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THE INFLATIONARY SCENARIO AND THE FORMATION OF STRUCTURE

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Introduction

The origin of the primordial inhomogeneities of the energy density which gave rise to the observable large scale structure of the universe is an old and yet unresolved problem in cosmology. In the frame of the hot Big Bang theory it is rather difficult to understand the origin of the fluctuations from which galaxies, clusters, superclusters and voids have been originated by gravitational collapse, because the matter which comprise a typical galaxy, for example, first came into causal contact about a year after the Big Bang. It is very hard to see how galaxy size fluctuations could have formed after that, but even harder to see how they could have formed earlier.

In the past few years, particle physics models have provided some possible explanations about how fluctuations in the energy density could have arisen during phase transitions processes occurring at very high energy, these are quantum fluctuations in inflationary universe models and cosmic strings. Within these theories the primordial spectrum of density fluctuations have been computed from first principles for the first time. I will focus here on the first of the two possibilities.

In chapter I, a review of the description of the energy density fluctuations is presented. The scale invariant spectrum of the primordial perturbations, which is predicted in most inflationary models, together with its predictions for baryon dominated, hot or cold dark matter dominated universes are discussed. Finally, some other proposals for the origin of density fluctuations are presented.

Chapter II deals with the gauge invariant formalism for studying the fluctuations in the energy density and the associated fluctuations in the metric in a self consistent way. Its application to the inflationary models and an approximate conservation law, which make it possible to relate the amplitude of the fluctuations in different eras in a simple way, is discussed in detail.

Chapter III is devoted to the inflationary universe scenario. Emphasis is put in the topics related to the study of the generation of density fluctuations.

Finally, in Chapter IV the computation of the spectrum of the density fluctuations generated by quantum fluctuations during the inflationary phase is performed. The results are applied to the particular case of the new inflationary model for the Higgs field which spontaneously breaks the symmetry in the SU(5) GUT with a Coleman Weimberg potential and to the chaotic inflationary model with a $\lambda\phi^4$ potential.

I- ENERGY DENSITY FLUCTUATIONS

I-1 General description and properties

In a universe composed by matter and radiation, the fluctuations in the energy density can be described in terms of the fluctuations of each component separately. So, we could consider fluctuations in the radiation density, while keeping the matter density fixed or vice-versa. Any density fluctuation will be a composition of both of them. However, it turns out that this is not the more convenient decomposition for perturbations. There exist a more physically meaningful one, which is generally used [1].

The two fundamental types of fluctuations considered are:

- Adiabatic fluctuations, which have the property that temperature fluctuations are proportional to density fluctuations, i.e. $\frac{\delta T}{T} \sim \frac{\delta \rho}{\rho}$. In this way, the ratio of the number of photons and baryons in a small volume remains everywhere the same. Since this ratio is in fact the entropy per baryon, adiabatic fluctuations leave the entropy per baryon constant everywhere, while the density may change from point to point.

- Isothermal fluctuations, corresponding to fluctuations in the matter density which keeps the photon density constant everywhere. These are isothermal in the sense that the temperature, as defined by photons, is constant in all the space: $\delta T = 0$. They are also called "entropy perturbations" because the cosmic entropy per baryon fluctuates.

The study of the distribution of the large scale structure is in a large part done statistically [2]. Since many catalogues list the positions of galaxies, or clusters of galaxies, or radio sources in the sky, a useful description is to consider them as a distribution of point-like objects and analyse it by giving their n-point correlation functions.

The probability that an object is found in the infinitesimal volume δV is

$$\delta P = n\delta V \tag{I.1}$$

where n is the mean number density.

The two point correlation function ξ is defined by the joint probability of finding an object in both of the volume elements δV_1 and δV_2 at a distance r_{12}

$$\delta P = n^2(1 + \xi(r_{12}))\delta V_1 \delta V_2 \tag{I.2}$$

where the hypothesis of large scale homogeneity and isotropy has been used, which implies that ξ depends only on the distance r_{12}. This hypothesis is not necessarily true, so also angular two point correlation functions are studied. In the same way, higher order correlation functions can be defined but I will not focus on them here.

If the distribution of objects is described by a continuous function, associated to it there is an energy density function $\rho(\mathbf{x}, t)$. Its deviation from the mean value $< \rho > (t)$ is defined by

$$\delta(\mathbf{x}, t) \equiv \frac{\delta \rho}{\rho} \equiv \frac{\rho(\mathbf{x}, t)}{\langle \rho \rangle (t)} - 1 \tag{I.3}$$

which has vanishing mean value $\langle \delta \rangle = 0$.

Note that this definition is not unique, it depends on the set of spacelike hypersurfaces choiced in which energy density fluctuations is measured. This problem will be treated in detail in chapter II.

One convenient measure of the irregularities in the space distribution is the dimensionless autocorrelation function ξ

$$\xi(\mathbf{r}) = \langle \delta(\mathbf{x})\delta(\mathbf{x} + \mathbf{r}) \rangle \tag{I.4}$$

It is usual to express density fluctuations in terms of a Fourier expansion

$$\delta(\mathbf{x},t) = \frac{1}{(2\pi)^{-3}} \int d^3\mathbf{k}\delta_{\mathbf{k}}(t)e^{-i\mathbf{k}.\mathbf{x}} \tag{I.5}$$

The physical wavelength and wavenumber associated to each mode are related to the comoving wavelength and wavenumber, λ and k, by

$$\lambda_{ph} = \frac{2\pi}{k}R(t) \equiv \lambda R(t) \tag{I.6.a}$$

$$k_{ph} = \frac{k}{R(t)} \tag{I.6.b}$$

It is helpful to think $\delta_{\mathbf{k}}$ in terms of the modulus and argument

$$\delta_{\mathbf{k}} = |\delta_{\mathbf{k}}|e^{i\epsilon_{\mathbf{k}}} \tag{I.7}$$

since simplifying assumptions about the statistics of the modulus $|\delta_{\mathbf{k}}|$ are currently done.

A significative quantity used to describe the perturbations is the variance of $|\delta_{\mathbf{k}}|$ at a given $\mathbf{k}$

$$P(\mathbf{k}) \equiv \langle|\delta_{\mathbf{k}}|^2\rangle \tag{I.8}$$

which is called the "power spectrum" of the fluctuation process $\delta(\mathbf{x},t)$.

An important relation between the power spectrum and the autocorrelation function $\xi(\mathbf{r})$ is that one is the Fourier transformed of the other

$$P(\mathbf{k}) = \int d^3r\xi(\mathbf{r})e^{i\mathbf{k}.\mathbf{r}} \tag{I.9.a}$$

$$\xi(\mathbf{r}) = \frac{1}{(2\pi)^3} \int d^3kP(\mathbf{k})e^{-i\mathbf{k}.\mathbf{r}} \tag{I.9.b}$$

The power spectrum contains the information about the amplitude of the fluctuations. The specification of the fluctuation process requires also to do some assumption about the phases $\epsilon_{\mathbf{k}}$. A usual one is to suppose that at early times the phases are random numbers, uniformly distributed on the interval $[0,\pi]$.

The relation between the wavelength and the wavenumber $\lambda = 2\pi k^{-1}$ enable as to translate from the notion of power spectrum in the Fourier space to the distribution of mass fluctuations on the real space. A density fluctuation in a patch of universe of scale R is made up of contributions from all the Fourier components which frequency exceeds $\frac{2\pi}{R}$. The mass associated with this patch is simply

$$M = \frac{4\pi}{3}\langle\rho\rangle R^3 \tag{I.10}$$

If we assume that the power spectrum follows a power law

$$P(k) \propto k^n \tag{I.11}$$

and we compute the statistics of the fluctuations δM of the mass contained within spheres of a given radius R

$$\langle(\frac{\delta M}{M})^2\rangle = \int d^3k\langle|\delta_{\mathbf{k}}|^2\rangle|W(Rk)|^2 \tag{I.12}$$

where the window function is given by

$$W(Rk) = \frac{\int_{S_R} d^3x e^{-i\mathbf{k}.\mathbf{x}}}{\int_{S_R} d^3x} \tag{I.13}$$

where S_R is a sphere of radius R about the origin. Since $W(Rk) \simeq 1(0)$ if $Rk < 1(Rk > 1)$, we obtain

$$\langle(\frac{\delta M}{M})^2\rangle \simeq \langle|\delta_{\mathbf{k}}|^2\rangle k^3 \propto k^{n+3} \tag{I.14}$$

A quantity of astrophysical interest is $\langle\frac{\delta M}{M}\rangle(k, t_H(k))$ which is the average relative rest mass excess on a comoving length scale $k^{-1} = R$ when this scale enters the horizon in the radiation or matter dominated phase at the time $t_H(k)$.

I-2 Scale invariant spectrum

A very simple hypothesis for the shape of $\langle\frac{\delta M}{M}\rangle(k, t_H(k))$ was proposed by Harrison [3] and Zeldovich[4], namely

$$\langle\frac{\delta M}{M}\rangle(k,t_H(k)) = const \qquad (I.15)$$

which correspond to take a spectral index in the power spectrum $(I.11)$ $n = 1$.

A scale invariant spectrum was originally postulated because it fits the experimental constraints fairly well and is the only power law spectrum to do so. The experimental constraints are twofold. First, the absence of anisotropies in the cosmic background radiation [5] impose an upper bound on the amplitude of primordial perturbations on large scales [6]

$$\langle\frac{\delta M}{M}\rangle(k,t_H(k)) < 10^{-4} \qquad for\ M \sim 10^{19} M_{\odot} \qquad (I.16)$$

(scales are labeled by the rest mass in a sphere of comoving radius k^{-1}).

On the other hand, clusters of galaxies can only form via nonlinear processes. Linear perturbation theory breaks down when relative perturbations become of order unity. Thus, knowing that perturbations on the scale of clusters of galaxies must have had time to grow to order 1 after horizon crossing impose a lower bound on small scales [7]

$$\langle\frac{\delta M}{M}\rangle(k,t_H(k)) > 10^{-6} \qquad for\ M \sim 10^{15} M_{\odot} \qquad (I.17)$$

This bound depends on the details of the cosmological model. In particular, the properties of the particles forming the dark matter of the universe will determine the length of the period during which perturbations on scales of interest grow, and those will influence the lower bound.

We see that equations (I.16) and (I.17) make a scale invariant spectrum an obvious candidate and restricts its amplitude to $O(10^{-(5\pm1)})$.

We will see that the most natural spectrum predicted by the inflationary scenario is a scale invariant one, which is a very attractive feature. The inflationary scenario will be reviewed in chapter III. According to it, the universe underwent a very fast expansion phase at the very early times, when its energy density was dominated by the potential energy of a scalar field (inflaton). Such an expansion gives a possible solution for the monopole, isotropy and flatness problems. After its invention, it was realized that it can also give rise to primordial energy density perturbations. The first point on analysing the resulting perturbations is to note that most perturbations which existed before inflation are not relevant for galaxy formation, since inflation act in the sense of washing out the initial inhomogeneities (their comoving scales are stretched and become exponentially larger than the comoving scale of the presently observable portion of the universe). At the same time, however, quantum fluctuations during inflation lead to the creation of adiabatic perturbations of the energy density with an almost flat (scale invariant) spectrum, as we will see in detail in chapter IV.

I-3 Evolution of the perturbations

The posterior evolution of the density perturbations in the early universe is determined by the influence of various phenomena. There are some physically important length scales describing the domain of influence of the different phenomena.

The Jeans length

$$\lambda_J = c_s \left[\frac{\pi}{G\rho} \right]^{\frac{1}{2}} \qquad (I.18)$$

determines the preponderance of the gravitation or pressure effects on the evolution of the density perturbations. For perturbation wavelengths larger than λ_J, the gravitational effect dominates and the amplitude of the perturbation grows. Instead, for perturbation wavelengths smaller than λ_J the pressure effect dominates and the amplitude of the perturbation oscillates as an acoustic wave. Associated to the Jeans length there is a mass scale

$$M_J = \frac{4\pi}{3} \rho \left[\frac{\lambda_J}{2} \right]^3 \qquad (I.19)$$

During matter dominated era, before recombination, matter was coupled to radiation via Compton scattering and $p = \frac{\rho_r}{3}$, the sound velocity was $c_s = \frac{c}{\sqrt{3}}[1 + \frac{3}{4}\frac{\rho_m}{\rho_r}]^{\frac{1}{2}}$ and $M_J \sim 10^{17} M_\odot$. After recombination, the radiation pressure is of no importance and $p = nkT$, M_J drops to about $10^6 M_\odot$ and decrease thereafter as temperature diminishes.

Adiabatic fluctuations are also influenced by dissipative phenomena. Photon diffusion can damp an adiabatic perturbation (Silk damping) if its characteristic wavelength is sufficiently small, so that the time necessary for photons to diffuse out of the perturbation region is smaller than one expansion time [8]. The Silk damping length is given by

$$d_s = \sqrt{H^{-1} c\, l_T} \qquad (I.20)$$

where l_T is the photon mean free path, $l_T = (\sigma_T n_e)^{-1}$ with σ_T the Thompson cross section and n_e the electron density.

If the initial adiabatic perturbation has wavelength smaller than d_s, in one expansion time H^{-1} it will be transformed in an isothermal perturbation whose amplitude is much smaller than that of the adiabatic perturbation it comes from. The mass scale associated to this attenuation is given by [9]

$$M_S \simeq 1,3 \cdot 10^{12} (\Omega h^2)^{-\frac{3}{2}} M_\odot \qquad (I.21)$$

Typical masses are of the order of $10^{13} - 10^{14} M_\odot$ or larger, which correspond to clusters of galaxies. Thus, galaxies ($10^{11} - 10^{12} M_\odot$) can form only after the collapse of large scale perturbations. These would preferentially collapse first in one dimension (pancake collapse).

In the gravitational collapse model, structure form when perturbations $\delta = \frac{\delta\rho}{\rho}$ grow to non linearity ($\delta \geq 1$), they cease to expand with the Hubble flow and subsequently collapse and virialize. The problem is to understand how fluctuations of galactic and cluster size can grow to nonlinearity by the present without violating the observational bounds on small angle fluctuations on cosmic microwave background radiation.

I will first consider a universe with no non baryonic dark matter[10]. Fluctuations grow linearly with the scale factor

$$\delta \propto S = (1+z)^{-1} \qquad (I.22)$$

once the universe becomes matter dominated, but fluctuations smaller than the Jeans mass ($\sim 10^{17} M_\odot$) cannot begin growing until recombination. Moreover, the growth of ρ slower when an open universe goes into free expansion ($z \leq \Omega^{-1}$). So, in a baryonic universe δ grows between recombination time ($z \simeq 10^3$) and $z = \Omega^{-1} \geq 10$, when the free expansion begins. During this period, the scale factor expands 10^2 times ($\frac{S_f}{S_i} = 10^2$) and so δ grows by a factor 10^2. In order to form structure at the present, we need $\delta_o \geq 1$, which requires that at recombination time $\delta_{rec} \geq 10^{-2}$. This constrains the value of temperature fluctuations

$$\frac{\delta T}{T} = \frac{1}{3}\frac{\delta \rho}{\rho} \geq 3 \cdot 10^{-3} \qquad for\ M \geq M_S \tag{I.23}$$

which correspond to fluctuations of few arc minutes today.

This result is more than one order of magnitude larger than present observational upper limits[11]. Note that this analysis apply only to adiabatic perturbations. Galaxies which can be formed from isothermal perturbations are not affected by Silk damping and avoid contradiction with present $\frac{\delta T}{T}$ limits.

When non baryonic dark matter is considered, the picture of structure formation is rather different. It depends on the type of dark matter considered, as I will briefly review now.

- Hot dark matter: It refers to particles, such as neutrinos, that were still in thermal equilibrium after the most recent phase transition in the hot early universe, the QCD confinement transition. In discussing the galaxy formation, I will refer specifically to neutrinos. If neutrinos have a rest mass $m_\nu \geq 10eV$, their contribution to the total energy density would exceed the baryon one [12].

The most salient feature of hot dark matter is the erasure of small scale fluctuations by free streaming [13]. Neutrinos of mass m_ν stream relativistically from decoupling until the temperature drops to m_ν, travelling a distance $d_\nu \sim M_P m_\nu^{-2}$. Thus, to survive free streaming the wavelength of the fluctuation $\lambda_\nu > d_\nu$. So, neutrinos exhibit an effective Jeans length. Correspondingly, the mass in neutrinos needed for a fluctuation to survive free streaming is

$$M_J(\nu) = 1,77 \cdot M_P^3 m_\nu^{-2} = 3,2 \cdot 10^{15} (\frac{m_\nu}{30eV})^{-2} M_\odot \tag{I.24}$$

which is the mass scale of superclusters.

Therefore, hot dark matter with a primordial scale free adiabatic fluctuation spectrum gives rise to perturbations which have a cutoff in the short wave region due to free streaming, is peaked at $\lambda \sim d_\nu$ and decrease for larger wavelengths because fluctuations with larger wavelengths have less time to grow. This spectrum leads to pancake structure formation: superclusters form first and smaller scale structure, such as galaxies, form after by fragmentation of larger structures.

