SISSA - SCUOLA INTERNAZIONALE SUPERIORE DI STUDI AVANZATI
TRIESTE
Strada Costiera 11

ISAS - INTERNATIONAL SCHOOL FOR ADVANCED STUDIES

Some aspects of representation theory of Krichever-Novikov algebras

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Candidate
M.Rinaldi

Supervisor
Prof. Loriano Bonora

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Introduction

Recent years have seen a growth of interest for conformal field theories in two dimensions.

This is due to the fact that in conformal field theories converge many physics and mathematics problems. From the physics point of view we have statistical mechanics models, which in two dimensions show up conformal invariance in the large distances range and also the ground state of string theory.

From the mathematical point of view the study of this subject has enforced the interst for Riemann surfaces, and for the representations of infinite-dimensional Lie algebras.

The connection between Physics and Mathematics becomes in conformal field theory extremely deep.

One of the open questions in two-dimensional field theory concerns the formulation of conformal field models on higher genus Riemann surfaces. In particular one can ask himself whether in higher genus, it is possible to accomodate a class of models larger than the one known in the zero genus case, particularly in regard to unitarity problems.

Moreover in genus 0, the classification of Virasoro's representations constitute a classifying tool for conformal models. The central charge and the highest weight of the Virasoro representations parametrize, so to speak, conformal field models in two dimensions (on the plane or on the sphere).

Recently Krichever and Novikov ([KN1],[KN2]) have introduced a generalization of the Virasoro algebra, which, from our viewpoint can completely replace the Virasoro algebra in its classifying rôle in higher genus.

Moreover the classification one can get with such a generalization is expected to be finer than the one given by the Virasoro algebra, in the sense that is expected to reveal the high degeneracy which arise in the genus 0 case. Our interest in this work will be primarly to construct representations of the higher genus generalization of the Virasoro algebra and try to understand its meaning and relevance in string theory.

The Krichever and Novikov ([KN1], [KN2]) formalism lends itself naturally to the study of conformal field theories on surfaces of non trivial genus. We describe now how they proceed.

Given a Riemann surface Σ, they fix two points P_+ and P_-, endow the Riemann surface with a set of contours which play the role of equitemporal surfaces and expand all relevant tensors over bases of tensorial fields, holomorphic outside $P_\pm$ and possibly with poles in $P_\pm$, of arbitrary weight λ.

The coefficients of such an expansion will become quantum operators, acting on a suitable Fock space. In such a way the theory can be quantized.

It is possible, following this procedure, to obtain algebras which generalize the usual genus 0 algebras to non trivial Riemann surfaces.

In particular we will describe how to generalize the Virasoro, the Heisenberg, the super Virasoro and the loops algebras. Moreover we will have at our disposal an operator Q_{BRST} genus dependent. If we try to impose its closure, when properly quantized via normal ordering, we get the critical dimensions $D = 26$ or $D = 10$ according to whether Q_{BRST} corresponds simply to the generalized Virasoro algebra or to the generalized super Virasoro algebra.

In this work we will devote our attention principally to the representations of such algebras, especially the ones realized on semiinfinite forms. These representations are interesting because of the fact that they are very close to the physical intuition.

In chapter **1** we will introduce definitions and notations accordingly to the work of Krichever and Novikov, in particular we construct bases for Krichever-Novikov tensor fields of any weight, integer or half integer, and possibly with branch cuts.

Attention is devoted also to the introduction of time on a genus g Riemann surface. Moreover we will introduce the hamiltonian vector field, which has extremely rich applications; in particular we will describe how it generates the classical time-evolution.

In chapter **2** we describe the Krichever Novikov algebra and its central extensions, realized on semiinfinite forms. Also some attention is paid to the generalized Heisenberg algebra. We construct representation modules for the Krichever algebra, not centrally extended.

In chapter **3** we introduce semiinfinite forms, according to the work of Feigin and

Fuks ([FF]). On the space of semiinfinite forms we realize the central extensions of the Krichever-Novikov algebra.

We also realize on the same space a representation of the Lie algebra a_∞.

The Krichever algebra will be viewed as a subalgebra of a_∞, and its representations as induced from a_∞. This is remarkable because also the Virasoro representations are induced from the same algebra, via a different inclusion.

Finally we introduce the Clifford representation whose "generators" are very close to the creation and annihilation operators of quantum field theory. We define in this context a mathematical normal ordering which turns out to be strictly related to the physical normal ordering. We end this chapter representing also the Heisenberg algebra on semiinfinite forms.

In chapter 4 we will see the relation between what has been done in the preceeding chapters and string theory. We show how to quantize a string in the Krichever-Novikov formalism and how to obtain the critical dimension $D = 26$ and a nilpotent Q_{BRST} operator.

In chapter 5 we introduce a generalized superalgebra, which we realize in terms of classical fields. Later we realize its central extension by means of quantum operators.

We find again the relevant critical dimension $D = 10$.

Finally we realize a representation of the superalgebra on semiinfinite forms and study the relation with its expression in terms of quantum operators. This subject is still not completely understood, at least at the level of computations.

The last chapter 6 is devoted to the study of Kac-Moody algebras and of the Sugawara construction in higher genus Riemann surfaces. This will give us another way of representing the Krichever algebra.

Chapter 1

Definitions and notations. The Krichever-Novikov work

1.1. Construction of bases for tensorial fields of weight λ

Consider a generic Riemann surface Σ of genus g. Let us fix to points P_+ and P_- in general position over Σ. We want to construct bases for tensorial fields of weight λ, holomorphic outside the two distinguished points $P_\mp$ and possibly with poles at $P_\pm$. To this end we will use the Riemann-Roch theorem. In what follows let K be the canonical bundle, K^n its powers and K^n_D the bundle K^n modified with the divisor D according to the usual prescription [FK]. We will consider the divisor

$$D = (S-n)P_+ + (S+n)P_- \tag{1.1.1}$$

Let $H^k(\Sigma, L)$, for a line bundle L over Σ be the k-th Cech cohomology group and let $h^k = dimH^k$. In particular $H^0(\Sigma, K^j_D)$ will be the dimension of the space of holomorphic section of K^j, with poles possibly in D and with order in D greater or equal to $-D$. Applied to this situation the Riemann-Roch theorem says:

$$h^0(\Sigma, K^j_D) = h^0(\Sigma, K \otimes {K^j_D}^{-1}) + 1 - g + j\, degK \tag{1.1.2}$$

If we require uniqueness of the sections, i.e. $h^0(\Sigma, K^j_D) = 1$, then we obtain

$$S = \frac{g}{2} + j(1-g)$$

being $h^0(\Sigma, K \otimes {K^j_D}^{-1}) = 0$. In fact its degree is $(1-j)degK - degD = g-2$; and for a general bundle L with $degL < g-1$ we have $h^0(\Sigma, L) = 0$.

It is possible, for instance by explicitly constructing the bases with the help of theta functions as done in [BLMR], to see that the behaviour of the bases in the neighborhood of $P_\pm$ is actually the one prescribed by the divisor (the maximum attainable), apart from a slight modification which occurs for functions and 1-forms.

So we can describe the basis elements simply by specifying their behaviour in a neighborhood of $P_\pm$. Let $z_\pm$ be the coordinate systems around $P_\pm$ with $z_\pm(P_\pm) = 0$. Then we have, for the basis elements for Krichever-Novikov tensor fields of weight λ, the following behaviour:

$$f^\lambda_n(z_\pm) = a_\pm z_\pm^{\pm n - S(\lambda)}(1 + \mathcal{O}(z_\pm))(dz_\pm)^\lambda, \tag{1.1.3}$$

where $\mathcal{O}(z)$ denotes higher order terms and n is integer or half-integer according to the parity of g, precisely n is integer if g is even and half integer if g is odd.

The coefficients a_+ can be normalized to one. We already said that the behavior is different for functions A_i and 1-forms ω^i, because of the Weierstrass theorem. Their behavior can be determined by explicit construction ([BLMR]) and can be taken to be:

$$A_i(z_\pm) = a_i^\pm z_\pm^{\pm i-\frac{g}{2}}(1+\mathcal{O}(z_\pm) \quad |i| \geq \frac{g}{2}+1 \tag{1.1.4}$$

$$A_i(z_\pm) = a_i^\pm z_\pm^{\pm i-\frac{g}{2}\pm {0 \atop -1}}(1+\mathcal{O}(z_\pm))i = -\frac{g}{2},\ldots,\frac{g}{2}-1 \tag{1.1.5}$$

$$A_{\frac{g}{2}} = 1 \tag{1.1.6}$$

and analogously for 1-forms

$$\omega^i(z_\pm) = b_i^\pm z_\pm^{\mp i+\frac{g}{2}-1}(1+\mathcal{O}(z_\pm))(dz_\pm) \quad |i| \geq \frac{g}{2}+1 \tag{1.1.7}$$

$$\omega^i(z_\pm) = b_i^\pm z_\pm^{\mp i+\frac{g}{2}\pm {0 \atop -1}}(1+\mathcal{O}(z_\pm))(dz_\pm) \quad i = -\frac{g}{2}\ldots,\frac{g}{2}-1 \tag{1.1.8}$$

$$\omega^{\frac{g}{2}} = dk, \tag{1.1.9}$$

where dk is the third kind differential with simple poles in $P_\pm$ with residues ± 1, normalized in such a way that its periods over cycles be all purely imaginary. This differential will play a crucial role in the following. The coefficient a_+ and b_+ can be taken to be one. Observe that the index i for 1-forms has sign opposed to the index i in the exponent. This change of sign is only technical. Its meaning will become clear later.

It is possible to generalize this stuff and consider tensor fields which have a brunch cut along a curve connecting the points P_+ and P_-, where possibly poles live, holomorphic elsewhere. If f is such a tensor field, we parametrize the discontinuity across the cut with a parameter x_0 in the following way. If f^+ and f^- denote the limiting function on the two sides of the cut, then

$$f^+ = e^{2\pi i x_0} f^-. \tag{1.1.10}$$

Analogously to what we have done before, we can describe such tensor fields by their behaviour in neighborhoods of $P_{\pm}$. It is possible to identify this behaviour

$$f_i^{\lambda,x_0} = a_{\pm} z_{\pm}^{\pm(i+x_0)-S(\lambda)}(1+\mathcal{O}(z_{\pm}))(dz_{\pm})^{\lambda}, \tag{1.1.11}$$

where as usual i assumes integer or half-integer values according to the parity of g. Notice that we worked until now with λ integer. But there are no problems in getting expressions for λ half-integer or, even more, rational. But, in what follows we will need only the expression for λ half-integer, which is very simple, being exactly the same as for λ integer, with only one small difference. Precisely the only difference is that the index i takes integer values if g is odd and half-integer values if g is even.

1.2. Introduction of time over a Riemann surface

We will now introduce the concept of time on a Riemann surface in such a way that the string is described in the infinite past and in the infinite future by the points P_- and P_+. The level curves of the time will describe the string at the fixed time. Choose an arbitrary point Q_0 on Σ and introduce the function

$$p(Q) = Re \int_{Q_0}^{Q} dk. \tag{1.2.1}$$

The function p is a univalent function on Σ due to the fact that dk has periods purely imaginary along any cycle on Σ. Its level curves $C_\tau = \{Q | p(Q) = \tau\}$ describe the position of the string at the time τ. If τ goes to $\pm\infty$, then C_τ become small circles around $P_\pm$ as is easy to prove in local coordinates. So the string enter at P_- for $\tau \to -\infty$ and exits at P_+ for $\tau \to +\infty$. Notice that the C_τ's are not always connected, but can split in two or more components. When we speak of C_τ we will always mean the union of all components. The introduction of τ permits us to introduce a duality between tensorial fields of weight λ and $1-\lambda$ (we will consider x_0 integer or half-integer). In fact the product of two tensorial fields, one of weight λ, the other one of weight $1-\lambda$ is a 1-form which can be integrated over the C_τ's. The integral is a number, which does not depend on the C_τ chosen being the integrand holomorphic in the "annulus " between two different C_τ. In particular to this duality corresponds the duality of the bases in the sense that (for x_0 integer or half integer)

$$\frac{1}{2\pi i} \int_{C_\tau} f_i^{\lambda,x_0} f_k^{1-\lambda,x_0} = \delta_{i+k+2x_0} \tag{1.2.2}$$

independently from the C_τ. This result follows immediately from the fact that the integral can be done in a small circle around P_+, or, with opposite sign in a small circle around P_-. Combining the results of the integration around P_+ and P_-, we get the result. Defining

$$f^{-i}_{\lambda,x_0} \equiv f_i^{\lambda,-x_0} \tag{1.2.3}$$

we get the duality relation among basis elements

$$\frac{1}{2\pi i} \int_{C_\tau} f^{i}_{\lambda,x_0} f_k^{1-\lambda,x_0} = \delta^i_k \tag{1.2.4}$$

Observe that the same change of relative sign as in (1.2.3) has already been taken into account in the case of functions and one-forms. We will omit from now on the index x_0, when not explicitly required.

We want now to mention an important theorem, which allows us to connect the Virasoro fields (defined as tensorial fields on the circle) with the Krichever fields.

We can restrict to C_τ any tensor field on Σ of any weight λ. Let i_τ denote this restriction map. Denote by F^λ the set of all Krichever-Novikov tensor fields of weight λ on Σ and by $F^\lambda_{C_\tau}$ the set of all holomorphic tensor field (of weight λ) on C_τ. Then we have

$$i_\tau(F^\lambda) \quad \text{is dense in } F^\lambda_{C_\tau} \tag{1.2.5}$$

In particular we can taken the C_τ's to be small circles around P_+ or P_-, with the result that we have density of the restriction of the Krichever fields in the space of the Virasoro tensor fields. So any tensor field of arbitrary weight, differentiable over C_τ can be developed over the relevant Krichever basis. Precisely let $F^\lambda(Q)$ a tensorial field differentiable over C_τ. The following relation holds:

$$F^\lambda(Q) = \sum_{n\in\mathbf{Z}} f^\lambda_n(Q)\,\frac{1}{2\pi i}\int_{C_\tau} f^n_{1-\lambda}(Q')F^\lambda(Q') \quad Q, Q' \in C_\tau \tag{1.2.6}$$

we will give no the proof of this theorem (see [KN1]).

In particular we can introduce the following objects:

$$\Delta^\lambda(Q, Q') \equiv \frac{1}{2\pi i}\sum_{n\in\mathbf{Z}} f^\lambda_n f^n_{1-\lambda}(Q') \quad Q, Q' \in C_\tau \tag{1.2.7}$$

so the relations (1.2.7) can be written

$$F^\lambda(Q) = \int_{C_\tau} \Delta^\lambda(Q, Q')F^\lambda(Q') \quad Q, Q' \in C_\tau \tag{1.2.8}$$

So the Δ^λ's play the rôle of delta-functions for tensorial fields of weight λ.

We will view their importance next, in quantizing a string theory.

1.3. Hamiltonian vector field

We want to introduce now the hamiltonian vector field over a non trivial Riemann surface.

This will take us away from the line of discussion, but being of great importance, we we deserve to it some attention.

The hamiltonian vector field in genus $g = 0$ is easy to characterize.

It is related to the vector field

$$e_0 \doteq z\partial/\partial z$$

hence we can expect it will be, in genus different from zero the vector field e_{g_0} or else something related to that. We will see that this is a naif expectation.

The hamiltonian vector field, we are going to describe, must be something dual to the "time", in the sense that, just as τ determines a set of equal time curves, we want the hamiltonian to determine a complementary set of curves on the Riemann surface, that describe time evolution.

In order to construct a suitable hamiltonian vector field, we have to use the third kind differential dk, already introduced. This is characterized by the behaviour around cycles (its periods are purely imaginary) and by the behaviour near $P_{\pm}$, which we recall:

$$dk(z_{\pm}) = \frac{1}{z_{\pm}}(1 + \mathcal{O}(z_{\pm}))dz_{\pm} \tag{1.3.1}$$

Analogously to dk, we can introduce $\overline{dk}$ characterized by the same properties of dk, but antiholomorphic

$$\overline{dk}(\bar{z}_{\pm}) = \frac{1}{\bar{z}_{\pm}}(1 + \mathcal{O}(\bar{z}_{\pm}))d\bar{z}_{\pm} \tag{1.3.2}$$

It is possible to introduce two vector fields X and $\bar{X}$, dual to dk and $\overline{dk}$ in the following sense

$$dk \otimes X = 1$$

$$\overline{dk} \otimes \bar{X} = 1$$

They are uniquely defined, being dk and $\overline{dk}$ uniquely defined.

Their behaviour near $P_\pm$, independent of the genus is

$$X = z^{\pm}(1 + \mathcal{O}(z_\pm))\{\partial/\partial z_\pm\}$$

$$\bar{X} = \bar{z}^{\pm}(1 + \mathcal{O}(\bar{z}_\pm))\{\partial/\partial \bar{z}_\pm\}$$

This behaviour is exactly the same as e_0, but there is a big difference. Having dk and $\overline{dk}$, $2g$ zeroes outside of $P_\pm$, in the same points, X and $\bar{X}$ will have $2g$ poles outside $P_\pm$, hence they will be neither Krichever fields, nor linear combinations of them.

Next, we will see the meaning of these $2g$ poles.

Remember the function $p(Q)$

$$p(Q) = Re \int_{Q_0}^{Q} dk = \frac{1}{2}\int_{Q_0}^{Q} dk + \overline{dk}$$

Take its Lie derivative with respect to the vector field $X + \bar{X}$. We get

$$\text{(1.3.3)} \qquad L_{X+\bar{X}}p(Q) = lim_{t\to 0}\frac{\{\int_{Q}^{Q+t(X+\bar{X})}\frac{1}{2}(dk + \overline{dk}\}}{t} = \frac{1}{2}[dk(X) + \overline{dk}(\bar{X})] = 1$$

Take now the derivative with respect to $X - \bar{X}$

$$\text{(1.3.4)} \qquad L_{X-\bar{X}}p(Q) = lim_{t\to 0}\frac{\{\int_{Q}^{Q+t(X-\bar{X})}\frac{1}{2}(dk + \overline{dk}\}}{t} = \frac{1}{2}[dk(X - \overline{dk}(\bar{X})] = 0$$

So the vector field

$$\Omega = X - \bar{X}$$

generates a flow which "rotates" the C_τ's and the vector field

$$H = X + \bar{X}$$

generates a flow which along τ. This can be seen more directly in the following way: consider the function

$$\text{(1.3.5)} \qquad f(t) = \int_{Q_0}^{\varphi_t^* Q} dk + \overline{dk}$$

where $\varphi_t(Q)$ is the flow of H. Then

$$f'(t) = 1,$$

hence

$$\text{(1.3.6)} \qquad f(t) = t + \tau$$

Observe that the flow is not univalent, in general, but the function (1.3.5) is.

