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"MAGISTER PHILOSOPHIAE"

THE MASSLESS DIPOLE FIELD

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CHAPTER 1.

1.1 INTRODUCTION.

Most of the conventional wisdom on gauge quantum field theory has been obtained by using local (renormalizable) gauges and therefore it is of some interest to explore the general mathemathical and physical properties of this kind of approach. It may be shown that both in the gauge symmetry breaking case as well as in the non-abelian unbroken case the Wightman functions of the field algebra exibit infrared singularities which are worse than measures and therefore they cannot satisfy the axiom of positivity.

It is believed that many of the interesting features emerging in the local formulations of gauge quantum field theories are related to the occurrence of such infrared singularities [1] [2].

The aim of this thesis is to consider one concrete model, the dipole field model, which isolates one of the characteristics infrared singularities of local gauge field theories .

Actually it has been rigorously proven that this type of singularity occurs in the abelian gauge symmetry breaking case , and is the quantum field theory version of the linearly rising potential in Q.C.D. .

The lack of positivity that depends on the infrared behavior of the correlation functions, is at the basis of the possibility of explaining certain unconventional mechanisms not shared by the standard Q.F.T. . From a more technical point of view, given a set of such correlation functions, the problem is to construct the corresponding Hilbert space of states. In the standard case this is an intrinsic content of the Wightman functions and is provided by the reconstruction theorem [3]. In the indefinite metric case the Wightman functions supply only a linear space (the local states) with an intrinsic indefinite inner product. A supplement of informations is needed in order to obtain a Hilbert space of states [4]; this is what will be called "Hilbert space structure condition" and plays the role of the axiom of positivity. This condition is usually given by assigning a set of Hilbert seminorms which majorize the Wightman functions.

Clearly there is some arbitrariness in this procedure because to different sets of seminorms correspond different Hilbert spaces of states. Among all the possible Hilbert structures are of particular interest those which generate a maximal Hilbert space,because they contain in a certain sense the maximum of information that is possible. In this case one recognizes that the Hilbert space of the theory has a Krein type structure [5] [6] and it is possible to show that at the one-particle level this structure is unique (up to isomorphisms).

When the construction of one Hilbert space has been performed it is necessary to identify in it a subspace in which the inner product defined in terms of the Wightman functions is positive and this for the probabilistic interpretation of the theory. The elements of this subspace will identify the physical states and the condition that selects them is called "subsidiary condition" . It is usual to give the subsidiary condition as an operator condition of the type $A\Psi=0$ (like the Gupta-Bleuler condition).

One could at this point wonder whether it would be possible to identify the physical states already at the level of the local states and then to get a Hilbert space by completion of this subspace in the topology induced by the intrinsic inner product which is positive on this subspace. In this way however one can at most obtain the vacuum sector; as a matter of fact the problem of existence and construction of charged states (confinement and Higgs phenomenon) cannot be solved in this way because charged states cannot be local and their construction requires some closure of the local states in a Hilbert topology

All these structures will be explicitely constructed for the dipole field, i.e. a field ϕ obeying the equation

$$\Delta^2\phi = 0 \tag{1.1}$$

where Δ is the d'Alembert operator $\partial_t^2 - \partial_x^2 - \partial_y^2 - \partial_z^2$.

This field that is in strict correlation with the Froissart model [7] has been studied in past by several people: in particular there are the contributions of Ferrari [8], Zwanziger [9], Narnhofer and Thirring [10]

However all the treatments known leave open non trivial questions of principle and therefore may be interesting to reconsider the problem from the very beginning.

Indeed in the discussion given by Zwanziger, the choice of the Hilbert structure is somewhat ad hoc and is not used to discuss the physical interpretation of the model. Actually Zwanziger concludes that there is no physical interpretation of the model. On the other hand the CCR approach to the model that can be found in the paper of Narnhofer and Thirring exhibit a quantization with non implementable time translations but it is not clear how general is this feature since many questionable ingredients have been used in the derivation.

The present work investigates some aspects of the dipole field without special or ad hoc a priori assumption. The general strategy will be that of keeping locality and covariance and eventually the physical interpretation will be obtained through a subsidiary condition. The main results that will be obtained are:

a) the rigorous control on the Krein -Hilbert structure associated to the Wightman functions of the dipole field.

b) a Fock-Krein representation of the field algebra and a discussion of the uniqueness of the translationally invariant states.

c) The existence of translationally invariant field operators belonging to the Krein strong closure of the local field algebra.

d) Some remarks about the breaking of the Poincare' group and a concrete realization of certain classes of physical spaces.

Possible future developments are a general characterization of the possible physical spaces and a study of the symmetries of the model and their implementation.

1.2 THE WIGHTMAN FUNCTIONS.

(The references for the general results of functional analysis are [11] [12] [13]).

A local and covariant quantization of the dipole ghost field is characterized by a set of Wightman functions satisfying the following axioms:

I. COVARIANCE

For any Poincare' transformation $\{a,\Lambda\}$ the n-points functions are invariant:

$$\mathcal{W}(\Lambda x_1 + a, \ldots, \Lambda x_n + a) = \mathcal{W}(x_1, \ldots, x_n) \tag{1.2}$$

II. LOCALITY

If $x_i - x_{i+1} = \xi_i$ is spacelike then

$$\mathcal{W}(x_1, \ldots, x_i, x_{i+1}, \ldots, x_n) = \mathcal{W}(x_1, \ldots, x_{i+1}, x_i, \ldots, x_n) \tag{1.3}$$

III. WEAK SPECTRAL PROPERTIES

The Fourier transforms $\hat{W}(k_1, \ldots, k_{n-1})$ of the distributions

$$W(\xi_1, \ldots, \xi_{n-1}) = \mathcal{W}(x_1, \ldots \ldots x_n) \tag{1.4}$$

have supports contained in the cones $(k_i)^2 = k_i^{\mu} k_{i\mu} \geq 0$, $k_{io} > 0$.

The Fourier transform of test functions belonging to $\mathcal{S}(R^4)$ is defined

$$\hat{f}(k) = (2\pi)^{-2} \int e^{ikx} f(x)\, d^4x \tag{1.5}$$

where kx is the Lorentz invariant scalar product

$$kx = k^{\mu} x_{\mu} = g_{ij} k^i k^j \tag{1.6}$$

with $g_{ij} = \mathrm{diag}(1,-1,-1,-1)$

If $T \in \mathcal{S}'(R^4)$ its Fourier transform is defined

$$\hat{T}(f) = T(\hat{f}) \tag{1.7}$$

Starting from a set of Wightman functions obeying axioms I-II-III it is only possible to recover a set of vector states which has a linear structure; one starts from the Borchers algebra $\mathcal{B}$ whose elements are finite sequences

$$\underline{f} = (f_0, \ldots, f_j, \ldots) \tag{1.8}$$

where f_0 is any complex number, $f_j \in \mathcal{S}(R^{4j})$, and all but a finite number of test functions are zero.

The Wightman functional $\underline{\mathcal{W}}$ defines a linear functional on $\mathcal{B}$ through

$$\underline{\mathcal{W}}(\underline{f}) = \sum \mathcal{W}_n(f_n) \tag{1.9}$$

Defining the tensor product in $\mathcal{B}$ as

$$(\underline{f} \times \underline{g})_n = \sum_{k+l=n} f_k \, g_l \tag{1.10}$$

one obtains the following inner product

$$<\underline{f},\underline{g}> = \underline{\mathcal{W}}\,(\underline{f}^* \times \underline{g}) \tag{1.11}$$

where $f^*(x_1,\ldots,x_n) = \bar{f}(x_n,\ldots,x_1)$ and the "-" means complex conjugation. The linear space D_0 is then defined as the set of equivalence classes $[\underline{f}] \in \mathcal{B}/I_W$ where

$$I_W = \{\underline{f} \in \mathcal{B} \; : \; <\underline{f},\underline{g}> = 0 \quad \forall \; \underline{g} \in \mathcal{B} \; \} \tag{1.12}$$

is the Wightman ideal. The field operators are defined on D_0 by

$$\phi(f)[\underline{g}] = [\underline{f} \times \underline{g}] \tag{1.13}$$

with $f \in \mathcal{S}(R^4)$ and $\underline{f} = (0, f, 0, \ldots)$.