Numerical simulations of dissipationless gravitational clustering originated by this spectrum predict regions of high density forming a network of filaments, with the highest density occurring at the intersections, and with voids between them [14,15]. The similarity of this picture with observations is cited as evidence in favour of this model. The limits on small angle $\frac{\delta T}{T}$ fluctuations are also compatible with this picture [10]. However,there exist many problems associated with the neutrino dominated galaxy formation scenario. Studies of nonlinear clustering indicates that supercluster collapse must have occurred recently, for $z < 2$. However, the best limits on galaxy ages indicates that galaxy formation took place before $z = 3$. Another problem is associated with the large scale (quadrupole) anisotropy of the cosmic microwave background radiation. Observations constrains it to be $(\frac{\delta T}{T})_Q \leq 3 \cdot 10^{-5}$ [16]. Theoretical predictions for the neutrino dominated universe are at the verge of contradictions with the observational limits [17]. There is also a problem due to the fact that the value of the dispersion of the velocities of galaxies $\sigma_v \sim 10^3 \frac{km}{s}$ exceeds the observed value.

- Cold dark matter: Some of the problems associated to the neutrino picture can be alleviated by supposing that the universe is dominated by particles whose mass is much larger than $10eV$ (thus reducing the free streaming damping mass, $M_D < 10^8 M_\odot$) and which decouples at an earlier time (so that the density is not too large). There exist different kinds of candidates as axions, photinos or primordial black holes. The spectrum predicted for energy density fluctuations when one consider primordial adiabatic scale free perturbations as in the hot dark matter case is quite different in this case . During the radiation dominated era, fluctuations grow as $\delta \sim S^2$ on scales larger than the horizon. When the fluctuation enters the horizon, the photons and charged particles oscillates as an acoustic wave and the non interacting neutrinos freely stream away. They are relativistic since in the cold dark matter case their mass $m_\nu << 30eV$. As a result the main driving terms for the growth of δ_{DM} disappears and the growth stagnates as the universe continues to expand until matter dominates.

Numerical computations of the cold dark matter fluctuation spectrum [10] show that it is relatively flat for $M < 10^9 M_\odot$ and then decreases for larger wavelengths. Therefore, for smaller mass fluctuations will become non linear and begin to collapse at earlier times than large mass fluctuations. Small mass systems are subsequently clustered within larger mass systems which become non linear at later time. This hierarchical clustering begins at the baryons Jeans mass scale ($M_J(b) \sim 10^5 M_\odot$ at recombination) and continues until the present. Cold dark matter yields an epoch for galaxy formation that accords well with observations. However, not all is completely satisfactory in this theory, also in this model dispersion of velocities is too large, $\sigma_v \sim 800 \frac{km}{s}$ [18]. Cluster- cluster correlation function is equally difficult to understand in this scenario.

Therefore, if galaxies trace the mass distribution, none of the models seems to be fully consistent with observations. However, if the hypothesis that galaxies formed only at the highest peaks of the initial density distribution is done, which is called the "biased" clustering, some of these problems can be eliminated.

I-4 Alternative proposals

The difficulties presented in both hot and cold dark matter scenarios with adiabatic scale free primordial spectrum have stimulated the search of an alternative source of initial density perturbations. A rather interesting possibility is based on the theory of cosmic strings [19,20,21]. These are one dimensional false vacuum defects which have been formed at a symmetry breaking phase transition at the epoch of grand unification. In this theory, galaxy formation is due to the accretion of matter around loops of strings. The starting point is a distribution of string loops about which matter begins to accrete after pressure becomes negligible at t_{eq} (when the universe becomes matter dominated). The idea is that the smaller loops develop into galaxies and the larger ones into clusters. Within this theory, two point galaxy-galaxy and cluster- cluster correlation functions are correctly explained [22,23]. Also a mechanism to create sheet-like density enhancement by moving infinite strings over large scales has been proposed [24]. This can give rise to a universe crossed by sheets of galaxies with large voids between them. Recent estimations of the string mass per unit length [22] which give rise to density perturbations sufficient for galaxy formation and consistent with cosmic microwave background radiation constraints requires $\mu \sim (2 \cdot 10^{16} GeV)^2$. There are controversial opinions about the compatibility of string galaxy formation and inflation due to the large value of μ[25,26].

Another alternative is to study the role of isothermal perturbations in the inflationary cosmology [25,27,28,29]. During inflation, not only adiabatic fluctuations can be produced, but also isothermal ones [27], and in a wide class of elementary particle theories these perturbations can become dominating at the later stages of the universe evolution. A very interesting fact is the possibility of obtaining isothermal perturbations with a variety of spectrums, which can differ substantially from the flat one [25,28,29].

The mechanism of generation of isothermal fluctuations during inflation can be rather easily understood by comparing it to the more usual theory of generation of adiabatic density perturbations in the inflationary cosmology. In this case, a single scalar field is considered (the inflaton field) which drives inflation. As the main contribution to the energy density of the universe during inflation is given by the potential energy density of the scalar field $U(\phi)$, fluctuations in this field originates perturbations in the total energy density and when the inflaton decays in light particles, these transform in fluctuations of density (and temperature) of the created particles. So, they are purely adiabatic.

However, in realistic elementary particle theories there exist many different scalar fields, and during inflation fluctuations in all of them were generated. In the inflationary period, their mean energy density was smaller than that of the inflaton, otherwise they would have driven inflation, but their fluctuations can be large. If some of them interact very weakly with the rest of the fields, their energy density would decrease more slowly than that of the inflaton field and its decay products during the expansion of the universe. So, the perturbation in the energy density associated with these fields may become dominating. As they do not originate fluctuations in the temperature, they can be called isothermal. The existence of these fields, very weakly interacting with the other fields is a typical fact in many theories of elementary particles. One case is that of the axion, which has been introduced to solve the strong CP problem. Another class is the one constituted by the so called "hidden sector" in supersymmetric theories. In many cases, they are also candidates of dark matter, and so their contribution to the present density of the universe could be more important than that of the matter produced by the inflaton decay.

The generation of isothermal fluctuations during inflation has been studied by many authors in the case of the axion field [29,30], but some other possibilities have also been considered by Linde [29] and by Koffman and Linde [25]. The main result in the axion case is the appearance of a cutoff of the spectrum at long wavelengths, which leads to the suppression of large scale fluctuations of the temperature of background radiation. The study of the other possibilities has shown that rather different kinds of spectrums of isothermal perturbations can be obtained by a proper modelling of the underlying theory of particle physics.

II-GAUGE INVARIANT PERTURBATIONS

II-1 Introduction

When we study the fluctuations in the energy density of the universe which give rise to the observed structure, we must consider also the associated fluctuations in the metric. If one tries to include them in the analysis, a problem arise in relation with the freedom of making gauge transformations. Because of this, the notion of density perturbations, for example, looses its direct physical significance due to the presence of coordinate gauge freedom inherent in general relativistic perturbation theories. It can be seen that the amplitude of perturbations in geometrical quantities can be comparable to or even greater than that of a density perturbation and one can assign practically any value to the latter by a suitable gauge transformation. In the earlier works on the subject, a particular gauge was chosen and some scheme was proposed to treat the gauge modes (modes representing only coordinate changes).

An entirely different solution to the same sort of problems was given by Bardeen [31]. His approach, based on previous works of Hawking [32] and Olson [33], presents a complete gauge invariant framework for studying the time evolution of perturbations in the matter content and in the metric, in which only variables that are invariant under the change of gauge are dealt with.

I will present first the Bardeen formalism, which deals only with a fluid as matter content of the universe, and then its generalization to the case of scalar fields [34,35]. Finally, I will present a first integral of Bardeen equations valid for large scales [36], which permits to relate the value of the amplitude of the perturbations in different epochs in a simple way.

II-2 Outline of the method

In describing perturbations, one is dealing with two space-times: the physical perturbed space-time and a fictitious background space-time, which will be described by a Robertson-Walker metric. Points in the background are labelled by coordinates x^k (latin indices ranging from 0 to 3 and greek ones from 1 to 3). A one-to-one correspondence between points in the background and points in the physical space-time carries these coordinates over into the physical space-time and defines a choice of gauge. A change in the correspondence, keeping the background coordinates fixed, is called a gauge transformation, to be distinguished from a coordinate transformation, which change the labeling of points in the background and in the physical spacetime together. The perturbation in some quantity is the difference between the value it has at a point in the physical space-time and the value at the corresponding point in the background space-time.

The unperturbed background metric may be written as

$$ds^2 = S^2(\tau)(-d\tau^2 + {}^3g_{\alpha\beta}dx^\alpha dx^\beta) \qquad (II.1)$$

where ${}^3g_{\alpha\beta}$ is the metric tensor for a 3-space of uniform spatial curvature κ, with Riemann tensor

$${}^3R_{\alpha\beta\gamma\delta} = \kappa({}^3g_{\alpha\gamma}\,{}^3g_{\beta\delta} - {}^3g_{\alpha\delta}\,{}^3g_{\beta\gamma}) \qquad (II.2)$$

The stress tensor of the background matter takes a perfect fluid form:

$$T^i_k = P_o\delta^i_k + (P_o + E_o)u^i\,u_k \qquad (II.3)$$

where P_o is the background pressure and E_o the background energy density, and the four velocity u^i has components

$$u^0 = \frac{1}{a(\tau)}, \qquad u^\alpha = 0 \qquad (II.4)$$

Lets define

$$w = \frac{P_o}{E_o}, \qquad c_s^2 = \frac{dP_o}{dE_o} \qquad (II.5)$$

The time evolution of the background is given by the usual Friedmann equations

$$(\frac{S'}{S})' = -\frac{1}{6}(E_o + 3P_o)S^2 \tag{II.6.a}$$

$$(\frac{S'}{S})^2 = \frac{1}{3}E_o S^2 - K \tag{II.6.b}$$

$$E_o' = -3\frac{S'}{S}(E_o + P_o) \tag{II.6.c}$$

where $S' = \frac{dS}{d\tau}$ and units have been chosen so $c = 8\pi G = 1$.

Perturbations in various quantities can be classified according to how they transform under spatial coordinate transformations in the background space-time, as spatial scalars, vectors and tensors. As the Robertson-Walker background is homogeneous and isotropic, a separation of the time dependence and the spatial dependence is possible. The spatial dependence is related to solutions of a generalized Helmholtz equation [37]. Scalar quantities can be expanded by a complete set of scalar harmonic functions $Q^{(0)}$ satisfying the equation

$$Q^{(0)\ |\alpha}{}_{|\alpha} + k^2 Q^{(0)} = 0 \tag{II.7}$$

The slash denotes the covariant derivative of a 3-tensor with respect to ${}^3g_{\alpha\beta}$ and k the wave number of the perturbation, which characterizes its spatial scale relative to comoving coordinates in the background. Scalar perturbations have a spatial dependence derived from one of the $Q^{(0)}$.

A vector quantity which is associated to a scalar perturbation must be constructed from covariant derivatives of $Q(0)$ and the metric

$$Q^{(0)}{}_{\alpha} \equiv -\frac{1}{k} Q^{(0)}{}_{|\alpha} \tag{II.8}$$

And also a traceless, symmetric second rank tensor is obtained by

$$Q^{(0)}{}_{\alpha\beta} \equiv \frac{1}{k^2} Q^{(0)}{}_{|\alpha\beta} + \frac{1}{3}\, {}^3g_{\alpha\beta}\, Q^{(0)} \tag{II.9}$$

Similarly, divergenceless spatial vectors are expanded by a complete set of vector harmonic functions $Q^{(1)\alpha}$ satisfying

$$Q^{(1)\alpha\ |\beta}{}_{|\beta} + k^2 Q^{(1)\alpha} = 0 \tag{II.10}$$

with $Q^{(1)\alpha}{}_{|\alpha} = 0$.

Vector type components of symmetric tensors are defined by

$$Q^{(1)\alpha\beta} = -\frac{1}{2k}(Q^{(1)\alpha\ |\beta} + Q^{(1)\beta\ |\alpha}) \tag{II.11}$$

which are necessarily traceless but not divergenceless.

Finally, tensor type components, namely divergenceless and traceless second rank symmetric spatial tensors, describing gravitational waves, are expanded by a complete set of tensor harmonic functions specified by

$$Q^{(2)\alpha\beta\ |\gamma}{}_{|\gamma} + k^2 Q^{(2)\alpha\beta} = 0 \tag{II.12.a}$$

with

$$Q^{(2)\alpha}{}_{\alpha} = 0 \tag{II.12.b}$$

$$Q^{(2)\alpha\beta}{}_{|\beta} = 0 \tag{II.12.c}$$

A perturbation on the metric or on the matter variables can be written as a linear combination of perturbations associated with individual spatial harmonics, with no coupling between each other [38]. Hence, one can treat each type of perturbation independently. The formalism is linearized in the perturbed quantities. The Einstein equations and matter evolution equations become ordinary differential equations for the time dependent coefficients of the expansion.

II.3 Perturbation of metric and stress tensor

A-Scalar perturbations

By a spatial coordinate transformation, the components of the metric tensor $g_{00}, g_{0\alpha}$ and $g_{\alpha\beta}$ transform as a scalar, a vector and a tensor respectively. Hence, for a scalar perturbation the perturbed metric tensor can be written in terms of independent functions of time A, B, H_L and H_T

$$g_{00} = -S^2(\tau)[1 + 2A(\tau)Q^{(0)}(x^\mu)] \qquad (II.13.a)$$

$$g_{0\alpha} = -S^2(\tau)B^{(0)}(\tau)Q^{(0)}{}_{\alpha}(x^\mu) \qquad (II.13.b)$$

$$g_{\alpha\beta} = S^2(\tau)[(1 + 2H_L(\tau)Q^{(0)}(x^\mu))^3 g_{\alpha\beta}(x^\mu) + 2H_T^{(0)}(\tau)Q^{(0)}{}_{\alpha\beta}(x^\mu)] \qquad (II.13.c)$$

In order to write down expressions for matter variables in terms of the harmonics, we choose first appropriate variables which describe matter at a perturbed state. We take the perturbed 4-velocity of matter u^i as the time-like eigenvector of the stress tensor with unit norm, and the perturbed energy density as the corresponding eigenvalue

$$T^i_k u^k = -E_o u^i \qquad (II.14)$$

$$u^i u_i = -1 \qquad (II.15)$$

The spatial stress tensor is given by

$$^3T_{ik} = P^j_i \; P^l_k \; T_{jl} \qquad (II.16)$$

where

$$P^j_i = \delta^j_i + u_i u^j \qquad (II.17)$$

Note that $^3T_{ik}$ is orthogonal to u^i from (II.15) and (II.17)

$$^3T_{ik}u^i = 0 \qquad (II.18)$$

Since E is a scalar, it can be expressed as

$$E = E_o(1 + \delta(\tau)Q^{(0)}(x^\mu)) \qquad (II.19)$$

The three independent degrees of freedom of u^i are represented by the spatial velocity which can be expressed as

$$\frac{u^\alpha}{u^0} = v^{(0)}(\tau)Q^{(0)\alpha}(x^\mu) \qquad (II.20)$$

To first order, the normalization condition (II.15) gives

$$u^{(0)} = S^{-1}[1 - A\ Q(0)] \qquad (II.21)$$

And u_i is expressed as

$$u_\alpha = S(v^{(0)} - B^{(0)})Q^{(0)}{}_\alpha \qquad (II.22.a)$$

$$u_0 = -S(1 + A\ Q^{(0)}) \qquad (II.22.b)$$

Using (II.17), (II.20), (II.21) and (II.22), ${}^3T_{\mu\nu}$ can be expressed as

$${}^3T_0^0 = 0 \qquad (II.23.a)$$

$${}^3T_\alpha^0 = P_o(v^{(0)} - B^{(0)})Q^{(0)}{}_\alpha \qquad (II.23.b)$$

$${}^3T_0^\alpha = -P_o\ v^{(0)}\ Q^{(0)\ \alpha} \qquad (II.23.c)$$

$${}^3T_\beta^\alpha = T_\beta^\alpha \qquad (II.23.d)$$

As T_β^α is a second rank symmetric tensor with respect to spatial coordinate transformations, it is expressed as

$$T_\beta^\alpha = P_o[\delta_\beta^\alpha + \pi_L(\tau)\delta_\beta^\alpha Q^{(0)} + \pi_T(\tau)Q^{(0)\alpha}{}_\beta] \qquad (II.24)$$

π_L is interpreted as the amplitude of an isotropic pressure perturbation and π_T as the amplitude of an anisotropic stress one.

