same methods and it requires only some technical complications ([6]).

Let $\varepsilon > 0$, V a Rollnik potential ([13]) , i.e.

$$\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|V(x)|\,|V(x')|}{|x-x'|^2}\,dx\,dx' < +\infty \qquad (1.3.1.)$$

with compact support and define the family of Hamiltonians

$$H_\varepsilon = -\Delta + \frac{\lambda(\varepsilon)}{\varepsilon^2} V\left(\frac{x}{\varepsilon}\right) = \frac{1}{\varepsilon^2} U_\varepsilon \left(-\Delta + \lambda(\varepsilon) V(x)\right) U_\varepsilon^{-1} \qquad (1.3.2.)$$

where $\lambda(\varepsilon)$ is a differentiable function with $\lambda(0)=1$ and U_ε is the unitary scaling operator in $L^2(\mathbb{R}^3)$ defined by

$$(U_\varepsilon g)(x) = \frac{1}{\varepsilon^{3/2}}\, g\left(\frac{x}{\varepsilon}\right) \qquad (1.3.3.)$$

It is apparent from (1.3.2.) that H_ε describe the dynamics of a particle moving in a potential which is concentrated, for ε small enough, only in a neighbourhood of the origin; so we may expect $H_\varepsilon \xrightarrow[\varepsilon\to 0]{} -\Delta_{\alpha,0}$, with $-\Delta_{\alpha,0}$ Schrödinger operator with point interaction centered in the origin, for some suitable value of α. The convergence is to be intended in the resolvent sense and so, making use of (1.3.2.) and of the resolvent equation, one has to study

$$\left(H_\varepsilon - k^2\right)^{-1} = \left(\frac{1}{\varepsilon^2} U_\varepsilon\left(-\Delta + \lambda(\varepsilon) V(x)\right) U_\varepsilon^{-1} - k^2\right)^{-1} =$$

$$= \varepsilon^2 U_\varepsilon \left(-\Delta + \lambda(\varepsilon) V(x) - (\varepsilon k)^2\right)^{-1} U_\varepsilon^{-1} =$$

$$= G_k - \lambda(\varepsilon) A_\varepsilon(k)\, \varepsilon \left(1 + \lambda(\varepsilon)\, u\, G_{\varepsilon k}\, v\right)^{-1} C_\varepsilon(k) \qquad (1.3.4.)$$

where $A_\varepsilon(k)$ and $C_\varepsilon(k)$ are integral operators with kernels

$$A_\varepsilon(k, x, x') = G_k(x, \varepsilon x')\, v(x') , \qquad (1.3.5.)$$

$$C_\varepsilon(k, x, x') = u(x)\, G_k(\varepsilon x, x') \qquad (1.3.6.)$$

and

$$u(x) = \left[\operatorname{sgn} V(x)\right] \left|V(x)\right|^{1/2} , \quad v(x) = \left|V(x)\right|^{1/2} \qquad (1.3.7.)$$

$A_\varepsilon(k)$ and $C_\varepsilon(k)$ are easily seen to converge for $\varepsilon \to 0$, in the Hilbert-Schmidt norm, to the integral operators $A(k)$, $C(k)$ with kernels

$$A(k, x, x') = G_k^0(x)\, v(x') \qquad (1.3.8.)$$

$$C(k, x, x') = u(x)\, G_k^0(x') \qquad (1.3.9.)$$

The crucial point is the convergence for $\varepsilon \to 0$ of the term

$$\varepsilon \left(1 + \lambda(\varepsilon)\, u\, G_{\varepsilon k}\, v\right)^{-1} \qquad (1.3.10.)$$

The form of (1.3.10.) suggests that the result is strictly related to the low energy spectral properties of $-\Delta + V$.

In particular suppose the operator $u\, G_0\, v$ has the eigenvalue -1 and let ϕ the corresponding eigenfunction. Then $\Psi = G_0\, v\, \phi$ satisfies ([6])

$$\Psi \in L^2_{loc}(\mathbb{R}^3) \quad , \quad \nabla \Psi \in L^2(\mathbb{R}^3) \quad , \quad (-\Delta + V)\Psi = 0 \qquad (1.3.11)$$

in the distributional sense and

$$0 \in \sigma(-\Delta + V) \qquad (1.3.12)$$

We say that Ψ is a zero-energy eigenstate of $-\Delta + V$ if $\Psi \in L^2(\mathbb{R}^3)$ and so $0 \in \sigma_p(-\Delta + V)$ while Ψ is a zero-energy resonance function of $-\Delta + V$ if $\Psi \notin L^2(\mathbb{R}^3)$ and 0 is a resonance of $-\Delta + V$.

Now there are four possible cases, giving different limits of H_ε for $\varepsilon \to 0$

1. -1 is not an eigenvalue of $u G_0 v$
2. -1 is a simple eigenvalue of $u G_0 v$ with $\Psi \notin L^2(\mathbb{R}^3)$
3. -1 is an eigenvalue (simple or not) of $u G_0 v$ with all functions $\Psi \in L^2(\mathbb{R}^3)$
4. -1 is a multiple eigenvalue of $u G_0 v$ with at least one $\Psi \notin L^2(\mathbb{R}^3)$.

The possible results of the limiting procedure are summarized in the following theorem ([6]) :

let V belonging to the Rollnik class with compact support and assume $\lambda'(0) \neq 0$ in cases 2. , 4. , then H_ε converges to $-\Delta_{\alpha,0}$ in the norm resolvent sense for $\varepsilon \to 0$, with α given by

$$\alpha = \begin{cases} +\infty & \text{in case } 1. \\ -\dfrac{\lambda'(0)}{|(v,\phi)|^2} & \text{in case } 2. \\ +\infty & \text{in case } 3. \\ -\dfrac{\lambda'(0)}{\sum_{\ell=1}^{N} |(v,\phi_\ell)|^2} & \text{in case } 4. \end{cases} \qquad (1.3.13)$$

where in case 2. ϕ is the eigenfunction of $u G_0 v$ with eigenvalue -1 and in case 4. ϕ_ℓ , $\ell = 1, \ldots, N$, are eigenfunctions of $u G_0 v$ with eigenvalue -1 such that $\Psi_\ell = G_0 v \phi_\ell \notin L^2(\mathbb{R}^3)$.

It tuns out that only in cases 2. , 4. H_ε converges to a non trivial point interaction as $\varepsilon \to 0$. In other words we get a point interaction in the limit $\varepsilon \to 0$ only if the potential V is such that $-\Delta + V$ has zero-energy resonance.

Making use of the above norm resolvent convergence one can prove convergence of many quantities related to H_ε , e.g. eigenvalues, resonances, scattering amplitude and scattering operator, to the corresponding quantities related to the limit $-\Delta_{\alpha,0}$ ([6]).

CHAPTER 2
QUADRATIC FORMS ASSOCIATED TO POINT INTERACTIONS

In this chapter we construct the quadratic form $F^{\alpha,y}$ associated to $-\Delta_{\alpha,y}$ in dimension three and then we obtain $F^{\alpha,y}$ as the Γ-limit of a sequence of approximating forms. Moreover the construction is extended to the two-dimensional case and to the many centers case.

2.1 QUADRATIC FORMS AND Γ-CONVERGENCE

We briefly summarize some elements of the theory of quadratic forms and associated selfadjoint operators. Definitions will be recalled with the least generality we need for our aims and the proofs of the theorems will be omitted; for a complete treatment we refer to [7] and references therein.

Let us consider a Hilbert space $\mathcal{H}$, with norm $\|\cdot\|$ and scalar product $(\cdot,\cdot)$, and a symmetric and bilinear form (in the sense that it is linear in the second element, anti-linear in the first)

$$B : D(B) \times D(B) \longrightarrow \mathbb{C} \qquad (2.1.1.)$$

where $D(B)$ is a dense subset of $\mathcal{H}$.

B is called positive if

$$B(u,u) \geq 0 \qquad \forall\, u \in D(B) \qquad (2.1.2)$$

and bounded from below by $\gamma \in \mathbb{R}$ if γ is the largest number such that

$$B(u,u) \geq \gamma \; \|u\|^2 \qquad \forall\, u \in D(B) \qquad (2.1.3)$$

A form B bounded from below by $\gamma \in \mathbb{R}$ is closed if $D(B)$ is a Hilbert space with respect to the scalar product

$$(u,v)_B = B(u,v) + (|\gamma|+1)(u,v) \qquad (2.1.4.)$$

The norm in $D(B)$ is obviously

$$\|u\|_B^2 = B(u,u) + (|\gamma|+1)\|u\|^2 \qquad (2.1.5.)$$

The definition of quadratic form adopted in the following will be:

DEFINITION 2.1.1. A map $F: \mathcal{H} \to \overline{\mathbb{R}}$ ($\overline{\mathbb{R}}$ is the set of extended real numbers) is a quadratic form if there exists $D(F)$, dense subset of $\mathcal{H}$, and B_F, symmetric and bilinear form with domain $D(F)$, such that

$$F(u) = \begin{cases} B_F(u,u) & \text{if } u \in D(F) \\ +\infty & \text{if } u \notin D(F) \end{cases} \qquad (2.1.6.)$$

We say that F is positive or bounded from below if the corresponding B_F is respectively positive or bounded from below. The notion of closure is substituted by that of lower semicontinuity:

DEFINITION 2.1.2. A quadratic form F is lower semicontinuous (l.s.c.) if

$$F(u) \leq \liminf_n F(u_n) \quad , \forall u \in \mathcal{H} \quad , \quad \forall u_n \xrightarrow[n]{} u \qquad (2.1.7.)$$

In fact lower semicontinuity and closure are equivalent

PROPOSITION 2.1.3. Let F be a quadratic form bounded from below, then F is l.s.c. iff B_F is closed.

Now we state the usual correspondence between quadratic forms and linear operators. Let F be a quadratic form bounded from below

DEFINITION 2.1.4. The operator A associated with F is the map with domain

$$D(A) = \{ u \in D(F) \mid \exists f \in \mathcal{H} : B_F(u,v) = (v,f) \ \forall v \in D(F) \} \qquad (2.1.8)$$

and such that Au is defined by

$$B_F(v,u) = (v, Au) \qquad \forall v \in D(F) \qquad (2.1.9)$$

Then it can be proved (see also [9], [14])

PROPOSITION 2.1.6. Let F be a quadratic form bounded from below and A its associated operator then

A is a linear operator, densely defined, bounded from below (with the same lower bound of F), symmetric;

if F is l.s.c. then A is selfadjoint and $D(A)$ is a dense subset of $(D(F), \|\cdot\|_{B_F})$.

If F_1, F_2 are quadratic forms, bounded from below, l.s.c. and A_1, A_2 their associated operators then

$$A_1 = A_2 \quad \text{implies} \quad F_1 = F_2$$

Let us now introduce the notion of Γ-convergence ([7]) for quadratic forms and let us estabilish the relation with convergence in the resolvent sense of the associated operators. Let F_n, F be quadratic forms

DEFINITION 2.1.6. F_n Γ-converges to F ($F_n \xrightarrow[n]{\Gamma} F$ or $F = \Gamma\text{-}\lim_n F_n$) if the following two conditions are satisfied:

(a) $\forall u \in D(F)$ $\exists$ (u_n) converging to u such that

$$F(u) = \lim_n F_n(u_n) \qquad (2.1.10)$$

(b) $\forall u \in \mathcal{H}$ and $\forall (u_n)$ converging to u it results

$$F(u) \leq \liminf_n F_n(u_n) \qquad (2.1.11)$$

REMARK 2.1.7. If F is l.s.c. proposition 2.1.5. allows us to substitute $D(F)$ with $D(A)$ in condition (a).