Clearly the vector $\Psi_0 = (f_0, 0, \ldots)$ is non zero, is such that

$<\Psi_0,\Psi_0>$ is greater than zero and is cyclic with respect to $\mathcal{F}$, the polynomial algebra generated by the fields $\phi(f)$. There is a linear representation of the restricted Poincare' group on D_0 defined by

$$U(a,\Lambda)P(\phi(f))\Psi_0= P(\phi(f_{\{a,\Lambda\}}))\Psi_0 \qquad (1.14)$$

where P denotes a polynomial and $f_{\{a,\Lambda\}}(x)=f(\Lambda^{-1}(x-a))$.
For more details about these structures see [4].
In the present case the field equation (1.1) is used to determinate the two-point function and the truncated n-points function are assumed to vanish (it is ever possible to take $\mathcal{W}_1=0$).The equation (1.1) imply that the two-points function must satisfy the equation

$$\Delta^2 W(x) = 0 \qquad (1.15)$$

The most general solution of this equation which agrees with the spectral condition is [8]

$$W(x)=b_1\ln[-(x_0- i\epsilon)^2+ \mathbf{x}^2] + b_2(x^2-i\epsilon x_0) + b_3 \qquad (1.16)$$

Clearly the relevant term is the first one in the R.H.S. because the other two terms correspond to a massless 4-dimensional free scalar field and this is an uninteresting solution of the equation (1.16). For motivations that will be clear in the following one chooses $b_1= -(4\pi)^{-2}a$.By the appendix 1A the Fourier transform of

$$W(x)=-(4\pi)^{-2}\ln[-(x_0-i\epsilon)^2+ \mathbf{x}^2] \quad \text{is:} \qquad (1.17)$$

$$\hat{W}(k)= (2\pi)^{-1}a\ \vartheta(k_0)\ \delta'(k^2) + \hat{f}(k) \qquad (1.18)$$

where $k^2\hat{f}(k)=0$.This again implies that $\hat{f}(k)$ describes a massless free field and may be transcured.

The hypothesis of factorization of the n-point function implies that the one-particle space contains already all the informations of the theory and the n-particle spaces may be obtained by (symmetric) tensorial products. Therefore one defines in $\mathscr{S}(R^4)$ the inner product

$$<f,g> = \int \bar{f}(x)\mathscr{W}(x,y)g(y)d^4xd^4y \qquad (1.19)$$

The Fourier transform of $\mathscr{W}(x,y)$ is

$$\hat{\mathscr{W}}(k,q) = (2\pi)^2\hat{W}(k)\delta(k+q) \qquad (1.20)$$

$$\text{with} \quad \hat{W}(k) = (2\pi)^{-2}\int e^{ikx}\, W(x)\, d^4x \qquad (1.21)$$

It follows that

$$<f,g> = \pi a \int \vartheta(k_o)\delta'(k^2)\bar{\hat{f}}(k)\hat{g}(k)d^4k \qquad (1.22)$$

Using the results of appendix 1.B this scalar product may be written

$$<f,g> = \tfrac{1}{4}\pi a \int d\mu(k)\{\bar{F}_1(k)G_1(k)-\bar{F}_2(k)G_2(k) + \bar{\hat{f}}(0)\hat{\chi}(k)G_1(k)$$

$$+ \bar{\hat{g}}(0)F_1(k)\hat{\chi}(k)\} \qquad (1.23)$$

They have been used the following definitions: χ is a real function of $\mathscr{S}(R^4)$ such that 1) $\hat{\chi}(0)=1$, 2) $k_o\partial_o\hat{\chi}(k)\big|_{C_+}=0$,

$$3)\int \hat{\chi}^2(k)d\mu(k) = 0 \qquad (1.24)$$

The integral in (1.24,3) has obviously a distributional meaning. C_+ is the mantle of the future cone V_+.

$$\hat{f}_o(k) = \hat{f}(k) - \hat{f}(0)\hat{\chi}(k) \qquad (1.25)$$

$$F_1(k) = \hat{f}_o(k) - D\hat{f}_o(k) \qquad (1.26)$$

$$F_2(k) = D\hat{f}_o(k) = k_o \partial_o \hat{f}_o(k) \qquad (1.27)$$

$$\omega^2 = k_1^2 + k_2^2 + k_3^2 \qquad (1.28)$$

$$d\mu(k) = \omega^{-3} \delta(k_o - \omega) d^4 k \qquad (1.29)$$

The definition(1.25) has a great importance in the procedure that will be exploited[4][14]. It identifies the part of a test function which is free from infrared singularities. From (1.25) follows the splitting

$$\mathcal{S}(R^4) = \mathcal{S}_o(R^4) + \{\chi\} \qquad (1.30)$$

$$\mathcal{S}_o(R^4) = \{ f \in \mathcal{S}(R^4) : \hat{f}(0) = 0 \} \qquad (1.31)$$

and the bad infrared singular part of the theory is singled out in the function χ.

APPENDIX 1.A

In this appendix it will be calculated the Fourier transform of the distribution

$$W(x) = -(4\pi)^{-2} a \ln [\mathbf{x}^2 - (x_o - i\epsilon)^2]$$

To this aim the subsequent formulae will be used [15]

$$\Delta \ln [\mathbf{x}^2 - (x_o - i\epsilon)^2] = -4 [\mathbf{x}^2 - (x_o - i\epsilon)^2]^{-1} \qquad \text{(A1.1)}$$

$$\int \vartheta(k_o)\, \delta(k^2) \exp[-i(x_o - i\epsilon)k_o + i\mathbf{xk}] d^4k = 4\pi [\mathbf{x}^2 - (x_o - i\epsilon)^2]^{-1} \qquad \text{(A1.2)}$$

Then inserting into (A1.2) the identity [16]

$$\delta(k^2) + k^2 \delta'(k^2) = 0 \qquad \text{(A1.3)}$$

and using (A1.1) one obtains

$$\Delta \int \vartheta(k_o)\, \delta'(k^2) \exp[-i(x_o - i\epsilon)k_o + i\mathbf{xk}]\, d^4k =$$

$$= -\pi \Delta \ln [\mathbf{x}^2 - (x_o - i\epsilon)] \qquad \text{(A1.4)}$$

This implies that

$$(2\pi)^{-2} \int \vartheta(k_o)\, \delta'(k^2) \exp[-i(x_o - i\epsilon)k_o + i\mathbf{xk}] d^4k =$$

$$-(4\pi)^{-1} \ln [\mathbf{x}^2 - (x_o - i\epsilon)^2] + f(x) \qquad \text{(A1.5)}$$

where $f(x)$ is solution of the equation $\Delta f=0$ and for this reason it is an uninteresting term. Now using the Fourier inversion theorem one has

$$\hat{W}(k) = (4\pi)^{-1} a\, \vartheta(k_o)\, \delta'(k^2) + \hat{f}(k) \qquad \text{(A1.6)}$$

with $k^2 f(k) = 0$.

APPENDIX 1.B

The aim of this appendix is to give a concrete definition of the distribution $\vartheta(k_o)\delta'(k^2)$. The methods used are those of [16].One exploits the following change of variables $(k_o,k_1,k_2,k_3) \Rightarrow (P,\omega,\gamma,\phi)$ with

$$P=k^2,\ \omega=(\vec{k}^2)^{1/2}, \gamma = \mathrm{artg}(k_1/k_2),\ \phi = \arccos(k_3/\omega) \qquad \text{(A1.7)}$$

This change of variables has perfectly meaning in an opportune neighborhood of the mantle C_+ of the future cone but the new frame is not orthogonal. One has that

$$d^4k = \frac{1}{2}(P+\omega)\ \omega^2\ dP\ d\omega\ d\Omega \qquad \text{(A1.8)}$$

$\frac{\partial}{\partial P}$ means here partial derivation with respect to P with ω,γ and ϕ held fixed ; it follows that

$$\frac{\partial}{\partial P} = \frac{\partial}{2k_o\partial k_o} \qquad \text{(A1.9)}$$

If f is such that f(0)=0 then one has

$$(\vartheta(k_o)\delta_1'(k^2)\ ,\ f(k)) =$$

$$= \frac{1}{2}\int (\frac{\partial}{\partial P}\ \delta(P))\ \vartheta(k_o)\ f(k)\ (P+\omega)^{1/2}\ \omega^2\ d\omega\ d\Omega\ dP =$$

$$= -\frac{1}{2}\int \delta(P)\ \vartheta(k_o)\ \frac{\partial}{\partial P}\ [f(k)\ (P+\omega)^{1/2}]\ \omega^2\ d\omega\ d\Omega\ dP =$$

$$= -\frac{1}{4}\int [\frac{1}{k_o}\frac{\partial}{\partial k_o}(\frac{f(k)}{k_o})]\Big|_{C_+}\ d^3k \qquad \text{(A1.10)}$$

With a different choice of the new coordinate system one obtains another realization:

$$(\vartheta(k_o)\delta_2'(k^2),f(k)) = \frac{1}{4}\int [\omega^{-1}\ \frac{\partial}{\partial\omega}\ (\omega\ f(k))]_{C_+}\ dk_o\, d\Omega \qquad \text{(A1.11)}$$

A procedure to regularize these formulae in the general case $f \in \mathcal{S}(R^4)$ will be provided below.