The four functions of time $v^{(0)}, \delta, \pi_L$ and π_T completely describe the perturbed stress tensor. Equation (II.16) can be written as

$$T_k^i = E_o\ u^i\ u_k +^3 T_k^i \qquad (II.25)$$

Substituting (II.19) to (II.24) in (II.25), we obtain T_0^0, T_0^α and T_α^0 in terms of these four quantities

$$T_0^0 = -E_o(1 + \delta(\tau)Q^{(0)}(x^\mu)) \qquad (II.26.a)$$

$$T_\alpha^0 = (E_o + P_o)(v^{(0)} - B^{(0)})Q^{(0)}{}_\alpha(x^\mu) \qquad (II.26.b)$$

$$T_0^\alpha = -(E_o + P_o)\, v^{(0)}\, Q^{(0)\,\alpha} \qquad (II.26.c)$$

The perturbed isotropic pressure need not to be related to the energy density in the same way as the background one. The difference between the fractional pressure perturbation and that expected from the background pressure-energy density relation will be called the entropy perturbation

$$\eta(\tau)Q^{(0)} = (\pi_L - \frac{E_o}{P_o}\frac{dP_o}{dE_o}\,\delta)Q^{(0)} = \frac{1}{\omega}(\omega\pi_L - c_s^2\delta)Q^{(0)} \qquad (II.27)$$

B-Vector perturbations

The description of vector quantities can be done analogously to the scalar ones, but the spatial dependence is given in terms of fundamental vector harmonics $Q^{(1)}{}_\alpha$.

We obtain for the metric tensor

$$g_{00} = -S^2(\tau) \qquad (II.28.a)$$

$$g_{0\alpha} = -S^2(\tau)B^{(1)}(\tau)Q^{(1)}{}_\alpha(x^\mu) \qquad (II.28.b)$$

$$g_{\alpha\beta} = S^2(\tau)[{}^3g_{\alpha\beta} + 2H_T^{(1)}(\tau)Q^{(1)}{}_{\alpha\beta}(x^\mu)] \qquad (II.28.c)$$

And for the matter variables

$$\frac{u^\alpha}{u^0} = v^{(1)}(\tau)Q^{(1)\alpha}(x^\mu) \qquad (II.29.a)$$

$$u^0 = S^{-1}(\tau) \qquad (II.29.b)$$

$$u_\alpha = S(\tau)[v^{(1)}(\tau) - B^{(1)}(\tau)]Q^{(1)}{}_\alpha(x^\mu) \qquad (II.29.c)$$

$$T_0^0 = -E_o \qquad (II.30.a)$$

$$T_0^\alpha = -(E_o + P_o)v^{(1)}Q^{(1)\alpha} \qquad (II.30.b)$$

$$T^0_\alpha = (E_o + P_o)(v^{(1)} - B^{(1)})Q^{(1)\alpha} \qquad (II.30.c)$$

$$T^\alpha_\beta = P_o[\delta^\alpha_\beta + \pi^{(1)}_T Q^{(1)\alpha}{}_\beta] \qquad (II.30.d)$$

Hence, a vector perturbation is described by two functions of time $B^{(1)}$ and $H^{(1)}_T$ for the metric and the two functions of time, $v^{(1)}$ and $\pi^{(1)}_T$ for the matter.

C-Tensor perturbations

Similarly, for a tensor perturbation we find

$$g_{00} = -S^2(\tau) \qquad (II.31.a)$$

$$g_{0\alpha} = 0 \qquad (II.31.b)$$

$$g_{\alpha\beta} = S^2(\tau)[{}^3g_{\alpha\beta} + 2H^{(2)}_T(\tau)Q^{(2)}{}_{\alpha\beta}(x^\mu)] \qquad (II.31.c)$$

$$u^0 = S^{-1}(\tau) \qquad u^\alpha = 0 \qquad (II.32)$$

$$T^0_0 = -E_o \qquad (II.33.a)$$

$$T^\alpha_0 = T^0_\alpha = 0 \qquad (II.33.b)$$

$$T^\alpha_\beta = P_o[\delta^\alpha_\beta + \pi^{(2)}_T Q^{(2)\alpha}{}_\beta] \qquad (II.33.c)$$

Thus, a tensor perturbation is described by one function of time $H^{(2)}_T$ for the metric and one function of time $\pi^{(2)}_T$ for the matter.

II-4 Gauge invariant variables

The variables introduced to describe perturbations in the metric and in the stress tensor change their values under a gauge transformation, which corresponds to a change of coordinates in the physical space-time, leaving fixed the coordinates in the background. In the linear perturbation theory, it is necessary only to consider first order effects of the coordinate transformation, which can be of the scalar or vector type. There exist no tensor type gauge transformation, as no scalar or vector can be constructed from $Q^{(2)}{}_{\alpha\beta}$.

Gauge transformations of each type are expanded by the corresponding harmonic functions, and different modes are decoupled from each other. From the study of gauge transformation properties of the perturbed variables, gauge invariant variables can be constructed in the following way.

A-Scalar perturbations

A scalar type gauge transformation can be expressed as the coordinate transformation

$$\tilde{\tau} = \tau + T(\tau)Q^{(0)}(x^{\mu}) \qquad (II.34.a)$$

$$\tilde{x}^{\alpha} = x^{\alpha} + L^{(0)}(\tau)Q^{(0)\alpha}(x^{\mu}) \qquad (II.34.b)$$

where $T(\tau)$ and $L^{(0)}(\tau)$ are arbitrary functions of τ.

The change of the perturbed metric tensor under the transformation (II.34) is given by

$$g_{ab}(\mathrm{x}) = \frac{\partial \tilde{x}^{k}}{\partial x^{a}}\frac{\partial \tilde{x}^{l}}{\partial x^{b}}\, \tilde{g}_{kl}(\tilde{\mathrm{x}}) \qquad (II.35)$$

The scale factors in $\tilde{g}_{kl}$ and g_{ab} are related by

$$S(\tilde{\tau}) = S(\tau)[1 + \frac{S'}{S}TQ^{(0)}] \qquad (II.36)$$

and

$$^{3}g_{\alpha\beta}(\tilde{x}^{\mu}) =^{3} g_{\alpha\beta}(x^{\mu}) + L^{(0)}Q^{(0)\alpha}\frac{\partial}{\partial x^{\mu}}[^{3}g_{\alpha\beta}(x^{\mu})] \qquad (II.37)$$

To first order, we obtain for the change in the metric perturbations

$$\tilde{A} = A - T' - \frac{S'}{S}T \qquad (II.38.a)$$

$$\tilde{B}^{(0)} = B^{(0)} + L^{(0)'} + kT \qquad (II.38.b)$$

$$\tilde{H}_{L}^{(0)} = H_{L}^{(0)} - \frac{k}{3}L^{(0)} - \frac{S'}{S}T \qquad (II.38.c)$$

$$\tilde{H}_{T}^{(0)} = H_{T}^{(0)} + kL^{(0)} \qquad (II.38.d)$$

Since the gauge transformation (II.34) have two arbitrary functions of time, two independent gauge invariants can be constructed from $A, B^{(0)}, H_L$ and H_T. One possible choice is [31]

$$\phi_A \equiv A + \frac{1}{k}B^{(0)\prime} + \frac{1}{k}\frac{S'}{S}B^{(0)} - \frac{1}{k^2}\left[H_T^{(0)\prime\prime} + \frac{S'}{S}H_T^{(0)\prime}\right] \tag{II.39}$$

and

$$\phi_H \equiv H_L + \frac{1}{3}H_T^{(0)} + \frac{1}{k}\frac{S'}{S}B^{(0)} - \frac{1}{k^2}\frac{S'}{S}H_T^{(0)\prime} \tag{II.40}$$

Now let us study the gauge transformation properties of matter variables. Since

$$\tilde{v}^{(0)}Q^{(0)\alpha} = \frac{d\tilde{x}^\alpha}{d\tilde{\tau}} \simeq \frac{dx^\alpha}{d\tau} + L^{(0)\prime}Q^{(0)\alpha} \tag{II.41}$$

A gauge transformation for $v^{(0)}$ is given by

$$\tilde{v}^{(0)} = v^{(0)} + L^{(0)\prime} \tag{II.42}$$

The energy density (II.19) transforms as

$$\tilde{E}(\tilde{\tau}) = E_o(\tilde{\tau})[1 + \tilde{\delta}Q^{(0)}] = E_o(\tau)\left[1 + (\tilde{\delta} + T\frac{E_o'}{E_o})Q^{(0)}\right] \tag{II.43}$$

Since E is a coordinate scalar, the energy density perturbation change by

$$\tilde{\delta} = \delta - \frac{E_o'}{E_o}T = \delta + 3\frac{S'}{S}T(1+w) \tag{II.44}$$

equation (II.6.c) has been used in the last equality.

Similarly, the isotropic pressure transforms as

$$\tilde{P} = P_o(\tilde{\tau})[1 + \tilde{\pi}_L Q^{(0)}] = P_o(\tau)\left[1 + (\tilde{\pi}_L + \frac{P_o'}{P_o}T)Q^{(0)}\right] \tag{II.45}$$

And for the isotropic pressure perturbations we obtain

$$\tilde{\pi}_L = \pi_L + 3(1+\omega)\frac{c_s^2}{\omega}\frac{S'}{S}T \tag{II.46}$$

The traceless part of the stress tensor is gauge invariant

$$\tilde{\pi}_T = \pi_T \qquad (II.47)$$

and also is gauge invariant the entropy perturbation given by (II.27)

$$\eta = \pi_L - \frac{c_s^2}{\omega}\delta \qquad (II.48)$$

$$\tilde{\eta} = \eta \qquad (II.49)$$

In order to construct gauge invariant quantities associated to v and δ, we must combine them with the geometrical quantities. The simplest gauge invariant matter "velocity" amplitude can be constructed using (II.42) and (II.38.d)

$$v_s^{(0)} = v^{(0)} - \frac{1}{k}H_T^{(0)'} \qquad (II.50)$$

It is easy to give a physical interpretation to $v_s^{(0)}$ in terms of the shear of the matter velocity field.

The shear tensor is [39]

$$\sigma_{ab} = \frac{1}{2}P_a^k(u_{k;l} + u_{l;k})P_b^l - \frac{1}{3}P_{ab}u^k_{;k} \qquad (II.51)$$

where the projector operator is given by (II.17).

Using the expressions for the perturbed Christoffel symbols given in Appendix A, we find that the only non vanishing term for the shear is

$$\sigma_{\alpha\beta} = S(H_T^{(0)'} - k\ v(0))Q^{(0)}{}_{\alpha\beta} = -S\ k\ v_s^{(0)}\ Q^{(0)}{}_{\alpha\beta} \qquad (II.52)$$

This suggests that $v_s^{(0)}$ is the most natural gauge invariant representing the perturbation in the velocity. Its geometrical meaning is that it gives the time dependence of the amplitude of the shear associated to the perturbation.

In contrast to the velocity, there exist no unique natural definition of a gauge invariant quantity corresponding to the density perturbations. One criterion to construct it is that the gauge invariant quantity reduce to δ when we consider perturbations with wavelength much smaller than the Hubble radius. If we limit ourselves to the simplest combinations, there are two possible choices. The first is

$$\epsilon_m = \delta + 3(1+\omega)\frac{1}{k}\frac{S'}{S}(v^{(0)} - B^{(0)}) \qquad (II.53)$$

If we use a gauge in which $v^{(0)} = B^{(0)}$, the amplitude ϵ_m equals δ. This is the case when the matter world lines are orthogonal to the $\tau = const$ spacelike hypersurfaces. The choice of ϵ_m to describe density perturbations is very convenient because it gives the amplitude of the perturbations in the energy density relative to the spacelike hypersurface which represents everywhere the matter local rest frame (comoving frame).

Another possibility is to consider

$$\epsilon_g = \delta - 3(1+\omega)\frac{1}{k}\frac{S'}{S}\left[B^{(0)} - \frac{1}{k}H_T^{(0)'}\right] \qquad (II.54)$$

As we have seen, $B^{(0)}$ corresponds to the 3-velocity amplitude of world lines orthogonal to the $\tau = const$ hypersurfaces, so ϵ_g measures the energy density relative to the hypersurfaces whose normal unit vectors have zero shear. This set of hypersurfaces is usually called "Newtonian slicing". One can also chose any linear combination of ϵ_m and ϵ_g or its time derivatives as gauge invariant variables.

The geometrical meaning of the gauge invariant potentials ϕ_A and ϕ_H corresponding to the perturbations in the metric variables becomes clear when one consider a Newtonian slicing

$$B^{(0)} - \frac{1}{k}H_T^{(0)'} = 0 \qquad (II.55)$$

In this particular case, equations (II.39) and (II.40) reduce to

$$\phi_A = A \qquad (II.56.a)$$

$$\phi_H = H_L + \frac{1}{3}H_T^{(0)} \qquad (II.56.b)$$

So, ϕ_A measures the spatial dependence of the proper time interval along the normals between two neighboring zero-shear hypersurfaces.

The geometrical meaning of ϕ_H is clear by noting from Appendix A that the intrinsic scalar curvature of zero-shear hypersurfaces is given by

$${}^3R_{zeroshear} = \frac{1}{S^2}[6K + 4(k^2 - 3K)\phi_H Q^{(0)}] \qquad (II.57)$$

Thus, ϕ_H represents the amplitude of the perturbation in the intrinsic curvature.

B-Vector perturbations

A vector type gauge transformation is expressed as

$$\tilde{\tau} = \tau \tag{II.58.a}$$

$$\tilde{x}^\alpha = x^\alpha + L^{(1)}(\tau) Q^{(1)\alpha}(x^\mu) \tag{II.58.b}$$

There exist only one gauge invariant combination, given by

$$\psi \equiv B^{(1)} - \frac{1}{k} H_T^{(1)'} \tag{II.60}$$

which represents the shear of the normals to the $\tau = const$ hypersurfaces. The gauge transformation law of the matter variables can be obtained as in the scalar case, but now there exist no perturbation of energy density and the isotropic pressure

$$\tilde{v}^{(1)} = v^{(1)} + L^{(1)'} \tag{II.61}$$

$$\tilde{\pi}_T^{(1)} = \pi_T^{(1)} \tag{II.62}$$

The amplitude of an anisotropic stress perturbation is again gauge invariant by itself.

In contrast to the scalar case, the gauge transformation laws (II.59) and (II.61) admits two natural gauge invariant combinations corresponding to a velocity perturbation. The first one, given by

$$v_s^{(1)} \equiv v^{(1)} - \frac{1}{k} H_T^{(1)'} \tag{II.63}$$

represent the shear of the matter velocity perturbation.

The other one is given by

$$v_c \equiv v^{(1)} - B^{(1)} \tag{II.64}$$

and is related to the vorticity tensor

$$\omega_{ab} = \frac{1}{2} P_a^k (u_{k;l} - u_{l;k}) P_b^l \qquad (II.65)$$

The only non zero components are

$$\omega_{\alpha\beta} = S(v^{(1)} - B^{(1)})(Q^{(1)}{}_{\alpha|\beta} - Q^{(1)}{}_{\beta|\alpha}) \qquad (II.66)$$

Thus, v_c gives the amplitude of vorticity of the matter velocity field.

C-Tensor perturbations

There exist no tensor type gauge transformation. Hence, all the quantities associated with a tensor perturbation $H_T^{(2)}$ and $\pi_T^{(2)}$ are gauge invariant by themselves.

II-5 Einstein equations for gauge invariant variables

Since a gauge transformation is formally an infinitesimal coordinate transformation in the perturbed spacetime, the general covariance of the Einstein equations guarantees that the perturbation of these

$$\delta G^\mu_\nu = \delta T^\mu_\nu \qquad (II.67)$$

can be written only in terms of the gauge invariant combinations of the original perturbation variables.

A-Scalar perturbations

Using the expression for δG^μ_ν in terms of A, B, H_L and H_T given in Appendix A, it follows that one can construct the following gauge invariant combinations

$$\delta G^0_0 - \frac{3}{k^2}\frac{S'}{S}(\delta G^0_\alpha)^{|\alpha} = -\frac{2}{S^2}(k^2 - 3K)\phi_H Q^{(0)} \qquad (II.68)$$

and

$$\delta G^{\alpha}_{\beta} - \frac{1}{3}\delta^{\alpha}_{\beta}\delta G^{\mu}_{\mu} = -\frac{k^2}{S^2}(\phi_A + \phi_H)Q^{(0)\alpha}{}_{\beta} \tag{II.69}$$

From (II.67) and making use (II.68), (II.24), (II.26) and (II.53) we obtain that

$$E_o\epsilon_m = 2(k^2 - 3K)\frac{\phi_H}{S^2} \tag{II.70}$$

and from (II.69) and (II.24)

$$P_o\pi_T^{(0)} = -\frac{k^2}{S^2}(\phi_A + \phi_H) \tag{II.71}$$

In this simple way the gauge invariant metric perturbation amplitudes are related to the gauge invariant perturbation amplitudes. Equation (II.71) implies that for a perfect fluid $\phi_A = \phi_H$.

The equation of motion for the matter variables $v_s^{(0)}$ and ϵ_m can be obtained from the Einstein equations (II.67) for the components δG^0_α and δG^β_α and (II.70), (II.71). They can be written in term of gauge invariant quantities as

$$v_s^{(0)\prime} + \frac{S'}{S}v_s^{(0)} = k\phi_A + k\frac{1}{1+\omega}(c_s^2\,\epsilon_m + \omega\eta) - \frac{2}{3}k(1 - \frac{3K}{k^2})\frac{\omega}{1+\omega}\pi_T^{(0)} \tag{II.72}$$

$$\epsilon_m' - 3\omega\frac{S'}{S}\epsilon_m = -(1 - \frac{3K}{k^2})(1+\omega)kv_s^{(0)} - 2(1 - \frac{3K}{k^2})\frac{S'}{S}\omega\pi_T^{(0)} \tag{II.73}$$

The physical interpretation of these equations is the following: the second term on the left hand side of (II.72) represents the slowing down of the velocity due to the cosmic expansion, the first term on the right hand side represents the gravitational force, the second term the force due to the pressure gradient and the last one represents the acceleration due to the anisotropic part of the stress tensor. The second term on the right hand side of (II.73) represents the change due to the cosmic expansion, while the first term on the right hand side, the compression due to the proper motion of the matter.