So the evolution for a time t, via the vector field H, moves C_τ to $C_{\tau+t}$, hence H describes properly time evolution.

It will be our hamiltonian vector field. There is still more: precisely on a Riemann surface, we can think of the complex structure as defined via an operator J, which is an endomorphism of the tangent space. What happens is that the vectors X and JX are orthogonal.

So on a Riemann surface the complex structure is a conformal structure.

From this it is easy to see that the vector fields Ω and X are orthogonal, for every metric G compatible with the complex structure: $G(X, JX) = 0$.

Now we want to make some comments on the meaning of the poles of H.

First we observe that $p(Q)$ is a Morse function on the Riemann surface.

First of all, it has non degenerated isolated critical points: in fact we have

$$dp = dk + \overline{d}k$$

so p has exactly $2g$ zeroes at the points where $dk + \overline{dk}$ is zero.

They are obviously isolated. Consider now the jacobian of p at the critical points. It must be non degenerate. To prove this fact we give a geometric description of the jacobian at a critical point Q of a function f.

Let X and Y be two vectors at Q. Extend them in a neighborhood and call their extension $\tilde{X}$ and $\tilde{Y}$. Then the jacobian H at Q can be defined in the following way:

$$H(X,Y) \equiv L_{\tilde{X}} L_{\tilde{Y}} f \tag{1.3.7}$$

It is symmetric and independent of the extensions. In fact

$$H(Y,X)f = L_{\tilde{Y}} L_{\tilde{X}} f = L_{\tilde{X}} L_{\tilde{Y}} f + L_{[\tilde{X},\tilde{Y}]} f$$

but

$$L_{[\tilde{X},\tilde{Y}]} f(Q) = i_{[\tilde{X},\tilde{Y}]} df(Q) = 0$$

being Q a critical point and i the interior product.

Moreover $H(X,Y)$ does not depend on $\tilde{X}$ and $H(Y,X)$ does not depend on $\tilde{Y}$; so, due to the symmetry, $H(X,Y)$ does not depend on the extensions $\tilde{X}$ and $\tilde{Y}$.

Let us now compute the jacobian in our case.

$$H(X,Y)p(Q) = L_{\tilde{X}} L_{\tilde{Y}} p(Q) = i_{\tilde{X}} d\, i_{\tilde{Y}} \{dk + \overline{dk}\} \mid_Q .$$

So in order for it to be non degenerate we have to impose

$$d i_{\tilde{Y}}\{dk + \overline{dk}\}|Q = 0 \Rightarrow Y = 0. \tag{1.3.8}$$

It is easy to convince ourselves in local coordinates, by using the fact that $\{dk + \overline{dk}\}(Q) = 0$, that (1.3.8) holds.

So we conclude that p is a Morse function. Hence it can be viewed as a height function on the Riemann surface. Its critical points will essentially correspond to the points where the Riemann surface has a hole, or more precisely the C_τ's split in two (or more) curves.

These critical points correspond finally to the poles of the hamiltonian; hence its flow in a neighborhood of these points will be double-valued, corresponding to the splitting of the C_τ's it describes.

Final remark concerns the hamiltonian function; it is not hard to relate it to τ, by choosing appropriately the Kaehler form on Σ.

We will not discuss this problem because it will take us away from our line of discussion.

Chapter 2

Krichever-Novikov algebra

2.1. Some notations and definitions

We introduce now the Krichever-Novikov algebra, which is the proper generalization of the Virasoro algebra, seen as algebra of vector fields on the circle, to non trivial Riemann surfaces. It is given exactly by the algebra of vector fields on the Riemann surface, that is that of tensor fields of weight -1. Its commutator is the geometrical commutator of vector fields, the Lie derivative. Before describing the algebra we recall what is usually meant for *k−graded algebras* or *k-graded modules.*

An algebra $G = \sum_{i\in\mathbf{Z}} G_i$ is said to be k-graded if $G_i \cdot G_j$ is contained in $\sum_{s=i+j-k}^{i+j+k} G_s$. Analogously k-graded modules are introduced. Let us put $e_i = f_i^{-1,0}$, $\Omega^j = f_{2,0}^j$. The basis of KN vector fields can then be written

$$e_i(z_\pm) = c_i^\pm z_\pm^{\pm i - g_0 + 1}(1 + \mathcal{O}(z_\pm))\frac{\partial}{\partial z_\pm} \qquad g_0 = \frac{3}{2}g \tag{2.1.1}$$

and analogously the basis of KN quadratic differentials Ω^i is

$$\Omega^i(z_\pm) = b^{i\pm} z_\pm^{\mp i + g_0 - 2}(1 + \mathcal{O}(z_\pm))(dz_\pm)^2 \qquad g_0 = \frac{3}{2}g. \tag{2.1.2}$$

It is now easy to compute the commutator of two vector fields in terms of the constants appearing in the tails $\mathcal{O}(z_\pm)$. But this will give us no relevant information. The important thing here is that the algebra is g_0-graded, as can be recognized simply by computing the leading term of the commutator in P_+ and in P_-. So it can be written

$$[e_i, e_j] = \sum_{s=-g_0}^{g_0} C'^s_{ij} e_{i+j-s}, \qquad g_0 \equiv \frac{3}{2}g \tag{2.1.3}$$

C'^s_{ij} being appropriate structure constants (in particular $C'^{g_0}_{ij} = j - i$).

Notice that assigning gradation j to e_j, this relation shows that the algebra of Krichever and Novikov vector fields is g_0 graded.

We prefer to put the previous commutation relation in another form, in which it does not appear obvious the g_0-gradation, but which is more useful in doing computations

$$[e_i, e_j] = \sum_k C^k_{ij} e_k, \qquad g_0 \equiv \frac{3}{2}g \tag{2.1.4}$$

where the structure constants can be computed, using the duality

$$C^k_{ij} = \int_{C_\tau} [e_i, e_j]\Omega^k \tag{2.1.5}$$

and are different from zero, as is easily seen only for

$$i + j - g_0 \geq k \geq i + j + g_0$$

so reproducing the g_0-gradation Let L^{Σ} denote the Krichever Novikov algebra. It is possible to introduce for $s \geq -1$ the vector spaces

$$L^s_{\pm} = \{e_i | \pm i \geq g_0 - s\}$$

which are subalgebras of L^{Σ}. In particular $L^{-1}_{\pm} \equiv \{e_i | \pm i \geq g_0 + 1\}$ are the subalgebras of holomorphic vector fields with possibly poles *or* in P_+ *or* in P_-. So if we write $L^{\Sigma} = L^{-1}_{+} + L_0 + L^{-1}_{-}$, then L_0 is the set of holomorphic vector fields with poles in P_+ *and* P_-, which so generate non trivial transformation of the Riemann surface. Hence L_0 can be identified with the tangent space to the moduli space. Notice that the dimension of L_0 is just $2g - 2$, exactly the dimension of the moduli space.

2.2. Central extension of the Krichever Novikov algebra

We want now to generalize the Krichever Novikov algebra.

To do that we need the notion of central extension of an algebra.

A complex-valued k-cocycle c for a Lie algebra A is just a k-linear, antisymmetric map from the algebra to $\mathbf{C}$, which is closed under a coboundary operator δ defined as follows: let a_i be elements of the algebra A, then

$$\delta c(a_1, ..., a_{k+1}) = \sum_{i \geq j} c([a_i, a_j], a_1, ...\hat{a}_i..\hat{a}_j..)(-1)^{i+j}$$

In particular, for k=2, closure under δ means the equality

$$c([x,y],z) - c([x,z],y) + c([y,z],x) = 0 \quad x,y,z \in A,$$

which is called the Jacobi identity for 2-cocycles.

A cocycle is said to be trivial, or cohomologous to zero, if it is in the image of δ. In particular a 2-cocycle c is trivial if it has the form

$$c(x,y) = b([x,y]) \quad x,y \in A$$

for some 1-cocycle b. With the help of two cocycles it is possible to define central extensions of the algebra A, that is to adjoin to the algebra a central generator t, modifying the commutation rules in the following way

$$[a_i, a_j]' = [a_i, a_j] + tc(a_i, a_j), [a_i, t] = 0.$$

It is easy to show that this definition preserves the Jacobi identity of the algebra and the anticommutativity, giving rise to a new Lie algebra.

Consider a projective representation ρ of the algebra A, that means a representation of the algebra A centrally extended

$$[\rho(x), \rho(y)] = \rho([x,y]) + c(x,y)$$

c being a trivial cocycle.

Redefining

$$\rho(x) \longrightarrow \rho'(x) = \rho(x) + b(x)$$

we have

$$[\rho(x) + b(x), \rho(y) + b(y)] = [\rho(x), \rho(y)] + b([x,y]) = (\rho + b)([x,y]) = \rho'([x,y])$$

So trivial cocycles are of no great importance in the representation theory of central extensions an algebra. They can be put away simply by redefining the generators. So, also in the case of the Krichever algebra, we will pay in general not too much attention to trivial cocycles.

We will define now, analogously to what happens for vector fields on S^1, a central extension of the Krichever-Novikov algebra. To the aim of introducing this central extension we need a particular geometrical object on the Riemann surface, namely a projective connection (see [HS]).

A projective connection R is an object which has the following transformation properties under change of coordinate system (or diffeomorphism) $z \to z'(z)$

$$R'(z')(\frac{\partial z'^2}{\partial^2 z}) = R(z) + \mathcal{S}(z'(z)) \tag{2.2.1}$$

where $\mathcal{S}$ is the schwartzian derivative, and has the following form

$$\mathcal{S}(z'(z)) = (\frac{\partial^3 z'}{\partial z^3})/(\frac{\partial z'}{\partial z}) - \frac{3}{2}\left((\frac{\partial^2 z'}{\partial z^2})/(\frac{\partial z'}{\partial z})\right)^2 . \tag{2.2.2}$$

We will take projective connections which are holomorphic out of P_+ and P_-, and with the following behaviour near $P_\pm$:

$$R(z_\pm) = \rho_\pm z^{-2}(1 + \mathcal{O}(z_\pm)).$$

Hence the difference of two projective connections will be a quadratic differential which at most a pole of order 2 in $P_\pm$. So this quadratic differential can be expanded in the basis Ω^j on the elements with $|j| \leq g_0$.

Hence the most general projective connection R' can be expressed

$$R' = R + \sum_{n=-g_0}^{g_0} a_n \Omega^n$$

R being a fixed projective connection and a_n being arbitrary constants. The central extension of the Krichever-Novikov algebra is now defined by means of the cocycle

$$\chi(e_i, e_j) = \frac{1}{24\pi i} \int_{C_\tau} \tilde{\chi}(e_i, e_j) \tag{2.2.3}$$

where $\tilde{\chi}(f, g)$ for any two Krichever vector fields $f = f(z)\frac{\partial}{\partial z}$ and $g = g(z)\frac{\partial}{\partial z}$, is given by

$$\tilde{\chi}(f,g) = \Big(\frac{1}{2}(f'''g - g'''f) - R(f'g - fg')\Big)dz \tag{2.2.4}$$

In doing computation it is helpful to integrate over small circles around $P_\pm$. The cocycle χ satisfies the properties

(i) $\chi(f,g) = -\chi(g,f)$ (antisymmetry)

(ii) it is independent of the coordinate system

(iii) it satisfies the cocycle condition (Jacobi identity):

$$\chi(f,[g,h]) + \chi(g,[h,f]) + \chi(h,[f,g]) = 0$$

iv) it is "local"

$$\chi_{i,j} \equiv \chi(e_i,e_j) = 0 \quad for \quad |i+j| > 3g.$$

i), ii) and iii) guarantee that this is a cocycle. Property iv) guarantees that it is unique ([KN1]).

The relation

$$[e_i,e_j] = \sum_k C_{ij}^k e_k + t\chi(e_i,e_j), \quad [e_i,t] = 0 \tag{2.2.5}$$

defines the extended Krichever-Novikov algebra, t being the generator of the centre. Our primary aim will be to construct representations of the algebra (2.2.5).

2.3. Generalized Heisenberg algebra

We will now recall an other algebra of particular interest, precisely the Heisenberg algebra, i.e. a trivial Lie algebra, centrally extended by means of some cocycle. Our Heisenberg algebra will be the Lie algebra of Krichever-Novikov functions on a Riemann surface, centrally extended.

The commutator of the pointwise product of functions C-valued is trivially 0, and give the structure of trivial Lie algebra to the ring of functions. We can centrally extend this algebra by introducing a suitable cocycle. If f and g are two Krichever-Novikov functions, i.e. functions holomorphic in Σ, with possibly poles in $P_{\pm}$, we can define a cocycle γ.

(2.3.1) $$\gamma(f,g) \equiv \int_{C_\tau} df g$$

where d is the exterior derivative. Like for the cocycle of vector fields, we have the properties

(i) $\gamma(f,g) = -\gamma(g,f)$

(ii) it is independent of the coordinate system

(iii) it satisfies the cocycle condition:

$$\gamma(f,[g,h]) + \gamma(g,[h,f]) + \gamma(h,[f,g]) = 0$$

(which holds trivially)

iv) it is "local"

$$\gamma_{i,j} \equiv \gamma(A_i, A_j) = 0 \quad for \quad |i+j| > g.$$

i), ii) and iii) guarantee that this is a cocycle. Property iv) guarantees that it is unique.

So we define the generalized extended Heisenberg algebra:

(2.3.2) $$[A_i, A_j] = t\gamma(A_i, A_j)$$

(2.3.3) $$[A_i, t] = 0.$$

We will devote some time later to the representation theory of this algebra.

2.4. Representation modules for the Krichever Novikov algebras not centrally extended

Before coming to the study of the representations of the Krichever Novikov algebras centrally extended we want to construct modules for the non centrally extended Krichever-Novikov algebras. This is very easy.

Vector fields in fact operate, via Lie derivative, over tensor fields of any weight integer (and in 2 dimensions also half-integer). In local coordinates, if f^λ is any meromorphic tensor field of weight λ and e is a meromorphic vector field, we have

$$L_e f^\lambda = e\frac{\partial f^\lambda}{\partial z} + \lambda f^\lambda \frac{\partial e}{\partial z} \tag{2.4.1}$$

Being

$$[L_e, L_f] = L_{[e,f]}$$

for any two vector field e and f, we have a representation of the Krichever-Novikov algebra on the space of Krichever tensor fields of any weight λ, F^λ. In particular, taking $\lambda = -1$ we get the adjoint representation. This representations are very important because the representations on semiinfinite forms of the extended KN algebra we will construct can be seen as "product" of infinite of them. It is easy to see how this action is represented on the bases. We will give only the structure:

$$e_i f_j^{\lambda,x_0} = \sum_k R_{i,j}^k(x_0) f_k^{\lambda,x_0} \tag{2.4.2}$$

where

$$R_{ij}^k(x_0) = \int_{C_\tau} L_{e_i}(f_j^{\lambda,x_0}) f_{1-\lambda,x_0}^k \tag{2.4.3}$$

and in particular

$$R_{ij}^{i+j-g_0}(x_0) = (j + x_0 - S(\lambda)) + \lambda(i - g_0 + 1) \quad \lambda \neq 0. \tag{2.4.4}$$

It is easy to show that the modules now defined are g_0-graded The case $\lambda = 0$ is slightly different because of the different definition of the bases.

Precisely there is no g_0-gradation for all i and j, although essentially for almost i and j the degree of diffusion is exactly equal to g_0. For more details see [KN1,KN2].

Analogously we can define a representation of the commutative algebra of functions on the space of tensor fields. The action is given by the tensorial product. In terms of bases we have

$$A_i f_j^{\lambda,x_0} = \sum_k \alpha_{ij}^k(x_0) f_k^{\lambda,x_0} \tag{2.4.5}$$

where

$$\alpha_{ij}^k(x_0) = \int_{C_\tau} A_i f_j^{\lambda,x_0} f_{1-\lambda,x_0}^k \tag{2.4.6}$$

Notice in particular that

$$\alpha_{\frac{g}{2}j}^k(x_0) = \delta_j^k \tag{2.4.7}$$

being $A_{\frac{g}{2}}$ the function 1.

A notational remark: sometimes when speaking of a pair of dual bases, we will use the notation

$$f_{1-\lambda}^k \equiv f_\lambda^{+k} \tag{2.4.8}$$

and moreover, when it makes no confusion, we will drop also the index λ.

Chapter 3

Representations of the Krichever-Novikov algebra centrally extended on semiinfinite forms

3.1. Physical meaning of semiinfinite forms

We already constructed representations of the KN algebra not centrally extended. We want in this section to construct representations of the central extension of the Krichever-Novikov algebra. Let us recall Dirac words, to give a meaning to the representations we are going to describe: *"...the wave equation for the electron admits twice as many solutions as it ought to, half of them referring to states with negative values for the kinetic energy...we are led to infer that the negative energy solutions.. refer to the motion of a new kind of particle having the mass of an electron and the opposite charge. Such particles have been observed experimentally and are called positrons... We assume that nearly all the negative-energy states are occupied, with one electron in each state in accordance with the exclusion principle of Pauli. An unoccupied negative energy state will now appear as something with a positive energy, since to make it disappear, i.e. to fill it up, we should have to add to it an electron with negative energy. We assume that these unoccupied negative-energy states are the positrons.*

These assumptions require there to be a distribution of electron of infinite density everywhere in the world. A perfect vacuum is a region where all the states of positive energy are unoccupied and all those of negative energy are occupied.... the infinite distribution of negative-energy electrons does not contribute to the electric field..there will be a contribution $-e$ *for each occupied state of positive energy and a contribution* $+e$ *for each unoccupied state of negative energy.*

The exclusion principle will operate to prevent a positive-energy electron ordinarily from making transitions to states of negative energy. It will still be possible, however, for such an electron to drop into an unoccupied state of negative-energy. In this case we should have an electron and positron disappearing simultaneously, their energy being emitted in the form of radiation. The converse process would consist in the creation of an electron and a positron from electromagnetic radiation". We will describe the "particle" by an element in $F^{\lambda}(x_0)$, i.e. a KN tensorial field of weight λ with discontinuity parametrized by x_0; the vector f_j^{λ} in this vector space will describe a state of energy $N-j$. So the energy is not always positive: to fix this Dirac requires an infinite number of such particles, satisfying the Pauli exclusion principle.

We are therefore naturally led to consider the space of semiinfinite forms $\Lambda^\infty F^\lambda(x_0) \equiv \mathcal{F}^\lambda(x_0)$, i.e. the space consisting of object of the form

$$f_{i_0} \wedge f_{i_1} \wedge ....f_{i_k} \wedge f_m \wedge f_{m+1} \wedge ...... \tag{3.1.1}$$

where the indices λ and x_0 have been suppressed and $i_0 < i_1 < .. < i_k < m$ and from m towards all indices are consecutive, we assume moreover $m - k = N$. Elements (3.1.1) are obviously antisymmetric under the exchange of any pair of indices.