Formulae (A1.10/11) may be rewritten in the following way

$$(\vartheta(k_o)\delta_1'(k^2), f(k)) = -\frac{1}{4}\int \frac{\partial}{k_o \partial k_o}\left(\frac{f(k)}{k_o}\right) \delta(k_o - \omega) d^4k \tag{A1.12}$$

$$(\vartheta(k_o)\delta_2'(k^2), f(k)) = \frac{1}{4}\int \omega^{-4}\, k_i \frac{\partial}{\partial k_i}(\omega f(k))\, \delta(k_o - \omega) d^4k \tag{A1.13}$$

It is easy to show that these two formulae give the same results when tested on functions which vanish at the origin. Therefore one may define the distribution

$$(\vartheta(k_o)\delta_c'(k^2), f(k)) = (\vartheta(k_o)(c\delta_1'(k^2) + (1-c)\delta_2'(k^2), f(k)) =$$

$$= \frac{1}{4}\int \omega^{-3}\, [f(k) - D_c f(k)]\delta(k_o - \omega)\, d^4k \tag{A1.14}$$

$$D_c = ck_o \frac{\partial}{\partial k_0} + (c-1)k_i \frac{\partial}{\partial k_i} \tag{A1.15}$$

Clearly (A1.14) is not dependent on c when $f(0)=0$. Consider now the scalar product (1.22) with $f, g \in \mathscr{S}_o(R^4)$. It follows from (A1.14) that

$$\langle f, g\rangle_c = \frac{1}{4}\pi a \int d\mu(k)\, (1 - D_c)\hat{f}(k)\hat{g}(k) \tag{A1.16}$$

with $d\mu$ defined in (1.29). Let now χ_c be a real function belonging to $\mathscr{S}(R^4)$ and satisfying the conditions:

$$1)\ \hat{\chi}_c(0) = 1,\quad 2)\ D_c\hat{\chi}_c(k)\big|_{C_+} = 0,\quad 3)\ \int \chi_c^2(k)\, d\mu(k) = 0 \tag{A1.17}$$

The integral in (A1.17,3) has to be intended in a distributional sense. Then given $f \in \mathscr{S}(R^4)$ one subtracts the bad infrared singular part according to

$$\hat{f}_{oc}(k) = \hat{f}(k) - \hat{f}(0)\chi_c(k) \tag{A1.18}$$

and obtains the final regularized formula

$$\langle f, g\rangle_c = \frac{1}{4}\pi a \int d\mu(k)\, [\,\overline{F}_{1c}(k)G_{1c}(k) - \overline{F}_{2c}(k)G_{2c}(k) +$$

$$\bar{\hat{f}}(0)\chi_c(k)G_{1c}(k) + \hat{g}(0)\bar{F}_{1c}(k)\chi_c(k)] \tag{A1.19}$$

with

$$F_{1c}(k) = \hat{f}_{oc}(k) - D_c\hat{f}_{oc}(k) \tag{A1.20}$$

$$F_{2c}(k) = D_c\hat{f}_{oc}(k) \tag{A1.21}$$

Formulae (1.23,24,25,26) follow by choosing c=1. The formulae used in [9] correspond to the choice c=1/2.

Finally, observing that

$$D_b = D_c + (b-c)\left(k_o \frac{\partial}{\partial k_o} + k_i \frac{\partial}{\partial k_i}\right) \tag{A1.22}$$

one concludes that

$$\vartheta(k_o)\delta_c'(k^2) = \vartheta(k_o)\delta'(k_b^2) + 2\pi^2 a(b-c)\delta^4(k) \tag{A1.23}$$

APPENDIX 1.C

In this appendix it will be verified that the inner product (1.23) is Lorentz invariant. The restricted Lorentz group is represented on the functions of the momentum variable as follows:

$$(U(\Lambda)f)(k) = f(\Lambda^{-1}k) \tag{A1.24}$$

This representation implies that the Lorentz invariance of (1.23) is equivalent to

$$\int \frac{d^3k}{\omega} \{[\bar{k}_o^{-2}(\bar{k}_o\bar{\partial}_o f(k)-f(k)] - [k_o^{-2}(k_o\partial_o f(k) - f(k)]\}\Big|_{C_+} \tag{A1.25}$$

$$\text{with} \quad \bar{k} = \Lambda\, k \tag{A1.26}$$

(for simplicity has been chosen a f such that f(0)=0).

One has that

$$\bar{k}_o\bar{\partial}_o = (\Lambda o^o)^2 k_o\partial_o + \Lambda o^o \Lambda o^i (k_o\partial_i + k_i\partial_o) + \Lambda o^i \Lambda o^j\, k_i\partial_j \tag{A1.27}$$

Taking into account that

$$\partial_i f(k)\Big|_{C_+} = \partial_i\, f(k)_L + [(k_i/k_o)\partial_o f(k)]\Big|_{C_+} \tag{A1.28}$$

$$\text{with} \quad f(k)_L = f(k)\Big|_{C_+} \tag{A1.29}$$

one obtains

$$\bar{k}_o^2 = [\,(\Lambda o^o)^2 + 2\Lambda o^o \Lambda o^i (k_i/k_o) + \Lambda o^i \Lambda o^j (k_i k_j/k_o^2)]\, k_o^2 \tag{A1.30}$$

$$\bar{k}_o\bar{\partial}_o f(k)\Big|_{C_+} = [\,(\bar{k}_o^2/k_o^2)k_o\partial_o f(k)]\,\Big|_{C_+} + \Lambda o^\mu \Lambda o^i k_\mu \partial_i f(k)_L \tag{A1.31}$$

The integral (A1.24) becomes

$$\int d^3k\, \{[\,(\Lambda o^\mu \Lambda o^i k_\mu \partial_i f(k)_L /\bar{k}_o^2) + (k_o^{-2} - \bar{k}_o^{-2}) f(k)]\,/k_o\}\Big|_{C_+} \tag{A1.32}$$

The integrand in (A1.31) may be written

$$\{[(k_o^{-3} - k_o^{-1}\bar{k}_o^{-2})f(k)]\Big|_{C_+} - f(k)\,{}_L\partial_i[k_o^{-1}\bar{k}_o^{-2}(\Lambda o^{\mu}\Lambda o^{i}k_{\mu})]_L +$$

$$+ \partial_i\ [\Lambda o^{\mu}\Lambda o^{i}k_{\mu}k_o^{-1}f(k)]_L\,\} \qquad (A1.33)$$

The last term of this expression is integrated to zero.It is possible to show after some manipulations that the first two terms in (A1.31) are equal and therefore the integral (A1.24) vanishes.From (A1.33) one has that

$$[\bar{k}_o^{-2}(\bar{k}_o\bar{\partial}_o - 1)f(k)]\Big|_{C_+} = [k_o^{-2}(k_o\partial_o - 1)f(k)]\Big|_{C_+} +$$

$$k_o\Lambda o^{i}\partial_i[k_o\bar{k}_o f(k)]_L \qquad (A1.34)$$

This formula will be useful in the statement of the subsidiary condition.

CHAPTER 2.

2.1 THE HILBERT SPACE STRUCTURE CONDITION.

As briefly stated in the introduction it is necessary, for the physical interpretation of an indefinite metric Q.F.T., to extend in some way the set of the local states D_o, as well as the local field algebra $\mathcal{F}$. The topology that seems the most natural to obtain these extensions is the so called Wightman topology τ_W; it is defined by the following set of seminorms:

$$p_{\underline{g}}(\Psi_{\underline{f}}) = |\langle \underline{g}, \underline{f} \rangle| \ , \ \Psi_{\underline{f}} \in D_o, \ \underline{g} \in \mathcal{B}. \qquad (2.1)$$

This topology is locally convex and separated on D_o [5],but unfortunately cannot be used to extend D_o and $\mathcal{F}$ because the inner product (1.11) is not jointly continuous w.r. to it and it would be problematic to extend (1.11) to the eventually obtained completion $\overline{D_o}^{\tau_W}$.

This should not be a surprise because also in the standard case the τ_W topology is too weak and does not identify a Hilbert space. The thing that seems the most natural at this point is to associate to the Wightman functions a Hilbert majorant topology with whose help one completes D_o into a Hilbert space. In any case the topology τ_W is the weakest one and can be used as a guide to define extensions of D_o and $\mathcal{F}$ (for more details on this argument see the illuminating discussion in [14]).