To illustrate the the definition given above we give an example of Γ-convergence in the trivial case in which $\mathcal{H} = \mathbb{R}$ and F_n is simply the sequence of functions with graphs shown in figure 1.

It is not hard to see that

$$\Gamma\text{-}\lim_n F_n = F$$

where

$$F(x) = \begin{cases} -1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$$

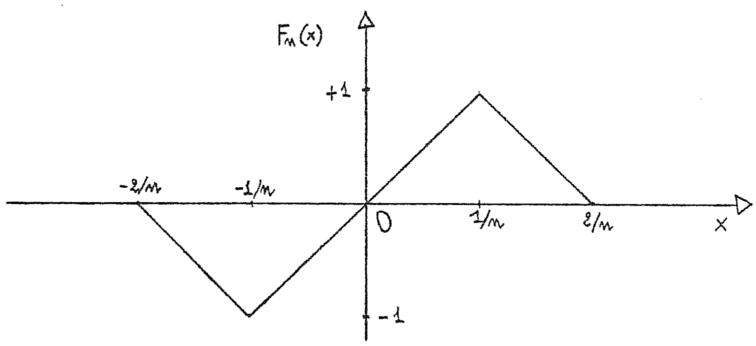

FIG. 1

This example shows that Γ-convergence is completely different from pointwise convergence (in fact $F_n(x) \xrightarrow[n]{} 0 \quad \forall x \in \mathbb{R}$) whereas it is related to the convergence of the minimum values and minimum points of F_n . It turns out that, under suitable assumption, Γ-convergence implies convergence of the minimum values and minimum points and this is the reason why Γ-convergence is so largely used in variational problems (as example of applicationn see [15]).

Now we list some properties of Γ-limits which will be useful in the sequel.

It holds a comparison theorem which in turn makes the Γ-limit unique:

PROPOSITION 2.1.8. If $F_n \xrightarrow[n]{\Gamma} F$, $G_n \xrightarrow[n]{\Gamma} G$, $F_n \leq G_n$ then $F \leq G$

In general the Γ-limit of the sum is not equal to the sum of the Γ-limits:

PROPOSITION 2.1.9. If $F_n \xrightarrow[n]{\Gamma} F$, $G_n \xrightarrow[n]{\Gamma} G$, $F_n + G_n \xrightarrow[n]{\Gamma} H$ then $F + G \leq H$

There is a particular case in which the equality sign holds:

PROPOSITION 2.1.10. If $F_n \xrightarrow[n]{\Gamma} F$, G continuous then $F_n + G \xrightarrow[n]{\Gamma} F + G$.

The following proposition shows that lower semicontinuity and Γ-convergence are strictly linked:

PROPOSITION 2.1.11. If $F_n \xrightarrow[n]{\Gamma} F$ then F is l.s.c.

Finally we state the connection between Γ-convergence of quadratic forms and convergence in the resolvent sense of the associated operators.

PROPOSITION 2.1.12. Let F_n , F be quadratic forms, uniformely bounded from below and l.s.c. and A_n , A the corresponding associated operators; then the following conditions are equivalent:

i) $F_n \xrightarrow[n]{\Gamma} F$ and $F(u) \leq \liminf_n F_n(u_n)$ $\forall u \in \mathcal{H}$ and $\forall\, (u_n)$ converging weakly to u .

ii) A_n converges to A in the strong resolvent sense.

2.2. POINT INTERACTION DEFINED AS QUADRATIC FORM

In this section we introduce a new definition of $-\Delta + \mu\,\delta_y$ in dimension three by means of suitable quadratic form. It is useful to start with the one-dimensional case in which such a definition is well known ([6], [9] pag. 168, [16]). Let

$\mathcal{H} = L^2(\mathbb{R}^3)$, $y \in \mathbb{R}$, $\alpha \in \mathbb{R}$ and

$$F^{\alpha,y}(u) = \begin{cases} \int_{\mathbb{R}} |\nabla u|^2 + \alpha\,|u(y)|^2 & \text{if } u \in H^1(\mathbb{R}) \\ +\infty & \text{if } u \notin H^1(\mathbb{R}) \end{cases} \qquad (2.2.1.)$$

$F^{\alpha,y}$ is a quadratic form with domain $D(F^{\alpha,y}) = H^1(\mathbb{R})$.

Note that definition $(2.2.1.)$ does make sense since $H^1(\mathbb{R})$ is embedded into the space of the continuous functions ([12]).

Observe that

$$F^{\alpha,y} = F_1^{\alpha,y} + F_2^{\alpha,y} \qquad (2.2.2.)$$

where $F_1^{\alpha,y}$ is a positive and l.s.c. quadratic form defining $-\Delta$ and

$$F_2^{\alpha,y}(u) = \begin{cases} \alpha\,|u(y)|^2 & \text{if } u \in H^1(\mathbb{R}) \\ +\infty & \text{if } u \notin H^1(\mathbb{R}) \end{cases} \qquad (2.2.3.)$$

It is not difficult to prove ([16]) that $F_2^{\alpha,y}$ is infinitesimally bounded with respect to $F_1^{\alpha,y}$, i.e. ([9] pag. 168) $\exists\ a>0$ arbitrarily small and $b>0$ s.t.

$$D(F_2^{\alpha,y}) \supseteq D(F_1^{\alpha,y}) \ , \ \left|F_2^{\alpha,y}(u)\right| \le a\ F_1^{\alpha,y}(u) + b\,\|u\|^2_{L^2(\mathbb{R}^3)} \qquad \forall u \in D(F_1^{\alpha,y})$$

So we can assert ([9] pag. 167) that $F^{\alpha,y}$ is bounded from below and l.s.c. with domain $H^1(\mathbb{R})$ and it is easily seen that the selfadjoint associated operator is $-\Delta_{\alpha,y}$, as defined in chapter 1.

We already know that, in dimension 1,2,3, " δ_y " is not a small perturbation in the sense of Kato ([9] pag. 162) with respect to $-\Delta$, since $D(-\Delta_{\alpha,y}) \supset D(-\Delta)$ (see section 1.1) ; more precisely $D(-\Delta_{\alpha,y})$ consists of the direct sum of $D(-\Delta) = H^2(\mathbb{R}^n)$ and the one-dimensional subspace spanned by G_k^y , $\operatorname{Im} k>0$, which is square-integrable in $\mathbb{R}^n$ but doesn't belong to $H^2(\mathbb{R}^n)$ for $n = 1,2,3$.

On the contrary the above construction of $F^{\alpha,y}$ in dimension one shows that " δ_y " is a small perturbation in the sense of the forms with respect to $-\Delta$. This can be interpreted as follows: the form domain of $-\Delta_{\alpha,y}$ still consists of the form domain $H^1(\mathbb{R})$ of $-\Delta$ plus $[G_k^y]$ but in this case we have $G_k^y \in H^1(\mathbb{R})$ and so the form domain of $-\Delta$ is not perturbed by " δ_y " . In dimension greater than one this is not true since $G_k^y \notin H^1(\mathbb{R}^n)$, $n > 1$, and so it reasonable to expect that the form domain of $-\Delta$ will be enlarged with the subspace spanned by G_k^y to obtain the form domain of $-\Delta_{\alpha,y}$.
Let us rewrite $F^{\alpha,y}$ in dimension one in such a way to make clear the role of G_k^y in the form domain. We will consider only purely imaginary values of k

$$k = i\sqrt{\lambda} \quad , \quad \lambda > 0 \tag{2.2.4.}$$

but we will see that this is not a restriction.

$$F^{\alpha,y}(u) = \int_{\mathbb{R}} \left|\nabla\left(u - c(u)G^y_{i\sqrt{\lambda}}\right)\right|^2 + \lambda \int_{\mathbb{R}} \left|u - c(u)G^y_{i\sqrt{\lambda}}\right|^2 + A(\lambda,\alpha)\,|c(u)|^2 -$$

$$-\lambda \int_{\mathbb{R}} |u|^2 \qquad \forall \lambda > 0 \ , \ \forall u \in H^1(\mathbb{R}) \tag{2.2.5.}$$

where

$$c(u) = -\alpha\, u(y) \tag{2.2.6.}$$

$$A(\lambda,\alpha) = -\frac{2\sqrt{\lambda} + \alpha}{2\sqrt{\lambda}\,\alpha} \tag{2.2.7.}$$

and

$$G^y_{i\sqrt{\lambda}}(x) = \left(-\frac{d^2}{dx^2} + \lambda\right)^{-1}(x,y) = \frac{e^{-\sqrt{\lambda}|x-y|}}{2\sqrt{\lambda}} \tag{2.2.8.}$$

is the Green function of $-\Delta + \lambda$ in dimension one.
Note that , appropriately defining $c(u)$, the ~~last two lines of~~ (2.2.5.) does make sense even if $G^y_{i\sqrt{\lambda}} \notin H^1(\mathbb{R}^n)$ but only $G^y_{i\sqrt{\lambda}} \in L^2(\mathbb{R}^3)$, which is just the case for

$n = 2, 3$; moreover if $c(u) = 0$ they reduce to the form defining $-\Delta$. If we confine ourself to $n = 3$ a reasonable definition of $c(u)$, when u is not continuous in general, seems

$$c(u) = \beta \lim_{R \to 0} R \fint_{B_R(y)} u \qquad \forall u \in L^2(\mathbb{R}^3) \text{ s.t. the limit exists} \qquad (2.2.9.)$$

where $\beta = \frac{8\pi}{3}$ and $\fint_{B_R(y)} u$ indicates the mean value of u over the sphere of radius R centered in y .

With this definition $c(u)$ is zero $\forall u \in H^1(\mathbb{R}^3)$:

$$\frac{R}{|B_R(y)|} \int_{B_R(y)} |u| \leq \frac{R}{|B_R(y)|} \left(\int_{B_R(y)} dx \right)^{5/6} \| u \|_{L^6(\mathbb{R}^3)} \leq \text{const. } R^{1/2} \| u \|_{H^1(\mathbb{R}^3)} \xrightarrow[R \to 0]{} 0 \qquad (2.2.10.)$$

where Holder inequality and a standard Sobolev embedding ([12]) have been used. Moreover $c(u)$ ~~it~~ is finite and different from zero if u has a singularity in y of the type of $G^y_{i\sqrt{\lambda}}$ and it is normalized in such a way that $c\left(G^y_{i\sqrt{\lambda}}\right) = 1$.