A second order form of the evolution equation is often used instead of the system (II.72) and (II.73). Such a second order form is obtained by eliminating $v_s^{(0)}$ from (II.72) and (II.73)

$$\epsilon_m'' - [3(2\omega - c_s^2) - 1]\frac{S'}{S}\epsilon_m' +$$

$$+3\left[(\frac{3}{2}\omega^2 - 4\omega - \frac{1}{2} + 3c_s^2)\left[\frac{S'}{S}\right]^2 + \frac{3\omega^2 - 1}{2}K + \frac{k^2 - 3K}{3}c_s^2\right]\epsilon_m = \theta \qquad (II.74)$$

where

$$\theta = -(k^2 - 3K)\omega\eta + 2\left[(3(\omega^2 + c_s^2) - 2\omega)(\frac{S'}{S})^2 + \omega(3\omega + 2)K + \frac{k^2 - 3K}{3}c_s^2\right](1 - \frac{3K}{k^2})\pi_T^{(0)}$$

$$-2(1 - \frac{3K}{k^2})\frac{S'}{S}\omega\pi_T^{(0)\prime} \qquad (II.75)$$

This shows that an entropy perturbation and an anisotropic stress perturbation act as sources for density perturbations. In order to study their effects, they must be expressed as functions of time explicitly or in terms of ϵ_m and $v_s^{(0)}$.

B-Vector perturbations

From the expression of δG^μ_ν for a vector perturbation given in Appendix A, it follows that the Einstein equation reduce to the following two gauge invariant equations

$$\frac{1}{2}\frac{(k^2 - 2K)}{S^2}\psi = (E_o + P_o)v_c \qquad (II.76)$$

$$v_c' = \frac{S'}{S}(3c_s^2 - 1)v_c - k\frac{\omega}{1+\omega}\pi_T^{(1)} \qquad (II.77)$$

Hence, the vorticity of matter is originated by a vector type anisotropic stress perturbation.

C-Tensor perturbation

For a tensor perturbation the Einstein equations reduce to a single gauge invariant equation

$$H_T^{(2)\prime\prime} + 2\frac{S'}{S}H_T^{(2)\prime} + (k^2 + 2K)H_T^{(2)} = S^2 P_o \pi_T^{(2)} \qquad (II.78)$$

Since $H_T^{(2)}$ corresponds to the divergenceless part of the metric tensor, it represents the amplitude of a gravitational wave. As expected its equation of motion is of wave type.

II-6 Perturbations in scalar field dominated systems

Up to now we have considered only fluctuations in a system which stress tensor is given by a fluid component exclusively. Since in the early universe quantum fields could have dominated the cosmic expansion, I will extend the formalism to deal with a scalar field [33,34]

The lagrangian determining the dynamics of a scalar field in curved spacetime is

$$\mathcal{L}(\psi) = -\frac{1}{2}\sqrt{-g}\,[g^{ab}\frac{\partial\psi}{\partial x^a}\frac{\partial\psi}{\partial x^b} + 2U(\psi)] \qquad (II.79)$$

This lagrangian yields the field equation

$$\nabla^a\nabla_a\psi - \frac{\partial U}{\partial\psi} = 0 \qquad (II.80)$$

and the stress tensor

$$T_b^a(\psi) = \nabla^a\psi\nabla_b\psi - \frac{1}{2}[\nabla^k\psi\nabla_k\psi + 2U]\delta_b^a \qquad (II.81)$$

Equation (II.80) guarantees the conservation of T_b^a

$$\nabla_a T_b^a(\psi) = 0 \qquad (II.83)$$

In actual situations, it often occurs that there are interactions of the scalar field with some other matter, for example that it can decay in radiation. In this case, the energy momentum tensor will have another component corresponding to radiation

$$T_{ab} = T_{ab}(\psi) + T_{ab}(rad) \qquad (II.83)$$

and $T_{ab}(\psi)$ will not be conserved by itself. The evolution of ϕ and ρ in the homogeneous case is given by

$$\phi'' + 4\frac{S'}{S}\phi' + S^2\frac{\partial U}{\partial \phi} + S^2\Gamma\phi' = 0 \qquad (II.84.a)$$

$$E_o'(rad) + 4\frac{S'}{S}E_o(rad) = \Gamma\phi'^2 \qquad (II.84.b)$$

where Γ takes into account the decay of ϕ into radiation.

The unperturbed energy density and pressure can be written as

$$E_o = -T_0^0 = E_o(rad) + \frac{1}{2S^2}\phi'^2 + U(\phi) \qquad (II.85.a)$$

$$P_o = \frac{1}{3}T_i^i = P_o(rad) + \frac{1}{2S^2}\phi'^2 - U(\phi) \qquad (II.85.b)$$

Among the three types of perturbations: Scalar, vector and tensor; since the scalar field transforms as a scalar under spatial coordinate transformations, we need only to consider scalar perturbations because this is the only one which couples to the scalar field

$$\psi(x^\alpha, \tau) = \phi(\tau) + \delta\phi(\tau)Q^{(0)}(x^\alpha) \qquad (II.86)$$

Under a gauge transformation

$$\tilde{\tau} = \tau + T(\tau)Q^{(0)}(x^\mu) \qquad (II.87.a)$$

$$\tilde{x}^\alpha = x^\alpha + L^{(0)}(\tau)Q^{(0)\alpha}(x^\mu) \qquad (II.87.b)$$

The scalar field transforms as

$$\tilde{\psi} = \phi(\tilde{\tau}) + \tilde{\delta\phi}Q^{(0)} = \phi(\tau) + [\phi'T + \tilde{\delta\phi}]Q^{(0)} \qquad (II.88.a)$$

Since it is a scalar, the scalar field perturbation changes by

$$\tilde{\delta\phi} = \delta\phi - \phi'T \qquad (II.88.b)$$

Comparing the gauge transformation properties of the metric perturbation variables (II.38) and of the scalar field perturbation (II.88.b), we can construct the following gauge invariant quantity characterizing the scalar field perturbations

$$\Delta\phi = \delta\phi + \frac{1}{k}(B^{(0)} - \frac{1}{k}H_T^{(0)'})\phi' \qquad (II.89)$$

For describing the perturbations in the stress tensor, we need to study the perturbations in the scalar field derivatives

$$\nabla_0 \phi = \phi' + (\delta\phi)' Q^{(0)} \tag{II.90.a}$$

$$\nabla_\alpha \phi = -k \ \delta\phi \ Q^{(0)}{}_\alpha \tag{II.90.b}$$

$$\nabla^0 \phi = -S^{-2}[\phi' + (\delta\phi)' Q^{(0)} - 2A \ \phi' \ Q^{(0)}] \tag{II.90.c}$$

$$\nabla^\alpha \phi = -S^{-2}[B \ \phi' + k \ \delta\phi] Q^{(0)\ \alpha} \tag{II.90.d}$$

and in the potential

$$U(\psi) = U(\phi) + U_\phi \ \delta\phi \ Q^{(0)} \tag{II.91}$$

where $U_\phi = \frac{\partial U}{\partial \phi}$.

Replacing them in (II.81), we obtain

$$T^0_0 = S^{-2}[-\frac{1}{2}\phi'^2 + (A\phi'^2 - \phi' \ (\delta\phi)')Q^{(0)}] - U(\phi) - U_\phi \ \delta\phi \ Q^{(0)} \tag{II.92.a}$$

$$T^0_\alpha = k \ S^{-2} \ \phi' \ \delta\phi \ Q^{(0)}{}_\alpha \tag{II.92.b}$$

$$T^\alpha_0 = -S^{-2}[B \ \phi'^2 + k \ \phi' \ \delta\phi] Q^{(0)\alpha} \tag{II.92.c}$$

$$T^\alpha_\beta = S^{-2}[\frac{1}{2}\phi'^2 - (A\phi'^2 - \phi' \ (\delta\phi)')Q^{(0)}]\delta^\alpha_\beta - [U(\phi) + U_\phi \ \delta\phi \ Q^{(0)}]\delta^\alpha_\beta \tag{II.92.d}$$

From this last equation, we see that the scalar field perturbations do not generate anisotropic perturbations in stress tensor ($\pi_T = 0$). It can be seen that this result is no more true when non minimal coupling of the scalar field with the curvature is considered[]. The absence of anisotropic perturbations in the stress tensor has as consequence that the gauge invariant metric perturbation variables must be related when the scalar field is the dominant contribution to the stress tensor as

$$\phi_A = -\phi_H \tag{II.93}$$

as can be seen from (II.69) by using (II.92.d) and (II.67).

The equations of motion for the gauge invariant variables $\Delta\phi$ and ϕ_A can be obtained as follows. From equation (II.68) and the Einstein equation (II.67)

$$(\Delta\phi)'\phi' + \Delta\phi\left[3\frac{S'}{S}\phi' + S^2\ U_\phi\right] = \phi_A[2k^2 + \phi'^2 - 6K] \qquad (II.94)$$

and perturbing the field equation (II.80)

$$(\Delta\phi)'' + 2\frac{S'}{S}\ (\Delta\phi)' + (k^2 + S^2\ U_{\phi\phi})\ \Delta\phi = 4\phi'_A\ \phi' - 2S^2\ U_\phi\ \phi_A \qquad (II.95)$$

which is a system of coupled equations for ϕ_A and $\Delta\phi$.

The way in which these perturbations in the scalar field are related to the perturbations in the stress tensor components, energy density and velocity, thought as a fluid will now be clarified. In first place, we will see which is the perturbation in the energy density δ originated by a perturbation in the scalar field $\Delta\phi$. By comparing (II.19) and (II.92.a) and using (II.89) and (II.85.a) it can be seen that

$$E_o(\phi)\ \delta_\phi = 3\frac{S'}{S}\frac{1}{k}\left[B^{(0)} - \frac{1}{k}H_T^{(0)'}\right](1+\omega)E_o(\phi) - \phi'^2 S^{-2}\phi_A + S^{-2}\phi'(\Delta\phi)' + U_\phi\ \Delta\phi \quad (II.96)$$

A gauge invariant quantity can be constructed from it, using (II.54)

$$E_o(\phi)\ \epsilon_g(\phi) = -\phi'^2\ S^{-2}\ \phi_A + S^{-2}\phi'(\Delta\phi)' + U_\phi\ \Delta\phi \qquad (II.97)$$

The perturbation in the velocity associated to a perturbation of the scalar field is obtained by comparing (II.26.b) and (II.26.c) with (II.92.b) and (II.92.c)

$$\phi'\ (v^{(0)} - \frac{1}{k}H_T^{(0)'}) = k\ \Delta\phi \qquad (II.98)$$

where

$$E_o(\phi) + P_o(\phi) = S^{-2}\ \phi'^2 \qquad (II.99)$$

has been used.

A gauge invariant velocity perturbation is obtained by using (II.50)

$$\phi'\ v_s^{(0)} = k\ \Delta\phi \qquad (II.100)$$

Finally, by comparing (II.24) and (II.92.d) we obtain

$$\pi_T(\phi) = 0 \tag{II.101}$$

and

$$P_o(\phi)\pi_L(\phi) = 3\frac{S'}{S}\frac{1}{k}\left[B^{(0)} - \frac{1}{k}H_T^{(0)'}\right]\frac{1+\omega}{\omega}E_o(\phi) - \phi'^2 S^{-2}\phi_A + S^{-2}\phi'(\Delta\phi)' - U_\phi\ \Delta\phi \tag{II.102}$$

which can be put in a gauge invariant form

$$P_o(\phi)\ \delta p_g(\phi) = -\phi'^2\ S^{-2}\ \phi_A + S^{-2}\phi'(\Delta\phi)' - U_\phi\ \Delta\phi \tag{II.103}$$

The following step is to study the relation between the evolution equation of the scalar field perturbations (II.94) and (II.95) and the evolution equations of matter variable perturbations (II.72) and (II.73). The evolution equation for the matter velocity (II.72) looks rather problematic for the case of a scalar field dominated system, as the two last terms in the right hand side have a term $(1+\omega)$ in the denominator. Remembering that when we consider quantum fields, the pressure can take negative values and that just in many theories which are considered in studying the inflationary scenario, the limit $\omega \rightarrow -1$ is the usual one, divergences could arise in that equation. Note that the third term in the right hand side vanishes in the case that we are considering, since $\pi_T = 0$. The second one can be computed

$$k\frac{1}{1+\omega}(c_s^2\ \epsilon_m + \omega\eta) = 3\frac{S'}{S}v_s^{(0)} + \frac{k}{E_o+P_o}[-\phi'^2\ S^{-2}\ \phi_A + S^{-2}\phi'(\Delta\phi)' + U_\phi\ \Delta\phi] \tag{II.104}$$

So (II.72) reduces to

$$v_s^{(0)'} - 2\frac{S'}{S}v_s^{(0)} = k\ \phi_A + \frac{k}{E_o+P_o}[-\phi'^2\ S^{-2}\ \phi_A + S^{-2}\phi'(\Delta\phi)' + U_\phi\ \Delta\phi] \tag{II.105}$$

By replacing (II.100) into it, it can be seen that (II.105) is satisfied automatically. So, equation (II.72) does not have any divergence problem and in fact it does not impose any new condition on scalar field perturbations.

For analysing the evolution equation of the energy density perturbation (II.73) in this case, we need to obtain the expression of the density perturbation ϵ_m . It can be obtained from (II.97) by using that

$$E_o\ \epsilon_m = E_o\ \epsilon_g + 3\frac{S'}{S}(E_o + P_o)\frac{1}{k}v_s \qquad (II.106)$$

The result is

$$E_o\ \epsilon_m = S^{-2}[-\phi'^2\ \phi_A + \phi'(\Delta\phi)' + S^2 U_\phi\ \Delta\phi + 3\frac{S'}{S}\phi'\Delta\phi] \qquad (II.107)$$

Inserting this expression for ϵ_m in equation (II.73), it can be seen with the help of (II.70) or (II.94) that the resulting equation is exactly (II.95), the equation of motion for the scalar field perturbation obtained by perturbing the Klein-Gordon equation equation of motion for the scalar field. So, we see that the treatment of the perturbations in the energy density originated by a scalar field as corresponding to a fluid with gauge invariant energy density perturbation given by (II.97) and (II.107) and velocity perturbation given by (II.100) is consistent with the equation of motion for the scalar field.

II-7 Evolution of the large scale density perturbations

In this section, the evolution equation for the gauge invariant energy density ϵ_m, (II.74) which is a second order differential equation is shown to be equivalent for large scales to a first order differential equation [36]. Instead of using the time variable τ (see (II.1)) used up to now, I will consider the variable t

$$dt = S\ d\tau \qquad (II.108)$$

Its associated Hubble constant is $H = \dot{S}/S$. Note that $HS = S'/S$.

For "large scales", I mean $\frac{SH}{k} > 1$. The parameter $\frac{SH}{k}$ turns out to be very important in the study of the evolution of density perturbations. It gives the ratio of the reduced wavelength $\frac{S}{k}$ to the Hubble distance H^{-1}. This one is usually referred as "effective horizon" or also "horizon", even if it does not correspond neither to the particle horizon nor to the event one[40]. This point has created some confusion among people working in this subject. The fact of speaking about a perturbation being larger than the effective horizon while $\frac{SH}{k} > 1$ makes sense because it is the value of this parameter, being larger or smaller than one which determines which term in the evolution equation for the perturbation are dominant as we will see. As during the evolution of the perturbation, $\frac{SH}{k}$ can be smaller than one for early times, grew to values larger than one and decrease again to values smaller than one, sometimes this is referred as perturbations leaving and reentering the effective horizon. This occur for example in the inflationary models.

It will be assumed that the stress tensor remains isotropic ($\pi_T = 0$), as it occurs for example when the evolution is dominated by a scalar field minimally coupled to the curvature or for a perfect fluid.

It is convenient to rewrite equation (II.74), which gives the evolution of ϵ_m, in terms of the new variable

$$Z = \frac{(H^2 - \frac{K}{S^2})}{(k^2 - 3K)} S^2 \ \epsilon_m \qquad (II.109)$$

It results

$$\ddot{Z} + \dot{Z}(4 + c_s^2)H + Z\left[E_o(c_s^2 - \omega) + \frac{3}{2}(1 - \omega)\frac{K}{S^2}\right] = -\frac{P_o}{3}\left[\eta + \frac{c_s^2}{\omega}\epsilon_m\right] \qquad (II.110)$$

The quantity in the right hand side is a gauge invariant and it is given by

$$\delta p_m = P_o\left[\eta + \frac{c_s^2}{\omega}\epsilon_m\right] = P_o\left[\pi_L + 3\frac{c_s^2}{\omega}(1 + \omega)\frac{1}{k}\frac{\dot{S}}{S}(v^{(0)} - B^{(0)})\right] \qquad (II.111)$$

which reduces to the perturbation of the isotropic pressure in the comoving gauge.