The following condition replaces the standard axiom of positivity:

IV HILBERT SPACE STRUCTURE CONDITION

There exists a set of Hilbert seminorms $\{p_n\}$, p_n defined on $\mathcal{S}(R^{4n})$ such that

$$|\underline{\mathcal{W}}(f_n^{*} \times g_m)| \leq p_n(g_n)\; p_m(g_m) \tag{2.2}$$

For reasons discussed in [4] the seminorms are required to be $\mathcal{S}$-continuous. Given the seminorms $\{p_n\}$ one may easily define a Hilbert seminorm p on D_o e.g. by putting

$$(\underline{f} \;,\; \underline{g} \;) = \sum_n \; (n+1)^2 \; (f_n, g_n)_n \tag{2.3}$$

where $(\;,\;)_n$ is the Hilbert scalar product defined by the seminorm p_n and $\underline{f}$ and $\underline{g}$ are representatives for $[\underline{f}]$ and $[\underline{g}]$. The closure of D_o in the Hilbert topology induced by p defines an Hilbert space $\mathcal{H}$ and a metric operator η such that

$$< \underline{f} \;,\; \underline{g} > \; = \; (\underline{f} \;,\; \eta\underline{g} \;) \tag{2.4}$$

Without loss of generality one may assume that p vanishes on the Wightman ideal I_W:

$$< \underline{f} \;,\; \underline{g} > \; = o \;\; \forall \;\underline{g} \in \mathcal{B} \text{ implies } p(\underline{f}) = 0 \tag{2.5}$$

It is worth to point out again that different choices of the seminorms p_n give rise to different Hilbert spaces and whereas in the standard case the Wightman functions uniquely fix the closure of D_o, in the indefinite metric case different closures are available corresponding to different Hilbert topologies.

2.2 THE KREIN TOPOLOGY.

Among all the possible Hilbert closures of D_o, those which associate a maximal set of states to the given set of Wightman functions are of particular interest .

A Hilbert space structure { η , $\mathcal{H}$ } associated to a set of Wightman functions , with non degenerate metric operator η , is said maximal if there is no other Hilbert space structure { $\bar{\eta}$, $\bar{\mathcal{H}}$ } such that $\bar{\eta}$ is non degenerate and $\bar{\mathcal{H}}$ contains $\mathcal{H}$ properly. This happens if and only if the operator η^{-1} is bounded; in this case one may redefine the metric operator in such a way that $\eta^2=1$ [4]. Inner product spaces with the property that $\eta^2=1$ are called Krein spaces [5][6]. In the case under examination it is possible a simple solution of the condition (2.2) and this again thanks to the hypothesis of factorization of the n-point function.

The first step consists in finding a seminorm p_1 which majorizes the two point function. To this aim consider the scalar product (1.23) with $f,g \in \mathcal{S}_o(R^4)$:

$$\langle f,g\rangle = \int d\mu(k) \{ \bar{F}_1(k)G_1(k) - \bar{F}_2(k)G_2(k) \} \tag{2.6}$$

The constant in front of the integral in (1.23) has been set equal to one because its value is irrelevant in this discussion. A Hilbert product majorizing (2.6) is immediately given by

$$[f,g] = \int d\mu(k) \{ \bar{F}_1(k)G_1(k) + \bar{F}_2(k)G_2(k) \} \tag{2.7}$$

One easily obtains that

$$|\langle f,g\rangle| \le [f,f]^{1/2}[g,g]^{1/2} \qquad f,g \in \mathcal{S}_o(R^4) \tag{2.8}$$

In the general case in which $f\in\mathcal{S}(R^4)$ one exploits the splitting (1.25) and defines the Hilbert scalar product

$$(f,g) = [f_o,g_o] + \langle f,\chi\rangle\langle\chi,g\rangle + \bar{f}(0)g(0) \tag{2.9}$$

If one rewrites (1.23) in the form

$$\langle f,g\rangle = \langle f_o,g_o\rangle + \bar{\hat{f}}(0)\langle\chi,g\rangle + \hat{g}(0)\langle f,\chi\rangle \tag{2.10}$$

easily obtaines

$$|\langle f,g\rangle| \le \|f\| \; \|g\| \tag{2.11}$$

$$\text{with } \|f\| = (f,f)^{1/2} = p_1(f) \tag{2.12}$$

Now it is possible to show that there exist constants C_n such that the seminorms

$$p_n(f_1.\ldots.f_n) = C_n\, p_1(f_1).\ldots.p_1(f_n) \tag{2.13}$$

satisfy the condition (2.2). Using (2.12) and the τ_W topology it is possible [14] to introduce the creation and annihilation operators and a Fock type realization of the theory. It is also possible using the so constructed Fock structure to improve the already simple topology defined in (2.12). One introduces the following set of vectors which generates the Borchers algebra:

$$\Psi^n_{f_1 \ldots f_n} = (n!)^{-1/2} :\phi(f_1)\ldots\phi(f_n): \Psi_o \qquad (2.14)$$

where $: \;:$ denotes the Wick ordered product defined in terms of the Wightman functions . The intrinsic scalar product of two such vectors is given by

$$\langle \Psi^n_{f_1 \ldots f_n} , \Psi^m_{g_1 \ldots g_m} \rangle = (n!)^{-1} \delta_{n,m} \sum_\pi <f_1, g_{i_1}> \ldots <f_n, g_{i_n}>$$

(2.15)

One then may introduces the following set of seminorms on this kind of vectors:

$$p_n(\Psi^n_{f_1 \ldots f_n}) = (n!)^{-1} \sum_\pi (f_1, f_{i_1}) \cdot \cdots \cdot (f_n, f_{i_n}) \qquad (2.16)$$

(2.16) defines the following Hilbert product:

$$(\Psi^n_{f_1,\ldots,f_n}, \Psi^m_{f_1,\ldots,f_m}) = \delta_{n,m} (n!)^{-1} \sum_\pi (f_1, g_{i_1}) \cdot \cdots \cdot (f_n, g_{i_n})$$

(2.16b)

The main result of this chapter is the proof that the closure of D_o with respect to the topology induced by the seminorms (2.15) is a Krein space. In order to achieve this result several intermediate steps are needed. In the following paragraph it is described an intermediate structure of a great utility.

2.3 THE SPACE $\mathcal{H}_o^1$.

Consider the following Hilbert space:

$$\mathcal{H}_o^1 = \overline{\mathcal{S}_o(R^4)}^{][} \;/\; \mathcal{N}\left(\overline{\mathcal{S}_o(R^4)}^{][}\right) \tag{2.17}$$

where

$$]f[\; = [f,f]^{1/2} \tag{2.18}$$

$$\mathcal{N}\left(\overline{\mathcal{S}_o(R^4)}^{][}\right) = \{ f \in \overline{\mathcal{S}_o(R^4)}^{][} :]f[\; = 0 \} \tag{2.19}$$

Theorem 2.1 : It is possible to extend the inner product (2.6) to the whole $\mathcal{H}_o^1$ and there exists a bounded and self-adjoint operator η_o^1 such that

$$< f , g > = [f , \eta_o^1 g] \tag{2.20}$$

Proof: it is an easy consequence of (2.8) and the representation theorem of sesquilinear forms [13].##

Theorem 2.2: $\quad (\eta_o^1)^2 = 1 \quad$ (2.21)

Proof: consider the linear space

$$\mathcal{S}_o^2(R^4) = \{ F = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} : f_i \in \mathcal{S}_o(R^4) \;,\; i=1,2 \} \tag{2.22}$$

Define in $\mathcal{S}_o^2(R^4)$ the following inner products :

$$\{ F , G \} = \int d\mu(k) \; [\overline{\hat{f}}_1(k)\hat{g}_1(k) + \overline{\hat{f}}_2(k)\hat{g}_2(k)] \tag{2.23}$$

$$<< F , G >> = \int d\mu(k) \; [\overline{\hat{f}}_1(k)\hat{g}_1(k) - \overline{\hat{f}}_2(k)\hat{g}_2(k)] \tag{2.24}$$

It immediately follows the identity

$$<< F , G >> = \{ F , \sigma_3 G \} \tag{2.25}$$

with σ_3 = diag (1,-1). (2.26)

In $\mathcal{S}_o^2(R^4)$ define the following subspaces :

$$\mathcal{D}_\pm = \{ F \in \mathcal{S}_o^2(R^4) \; : \; \hat{f}_1(k) = (1-D)\hat{f}(k) \;, \; \hat{f}_2(k) = \pm D\hat{f}(k)$$

$$\text{with } f \in \mathcal{S}_o(R^4) \; \} \tag{2.27}$$

Besides define the following operators :

$$U_\pm : \mathcal{S}_o(R^4) \longrightarrow \mathcal{D}_\pm \quad , \; U_\pm f = \begin{Bmatrix} (1-D)\hat{f}(k) \\ \pm D\hat{f}(k) \end{Bmatrix} \tag{2.28}$$