In accordance with Dirac theory moreover the perfect vacuum will be

$$\varphi_0^N \equiv f_N \wedge f_{N+1} \wedge f_{N+2}...... \tag{3.1.2}$$

The condition $m-k = N$ ensures that the generic vector (3.1.1) has an equal number of particles and antiparticles, so that it can be obtained from the vacuum via elementary excitations. Having understood the physical meaning of the object we are dealing with, let us come back to the problem of representing the KN algebra on that object. The easiest way to follow is to define an action of the Krichever fields e_i's on element of the form (3.1.1) via the Leibnitz rule. But there are problems, as we will see.

Notice, that acting in this way the e_i's with $|i| > g_0$ applied to the vacuum (3.1.2) give 0. In fact, they reproduce pieces already present in the vacuum semiinfinite form, because $e_i f_j$ contains f_k with $k \geq i + j - g_0 \geq j$. So

$$e_i \varphi_0 = 0 \quad i > g_0 \tag{3.1.3}$$

The problems arise from the fact acting on infinite product e_i give a infinite sum, except when, due to antisymmetry, the sum reduces to a finite one. It is easy to understand that if $e_i f_j$ contains a term proportional to f_j, this always contributes to the sum. Remembering the g_0-gradation it is easy to see that this happen for $|i| \leq g_0$ and does not happen otherwise. Moreover for $e_i f_k$, if k is enough big (depending on i), to contribute to the sum we need it contains a term proportional to f_k (the other pieces get cancelled due to the g_0-gradedness and the fact that from a certain point all indices are consecutive). So we get that the action is ill defined if $|i| \leq g_0$ and works well in the other sector. It is possible to redefine the action for $|i| \leq g_0$ as we will see, but this will give rise to a projective representation of KN or, equivalently, to a representation of its central extension. This can be seen in many ways. First we will see this by noticing that the action of $e_i, |i| \leq g_0$ can be defined in terms of the action of commutators $[e_k, e_l]$ with $|k|, |l| > g_0$ which is well defined. Before proceeding observe that we naturally subdivided the KN algebra in three pieces $L^\Sigma = L^+ + L^0 + L^-$, $L^+ = \{e_i | i > g_0\}$, $L^- = \{e_i | i < -g_0\}$ and $L^0 = \{e_i || i| \leq g_0\}$. We will give now an example of how to compute the action of L^0 on semiinfinite forms, in the specific case of e_{g_0}. Let us take $i > 3g_0$ so that $2g_0 - i < -g_0$.

Then

$$[e_i, e_{2g_0-i}]\varphi_0^N = e_i e_{2g_0-i}\varphi_0^N = C_{ij}^k e_k \varphi_0^N + t\chi_{i,2g_0-i}\varphi_0^N \tag{3.1.4}$$

but due to (3.1.3) this reduces to

$$C_{ij}^{i+j-g_0} e_{g_0}\varphi_0^N + t\chi_{i,2g_0-i}\varphi_0^N \tag{3.1.5}$$

Compute now $e_i e_{2g_0-i}\varphi_0^N$

$$\begin{aligned} e_i e_{2g_0-i}\varphi_0^N &= e_i \sum_{k\geq 0} R_{2g_0-i,N+k}^{N+p} f_N \wedge f_{N+1} \wedge \ldots . f_{p+N}^{(k+N)} \wedge f_{k+1} \ldots \\ &= \sum_{\substack{k\geq 0\\ p<0}} R_{2g_0-i,N+k}^{N+p} R_{i,N+p}^{N+k} \varphi_0^N \end{aligned} \tag{3.1.6}$$

where $f_p^{(k)}$ means that at the place k, there stays the tensor field f_p.

Recalling the g_0 gradedness and (2.4.4) we get

$$\begin{aligned} [e_i, e_{2g_0-i}] = \sum_{\substack{k\geq 0\\ p<0}} R_{2g_0-i,N+k}^{(N+p)} R_{i,N+p}^{N+k} &= [(i-g_0)^3 - (i-g_0)](-\lambda^2 + \lambda - \frac{1}{6}) \\ &+ (i-g_0)[N + x_0 - S(\lambda))](N + x_0 - S(\lambda) - 1 + 2\lambda) \end{aligned} \tag{3.1.7}$$

but this, acting on the vacuum, must also be equal to

$$C_{i,2g_0-i}^{g_0} e_{g_0} + c\chi_{i,2g_0-i} \tag{3.1.8),}$$

where c is the value of the central operator t and is called the *central charge*.

Before proceeding in the computation we have to digress slightly on the computations of object of the kind of (3.1.7) in terms of the coefficients appearing in the expansions of the various fields, in a neighborhood of $P_\pm$.

We can expand each field in P_+ in the following way:

$$f_j^{\lambda,x_0} \equiv= \sum_{p\geq j} f_j^p z^{p+x_0-S(\lambda)} \quad a_j^j = 1 \tag{3.1.9}$$

$$f^{+j}_{\lambda,x_0} \equiv f^{j}_{1-\lambda,x_0} = \sum_{k\leq j} f^{+j}_{k} z^{-k-x_0+S(\lambda)-1} \quad b^j_j = 1 \tag{3.1.10}$$

To find relations among the coefficients a and b, for each pair of dual bases, we have to use the relation of *duality* and *completeness*. It is easy, starting from the duality relation

$$\frac{1}{2\pi i}\int_{C_\tau} f^{\lambda,x_0}_{j} f^{k}_{1-\lambda,x_0} = \delta^k_j \tag{3.1.11}$$

to get

$$\sum_{k\geq p\geq j} f^{p}_{j} f^{+k}_{p} = \delta^k_j. \tag{3.1.12}$$

Analogously from the completeness relation, which corresponds to (1.2.5), we can get

$$\sum_{m\geq j\geq k} f^{m}_{j} f^{+j}_{k} = \delta^m_k \tag{3.1.13}$$

These two relations, relating the expansion coefficients will be useful very often.

In term of relations (3.1.12) and (3.1.13), we can give a general expression for the cocycle $\chi(e_i, e_j)$ given in eq.(2.2.3). We will use the expansions:

$$e_i = \sum_{p\geq i} e^p_i z^{p-g_0+1} \qquad e_j = \sum_{q\geq j} e^q_j z^{q-g_0+1} \tag{3.1.14}$$

$$R = \sum_{j\geq 0} R^j z^{j-2} \tag{3.1.15}$$

So that

$$e_i''' = \sum_{p\geq i} [(p-g_0)^3 - (p-g_0)] e^p_i z^{p-g_0-2} \tag{3.1.16}$$

Hence the cocycle (2.2.3) can be expressed

$$\frac{1}{12}\sum_{p\geq i}\sum_{q\geq j} e^p_i e^q_j \{[(p-g_0)^3 - (p-g_0)]\delta_{p+q-2g_0} + \sum_{k\geq 0}^{4g_0} (p-q) R^k \delta_{p+q+k-2g_0}\} \tag{3.1.17}$$

In the case where $j = 2g_0 - i$ this expression simplifies to

$$\frac{1}{12}\{[(i-g_0)^3 - (i-g_0)] + 2(g_0 - i) R_0\} \tag{3.1.18}$$

Comparing now (3.1.7) and (3.1.8) we get

$$e_{g_0}\varphi_0 = h_+\varphi_0$$

$$t\varphi_0 = c\varphi_0 \tag{3.1.19}$$

with

$$c = c(\lambda) = -2(6\lambda^2 - 6\lambda + 1) \tag{3.1.20}$$

and

$$h_+ = -\frac{1}{2}\{(N + x_0 - S)(N + x_0 - S - 1 + 2\lambda) + \frac{1}{6}R_0 c\}. \tag{3.1.21}$$

h_+ is called *highest weight* of the representation we are dealing with. The central charge and the highest weight classify almost completely the representation. So recalling

$$e_i \varphi_0^N = 0 \quad i > g_0$$

and

$$e_{g_0} \varphi_0^N = h_+ \varphi_0^N$$

we realize immediately that we are dealing with a highest weight representation of the Krichever-Novikov algebra, with highest weight vector φ_0^N and highest weight h_+. Of this point we will discuss later. We will see also that this representation is induced from an inclusion of the Krichever-Novikov algebra in a bigger Lie algebra.

Now we want to generalize the previous computation. Precisely we want to compute the term proportional to the vacuum of $[e_i, e_j]\varphi_N^0$ for generic i and j.

If all works, in the sense that we have defined things exactly, this must turn out to be proportional to $\chi_{i,j} = \chi(e_i, e_j)$ plus possibly trivial cocycles times the vacuum. In fact

$$[e_i, e_j]\varphi_0^N = C_{ij}^k e_k \varphi_0^N + t\, \chi_{ij} \varphi_0^N \tag{3.1.22}$$

Writing

$$e_k \varphi_0^N \mid_{\varphi_0^N} \equiv S_k \varphi_0^N$$

we get that the term proportional to the vacuum in (3.1.22) is

$$C_{ij}^k S_k + c\, \chi_{ij} \tag{3.1.23}$$

times the vacuum.

Now remember what a trivial 2-cocycle is for the Krichever-Novikov algebra: it is something of the form

$$\tilde{\chi}(e_i, e_j) = b([e_i, e_j]) = C_{ij}^k b(e_k)$$

b being a 1-cocycle. Defining $b(e_k) \equiv S_k$, we immediately get that (3.1.22) must be cohomologous to the Krichever-Novikov cocycle times the vacuum. We will show now that the action of $[e_i, e_j]$ on φ_0^N reproduces exactly the cocycle more possibly a trivial term.

Consider in fact the what we get applying $[e_i, e_j]$ to the vacuum vector, under the simplifying assumption $i > g_0$. This is precisely equal to

$$\sum_{\substack{k \geq N \\ p < N}} R_{ip}^k R_{jk}^p \varphi_o^N$$

We want now to compute this object.

We suppose for simplicity $x_0 = 0$. By employing the expansions (3.1.9) and (3.1.10) we easily get

$$\begin{aligned} \sum_{\substack{k \geq N \\ p < N}} R_{ip}^k R_{jk}^p = \sum_{\substack{k \geq N \\ p < N}} e_i^a f_p^b f_c^{+k} e_j^d f_k^f f_h^{+p} \times (b - S(\lambda) + \lambda(a - g_0 + 1)) \\ \times (f - S(\lambda) + \lambda(d - g_0 + 1))\delta_{a+b-c-g_0}\delta_{d+f-h-g_0}, \end{aligned} \tag{3.1.24}$$

where the other sum to be done are understood.

If $c \geq N$ the sum over k can be easily done, using the duality relation, the same for $b < N$ for the sum over p. So the sum splits in four sums:

$$\sum_{\substack{c \geq N \\ b < N}} + \sum_{\substack{c < N \\ b < N}} + \sum_{\substack{c \geq N \\ b \geq N}} + \sum_{\substack{c < N \\ b \geq N}} \tag{3.1.25}$$

The last one is zero due to the δ, whereas the first one is very easy to compute: we will do this computation later. Consider now the two central pieces. Their sum is, as it is easy to verify

$$\begin{aligned} &\sum_{\substack{c \geq N, b \geq N \\ p < N}} e_i^a e_j^d f_p^b f_h^{+p} \delta_{cf}(b - S(\lambda) + \lambda(a - g_0 + 1))(f - S(\lambda) + \lambda(d - g_0 + 1)) \times \\ &\times \delta_{a+b-c-g_0}\delta_{d+f-h-g_0} + \sum_{\substack{b < N, c < N \\ p \geq N}} e_i^a e_j^d f_p^f f_c^{+p} \delta_{bh}(b - S(\lambda) + \lambda(a - g_0 + 1)) \times \\ &\times (f - S(\lambda) + \lambda(d - g_0 + 1))\delta_{a+b-c-g_0}\delta_{d+f-h-g_0} \end{aligned}$$

and can also be written

$$\sum_{\substack{b\geq N\\ p<N}} e_i^a e_j^d f_p^b f_h^{+p}(b-S(\lambda)+\lambda(a-g_0+1))(h-d+g_0-S(\lambda)+\lambda(d-g_0+1))$$

$$\delta_{d+a+b-h-2g_0}+\sum_{\substack{h<N\\ p\geq N}} e_i^a e_j^d f_p^b f_h^{+p}(h+g_0-a-S(\lambda)+\lambda(a-g_0+1))$$

$$(b-S(\lambda)+\lambda(d-g_0+1))\delta_{a+d+b-h-2g_0}$$

or

$$\sum_{\substack{b\geq N\\ p<N}} e_i^a e_j^d f_p^b f_h^{+p}(a-d)(\lambda(h+S(1-\lambda))+(1-\lambda)(b-S(\lambda))\delta_{a+b+d-h-2g_0}$$

(3.1.26)

$$=\sum_{\substack{b\geq N\\ p<N}} f_p^b f_h^{+p} e_k^{h+g_0-b}(\lambda(h+S(1-\lambda))+(1-\lambda)(b-S(\lambda))C_{ij}^k\,.$$

Observe here the particular form of the coefficient: if the sum had to be done over all b it would be exactly equal to that of $[e_i,f_p]f^{+p}$. But this is not the case.

Consider now the first term in (3.1.25).

It can be written

$$\sum_{c\geq m,b<N} e_i^a e_j^d(b-S+\lambda(a-g_0+1))(c-S+\lambda(d-g_0+1))\times$$

(3.1.27)

$$\delta_{a+d-2g_0}\times\times\delta_{a+b-c-g_0}$$

$$=\sum_{b=1}^{a-g_0} e_i^a e_j^d\delta_{a+d-2g_0}\times(b+N-g_0-a-1-S+\lambda(a-g_0+1))$$

$$(b+N-1-S+\lambda(g_0-a+1))$$

which turns out to be equal to

$$\sum_{a,d} e_i^a e_j^d\delta_{a+d-2g_0}\{(a-g_0)^3-(a-g_0)\}(-\lambda^2+\lambda-\frac{1}{6})+(a-g_0)(N-S)(N-S-1+2\lambda)\}$$

So we have proven

(3.1.28)

$$\sum_{\substack{k\geq N\\ p<N}} R_{ip}^k R_{jk}^p=\sum_{a,d} e_i^a e_j^d\delta_{a+d-2g_0}\Big[\{(a-g_0)^3-(a-g_0)\}$$

$$(-\lambda^2+\lambda-\frac{1}{6})+(a-g_0)(N-S)(N-S-1+2\lambda)\Big]$$

$$+\sum_{\substack{b\geq N\\ p<N}} f_p^b f_h^{+p} e_k^{h+g_0-b}(\lambda(h+S(1-\lambda))+(1-\lambda)(b-S(\lambda))C_{ij}^k$$

Which is equal to

$$
\begin{aligned}
&- 2(6\lambda^2 - 6\lambda + 1)\chi_{i,j} - \sum_{p,q} e_i^p e_j^q \Big\{ [(p-q)h_+ \delta_{p+q-2g_0} \\
(3.1.29) \qquad &- \sum_{k \geq 0} (p-q) \frac{c(\lambda)}{12} R^k \delta_{p+q+k-2g_0} \Big\} \\
&+ \sum_{\substack{b \geq N \\ p < N}} f_p^b f_h^{+p} e_k^{h+g_0-b} (\lambda(h + S(1-\lambda)) + (1-\lambda)(b - S(\lambda)) C_{ij}^k
\end{aligned}
$$

or else

$$
\begin{aligned}
&- 2(6\lambda^2 - 6\lambda + 1)\chi_{i,j} + C_{ij}^k \Big\{ h_+ + c(\lambda)/12 e_k^m R_{m+g_0} + \\
(3.1.30) \qquad &\sum_{\substack{b \geq N \\ p < N}} f_p^b f_h^{+p} e_k^{h+g_0-b} (\lambda(h + S(1-\lambda)) + (1-\lambda)(b - S(\lambda)) \Big\}
\end{aligned}
$$

that is it has the form of the cocycle $\chi(e_i, e_j)$ plus a trivial cocycle, the projective connection R being combination of the coefficients of f and f^+.

So we have proved that the term proportional to the vacuum of $[e_i, e_j]\varphi_0^m$ coincides with a multiple of the cocycle $\chi(e_i, e_j)$ for a suitable choice of the projective connection.

Or, which is the same, for any choice of the projective connection, differs from a multiple of χ_{ij} only by a trivial cocycle, or else by a redefinition of the action at the $e_i \in \mathrm{L}^0$.

In the next section we will describe the kind of representation we are dealing with, from another point of view.

3.2. Representations of a_∞

We want now to discuss another way of studying the representation described in the previous section. Properly we will see it as a representation induced from a representation of a_∞. Introduce the infinite complex matricial group

$$GL_\infty = \{A = (a_{ij})_{i,j\in\mathbf{Z}} | A \quad \text{is invertible and only a}$$
$$\text{a finite number of the differences} \quad a_{ij} - \delta_{ij} \quad \text{is different from } 0\};$$

let gl_∞ be its Lie algebra

$$gl_\infty = \{A = (a_{ij})_{i,j\in\mathbf{Z}} | A \quad \text{has only a finite}$$
$$\text{number of} \quad a_{ij} \quad \text{different from } 0\}$$

We have a base in GL_∞, $\{E_{ij}\}_{i,j\in\mathbf{Z}}$ consisting of matrices everywhere 0, apart for the (i,j)-entry which assumes value 1. Define the following action on $F^\lambda(x_0)$

(3.2.1) $$E_{ij} f_j^{\lambda,x_0} = f_i^{\lambda,x_0}$$

and extend it linearly.

From now on when not necessary, we will omit the indices λ and x_0, being irrelevant in this context.

We can so define representations $R^{\mathcal{F}}$ and $r^{\mathcal{F}}$ of the group GL_∞ and of its Lie algebra on $\mathcal{F}$ by the following formulas

(3.2.2) $$R^{\mathcal{F}}(A)(f_{i_0} \wedge f_{i_1} \wedge ...) = Af_{i_0} \wedge Af_{i_1} \wedge .... \quad A \in GL_\infty$$

and

$$r^{\mathcal{F}}(A)(f_{i_0} \wedge f_{i_1} \wedge ...) = Af_{i_0} \wedge f_{i_1} \wedge .... + f_{i_0} \wedge Af_{i_1} \wedge .... + ... \quad A \in gl_\infty$$

$i_0 < i_1 < ....$, and multilinearity and anticommutativity are assumed.