When $\frac{SH}{k} > 1$, the source term (right hand side) of (II.111) result to be $\mathcal{O}(\frac{S^2H^2}{k^2})$ smaller than the terms in the left hand side, and the evolution of the perturbation is essentially kinematic. This is the reason why it is usually said that for wavelength larger than the effective horizon ($\frac{SH}{k} > 1$), microphysical processes cannot affect the amplitude of the perturbations.

We shall see now that equation (II.110) has a first integral for large wavelengths. For $\frac{SH}{k} > 1$ the terms in the right hand side can be neglected, and assuming a flat background, (II.98) can be written as

$$\ddot{Z} + (1-\gamma)\ H\ \dot{Z} - [(E_o + P_o) + \frac{1}{3}\gamma\ E_o]Z = 0 \qquad (II.112)$$

where

$$\gamma \equiv -3(1 + c_s^2) \qquad (II.113)$$

This equation posses an exact integral

$$Z + \frac{2}{3}\frac{1}{(1+\omega)}(Z + H^{-1}\dot{Z}) - \frac{\delta K}{k^2} = 0 \qquad (II.114)$$

where δK is a constant.

This first order differential equation for Z becomes specially simple during periods in which ω is constant, as for example

$$\omega = 0 \qquad (matter\ dominated) \qquad (II.115.a)$$

$$\omega = \frac{1}{3} \qquad (radiation\ dominated) \qquad (II.115.b)$$

$$\omega \simeq -1 \qquad (inflation) \qquad (II.115.c)$$

In this cases, equation (II.114) has a constant solution

$$Z = [1 + \frac{2}{3}\frac{1}{1+\omega}]^{-1} \qquad (II.116)$$

plus a decaying mode, which will become negligible a few Hubble times after the beginning of the era in question. This fact makes it possible to compare the values of Z in different eras of constant ω in a simple way, without necessity of making assumptions of what happened in the intermediate periods except that the behaviour should not be such as to cause the decaying mode to dominate.

Specifically, we obtain that

$$\frac{5}{3}Z(matter) = \frac{3}{2}Z(radiation) = \frac{2}{3}\frac{1}{1+\omega}Z(inflation) \qquad (II.116)$$

This result is valid for large scale perturbations ($\frac{SH}{k} > 1$). During the radiation and matter dominated eras, this quantity decreases with time and eventually becomes smaller than one (the scale enters the "effective horizon").

III-THE INFLATIONARY SCENARIO

III-1 General description

I will present in this section a brief review of inflationary models[41,42,43] and I will treat more carefully the topics which are more important for studying the density fluctuations problem. The inflationary scenario is a modification of the standard Big Bang model, born with the scope of solving the so called "horizon problem", "flatness problem" and "monopole problem" which arise when one extrapolates it back to the initial time[44].

The horizon problem is related to the fact that the homogeneity and isotropy of the cosmic microwave background radiation indicates that the regions where photons coming from different regions in the sky last scattered at recombination time were at the same temperature. This cannot be explained in the standard Big Bang model because of causality, as that regions were not causally connected at recombination time.

The flatness problem corresponds to the fact that Ω parameter defined by $\Omega = \frac{\rho_c}{\rho}$, where $\rho_c = 3H^2$ is the critical density necessary for the universe to be flat, is measured to be of order one. It is easy to see that $|1 - \Omega^{-1}|$ increases as S^2 when the universe expands, so that to have $\Omega \sim \mathcal{O}(1)$ today requires to fine tune its value extremely near to one as initial condition.

The monopole problem is connected to the fact that in the context of grand unified theories, the standard Big Bang model predicts a large overproduction of monopoles, which are topologically stable knots in the Higgs field vacuum expectation value. This is in contraction with observations, so an incompatibility of grand unified theories and the standard Big Bang model arise.

The idea underlying the solution of these problems in the inflationary model is to assume a period of very fast expansion of the scale factor $S(t)$ in the very early universe.

From the Einstein equation

$$\ddot{S} = -\frac{1}{6}(E_o + 3P_o)S \qquad (III.1)$$

we see that in order that $\ddot{S} > 0$, it is necessary that

$$P_o < -\frac{1}{3}E_o \qquad (III.2)$$

Within the classical description of matter, the pressure is always positive, so this inflationary expansion does not occur. But this is not the case when matter is described in terms of quantum fields, as we will now see.

Consider a simple example consisting in a scalar field ϕ with a double well potential.

$$U(\phi) = \lambda[\phi^2 - \sigma^2]^2 \qquad (III.3)$$

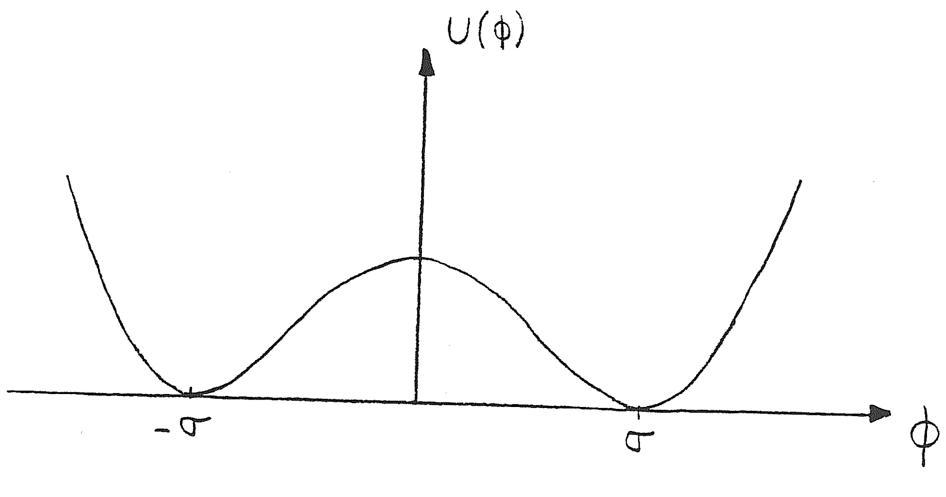

Figure 1: The potential energy density of the scalar field ϕ

This scalar field is of the type introduced in particle physic theories to induce a spontaneous symmetry breaking [45]

From (II.81) we obtain

$$E_o(\phi) = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2S^2}(\nabla\phi)^2 + U(\phi) \qquad (III.4.a)$$

$$P_o(\phi) = \frac{1}{2}\dot{\phi}^2 - \frac{1}{6S^2}(\nabla\phi)^2 - U(\phi) \qquad (III.4.b)$$

Assuming that at some early time $\phi(\mathbf{x},t) = 0$, the contribution of the scalar field to the equation of state is $P_o(\phi) = -E_o(\phi)$. If at this initial time, the energy density in radiation $E_o(rad)$ is larger than the corresponding to the scalar field $E_o(\phi)$, the universe will expand as $S(t) \sim t^{\frac{1}{2}}$ and $E_o(rad)$ will decrease as $E_o(rad) \sim t^{-2}$, meanwhile $E_o(\phi)$ stays constant provided that $\phi(\mathbf{x},t) = 0$. If this holds for enough time, then the scalar field will dominate the equation of state and $E_o \simeq -P_o$. So, an equation of state with negative pressure, satisfying the constraint (III.2) is obtained. In the particular case in which $E_o = -P_o$, from (II.6.b) we find $H \simeq const$ and

$$S(t) = S_o\, e^{Ht} \qquad (III.5)$$

(The curvature term can be neglected after some time because it decreases exponentially with respect to the energy one). This is the usual mechanism for inflation.

The question now is to explain why should the initial scalar field configuration be $\phi(\mathbf{x},t) = 0$, a configuration which certainly does not minimize the potential energy. The key point is that at finite temperature, the equilibrium state of a system does not correspond to a minimum of the potential energy, but to a minimum of the free energy (which at $T = 0$ coincide with the potential energy). In quantum field theory, this means that the equilibrium configuration of the field at a temperature T is obtained by minimizing the effective potential $U_{eff}(\phi, T)$ which takes into account quantum and thermal effects [46].

The temperature dependence of $U_{eff}(\phi, T)$ is given by [47]

$$U_{eff}(\phi, T) = U_{eff}(\phi, 0) + A\,\phi^2\,T^2 + B\,T^2 + C\,T^4 \qquad (III.6)$$

At sufficiently high temperature, it has only one minimum at $\phi = 0$, due to the $A\,\phi^2\,T^2$ term, and thus the scalar field configuration will be $\phi(\mathbf{x},t) = 0$. There is a critical temperature T_c for which the minimum $\phi = 0$ becomes unstable and for $T < T_c$, the global minima are at $\phi = \pm\sigma$.

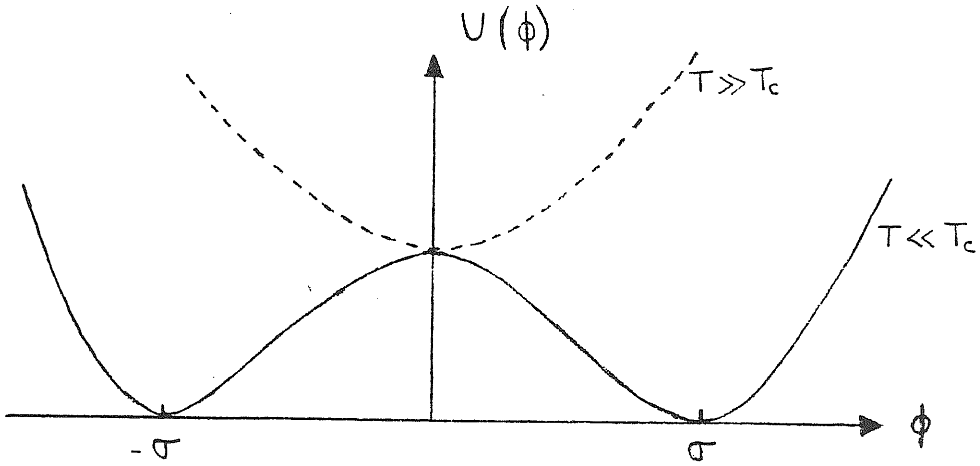

Figure 2: The finite temperature effective potential for ϕ at high and low temperatures

The temperature dependence in the effective potential acts in the way of producing the initial configuration required for inflation. It also gives rise to a phase transition. The way in which it proceeds will depend on the particular shape of the potential [43]. If the minimum of the effective potential changes continuously from $\phi = 0$ to $\phi = \pm\sigma$ as temperature decreases, the scalar field will evolve classically to the global minimum. However, in some cases the minimum of the effective potential changes discontinuously as the temperature decreases, as when there is a potential barrier separating both minima in the effective potential. In these cases, the scalar field evolves to the global minimum by quantum tunnelling the potential barrier. This is the case in the first inflationary model proposed by Guth [44]. Bubbles of the true vacuum state expand in a sea of false vacuum. This model has the problem that it leads to an extremely large inhomogeneity and anisotropy of the universe and it was discarded [48].

III-2 New inflationary scenario

A new inflationary scenario was proposed by Linde [49] and Albretch and Steinhardt [50] in which the phase transition proceeds by classical rolling. The idea is that the effective potential is rather flat near $\phi = 0$, so that the phase transition occurs gradually and significant inflation can take place, producing huge regions of homogeneous space, and we would be living today deep inside one of these regions.

In the usually accepted picture, just after the Big Bang the temperature was very large and the stress tensor is dominated by the radiation component. The scale factor grows as $S(t) \sim t^{\frac{1}{2}}$. Thermal effects confine $\phi(\mathbf{x}, t)$ to be zero. Meanwhile, temperature is decreasing and at some point the potential energy of the scalar field becomes the dominating term in $T_{\mu\nu}$. So, the equation of state changes to $P_o \simeq -E_o$, the universe begins to expand exponentially, $S(t) \sim e^{Ht}$ and the temperature of radiation drops exponentially, $T \sim e^{-Ht}$. The scalar field configuration remains near $\phi \sim 0$ in the flat part of the potential.This epoch is called the de Sitter phase and it can last for many Hubble expansion times (H^{-1}).

During the de Sitter phase, the temperature confining effect for ϕ decays as temperature decreases. Therefore $\phi(\mathbf{x}, t)$ begins to roll down the potential, making a transition to the new global minimum of the effective potential. During it, the potential energy of ϕ is released as radiation. This process is called reheating. It produces a hot gas of particles which is the initial state postulated by the standard hot Big Bang model. After the reheating, the stress tensor is dominated by radiation and the evolution joins the Standard Big Bang model.

The evolution of the scalar field and radiation is studied using the energy momentum conservation equation which leads to

$$\dot{\phi}[\ddot{\phi} + 3H\dot{\phi} + U_\phi] = -\dot{E}_o(rad) - 4HE_o(rad) \qquad (III.7)$$

This equation can be splited in a couple of coupled equations describing the evolution of ϕ and $E_o(rad)$ respectively

$$\dot{E}_o(rad) + 4HE_o(rad) = \Gamma\dot{\phi}^2 \qquad (III.8.a)$$

$$\ddot{\phi} + 3H\dot{\phi} + \Gamma\dot{\phi} = -U_\phi \qquad (III.8.b)$$

In the last equation, the second term in the left hand side is a friction term due to the expansion of the universe, meanwhile the third one is due to the creation of radiation.

The inflationary era begins when the temperature decrease up to a value such that $E_o(rad) < U(\phi = 0)$. The evolution of ϕ has essentially two different periods:

a) Slow rollover: During this period, the terms $\ddot{\phi}$ and $\Gamma\dot{\phi}$ in (III.8.b) are negligible

$$\ddot{\phi}, \Gamma\dot{\phi} << 3H\dot{\phi}, U_\phi \tag{III.9}$$

For power type potentials, the condition for neglecting them is

$$|U_{\phi\phi}| << 9H^2 \tag{III.10}$$

So, during this period $E_o \simeq U(\phi)$ and $U(\phi)$ is approximately constant during this first part of the evolution and this produce an inflationary expansion.

The slow rollover process finishes when the $\ddot{\phi}$ and $\Gamma\dot{\phi}$ terms becomes important.

b) Reheating: it corresponds to the last part of the way of ϕ to the global minimum $\phi = \pm\sigma$. As the $\phi \simeq 0$ configuration has positive potential energy and $\phi = \pm\sigma$ has vanishing one, during the phase transition, vacuum energy is converted into thermal energy. During this process of conversion of an ordered type of energy into an unordered one, the present entropy of the universe is generated. This process is taken in account by the Γ term, which couples the scalar field and radiation. As ϕ approaches σ and begins to oscillate around the minimum of the potential, it decays in lighter particles and so, its energy is converted into radiation.

The evolution of the temperature is as follows: During the inflationary expansion the temperature of the original thermal state decreases exponentially. Then, when the phase transition of scalar field is produced, the vacuum energy is converted into thermal energy. During this reheating process, the temperature increases rapidly. The temperature after reheating is of the same order of magnitude as the temperature before inflation. Figure 3 shows the time evolution of the temperature and the scalar field.

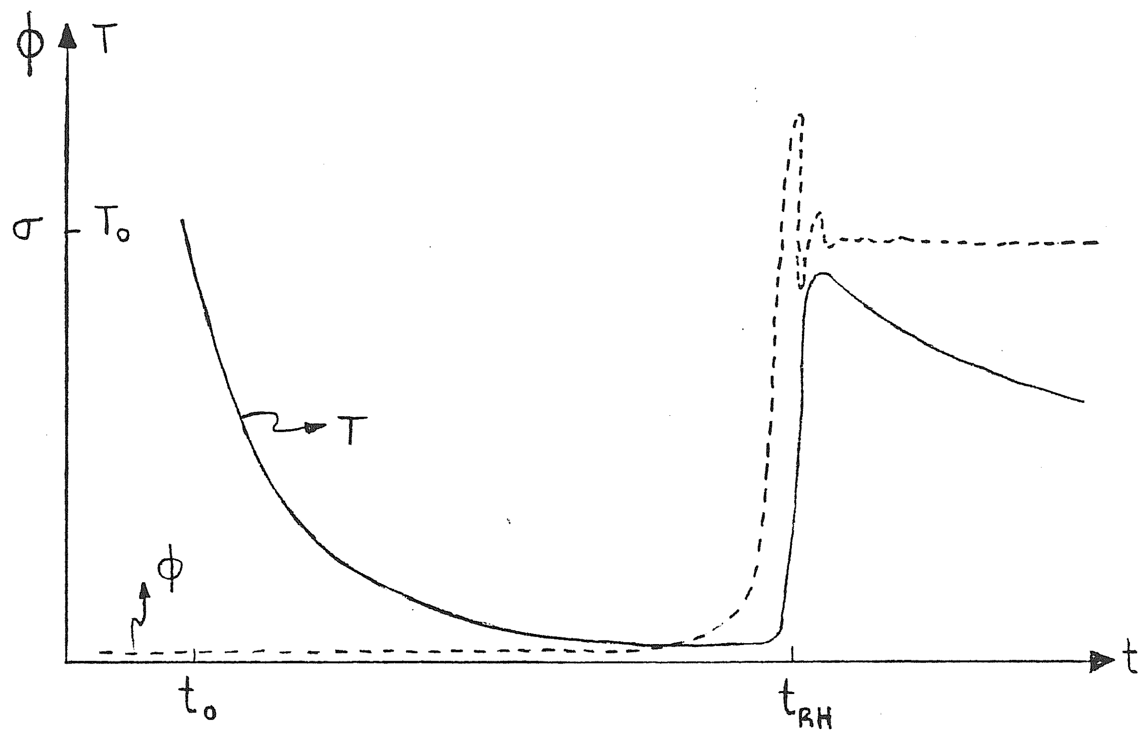

Figure 3: Sketch of the phases in the new inflationary scenario. During most of the inflationary phase ϕ remains close to zero and T decreases exponentially. At reheating, ϕ increases to its ground state value σ and the universe reheats to almost the same temperature as before the inflationary phase

This model permits to solve the horizon and flatness problems provided that the period of inflation is sufficiently long, namely that $e^{H\Delta T} > 10^{29}$, which impose restrictions on the shape of the effective potential [51]. It also permits to solve the monopole problem, as all our observable universe is inside one of the homogeneous scalar field regions, there are no topological defect inside it.