The following identities are easily obtained

$$\{ U_\pm f \;,\; U_\pm g \} = [f \;,\; g] \tag{2.29}$$

$$\ll U_\pm f \;,\; U_\pm g \gg = < f \;,\; g > \tag{2.30}$$

Consider now the following Hilbert spaces :

$$\overline{\mathcal{S}_o^2(R^4)}^{]\,[_2} \,/\, \mathcal{N}\, (\overline{\mathcal{S}_o^2(R^4)}^{]\,[_2}) = L^2(R^4,\, C^2,\, d\mu) \tag{2.31}$$

$$\bar{\mathcal{D}}_\pm = \overline{\mathcal{D}_\pm}^{]\,[_2} \,/\, \mathcal{N}\, (\overline{\mathcal{D}_\pm}^{]\,[_2}) \tag{2.32}$$

with $]\, F \,[_2 = \{ F \;,\; F \}^{1/2}$

Let $f \in \mathcal{H}_o^1$ and f_n a sequence in $\mathcal{S}_o(R^4)$ such that

$$\lim \;]f_n - f[\longrightarrow 0 \tag{2.33}$$

One has that the sequences $U_\pm f_n$ are Cauchy's sequences; indeed

$$]\, U_\pm f_n - U_\pm f_m [_2 = \,]\, f_n - f_m [\longrightarrow 0 \tag{2.34}$$

Then they exist $F_\pm \in \bar{\mathcal{D}}_\pm$ such that

$$\lim U_\pm f_n = F_\pm \;\text{ in } \mathcal{D}_\pm \tag{2.35}$$

Therefore it is possible to extend the operators $U_{\pm}$ to two operators $\overline{U}_{\pm}: \mathcal{H}_o^1 \longrightarrow \overline{\mathcal{D}}_{\pm}$ in such a way that

$$\overline{U}_{\pm} f = F_{\pm} \tag{2.36}$$

As a consequence of (2.36) one has that it is possible to extend (2.29) to the whole $\mathcal{H}_o^1$:

$$\{ \overline{U}_{\pm} f , \overline{U}_{\pm} g \} = [f , g] , \quad f,g \in \mathcal{H}_o^1 \tag{2.37}$$

Therefore $\overline{U}_{\pm}$ are isometric operators. It is clear that the form (2.24) may be extended to the whole $L^2(R^4,C^2,d\mu)$ and there exists a bounded and self-adjoint operator, that coincides with σ_3 itself , such that for F and G in $L^2(R^4, C^2, d\mu)$ one has

$$\ll F , G \gg = \{ F , \sigma_3 G \} \tag{2.38}$$

Then (2.20),(2.35) and (2.37) imply that for $f,g \in \mathcal{H}_o^1$

$$[f , \eta_o^1 g] = < f , g > = \ll \overline{U}_{\pm} f , \overline{U}_{\pm} g \gg = \{ U_{\pm} f , \sigma_3 U_{\pm} g \} \tag{2.39}$$

Before concluding the proof it is necessary to enunciate and demonstrate the important

<u>Lemma</u> <u>2.3</u> : $\overline{\mathcal{D}}_+ = \overline{\mathcal{D}}_- = \overline{\mathcal{D}}$ (2.40)

<u>Proof</u>: let $F = \{f_1 , f_2\}^T$ be an element of $\overline{\mathcal{D}}_+$. It will be shown that it is possible to construct a sequence of elements of $\overline{\mathcal{D}}_-$ converging to F. Consider indeed the following sequences of elements of $\mathcal{H}_o^1$:

$$\hat{g}_n(k) = -D\hat{f}_n(k)\ k^2 k_o^{-2} \qquad (2.41)$$

$$\hat{h}_n(k) = -2D\hat{f}_n(k) + D^2\hat{f}_n(k)\ k^2 k_o^{-2} \qquad (2.42)$$

with

$$\lim\ ]\ U_+ f_n - F\ [_2 \longrightarrow 0 \qquad (2.43)$$

One has that

$$\hat{g}_n(k)\lfloor_{C_+} = 0\ , \quad D\hat{g}_n(k)\lfloor_{C_+} = -2D\hat{f}_n(k)\lfloor_{C_+} \qquad (2.44)$$

$$\hat{h}_n(k)\lfloor_{C_+} = -2D\hat{f}_n(k)\lfloor_{C_+}\ , \quad D\hat{h}_n(k)\lfloor_{C_+} = 0 \qquad (2.45)$$

From these identities it follows that

$$\lim\ ]\ \overline{U}_-(f_n + g_n + h_n) - F\ [\ \longrightarrow 0 \qquad (2.46) \quad \#\#$$

<u>End of the proof of theorem 2.2</u>:

let $f,g \in \mathcal{H}_o^1$. One has from (2.37) and (2.39) that

$$[\ f\ ,\ \eta_o^1 g\ ] = \{\ \overline{U}_+ f\ ,\ \overline{U}_+(\eta_o^1 g)\ \} = \{\ \overline{U}_+ f\ ,\ \sigma_3 \overline{U}_+ g\ \} \qquad (2.47)$$

It is clear that $\sigma_3 \mathcal{D}_+ = \mathcal{D}_-$. By lemma 2.3 one then obtains that $\sigma_3 \mathcal{D}_+ \subseteq \mathcal{D}_+$. This allows to exploit the formula (2.47) twice and to obtain

$$[\ f\ ,\ (\eta_o^1)^2 g\ ] = \{\ \overline{U}_+ f\ ,\ \overline{U}_+(\eta_o^1)^2 g\ \} = \{\ \overline{U}_+ f\ ,\ \sigma_3^2\ \overline{U}_+ g\ \} =$$

$$\{\ \overline{U}_+ f\ ,\ \overline{U}_+ g\ \} = [\ f\ ,\ g\ ] \quad \forall\ f,g \in \mathcal{H}_o^1. \qquad (2.48)$$

This implies that $(\eta_o^1)^2 = 1$ ##

<u>Corollary</u> <u>2.4</u>: The spaces $\overline{\mathcal{D}}$, $\mathcal{H}_o^1$ and $L^2(R^4, C^2, d\mu)$ are isomorphic.

Proof: the isomorphism of $\mathcal{H}_o^1$ has been already shown with the construction of the operators $\overline{U}_{\pm}$.

The inclusion $\overline{\mathcal{D}} \subseteq L^2(R^4, C^2, d\mu)$ is obvious. The opposite inclusion may be proved in the same way of lemma 2.3 by showing that $\forall\ F \in L^2(R^4, C^2, d\mu)$ there exists a sequence of elements of $\overline{\mathcal{D}}$ converging to F. The completeness of $\overline{\mathcal{D}}$ then obtains the desired inclusion. ##

Therefore the space $\mathcal{H}_o^1$ may be concretely realized as $L^2(R^4, C^2, d\mu)$, that is space of complex two-component functions depending on the momentum variables k , square-integrable with respect to the measure $d\mu$; this is the same of the space of complex two-component functions defined on C_+ with its tip deleted, square-integrable with respect to the measure $\omega^{-3}d^3k$, denoted with $L^2(C_+,C^2,\omega^{-3}d^3k)$.

Corollary 2.5: a possible explicit representation of the action of the operator η_o^1 on test function in $\mathcal{S}_o(R^4)$ is given by the formula

$$(\eta_o^1 f)^{\wedge} = \hat{f}(k) - (2 + k^2 k_o^{-2}) D\hat{f}(k) + k^2 k_o^{-2} D^2 \hat{f}(k) \qquad (2.49)$$

Proof: again one follows the proof of lemma 2.3. Besides it is easy to verify the correctness of (2.49) exploiting it on the formula (2.20). ##

2.4 The one-particle space K^1 and the Fock-Krein-Hilbert space K.

The one particle space K^1 is given by

$$K^1 = \overline{\mathcal{S}(R^4)}^{\|\ \|} / \mathcal{N}\left(\overline{\mathcal{S}(R^4)}^{\|\ \|} \right) \tag{2.50}$$

Using (2.11) one gets that the scalar product (1.23) may be extended to the whole K^1 and it exists a bounded and self-adjoint operator η^1 such that $\forall\ f,g \in K^1$ one has:

$$< f , g > = (f , \eta^1 g) \tag{2.51}$$

The special role played by the function χ becomes evident in some of the subsequent lemmas.