Then we define $\forall \lambda > 0$

$$D(F^{\alpha,y}) = \left\{ u \in L^2(\mathbb{R}^3) \,\middle|\, c(u) \text{ exists finite}, \; u - c(u) G^y_{i\sqrt{\lambda}} \in H^1(\mathbb{R}^3) \right\} \qquad (2.2.11.)$$

and

$$F^{\alpha,y}(u) = \begin{cases} \int_{\mathbb{R}^3} |\nabla(u - c(u) G^y_{i\sqrt{\lambda}})|^2 + \lambda \int_{\mathbb{R}^3} |u - c(u) G^y_{i\sqrt{\lambda}}|^2 + A(\lambda,\alpha) |c(u)|^2 - \lambda \int_{\mathbb{R}^3} |u|^2 & \text{if } u \in H^1(\mathbb{R}^3) \\ +\infty & \text{if } u \notin H^1(\mathbb{R}^3) \end{cases} =$$

$$= F^{\alpha,y,\lambda}(u) - \lambda \int_{\mathbb{R}^3} |u|^2 \qquad (2.2.12.)$$

where

$$A(\lambda,\alpha) = \frac{\sqrt{\lambda}}{4\pi} + \alpha \qquad (2.2.13.)$$

(the reason of this position will be clear in the following). The definition is given in terms of the parameter $\lambda > 0$ but, as we will show in the next section (see remark 2.3.2.), it is independent of λ . In fact if we put $\lambda = 0$ the second integrand in (2.2.12.) is not summable when $|x| \to \infty$, so the presence of a positive λ is useful to regularize the behavior of the integrand at infinity, i.e. to make the integral finite. It is not difficult to prove that $F^{\alpha,Y}$ has the right properties for our aims.

PROPOSITION 2.2.1. $F^{\alpha,Y}$ is a quadratic form, with domain $D(F^{\alpha,Y})$, which is positive for $\alpha \geq 0$, bounded from below by $-(4\pi\alpha)^2$ for $\alpha < 0$ and l.s.c.

PROOF. It is clear that $F^{\alpha,Y}$ is a quadratic form with dense domain $D(F^{\alpha,Y})$.
Let $\alpha \geq 0$ then $\forall u \in D(F^{\alpha,Y})$

$$F^{\alpha,Y}(u) + \lambda \int_{\mathbb{R}^3} |u|^2 \geq 0 \qquad \forall \lambda > 0 \tag{2.2.14.}$$

which implies positivity of $F^{\alpha,Y}$.
Lower boundness for $\alpha < 0$ is obtained in the same way.
It remains to prove lower semicontinuity. Taking into account that the term $\lambda \int_{\mathbb{R}^3} |u|^2$ is continuous, it is sufficient to prove semicontinuity of $F^{\alpha,Y,\lambda}$.
Let $u \in L^2(\mathbb{R}^3)$ and (u_n) converging to u ; if $F^{\alpha,Y,\lambda}(u_n) \to +\infty$ then inequality (2.1.7.) is trivially true. Otherwise there exists a subsequence (u'_n) such that

$$F^{\alpha,Y,\lambda}(u'_n) < \text{const.} \tag{2.2.15.}$$

which trivially implies

$$\| u'_n - c(u'_n)\, G^Y_{i\sqrt{\lambda}} \|_{H^1(\mathbb{R}^3)} < \text{const.} \tag{2.2.16.}$$

$$| c(u'_n) | < \text{const.} \tag{2.2.17.}$$

A bounded sequence in a Banach space is weakly convergent and so there exists $w \in H^1(\mathbb{R}^3)$, $c \in \mathbb{C}$ such that

$$u'_n \underset{n}{\longrightarrow} w + c\, G^Y_{i\sqrt{\lambda}} \qquad \text{in } L^2(\mathbb{R}^3) \tag{2.2.18.}$$

This means $u = w + c\, G^{y}_{i\sqrt{\lambda}}$ and so $u \in D(F^{\alpha,y})$, $c = c(u)$.
In a Banach space the norm is l.s.c. with respect to the weak topology so we have

$$\int_{\mathbb{R}^3} \left|\nabla\left(u - c(u)\, G^{y}_{i\sqrt{\lambda}}\right)\right|^2 + \lambda \int_{\mathbb{R}^3} \left|u - c(u)\, G^{y}_{i\sqrt{\lambda}}\right|^2 \leq \liminf_{n} \left(\int_{\mathbb{R}^3} \left|\nabla\left(u'_n - c(u'_n) G^{y}_{i\sqrt{\lambda}}\right)\right|^2 + \lambda \int_{\mathbb{R}^3} \left|u'_n - c(u'_n)\, G^{y}_{i\sqrt{\lambda}}\right|^2 \right) \qquad (2.2.19.)$$

which implies inequality $(2.1.7)$. ∎

Now it remains to verify that our quadratic form $F^{\alpha,y}$ really defines $-\Delta_{\alpha,y}$ in dimension three. Let $T_{\alpha,y}$ be the selfadjoint operator associated to $F^{\alpha,y}$ then

PROPOSITION 2.2.2.

i) $D(T_{\alpha,y}) = \left\{ u \in D(F^{\alpha,y}) \,\middle|\, u - c(u) G^{y}_{i\sqrt{\lambda}} \in H^2(\mathbb{R}^3) \,,\; \left(u - c(u)\, G^{y}_{i\sqrt{\lambda}}\right)(y) = A(\lambda,\alpha)\, c(u) \right\} = D(-\Delta_{\alpha,y})$

ii) $(T_{\alpha,y} + \lambda)\, u = (-\Delta + \lambda)\left(u - c(u)\, G^{y}_{i\sqrt{\lambda}}\right) \qquad \forall \lambda > 0 \quad , \; \forall u \in D(T_{\alpha,y})$

iii) $(T_{\alpha,y} + \lambda)^{-1} f = G_{i\sqrt{\lambda}} f + \dfrac{1}{A(\lambda,\alpha)} \left(G_{i\sqrt{\lambda}} f\right)(y)\, G^{y}_{i\sqrt{\lambda}} \qquad \forall \lambda > 0$ if $\alpha \geq 0$, $\forall \lambda > (4\pi\alpha)^2$ if $\alpha < 0$, $\forall f \in L^2(\mathbb{R}^3)$

and so $T_{\alpha,y} = -\Delta_{\alpha,y}$

PROOF. If $u \in D(T_{\alpha,y})$ then $\exists f \in L^2(\mathbb{R}^3)$ such that

$$\int_{\mathbb{R}^3} \overline{\nabla v} \cdot \nabla\left(u - c(u)\, G^{y}_{i\sqrt{\lambda}}\right) + \lambda \int_{\mathbb{R}^3} \bar{v}\left(u - c(u)\, G^{y}_{i\sqrt{\lambda}}\right) = \int_{\mathbb{R}^3} \bar{v} f \qquad \forall v \in H^1(\mathbb{R}^3) \qquad (2.2.20.)$$

which, by regularity theorems ([17]), implies

$$u - c(u)\, G^{y}_{i\sqrt{\lambda}} \in H^2(\mathbb{R}^3) \qquad (2.2.21.)$$

$$(-\Delta+\lambda)\left(u-c(u)G^{y}_{i\sqrt{\lambda}}\right) = f \qquad (2.2.22.)$$

If $v \in D(F^{\alpha,y})$ with $c(v) \neq 0$ then, taking into account of (2.2.20.) ,we have

$$A(\lambda,\alpha)\,c(u) = \int_{\mathbb{R}^3} G^{y}_{i\sqrt{\lambda}} f = \left(u - c(u)G^{y}_{i\sqrt{\lambda}}\right)(y) \qquad (2.2.23.)$$

Viceversa if $u \in D(F^{\alpha,y})$ with $u - c(u)G^{y}_{i\sqrt{\lambda}} \in H^2(\mathbb{R}^3)$ and $\left(u-c(u)G^{y}_{i\sqrt{\lambda}}\right)(y) = A(\lambda,\alpha)c(u)$ then

$$B_F(v,u) = \int_{\mathbb{R}^3} \bar{v}\left[(-\Delta+\lambda)\left(u-c(u)G^{y}_{i\sqrt{\lambda}}\right)\right] \qquad \forall v \in D(F^{\alpha,y}) \qquad (2.2.24.)$$

If we define $f = (-\Delta+\lambda)\left(u-c(u)G^{y}_{i\sqrt{\lambda}}\right)$ we obtain the proof of the first inequality in i). The second inequality in i) is easily verified: if $u \in D(T_{\alpha,y})$ fix

$$\Phi_k(x) = u(x) - c(u)\, e^{i\,\mathrm{Re}\,k|x-y|}\, G^{y}_{i\sqrt{\lambda}} \qquad (2.2.25.)$$

with $\mathrm{Im}\,k = \sqrt{\lambda}$ and so $u \in D(-\Delta_{\alpha,y})$; conversely if $u \in D(-\Delta_{\alpha,y})$ it results

$$c(u) = \frac{\Phi_k(y)}{\alpha - \frac{ik}{4\pi}} \qquad (2.2.26.)$$

and so $u \in D(T_{\alpha,y})$.

The statement ii) is an easy consequence of i).

Finally we define

$$u = G^{y}_{i\sqrt{\lambda}} f + \frac{\overline{(G^{y}_{i\sqrt{\lambda}} f)(y)}}{A(\lambda,\alpha)}\, G^{y}_{i\sqrt{\lambda}} \qquad \forall f \in L^2(\mathbb{R}^3) \qquad (2.2.27.)$$

then using ii) we obtain

$$(T_{\alpha,y}+\lambda)\,u = f \qquad (2.2.28.)$$

which implies iii). ∎

2.3. POINT INTERACTION AS Γ-LIMIT OF APPROXIMATING FORMS

Here we show that the quadratic form $F^{\alpha,y}$, defining $-\Delta_{\alpha,y}$ in dimension three, can be obtained as the Γ–limit of approximating forms. As usual we start with the one-dimensional case where a natural choice of approximating forms for $F^{\alpha,y}$, given by (2-2-1), is

$$F_R^{\alpha,y}(u) = \begin{cases} \int_{\mathbb{R}} |\nabla u|^2 + \alpha \left| \fint_{B_R(y)} u \right|^2 & \text{if } u \in H^1(\mathbb{R}) \\ +\infty & \text{if } u \notin H^1(\mathbb{R}) \end{cases} \qquad (2.3.1.)$$

with $D(F^{\alpha,y}) = H^1(\mathbb{R})$.

We know that $F^{\alpha,y}$ can also be written as in (2.2.5.) which is a more useful form in view of the extension to the three dimensional case. This suggests to rewrite $F_R^{\alpha,y}$ to make clear that it is an approximating form for (2-2-5.). To this aim we introduce an approximate Green function ([18]) for $-\Delta+\lambda$, defined $\forall R>0$ as the solution of the problem

$$\begin{cases} (-\Delta+\lambda)\, G^y_{i\sqrt{\lambda},R} = \dfrac{1_{B_R(y)}}{|B_R(y)|} \\ G^y_{i\sqrt{\lambda},R} \in H^1(\mathbb{R}) \end{cases} \qquad (2.3.2.)$$

where $|B_R(y)|$ is the measure of the sphere $B_R(y)$. An explicit calculation shows that

$$G^y_{i\sqrt{\lambda},R}(x) = \begin{cases} \dfrac{e^{-\sqrt{\lambda}|x-y|}}{2\sqrt{\lambda}} \cdot \dfrac{\sinh \sqrt{\lambda}R}{\sqrt{\lambda}R} & \text{if } |x-y|>R \\ \dfrac{1}{2\lambda R} - \dfrac{1}{2\lambda R} e^{-\sqrt{\lambda}R} \cosh \sqrt{\lambda}|x-y| & \text{if } |x-y|\le R \end{cases} \qquad (2.3.3.)$$

$G^y_{i\sqrt{\lambda},R}$ is an everywhere differentiable function and it's not difficult to verify that

$$G^y_{i\sqrt{\lambda},R} \xrightarrow[R\to 0]{} G^y_{i\sqrt{\lambda}} \quad \text{in } H^1(\mathbb{R}) \qquad (2.3.4.)$$

(in fact one has pointwise convergence $\forall x \in \mathbb{R}$).