As it has been pointed out in chapter I, the inflationary scenario provides also a possible explanation of the origin of the density fluctuations which gives rise to the observed structure. However, as we will see in the next chapter, in order to reconcile the predicted spectrum with the observational limits, severe restrictions on the magnitude of the potential coupling constant arise. This is the why the scalar field responsible for the spontaneous symmetry breaking in the minimal SU(5) grand unified theory with a Coleman-Weimberg potential was discarded as possible candidate for the inflaton [34,52]. This point will be discussed with more detail in the next chapter. There are more restrictions that must be satisfied by the scalar field in order to obtain a successful new inflationary scenario , as for example on the reheating temperature. The constraints come essentially from Big Bang nucleosynthesis and baryogenesis [52,53]. Another inflationary scenario, which can occur for a much more general type of scalar field potentials has been proposed by Linde [54,55]. I will review it now

III-3 Chaotic inflationary scenario

The starting point of chaotic inflation is the observation that the new inflationary scenario is based on the assumption that the universe initially was in the state corresponding to a minimum of the effective potential $U_{eff}(\phi, T)$. Such an assumption at a first sight seems absolutely natural, since any non equilibrium configuration of the field will eventually evolve to the minimum of the effective potential. However, investigating this question in more detail, it can be seen that it is not so.

A typical curvature of the effective potential, which arise due to high temperature effects in a $\lambda\phi^4$ theory is

$$m^2(T) = \frac{d^2U}{d\phi^2} = \frac{\lambda}{4}T^2 \qquad (II.11)$$

The time necessary for the field ϕ to drop to the minimum of $U_{eff}(\phi, T)$ exceeds

$$\tau = m^{-1}(T) = \frac{2}{\sqrt{\lambda}}\frac{1}{T} \qquad (III.12)$$

On the other hand, the age of the hot universe t is given by

$$t = \frac{1}{4\pi}\sqrt{\frac{45}{\pi N}}\,\frac{M_P}{T^4} \qquad (III.13)$$

where N is the number of degrees of freedom associated to relativistic particles. For the case $N \geq 200$, which is a reasonable lower limit for high temperatures

$$t \leq \frac{1}{50}\frac{M_P}{T^2} \qquad (III.14)$$

By comparison of t and τ, one concludes that the field ϕ can be influenced by high temperature effects only for $t > \tau$ which implies

$$T \leq T^* \sim 10^{-2}\frac{\sqrt{\lambda}}{2}M_P \qquad (III.15)$$

or equivalently, at the moment at which the energy density of the hot matter E_o becomes sufficiently small

$$E_o \leq E_o^* \sim 10^{-6}\lambda^2 M_P^4 \qquad (III.16)$$

However, if the effective potential $U_{eff}(\phi, T)$ is sufficiently flat, and if the universe initially was in a state with $U(\phi) \leq E_o^*$, the universe becomes exponentially expanding and the temperature decreases considerably before it could have any effect on the value of the field ϕ. This means that in the theories in which $U_{eff}(\phi)$ can take initially a large value and with sufficiently small coupling constants (as required to have sufficiently small density fluctuations), the inflationary scenario cannot proceed in the usual way, based on the theory of high temperature phase transitions. But in those cases another scenario, the chaotic inflation is possible.

To understand the main idea of the new scenario, we will see how the classical field $\phi(\mathbf{x}, t)$ could be distributed in the early universe. The value of the effective potential at the Planck time $t_P = M_P^{-1}$ at which the classical description of spacetime becomes possible is defined with an accuracy of $\mathcal{O}(M_P^4)$ due to the uncertainty principle. Therefore one may expect that in the hot universe at $t \sim t_P$ any value of the field ϕ such that $U_{eff}(\phi) \leq M_P^4$ and $(\partial_\mu\phi)^2 \leq M_P^4$ can appear in a point $\mathbf{x}$ with an almost ϕ independent probability.

Lets study the evolution of such an initial distribution of the field ϕ in the simplest model $U(\phi) = \frac{\lambda}{4}\phi^4$ with $\lambda << 1$. We will be specially interested in the evolution of the domains of the universe in which the field was initially sufficiently homogeneous (on a scale $\geq H^{-1}$), $(\partial_\mu\phi)^2 \leq U(\phi)$ and sufficiently large, $\phi \geq M_P$.

The equation of motion of ϕ inside one of such domains is

$$\ddot{\phi} + 3H\dot{\phi} = -U_\phi = -\lambda\phi^3 \qquad (III.17)$$

The contribution to the energy density is essentially $E_o \simeq U(\phi)$, which in this case is not necessarily constant. So, the "hubble constant", which is given by

$$H \simeq \left[\frac{8\pi}{3M_P^2}U(\phi)\right]^{\frac{1}{2}} = \left[\frac{2\pi}{3}\lambda\right]^{\frac{1}{2}} \frac{\phi^2}{M_P} \qquad (III.18)$$

is no more constant.

Equation (III.17) can be written as

$$\ddot{\phi} + \sqrt{6\pi\lambda}\frac{\phi^2}{M_P}\dot{\phi} = -\lambda\phi^3 \qquad (III.19)$$

In the cases that one can neglect $\ddot{\phi}$ against $3H\dot{\phi}$, the solution is

$$\phi = \phi_o\ exp(-\sqrt{\frac{\lambda}{6\pi}}M_P t) \qquad (III.20)$$

where $\phi_o = \phi(t=0)$.

This condition is valid for

$$\phi^2 >> \frac{M_P^2}{6\pi} \qquad (III.21)$$

Meanwhile, the domain expands with scale factor

$$R(t) = R_0\ exp\left[\int_0^t H(t')\ dt'\right] \simeq R_0\ exp\left[\frac{\pi}{M_P^2}(\phi_o^2 - \phi^2)\right] \qquad (III.22)$$

The expansion will be quasi exponential in the case that $H^2 >> \dot{H}$. In fact the condition for this to be valid is essentially the same condition (III.21) that we have imposed on the initial value of ϕ. Hence, the result is that if $\phi >> M_P$, the space inside the domain will expand quasi exponentially.

During the time of quasiexponential expansion, the domain will expand approximately $exp(\frac{\pi\phi_o^2}{M_P^2})$ times. If $\phi_o > 5M_P$, the universe expands more than e^{70} times, the value needed to solve the horizon and flatness problems.

As stated before, the only constraint in the initial value of ϕ is the condition $U(\phi) = \frac{\lambda}{4}\phi^4 \leq M_P^4$. The value $\phi_o = 5M_P$ is quite possible if $\lambda \leq 10^{-2}$, which can be satisfied in in many reasonable theories.

From this point of view, inflation is not a peculiar desirable phenomenon in those theories, but is a natural consequence of the chaotic initial conditions in the very early universe which will arise in some domains of the universe.

When ϕ rolls down to the region $\phi \leq \frac{M_P}{3}$, it begins to oscillate around the minimum of $U_{eff}(\phi)$ and the potential energy is converted into radiation. The reheating temperature may be as large as $\mathcal{O}(\lambda^{\frac{1}{4}} M_P)$ or smaller. It does not depend on the value of ϕ_o. Only the ratio of the scale factor before and after inflation depends on ϕ_o.

In the chaotic inflationary scenario, as in the new inflationary one, the most severe restrictions on the strength of the scalar field interactions come from the spectrum of density fluctuations predicted by the model. Also in the case of chaotic inflation, it is necessary that the scalar field have very weak interactions, as we will see in the next chapter.

In realistic theories of elementary particles, there exist many scalars fields ϕ_i, with different values of the coupling constants. For the fields having larger values of the coupling constant, the corresponding effective potentials are more curved than those smaller coupling constant. Therefore they roll down to the minimum of the effective potential more rapidly, and the last stages of inflation are driven by the field ϕ which has a more flat effective potential. Thus, the chaotic inflationary scenario can proceed if the conditions necessary for inflation are satisfied by at least one of the scalar fields.

IV- FLUCTUATIONS IN INFLATIONARY UNIVERSE MODELS

IV-1 Introduction

The problem of explaining how did the inhomogeneities that we observe today in the universe on different length scales, as clusters of galaxies, galaxies and stars, arise have proven to be a very difficult problem in cosmology.

One the major successes of the inflationary universe models, first realized by Press [56], Sato [57], Lukash [58] and Chivisov and Mukhanov [59], and subsequently investigated quantitatively for the new inflationary scenario by Guth and Pi [60], Hawking [61], Starobinsky [62], Bardeen Steinhardt and Turner [34] and Brandenberger and Kahn [52] is a possible solution to the problem of the origin of fluctuations. The same mechanism that solves the horizon problem, an exponential expansion of the universe for a finite period, naturally explains that perturbations on cosmologically interesting scales originate inside the Hubble radius at some point in the inflationary expansion phase.

As we have seen in chapter II, the analysis of the evolution of perturbations of the energy density can be done in the linearized theory for each Fourier mode of the gauge invariant quantity ϵ_m independently. Its evolution separates into two qualitatively different regimes, depending on whether the associated wavelength λ_{ph} is larger or smaller than the Hubble radius. When $\lambda_{ph} < H^{-1}$, microphysical processes such as pressure support, free streaming of particles or quantum mechanical effects can affect its evolution. Instead, when $\lambda_{ph} > H^{-1}$, these processes do not affect the evolution of the perturbations.

In the standard cosmology λ_{ph} and H^{-1} crosses only once, and for early times λ_{ph} is always larger than H^{-1}. For this reason, it is not possible to create density perturbations by processes acting at early times. Instead in the inflationary cosmology, λ_{ph} and H^{-1} crosses twice, λ_{ph} is initially smaller than H^{-1}, then it becomes larger than H^{-1} during the inflationary era and again it becomes smaller than H^{-1} during the radiation or matter dominated era. This implies that microphysical processes occurring at early times can originate perturbations of astrophysical interesting size.

The idea is that quantum fluctuations of the inflaton field during the inflationary era give rise to the density fluctuations in which we are interested.

IV-2 Quantum fluctuations

Lets discuss briefly the nature of quantum fluctuations and its main characteristics in the case of an inflationary scenario, as these will be the seeds for galaxy formation. According to quantum field theory, empty space is not entirely empty. It is filled with quantum fluctuations of all types of physical fields. These fluctuations can be regarded as waves of physical fields with all possible wavelength, moving in all possible directions. If the values of these fields, averaged over some macroscopically large time vanish, then the space filled with these fields seems to us empty, and is called the vacuum. Another usual way for visualize quantum fluctuations is in terms of particles which quantum fluctuates between being and disappearing. They can come into existence for a small fraction of time before they annihilate each other, leaving nothing behind. The corresponding changes on the strength of the fields microscopically takes random directions and average to zero.

Nevertheless, these fluctuations still carry energy and for a brief interval of time they can create material particles, which disappear rapidly as the fluctuation dies.

In the exponentially expanding universe, the vacuum structure have some particular characteristics. The wavelength of all vacuum fluctuations of the inflaton field ϕ grow exponentially with the expansion of the universe. When the wavelength of a particular fluctuation becomes greater than H^{-1}, this fluctuation stops propagating, and its amplitude freezes at some non zero value $\delta\phi(\mathbf{x})$ because of the large friction term $3H\dot{\phi}$ in the equation of motion of the field ϕ. Then, the amplitude of this fluctuation remains nearly unchanged, meanwhile its wavelength grows exponentially. Therefore the appearance of such a frozen fluctuation is equivalent to the appearance of a classical field $\delta\phi(\mathbf{x})$ that does not vanish after averaging over macroscopic intervals of space and time. As the vacuum contains fluctuations of all the wavelengths, inflation leads to a continuum creation of perturbations, as more and more wavelengths become larger than H^{-1}.

The spectrum of the perturbations generated results to be almost scale invariant. A qualitative argument to support this has been given by Bardeen, Steinhardt and Turner [34]. The general idea is the following one: First, assume some mechanism that generates the fluctuations inside the Hubble radius in the de Sitter phase. By time translation invariance of the de Sitter phase, the evolution of the fluctuations on two different scales k_1 and k_2 up to the time when they leave the Hubble radius will be identical up to time translations, and we expect a scale invariant spectrum. As when the wavelength of the perturbation becomes larger than the Hubble radius, the perturbation evolves kinematically, they argue that the shape of the spectrum should remain unchanged up to the time when the perturbation reenters the Hubble radius during the radiation or matter dominated era, and this is the quantity in which we are interested. This qualitative argument gives the underlying idea explaining the scale invariance of the spectrum but it has its loopholes, for example the amplitude of the energy perturbations does not remains constant outside the Hubble radius, so it should be proved that the amplification factor is independent of the scale.

In the following sections, I will present a detailed computation of the energy density fluctuations generated in the inflationary universe model.

IV-3 Generation of the perturbations

Lets consider a scalar field theory with an effective potential

$$U_{eff}(\phi) = U(0) + \frac{1}{2}m^2\ \phi^2 + \frac{1}{4}\lambda\ \phi^4 \qquad (IV.1)$$

in an exponentially expanding background

$$ds^2 = dt^2 - exp(2Ht)[dx^2 + dy^2 + dz^2] \qquad (IV.2)$$

Its evolution is governed by the lagrangian

$$\mathcal{L} = e^{3Ht}[\frac{1}{2}\dot{\phi}^2 - \frac{1}{2}e^{-2Ht}(\nabla\phi)^2 - U(\phi)] \qquad (IV.3)$$

The equation of motion is given by

$$\nabla^a\nabla_a\phi = -U_\phi(\phi) \qquad (IV.4)$$

or more explicitly

$$\ddot{\phi} + 3H\ \dot{\phi} - e^{-2Ht}\nabla^2\phi = -m^2\phi - \lambda\phi^3 \qquad (IV.5)$$

To quantize the system lets introduce the canonical momentum density

$$\pi(\mathbf{x}) = i\hbar\frac{\partial\mathcal{L}}{\partial\dot{\phi}} = i\hbar e^{3Ht}\dot{\phi} \qquad (IV.6)$$

and impose the canonical equal time commutation relations

$$[\phi(\mathbf{x},t), \pi(\mathbf{y},t)] = \delta^3(\mathbf{x}-\mathbf{y}) \qquad (IV.7)$$

This interacting quantum field theory is not exactly soluble. A usual approximation [63] is to consider λ very small and neglect the $\lambda\phi^4$ term in the potential. Note that the interaction λ is taken into account in the mass term m which includes the leading temperature correction to the effective potential $\frac{\lambda}{8\hbar}T^2\phi^2$

$$m^2 = m_o^2 - \frac{\lambda}{4\hbar}T_o^2 e^{-2H(t-t_o)} \qquad (IV.8)$$

where $T = T_0 e^{-H(t-t_o)}$ has been assumed.

So, the problem has been reduced to a free field theory. We can use a Fourier decomposition in order to obtain decoupled degrees of freedom

$$\phi(\mathbf{x},t) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3\mathbf{k}[a(\mathbf{k})\varphi_k(t)e^{i\mathbf{k}.\mathbf{x}} + a^+(\mathbf{k})\varphi_k^*(t)e^{-i\mathbf{k}.\mathbf{x}}] \tag{IV.9}$$

The equation of motion for the Fourier components is

$$[\partial_t^2 + 3H\partial_t + (k^2+\gamma^2)e^{-2H(t-t_o)} - m^2]\varphi_k(t) = 0 \tag{IV.10}$$

where I have defined

$$\gamma^2 = \frac{\lambda T_o^2}{4\hbar} \tag{IV.11}$$

By defining

$$z = \frac{(k^2+\gamma^2)^{\frac{1}{2}}}{H} e^{-H(t-t_o)}$$

$$\nu = \left[\frac{9}{4} - \frac{m^2}{H^2}\right]^{\frac{1}{2}} \tag{IV.12}$$

equation (IV.10) can be written as

$$\left[z^2\frac{\partial^2}{\partial z^2} - 2z\frac{\partial}{\partial z} + z^2 - \nu^2 + \frac{9}{4}\right]\varphi_k(t) = 0 \tag{IV.13}$$

Which has the form of a Bessel equation. Its solutions can be written as

$$\varphi_k(t) \propto z^{\frac{3}{2}} H_\nu^{(1,2)}(z) \tag{IV.14}$$

where $H_\nu^{(1)}$ and $H_\nu^{(2)}$ denote the Haenckel functions (some useful properties of these are listed in Appendix B).