Lemma 2.6: the linear functional defined on K^1 by

$$F_\chi(f) = <\chi, f> \tag{2.52}$$

has norm equal to one; therefore it defines a $v \in K^1$ which has norm equal to one and such that

$$(v , f) = < \chi , f > \tag{2.53}$$

Proof: it is immediate that

$$| F_\chi(f) | / \|f\| \leq 1. \tag{2.54}$$

Consider now a sequence f_n^χ in $\mathcal{S}_o(R^4)$ such that

$$\hat{f}_n^\chi (k) = \vartheta_n(\omega)\ \hat{\chi}(k) \tag{2.55}$$

with ϑ_n a real, infinitely differentiable and non decreasing function such that

$$\vartheta_n(\omega) = \begin{cases} 0 & \text{if } \omega \leq 0 \\ 1 & \text{if } \omega \geq 1/n \end{cases} \tag{2.56}$$

It follows that

$$|< \chi , f_n^\chi >|^2 / \| f_n^\chi \|^2 = (1 + \| f_n^\chi \|^2 |<\chi, f_n^\chi>|^{-2})^{-1} \qquad (2.57)$$

One has that [14] $\| f_n^\chi \|^2 |<\chi, f_n^\chi>|^{-2} =$

$$(\int d^3k\ \omega^{-3} \vartheta_n^2(\omega) \chi^2(k)\,_L) \; (|\int {}^3k\ \omega^{-3} \vartheta_n(\omega)\ \chi^2(k)\,_L\ |)^{-2} \longrightarrow 0 \qquad (2.58)$$

It follows that the expression (2.57) tends to one. The assertion (2.53) is now a consequence of the Riesz theorem. ##

An important consequence of lemma 2.6 is that the space K^1 is not a functional space and indeed it contains an infinitely delocalized state that is v. This is seen in the following corollary

<u>Corollary</u> <u>2.7</u>: the sequence $v_n = (<\chi, f_n^\chi>)^{-1} f_n^\chi$ (2.59)

converges strongly in K^1 to the vector v.

<u>Proof</u>: it is an immediate consequence of the proof of lemma 2.6. ##

The interpretation of v as an infinitely delocalized state follows from that fact that the sequence v_n converges pointwise to zero everywhere. This type of phenomenon is directly related to the infrared singularities of the Wightman functions and to the minimality of the topology chosen to costruct the Hilbert space (i.e a topology that yields a maximal Hilbert structure) [14].

Consider now the following subspaces of K^1:

$$K_o^1 = \overline{\mathcal{S}_o(R^4)}^{\|\ \|} \, / \, \mathcal{N}\left(\overline{\mathcal{S}_o(R^4)}^{\|\ \|}\right) \qquad (2.60)$$

$$K_{oo}^1 = \{ \, f \in K_o^1 \; : \; \langle \chi , f \rangle = 0 \, \} \qquad (2.61)$$

Lemma 2.8: K^1 decomposes into orthogonal spaces:

$$K^1 = K_{oo}^1 + \{ \chi \} + \{ v \} \qquad (2.62)$$

Proof: let $f \in K_{oo}^1$. By definition

$$(v,f) = \langle \chi , f \rangle = 0 \qquad (2.63)$$

Besides if f_n is a sequence of elements of $\mathcal{S}_o(R^4)$ converging to f in K^1 then

$$(f, \chi) = \lim (f_n, \chi) = 0 \qquad (2.64)$$

Finally

$$(v , \chi) = \langle \chi , \chi \rangle = 0 \qquad (2.65) \ \#\#$$

The action of the metric operator η^1 on the 2-dimensional subspace of K^1 generated by v and χ is given by the following

Lemma 2.9: 1) $\eta^1 v = \chi$, 2) $\eta^1 \chi = v$ (2.66)

Proof: let $f \in K^1$. One has

$$(v , f) = \langle \chi , f \rangle = (\eta^1 \chi , f) \qquad (2.67)$$

and this implies (2.66,1).Let now $f \in \mathcal{S}(R^4)$. One has that

$$(\chi , f) = \hat{f}(0) \qquad (2.68)$$

Besides from lemma (2.7) it descends that

$$< v , f > = \lim < v_n , f > = \hat{f}(0) \tag{2.69}$$

Therefore

$$(\chi , f) = < v , f > = (\eta^1 v , f) \tag{2.70}$$

The density of $\mathcal{S}(R^4)$ finally yields (2.66,2). ##

It is finally possible to enunciate and demonstrate the

<u>Theorem</u> <u>2.10</u>: the space K^1 is a Krein space i.e the metric operator η^1 satisfy the equation $(\eta^1)^2 = 1$ (2.71)

<u>Proof</u>: let P be the projector on the subspace K^1_{oo}. By lemma 2.8 it is possible to decompose a generical $f \in K^1$ in the following way :

$$f = Pf + (v,f)v + (\chi,f)\chi \tag{2.72}$$

Lemma 2.9 then imply that

$$< f , g > = (Pf,\eta^1 Pg) + (f,v)(\chi,g) + (f,\chi)(v,g) \tag{2.73}$$

It is easy to understand that

$$P\eta^1 P = P\eta^1_o P \tag{2.74}$$

Define $v_{\pm} = 2^{-1/2} (v \pm \chi)$ (2.75)

and let $P_{\pm}$ be the corresponding projectors. One has

$$<f,g> = (f,\eta^1 g) = (f , \{ P\eta^1_o P + P_+ - P_- \} g) \tag{2.76}$$

$\forall$ f,g $\in K^1$. This implies that

$$\eta^1 = P\eta^1_o P + P_+ - P_- \tag{2.77}$$

Using (2.77) and theorem 2.2 one can easily show that the (2.71) is true and K^1 is a Krein space. ##

Corollary 2.11: the one-particle space K^1 may be concretely realized in terms of functions that depend on the momentum variables as follows:

$$K^1 = L^2(R^4,\ C^2,\ d\mu) + \{\ v\ \} + \{\ \hat{\chi}\ \} \tag{2.78}$$

Proof: the proof is easily obtained by lemma 2.8 and by the observation that the norm (2.18) is equal to the norm (2.12) on functions of the form $h = f - (v,f)v$, with $f \in \mathcal{S}_o(R^4)$.##

The closure of D_o with respect to the topology induced by (2.16) is called K. Clearly K has the form of a direct sum of tensor products:

$$K = \sum_n(\otimes_s K^1) = \sum_n K^n \tag{2.79}$$

Theorem 2.12: the space K is a krein space i.e. the metric operator η satisfies the equation $\eta^2=1$.

Proof: by equations (2.15) and (2.16) it follows that

$$\eta\ K^n \subset K^n \tag{2.80}$$

and therefore

$$\eta^n = \eta^1\otimes \eta^1 \cdots\otimes \eta^1 \tag{2.81}$$

this concludes the theorem. ##

2.5 THE OPERATORS $U(a,\Lambda)$ AND THE ESSENTIAL UNIQUENESS OF THE VACUUM.

The representation of the Poincare' group defined in (1.14) induces the following representation on the test functions depending on the momentum variable:

$$U(a,\Lambda)\hat{f}(k) = \exp(ika)\hat{f}(\Lambda^{-1}k) \tag{2.82}$$

Using this equation it is possible to show that the operators $U(\Lambda) = U(0,\Lambda)$ are bounded when restricted to the n-particle space K^n, but their norm is greater than one. This implies that $U(\Lambda)$ is unbounded on K.

The situation is different for what concerns the operators $U(a) = U(a,1)$. Indeed it is easy to see that they are unbounded also when restricted to the space K^n. However the unboundedness of $U(a)$ is due to the ultraviolet region and therefore it is not a serious problem in this context.

An important feature of the Krein space is given by the following

Theorem 2.13: The space K contains an infinite dimensional subspace V of vectors which are invariant under Poincare' transformations.

Proof: the first step consists in proving that the vector $v \in K^1$ is invariant under $U(a,\Lambda)$. Since $U(a)$ is unbounded it is necessary to check that v belongs to its domain. This is true because the sequence $U(a)v_n$ is a Cauchy sequence in

K^1 with v_n defined in (2.59).

Now it is sufficent to observe that for $f \in \mathcal{S}(R^4)$ one has

$$<v,f> = (\chi,f) = \hat{f}(0) \qquad (2.83)$$

This implies that

$$< U(a,\Lambda)v - v , f > = < v , f_{\{a,\Lambda\}} - f > = 0 \qquad (2.84)$$

The density of $\mathcal{S}(R^4)$ and the non degeneracy of the intrinsic inner product together with (2.84) allow one to proof the Poincare' invariance of the vector v. Clearly all the vectors

$$v^n = \otimes_s^n v \qquad (2.85)$$

are Poincare' invariant and this concludes the theorem.##

At this point it is worth to note that the vacuum is essentialy unique in the sense that the η-norm of the vectors (2.85) is zero. These features of the Krein-Hilbert space of the dipole field are very close to the corresponding ones of the two-dimensional massless scalar field [14]. This is not a surprise because the x-space two-point function has the same form in the two cases. It is perhaps useful to say again that it is possible to discover these structures and in particular the infinitely delocalized and Poincare' invariant vector v only by making use of a minimal topology to majorize the Wightman functions.