With the aid of (2-3-3) $F_R^{\alpha,y}$ becomes

$$F_R^{\alpha,y}(u) = \begin{cases} \int_{\mathbb{R}} \left|\nabla\left(u - c_R(u) G^y_{i\sqrt{\lambda},R}\right)\right|^2 + \lambda \int_{\mathbb{R}} \left|u - c_R(u) G^y_{i\sqrt{\lambda},R}\right|^2 + A_R(\lambda,\alpha)|c_R(u)|^2 - \lambda \int_{\mathbb{R}} |u|^2 & \text{if } u \in H^1(\mathbb{R}) \\ +\infty & \text{if } u \notin H^1(\mathbb{R}) \end{cases} \tag{2.3.5.}$$

where

$$c_R(u) = -\alpha \fint_{B_R(y)} u \tag{2.3.6.}$$

$$A_R(\lambda,\alpha) = -\frac{1}{\alpha} - \fint_{B_R(y)} G^y_{i\sqrt{\lambda},R} = \frac{1}{2\lambda R} \cdot \frac{e^{-\sqrt{\lambda}R} - 1 + 2\sqrt{\lambda}R}{2\sqrt{\lambda}R} \tag{2.3.7.}$$

Now it is an easy exercise to prove Γ–convergence of $F_R^{\alpha,y}$ to $F^{\alpha,y}$.

In the three dimensional case we have to take into account that a function in $H^1(\mathbb{R}^3)$ is not continuous in general, so we still define the quadratic form

$$F_R^{\alpha,y}(u) = \begin{cases} \int_{\mathbb{R}^3} |\nabla u|^2 - a_R \left| \fint_{B_R(y)} u \right|^2 & \text{if } u \in H^1(\mathbb{R}^3) \\ +\infty & \text{if } u \notin H^1(\mathbb{R}^3) \end{cases} \tag{2.3.8.}$$

with domain $D(F_R^{\alpha,y}) = H^1(\mathbb{R}^3)$, but the coefficient a_R ,depending on α , must be infinitesimal in R in such a way to compesate the divergence of the term $\left|\fint_{B_R(y)} u\right|^2$ for $R \to 0$. In the following it will be clear that the right choice is

$$a_R = \eta R\left(1 - \eta R\alpha\right) \quad , \quad \eta = \frac{10\pi}{3} \tag{2.3.9.}$$

It is easy to check that $F_R^{\alpha,y}$ is bounded from below and l.s.c. $\forall R > 0$.

Now we proceed following the same steps of the one dimensional case. Let us define an approximate Green function for $-\Delta + \lambda$ as the solution $\forall R > 0$ of the problem

$$\left(-\Delta + \lambda\right) G^y_{i\sqrt{\lambda},R} = \frac{1_{B_R(y)}}{|B_R(y)|} \tag{2.3.10.}$$

$$G^y_{i\sqrt{\lambda},R} \in H^1(\mathbb{R}^3)$$

Again we explicitely solve $(2.3.10.)$ and we find

$$G^{y}_{i\sqrt{\lambda},R}(x) = \begin{cases} \dfrac{1}{\lambda|B_R(y)|} - \dfrac{\sqrt{\lambda}R+1}{\lambda|B_R(y)|}\, e^{-\sqrt{\lambda}R}\, \dfrac{\sinh\sqrt{\lambda}|x-y|}{\sqrt{\lambda}\,|x-y|} & \text{if } |x-y| \le R \\ \dfrac{\sqrt{\lambda}R\cosh\sqrt{\lambda}R - \sinh\sqrt{\lambda}R}{\lambda|B_R(y)|} \cdot \dfrac{e^{-\sqrt{\lambda}|x-y|}}{\sqrt{\lambda}\,|x-y|} & \text{if } |x-y| > R \end{cases} \tag{2.3.11.}$$

In fact $G^{y}_{i\sqrt{\lambda},R}$ is a continuous function and

$$G^{y}_{i\sqrt{\lambda},R} \xrightarrow[R\to 0]{} G^{y}_{i\sqrt{\lambda}} \quad \text{in} \quad L^2(\mathbb{R}^3) \tag{2.3.12.}$$

(we cannot have convergence in $H^1(\mathbb{R}^3)$ because $G^{y}_{i\sqrt{\lambda}} \notin H^1(\mathbb{R}^3)$) and the following relation holds

$$\int_{B_R(y)} G^{y}_{i\sqrt{\lambda},R} = \frac{3}{10\pi R} - \frac{\sqrt{\lambda}}{4\pi} + f^{\lambda}(R) \tag{2.3.13.}$$

where f^{λ} is analytic and $f^{\lambda}(0) = 0$.

Moreover it will be useful the weak formulation of (2.3.10.)

$$\int_{\mathbb{R}^3} \nabla G^{y}_{i\sqrt{\lambda},R}\cdot\overline{\nabla v} + \lambda\int_{\mathbb{R}^3} G^{y}_{i\sqrt{\lambda},R}\,\overline{v} = \int_{B_R(y)} \overline{v} \qquad \forall v \in H^1(\mathbb{R}^3) \tag{2.3.14.}$$

and in particular if we fix $v = G^{y}_{i\sqrt{\lambda},R}$

$$\int_{\mathbb{R}^3} |\nabla G^{y}_{i\sqrt{\lambda},R}|^2 + \lambda\int_{\mathbb{R}^3} |G^{y}_{i\sqrt{\lambda},R}|^2 = \int_{B_R(y)} G^{y}_{i\sqrt{\lambda},R} \tag{2.3.15.}$$

Now defining $\forall\, u \in H^1(\mathbb{R}^3)$

$$C_R(u) = a_R \int_{B_R(y)} u \tag{2.3.16.}$$

and

$$A_R(\lambda,\alpha) = \frac{1}{a_R} - \int_{B_R(y)} G^{y}_{i\sqrt{\lambda},R} = \frac{\alpha + \sqrt{\lambda}/4\pi}{1 - 10/3\,\pi R\alpha} - \frac{\sqrt{\lambda}/4\pi\;\; 10/3\,\pi R\alpha}{1 - 10/3\,\pi R\alpha} - f^{\lambda}(R) \tag{2.3.17.}$$

and using (2.3.14.) , (2.3.15.) we can rewrite $F^{\alpha,y}_{R}$ as

$$F_R^{\alpha,y}(u) = \int_{\mathbb{R}^3} \left|\nabla\left(u - c_R(u) G^y_{i\sqrt{\lambda},R}\right)\right|^2 + \lambda \int_{\mathbb{R}^3} \left|u - c_R(u) G^y_{i\sqrt{\lambda},R}\right|^2 + A_R(\lambda,\alpha)\left|c_R(u)\right|^2 - \lambda \int_{\mathbb{R}^3} |u|^2 =$$

$$= F_R^{\alpha,y,\lambda}(u) - \lambda \int_{\mathbb{R}^3} |u|^2 \qquad \forall u \in H^1(\mathbb{R}^3) \qquad (2.3.18)$$

Finally we can prove

PROPOSITION 2.3.1.

i) $\forall u \in D(F^{\alpha,y}) \quad \exists u_R \xrightarrow[R\to 0]{} u$ in $L^2(\mathbb{R}^3)$ such that

$$F^{\alpha,y}(u) = \lim_{R\to 0} F_R^{\alpha,y}(u_R)$$

ii) $\forall u \in L^2(\mathbb{R}^3)$ and $\forall u_R \xrightarrow[R\to 0]{} u$ in $L^2(\mathbb{R}^3)$ it is

$$F^{\alpha,y}(u) \leq \liminf_{R\to 0} F_R^{\alpha,y}(u_R)$$

then in particular $\qquad F^{\alpha,y} = \Gamma\text{-}\lim_{R\to 0} F_R^{\alpha,y}$

PROOF. First we observe that it is sufficient to prove statements i), ii), with $F^{\alpha,y}$ and $F_R^{\alpha,y}$ replaced by $F^{\alpha,y,\lambda}$ and $F_R^{\alpha,y,\lambda}$ (choosing $\lambda > 0$ for $\alpha \geq 0$ and $\lambda > (4\pi\alpha)^2$ for $\alpha < 0$). Let us prove i). Remark 2.1.7. allows us to substitute $D(F^{\alpha,y})$ with $D(-\Delta_{\alpha,y})$, then for $u \in D(-\Delta_{\alpha,y})$ we define

$$u_R = u - c(u) G^y_{i\sqrt{\lambda}} + c(u) G^y_{i\sqrt{\lambda},R} = w + c(u) G^y_{i\sqrt{\lambda},R} \qquad (2.3.19)$$

which clearly converges to u in $L^2(\mathbb{R}^3)$. Inserting u_R in $F_R^{\alpha,y,\lambda}$ and using (2.3.14), (2.3.15) one has

$$F_R^{\alpha,y,\lambda}(u_R) - F^{\alpha,y,\lambda}(u) = \left|c(u) - c_R(u_R)\right|^2 \int_{B_R(y)} G^y_{i\sqrt{\lambda},R} + 2\,\mathrm{Re}\left[\left(c(u) - c_R(u_R)\right)\int_{B_R(y)} \bar{w}\right] +$$

$$+ A_R(\lambda,\alpha)\left|c_R(u_R)\right|^2 - A(\lambda,\alpha)\left|c(u)\right|^2 \qquad (2.3.20)$$

Moreover it is easy to verify that

$$\lim_{R\to 0} \frac{|c_R(u_R) - c(u)|}{R^\varepsilon} = 0 \qquad \forall\, \varepsilon < 1 \tag{2-3-21-}$$

Now (2-3-21-),(2-3-13-) ,(2-3-17-) and the continuity of $w \in H^2(\mathbb{R}^3)$ imply i).
To prove ii) consider $u_R \xrightarrow[R\to 0]{} u$ in $L^2(\mathbb{R}^3)$. If $F_R^{\alpha,\gamma,\lambda}(u_R) \xrightarrow[R\to 0]{} +\infty$ then ii) is trivially true, otherwise there exists a subsequence (u'_R) such that

$$\| u'_R - c_R(u'_R)\, G^{\gamma}_{i\sqrt{\lambda},R} \|_{H^1(\mathbb{R}^3)} < \text{const.} \tag{2-3-22-}$$

$$| c_R(u'_R) | < \text{const.} \tag{2-3-23-}$$

Then there exist $w \in H^1(\mathbb{R}^3)$ and $c \in \mathbb{C}$ such that

$$u'_R - c_R(u'_R)\, G^{\gamma}_{i\sqrt{\lambda},R} \xrightarrow[R\to 0]{} w \quad \text{in } H^1(\mathbb{R}^3) \quad , \quad c_R(u'_R) \xrightarrow[R\to 0]{} c \tag{2-3-24-}$$

By our hypotheses and by uniqueness of the weak limit we have

$$u = w + c\, G^{\gamma}_{i\sqrt{\lambda}} \tag{2-3-25-}$$

which implies

$$u \in D(F^{\alpha,\gamma}) \quad , \quad c = c(u) \tag{2-3-26-}$$

Now we easily obtain statement ii)

$$F^{\alpha,\gamma,\lambda}(u) \le \liminf_{R\to 0} \Big[\int_{\mathbb{R}^3} \big| \nabla\big(u'_R - c_R(u'_R)\, G^{\gamma}_{i\sqrt{\lambda},R}\big)\big|^2 + \lambda \int_{\mathbb{R}^3} \big| u'_R - c_R(u'_R)\, G^{\gamma}_{i\sqrt{\lambda},R}\big|^2 +$$

$$+ A_R(\lambda,\alpha)\, |c_R(u'_R)|^2 \Big] = \liminf_{R\to 0} F_R^{\alpha,\gamma,\lambda}(u_R) \tag{2-3-27-}$$

where we used lower semicontinuity of the norm with respect to the weak topology and the convergence $A_R(\lambda,\alpha) \xrightarrow[R\to 0]{} A(\lambda,\alpha)$. ∎

REMARK 2.3.2. Proposition 2.3.1. states that $F^{\alpha,\gamma}$ is the (unique) Γ-limit of $F_R^{\alpha,\gamma}$ which is independent of λ and so $F^{\alpha,\gamma}$ itself is independent of λ .