Notice that the representations so defined are reducible. Precisely the space $\mathcal{F}$ has the following decomposition

(3.2.3) $$\mathcal{F} = \oplus_m \mathcal{F}^m,$$

where $\mathcal{F}^m$ consists of those semiinfinite forms of the form

(3.2.4) $$f_{i_0} \wedge f_{i_1} \wedge .... \wedge f_{i_k} \wedge f_{m+k+1} \wedge f_{m+k+2} \wedge ...$$

so that they differ only by a finite change of indices from

(3.2.5) $$\varphi_0^m \equiv f_m \wedge f_{m+1} \wedge ....$$

It is obvious that the actions defined before preserve this decomposition. The representations $R^{\mathcal{F}^m}$ and $r^{\mathcal{F}^m}$, defined as the restrictions of the representations $r^{\mathcal{F}}$ to $\mathcal{F}^m$ are, as it is easy to see, irreducible. Moreover we can get that of the group by exponentiating that of the Lie algebra. We can moreover introduce a gradation on $gl_\infty = \oplus_{j\in\mathbf{Z}} g_j$ called the principal gradation,

$$degE_{ij} = j - i$$

so that

$$[g_i, g_j] = g_{i+j}$$

Define $n_+ \equiv \oplus_{j>0} g_j$, then $r^m(n_+)\varphi_0^m = 0$.

It is convenient, in doing computations to work with a bigger group (and correspondingly a bigger algebra)

$$\overline{GL}_\infty = \{A = (a_{ij})_{i,j\in\mathbf{Z}} | A \quad \text{is invertible and all but a finite number of the}$$
$$\text{differences} \quad a_{ij} - \delta_{ij} \quad \text{with} \quad i \leq j \quad \text{is} \quad 0\}$$

$$\overline{gl}_\infty = \{A = (a_{ij})_{i,j\in\mathbf{Z}} | A \quad \text{has only a finite number of}$$
$$a_{ij}, i \leq j \quad \text{different from } 0\}$$

Both $\overline{GL}_\infty$ and $\overline{gl}_\infty$ act on the completion $\overline{F} = \{\sum_i c_i f_i | c_i = 0 \quad i \ll 0\}$ of F. But the representations $R^{\mathcal{F}}$ and $r^{\mathcal{F}}$ extend to representations on the same spaces. We now are ready to discuss a little on *highest weight representations of* GL_∞

Given a collection of number $h = \{(h_i)_{i\in\mathbf{Z}}\}$ we define the *highest weight representation* π_h of gl_∞ as an irreducible representation on a vector space $L(h)$ which admits a non zero vector v_h, called *highest weight vector*, such that

$$\pi_h(E_{ij})v_h = 0 \quad \text{for} \quad i > j; \pi_h(E_{ii})v_h = h_i v_h \tag{3.2.6}$$

It is known that for every h, the triple (L_h, π_h, v_h) satisfying (3.2.6) exists and is unique (up to isomorphism).

In particular the representations r_m^F of gl_∞ on $\mathcal{F}_m$ are highest weight representations with highest weight $h_m = \{h_i = 1, i \geq m, h_i = 0, i < m\}$ and highest weight vector φ_0^m.

These particular representations are called *fundamental representations* of gl_∞.

Introduce now a yet bigger Lie algebra

$$\bar{a}_\infty = \{(a_{ij})_{i,j\in\mathbf{Z}} | \text{for each} \quad k \quad \text{the number of non-zero} \tag{3.2.7}$$
$$a_{ij} \text{ with } i \geq k \quad \text{and } j \leq k \quad \text{is finite.}\}$$

Matrices in $\bar{a}_\infty$ have a finite number of non-zero diagonals and so they can be written as finite linear combinations of matrices of the form

$$a_k = \sum_i \lambda_i E_{i,i+k}$$

Clearly $\bar{a}_\infty$ acts on $\bar{F}$, but it is not possible to extend the representations $r^{\mathcal{F}}$. For instance $r^{\mathcal{F}^m}(\sum_i E_{ii}\lambda_i)\varphi_0^m = (\lambda_m + \lambda_{m+1} + ...)\varphi_0^m$ which in general diverges. To remove this infinity, we can change the definition on the diagonal:

$$\hat{r}^{\mathcal{F}^m}(E_{ij}) = r^{\mathcal{F}^m}(E_{ij}) \quad if \quad i \neq j \quad \text{or } i = j < N \tag{3.2.8}$$

$$\hat{r}^{\mathcal{F}^m}(E_{ij}) = r^{\mathcal{F}^m}(E_{ij}) - I \quad if \quad i = j \quad \text{and} \quad i \geq N \tag{3.2.9}$$

Extending this definition by linearity, we realize a projective representation of the Lie algebra $\bar{a}_\infty$, or equivalently a representation of its central extension a_∞, realized by means of a cocycle α, which can be described as follows:

$$\alpha(E_{ij}, E_{ji}) = 1 \quad \text{if} \quad i \geq N, j < N,$$
$$\alpha(E_{ij}, E_{ji}) = -1 \quad \text{if} \quad j \geq N, i < N, \tag{3.2.10}$$
$$\alpha(E_{ij}, E_{mn}) = 0 \quad \text{otherwise}$$

The algebra now read

$$[a, b] = ab - ba + \alpha(a, b)c, a, b \in \bar{a}_\infty$$

and defining $\hat{r}^{\mathcal{F}^m}(c) = I$ we can say that $\hat{r}^{\mathcal{F}^m}$ is a representation of a_∞.

After having digressed on the representations of a_∞, we come back to Krichever.

We will see that the Krichever Novikov algebra can be realized as a subalgebra of a_∞, and, as such, we can construct representations for them. These representations will turn out to be nothing but the representations we already constructed through Lie derivative and Leibnitz rule on semiinfinite forms. From this fact follows that all what we did before is well defined. Consider in fact the action of e_i on f_j. This can be expressed as in (2.4.2). Hence e_i can be seen as a particular element of $\bar{a}_\infty$, namely

$$e_i = \sum_{k,j} E_{k,j} R^k_{ij}$$

Hence, we see immediately that the representation on semiinfinite forms of the e_i's can be thought as induced from the representations of a_∞. We can redefine the e_i's corresponding to the redefinition of E_{ij} and we get a representation of the extended KN algebra induced from a_∞. It is obvious that (3.2.4) can also be written

$$r(E_{i_0 m}) r(E_{i_1, m+1}) ... r(E_{i_k, m+k}) \varphi_0^m$$

We can introduce an interior product $< \ \ >$ on $\mathcal{F}^m$ by declaring objects of this kind to be an orthonormal basis. Precisely let

$$e_{i_1,\ldots,i_k} \varphi_0^m \equiv E_{i_1,m} .... E_{i_k, m+k-1} \varphi_0^m \quad i_1 < ... i_k < g_0 \tag{3.2.11}$$

We can declare such a set to form an orthonormal basis of $\mathcal{F}^m$. This is well defined. Now we want to see what the adjoint e_i^+ of the operator e_i, is. Before inducing this notion from a_∞ let us consider the simpler case in which we consider the action of e_i on F (not on semiinfinite product). Then, declaring f_j to be an orthonormal basis for F we have the following definition

$$< e_i f_j, f_k > \equiv < f_j, e_i^+ f_k >$$

but the left hand side is equal to

$$R^k_{ij}$$

so

$$e_i^+ f_k = \overline{R}^k_{ip} f_p.$$

We will see that this definition extends to $\mathcal{F}^\lambda$, from another viewpoint. We have a standard linear antiinvolution of gl_∞ defined as the hermitian conjugation $a \longrightarrow a^*$, $a \in gl_\infty$. It is easy to show, that

$$< r(a)\psi, \psi' > = < \psi, r(a^+)\psi' > \ \psi, \psi' \in \mathcal{F}^\lambda, a \in gl_\infty$$

so that r^m are irreducible unitary representations of gl_∞. Now, when we pass to a_∞, if we define $c^+ = c$, the same relation holds. So the representations $\hat{r}^m$ are also unitary for a_∞. Now recalling $E^*_{ij} = E_{ji}$ and

$$e_i = \sum_{jk} E_{kj} R^k_{ij}$$

we have

$$e^+_i = \sum_{jk} E_{kj} \overline{R}^j_{ik}$$

which is exactly the adjoint we defined previously.

Observe the form ω

$$\omega(e_i) = e^+_i$$

has the properties

$$\omega([x,y]) = [(\omega(y), \omega(x)]$$

$$\omega(\lambda x) = \bar{\lambda}\omega(x)$$

which mean that it is a hermitian controvariant form (see [KR] for more details).

We want now to construct explicitly the cocycle from a_∞. Let us denote the cocycle induced from a_∞, $\tilde{\chi}$.

We have immediately

$$\tilde{\chi}(e_i, e_j) = \sum_{k,l} \sum_{r,s} \alpha(E_{kl}, E_{rs}) R^k_{il} R^r_{js}$$

$$(3.2.12) \qquad = \{ \sum_{k \geq N, l < N} - \sum_{l \geq N, k < N} \} R^k_{il} R^l_{jk}$$

Notice that, from the inclusion of the Krichever algebra in a_∞ it follows that this is a 2-cocycle for the Krichever algebra. So, because its uniqueness it must have the form of $\chi_{i,j}$. And in fact it has: remembering

$$[e_i, e_j] = C^k_{ij} e_k + t\chi_{i,j}$$

and observing that acting on φ^m_0 the commutator gives exactly, as contribute proportional to φ^m_0 the result (3.2.12), we obtain that $\chi_{i,j}$ and (3.2.12) can differ only by a term of the form

$$C^k_{ij} S_k$$

when properly normalized. But this term is trivial. So we get equality, modulo trivial cocycles.

3.3. Clifford representation of the Krichever algebra

We will see from another point of view the representation of the KN algebra constructed. Consider $\mathcal{F}^{\lambda}$, meant as the union of the various $\mathcal{F}^{\lambda}_{m}$. Because there is a pairing (duality) between λ and $1-\lambda$ tensor fields which we will denote $<,>$, we can construct the Clifford algebra associated with that pairing. It turns out, that the representation of this Clifford algebra on semiinfinite forms, is very close to the one constructed previously, in the sense that, using the Clifford algebra generators it is easy to construct the representations of KN. Moreover we will come very close to the setting of string theory. Consider the operators defined as follows on semiinfinite forms

$$\epsilon(x) f_{i_0} f_{i_1} \dots \equiv x \wedge f_{i_0} \wedge f_{i_1} \dots, \quad x \in \mathcal{F}^{\lambda} \tag{3.3.1}$$

$$i(x') f_{i_0} \wedge f_{i_1} \dots \equiv \sum_{k \geq 0} (-1)^k < x', f_{i_k} > f_{i_0} \wedge f_{i_1} .. \wedge \hat{f}_{i_k} .., \quad x' \in \mathcal{F}^{1-\lambda} \tag{3.3.2}$$

The following commutation relations holds for the operators now introduced

$$\{i(x'), i(y')\} = \{\epsilon(x), \epsilon(y)\} = 0$$

$$\{i(x'), \epsilon(x)\} = < x', x >$$

where $\{,\}$ denotes the anticommutator. It is easy to see that the action of non redefined E_{ij} on $\mathcal{F}^{\lambda}$ corresponds to the action of $\epsilon(f_i) i(f_j^{+})$. So with the help of the Clifford representation, and of that we said before we can construct a representations for the e_i's. What happens is that $e_i = \sum_{k,j} E_{k,j} R^k_{ij}$, expressed like element of $\bar{a}_{\infty}$, as we already said.

But to get a central extension of $\bar{a}_{\infty}$ we need a redefinition of E_{ij} according to (3.2.8) and (3.2.9).

Correspondingly we get a redefinition of the e_i's.

Now the redefinition of the e_i's for $i \leq g_0$ will correspond to a redefinition of the ordering of ϵ and i. In fact the redefinition of E_{ij} implies the following definition of the e_i's:

$$e_i = \sum_{k \neq j} E_{k,j} R^k_{ij} + \sum_{k=j<N} E_{k,j} R^k_{ij} + \sum_{j \geq N} \{E_{jj} R^j_{ij} - R^j_{ij}\}$$

or in terms of i and ϵ

$$e_i = \sum_{k\neq j} \epsilon(f^k)i(f^{+j})R^k_{ij} + \sum_{k=j<N} \epsilon(f_k)i(f^{+j})R^k_{ij} - \sum_{j\geq N} i(f^{+j})\epsilon(f^j)R^j_{ij}$$

This corresponds simply to changing the order of $\epsilon(j)$ and $i(j)$, with respect to the normal definition, in the open interval $j \geq m$. We can see differently this phenomenon: define the *normal ordering* of $i(f^{+j})$ and $\epsilon(f_k)$ in the following way:

$$:\epsilon(f_i)i(f^{+j}) := \begin{cases} \epsilon(f_i)i(f^{+j}) & j < M \\ -i(f^{+j})\epsilon(f_i) & j \geq M \end{cases}$$

Then we can write the redefined e_i's in the following way

$$e_i = R^k_{ij} : \epsilon(f_k)i(f^{+j}) : \tag{3.3.3}$$

The redefinition is effective only when $k = j$, that is, being k, due to the g_0-gradation limited

$$i + j - g_0 \leq k \leq i + j + g_0,$$

only for

$$i - g_0 \leq 0 \leq i + g_0,$$

so only in the subspace L_0.

We can use, in working with this notion of normal ordering the usual Wick theorem, which will allow us to simplify calculations.

Let us recall now what Wick theorem is.

Consider two linear operators A and B, bosonic or fermionic. Their normal ordering will differ by the simple product by some scalar operator which we denote with a brace connecting the two operators and which will be called contraction of the two operators:

$$AB =: AB : + \underbrace{AB}.$$

With the help of this notion of contraction, we can express the Wick theorem. Let us define now the normal product for more than two operators

$$: a_1 \ldots . a_n :$$

in the following way.

Define first normal ordering with pairing

$$: a_1 ... \underbrace{a_j .... a_k} ... a_n := \sigma_{jk} \underbrace{a_j a_k} : a_1 ... \hat{a}_j ... \hat{a}_k .. a_n :$$

where σ_{jk} is the signature of the permutation which leads $\{1, ..., n\}$ to $\{j, k, 1, ..\hat{j}..\hat{k}..., n\}$ (only the permutations of the fermionic operators are taken into account). Then we define

$$a_1 .... a_n$$

as the sum of

$$: a_1 .... a_n :$$

with the set of all possible contraction of $: a_1 ... a_n :$. This gives a inductive definition of the normal ordering for more than two operators.

This definition make so that any two operators inside the normal ordering commute if they are bosonic and anticommute if they are fermionic.

More generally we can define the product of normally ordered products of linear operators:

$$(: a_1 .... a_{i_1} :)(: a_{i_1+1} ..... a_{i_2} :)...(: a_{i_n-1} ... a_{i_n} :)$$

is equal to the sum of all normal ordered product of these operators with all possible pairings between elements of the n sets. A sign has to be taken into account in the case of fermionic operators, like previously.

Let now come back to the normal ordering defined before for the two anticommuting operators $i(f^+)$ and $\epsilon(f)$. Being anticommuting operators they will anticommute inside the normal ordering.

Moreover we have

$$\epsilon(f_j) i(f^{+i}) =: \epsilon(f_j) i(f^{+i}) : + \delta_{ij} \theta^+(i - M)$$

and

$$i(f^{+i}) \epsilon(f_j) =: i(f^{+i}) \epsilon(f_j) : + \delta_{ij} \theta^+(M + 1 - i)$$

where

$$\theta^+(i) = 1, \, i \geq 0, \, 0 \quad otherwise$$

in such a way that

$$\underbrace{i(f^{+i})\epsilon(f_j)} = \delta_{ij}\theta^+(M+1-i) \tag{3.3.4}$$

$$\underbrace{\epsilon(f_j)i(f^{+i})} = \delta_{ij}\theta^+(i-M) \tag{3.3.5}$$

(3.3.4) and (3.3.5) will be useful next, in computing for example commutators of L_i with L_j.

The normal ordering now defined will turn out to be meaningful also from a physical point of view.

When we will view the algebra of the e_i's as the algebra of the components L_i of the energy-momentum tensor, then the L_i's will have a decomposition very close to (3.3.3) in terms of fields operators.

The normal ordering now introduced from a mathematical point of view, i.e. to make sense to a representation, will then reproduce the usual normal ordering between creation and annihilation operators.

3.4. Representation of the commutation rules $[\alpha_m, \alpha_n]$ on semiinfinite forms

Recall that in chapter **2** we defined a central extension of the trivial algebra of functions, by means of a 2-cocycle γ.

$$[f, g] = \gamma(f, g)$$

We want now to represent this algebra on semiinfinite forms. This algebra will in fact be very useful in string theory. due to the fact that, in string theory, the Virasoro (Krichever) generators can be expressed as operators quadratic in the α's. This decomposition of the Virasoro (Krichever) generators corresponds to the bosonic part of the energy-momentum tensor.

If we try to represent this algebra on semiinfinite forms, like we did for the algebra of vector fields, we meet the same problem. Precisely the action is ill defined for some interval of the indices (in this case for $|i| \leq \frac{g}{2}$). Hence we need a redefinition of the action of some A_i. This redefinition can be done in such a way to have well defined results, but the algebra no more closes. In fact, in this way, we get only projective representations of the Heisenberg algebra, or else representations of his central extension.