The general solution will be a linear combination of both of them with coefficients c_1 and c_2 satisfying

$$|c_1|^2 - |c_2|^2 = 1 \tag{IV.15}$$

Different choices of the constants c_1 and c_2 leads to different choices of the positive and negative frequency modes and can be interpreted as different choices of the vacuum state of the quantum field theory.

The choice of the initial quantum state of the field is based on the following considerations of the behaviour of the quantum field for early times. The mode $\varphi_k(t)$ describe the evolution of a perturbation of physical wavelength $(\frac{2\pi}{k})e^{Ht}$, and thus, for sufficiently early times the wavelength is very small compared to H^{-1} and at such short distance scales, the de Sitter space is indistinguishable from the Minkowsky space. This short wavelength limit corresponds to large values of z. The behaviour of the Haenckel functions for large z is given in Appendix B. For early times, equation (B.2) says that the Haenckel functions behave as

$$H_\nu^{1(2)}(z) \propto e^{-(+)i\omega\Delta t} \qquad (IV.16)$$

where

$$\omega = (k^2 + \gamma^2)^{\frac{1}{2}} \qquad (IV.17)$$

The choice of the initial state which corresponds to positive frequency modes in the flat space limit corresponds to $c_1 \to 1$, $c_2 \to 0$.

The normalization of the solutions follows from requiring that

$$\varphi_k \frac{\partial \varphi_k^*}{\partial t} - \varphi_k^* \frac{\partial \varphi_k}{\partial t} = i\hbar e^{-3Ht} \qquad (IV.18)$$

from which we obtain

$$\varphi_k(t) = \frac{1}{2}\sqrt{\frac{\pi}{H}} e^{-\frac{3}{2}H(t-t_o)} H_\nu^{(1)}(z) \qquad (IV.19)$$

If it is assumed that before the inflationary phase the universe was in a hot radiation dominated Friedmann phase, a reasonable assumption [63] is that the initial state of the system can be described by a thermal state at the background temperature $T = T_o\, e^{-H(t-t_o)}$. It will be denoted $|\psi_o>$. The expectation value of the number of particles operator in this state is given by

$$< \psi_o|a^+(\mathbf{k})a(\mathbf{k}')|\psi_o >= \frac{1}{e^{\theta_k} - 1}\delta^3(\mathbf{k} - \mathbf{k}') \qquad (IV.20)$$

where

$$\theta_k = \frac{\hbar}{T_0}(k^2+\gamma^2)^{\frac{1}{2}} \qquad (IV.21)$$

Note that if the temperature T_0 is taken to be zero, the resulting state would be the standard de Sitter space vacuum proposed by Gibbons and Hawking [64] and by Bunch and Davies [65].

Now we can proceed to compute the spectrum of fluctuations of the scalar field

$$|\delta\phi(\mathbf{x}-\mathbf{y})|^2 =< \psi_o|\phi(\mathbf{x},t)\phi(\mathbf{y},t)|\psi_o > \qquad (IV.22)$$

Replacing (IV.9) and using that the expectation values of the products of two creation or annihilation operators $a^+(\mathbf{k})$ and $a(\mathbf{k})$ are given by (IV.20) and

$$< \psi_o|a(\mathbf{k})a(\mathbf{k}')|\psi_o >= 0 \qquad (IV.23.a)$$

$$< \psi_o|a^+(\mathbf{k})a^+(\mathbf{k}')|\psi_o >= 0 \qquad (IV.23.b)$$

we obtain that

$$|\delta\phi(\mathbf{x}-\mathbf{y})|^2 = \frac{1}{32\pi^2}\frac{\hbar}{H}e^{-3H(t-t_o)}\int d^3k\ e^{i\mathbf{k}.(\mathbf{x}-\mathbf{y})}|H_\nu^{(1)}(z)|^2 coth(\frac{\theta_k}{2}) \qquad (IV.24)$$

The asymptotic behaviour for large times corresponds to the limit $z \longrightarrow 0$ and using equation (B.3) of Appendix B we find that

$$|\delta\phi(\mathbf{x}-\mathbf{y})|^2 \simeq \frac{1}{32\pi^2}\frac{\hbar}{H}e^{-3H(t-t_o)}\int d^3k\ e^{i\mathbf{k}.(\mathbf{x}-\mathbf{y})}\frac{\Gamma^2(\nu)}{\pi^2}\left[\frac{(k^2+\gamma^2)^{\frac{1}{2}}e^{-Ht}}{2H}\right]^{-2\nu} coth(\frac{\theta_k}{2}) \qquad (IV.25)$$

In the limit $m^2 << H^2$, we have that

$$\nu \simeq \frac{3}{2} - \frac{m^2}{3H^2} \qquad (IV.26)$$

and

$$\Gamma^2(\nu) \simeq \Gamma^2(\frac{3}{2}) = \frac{\pi}{4} \qquad (IV.27)$$

So, the scalar field fluctuation takes the form for $\mathbf{x}=\mathbf{y}$

$$\sigma^2 = \frac{1}{4\pi^2}\frac{\hbar}{H} exp\left[-\frac{2m^2}{3H}(t-t_o)\right] \int dk\ k^2 \left[\frac{(k^2+\gamma^2)^{\frac{1}{2}}}{H}\right]^{-3+\frac{2m^2}{3H^2}} coth(\frac{\theta_k}{2}) \qquad (IV.28)$$

In the limit $T_o \rightarrow 0$ (after several Hubble times the thermal contribution to $< \phi^2 >$ becomes negligible), it reduces to the generally used result [66,62]

$$\sigma^2 = \frac{\hbar}{4\pi^4} h^2 exp\left[-\frac{2m^2}{3H}(t-t_o)\right] \int_H^{He^{H(t-t_o)}} d\ lnk \left[\frac{k}{H}\right]^{\frac{2m^2}{3H^2}} =$$
$$= \frac{3}{8\pi^2}\frac{H^4}{m^2}\left[1 - exp\left[-\frac{2m^2}{3H}(t-t_o)\right]\right] \qquad (IV.29)$$

The upper integration limit is fixed by the last wavelength which crosses the Hubble radius at time t. The lower integration limit, being different from zero, takes into account that inflation starts at time t_o; and so it must correspond to the first wavelength which crossed the Hubble radius when inflation began. From (IV.29) we see that the contribution to $< \phi^2 >$ from fluctuations in the logarithmic interval of k, $\Delta lnk = 1$, is given by

$$\delta\phi(k) = \frac{H}{\sqrt{2\pi}}\left[\frac{k}{H}\right]^{\frac{m^2}{3H^2}} exp\left[-\frac{2m^2}{3H}(t-t_o)\right] \qquad (IV.30)$$

An important feature of $\delta\phi(k)$ is that it is scale independent for $m^2 << H^2$, $\delta\phi(k) \sim \frac{H}{\sqrt{2\pi}}$.

IV-4 Classical evolution of the perturbations

In chapter II, the evolution of the perturbations in the energy density have been discussed. An approximate conservation law (II.115), which is valid for wavelength larger than the Hubble radius was derived. By applying it, we can easily relate the amplitude of the energy density fluctuation at the final Hubble radius crossing $t_f(k)$ in the radiation or matter dominated era with its amplitude at the initial Hubble radius crossing at $t_i(k)$ in the inflationary phase. In order to use equation (II.116), we need to compute the quantity $(1+\omega(t_i))$ in the inflationary era. From (II.85) and neglecting the radiation contribution it results

$$1 + \frac{P_o(t_i)}{E_o(t_i)} = \frac{\dot{\phi}^2(t_i)}{E_o(t_i)} \qquad (IV.31)$$

If the perturbation wavelength crosses the Hubble radius in the radiation dominated phase, we have

$$\frac{3}{2} Z(rad) = \frac{2}{3} \frac{E_o(t_i)}{\dot{\phi}^2(t_i)} Z(inf) \qquad (IV.32)$$

From it, we can deduce the amplitude of the perturbation in ϵ_m at the Hubble radius crossing in the radiation dominated era at t_f

$$\epsilon_m|_H(t_f) = \frac{4}{9} \frac{E_o(t_i)\epsilon_m(t_i)}{\dot{\phi}^2(t_i)} \qquad (IV.33)$$

The fluctuation in the energy density $\epsilon_m(t_i)$ at the Hubble radius crossing in the inflationary era are those generated by the fluctuations in the scalar field. Quantum fluctuations in the scalar field during the inflationary era have been computed in the last section. The hypothesis is that they give rise to classical fluctuations in the energy density when the physical wavelength associated to the fluctuation become larger than the Hubble radius. As the quantization of the scalar field has been carried out by imposing commutation relations to the field and its canonical momentum in a $t = const$ hypersurface, the result obtained for the amplitude of quantum field fluctuations corresponds to its value in a synchronous gauge. So, I will compute the fluctuations in the energy density during inflation $\epsilon_m(t_i)$ in terms of the fluctuations in the scalar field $\delta\phi$ in a synchronous gauge ($A = B^{(0)} = 0$).

From (II.26.a) and (II.91.a) we have that

$$E_o\ \delta = \dot{\phi}\ \delta\dot{\phi} + U_\phi\ \delta\phi \qquad (IV.34)$$

and from (II.26.c) and (II.91.c)

$$E_o(1+\omega)v^{(0)} = \frac{k}{S}\ \dot{\phi}\ \delta\phi \qquad (IV.35)$$

Replacing these expressions in (II.53) we obtain that

$$E_o\ \epsilon_m = \dot{\phi}\ \delta\dot{\phi} + U_\phi\ \delta\phi + 3\frac{\dot{S}}{S}\dot{\phi}\ \delta\phi \qquad (IV.36)$$

and using the equation of motion for the scalar field

$$E_o \, \epsilon_m = \dot{\phi} \, \delta\dot{\phi} - \ddot{\phi} \, \delta\phi \tag{IV.37}$$

Thus, from (IV.33) we obtain that

$$\epsilon_m|_H(t_f) = \frac{4}{9} \frac{\dot{\phi}(t_i) \, \delta\dot{\phi}(t_i) - \ddot{\phi}(t_i) \, \delta\phi(t_i)}{\dot{\phi}^2(t_i)} \tag{IV.38}$$

When the slow rolling approximation is valid, $\ddot{\phi}(t_i)$ is negligible and the first term in (IV.38) dominates and we obtain

$$\epsilon_m|_H(t_f) = \frac{4}{9} \frac{\delta\dot{\phi}(t_i)}{\dot{\phi}(t_i)} \tag{IV.39}$$

In order to estimate $\delta\dot{\phi}$, we look for the equation of motion of $\delta\phi$

$$\delta\ddot{\phi} + 3H\delta\dot{\phi} = \frac{k^2}{S^2}\delta\phi - U_{\phi\phi}\delta\phi \tag{IV.40}$$

In the slow rolling approximation, the term $\delta\ddot{\phi}$ is negligible. When we specify the right hand side at the time of Hubble radius crossing , we obtain that the first term becomes $H^2\delta\phi$, so that the second term is negligible in the slow rolling approximation ($U_{\phi\phi} << H^2$), and

$$\delta\dot{\phi}(t_i) = \frac{H \; \delta\phi(t_i)}{3} \tag{IV.41}$$

Thus, the amplitude of the energy density perturbation at Hubble radius crossing becomes

$$\epsilon_m|_H(t_f) = \frac{4}{9} \frac{H\delta\phi(t_i)}{3\dot{\phi}(t_i)} \tag{IV.42}$$

When the slow rolling approximation is not valid and the second term in the right hand side of (IV.38) dominates, we have

$$\epsilon_m|_H(t_f) = \frac{4}{9} \frac{U_\phi[\phi(t_i)] \; \delta\phi(t_i)}{\dot{\phi}^2(t_i)} \tag{IV.43}$$

For perturbations which wavelength crosses the Hubble radius in the matter dominated era , the factor $\frac{4}{9}$ in the right hand side of (IV.42) and (IV.43) must be replaced by a factor

$\frac{2}{5}$. These equations allow us to compute the spectrum of the perturbations at Hubble radius crossing in a variety of inflationary universe models.

IV-5 Density perturbations in Coleman Weimberg SU(5) GUT model

One of the first models of new inflationary scenario considered was the SU(5) model with a Coleman Weimberg potential for the scalar field which spontaneously breaks the symmetry . This model was then discarded; one of the main reasons was that it predicts a spectrum for the energy density perturbations nearly scale invariant, but of a too much large amplitude [34,52]. As a first example, I will apply the results of the last section to evaluate the fluctuations in this model.

In this case, the zero temperature effective potential including one loop radiative corrections can be written as [67]

$$U_{eff}(\phi) = \frac{25}{16}\alpha^2 \left[\phi^4 \, ln(\frac{\phi^2}{\sigma^2}) + \frac{1}{2}(\sigma^4 - \phi^4)\right] \qquad (IV.44)$$

The true vacuum is at $\phi = \pm\sigma$, $\alpha = \frac{g^2}{4\pi} \simeq \frac{1}{45}$, where g is the gauge field coupling constant and $\sigma \simeq 1,2 \cdot 10^{15} Gev$.

The equation of motion for the scalar field becomes

$$\ddot{\phi} + 3H\dot{\phi} = -U_\phi = \frac{25}{4}\alpha^2\phi^3 |ln(\frac{\phi^2}{\sigma^2})| \qquad (IV.45)$$

The logarithm is a slowly varying function compared to ϕ^3 and the coupling constant α is "running", that is to say that its value depends on the choice of the renormalization scale, which is in principle arbitrary. By taking the renormalization scale equal to ϕ, it can be seen that the dependence on ϕ of α is also slowly varying compared to ϕ^3, so the term $\alpha^2 |ln(\frac{\phi^2}{\sigma^2})|$ may be approximated by a constant and the equation of motion for the scalar field, neglecting the acceleration term is written as

$$3H\dot{\phi} = a \; \phi^3 \qquad (IV.46)$$

where $a \simeq \frac{1}{2}$ for $\phi \simeq H$. Its solution is

$$\phi^2 = \frac{3H}{2a(t_{RH} - t)} \qquad (IV.47)$$

where t_{rh} denotes the reheating time, representing the final time for the inflationary era. In order to apply the equation (IV.42) of the last section to evaluate the spectrum of perturbations generated, we need to compute the time of Hubble radius crossing during the inflationary era t_i. It is defined by

$$k\ e^{-Ht_i} = H \qquad (IV.48)$$

From which we obtain

$$t_{RH} - t_i = \frac{1}{H} ln \left[\frac{H}{k} e^{Ht_{RH}} \right] \qquad (IV.49)$$

In order to use this relation, we need to express the wavenumber k in terms of the physical distance scale measured at the present time λ_o.

$$\lambda_o = S_o\ k^{-1} = \frac{T_{RH}}{T_o} H^{-1} \left[\frac{H}{k} e^{Ht_{RH}} \right] \qquad (IV.50)$$

Thus

$$\frac{H}{k} e^{Ht_{RH}} \simeq \frac{\lambda_o}{b} \qquad (IV.51)$$

where

$$b = \frac{T_{RH}}{T_o} H^{-1} \simeq 10^{-15} ly \qquad (IV.52)$$

A galactic scale corresponds to about $10^6 ly$, so $\frac{\lambda_o}{b} \simeq 10^{21}$.

So, from (IV.46) we can compute

$$\dot{\phi}(t_i) = \sqrt{\frac{3}{8a}} H^2 \frac{1}{ln^{\frac{3}{2}}(\frac{\lambda_o}{b})} \qquad (IV.53)$$

Replacing (IV.53) into (IV.40) and using the results of section IV-3, we obtain that

$$\epsilon_m|_H(t_f) \simeq \frac{8}{27\sqrt{6\pi}} ln^{\frac{3}{2}}(\frac{\lambda_o}{b}) \simeq 1,3 \qquad (IV.54)$$

This value is much too large (four orders of magnitude larger than the observational limits) to be acceptable. So, it seems clear that this theory is inconsistent with the observed universe.

Another point against this model is the fact that it has been shown that the scalar field would not remain at the top of the potential hill for long enough to provide an adequate amount of inflation [66,68].

Both problems are due to the fact that quantum fluctuations of the scalar field are too large. As these fluctuations are determined by general principles of quantum field theory and are not much dependent on the detailed shape of the potential, what is needed to solve them is a potential for which the classical solution gives a value of $\dot{\phi}(t)$ large enough to suppress the effect of these fluctuations.

Once the failure of the minimal SU(5) model was discovered, efforts turn towards the construction of particle theories that have the properties necessary for the new inflationary cosmology. Some conditions that must be fulfilled for a successful new inflation have been listed in reference [51]. Lots of models can be found in the literature, but none of them have been generally accepted up to the present.

Another approach has been the proposal of a different version of the inflationary scenario: the chaotic inflation, which have been discussed at the end of the last chapter. I will analyse the density fluctuations in this case.