REMARK 2.3.3. The above result implies also the strong resolvent convergence to $-\Delta_{\alpha,\gamma}$ of the opeartor associated to $F_R^{\alpha,\gamma}$ (see proposition 2.1.12).

2.4 SOME EXTENSIONS

We start with the construction of $-\Delta + \mu\,\delta_y$ in dimension two as quadratic form, following the same steps of section 2.2.. It's well known that the Green function for $-\Delta + \lambda$ in dimension two is

$$G_{i\sqrt{\lambda}}(x,x') = \frac{1}{2\pi}\, K_0\left(\sqrt{\lambda}\,|x-x'|\right) \qquad (2.4.1.)$$

where $K_0(\cdot)$ is the Mc Donald function of zero order ([19]). The first orders of the asymptotic expansion near $|x-x'| = 0$ are

$$G_{i\sqrt{\lambda}}(x,x') \simeq \frac{1}{2\pi}\frac{1}{|x-x'|} - \frac{1}{2\pi}\left(\gamma + \log\frac{\sqrt{\lambda}}{2}\right) + \cdots \qquad (2.4.2.)$$

where γ is the Euler constant, and the behavior for $|x-x'| \to +\infty$ is exponentially decreasing. Again we have $G^y_{i\sqrt{\lambda}} \in L^2(\mathbb{R}^2)$ but $G^y_{i\sqrt{\lambda}} \notin H^1(\mathbb{R}^2)$ therefore as in dimension three we expect that the form-domain of $-\Delta_{\alpha,y}$ consists of $H^1(\mathbb{R}^2) \oplus \left[G^y_{i\sqrt{\lambda}}\right]$. Then analogously to $(2.2.9.)$ we define

$$c(u) = \beta \lim_{R\to 0} |\log R| \oint_{B_R(y)} u \qquad \forall u \text{ s.t. the limit exists,} \qquad (2.4.3.)$$

where β is such that $c\left(G^y_{i\sqrt{\lambda}}\right) = 1$.

Now, for $\alpha \in \mathbb{R}$ and $y \in \mathbb{R}^2$, our quadratic form can be defined by

$$D(F^{\alpha,y}) = \left\{u \in L^2(\mathbb{R}^2) \,\middle|\, c(u) \text{ exists finite}, \; u - c(u)G^y_{i\sqrt{\lambda}} \in H^1(\mathbb{R}^2)\right\} \qquad (2.4.4.)$$

$$F^{\alpha,y}(u) = \begin{cases} \displaystyle\int_{\mathbb{R}^2} \left|\nabla\left(u - c(u)G^y_{i\sqrt{\lambda}}\right)\right|^2 + \lambda\int_{\mathbb{R}^2}\left|u - c(u)G^y_{i\sqrt{\lambda}}\right|^2 + A(\lambda,\alpha)\,|c(u)|^2 - \lambda\int_{\mathbb{R}^2}|u|^2 & \text{if } u \in D(F^{\alpha,y}) \\ +\infty & \text{if } u \notin D(F^{\alpha,y}) \end{cases} \qquad (2.4.5.)$$

where ,

$$A(\alpha,\lambda) = \alpha + \frac{1}{2\pi}\left(\gamma + \log\frac{\sqrt{\lambda}}{2}\right) \quad , \quad \lambda > 0 \qquad (2.4.6.)$$

Following the same line of the proof of proposition 2.2.1. one can show that is a l.s.c. quadratic form, bounded from below by $-\left(2\, e^{-2\pi\alpha-\gamma}\right)^2, \; \forall \alpha \in \mathbb{R}$.

Then the associated operator $T_{\alpha,y}$ coincides with $-\Delta_{\alpha,y}$ and in particular one can explicitely construct the resolvent (the proof is similar to that of proposition 2.2.2.).

Furthermore, as in section 2.3. , we can obtain $F^{\alpha,y}$ as Γ–limit of approximating forms. In this case we can choose

$$F_R^{\alpha,y}(u) = \begin{cases} \int_{\mathbb{R}^2} |\nabla u|^2 - a_R \left| \fint_{B_R(y)} u \right|^2 & \text{if } u \in H^1(\mathbb{R}^2) \\ +\infty & \text{if } u \notin H^1(\mathbb{R}^2) \end{cases} \qquad (2\text{-}4\text{-}7\text{-})$$

where now one has to fix

$$a_R = \left(\frac{1}{2\pi} \log \frac{1}{R}\right)^{-1} \left[1 - \alpha \left(\frac{1}{2\pi} \log \frac{1}{R}\right)^{-1}\right] \qquad (2\text{-}4\text{-}8\text{-})$$

Again we introduce $G^y_{i\sqrt{\lambda},R}$, the approximate Green function, defined as in $(2\text{-}3\text{-}10\text{-})$ which is easily seen to converge for $R \to 0$ to $G^y_{i\sqrt{\lambda}}$ in $L^2(\mathbb{R}^2)$. In terms of $G^y_{i\sqrt{\lambda},R}$ $(2\text{-}4\text{-}7\text{-})$ becomes

$$F_R^{\alpha,y}(u) = \begin{cases} \int_{\mathbb{R}^2} \left|\nabla\left(u - e_R(u)\, G^y_{i\sqrt{\lambda},R}\right)\right|^2 + \lambda \int_{\mathbb{R}^2} \left|u - e_R(u) G^y_{i\sqrt{\lambda},R}\right|^2 + A_R(\lambda,\alpha) \left|e_R(u)\right|^2 - \lambda \int_{\mathbb{R}^2} |u|^2 & \text{if } u \in H^1(\mathbb{R}^2) \\ +\infty & \text{if } u \notin H^1(\mathbb{R}^2) \end{cases} \qquad (2\text{-}4\text{-}9\text{-})$$

where

$$e_R(u) = a_R \fint_{B_R(y)} u \qquad (2\text{-}4\text{-}10\text{-})$$

$$A_R(\alpha,\lambda) = \frac{1}{a_R} - \fint_{B_R(y)} G^y_{i\sqrt{\lambda},R} \xrightarrow[R \to 0]{} A(\lambda,\alpha) \qquad (2\text{-}4\text{-}11\text{-})$$

Now we have all the ingredients to obtain Γ-convergence of $F_R^{\alpha,y}$ to $F^{\alpha,y}$ as in dimension three and the proof can be carried out along the line of proposition 2.3.1..

Finally we briefly consider the many centers case. If we define

$$c_i(u) = \lim_{R \to 0} \beta R \fint_{B_R(y_i)} u \ , \quad i = 1,\dots,N \ , \quad \forall u \in L^2(\mathbb{R}^3) \text{ s.t. the limits exist} \qquad (2\text{-}4\text{-}12\text{-})$$

and introduce the vector notation

$$\underline{c}(u) = (c_1(u), \ldots, c_N(u)) \qquad (2.4.13.)$$

$$\underline{G}_{i\sqrt{\lambda},x} = \left(G^{y_1}_{i\sqrt{\lambda}}(x), \ldots, G^{y_N}_{i\sqrt{\lambda}}(x)\right) \qquad (2.4.14.)$$

the the most natural extension of $(2.2.11.)$, $(2.2.12.)$ to the case of N point interactions is

$$D(F^{\alpha^{(N)},Y^{(N)}}) = \left\{ u \in L^2(\mathbb{R}^3) \,\middle|\, c_i(u)\ ,\ i=1,\ldots,N\ ,\ \text{exist finite}\ ,\ u - \langle \underline{c}(u), \underline{G}_{i\sqrt{\lambda},\cdot} \rangle \in H^1(\mathbb{R}^3) \right\} \qquad (2.2.15.)$$

$$F^{\alpha^{(N)},Y^{(N)}}(u) = \begin{cases} \displaystyle\int_{\mathbb{R}^3} \left|\nabla(u - \langle \underline{c}(u), \underline{G}_{i\sqrt{\lambda},\cdot}\rangle)\right|^2 + \lambda \int_{\mathbb{R}^3} \left|u - \langle \underline{c}(u), \underline{G}_{i\sqrt{\lambda},\cdot}\rangle\right|^2 + \langle \underline{c}(u), \underline{\underline{\Gamma}}_{\alpha^{(N)},Y^{(N)}}(i\sqrt{\lambda})\, \underline{c}(u) \rangle - \\ \displaystyle\qquad - \lambda \int_{\mathbb{R}^3} |u|^2 & \text{if } u \in D(F^{\alpha^{(N)},Y^{(N)}}) \\ +\infty & \text{if } u \notin D(F^{\alpha^{(N)},Y^{(N)}}) \end{cases} \qquad (2.2.16.)$$

where $\langle \cdot , \cdot \rangle$ indicates the scalar product in $\mathbb{C}$.

We observe that $\underline{\underline{\Gamma}}_{\alpha^{(N)},Y^{(N)}}(i\sqrt{\lambda})$ is a symmetric matrix whose eigenvalues $\gamma_1(\lambda), \ldots, \gamma_N(\lambda)$ are all strictly increasing in λ ([6]) . Therefore there are at most N values of λ such that $\underline{\underline{\Gamma}}_{\alpha^{(N)},Y^{(N)}}(i\sqrt{\lambda})$ has eigenvalue zero. If $\det \underline{\underline{\Gamma}}_{\alpha^{(N)},Y^{(N)}}(i\sqrt{\lambda}) \neq 0 \quad \forall \lambda > 0$ then all the eigenvalues are strictly positive and $\underline{\underline{\Gamma}}_{\alpha^{(N)},Y^{(N)}}(i\sqrt{\lambda})$ is $\forall \lambda > 0$ a positive, non degenerate, symmetric matrix defining a norm in $\mathbb{C}^N$. If λ_0 is the maximum value of λ such that $\det \underline{\underline{\Gamma}}_{\alpha^{(N)},Y^{(N)}}(i\sqrt{\lambda}) = 0$ then $\forall \lambda > \lambda_0$ $\underline{\underline{\Gamma}}_{\alpha^{(N)},Y^{(N)}}(i\sqrt{\lambda})$ is again a norm in $\mathbb{C}^N$.