The vector v has an interesting counterpart at the level of the field algebra. Here too there is an important difference with respect to the standard case : the strong closure of the field algebra contains infinitely delocalized field operators which are translational invariant. Operators of this kind have been introduced in past, in different contexts, as ad hoc ingredients. On the contrary they appear in the present treatment as naturally associated to the infrared structure of the theory,and it is believed that they are proper of every Q.F.T. with non positive infrared singularities.The dipole shares also this characteristics with the massless scalar field (see [14] for more details about the next paragraph).

2.6 EXTENSION OF THE FIELD ALGEBRA. INFRARED OPERATORS

The representation of the fields is given on a dense set by the formula

$$(\phi(f)\Psi)^n = \{ (n+1)^{1/2} \pi a \int d^4k \ \vartheta(k_o)\delta'(k^2)\hat{f}(-k)\Psi^{n+1}(k,k_1,..,k_n)$$

$$+ n^{-1/2} \sum_{j=1}^{n} \hat{f}(k) \ \Psi^{n-1}(k_1,..k_{j-1},k_{j+1},..k_n) \qquad (2.86)$$

where Ψ^n are symmetric functions in $\mathscr{S}(R^{4n})$.

It is simple to check that the field $\phi(f)$ transforms covariantly under the action of $U(a,\Lambda)$, i.e.

$$U(a,\Lambda) \ \phi(f) \ U(a,\Lambda) = \phi(f_{\{a,\Lambda\}}) \qquad (2.87)$$

The norm of a vector belonging to K is defined by making use of the definition (2.16), and is denoted by $\| \ \|_K$. Using (2.86) one obtains for $\Psi \in K^m$:

$$\| \phi(f)\Psi \|_K \leq (m+1)^{1/2} \ || \ f \ || \ \| \Psi \|_K \tag{2.88}$$

with

$$|| \ f \ || = \| f \| + \| f^- \| \tag{2.89}$$

and $\hat{f}^-(k) = \hat{f}(-k)$ (2.90)

It is clear that the sequence of vectors

$$: \phi \ (f_{n_1}, \ldots\ldots, f_{n_k}) : \Psi_o \tag{2.91}$$

converges strongly in K if each of the sequences f_{n_i} and $f^-_{n_i}$ converges strongly in K^1. This in turn implies that field ϕ has a strongly continuous extension to the closure of $\mathcal{S}(R^4)$ w.r. to the topology defined by the norm $|| \ ||$.

Lemma 2.14: $\overline{\mathcal{S}(R^4)}^{|| \ ||} / \mathcal{N} \ (\overline{\mathcal{S}(R^4)}^{|| \ ||})$

$$= L^2(C_+ \cup C_-, \ C^2, \ \omega^{-3}d^3k) + \{ \ v \ \} + \{ \ v^- \ \} + \{ \ \chi \ \} \tag{2.92}$$

Proof: for the proof one has to repeat the steps leading to corollary 2.11. The vector v^- is obtained as the limit of the sequence v^-_n. ##

Corollary 2.15: The strong closure of the local field algebra contains two infinitely delocalized operators. Their explicit representation is the following:

$$(\phi(v)\psi)^n = \ (n)^{-1/2} \sum_{j=1}^{n} \ v \ \ \psi^{n-1}(k_1,..k_{j-1},k_{j+1},..,k_n) \tag{2.93}$$

$$(\phi(v^-)\psi)^n = (n+1)^{1/2}\ \psi^{n+1}(0,k_1,\ldots\ldots,k_n) \qquad (2.94)$$

They are invariant under Poincare' transformations and satisfy the following commutation relations

$$[\ \phi(v)\ ,\ \phi(f)\ ] = -\ \hat{f}(0) \qquad (2.95)$$

$$[\ \phi(v^-)\ ,\ \phi(f)\ ] = \hat{f}(0) \qquad (2.96)$$

for each f belonging to $\mathcal{S}(R^4)$.

Proof: The existence of the operators $\phi(v)$ and $\phi(v^-)$ follows easily from lemma 2.14. Their Poincare' invariance may be obtained using the covariance of the fields and the Poincare' invariance of the vectors v and v^-. The commutation relations are easily computed as their own vacuum expectation values.##

A final remark is in order: a result of Garding and Malgrange [16] guarantees that if ϕ is a tempered solution of a hyperbolic wave equation then it is sufficent to smear it either in the space or in the time variables; therefore they make sense the fixed-time fields

$$\phi(t,f) = \int \phi(t,\mathbf{x})\ f(\mathbf{x})\ d^3\mathbf{x} \qquad (2.97)$$

and the fixed space fields

$$\phi(g,\mathbf{x}) = \int \phi(t,\mathbf{x})\ g(t)\ dt \qquad (2.98)$$

and they are infinitely differentiable of the time and respectively of the space variables. This result is the key for a possible canonical formulation of the theory (which will not be performed here).

In particular thanks to equation (A1.1) one easily understands that the fields ϕ and $\partial_o \Delta\phi$, as well as $\partial_o \phi$ and $\Delta\phi$ are canonically conjugated.

CHAPTER 3.

3.1 The physical states.

The structure that has been constructed in the previous chapter may be used here as an intermediate step toward the identification of the physical Hilbert space of states.
The reasons that have been exposed in the introduction as well as the unconventional features of the space K revealed in chapter two should be enough to convince everyone of the convenience (if not necessity) of this intermediate step.
Arrived at this point one must identify a subspace $K' \subset K$ such that

$$< \Psi , \Psi > \geq 0 \qquad \text{if } \Psi \in K' \tag{3.1}$$

$$\Psi_o \in K' \tag{3.2}$$

The subspace of K' whose elements have η-norm equal to zero is denoted by

$$K'' = \{ \Psi \in K' : \; < \Psi , \Psi > = 0 \} \tag{3.3}$$

The positivity of the intrinsic inner product when restricted to the space K' gives sense to the quotient space K' / K''.
Then by completing this space in the topology defined by the

intrinsic inner product (which defines a norm on it) one obtains the physical Hilbert space of the theory:

$$K_{phys} = \overline{(K' / K'')}^{<\,,\,>} \qquad (3.4)$$

As before it will be possible at first to look for a one-particle physical space, and then to obtain the complete physical space by a Fock procedure.

3.2 Some considerations on the breaking of the Poincare' group.

Before making the concrete construction of some particular physical spaces it is worth to explore the possibility of constructing one physical space which is invariant under the Poincare' group. Actually the complete solution to this problem is not yet known but some considerations are possible.It is considered only the explicit action of the time translations.

Consider a function $f \in \mathcal{S}(R^4)$ such that

$$<f,f> = \int (1-D)\ |\hat{f}|^{2} d\mu \geq 0 \qquad (3.5)$$

where the expression (3.5) has obviously the meaning that was given to it in chapter 1. Clearly if f has a chance to represent a physical state must satisfy condition (3.5).

The time translations are represented on f in the usual way:

$$\hat{f}_a(k) = \exp(ik_o a)\ \hat{f}(k) \tag{3.6}$$

It is useful for the following to compute the expression

$$h_n(a) = \mathrm{Re} <f, f_{na}> =$$

$$= \mathrm{Re} \int ((1-D)|\hat{f}|^2 - inak_o|\hat{f}|^2)\exp(inak_o)\ d\mu =$$

$$= \int (\ \cos(nak_o)\ (1-D)|\hat{f}|^2 + nak_o \sin(nak_o)|\hat{f}|^2)d\mu \tag{3.7}$$

Define

$$h_n^{(m)}(a) = (\frac{d^m}{da^m}\ h)(a) \tag{3.8}$$

It is possible to show with a recurrence argument that

$$h_n^{(2m+1)}(0) = 0 \tag{3.9}$$

$$h_n^{(2m)}(0) = n^{2m} \{(-1)^m \int \omega^{2(m-1)} ((1-2m-D)|\hat{f}|^2)\big|_{C_+} \frac{d^3k}{\omega} \} =$$

$$= n^{2m}\ h_1^{(2m)}(0) \tag{3.10}$$

Consider now the expression

$$F_1(a) = <f-f_a, f-f_a> = 2<f,f> - 2h_1(a) \tag{3.11}$$

By construction one has that

$$F_1(0) = 0 \tag{3.12}$$

Besides (3.9) implies that

$$F_1^{(1)}(0) = 0 \tag{3.13}$$

Therefore the point a=0 is an extremal point for the function $F_1(a)$. If one requires the non negativity of the function

$F_1(a)$ then he must impose the condition

$$F_1^{(2)}(0) = -2\int \{|\hat{f}|^2 + D|\hat{f}|^2\}\Big|_{C_+} \frac{d^3k}{\omega} \geq 0 \qquad (3.14)$$

Note that now the expression in the R.H.S. has a perfect meaning in itself.