IV-6 Density fluctuations in chaotic inflation

In the chaotic inflationary scenario, an inflationary era occurs for a rather general type of potentials for the scalar field. However, we shall see that if we want that the amplitude of the resulting perturbations in the energy density be compatible with observations, we need to impose constraints on the strength of the scalar field interactions. To this end, I will apply the formalism of section (IV.4) to the case of a scalar field with a potential

$$U(\phi) = \frac{\lambda}{4}\phi^4 \qquad (IV.55)$$

In the chaotic scenario, H is no more nearly constant during inflation, but is given by

$$H^2 = \frac{8\pi}{3M_P^2}U(\phi) \qquad (IV.56)$$

By another hand, from the slow rolling condition we have

$$\dot{\phi} = -\frac{1}{3H}U_\phi = -\frac{1}{3H}\lambda\phi^3 \qquad (IV.57)$$

At Hubble radius crossing

$$\frac{k}{S(t_i)} = H(t_i) = \sqrt{\frac{2\pi\lambda}{3}}\frac{\phi^2(t_i)}{M_P} \qquad (IV.58)$$

A typical wavenumber associated with a galactic scale corresponds to

$$\lambda_o = \frac{T_{RH}S_{RH}}{T_o}\frac{1}{k} \qquad (IV.59)$$

So, the value of the scalar field corresponding to the time at which galactic scales crosses the Hubble radius is given by

$$\phi^2(t_i)exp\left[\frac{\pi}{M_P^2}(\phi^2(t_{RH}) - \phi^2(t_i))\right] = \frac{T_{RH}}{T_o}\sqrt{\frac{3}{2\pi}\frac{1}{\lambda_o}\frac{1}{\sqrt{\lambda}}} \qquad (IV.60)$$

with $\lambda_o \simeq 10^6 ly, T_{RH} = 10^{14}Gev, T_o = 3 \cdot 10^{-13}Gev, \phi_{RH} \simeq 3M_P$

Now we can apply equation (IV.42) to evaluate the magnitude of the energy density fluctuations predicted, with the help of (IV.57) and (IV.30)

$$\epsilon_m|_H(t_f) \simeq \frac{8}{27}\sqrt{\frac{\pi\lambda}{3}}\frac{\phi^3}{M_P^3} \qquad (IV.61)$$

In order that the predicted amplitude be compatible with observations, the expression in (IV.61) must take a value of order $\mathcal{O}(10^{-4})$. This condition and equation (IV.60) form a system of coupled equations for the variables $\phi(t_i)$ and the coupling constant λ. By solving it ,we obtain that

$$\phi(t_i) \simeq 5,3M_P \qquad (IV.62)$$

and

$$\lambda \simeq 5 \cdot 10^{-12} \qquad (IV.63)$$

So, we see that the requirement of having sufficiently small density fluctuations restrict the coupling constant of the scalar field to be very small.

IV-7 Remarks

I have presented in this chapter a sketch of the computation of the energy density fluctuations originated by the quantum fluctuations of the inflaton field during the inflationary phase. This is a very important result because it gives the possibility of making quantitative predictions about the spectrum of the fluctuations generated by a given model. This feature is attractive by one side because the fluctuation spectrum can be computed without doing any assumption about its initial value, and by other side because the comparison of the predicted results with the observed values provide a test for the inflationary models.

Nevertheless, some remarks need to be done about the methodology used. In first place, there is no general agreement about how the "classical field", that which roll down the potential hill following the classical equation of motion, is related to the real quantum field. Hawking and Moss [69] have pointed out that if the system begins in a thermal state which has an exact symmetry $\phi \to -\phi$ and the dynamics is also consistent with this symmetry, $\langle\phi(\mathbf{x},t)\rangle$ will be zero for all times (the field will roll down the hill, but as the probability is the same in any direction, the expectation value will remain zero). A number of authors [69,70] have suggested that $\sqrt{\langle\phi^2(\mathbf{x},t)\rangle}$ can play the role of $\phi(t)$, but this identification is problematic, as it is not evident why it should evolve following the classical equation of motion. Guth and Pi [63] instead proposed to "smear" the quantum field over some finite volume in order to obtain a measurable operator, then decompose the Fourier expansion into the contribution of wavelength larger and smaller than the size of the observed universe and identify $\phi(t)$ with the term corresponding to the larger wavelengths, as they can be considered homogeneous for astronomical purposes. In the last years, a different approach to the study of the dynamics of scalar field phase transition has been developed by several authors [71,72,73]. The idea is to consider inflation as a stochastic process. The realization that stochastic processes can be applied to the theory of inflation came in part from the observation that the equation of motion for the scalar field in the slow rolling approximation yields a Langevin equation. The scalar is decomposed in a "coarse grain" and a "fine grain" contribution, which correspond to wavelengths larger or smaller than H^{-1}. The equation of motion of the coarse grain field is of the Langevin type (or Fokker-Planck type) with a noise term given by the fine grain field (corresponding to quantum fluctuations). In this way, the dynamics of the large scale quasi homogeneous scalar field producing the inflationary stage is affected by small scale quantum fluctuations which can produce classical perturbations as they expand beyond the Hubble radius.

Another point which need to be examined in more detail is the hypothesis that quantum fluctuations of the scalar field give rise to classical fluctuations when their wavelength become larger than the Hubble radius. Such kind of hypothesis is unavoidable in the frame of the semiclassical theory considered. Guth and Pi [63] have shown that the product of the quantum uncertainties in the scalar field and its canonical momentum becomes much smaller than the product of the variables themselves for wavelength much smaller than the Hubble radius. Lyth [36] has shown that a few Hubble times after the field fluctuations leave the Hubble radius, they can be considered "classical" in the sense that it becomes possible to form a wave packet in the field variable which does not spread appreciably. These results have been criticized by Sasaki [74] because they are based on the study of the behaviour of the scalar fluctuations for long wavelengths, but neglecting the fluctuations in the metric. He argue that the criterion for the validity of the classical description of a quantum system, as the product $\langle\varphi|X^2|\varphi\rangle\langle\varphi|P^2|\varphi\rangle$ is not invariant under canonical transformations of the dynamical variables X and P is problematic and, due to the arbitrariness in the choice of coordinate gauge on which the perturbation variables depend, there is no natural choice for them. He suggested that it would be necessary to extend the quantum mechanical analysis up to the reheating era. Another proposal was made by Boucher and Trashen [75], who suggested to modify the usual semiclassical theory used to study the dynamics of the gravitational field (treated classically) and of the quantum matter fields. In this one, the gravitational field couples to the quantum fields only through the expectation value of the stress energy tensor in some quantum state. They proposed to modify it in such a way that quantum fluctuations acts also as a source for the classical variables. But no results are yet available within this scheme.

Conclusions

Inflationary universe models provide a mechanism which, for the first time, explains from first principles the origin of the primordial energy density fluctuations required as initial conditions in theories of galaxy formation. Quantum fluctuations of the scalar field which drives inflation during the inflationary phase give rise to adiabatic energy density fluctuations which spectrum at the time at which the perturbation wavelength cross the Hubble radius is roughly scale invariant. In order that the amplitude of the perturbations be compatible with the observations, the scalar field must be very weakly coupled. The prediction of a scale invariant spectrum is considered a nice feature in inflationary models, because this type of spectrum has been widely used well.

Nevertheless, in the last times, some problems associated to the use of a primordial scale invariant spectrum have been pointed out, and it seems very difficult that the multitude of different objects in the universe can be constructed exclusively from it. For some time, no other perturbation of a sufficiently large magnitude generated during inflation was known, but recently some other processes, as for example strings models of galaxy formation and isothermal perturbations during inflation, have added new possibilities. Each of these processes generate different kinds of fluctuations, and one should not necessarily expect that all the structures originate from any single effect. The largest scale features (such as giant voids and filaments) and the galaxy correlation function may well derive from completely different mechanisms.

APPENDIX A

Perturbation formulae for geometrical quantities

1- Scalar perturbations

- Christoffel symbols

$$
\begin{aligned}
\delta\Gamma^{0}_{00} &= A' Q^{(0)} \\
\delta\Gamma^{0}_{0\alpha} &= -[kA + \tfrac{S'}{S} B] Q^{(0)}{}_{\alpha} \\
\delta\Gamma^{\alpha}_{00} &= -[kA + B' + \tfrac{S'}{S} B] Q^{(0)\alpha} \\
\delta\Gamma^{\alpha}_{0\beta} &= H_L' \, \delta^{\alpha}_{\beta} Q^{(0)} + H_T' Q^{(0)\alpha}{}_{\beta} \\
\delta\Gamma^{0}_{\alpha\beta} &= \left[-2(\tfrac{S'}{S})A + \tfrac{k}{3}B + \tfrac{(S^2 H_L)'}{S^2}\right] {}^{3}g_{\alpha\beta} Q^{(0)} + \left[-kB + \tfrac{(S^2 H_T)'}{S^2}\right] Q^{(0)}{}_{\alpha\beta} \\
\delta\Gamma^{\alpha}_{\beta\gamma} &= -kH_L \left[\delta^{\alpha}_{\beta} Q^{(0)}{}_{\gamma} + \delta^{\alpha}_{\gamma} Q^{(0)}{}_{\beta} - {}^{3}g_{\beta\gamma} Q^{(0)\alpha}\right] + \tfrac{S'}{S} B \, {}^{3}g_{\beta\gamma} Q^{(0)\alpha} \\
&\qquad + H_T \left[Q^{(0)\alpha}{}_{\beta|\gamma} + Q^{(0)\alpha}{}_{\gamma|\beta} - Q^{(0)}{}_{\beta\gamma}{}^{|\alpha}\right]
\end{aligned}
$$

- Einstein tensor

$$
\begin{aligned}
\delta G^{0}_{0} &= \tfrac{2}{S^2}\left[3(\tfrac{S'}{S})^2 A - \tfrac{S'}{S} kB - 3\tfrac{S'}{S} H_L' - (k^2 - 3K)(H_L + \tfrac{H_T}{3})\right] Q^{(0)} \\
\delta G^{0}_{\alpha} &= \tfrac{2}{S^2}\left[k\tfrac{S'}{S} A - K\ B - k\ H_L' - \tfrac{k^2 - 3K}{3k} H_T'\right] Q^{(0)}{}_{\alpha} \\
\delta G^{\alpha}_{0} &= \tfrac{2}{S^2}\left[-k\tfrac{S'}{S} A + [(\tfrac{S'}{S})' - (\tfrac{S'}{S})^2]B + k\ H_L' + \tfrac{k^2 - 3K}{3k} H_T'\right] Q^{(0)\alpha} \\
\delta G^{\alpha}_{\beta} &= \tfrac{2}{S^2}[\tfrac{S'}{S} A' + [\tfrac{2S''}{S} - (\tfrac{S'}{S})^2]A - \tfrac{k^2}{3} A - \tfrac{k}{3}(B' + 2\tfrac{S'}{S} B) - \tfrac{1}{S}(S\ H_L')' - \\
&\qquad - \tfrac{S'}{S} H_L' - \tfrac{1}{3}(k^2 - 3K)(H_L + \tfrac{H_T}{3})] \delta^{\alpha}_{\beta} Q^{(0)} + \\
&+ \tfrac{1}{S^2}\left[-k^2 A - k(B' + \tfrac{S'}{S} B) + \tfrac{1}{S}(S\ H_T')' + \tfrac{S'}{S}(H_T' - kB) - k^2(H_L + \tfrac{H_T}{3})\right] Q^{(0)\alpha}{}_{\beta}
\end{aligned}
$$

2- Vector perturbations

- Christoffel symbols

$$\delta\Gamma^0_{00} = 0$$
$$\delta\Gamma^0_{0\alpha} = -\frac{S'}{S} B Q^{(1)}{}_\alpha$$
$$\delta\Gamma^\alpha_{00} = -\left[B' + \frac{S'}{S}B\right] Q^{(1)\alpha}$$
$$\delta\Gamma^\alpha_{0\beta} = H_T' Q^{(1)\alpha}{}_\beta + \frac{1}{2}B\left[Q^{(1)}{}_\beta{}^{|\alpha} - Q^{(1)\alpha}{}_{|\beta}\right]$$
$$\delta\Gamma^0_{\alpha\beta} = \left[-kB + \frac{(S^2 H_T)'}{S^2}\right] Q^{(1)}{}_{\alpha\beta}$$
$$\delta\Gamma^\alpha_{\beta\gamma} = \frac{S'}{S} B\, {}^3g_{\beta\gamma} Q^{(1)\alpha} + H_T\left[Q^{(1)\alpha}{}_{\beta|\gamma} + Q^{(1)\alpha}{}_{\gamma|\beta} - Q^{(1)}{}_{\beta|\gamma}{}^{|\alpha}\right]$$

- Einstein tensor

$$\delta G^0_0 = 0$$
$$\delta G^0_\alpha = \frac{2K-k^2}{2S^2k}[H_T' - k\ B]Q^{(1)}{}_\alpha$$
$$\delta G^\alpha_0 = \frac{1}{S^2}\left[-\left[\frac{2K-k^2}{2} + 2[-(\frac{S'}{S})' + (\frac{S'}{S})^2]\right] B - \frac{2K-k^2}{2k} H_T'\right] Q^{(1)\alpha}$$
$$\delta G^\alpha_\beta = \frac{1}{S^2}\left[-k(B' + \frac{S'}{S}B) + \frac{1}{S}(S\ H_T')' + \frac{S'}{S}(H_T' - kB)\right] Q^{(1)\alpha}{}_\beta$$

3- Tensor perturbations

- Christoffel symbols

$$\delta\Gamma^0_{00} = \delta\Gamma^0_{0\alpha} = \delta\Gamma^\alpha_{00} = 0$$
$$\delta\Gamma^\alpha_{0\beta} = H_T' Q^{(2)\alpha}{}_\beta$$
$$\delta\Gamma^0_{\alpha\beta} = \frac{(S^2 H_T)'}{S^2} Q^{(2)}{}_{\alpha\beta}$$
$$\delta\Gamma^\alpha_{\beta\gamma} = H_T\left[Q^{(2)\alpha}{}_{\beta|\gamma} + Q^{(2)\alpha}{}_{\gamma|\beta} - Q^{(2)}{}_{\beta\gamma}{}^{|\alpha}\right]$$

- Einstein tensor

$$\delta G^0_0 = \delta G^0_\alpha = \delta G^\alpha_0 = 0$$
$$\delta G^\alpha_\beta = \frac{1}{S^2}\left[\frac{1}{S}(S\ H_T')' + \frac{S'}{S}H_T' + (k^2 + 2K)H_T\right] Q^{(2)\alpha}{}_\beta$$

4- Intrinsic curvature perturbations

For each time slicing, there is associated an intrinsic scalar curvature ${}^3R$ of each constant time hypersurface. From the perturbed intrinsic metric $g_{\alpha\beta}$, the perturbed spatial Riemann curvature tensor and Ricci tensor can be computed for general perturbations. The result obtained for this last one is

$$
\begin{aligned}
{}^3R^\alpha_\beta = \frac{1}{S^2}\left[2K + \tfrac{4}{3}(k^2 - 3K)(H_L + \tfrac{1}{3}H_T^{(0)})Q^{(0)}\right]\delta^\alpha_\beta \\
-\frac{k^2}{S^2}(H_L + \tfrac{1}{3}H_T^{(0)})Q^{(0)\alpha}{}_\beta + \frac{(k^2+2K)}{S^2}H_T^{(2)}Q^{(2)\alpha}{}_\beta
\end{aligned}
$$

Contracting indices, we obtain for the intrinsic scalar curvature

$$
{}^3R = \frac{1}{S^2}\left[6K + 4(k^2 - 3K)(H_L + \tfrac{1}{3}H_T^{(0)})Q^{(0)}\right]
$$

APPENDIX B

Some useful properties of Bessel functions

I summarize here some properties of Bessel functions [76], which are useful in chapter IV computations.

Bessel functions of the first kind are denoted $J_\nu(z)$, Bessel functions of the second kind (also called Neuman functions) are denoted by $N_\nu(z)$ and Bessel functions of the third kind (also called Haenckel functions) are denoted by $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$.

The Haenckel functions are related to $J_\nu(z)$ and $N_\nu(z)$ by

$$H_\nu^{(1)}(z) = J_\nu(z) + iN_\nu(z) \qquad (B.1.a)$$

$$H_\nu^{(2)}(z) = J_\nu(z) - iN_\nu(z) \qquad (B.1.b)$$

The behaviour of the Haenckel functions for asymptotically large z is given by

$$H_\nu^{(k)}(z) \sim \left[\frac{2}{\pi z}\right]^{\frac{1}{2}} exp\left[\pm i(z - \frac{\pi}{2}\nu - \frac{\pi}{4})\right]\left[1 + \mathcal{O}\left[\frac{1}{|z|}\right]\right] \qquad (B.2)$$

where the plus and minus signs holds for $k = 1$ and $k = 2$ respectively. For small z, one has the following asymptotic forms

$$H_\nu^(k)(z) = \mp\frac{i}{\pi}\Gamma(\nu)\left[\frac{z}{2}\right]^{-\nu} \qquad (B.3)$$

where the minus and plus signs holds for $k = 1$ and $k = 2$ respectively.

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