The above discussion shows that $F^{\alpha^{(N)},Y^{(N)}}$ is a positive quadratic form if $\det \underline{\underline{\Gamma}}_{\alpha^{(N)},Y^{(N)}}(i\sqrt{\lambda}) \neq 0$ $\forall \lambda > 0$ while $F^{\alpha^{(N)},Y^{(N)}}$ is bounded from below by $-\lambda_0$ if λ_0 is the maximum value of λ such that $\det \underline{\underline{\Gamma}}_{\alpha^{(N)},Y^{(N)}}(i\sqrt{\lambda}) = 0$.

Using the same methods of the proofs of propositions 2.2.1. , 2.2.2. one can prove that $F^{\alpha^{(N)},Y^{(N)}}$ is l.s.c. and the associated selfadjoint operator is $-\Delta_{\alpha^{(N)},Y^{(N)}}$ as defined in section 1.2..

The extension to the many centers case of $F^{\alpha,Y}$ in dimension two can be carried out by following the same line of reasoning and therefore we don't go into details.

CHAPTER 3.
POINT INTERACTIONS AND BOUNDARY VALUE PROBLEMS

The aim of this chapter is to show how the Schrödinger operator with point interactions is related to a particular boundary value problem or, more precisely, to the laplacian with mixed boundary conditions on the surface of a sphere shrinking to a point. This connection is mentioned for the N centers case in ([8]). Here we provide a rigorous proof only for the simpler case of one center point interaction (an analogous case is treated in ([20])).

3.1. A MIXED BOUNDARY VALUE PROBLEM

We start defining the laplace operator in an exterior domain with mixed boundary conditions. Let B a bounded region of $\mathbb{R}^3$ with C^1 boundary ∂B and σ a continuous, bounded and real valued function defined on ∂B. Let us consider the quadratic form defined in $L^2(\mathbb{R}^3 \setminus B)$

$$F_\sigma(u) = \begin{cases} \int_{\mathbb{R}^3\setminus B} |\nabla u|^2 + \int_{\partial B} \sigma |u|^2 \, dS & \text{if } u \in H^1(\mathbb{R}^3\setminus B) \\ +\infty & \text{if } u \notin H^1(\mathbb{R}^3\setminus B) \end{cases} \qquad (3.1.1)$$

Using a standard technique based on trace theorems and Sobolev embeddings ([22]) it is not difficult to prove

PROPOSITION 3.1.1. F_σ is a l.s.c. and bounded from below.

Now the unique selfadjoint operator $-\Delta_\sigma$ associated to F_σ is by definition the laplacian in the exterior domain $\mathbb{R}^3 \setminus B$ with mixed boundary condition on ∂B. Proposition 3.1.1. is rather general but does not give any information on the lower bound of F_σ or $-\Delta_\sigma$ as a function of ∂B. To get such informations we restrict ourself to the simpler case in which B is the sphere $B_R(0)$ of radius R centered in the origin and σ is a smooth function of R. The first estimate concerns the norm of the trace on the surface of the sphere.

PROPOSITION 3.1.2.

$$\int_{\partial B_R(0)} |f|^2 \, dS \leq R \int_{\mathbb{R}^3 \setminus B_R(0)} |\nabla f|^2 \qquad \forall f \in H^1\left(\mathbb{R}^3 \setminus B_R(0)\right) \qquad (3.1.2.)$$

PROOF. Let us fix $f \in C_0^1\left(\mathbb{R}^3 \setminus B_R(0)\right)$. Introducing spherical coordinates we have

$$f(R,\vartheta,\varphi) = -\int_R^{+\infty} \frac{\partial f}{\partial \xi}(\xi,\vartheta,\varphi)\, d\xi \qquad (3.1.2.)$$

Using Schwartz inequality we easily get

$$\left|f(R,\vartheta,\varphi)\right|^2 \leq \left(\int_R^{+\infty} \frac{1}{\xi} \cdot \left|\frac{\partial f}{\partial \xi}(\xi,\vartheta,\varphi)\right| \xi \, d\xi\right)^2 \leq \frac{1}{R} \int_R^{+\infty} \left|\frac{\partial f}{\partial \xi}(\xi,\vartheta,\varphi)\right|^2 \xi^2 \, d\xi \qquad (3.1.3.)$$

Then integrating over the surface $\partial B_R(0)$ and making use of a density argument ([17] pag. ~~7~~ 141) we obtain (3.1.2.) . ∎

Inequality (3.1.2.) can be written as

$$\int_{\mathbb{R}^3 \setminus B_R(0)} |\nabla f|^2 - \frac{1}{R} \int_{\partial B_R(0)} |f|^2 \, dS \geq 0 \qquad \forall f \in H^1\left(\mathbb{R}^3 \setminus B_R(0)\right) \qquad (3.1.4.)$$

and (3.1.4.) suggests to fix

$$\sigma(R) = -\frac{1}{R} + 4\pi\alpha \quad , \quad \alpha \in \mathbb{R} \qquad (3.1.5.)$$

It will be useful in the sequel to define $\forall \lambda \in \mathbb{R}$

$$F^{\lambda}_{\sigma(R)}(u) = F_{\sigma(R)}(u) + \lambda \int_{\mathbb{R}^3 \setminus B_R(0)} |u|^2 \qquad (3.1.6.)$$

then, depending on the sign of α , we have the following estimates

PROPOSITION 3.1.3. Let $\alpha > 0$, $\lambda > 0$, $u \in H^1(\mathbb{R}^3 \setminus B_R(0))$, then

(a) $\lambda \| u \|^2_{L^2(\mathbb{R}^3\setminus B_R(0))} \leq F^{\lambda}_{\sigma(R)}(u) \leq \max\{\lambda, 1+4\pi\alpha R\} \| u \|^2_{H^1(\mathbb{R}^3\setminus B_R(0))}$ (3.1.7.)

(b) $\displaystyle\int_{\partial B_R(0)} |u|^2 dS \leq \frac{1}{4\pi\alpha} F^{\lambda}_{\sigma(R)}(u)$ (3.1.8.)

moreover let $\alpha \leq 0$, $\lambda > (4\pi\alpha)^2$, $u \in H^1(\mathbb{R}^3 \setminus B_R(0))$, then

(c) $\left(\lambda - (4\pi\alpha)^2\right) \| u \|^2_{L^2(\mathbb{R}^3\setminus B_R(0))} \leq F^{\lambda}_{\sigma(R)}(u) \leq \max\left\{2, \lambda + 3(4\pi\alpha)^2 + \frac{4}{R^2}\right\} \| u \|^2_{H^1(\mathbb{R}^3\setminus B_R(0))}$ (3.1.9.)

(d) $\displaystyle\int_{\partial B_R(0)} |u|^2 dS \leq \frac{1}{\sqrt{\lambda} + 4\pi\alpha} F^{\lambda}_{\sigma(R)}(u)$ (3.1.10.)

PROOF. Taking (3.1.4.) into account the proof of (a) and (b) is trivial. To prove the first inequality in (c) we define

$$\phi(x) = \frac{e^{4\pi\alpha|x|}}{|x|} \qquad (3.1.11.)$$

and observe that

$$\left(-\frac{1}{R} + 4\pi\alpha\right) \int_{\partial B_R(0)} u^2 dS = -\int_{\partial B_R(0)} u^2 \frac{\partial}{\partial n} \log\phi \, dS =$$

$$= -\int_{\mathbb{R}^3\setminus B_R(0)} \nabla u^2 \cdot \nabla \log\phi - \int_{\mathbb{R}^3\setminus B_R(0)} u^2 \Delta \log\phi \qquad (3.1.12.)$$

where we used the Green identity to transform the surface integral. Using (3.1.12.) our quadratic form can be written as

$$F^{\lambda}_{\sigma(R)}(u) = \int_{\mathbb{R}^3\setminus B_R(0)} |\nabla u - u \nabla \log\phi|^2 + \lambda \int_{\mathbb{R}^3\setminus B_R(0)} u^2 - (4\pi\alpha)^2 \int_{\mathbb{R}^3\setminus B_R(0)} u^2 \qquad (3.1.13.)$$

which trivially implies the first part of (c). An analogous integration by parts allows us to prove also inequality (d). It remains to verify the second inequality in (c). Starting from (3.1.13.) we easily get

$$F^{\lambda}_{\sigma(R)}(u) \leq 2\int_{\mathbb{R}^3\setminus B_R(0)} |\nabla u|^2 + \int_{\mathbb{R}^3\setminus B_R(0)} \left(\lambda - (4\pi\alpha)^2 + 2|\nabla \log \phi|^2\right) u^2 \leq$$

$$\leq 2\int_{\mathbb{R}^3\setminus B_R(0)} |\nabla u|^2 + \left(\lambda + 3(4\pi\alpha)^2 + \frac{4}{R^2}\right)\int_{\mathbb{R}^3\setminus B_R(0)} u^2 \qquad (3.1.14.)$$

and this completes the proof. ∎

REMARK 3.1.4. As a consequence of ~~the~~ proposition 3.1.1. $F^{\lambda}_{\sigma(R)}(\cdot)$, with $\lambda > 0$ if $\alpha > 0$ and $\lambda > (4\pi\alpha)^2$ if $\alpha \leq 0$,is a norm in $H^1(\mathbb{R}^3\setminus B_R(0))$ and the associated bilinear form $B^{\lambda}_{\sigma(R)}(\cdot,\cdot)$ is the corresponding scalar product. The closure of $H^1(\mathbb{R}^3\setminus B_R(0))$ with respect to the norm $F^{\lambda}_{\sigma(R)}(\cdot)$ is denoted by ${}^*H^1(\mathbb{R}^3\setminus B_R(0))$ and it will be useful in the next section.

Now we turn to selfajoint operator $-\Delta_{\sigma(R)}$ associated with $F_{\sigma(R)}$. Clearly the domain is

$$D(-\Delta_{\sigma(R)}) = \left\{ u \in H^2(\mathbb{R}^3) \;\middle|\; \frac{\partial u}{\partial n} + \sigma(R)\, u = 0 \quad \text{on } \partial B_R(0) \right\} \qquad (3.1.15.)$$

Because of the radial symmetry we can decompose the Hilbert space with respect to angular momenta and obtain

$$-\Delta_{\sigma(R)} = \bigoplus_{l=0}^{+\infty} U^{-1} h_{l,\sigma(R)} U \otimes \mathbb{1} \qquad (3.1.16.)$$

where U is defined in $(1.1.4.)$ and

$$h_{l,\sigma(R)} = -\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} \quad , \quad D(h_{l,\sigma(R)}) = \Big\{ u \in L^2([R,+\infty)) \;\Big|\; u, u' \in AC_{loc}([R,+\infty)),$$

$$-u'' + \frac{l(l+1)}{r^2} u \in L^2([R,+\infty)), \; -u'(R) + \left(\sigma(R) + \frac{1}{R}\right) u(R) = 0 \Big\} \qquad (3.1.17.)$$

It is not difficult to compute the resolvent of $h_{l,\sigma(R)}$.