We will not discuss too much this problem within the framework of chapter **2** but our viewpoint will be Clifford representation and normal ordering. We can define the following operators

$$\alpha_m = \sum_k : \epsilon(A_m \otimes f_k) i(f^{+k}) := \sum_k \alpha^p_{mk} : \epsilon(f_p) i(f^{+k}) : \tag{3.4.1}$$

where normal ordering is defined as usual:

$$: \epsilon(f_i) i(f^{+j}) := \begin{cases} \epsilon(f_i) i(f^{+j}) & j < M \\ -i(f^{+j})\epsilon(f_i) & j \geq M \end{cases}$$

By using the relation

$$[AB, CD] = AD[B, C] - [A, C]BD + CA[B, D] - C[A, D]B$$

and the Wick theorem we get

$$
\begin{aligned}
[\alpha_m, \alpha_n] = \{ &\sum_{k<M,p<M} + \sum_{k\geq M,p<M} \}\{\epsilon(A_m f_k) i(f^{+p}) < f^{+k}, A_n f_p > \\
&- \epsilon(A_n f_p) i(f^{+k}) < f^{+p}, A_m f_k >\} \\
&-\{ \sum_{k\geq M,p\geq M} - \sum_{k<M,p\geq M} \}\{ i(f^{+p})\epsilon(A_m f_k) < f^{+k}, A_n f_p > - \\
&i(f^{+k})\epsilon(A_n f_p) < f^{+p}, A_m f_k >\} \\
&= \{- \sum_{k\geq M,p<M} + \sum_{k<M,p\geq M} \}\{< f^{+p}, A_m f_k >< f^{+k}, A_n f_p >
\end{aligned}
\tag{3.4.2}
$$

where the notation $<,>$ refers to the duality relation previously introduced among tensor fields. We want to show that this object is just the right cocycle, independently of M, because for functions there are no trivial cocycles. The proof is lengthy and tedious, we will reproduce only some points. Suppose for simplicity $m > \frac{g}{2}$, then the cocycle reduces to

$$
\sum_{k<M,p\geq M} < f^{+p}, A_m f_k >< f^{+k}, A_n f_p > \equiv \sum_{k<M,p\geq M} \alpha^{p}_{mk} \alpha^{k}_{np}
\tag{3.4.3}
$$

using the usual conventions (3.4.3) can be expressed

$$
\sum_{k<M,p\geq M} a^e_m a^b_n f^c_p f^{+p}_d f^f_k f^{+k}_a \delta_{e+f-d-\frac{g}{2}} \delta_{c+b-a-\frac{g}{2}}
\tag{3.4.4}
$$

the other sums being understood. The sum over f and d can be divided in four sectors

$$
\sum_{\substack{f\geq M\\ d<M}} + \sum_{\substack{f<M\\ d<M}} + \sum_{\substack{f\geq M\\ d\geq M}} + \sum_{\substack{f<M\\ d\geq M}} .
$$

It is easily seen that the first region does not give any contribution, due to the δ, whereas the contribution of the second and third is:

$$
\sum_{\substack{f\geq M, d\geq M\\ p<M}} a^b_n a^e_m \delta_{-a+b+d-\frac{g}{2}} \delta_{e+f-d-\frac{g}{2}} f^{+p}_a f^f_p + \sum_{\substack{f<M, d<M\\ p\geq M}} a^b_n a^e_m \delta_{-f+b+c-\frac{g}{2}} \delta_{e+f-d-\frac{g}{2}} f^{+p}_d f^c_p
$$

which is also equal to

$$
\sum_{\substack{f\geq M, a<M\\ p<M}} a^b_n a^e_m \delta_{b+e+f-a-g} f^{+p}_a f^f_p + \sum_{\substack{f\geq M, a<M\\ p\geq M}} a^b_n a^e_m \delta_{e+f+b-a-g} f^{+p}_a f^f_p
\tag{3.4.5}
$$

which is easily shown to be 0.

So what remains is the last term, in which we can easily sum getting the right cocycle, being equal to

$$(3.4.6)\qquad \sum_{\substack{a<M\\ c\geq M}} a^b_n a^e_m \delta_{+e+a-c-\frac{g}{2}} \delta_{a+b-g} = (e-\frac{g}{2}) a^e_m a^b_n \delta_{e+b-g} = \int_{C_\tau} dA_m\, A_n .$$

The last identity will turn out to be useful. It can be written in another form:

$$(3.4.7)\qquad \begin{aligned} &\{ \sum_{k<M,p\geq M} - \sum_{k<M,p\geq M} \} \frac{1}{2\pi i} \int A_m(Q) f_k(Q) f^{+p}(Q) \times \\ &\frac{1}{2\pi i} \int A_n(Q') f_p(Q') f^{+k}(Q') = \frac{1}{2\pi i} \int dA_m(Q) A_n(Q) \end{aligned}$$

for this to hold, we must have

$$(3.4.8)\qquad \begin{aligned} &\{ \sum_{k\geq M,p<M} - \sum_{k<M,p\geq M} \} f_k(Q) f^{+p}(Q) f_p(Q') f^{+k}(Q') = \\ &\qquad = d \sum_k A_k(Q) \omega^k(Q') = d\Delta^0(Q,Q') \end{aligned}$$

where Δ^0 which has been already introduced in chapter **1**, plays the rôle of delta-function over C_τ for functions.

Observe that for functions there cannot be trivial cocycles, in fact a trivial cocycle will be of the form

$$\tilde{\gamma}_{mn} = b([A_m, A_n]) = 0$$

b being a 1-cocycle, so a linear map. Because the commutator for functions is zero, the linear map b from the space of functions will be zero, on the zero function.

Chapter 4

Strings and Krichever-Novikov algebras

4.1. Quantizing a string in the Krichever formalism

Let us now make contact with string theory.

We would like now to show how to work to quantize a conformal field theory on higher genus Riemann surface according to the Krichever and Novikov formalism.

In quantizing a string theory via the operatorial formalism in higher genus Riemann surfaces, we will naturally realize the above Krichever-Novikov algebra as algebras of the "momenta" of the energy momentum tensor ensuing from the classical Poisson brackets in a string theory. We will proceed in the following way.

Any field can be expanded in the Krichever basis corresponding to its weight.

At the classical level, the Poisson brackets of the fields involved induce Poisson brackets for the expansion coefficients.

When the theory is quantized the expansion coefficients become quantum operators acting on suitable Fock spaces. To obtain well defined results (i.e. to remove the infinities we encounter) we have to introduce normal ordering between these quantum operators. These operators will not any more realize the previous algebra, but a central extension of that. Precisely the algebra they realize, like in genus 0 is the Virasoro one, in higher genus is precisely the Krichever-Novikov algebra (centrally extended).

In order to define a nilpotent BRST charge we will need to introduce suitable ghosts. The matching of the ghost contribution with the "matter" contribution to yield a nilpotent charge will be, as usual, the origin of the critical (D=26) dimension for the string theory.

Later we will introduce the superalgebra and we will recover the dimension $D = 10$.

We want so to realize (find the analogues of, according to the lexicon of [KN1,KN2]) the KN algebra as intrinsic algebras of string theories, first from a classical and then from a quantum point of view.

We start considering a closed string in a flat background, described by the action, in local coordinates

$$\mathcal{S} = -\frac{1}{2}\int d\sigma d\tau(-\dot{X}^{\mu}\dot{X}^{\nu} + X'^{\mu}X'^{\nu})\eta_{\mu\nu} \tag{4.1.1}$$

where the dot denotes differentiation with respect to the coordinate τ and the prime with respect to the coordinate σ.

By introducing a Wick rotation $\sigma \longrightarrow i\sigma$, and in terms of the coordinates $z = \tau + i\sigma$, $\bar{z} = \tau - i\sigma$ the canonical momentum is

$$P_\mu = i\dot{X}^\mu$$

and the non vanishing components of the energy-momentum tensor are

$$T^X_{++} = -\partial X^\mu \partial X_\mu = \frac{1}{4}(dX^\mu + P^\mu)(dX_\mu + P_\mu) \tag{4.1.2}$$

and

$$T^X_{--} = -\bar{\partial} X \bar{\partial} X = \frac{1}{4}(dX^\mu - P^\mu)(dX_\mu - P_\mu) \tag{4.1.3}$$

We will globally interpret X^μ as a Krichever-Novikov function and P_μ as a Krichever-Novikov 1-form.

We have to add to this energy-momentum tensor the ghost energy momentum tensor T^{gh} which has as before only two non vanishing components. Because all is written in terms of holomorphic or antiholomorphic coordinates, we will work only in the holomorphic sector. So the total (holomorphic part of the) energy-momentum tensor reads $T = T^X_{++} + T^{gh}_{++}$

$$T^X_{++} \equiv -\partial X^\mu \partial X_\mu, \quad T^{gh}_{++} \equiv c\partial b + 2\partial c b \tag{4.1.4}$$

$X^\mu(Q)$ is a field of weight 0. $b(Q)$ and $c(Q)$ are anticommuting ghost fields of weight 2 and -1 respectively. T has so weight 2. Observe that also the ghost contribution has so a global meaning. It is just the Lie derivative along the ghost c of the antighost b.

We can use the bases $\{e_i\}, \{\omega^i\}$, etc., introduced above in order to expand these fields. The coefficients will be later interpreted as creation and annihilation operators in suitable Fock spaces. We expand

$$\lambda = -1: \quad c(Q) = \sum_i c^i e_i(Q), \quad c^i = \frac{1}{2\pi i}\oint_{C_\tau} c(Q)\Omega^i(Q) \tag{4.1.5}$$

$$\lambda = 0: \quad X^\mu(Q) = \sum_i X^\mu_i A_i(Q), \quad X^\mu_i = \frac{1}{2\pi i}\oint_{C_\tau} X^\mu(Q)\omega^i(Q) \tag{4.1.6}$$

(4.1.7) $$\lambda = 2: \quad b(Q) = \sum_i b_i \Omega^i(Q), \quad b_i = \frac{1}{2\pi i} \oint_{C_\tau} b(Q) e_i(Q)$$

(4.1.8) $$\lambda = 1: \quad P^\mu(Q) = \sum_i P_i^\mu \omega^i(Q), \quad P_i^\mu = \frac{1}{2\pi i} \oint_{C_\tau} P^\mu(Q) A_i(Q)$$

(4.1.9) $$\lambda = 1: \quad dX^\mu(Q) + P^\mu(Q) = \sqrt{2} \sum_i \alpha_i^\mu \omega^i(Q),$$

$$\sqrt{2}\alpha_i^\mu = \frac{1}{2\pi i} \oint_{C_\tau} (dX^\mu(Q) + P^\mu(Q)) A_i(Q)$$

where P^μ is the conjugate momentum of X^μ. Now let us introduce the Poisson brackets

(4.1.10) $$[X^\mu(Q), P^\nu(Q')] = 2\pi\eta^{\mu\nu} \Delta_\tau(Q, Q'), \quad Q, Q' \in C_\tau$$

(4.1.11) $$\{c(Q), b(Q')\} = 2\pi D_\tau(Q, Q'),$$

where

(4.1.12) $$\Delta_\tau(Q, Q') = \frac{1}{2\pi i} \sum_i A_i(Q) \omega^i(Q')$$

(4.1.13) $$D_\tau(Q, Q') = \frac{1}{2\pi i} \sum_i e_i(Q) \Omega^i(Q')$$

These objects play the rôle of δ-functions over C_τ for smooth tensors of weight 0 and -1, as already noted in chapter **1**. For example for a generic smooth function $f(Q)$ over C_τ we have

$$f(Q) = \oint_{C_\tau} \Delta_\tau(Q, Q') f(Q') \quad Q, Q' \in C_\tau.$$

As a consequence of eqs. (4.1.10), (4.1.11) we have the following Poisson brackets for the coefficients of the expansion:

(4.1.14) $$[X_i^\mu, P_j^\nu] = -i\eta^{\mu\nu} \delta_{ij}$$

(4.1.15) $$[\alpha_i^\mu, \alpha_j^\nu] = -i\gamma_{ij} \eta^{\mu\nu}$$

(4.1.16) $$\{b_i, c_j\} = -i\delta_{ij}$$

Now let us consider $L_i = L_i^X + L_i^{gh}$ defined by:

(4.1.17) $$T(Q) = \sum_i L_i \Omega^i(Q)$$

So

$$L_i^X = -\frac{1}{2}\sum_{jk} l^i_{jk}\alpha_i \cdot \alpha_k \tag{4.1.18}$$

$$L_i^{gh} = \sum_j \sum_k C^k_{ij} c_j b_k$$

where l^i_{jk} is defined as

$$l^i_{jk} = \frac{1}{2\pi i}\oint_{C_\tau} e_i \omega^j \omega^k \tag{4.1.19}$$

and the dot in $\alpha_j \cdot \alpha_k$ means summation over the greek index μ understood. Then the Poisson brackets for the L_i's are:

$$[L_i, L_j] = -i\sum_k C^k_{ji} L_k \tag{4.1.20}$$

These are a replica of the commutation relations for KN vector fields apart from the opposite sign of the structure constants and the $-i$ factor.

The next step is quantization. All the classical quantities considered so far are promoted to operators acting in a Fock space. The Poisson brackets are replaced by quantum commutators according to the recipe: $[\,,\,]_{PB} \rightarrow -i[\,,\,]_{quantum}$. In order to avoid ambiguities we have to define normal ordering. For the ghost sector it is defined by considering as annihilation operators b_i for $i > N$ and c_i for $i \leq N$ and as creation operators the complementary set. As for the α_i's, the normal ordering prescription requires a little care. We have seen, representing them on semiinfinite forms, that the α_j's annihilate the vacuum for $j > \frac{g}{2}$. So these α's are to be considered as annihilation operators.

But we cannot now define the normal ordering of the α's by declaring that all the annihilation operators stay on the right and all the creation operators stay on the left, because this is ill defined, due to the fact that two creation operators α_m and α_n, $m, n \leq \frac{g}{2}$, $m + n > -g$ do not commute.

So we have to be more carefully. Let us proceed now how Krichever and Novikov.

First notice that the ordering must be defined essentially on the strip of the (m, n) plane $|m + n| \leq g$ because elsewhere the α's commute. In order to get finite results we have to divide the plane (m, n) in two regions Σ_+ and Σ_- such that each one differ only by a finite number of (integer or half-integer) points from the regions $\tilde{\Sigma}_+ = \{(i,j) \mid i < j\}$ and $\tilde{\Sigma}_- = \{(i,j) \mid i \geq j\}$ respectively, plus possibly points (m,n) where the corresponding α's commute. Now define the normal ordering like follows:

$$: \alpha_i \alpha_j := \begin{cases} \alpha_i \alpha_j & (i,j) \in \Sigma_+ \\ \alpha_j \alpha_i & (i,j) \in \Sigma_- \end{cases}$$

Observe that with this definition two α's inside the normal ordering will not in general commute (a part from the case $\Sigma^+ = \tilde{\Sigma}^+$).

This definition anyway removes the infinite sum in $l^i_{jk}\alpha^j\alpha^k$. In fact suppose we act on the vacuum, annihilated for $k > \frac{g}{2}$. Then $l^i_{jk} : \alpha^j \alpha^k : \varphi_0 = 2\sum_{j<k\leq \frac{g}{2}} l^i_{jk}\alpha_j\alpha_k\varphi_0 +$ *finite sums*. But recalling that $l^i_{jk} = 0$ for $|j+k-i| > \frac{g}{2}$ (this is not precise, because of the different definition of A_i and ω^i in the range $i = -\frac{g}{2}, ..., \frac{g}{2}$, but essentially works) for each fixed i only a finite number of possibilities for the indices j and k survive. So with this definition the sum is finite.

Now we need to calculate the algebra of the L_i's. This is an easy task for what concerns the ghost part in view of what has been done previously.

In fact for the ghost part we can realize the operators b_i and c_j on the space $\mathcal{F}^2_N$by identifying

$$b_i = i(e_i)$$

$$c_i = \epsilon(\Omega^i).$$

Hence we realize immediately that the $: L_i^{gh} :$'s give a realization of the KN algebra (with a sign reversed) with central charge

$$-2(6\lambda^2 - 6\lambda + 1)$$

with $\lambda = 2$, that is with central charge $C = -26$. The task is more difficult for what regards the $: L_i^X :$'s.

We can do computations by using only commutation relation and normal ordering prescriptions and get in such a way that the $: L_i^X :$'s realize a KN algebra with central charge D.

But we prefer to understand better things and to follow another way.

The α's can be realized on a Fock space, but we have at our disposal a representation in terms of semiinfinite forms.

We will not discuss the index μ because essentially it reproduces D copies of the same algebra. So it will only multiply the central charge by D. We want now to see what (4.1.18) is in terms terms of semiinfinite forms.

Now we try to compute $l^i_{jk} : \alpha_j \alpha_k :$ on all generality

$$\begin{aligned}\sum_{jk} l^i_{jk} : \alpha_j \alpha_k : &= \sum_{(j,k)\in\Sigma_+} l^i_{jk}\alpha_j\alpha_k + \sum_{(j,k)\in\Sigma_-} l^i_{jk}\alpha_k\alpha_j \\ &= \sum_{(j,k)\in\Sigma_+} l^i_{jk}[: \epsilon(A_j f_p) i(f^{+p}) :: \epsilon(A_k f_q) i(f^{+q}) :] \\ &+ \sum_{(j,k)\in\Sigma_-} l^i_{jk}[: \epsilon(A_k f_q) i(f^{+q}) :: \epsilon(A_j f_p) i(f^{+p}) :]\end{aligned} \tag{4.1.21}$$

Consider now for instance the sum in Σ_+.

By applying Wick theorem we get:

$$\begin{aligned}&\sum_{(j,k)\in\Sigma_+} l^i_{jk}[: \epsilon(A_j f_p) i(f^{+p}) \epsilon(A_k f_q) i(f^{+q}) :] \\ &+ \sum_{(j,k)\in\Sigma_+} l^i_{jk}[: \epsilon(A_j f_p) i(f^{+q}) : \underbrace{i(f^{+p})\epsilon(A_k f_q)}] \\ &- \sum_{(j,k)\in\Sigma_+} l^i_{jk}[: \epsilon(A_k f_q) i(f^{+p}) : \underbrace{\epsilon(A_j f_p) i(f^{+q})}] \\ &+ \sum_{(j,k)\in\Sigma_+} l^i_{jk}[\underbrace{\epsilon(A_j f_p) i(f^{+q})}\, \underbrace{i(f^{+p})\epsilon(A_k f_q)}]\end{aligned} \tag{4.1.22}$$

the quadrilinear term get cancelled due to the (j,k) symmetry in the coefficient when we add the contribute of the sum over Σ_-. Let us now take a particular case. Let in fact the f be functions and the f^+ be 1-forms. Moreover let us take the following definition of Σ_+

$$(j,k) \in \Sigma_+ \quad if \quad j < M$$

Then the total sum of the bilinear terms can be decomposed in four sums

$$\left\{ \sum_{\substack{j<M \\ p<M}} - \sum_{\substack{k<M \\ p\geq M}} + \sum_{\substack{k\geq M \\ p<M}} - \sum_{\substack{p\geq M \\ j\geq M}} \right\} l^i_{jk} R^s_{jp} R^p_{kq} : \epsilon(A_s) i(\omega^q) :$$

and, due to the identity already proven

$$\{-\sum_{\substack{f\geq M\\ d<M}}+\sum_{\substack{f<M\\ d\geq M}}\}A_f(Q)\omega^d(Q)A_d(Q')\omega^f(Q')=\sum_m d'A_n(Q')\omega^n(Q) \tag{4.1.23}$$

we get for the sum of the two central pieces:

$$\int -e_i(Q)\omega^s(Q)dA_q(Q) : \epsilon(A^s)i(\omega^q) :$$

which is just the expression for the L_i's acting on the weight 0 semiinfinite forms. Analogously the sum of the first and fourth piece can be done by using an analogous relation. We do not proceed now in the computation of the total expression of the L_i^X's. In fact it will be rather tedious and will give no clarification. What is important here is that we have realized the L_i^X's bilinearly in terms of interior and exterior product as acting on semiinfinite forms. It is anyway possible to show (see [KN2]) that the L_i^X's realize a KN algebra with a sign reversed. The corresponding cocycle has been calculated by Krichever and Novikov and with our choice of normal ordering for the α's it can be expressed as

$$\chi_{ij}^X=\frac{1}{2}\{\sum_{n<N,k\geq N}-\sum_{n\geq N,k<N}\}l_{nm}^i l_{ks}^j\gamma_{ns}\gamma_{km} \tag{4.1.24}$$

or also

$$\chi_{ij}^X=\frac{1}{2}\{\sum_{n<N,k\geq N}-\sum_{n\geq N,k<N}\}\alpha_{nk}^i\alpha_{kn}^j \tag{4.1.25}$$

where

$$\alpha_{nk}^i\equiv\frac{1}{2\pi i}\int e_i\omega^n dA_k \tag{4.1.26}$$

are just the structure constants for the action of the e_i's on the functions.