$$\left(h_{l,\sigma(R)} - k^2\right)^{-1}(r,r') = g_{k,l}(r,r') - \frac{\pi i}{2} B_l(k,\sigma(R),R)\, r'^{1/2} H^{(1)}_{l+1/2}(kr')\, r^{1/2} H^{(1)}_{l+1/2}(kr) \;,$$

$$k^2 \in \rho\left(h_{\ell,\sigma(R)}\right)\ ,\ \operatorname{Im} k > 0 \qquad (3.1.18.)$$

where

$$g_{k,\ell}(r,r') = \begin{cases} i\frac{\pi}{2}\, r^{1/2}\, H^{(1)}_{\ell+1/2}(kr)\ r'^{1/2}\, J_{\ell+1/2}(kr') & \text{if } r' \leq r \\ i\frac{\pi}{2}\, r'^{1/2}\, H^{(1)}_{\ell+1/2}(kr')\ r^{1/2}\, J_{\ell+1/2}(kr) & \text{if } r' > r \end{cases} \qquad (3.1.19.)$$

is the integral kernel of the free resolvent and

$$B_\ell(k,\sigma(R),R) = \frac{(\ell-\sigma(R)R)\, J_{\ell+1/2}(kR) - kR\, J_{\ell+3/2}(kR)}{(\ell-\sigma(R)R)\, H^{(1)}_{\ell+1/2}(kR) - kR\, H^{(1)}_{\ell+3/2}(kR)} \qquad (3.1.20.)$$

($H^{(\cdot)}_{\cdot}(\cdot)$, $J_{\cdot}(\cdot)$ denote Bessel functions ([19])).
From (3.1.18.) we can reconstruct the resolvent of $-\Delta_{\sigma(R)}$

$$\left(-\Delta_{\sigma(R)} - k^2\right)^{-1}(r,\theta,\varphi;r',\theta',\varphi') = G_k(r,\theta,\varphi;r',\theta',\varphi') +$$

$$+ \bigoplus_{\ell=0}^{+\infty} \bigoplus_{m=-\ell}^{\ell} B_\ell(k,\sigma(R),R)\ \frac{H^{(1)}_{\ell+1/2}(kr)}{r^{1/2}}\, Y_{\ell m}(\theta,\varphi)\ \frac{H^{(1)}_{\ell+1/2}(kr')}{r'^{1/2}}\, Y_{\ell m}(\theta',\varphi')\ ,\ k^2 \in \rho(-\Delta_{\sigma(R)}),\ \operatorname{Im} k > 0. \qquad (3.1.21.)$$

The knowledge of the explicit expression of the resolvent allows us to evaluate all the physical relevant quantities related to $-\Delta_{\sigma(R)}$, i.e. eigenvalues, bound-states, resonances, scattering data. We shall limit ourself to calculate some scattering quantities which will be useful in the next section. The scattering wave functions $\phi_{\ell,\sigma(R)}(k,r)$, $k \geq 0$, associated with $h_{\ell,\sigma(R)}$ must satisfy

$$\phi_{\ell,\sigma(R)}(k,r) = \lim_{\varepsilon \to 0} \lim_{r' \to \infty} e^{-i(k+i\varepsilon)r'} \left[h_{\ell,\sigma(R)} - (k+i\varepsilon)^2\right]^{-1}(r,r') \qquad (3.1.22.)$$

$$-\phi''_{\ell,\sigma(R)}(k,r) + \frac{\ell(\ell+1)}{r^2}\,\phi_{\ell,\sigma(R)}(k,r) = k^2\,\phi_{\ell,\sigma(R)}(k,r) \qquad (3.1.23.)$$

$$-\phi'_{\ell,\sigma(R)}(k,R) + \left(\sigma(R) + \frac{1}{R}\right)\phi_{\ell,\sigma(R)}(k,R) = 0 \qquad (3.1.24.)$$

We easily find

$$\phi_{\ell,\sigma(R)}(k,r) = e^{-i\frac{\pi}{2}\ell}\sqrt{\frac{\pi}{2k}}\ r^{1/2}\left[J_{\ell+1/2}(kr) - B_\ell(k,\sigma(R),R)\, H^{(1)}_{\ell+1/2}(kr)\right] \qquad (3.1.25.)$$

The asymptotic behaviour of the scattering wave functions defines the phase shifts $\delta_{\ell,\sigma(R)}(k)$

$$\Phi_{\ell,\sigma(R)}(k,r) \underset{r\to\infty}{\simeq} \frac{e^{-i\frac{\pi}{2}\ell}}{k}\left[\sin\left(kr-\frac{\pi}{2}\ell\right) + i\, B_\ell(k,\sigma(R),R)\, e^{-i\left(kr-\frac{\pi}{2}\ell\right)}\right] =$$

$$= \frac{1}{k}\, e^{i\left(-\frac{\pi}{2}\ell + \delta_{\ell,\sigma(R)}(k)\right)} \sin\left(kr - \frac{\pi}{2}\ell + \delta_{\ell,\sigma(R)}(k)\right) \qquad (3.1.26.)$$

where

$$\delta_{\ell,\sigma(R)}(k) = \mathrm{tg}^{-1}\left\{2i\frac{(\ell-\sigma(R)R)\, J_{\ell+1/2}(kR) - kR\, J_{\ell+3/2}(kR)}{(\ell-\sigma(R)R)\left[H^{(1)}_{\ell+1/2}(kR)-H^{(2)}_{\ell+1/2}(kR)\right]+kR\left[H^{(1)}_{\ell+3/2}(kR)-H^{(2)}_{\ell+3/2}(kR)\right]}\right\} \qquad (3.1.27.)$$

Now the scattering matrix can be expressed in term of the phase shifts

$$S_{\ell,\sigma(R)}(k) = e^{2i\,\delta_{\ell,\sigma(R)}(k)} =$$

$$= -\frac{(\ell-\sigma(R)R)\, H^{(2)}_{\ell+1/2}(kR) - kR\, H^{(2)}_{\ell+1/2}(kR)}{(\ell-\sigma(R)R)\, H^{(1)}_{\ell+1/2}(kR) - kR\, H^{(1)}_{\ell+1/2}(kR)} \qquad (3.1.28.)$$

REMARK 3.1.5. We observe that all the statements about $-\Delta_{\sigma(R)}$, i.e. (3.1.18.), (3.1.21.), (3.1.25.), (3.1.27.), (3.1.28.), are independent of the function $\sigma = \sigma(R)$.

3.2 CONVERGENCE TO POINT INTERACTION

Our aim is now to prove the convergence of $-\Delta_{\sigma(R)}$, with a suitable choice of $\sigma(R)$ to the Schrödinger operator with point interactions in the origin for $R \to 0$. From chapter 1 we know that $-\Delta_{\alpha,Y}$ defines an s-wave interaction and the parameter α is related to the scattering lenght a_α associated to $-\Delta_{\alpha,Y}$ via the relation

$$a_\alpha = -\frac{1}{4\pi\alpha} \tag{3.2.1.}$$

This implies that, in the limit $R \to 0$, only the phase shift of $h_{\ell,\sigma(R)}$ with $\ell = 0$ will be different from zero. Let us consider the effective range expansion ([21] pag. 309) for $h_{0,\sigma(R)}$

$$k \operatorname{ctg} \delta_{0,\sigma(R)} = -\frac{1}{R} \frac{(\sigma(R)R+1)kR\cos kR + k^2R^2 \sin kR}{(\sigma(R)R+1)\sin kR - kR\cos kR} =$$

$$= -\frac{\sigma(R)R+1}{\sigma(R)R^2} + \frac{2}{3}\frac{1-\sigma(R)R+\sigma^2(R)R^2}{\sigma^2(R)R^3}\cdot\frac{k^2}{2} + O(k^4) \tag{3.2.2.}$$

This expansion defines the scattering lenght $a_{0,\sigma(R)}$ and the effective range parameter $r_{0,\sigma(R)}$ associated with $h_{0,\sigma(R)}$

$$a_{0,\sigma(R)} = \frac{\sigma(R)R^2}{\sigma(R)R+1} \tag{3.2.3.}$$

$$r_{0,\sigma(R)} = \frac{2}{3}\frac{1-\sigma(R)R+\sigma^2(R)R^2}{\sigma^2(R)R^3} . \tag{3.2.4.}$$

It is clear from (3.2.3.) that the only way to obtain the limiting scattering length (3.2.1.) from $a_{0,\sigma(R)}$ when $R \to 0$ is to fix

$$\sigma(R) = -\frac{1}{R} + 4\pi\alpha \quad , \quad \alpha \in \mathbb{R} \tag{3.2.5.}$$

This chioce is in agreement with position (3.1.5.) of the preceding section. Now our claim is that, if $\sigma(R)$ is given by (3.2.5.) , the quadratic form $F_{\sigma(R)}$ and the associated operator $-\Delta_{\sigma(R)}$ converge respectively to $F^{\alpha,0}$ and $-\Delta_{\alpha,0}$ when

$R \to 0$. More precisely we have the following results

PROPOSITION 3.2.1.

$\forall f \in L^2(\mathbb{R}^3 \setminus B_R(0))$

$$\lim_{R\to 0} \left\| \left(-\Delta_{\sigma(R)}+\lambda\right)^{-1} f - \left(-\Delta_{\alpha,0}+\lambda\right)^{-1} f \right\|_{L^2(\mathbb{R}^3 \setminus B_R(0))} = 0 \qquad (3\text{-}2\text{-}6)$$

for $\lambda > 0$ if $\alpha \geq 0$ and $\lambda > (4\pi\alpha)^2$ if $\alpha < 0$

This result can be easily extended to $L^2(\mathbb{R}^3)$

PROPOØSITION 3.2.2.

$\forall f \in L^2(\mathbb{R}^3)$

$$\lim_{R\to 0} \left\| 1_{\mathbb{R}^3\setminus B_R(0)} \left(-\Delta_{\sigma(R)}+\lambda\right)^{-1} 1_{\mathbb{R}^3\setminus B_R(0)} f - \left(-\Delta_{\alpha,0}+\lambda\right)^{-1} f \right\|_{L^2(\mathbb{R}^3)} = 0 \qquad (3\text{-}2\text{-}7)$$

for $\lambda > 0$ if $\alpha \geq 0$ and $\lambda > (4\pi\alpha)^2$ if $\alpha < 0$.