We realize immediately that (4.1.25), up to trivial cocycles, is equal to $\chi(e_i,e_j)$, due to what has been done in chapter **2**, and hence that the corresponding central charge is 1.

Since we have D copies the total central charge will be D.

The same computation can be done if we take instead of functions and 1-forms any pair of dual tensorial fields.

The only difference being that computations are more difficult.

So, for the total normally ordered L_i we have

$$[: L_i :, : L_j :] = \sum_k C_{ji}^k : L_k : + (D - 26)\chi_{ij} \tag{4.1.27}$$

(modulo trivial cocycles). We want now to describe what the $BRST$ charge is. We have seen that incorporation of ghosts in the energy momentum tensor gives us Krichever generators, obeying an algebra without central extension. So it makes sense to search for states $|\chi>$ satisfying $L_i|\chi>=0$ for all i. But the problem is the fact we are working in a bigger Fock space containing ghosts and antighosts excitations.

BRST quantization permits us to answer this question. We deserve now some attention to the construction of the BRST charge.

Consider an arbitrary Krichever-Novikov module V. Denote by π the representation of the Krichever-Novikov algebra on V. Now we can define a differential d on $V \otimes \mathcal{F}^m$ in the following way, supposing there are no central extensions

$$d = \pi(e_i)\epsilon(\Omega^i) + \sum_{i<j} i([e_i, e_j])\epsilon(\Omega^i)\epsilon(\Omega^j)$$

This object satisfies $d^2 = 0$. With the help of this operator we can compute the cohomology groups $H^*_\infty(L^\Sigma, V)$, defined in the usual way in terms of d, which will be called semi-infinite cohomology of the Krichever-Novikov algebra, with coefficients in V.

Things are more subtle when we have to face with central extensions. Precisely the same definition holds, but in general $d^2 \neq 0$. Now this stuff can be applied to our case, when the space V is the representation space for the L_i^X's. Suppose first there are no central extensions. We call d in this case Q, and identify the interior product and the exterior product with the the antighost and the ghost respectively. Then we can write

$$Q = \sum_i L_i^X c_i + \frac{1}{2} \sum_{i,j} \sum_k C_{ij}^k c_i c_j b_k \tag{4.1.28}$$

which can be viewed also as

$$Q = \frac{1}{2\pi i} \oint_{C_\tau} (T^X(Q)c(Q) + \frac{1}{2} b(Q)[c(Q), c(Q)] \tag{4.1.29}$$

or equivalently

$$Q = \frac{1}{2\pi i} \oint_{C_\tau} \{c\partial X^\mu \partial X_\mu + \partial c c b\} \tag{4.1.30}$$

The integrand in eq. (4.1.30) is understood in a local coordinate patch. The integrand in eq. (4.1.29) is a global expression and the commutators are geometrical commutators (Lie derivatives). After quantization we have to consider $\hat{Q} =: Q :$. We obtain

$$\hat{Q}^2 = \{\hat{Q}, \hat{Q}\} = \sum_{i,j} \hat{\chi}_{i,j} : c_i c_j :$$

From eq. (4.1.27) we have that up to trivial cocycles $\hat{Q}^2 = 0$ for $D = 26$.

4.2. Fock representation of the oscillator algebra

We have seen how to represent the oscillator algebra on the space of semiinfinite forms.

Usually one meet instead another kind of representation called the Fock space representation. It is a highest weight representation defined simply by the commutation relations and from a vacuum state in the following way: let E be a real finite dimensional vector space of dimension D carrying a non degenerate quadratic form g of signature (r,s), r being the number of positive eigenvalues of g, s the number of negative eigenvalues of g,$r+s=D$. Let moreover p be a linear functional on E

$$p(\alpha_\mu) = p_\mu$$

if α_μ is a basis of E.

Consider infinite copies of E, call each one E_n and define the infinite dimensional vector space

$$\hat{E} = \sqcup_{n\in\mathbf{Z}} E_n \oplus \mathbf{C}\,.$$

Define a Lie algebra structure on $\hat{E}$ by imposing the following commutation relations:

$$[\alpha_n^\mu, \alpha_m^\nu] = \gamma_{nm} g^{\mu\nu} \tag{4.2.1}$$

So $\hat{E}$ becomes an Heisenberg Lie algebra.

Denote by $V(E,p)$ the representation ρ of $\hat{E}$ isomorphic to the symmetric algebra $Sym \sqcup_{n>0} E_n$ with the property, that there exists a state $|p>$ called the vacuum such that

$$\rho(\alpha_m^\mu)|p>=\delta_{\frac{g}{2},m} p^\mu |p> \qquad m \geq \frac{g}{2} \tag{4.2.2}$$

In our context we can take the following vacuum

$$|p>=1 \tag{4.2.3}$$

and define the following operators

$$\tilde{\alpha}_i^\mu := \begin{cases} \jmath(\alpha_i^\mu) & i>0 \\ p^\mu I \quad i=0 \\ \tilde{\alpha}_i^\mu = \quad -i\alpha_{-i}^\mu \vee & i<0 \end{cases}$$

where

(4.2.4)
$$\jmath(\alpha_m^\mu)\alpha_i^\nu \equiv \eta^{\mu\nu}\delta_{m+i}$$

We have

$$[\tilde{\alpha}_i^\mu, \tilde{\alpha}_j^\nu] = \eta^{\mu\nu} i\delta_{i+j}$$

Now we can use the relation with a_∞ of the functions algebra and define

(4.2.5)
$$\rho(\alpha_i^\mu) \equiv A_m^i \tilde{\alpha}^\mu_{m-\frac{g}{2}}$$

where A_m^i are just the constants appearing in the tail $\mathcal{O}(z)$ for the functions and are so zero for $i > m$. Hence the α's so defined annihilate the vacuum for $m > \frac{g}{2}$ which implies $i > \frac{g}{2}$. It is moreover an easy computation to verify that

(4.2.6)
$$[\alpha_i^\mu, \alpha_j^\nu] = \eta^{\mu\nu}\gamma_{mn}$$

Usually the α's are realized on the symmetric space described here. What we have previously done is to realize them on semiinfinite forms. The relation between the two representations is more general from what can appear here. In fact one can establish an isomorphism, which is called *boson-fermions correspondence* between the Fock space representations and the semiinfinite forms representations. For more details, in the genus zero case see [KR].

Chapter 5

Superalgebra on genus g Riemann surfaces

5.1. Construction of the superalgebra

We would like to show now that one can extend the construction of Krichever and Novikov and of [BBCM] so as to generalize the Neveu-Schwarz and Ramond algebras to Riemann surfaces of arbitrary genus, construct a corresponding BRST charge and recover the expected (D=10) critical dimension in the superstring case.

To the end of generalizing the superVirasoro algebra, we need, a part from what we used before in constructing the Krichever algebra, tensorial fields with half-integer weight $\frac{1}{2}$ and $x = \frac{1}{2}$ or $x = 0$ (the latter characterization will define, respectively, the Neveu-Schwarz and Ramond sector). Let us set $g_\alpha \equiv f_j^{(-\frac{1}{2},0)}$ with $\alpha = j$ integer and $g_\alpha \equiv f_{j+\frac{1}{2}}^{(-\frac{1}{2},\frac{1}{2})}$ with $\alpha = j + \frac{1}{2}$ half integer. That is, collectively,

$$g_\alpha(z_\pm) = a_\alpha^\pm z_\pm^{\pm\alpha - g + \frac{1}{2}}(1 + \mathcal{O}(z_\pm))(dz_\pm)^{-\frac{1}{2}} \tag{5.1.1}$$

With α integer g_α is holomorphic outside $P_\pm$ (Ramond sector) while when α is half integer g_α has a branch cut from P_+ to P_- (Neveu-Schwarz sector). For later use, we define also $k^\alpha \equiv f_j^{(\frac{3}{2},0)}$ with $\alpha = -j$ integer and $k^\alpha \equiv f_{j+\frac{1}{2}}^{(\frac{3}{2},\frac{1}{2})}$ with $\alpha = -j - \frac{1}{2}$ half integer, and $h_{-\alpha} = f_j^{(\frac{1}{2},0)}$ with $\alpha = j$ integer and $h_{-\alpha} = f_{j+\frac{1}{2}}^{(\frac{1}{2},\frac{1}{2})}$ with $\alpha = j + \frac{1}{2}$ half integer. The usual duality relation will hold now between the basis g_α and k^α and between h_α and $h^{+\alpha}$.

The superalgebra will be realized by adding to the e_i's the g_α', which being sections of $K^{-\frac{1}{2}}$ are spinor fields, and so the right ingredient to construct a superalgebra.

We have to introduce some operations between these fields in order to get a superalgebra.

Apart form the commutator of the e_i's we need to introduce something which play the rôle of commutator for e_i and g^α and finally something which plays the rôle of anticommutator among the g^α's. The task is easy: we can take the following operations: the Lie derivative along a vector field of a spinor field

$$[e_i, g_\alpha] \equiv L_{e_i} g_\alpha \tag{5.1.2}$$

and the tensor product of sections

$$\{g_\alpha, g_\beta\} \equiv g_\alpha g_\beta + g_\beta g_\alpha \tag{5.1.3}$$

Observe that (5.1.2) is defined (on Riemann surfaces) as $L_e g = \big(\epsilon(z)\partial\gamma(z) + \lambda\gamma(z)\partial\epsilon(z)\big)(dz)^\lambda$ in a local patch where $e = \epsilon(z)\frac{\partial}{\partial z}$ and $g = \gamma(z)(dz)^\lambda$). For integer λ, L_e reduces to the Lie derivative along the vector field e. Then

$$[e_i, e_j] = \sum_k C_{ij}^k e_k \tag{5.1.4}$$

$$[e_i, g_\alpha] = \sum_\beta H_{i\alpha}^\beta g_\beta \tag{5.1.5}$$

$$\{g_\alpha, g_\beta\} = \sum_p B_{\alpha\beta}^p e_p, \tag{5.1.6}$$

where from an analysis of the singularities in $P_\pm$ we obtain that the anticommutator has a $\frac{g}{2}$-gradation, whereas the commutators have a g_0-gradation.

The coefficients

$$H_{i\alpha}^\beta = \frac{1}{2\pi i}\int [e_i, g_\alpha]k^\beta, \tag{5.1.7}$$

$$B_{\alpha\beta}^p = \frac{1}{2\pi i}\int \{g_\alpha, g_\beta\}\Omega^p \tag{5.1.8}$$

can be calculated from the constants appearing in the expansion of e_i and g_α near $P_\pm$. For example, in the simplest case, we have $H_{i\alpha}^{i+\alpha-g_0} = \alpha - \frac{i}{2} - g + \frac{g_0}{2}$, $B_{\alpha\beta}^{\alpha+\beta-g_0} = 2$.

Eqs.(5.1.4), (5.1.5), (5.1.6) define the NS-KN superalgebra or the R-KN superalgebra for $\alpha, \beta, \gamma, \ldots$ integer or half integer, respectively. In both cases we will denote by $\mathcal{A}_\Sigma$ the superalgebra generated by the e_i's and the g_α's through eqs.(5.1.4), (5.1.5) (5.1.6).

If one wishes one can introduce a Grassmann variable θ and define the superalgebra by means of objects defined on super Riemann surfaces.

The algebra $\mathcal{A}_\Sigma$ splits according to

$$\mathcal{A}_\Sigma = \mathcal{A}_\Sigma^+ + \mathcal{A}_\Sigma^- + \mathcal{A}_\Sigma^0$$

where $\mathcal{A}_\Sigma^\pm$ are the subalgebras generated by e_i with $\pm i \geq g_0 - 1$ and g_α with $\pm\alpha \geq g - \frac{1}{2}$. These generate superconformal transformations. The complement $\mathcal{A}_\Sigma^0$ generated by e_i with $|i| \leq g_0 - 2$ and g_α with $|\alpha| < g - \frac{1}{2}$ correspond to deformations that change the superconformal structure. Since any deformation can be generated by an element of $\mathcal{A}_\Sigma$, then $\mathcal{A}_\Sigma^0$ is naturally identified with the tangent space to the supermoduli space.

One can easily see that the complex dimension of $\mathcal{A}^0_\Sigma$ is $3g-3+2g-2$, the dimension of the supermoduli space.

Let us now come to the central extension of the $GS-KN$ and $R-KN$ superalgebras. To this end we introduce the following cocycle

$$\varphi(g_\alpha, g_\beta) = \frac{1}{6\pi i}\oint \tilde{\varphi}(g_\alpha, g_\beta) \tag{5.1.9}$$

where the integral is over a contour surrounding P_+ and $\tilde{\varphi}$ is defined as follows.

Let $g = g(z_+)\frac{\partial}{\partial z_+}$ near P_+, then Let ρ and σ have weight $-\frac{1}{2}$ and be holomorphic on Σ except possibly for poles or branch points in $P_\pm$ (with associated branch cut), and let $\rho = \rho(z_+)(dz_+)^{-\frac{1}{2}}$, $\sigma = \sigma(z_+)(dz_+)^{-\frac{1}{2}}$. Then

$$\tilde{\varphi}(\rho,\sigma) = \rho'\sigma' dz_+ \tag{5.1.10}$$

It is immediate to see that $\varphi(g_\beta, g_\alpha)$ verifies the following properties:

(ii) it is independent of the coordinate system

(iii) it satisfies the following cocycle condition:

$$\varphi(\rho,[\sigma,f]) - \varphi(\sigma,[f,\rho]) + \chi(f,\{\rho,\sigma\}) = 0 \tag{5.1.11}$$

iv) it is "local", in the sense that

$$\varphi(g_\alpha, g_\beta) = 0 \quad for \quad |\alpha+\beta| > 2g \tag{5.1.12}$$

as follows from an elementary computation of the zeroes and poles in $P_\pm$.

In terms of expansion coefficients the cocycle can be expressed

$$-\frac{1}{3}\sum_{a,b} g^a_\alpha g^b_\beta \delta_{a+b-g}\left[(\alpha-g)^2 - \frac{1}{4}\right] \tag{5.1.13}$$

Finally we can centrally extend both NS-KN and R-KN superalgebras as follows

$$[e_i, e_j] = \sum_k C^k_{ij} e_k + t\chi(e_i, e_j) \tag{5.1.14}$$

$$[e_i, g_\alpha] = \sum_\beta H^\beta_{i\alpha} g_\beta \tag{5.1.15}$$

$$\{g_\alpha, g_\beta\} = \sum_p B^p_{\alpha\beta} e_p + t\varphi(g_\alpha, g_\beta) \tag{5.1.16}$$

$$[e_i, t] = [g_\alpha, t] = 0. \tag{5.1.17}$$

A few final remarks:

- The cocycles φ is easily calculated in a few cases. For example,

$$\varphi(g_\alpha, g_{2g-\alpha}) = -\frac{1}{3}(\alpha - g)^2 + \frac{1}{12} \tag{5.1.18}$$

- It is possible to show that up to trivial cocycles there is only one cocycle satisfying the "locality" condition (5.1.12)
- the above superalgebras reduce to the usual Neveu-Schwarz and Ramond superalgebras in the genus 0 case.

Now we want to realize the above superalgebra as intrinsic algebras of superstring theories, like we did before for the KN algebra. Our standing point will be now the following energy momentum tensor $T = T^X + T^\psi + T^{gh} + T^{\beta\gamma}$ where T^X and T^{gh} have been defined in the preceeding chapter, whereas

$$T^\psi \equiv -\frac{1}{2}\partial\psi^\mu\psi_\mu, \quad T^{\beta\gamma} \equiv -\frac{1}{2}\gamma\partial\beta - \frac{3}{2}\partial\gamma\beta \tag{5.1.19}$$

moreover we have the supersymmetric current $J = J^{X\psi} + J^{gh}$

$$J^{X\psi} = \psi_\mu \partial X^\mu, \quad J^{gh} = 2c\partial\beta + 3\partial c\beta - \gamma b \tag{5.1.20}$$

$\psi^\mu(Q)$ are fields of weight $\frac{1}{2}$ while $\beta(Q)$ and $\gamma(Q)$ are commuting ghost fields of weight $\frac{3}{2}$ and $-\frac{1}{2}$ respectively. T and J have weight 2 and $\frac{3}{2}$. The geometrical meaning of T and J will become clear later.

We can use the bases $\{e_i\}, \{\omega^i\}, \{g_\alpha\}, \{h^\alpha\}$,etc., in order to expand these fields, giving rise, as usual, to expansion coefficients to be seen later as creation and annihilation operators in suitable Fock spaces.