If we define the obvious extension $\overline{F}_{\sigma(R)}$ of $F_{\sigma(R)}$ to $L^2(\mathbb{R}^3)$

$$\overline{F}_{\sigma(R)}(u) = \begin{cases} \displaystyle\int_{\mathbb{R}^3\setminus B_R(0)} |\nabla u|^2 + \sigma(R) \int_{\partial B_R(0)} |u|^2 dS & \text{if } u \in H^1(\mathbb{R}^3) \\ +\infty & \text{if } u \notin H^1(\mathbb{R}^3) \end{cases} \qquad (3\text{-}2\text{-}8)$$

then propositions 2.1.12. and 3.2.2. trivially imply

COROLLARY 3.2.3.

$$\Gamma\text{-}\lim_{R\to 0} \overline{F}_{\sigma(R)} = F^{\alpha,0} \qquad (3\text{-}2\text{-}9)$$

PROOF OF PROPOSITION 3.2.1. $\forall f \in L^2(\mathbb{R}^3 \setminus B_R(0))$ the two functions

$$u_{\sigma(R)} = \left(-\Delta_{\sigma(R)} + \lambda\right)^{-1} f \tag{3.2.10.}$$

and

$$v_\alpha = \left(-\Delta_{\alpha,0} + \lambda\right)^{-1} f \tag{3.2.11.}$$

are obviously weak solutions of the following mixed boundary value problems

$$\begin{cases} (-\Delta + \lambda)\, u_{\sigma(R)} = f & \text{in } \mathbb{R}^3 \setminus B_R(0) \\ \dfrac{\partial u_{\sigma(R)}}{\partial n} + \sigma(R)\, u_{\sigma(R)} = 0 & \text{on } \partial B_R(0) \end{cases} \tag{3.2.12.}$$

$$\begin{cases} (-\Delta + \lambda)\, v_\alpha = f & \text{in } \mathbb{R}^3 \setminus B_R(0) \\ \dfrac{\partial v_\alpha}{\partial n} + \sigma(R)\, v_\alpha = \varphi^f_{\sigma(R),\alpha} & \text{on } \partial B_R(0) \end{cases} \tag{3.2.13.}$$

where

$$\varphi^f_{\sigma(R),\alpha} = \left(\frac{\partial}{\partial n} + \sigma(R)\right)\left(-\Delta_{\alpha,0} + \lambda\right)^{-1} f \Big|_{\partial B_R(0)} \tag{3.2.14.}$$

then the difference

$$W_{\sigma(R),\alpha} = v_\alpha - u_{\sigma(R)} \tag{3.2.15.}$$

is the weak solution of

$$\begin{cases} (-\Delta + \lambda)\, W_{\sigma(R),\alpha} = 0 & \text{in } \mathbb{R}^3 \setminus B_R(0) \\ \dfrac{\partial W_{\sigma(R),\alpha}}{\partial n} + \sigma(R)\, W_{\sigma(R),\alpha} = \varphi^f_{\sigma(R),\alpha} & \text{on } \partial B_R(0) \end{cases} \tag{3.2.16.}$$

This means that $W_{\sigma(R),\alpha}$ is the unique function belonging to $H^1(\mathbb{R}^3 \setminus B_R(0))$ such that

$$\left(z, W_{\sigma(R),\alpha}\right)_{*H^1(\mathbb{R}^3\setminus B_R(0))} = \int_{\partial B_R(0)} \bar{z}\, \varphi^f_{\sigma(R),\alpha}\, dS \tag{3.2.17.}$$

where we used the notation introduced in remark 3.1.4. Taking into account that $H^1(\mathbb{R}^3\setminus B_R(0))$ is dense in $^*H^1(\mathbb{R}^3\setminus B_R(0))$ we can write

$$\| W_{\sigma(R),\alpha}\|_{*H^1(\mathbb{R}^3\setminus B_R(0))} = \sup_{z\in H^1(\mathbb{R}^3\setminus B_R(0))} \frac{\left|(z, W_{\sigma(R),\alpha})_{*H^1(\mathbb{R}^3\setminus B_R(0))}\right|}{\|z\|_{*H^1(\mathbb{R}^3\setminus B_R(0))}} = \sup_{z\in H^1(\mathbb{R}^3\setminus B_R(0))} \frac{\left|\int_{\partial B_R(0)} \bar{z}\,\varphi^f_{\sigma(R),\alpha}\, dS\right|}{\|z\|_{*H^1(\mathbb{R}^3\setminus B_R(0))}} \leq$$

$$\leq \|\varphi^f_{\sigma(R),\alpha}\|_{L^2(\partial B_R(0))} \cdot \sup_{z\in H^1(\mathbb{R}^3\setminus B_R(0))} \frac{\|z\|_{L^2(\mathbb{R}^3\setminus B_R(0))}}{\|z\|_{*H^1(\mathbb{R}^3\setminus B_R(0))}} \tag{3.2.18.}$$

Making use of the estimates (3.1.7.), (3.1.8.), (3.1.9.), (3.1.10.) then (3.2.11.) implies

$$\|W_{\sigma(R),\alpha}\|_{L^2(\mathbb{R}^3\setminus B_R(0))} \leq \begin{cases} \dfrac{1}{\sqrt{4\pi\alpha\,\lambda}}\; \|\varphi^f_{\sigma(R),\alpha}\|_{L^2(\partial B_R(0))} & \text{if } \alpha>0 \\[2ex] \dfrac{1}{\sqrt{[\lambda-(4\pi\alpha)^2][\sqrt{\lambda}+4\pi\alpha]}}\; \|\varphi^f_{\sigma(R),\alpha}\|_{L^2(\partial B_R(0))} & \text{if } \alpha\leq 0 \end{cases} \tag{3.2.19.}$$

It remains to estimate the norm in $L^2(\partial B_R(0))$ of

$$\varphi^f_{\sigma(R),\alpha}(R,\vartheta,\varphi) = \left(\frac{\partial}{\partial n} G_{i\sqrt{\lambda}}f\right)(R,\vartheta,\varphi) + 4\pi\alpha\left(G_{i\sqrt{\lambda}}f\right)(R,\vartheta,\varphi) + \frac{1}{R}\left[\left(G_{i\sqrt{\lambda}}f\right)(0)\, e^{-\sqrt{\lambda}R} - \left(G_{i\sqrt{\lambda}}f\right)(R,\vartheta,\varphi)\right] \tag{3.2.20.}$$

If $f\in L^2(\mathbb{R}^3\setminus B_R(0))$ then $G_{i\sqrt{\lambda}}f\in H^2(\mathbb{R}^3\setminus B_R(0))$ and so we can write

$$\|\varphi^f_{\sigma(R),\alpha}\|_{L^2(\partial B_R(0))} \leq R^{1/2}\|\Delta G_{i\sqrt{\lambda}}f\|_{L^2(\mathbb{R}^3\setminus B_R(0))} + 4\pi|\alpha|R^{1/2}\|\nabla G_{i\sqrt{\lambda}}f\|_{L^2(\mathbb{R}^3\setminus B_R(0))} +$$

$$+\sqrt{4\pi}\,R^{\gamma} + \sqrt{4\pi}\left|\left(G_{i\sqrt{\lambda}}f\right)(0)\right|\left(1-e^{-\sqrt{\lambda}R}\right), \quad \gamma<1/2 \tag{3.2.21.}$$

where we used the estimate (3.1.2.) for $\|\frac{\partial}{\partial n}G_{i\sqrt{\lambda}}f\|_{L^2(\partial B_R(0))}$, $\|G_{i\sqrt{\lambda}}f\|_{L^2(\partial B_R(0))}$ and the Holder continuity of order $\gamma<1/2$ of the functions belonging to $H^2(\mathbb{R}^3\setminus B_R(0))$ ([14]).

Finally from (3.2.19.), (3.2.21.) we get the thesis. ∎

REMARK 3.2.4. Proposition 3.2.1. could be proved in a more direct way by exploiting the limit of the resolvent of $-\Delta_{\sigma(R)}$ which is explicitely given by (3.2.21.). We have preferred another method which, at least in principle, can be generalized to more complicate situations, such as the many centers case, where the resolvent is not explicitly known.

PROOF OF PROPOSITION 3.2.2. We observe that

$$\left\| 1_{\mathbb{R}^3\setminus B_R(0)} \left(-\Delta_{\sigma(R)}+\lambda\right)^{-1} 1_{\mathbb{R}^3\setminus B_R(0)} f - \left(-\Delta_{\alpha,0}+\lambda\right)^{-1} f \right\|_{L^2(\mathbb{R}^3)} \le$$

$$\le \left\| \left(-\Delta_{\sigma(R)}+\lambda\right)^{-1} 1_{\mathbb{R}^3\setminus B_R(0)} f - \left(-\Delta_{\alpha,0}+\lambda\right)^{-1} 1_{\mathbb{R}^3\setminus B_R(0)} f \right\|_{L^2(\mathbb{R}^3\setminus B_R(0))} + 2 \left\| \left(-\Delta_{\alpha,0}+\lambda\right)^{-1} 1_{B_R(0)} f \right\|_{L^2(\mathbb{R}^3)} \qquad (3.2.22.)$$

Because of proposition 3.2.1. we have only to estimate the last norm in (3.2.22.)

$$\left\| \left(-\Delta_{\alpha,0}+\lambda\right)^{-1} 1_{B_R(0)} f \right\|_{L^2(\mathbb{R}^3)} \le \left\| G_{i\sqrt{\lambda}}\left(1_{B_R(0)} f\right) \right\|_{L^2(\mathbb{R}^3)} + \frac{1}{\left|\frac{\sqrt{\lambda}}{4\pi}+\alpha\right|} \left| \left[G_{i\sqrt{\lambda}}\left(1_{B_R(0)} f\right)\right](0) \right| \left\| \mathring{G}_{i\sqrt{\lambda}} \right\|_{L^2(\mathbb{R}^3)} \le$$

$$\le \text{const.} \left\| 1_{B_R(0)} f \right\|_{L^1(\mathbb{R}^3)} + \frac{1}{\left|\frac{\sqrt{\lambda}}{4\pi}+\alpha\right| \sqrt{4\pi\sqrt{\lambda}}} \left\| 1_{B_R(0)} \mathring{G}_{i\sqrt{\lambda}} \right\|_{L^2(\mathbb{R}^3)} \left\| f \right\|_{L^2(\mathbb{R}^3)} \le$$

$$\le \text{const.}\, R^{3/2} \left\| f \right\|_{L^2(\mathbb{R}^3)} + \frac{1}{\left|\frac{\sqrt{\lambda}}{4\pi}+\alpha\right| 4\pi\sqrt{\lambda}} \left(1 - e^{-\sqrt{\lambda} R}\right) \left\| f \right\|_{L^2(\mathbb{R}^3)} \qquad (3.2.23.)$$

where a standard potential estimate ([23]) and a Schwartz inequality have been used. For $R \to 0$ we obtain the desired result from (3.2.23.) . ∎

3.3. POSSIBLE DEVELOPMENTS

We list some possible applications and developments of the material exposed in chapters 2., 3..

- First we observe that the construction of the quadratic form associated to $-\Delta + \mu\,\delta_y$ is essentially based on the singularity of the free Green function $G^{y}_{i\sqrt{\lambda}}$. This means that the same construction should be possible even if $-\Delta$ is replaced by any second order, strictly elliptic, differential operator in divergence form $\sum_{ij} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\,\frac{\partial}{\partial x_j}\right)$ with smooth $a_{ij}(x)$.

- Another possible extension concerns the definition of $-\Delta + \mu\,\delta_y + V$, where V is a potential which can be singular in y . Such operators have been studied in the literature making use of the theory of the extension of symmetric operators (see e.g. [24],[25]). Using the limiting procedure outlined in section 2.3. with a suitable choice of the coupling constant a_R it is possible to compensate the singular character of the potential in y and to obtain in the limit $R \to 0$ the quadratic form associated to $-\Delta + \mu\,\delta_y + V$.

-Any Schrödinger operator with potential in $L^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$ is a limit of a sequence of Schrödinger operators with randomly distributed point interactions. Moreover the fluctuations around the limit operator can be completely characterized (see [8]). This result is obtained using an image-charge technique developed essentially in [26],[27],[28]. It should be possible to translate this problem in the language of quadratic forms and Γ-convergence and to obtain the same result for a more general potential.

- The connection between point interactions and boundary value problems can be further exploited by proving convergence to $-\Delta_{\alpha,0}$ of the laplacian with mixed boundary condition on the surface of a non-spherical obstacle, when the linear size of the obstacle is going to zero, and then by extending the proof to the many centers case.

The crucial point to get such results is to obtain estimates analogous to (3.1.7.) .... (3.1.10.) which are the fundamental ingredients in the proof of the convergence.

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