We have the following expansions

$$\lambda = -\frac{1}{2}: \quad \gamma(Q) = \sum_\alpha \gamma_\alpha g_\alpha(Q), \quad \gamma_\alpha = \frac{1}{2\pi i}\oint_{C_\tau} \gamma(Q) k^\alpha(Q) \tag{5.1.21}$$

$$\lambda = \frac{1}{2}: \quad \psi^\mu(Q) = \sum_\alpha d^\mu_\alpha h_\alpha(Q), \quad d^\mu_\alpha = \frac{1}{2\pi i}\oint_{C_\tau} \psi^\mu(Q) h^{+\alpha}(Q) \tag{5.1.22}$$

$$\lambda = \frac{3}{2}: \quad \beta(Q) = \sum_\alpha \beta_\alpha k^\alpha(Q), \quad \beta_\alpha = \frac{1}{2\pi i}\oint_{C_\tau} \beta(Q) g_\alpha(Q) \tag{5.1.23}$$

Recall that, in the literature, when α is half-integer it is more customary to replace d_α^μ by b_α^μ. Now let us introduce the Poisson brackets

(5.1.24) $$\{\psi^\mu(Q), \psi^\nu(Q')\} = 2\pi\eta^{\mu\nu}\delta_\tau(Q, Q'),$$

(5.1.25) $$[\gamma(Q), \beta(Q')] = 2\pi d_\tau(Q, Q'),$$

where

(5.1.26) $$\delta_\tau(Q, Q') = \frac{1}{2\pi i}\sum_\alpha h_\alpha(Q)h^{+\alpha}(Q')$$

(5.1.27) $$d_\tau(Q, Q') = \frac{1}{2\pi i}\sum_\alpha g_\alpha(Q)k^\alpha(Q')$$

are the usual delta-functions for fields of weight $\frac{1}{2}$ and $-\frac{1}{2}$ respectively. As a consequence of eqs. (5.1.24), (5.1.25) before we have following Poisson brackets for the coefficients of the expansions:

(5.1.28) $$\{d_\alpha^\mu, d_\beta^\nu\} = -i\eta^{\mu\nu}\delta_{\alpha+\beta,0}$$

(5.1.29) $$[\gamma_\alpha, \beta_\beta] = -i\delta_{\alpha\beta}$$

Now let us consider $L_i = L_i^X + L_i^\psi + L_i^{bc} + L_i^{\beta\gamma}$ and $G_\alpha = G_\alpha^{X\psi} + G_\alpha^{gh}$ defined by:

(5.1.30) $$T(Q) = \sum_i L_i\Omega^i(Q), \quad J(Q) = \sum_\alpha G_\alpha k^\alpha(Q)$$

Now L_i^X and L_i^{bc} have been already expressed when we were treating the non supersymmetric case, whereas

(5.1.31) $$L_i^\psi = \frac{1}{2}\sum_{\alpha\beta} d_\alpha \cdot d_{-\beta}F^\beta_{i\alpha}$$

$$L_i^{\gamma\beta} = -\sum_\alpha\sum_\beta H^\beta_{i\alpha}\gamma_\alpha\beta_\beta$$

and

(5.1.32) $$G_\alpha^{X\psi} = \sum_{\beta j} d_{-\beta}\cdot\alpha_j D^\alpha_{\beta j}$$

$$G_\alpha^{gh} = -2\sum_j\sum_k c_j\beta_k H^k_{j\alpha} - \frac{1}{2}\sum_\beta\sum_p B^p_{\alpha\beta}\gamma_\beta b_k$$

where

$$F^{\beta}_{i\alpha} = \frac{1}{2\pi i}\oint_{C_\tau} 2(h^{+\beta}\partial h_\alpha - h_\alpha \partial h^{+\beta})e_i = \frac{1}{2\pi i}\int L_{e_i} h_\alpha h^{+\beta} \tag{5.1.33}$$

$$D^{\alpha}_{\beta j} = \frac{1}{2\pi i}\oint_{C_\tau} h^{+\beta}\omega^j g_\alpha \tag{5.1.34}$$

and we recall that α, β are either both integers (Ramond sector) or both half-integer (Neveu-Schwarz sector). Moreover $h^{+\beta} = h_{-\beta}$. Then the Poisson brackets for L_i and G_α are:

$$[L_i, L_j] = -i\sum_k C^k_{ji} L_k \tag{5.1.35}$$

$$[L_i, G_\alpha] = -i\sum_k H^{\beta}_{i\alpha} G_\beta \tag{5.1.36}$$

$$\{G_\alpha, G_\beta\} = -i\sum_p B^p_{\alpha\beta} L_p \tag{5.1.37}$$

These are a replica of eqs. (5.1.4), (5.1.5) (5.1.6), apart from the opposite sign in the first equation and the $-i$ factor.

5.2. Quantization of the superalgebra

Like we did in chapter 4 for the KN algebra, we proceed now to quantizing the superalgebra according to the usual recipe.

We need so to introduce a normal ordering for the relevant operators .

We will take as annihilation operators d_α and γ_α for $\alpha l N$ and β_α for $\alpha \geq N$, and as creation operators the complementary set. With this prescription the algebra of $: L_i :$ and $: G_\alpha :$ can be expressed:

$$[: L_i :, : L_j :] = \sum_k C^k_{ji} : L_k : + \hat{\chi}_{ij} \tag{5.2.1}$$

$$[: G_\alpha :, : L_i :] = \sum_\beta H^\beta_{i\alpha} : G_\beta : \tag{5.2.2}$$

$$\{: G_\alpha :, : G_\beta :\} = \sum_k B^k_{\alpha\beta} : L_k : + \hat{\varphi}_{\alpha\beta} \tag{5.2.3}$$

$\hat{\chi}_{ij}$ can be decomposed as:

$$\hat{\chi}_{ij} = D\chi^X_{ij} + \chi^{bc}{}_{ij} + D\chi^\psi_{ij} + \chi^{\beta\gamma}_{ij} \tag{5.2.4}$$

D is the target space dimension., χ^X_{ij} is eq. (4.1.25) of chapter 4 and χ^{gh}_{ij} has been shown in chapter 4 to be equivalent to $-26\chi(e_i, e_j)$. The last two terms can be calculated obtaining:

$$\chi^\psi_{ij} = \frac{-1}{2} \sum_{\alpha \leq N} \sum_{\beta > N} (F^\alpha_{i,\beta} F^\beta_{j\alpha} - F^\alpha_{j,\beta} F^\beta_{i\alpha}) \tag{5.2.5}$$

$$\chi^\gamma_{ij} = \sum_{\substack{\alpha < N \\ \beta \geq N}} (-H^\beta_{i,\alpha} H^\alpha_{j,\beta} + H^\beta_{j,\alpha} H^\alpha_{i,\beta}). \tag{5.2.6}$$

Similarly $\hat{\varphi}_{\alpha\beta}$ can be decomposed according to

$$\hat{\varphi}_{\alpha\beta} = D\varphi^{X\psi}_{\alpha\beta} + \varphi^{gh}_{\alpha\beta} \tag{5.2.7}$$

with

$$\varphi^{X\psi}_{\alpha\beta} = \sum_\gamma \sum_{ij} D^\alpha_{-\gamma j} D^\beta_{\gamma i} (\gamma_{ji} \Theta^+(\Sigma^-) - \gamma_{ji} \Theta_+(\gamma - N)) \tag{5.2.8}$$

$$\varphi^{gh}_{\alpha\beta} = (\sum_{i>0} \sum_{\gamma \leq 0} - \sum_{\gamma > 0} \sum_{i \leq 0})(H^\gamma_{i\alpha} B^i_{\gamma\beta} + \{\alpha \leftrightarrow \beta\}) \tag{5.2.9}$$

where $\theta^+(\Sigma_-)$ is 1 when $(i,j) \in \Sigma_-$ and 0 elsewhere. Observe that (5.2.8) can also be written in the case $\Sigma_+ = \{\{i,j\}, \quad i < N\}$ as

$$\sum_{j \geq N} \frac{1}{2\pi i} \int h_\gamma \omega^j g_\alpha \frac{1}{2\pi i} \int dA_j h^{+\gamma} g_\beta - \sum_{\gamma \geq N} \frac{1}{2\pi i} \int [g_\beta, A_j] h^{+\gamma} \frac{1}{2\pi i} \int \{g_\alpha, h_\gamma\} \omega^j$$

or also

$$\{ \sum_{\gamma < N, j \geq N} - \sum_{\gamma \geq N, j < N} \} \frac{1}{2\pi i} \int g_\beta dA_j h^+ \gamma \frac{1}{2\pi i} \int g_\alpha h_\gamma \omega^j \tag{5.2.10}$$

The cocycles just defined, are up to trivial cocycles, proportional to the cocycles $\chi(e_i, e_j)$ and $\varphi(g_\alpha, g_\beta)$. In fact they have just the right structure for a cocycle. The proportionality constants can be computated simply by observing what they are in a particular case. Considering for χ_{ij} the case $i + j = 2g_0$ and for $\varphi_{\alpha,\beta}$ the case $\alpha + \beta = 2g$ we obtain immediately that up to trivial cocycles

$$\hat{\chi}_{ij} = [(\frac{D}{2} + D) - 26 + 11]\chi_{ij} = (\frac{3}{2}D - 15)\chi(e_i, e_j), \tag{5.2.11}$$

and

$$\hat{\varphi}_{\alpha\beta} = -(\frac{3}{2}D - 15)\varphi(g_\alpha, g_\beta) \tag{5.2.12}$$

This is not satisfactory from our point of view. We will see next that this numerology can be understood more clearly from the point of view of the representations of the superalgebra.

Now without entering in too much detail we can easy generalize the BRST operator already defined to the case of the super algebra. We define

$$\begin{aligned} Q = \frac{1}{2\pi i} \oint_{C_\tau} &\{c\partial X^\mu \partial X_\mu + \frac{1}{2}(c\partial\psi^\mu\psi_\mu - \gamma\psi_\mu\partial X^\mu) \\ &+ \partial c c b - \frac{3}{2}\partial c \gamma \beta - \gamma c \partial\beta + \gamma^2 b\} \end{aligned} \tag{5.2.13}$$

or equivalently

$$\begin{aligned} Q = \frac{1}{2\pi i} \oint_{C_\tau} &(\{T^X + T^\psi\}(Q)c(Q) + J^{X,\psi}(Q)\gamma(Q) + \frac{1}{2}B(Q)[C(Q), C(Q)] \\ &-\beta(Q)[C(Q), \gamma(Q)] - \frac{1}{2}\{\gamma(Q), \gamma(Q)\}b(Q)) \end{aligned} \tag{5.2.14}$$

In terms of expansion coefficients

$$Q = \sum_i L_i^{X\psi} c_i + \sum_\alpha \gamma_\alpha G_\alpha^{X\psi} + \frac{1}{2} \sum_{i,j} C_{ij}^k c_i c_j b_k + \tag{5.2.15}$$

$$- \sum_{i,\alpha} H_{i\alpha}^k c_i \gamma_\alpha \beta_k - \frac{1}{2} \sum_{\alpha\beta} B_{\alpha\beta}^p \gamma_\alpha \gamma_\beta b_p$$

After quantization we have to consider $\hat{Q} =: Q :$. We obtain

$$\hat{Q}^2 = \{\hat{Q}, \hat{Q}\} = \sum_{i,j} \hat{\chi}_{i,j} : c_i c_j : + \sum_{\alpha,\beta} \hat{\varphi}_{\alpha\beta} : \gamma_\alpha \gamma_\beta : \tag{5.2.16}$$

From eq. (5.2.11), (5.2.12) we have that up to trivial cocycles $\hat{Q}^2 = 0$ for $D = 10$.

5.3. Representations of the superalgebra on semiinfinite forms

We want now to construct representations space for the superalgebra. In order to do that we have to define binary operations among the various KN tensorial fields.

First we want to define a coupling between forms of arbitrary weight which generalizes the usual Lie derivative, on Riemann surfaces. Let Λ and Ω two meromorphic tensor fields of weight λ and ω respectively.

Define

$$L_\Lambda \Omega \equiv [\Lambda, \Omega] \equiv -\lambda\Lambda\partial\omega + \omega\Omega\partial\Lambda \tag{5.3.1}$$

where ∂ denotes differentiation with respect to the holomorphic coordinate . $[\Lambda, \Omega]$ is a globally defined $(\lambda + \omega + 1)$ meromorphic tensor field.

Moreover $[\Lambda, \Omega]$ is obviously antisymmetric, and reduces to the usual Lie derivative, when Λ is a vector field.

In particular we can take Λ to be a spinor field, i.e. a tensor field of weight $-\frac{1}{2}$. The action of such a Λ on a tensor field will increase its weight by $\frac{1}{2}$.

This operation enjoys of the properties

$$L_\varphi(\tau \otimes \omega) = L_\varphi \tau \otimes \omega + \tau \otimes L_\varphi \omega \tag{5.3.2}$$

and

$$[L_\rho, L_\tau] = L_{[\rho,\tau]} \tag{5.3.3}$$

The first follows immediately.

For the second we have to compare

$$\begin{aligned} L_\rho L_\tau \omega = & L_\rho\{-\tau\partial\omega deg(\tau) + \omega\partial\tau deg(\omega)\} = -\rho deg(\rho)\times \\ & \times(-\tau\partial^2\omega deg(\tau) - \partial\tau\partial\omega deg(\tau) + \partial\omega\partial\tau deg(\omega) + \omega\partial^2\tau deg(\omega)) + \\ & +\partial\rho(deg(\tau + deg(\omega) + 1)(-\tau\partial\omega deg(\tau) + \omega\partial\tau deg(\omega)) + \{\rho \leftrightarrow \tau\} \end{aligned} \tag{5.3.4}$$

with

$$\begin{aligned} L_{[\rho,\tau]}\omega = & L_{\{-\rho deg(\rho)\partial\tau + \tau deg(\tau)\partial\rho\}}\omega = \\ & = (deg(\rho + deg\tau + 1)(-\rho deg(\rho\partial\tau + \tau deg(\tau\partial\rho)\partial\omega + \\ & +deg(\omega\{-\rho deg(\rho)\partial^2\tau - \partial\rho\partial\tau deg(\rho) + \partial\tau deg(\tau)\partial\rho + \tau deg(\tau)\partial^2\rho\} \end{aligned} \tag{5.3.5}$$

which a little rearrangement show to be equal.

There is moreover the connecting property with the tensorial product:

(5.3.6) $$2g \otimes L_{g'}\omega + 2g' \otimes L_g\omega = L_{\{g,g'\}}\omega$$

This properties guarantee that we can represent the superalgebra in the space $F^\lambda \oplus F^{\lambda+\frac{1}{2}}$, λ integer, in the following way:

$$g_\alpha\omega \equiv L_{g_\alpha}\omega$$

if $\omega \in F^\lambda$

$$g_\alpha\omega \equiv 2g_\alpha \otimes \omega$$

if $\omega \in F^{\lambda+\frac{1}{2}}$. Defining $\hat{f}_p = f_p^{\lambda+\frac{1}{2}}$ and omitting the index λ we have

(5.3.7) $$[g_\alpha, f_p] = S^\beta_{\alpha,p}\hat{f}_\beta$$

and

(5.3.8) $$\{g_\alpha, \hat{f}_\beta\} = T^p_{\alpha,\beta} f_p$$

where

(5.3.9) $$S^\beta_{\alpha,p} = \frac{1}{2\pi i}\int [g_\alpha, f_p]\hat{f}^{+\beta}$$

and

(5.3.10) $$T^p_{\alpha,\beta} = \frac{1}{2\pi i}\int \{g_\alpha, \hat{f}_\beta\} f^{+p}.$$

In particular $S^{\alpha+j-g_0}_{\alpha j} = \frac{1}{2}(j - S(\lambda)) + \lambda(\alpha - S(-\frac{1}{2}))$ and $T^k_{\alpha j} = 2$. As usual notice the g_0-gradation of the commutator and the $\frac{g}{2}$ gradation of the anticommutator.

We are now ready to discuss the representation of the central extension of the superalgebra, or else the representation of the superalgebra on semiinfinite forms.

The task is not easy . But anyway we will search to explain our understanding of the problem.

We could consider the obvious generalization of the usual construction on semi-infinite forms and take as vacuum of our theory the following object

(5.3.11) $$\varphi_m = f_m \wedge f_{m+1} \wedge \ldots$$

But this does not make any sense, because when we try to apply g_α on objects like this we obtain infinities.

To obtain a meaningful result we have to consider the following vacuum

(5.3.12) $$\varphi_M = f_M \bigodot \hat{f}_M \wedge f_{M+1} \bigodot \hat{f}_{M+1} \wedge ....$$

where $\bigodot$ denotes a sort of $\mathbf{Z}_2$-graded tensor product as will become clear.

We now define the action of g_α on an elementary block in the semiinfinite product

$$g_\alpha(h \bigodot k) = g_\alpha h \bigodot k + (-1)^{2k} h \bigodot g_\alpha k)$$

where

$$k = deg(h) + deg(k)$$

In particular

(5.3.13) $$\{g_\alpha, g_\beta\}(f_p \bigodot \hat{f}_p) = \{g_\alpha, g_\beta\} f_p \bigodot \hat{f}_p - f_p \bigodot \{g_\alpha, g_\beta\} \hat{f}_p\}$$

The cocycle corresponding to this definition is

(5.3.14) $$\sum_{k \geq M, p < M} (-T^p_{\beta k} S^k_{\alpha p} + S^p_{\beta k} T^k_{\alpha p}) + \{\alpha \leftrightarrow \beta\}$$

We can compute it. For instance, with the usual tricks

$$\sum_{k \geq M, p < M} S^p_{\beta k} T^k_{\alpha p} = (\frac{1}{2\pi i})^2 \int 2 g_\alpha \hat{f}_\beta f^{+k} \int \frac{1}{2}(g_\beta \partial f_k + \lambda \partial g_\beta f_k) \hat{f}^{+p} =$$

$$\sum_{k \geq M, p < M} g^a_\alpha g^d_\beta \hat{f}^e_k \hat{f}^{+k}_c f^b_p f^{+f}_p \delta_{a+b-c-\frac{g}{2}} \delta_{d+e-f-g_0} (e - S + 2\lambda(d - g + \frac{1}{2}))$$

which can be shown to be equal to

$$\sum_{c \geq M, b < M} g^a_\alpha g^d_\beta \delta_{a+b-c-\frac{g}{2}} \delta_{d+c-b-g_0} (c - S + 2\lambda(d - g + \frac{1}{2}))$$

or else to

$$\sum_{b=1}^{a-\frac{g}{2}} g^a_\alpha g^d_\beta \delta_{a+d-2g} (b + M - 1 - S + 2\lambda(g - a - 1/2))$$

Summing with the other contributes we obtain

$$
\begin{aligned}
&\sum_{a,b}\{g^a_\alpha g^d_\beta \delta_{a+d-2g}(\frac{g}{2}-a)^2[\frac{1}{2}-2\lambda]-(\frac{g}{2}-a)\times\\
&[\frac{1}{2}+M-1-S+\lambda+\lambda g+(\frac{g}{2}-d)^2[\frac{1}{2}-2\lambda]-(\frac{g}{2}-d)\times\\
&[\frac{1}{2}+M-1-S+\lambda+\lambda g]\}=\sum_{a,b} g^a_\alpha g^d_\beta \delta_{a+d-2g}\times\\
&(g-a)^2[1-4\lambda]+\frac{g^2}{4}(1-4\lambda)+g(\frac{1}{2}+M-1-S+\lambda(1+g))
\end{aligned}
\tag{5.3.15}
$$

which a part from a trivial cocycle is just the right supersymmetric cocycle with central charge $-3(1-4\lambda)$

We omitted an important point. In fact this must be a representation of the algebra of the e_i's. But this is not true, naifely speaking. In fact this representation gives no representation of the e_i's, but corresponds to the difference of the two representations of the e_i's on semiinfinite forms of weight λ and $\lambda+\frac{1}{2}$ respectively.

The corresponding central charge will be in fact

$$
-2(6\lambda^2-6\lambda+1)+2(6(\lambda+\frac{1}{2})^2-6(\lambda+\frac{1}{2})+1)=12\lambda-3. \tag{5.3.16}
$$

Now we want to understand better the numerology hidden in the cocycles (5.2.5), etc.. Recall that if R^q_{ip} are the structure constants for the representation of the e_i's on weight λ tensor fields, up to trivial cocycle, for each N we have

$$
-2(6\lambda^2-6\lambda+1)\chi_{ij}\approx\{\sum_{k\geq N,l<N}-\sum_{l\geq N,k<N}\}R^k_{il}R^l_{jk} \tag{5.3.17}
$$

hence cocycles (5.2.5) and (5.2.6) have exactly the prescribed form, a part from a factor of $-\frac{1}{2}$ in (5.2.5), with λ respectively $\frac{1}{2}$ and $-\frac{1}{2}$ The corresponding proportionality constants will therefore be $-\frac{1}{2}\times-1=\frac{1}{2}$ and -11 respectively.

Analogous considerations hold for the cocycle (5.2.9). It has just the form prescribed for a superalgebra cocycle with $\lambda=-1$, being in this case

$$
S^\beta_{\alpha,p}=\frac{1}{2\pi i}\int[g_\alpha,e_p]k^\beta \tag{5.3.18}
$$

and

$$
T^p_{\alpha,\beta}=\frac{1}{2\pi i}\int\{g_\alpha,g_\beta\}\Omega^p \tag{5.3.19}
$$

So it will be proportional to $\varphi_{\alpha\beta}$, the proportionality constant being $-3+12\lambda=-15$.

What is left is precisely the cocycle (5.2.8). In order to prove that it is proportional to the right cocycle we have to be a little careful.

In the case of $\lambda = 0$ we have the following coefficients to make the cocycle for the superalgebra.

$$S^{\beta}_{\alpha p} = \frac{1}{2\pi i}\int [g_\alpha, A_p]h^{+\beta} \tag{5.3.20}$$

and

$$T^{p}_{\alpha\beta} = \frac{1}{2\pi i}\int \{g_\alpha, h_\beta\}\omega^{p} \tag{5.3.21}$$

It is now easy noting that we have at our disposal also the expression (5.2.10) that (5.2.8) is a cocycle and the proportionality constants connecting them with $\varphi_{\alpha\beta}$ is just $-\frac{3}{2}$. So we have understood the numerology.

Final remark.

One can ask why precisely what has been described happens.

We don't understood completely the mechanism. But let us describe, for instance, what happen for the superghost part.

The quantum operators β and γ realize commutation relations. However when we take commutators of objects bilinear in γ and β we can substitute to commutation relations, anticommutation relations getting the same algebra, due to the following identity

$$\begin{aligned}&[\beta_\alpha\gamma_\delta, \beta_\rho\gamma_\tau] = \beta_\alpha\gamma_\tau[\gamma_\delta, \beta_\rho] - \beta_\rho\gamma_\delta[\gamma_\tau, \beta_\alpha] = \\ &\beta_\alpha\gamma_\tau[\gamma_\delta, \beta_\rho]_+ - \beta_\rho\gamma_\delta[\gamma_\tau, \beta_\alpha]_+\end{aligned} \tag{5.3.22}$$

So we can get easily the same algebra by representing β and γ as anticommuting object on semiinfinite forms. This is easily done as usual for ghosts and antighosts.

The $L_i^{\beta\gamma}$'s constructed in this way is exactly the same as the L_i's we can get representing the e_i's on semiinfinite forms in $\mathcal{F}^{\frac{3}{2}}$ with the help of interior and exterior product. Hence it is obvious we get the corresponding cocycle.

Chapter 6

Kac Moody algebras over Riemann surfaces

6.1. Some basic facts about Kac Moody algebras and higher genus generalization

We describe now a Kac-Moody algebra and Sugawara construction, and in the next section we will generalize it to non trivial Riemann surfaces.

This construction is very important because it gives a criterion for extablishing the unitarity of the Virasoro representations.

Let's recall what happens in genus $g = 0$, before going to the study of non trivial Riemann surfaces.

Consider a Lie algebra $g \subset gl(n, \mathbf{C})$. We can consider the space of loops from S^1 to g, i.e. the space of maps with finite Fourier series from S^1 to g with associative product defined pointwise.

We can give to this space the structure of Lie algebra (infinite dimensional) simply by taking as commutator the algebraic commutator of the associative product defined before.

We call this algebra $\widetilde{g}$.

A typical element will have the form

$$a(t) = \sum_k z^k a_k \quad a_k \in gl(n) \tag{6.1.1}$$

where the sum over k runs over a finite subset of $\mathbf{Z}$. Let T_a, $a = 1, ..., m$ be a system of generators of g. Then a system of generators for $\tilde{g}$ is easily given:

$$a(k) \equiv T^a z^k$$

If the Lie algebra has structure constants f^{ab}_c

$$[T^a, T^b] = f^{ab}_c T^c$$

then the commutation relations for the generators of $\tilde{g}$ are

$$[a(k), b(m)] = [a, b](k + m) = f^{ab}_c \delta_{p,k+m} c(p)$$

We can construct a central extension of $\tilde{g}$ with the help of the two cocycle α defined on generators as follows

$$\alpha(a(k), b(m)) = k\delta_{k+m} tr(T^a T^b) \tag{6.1.2}$$

It is obviously a two cocycle being antisymmetric and satisfying the Jacobi identity, due to the properties of the trace.

So the centrally extended algebra $\hat{g}$, which is called an affine Kac-Moody algebra, read

$$[a(k), b(m)] = [a, b](k+m) + c\delta_{k+m} k tr(T^a T^b) \tag{6.1.3}$$

$$[a(k), c] = 0$$

c being the generator of the centre.

Notice that, for g abelian the algebra reduces to an oscillator algebra. Out of the algebra $\hat{g}$, we can reproduce Virasoro generators in the following way: define the currents $J^a(z) = a(q) z^{-1-q}$. We can then construct the Sugawara energy-momentum tensor.

$$T(z) = \frac{1}{2} : J^a(z) J^a(z) :$$

it is a 2-differential field. The normal ordering is defined on the components T_k defined as

$$T_k = \int J^a(z) z^{k+1}$$

Within this definition they will be

$$T_k \equiv \frac{1}{2} \sum_{a,j} : a(-j) a(j+k) :$$

the normal ordering being defined in the following way ($N \in \mathbf{Z}$):

$$: a(p) b(q) := \begin{cases} a(p) b(q) & p < N \\ b(q) a(q) & p \geq N \end{cases} \tag{6.1.4}$$

For any element $x \in g$ the operators just defined satisfy (for $N = 0$)

$$[x(n), T_k] = (c_v + g)\, n\, x(n+k)$$

and

$$[T_n, T_k] = (c_v + g)(n-k) T_{n+k} + \delta_{n+k} \frac{n^3 - n}{12} dimg\, c_v\, (c_v + g).$$

Here c_v is the second Casimir of the adjoint representation $c_v \delta_{ab} = f^{acd} f^{bcd}$ and g is defined as $g\delta_{ab} = tr(t^a t^b)$ where t^a are the Lie algebra generators in the matrix representation we are considering.

Hence we can define

$$L_k = \frac{-1}{c_v + g} T_k$$

in such a way that the L_k satisfy the Virasoro algebra with central charge

$$\frac{c_v \, dim \, g}{c + g} \tag{6.1.5}$$

In particular, if the representation space on which $\hat{g}$ acts is unitary, then the Virasoro representation is unitary.

This extablishes a very useful criterion in determining the unitarity of the Virasoro representations and has allowed Kac and others to prove the determinant Kac formula. This construction of the Virasoro generators out of generators of $\hat{g}$ is called Sugawara construction.

Notice the strict analogy with the oscillator algebra.

Let us now come back to Krichever tecniques.

We will in fact construct now the analogous of affine Kac-Moody algebra and of Sugawara construction for non trivial Riemann surfaces.

Let $\mathcal{A}^{\Sigma}$ be the algebra of Krichever-Novikov functions.

We can tensorize it with g to get an infinite dimensional Lie algebra $\mathcal{A}^g$, whose commutator is the algebraic commutator of the pointwise defined product.

$\mathcal{A}^{\Sigma} \otimes g$ is the generalized affine Kac-Moody algebra.

By defining generators

$$a(n) \equiv T^a \otimes A_n$$

we obtain the following structure for the algebra just defined:

$$[a(k), b(m)] = [a, b](p) \alpha^p_{km} = f^{ab}_c \alpha^p_{km} c(p)$$

We can construct a central extension of $\mathcal{A}^g$ with the help of the two cocycle α defined on generators as follows

$$\alpha(a(k), b(m)) = \gamma_{km} tr(T^a T^b) \tag{6.1.6}$$

It is obviously a two cocycle being antisymmetric and satisfying the Jacobi identity, due to the properties of the trace and to the fact that γ_{mn} is just the cocycle for the functions space.

So the centrally extended algebra $\hat{\mathcal{A}}^g$, reads

(6.1.7)
$$[a(k), b(m)] = [a,b](p)\alpha^p_{km} + c\gamma_{km} tr(T^a T^b)$$
$$[a(k), c] = 0$$

c being the generator of the centre.

Observe that we have taken the right cocycle, without thinking about, and understand how to represent this algebra.

It is possible to follow another way. We can construct a representation of the $\alpha(p)$'s on semiinfinite forms as follows.

For simplicity assume now $g = gl(M)$.

Let $V = C^M$ be the fundamental representation space of g.Take the canonical basis of V, $v_1, ..., v_M$ and a system of generators of $gl(M)$

$$\{e_{ij}\}_{kl} = \delta_{ik}\delta_{jl}.$$

$\mathcal{A}^g$ acts on $\mathcal{A} \otimes V$ in an obvious way. Define the basis $\nu_{Mk+j} = \mathcal{A}_k v_j$ in $\mathcal{A} \otimes V$. So

$$E_{ij}(p)\nu_{Mk+j} = \nu_{Mp+i}\alpha^q_{pk}.$$

$e_{ij}(p)$ being defined, according to the usual notation as $E_{ij} \otimes \mathcal{A}_p$

There is, and it is easy to see from this expression, an inclusion of $\mathcal{A}^g$ in a_∞. Let us denote by τ this inclusion. We have trivially

$$\tau(E_{ij}(p)) = \sum_{k,q} E_{Mq+i, Mk+j}\alpha^q_{pk}$$

The corresponding cocycle will be

(6.1.8)
$$\alpha(\tau(E_{ij}(p)), \tau(E_{rs}(q)) = \sum_{k,l,m,t} \alpha(E_{Mk+i, Ml+j}, E_{Mm+r, Mt+s}) = tr(E_{ij}E_{rs})\gamma_{mn}$$

so that extending by linearity: $\alpha(f(p), g(q)) = \int df\, g\, tr(ab)$ So the cocycle we had chosen coincides with this one. It is possible to construct the same representation on semiinfinite forms in a way that may be clearer.

Define the following vacuum semiinfinite form

$$\varphi_0^{mV} \equiv f_m \otimes v_1 \wedge f_m \otimes v_2 \wedge ...... \wedge f_m \otimes v_M \wedge f_{m+1} \otimes v_1 \wedge f_{m+1} \otimes v_2 \wedge ....$$

and define the action of $a(m)$ on it by the Leibnitz rule. It is easy to verify, similarly to what happens for the Heisenberg algebra, that this definition can be corrected, via normal ordering for instance, in such a way to get a representation of $\mathcal{A}^g$ with cocycle given by $\gamma_{mn} tr(\theta^a \theta^b)$, θ_i being the generators of the representation of g we are considering.

Notice that, for g abelian the algebra reduces to the Heisenberg algebra, which we discussed previously. Out of the algebra $\hat{\mathcal{A}}^g$, we can reproduce Krichever generators generalizing the Sugawara construction in the following way.

We consider again the Sugawara energy momentum tensor ([Sug])

$$T(Q) = \frac{1}{2} : J^a(Q) J^a(Q) : \tag{6.1.9}$$

where now

$$J^a(Q) \equiv \sum_q a(q) \omega^q$$

and its "momenta"

$$T_i = \frac{1}{2\pi i} \oint T(Q) e_i(Q) = \frac{1}{2} l_i^{pq} : a(p) a(q) : \tag{6.1.10}$$

l_i^{pq} has been already defined by

$$l_i^{pq} = \frac{1}{2\pi i} \oint e_i \omega^p \omega^q \tag{6.1.11}$$

whereas the normal ordering is defined as in the genus 0 case. Let us first compute $[T_i, b(k)]$:

$$[T_i, b(k)] = c_v \Theta_{ik}^l b(l) - g S_{ik}^l b(l) \tag{6.1.12}$$

where

$$S_{ik}^l = \frac{1}{2\pi i} \oint \omega^l e_i dA_k, \quad \Theta_{ik}^l = \Bigg(\sum_{\substack{p \geq N \\ q < N}} - \sum_{\substack{p < N \\ q \geq N}} \Bigg) l_i^{pr} \alpha_{pk}^q \alpha_{rq}^l \tag{6.1.13}$$

It has been already proven that

$$\Bigg(\sum_{\substack{p \geq N \\ q < N}} - \sum_{\substack{p < N \\ q \geq N}} \Bigg) \omega^p(Q) \omega^q(Q') A_p(Q') A_q(Q) = d' \Delta(Q', Q) \tag{6.1.14}$$

As a consequence of eq. (6.1.14) $\Theta_{ik}^l = -S_{ik}^l$, and

$$[T_i, b(k)] = -(c_v + g) S_{ik}^l b(l) \tag{6.1.15}$$

Using eq. (6.1.15) it is now not hard to find

$$[T_i, T_j] = (c_v + g)\Big(\big(l_i^{pq} S^k_{jq} - l_j^{pq} S^k_{iq}\big) : b(p)b(k) : + c_v \, dim \, g \hat{\chi}_{ij} \Big) \tag{6.1.16}$$

where

$$\hat{\chi}_{ij} = \frac{1}{2}\Big(\sum_{\substack{p \geq N \\ q < N}} - \sum_{\substack{p < N \\ q \geq N}} \Big) S^q_{ip} S^p_{jq} \tag{6.1.17}$$

hence χ_{ij} coincides with the usual cocycle for vector fields for a suitable choice of the projective connection. In fact S^l_{ik} are just the "structure constants" for the representations of the e_i's on the functions via Lie derivative. We proved, in chapter **2** that objects like (6.1.17) are cohomologous (in the sense of Lie algebra cohomology), apart from a proportionality constant, to the usual cocycle for vector fields for *any* choice of projective connection.

The proportionality constant is in general

$$-2\{6\lambda^2 - 6\lambda + 1\}$$

λ being the weight of the tensor fields on which the e_i's act.

In this case, $\lambda = 0$, due also to the $\frac{1}{2}$ factor in (6.1.17) the proportionality constant is just 1.

But using the definitions (6.1.11) and (6.1.13) one easily verifies that

$$l_i^{pq} S^k_{jq} - l_j^{pq} S^k_{iq} = -C^s_{ij} l^{pk}_{i+j-s}$$

So eq. (6.1.16) becomes

$$[T_i, T_j] = (c_v + g) C^s_{ji} T_{i+j-s} + c_v dim g (c_v + g) \chi_{ij} \tag{6.1.18}$$

By using the same redefinition as in the genus 0 case

$$L_k = \frac{-1}{c_v + g} T_k$$

we get that the L_i satisfy the Krichever algebra with central charge

$$\frac{c_v \, dim \, g}{c_v + g} \tag{6.1.19}$$

This completes our construction of a Virasoro-like algebra over a Riemann surface of genus g by means of the Sugawara Ansatz.

Notice that the same can be done by constructing the Sugawara energy-momentum tensor out of femionic fields, but because this will give us no more information we will skip this point.

Conclusions

This conclusion will not be a summary of what has been done.

It will be essentially a list of open problems.

The first point to be considered is the relevance of the Krichever-Novikov algebras and of the Krichever-Novikov formalism for quantizing a string theory.

It turn out that the numerology we discover by working in the Krichever-Novikov approach is essentially (for instance look at central charges and critical dimensions) the same as for the Virasoro algebra and the like. What has to be understood now is what kind of further or different information contains the Krichever-Novikov approach. This is not easy, also due to the complications in doing computations in the Krichever-Novikov approach.

After all one can ask himself whether there is an isomorphism between the Virasoro and Krichever algebras. This question can arise from the fact that these two algebras can be viewed as subalgebras of a bigger Lie algebra a_∞. We can easily answer by saying that, if from one side, they are subalgebras of the same algebra, from the other side the corresponding inclusion are different.

Another aspect of the difference between Krichever and Virasoro can be inferred by thinking about what we said concerning the hamiltonian vector field. The genus zero case hides the nature of the hamiltonian vector field. The true nature of this field does not appear. There is so much degeneracy that this field is confused with a Virasoro field. This does not happen in higher genus.

So our opinion is that a lot more can be obtained in the Krichever-Novikov formalism.

Another aspect that need clarification is the importance (or not) of trivial cocycles. In this work we always skipped this problem. If it is true from a mathematical point of view, that trivial cocycles do not matter in representation theory of centrally extended algebras, from a physical point of view, things may be different.

In fact when we require the total central charge to be zero, we have to take into account also trivial cocycles if we want to know the exact redefinition of the L_i's, which makes the algebra exactly close.

The exact definition of the L_i's might determine their expectation values, and so can be physical relevant.

Another point to which we did not address our attention is that of the *boson-fermion correspondence.*

Almost all work we have done in representing the Krichever-Novikov algebra is on semiinfinite forms. We have seen, just briefly, that another kind of representation can be considered. If, from one side, we know that there is an isomorphism between the two representations, on the other side, we have not realized explicitly this correspondence, due also the complications in the calculations, which arise in higher genus.

There is still more. We do not treated the vertex representation which plays a crucial rôle in Physics, for instance in the BPZ program.

A remark concerns also the superalgebra representations. We understood how to represent it on semiinfinite forms but we skipped the important problem of how to represent the quantum operators in which it can be decomposed on semiinfinite forms, especially for the matter part. Now we recall one limit of what has been done, in connection with string theory. Precisely in constructing the algebra starting from the energy-momentum tensor we worked in a flat target manifold. The generalization to non flat target reveals a lot of difficulties from one side in making computations, from the other side in understanding the results.

Also the unitarity problem for the KN algebra representations, that needs serious attention has not found place in our work. Anyway let's conclude with some optimistic words.

Our opinion is that, a part from all difficulties one can meet in generalizing things from the genus zero case via the Krichever formalism, the Krichever-Novikov (and also our) method is from one side physically appealing and from the other side mathematically consistent. Moreover it seem to us the more direct way to generalize the zero genus case to any non trivial Riemann surface and as such it needs attention.

We hope we have elucitaded with this work some aspects which can be useful in progress our understanding.

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