Consider then the expression

$$F_2(a) = \langle f-2f_a+f_{2a} , f-2f_a+f_{2a}\rangle = 6\langle f,f\rangle - 8h_1(a) + 2h_2(a) \qquad (3.15)$$

Again by construction $F_2(0) = 0$ and by (3.9) $F_2^{(1)}(0) = 0$.

One has that

$$F_2^{(2)}(0) = -8h_1^{(2)}(0) + 2h_2^{(2)}(0) = -8h_1^{(2)}(0) + 2[4h_1^{(2)}(0)] = 0 \qquad (3.16)$$

The third derivative of F_2 vanishes thanks once more to (3.9) and therefore one has to compute the fourth derivative.

$$F_2^{(4)}(0) = -8h_1^{(4)}(0) + 2h_2^{(4)}(0) = 24\, h_1^{(4)}(0) \qquad (3.17)$$

The local minimum (i.e. positivity) condition now is written

$$h_1^{(4)}(0) = - \int \omega^2 \{ 3|\hat{f}|^2 + D|\hat{f}|^2\}\Big|_{C_+} \frac{d^3k}{\omega} \geq 0 \qquad (3.18)$$

The next expression to deal with would be

$$F_3(a) = \langle f-3f_a+3f_{2a}-f_{3a} , f-3f_a+3f_{2a}-f_{3a}\rangle =$$

$$=20\langle f,f\rangle - 30h_1(a) + 12h_2(a) - 2h_3(a) \qquad (3.19)$$

One may direcly check that all the derivatives of $F_3(a)$ up to the fifth order vanish when computed in a=0. One has

$$F_3^{(6)}(0) = -30h_1^{(6)}(0) + 12h_2^{(6)}(0) - 2h_3^{(6)}(0) =$$

$$= -30h_1^{(6)}(0) + 12[64h_1^{(6)}(0)] - 2[729h_1^{(6)}(0)] =$$

$$= -720h_1^{(6)}(0) \tag{3.20}$$

In this case one has that the f must satisfy the condition

$$-h_1^{(6)}(0) = -\int \omega^4\{5|\hat{f}|^2 + D|\hat{f}|^2\}\Big|_{C_+} \frac{d^3k}{\omega} \geq 0 \tag{3.21}$$

Going on in this way one arrives at the n-th step in which takes in consideration the function

$$F_n(a) = < \sum_{j=0}^{n} \binom{n}{j}(-1)^j f_{ja} \, , \sum_{k=0}^{n} \binom{n}{k}(-1)^k f_{ka} > \tag{3.22}$$

One has that all the derivatives of $F_n(a)$ up to the (2n-1)-th vanish while the 2n-th one gives the condition

$$\int \omega^{2n-2}\{(1-2n)|\hat{f}|^2 - D|\hat{f}|^2\}\Big|_{C_+} \frac{d^3k}{\omega} \geq 0 \tag{3.23}$$

They have been obtained in this way a set of necessary conditions that a function must satisfy if the linear span constructed with the function itself and its time translated is positive.These conditions will be used in the next section.

3.3 The physical spaces.

It is now time to describe concretely the structure of some possible physical spaces and their behavior under the Poincare' group.

Theorem 3.1: There are three families of maximal subspaces of K^1 in which the intrinsic inner product is positive semidefinite; they are given by

a) $K'^{1}_{z,v} = L^2_z(R^4,\ C^2,\ d\mu) + \{v\}$ (3.24)

b) $K'^{1}_{z,\chi} = L^2_z(R^4,\ C^2,\ d\mu) + \{\chi\}$ (3.25)

c) $K'^{1}_{z,v_+} = L^2_z(R^4, C^2,\ d\mu) + \{v_+\}$ (3.26)

where z is a complex number such that

$Re z < 1/2$ (3.27)

$L^2_z(R^4, C^2,\ d\mu)$ is the subspace of $L^2(R^4, C^2, d\mu)$ whose functions are of the form $\{(1-z)\Psi\ ,\ z\Psi\}$ (3.28)

These spaces are the completion in the norm defined by the Krein inner product of the dense sets

a1) $\mathcal{D}'^{1}_{v} = \{f \in \mathcal{S}(R^4)\ :\ D\hat{f}_o(k)\lfloor_{C_+} = z\hat{f}_o(k)\ ,\ \hat{f}_o(k) = 0\}$ (3.29)

b1) $\mathcal{D}'^{1}_{\chi} = \{f \in \mathcal{S}(R^4)\ :\ D\hat{f}_o(k)\lfloor_{C_+} = z\hat{f}_o(k)\ ,\ \langle\chi, f\rangle = 0\}$ (3.30)

c1) $\mathcal{D}'^{1}_{v_+} = \{f \in \mathcal{S}(R^4):\ D\hat{f}_o(k)\lfloor_{C_+} = z\hat{f}_o(k)\ ,\ \hat{f}(0) = \langle\chi, f\rangle\}$ (3.31)

Proof: By lemma 2.8 a generic function $f \in K^1$ may be written

$$f = Pf + (v,f)v + (\chi,f)\chi \qquad (3.32)$$

Therefore f has positive η-norm if

$$<Pf,Pf> + 2Re\{\overline{(v,f)}(\chi,f)\} \geq 0 \qquad (3.33)$$

This may be verified if

$$<Pf,Pf> \geq 0 \qquad (3.34)$$

and

$$2\ Re\ \overline{(v,f)}(\chi,f) \geq 0 \qquad (3.35)$$

The (3.35) is verified in one of the following case:

1) $(v,f) = 0$ (3.36)

2) $(\chi,f) = 0$ (3.37)

3) $(v,f) = (\chi,f)$ (3.38)

The interpretation of the first two of these conditions is immediate; in the third case f has a component in the space generated by v and χ which is directed along the vector v_+ which has been defined in (2.75)

Condition (3.33) may be rewritten

$$<Pf,Pf> = \int (|f_1|^2 - |f_2|^2)\, d\mu \geq 0 \qquad (3.38)$$

where $\{ f_1, f_2\}^T = \overline{U}_+(Pf)$ (3.39)

and is solved by the spaces (3.28). It is easy to see that these spaces are maximal in the sense that no one of them is properly contained in a largest positive semidefinite subspace of $L^2(R^4, C^2, d\mu)$. It is also clear that they are isomorphic to $L^2(R^4, C, d\mu)$. This concludes the first part of the theorem. To obtain the second part one starts from the dense set $\mathcal{S}(R^4)$. Again one asks that

$$<f,f> = <f_o,f_o> + 2\text{Re}\ \check{f}(0)\ <\chi,f> \geq 0$$

This condition is splitted into the two inequalities:

$$<f_o,f_o> = \int (|\hat{f}_o - D\hat{f}_o(k)|^2 - |D\hat{f}_o(k)|^2) d\mu \geq 0 \tag{3.40}$$

$$\text{Re}\ \check{f}(0)\ <f,\chi> \geq 0 \tag{3.41}$$

A linear condition which solves (3.40) is written

$$D\hat{f}_o(k)\lfloor_{C_+} = z\ \hat{f}_o(k)\lfloor_{C_+} \tag{3.42}$$

(3.41) is solved by the one of the conditions

$$\hat{f}(0) = 0 \tag{3.43}$$

$$<\chi,f> = 0 \tag{3.44}$$

$$\hat{f}(0) = <\chi,f> \tag{3.45}$$

First of all notice that there are solutions to the condition (3.43) in $\mathcal{S}_o(R^4)$; indeed for instance

$$\hat{f} = \omega^2\hat{f} - 1/2\ k^2(D\hat{f} - z\hat{f}) \qquad f\in\mathcal{S}(R^4) \tag{3.46}$$

explicitely solves (3.42). A more interesting class of solutions of the (3.43) may be written in the following way:

$$\hat{g} = \vartheta_n(\omega)\ [\hat{f} - k^2(D\hat{f} - z\hat{f})/2\omega^2] \qquad f\in\mathcal{S}(R^4) \tag{3.47}$$

It is now clear that the Krein closures of the sets (3.29), (3.30), (3.31) are the sets (3.24), (3.25) and (3.26).##

The one particle physical spaces that one obtains exploiting the procedure (3,4) are

$$K^{1}_{phys\ z,v} = K'^{1}_{z,v} \ / \ \{v\} \tag{3.48}$$

$$K^{1}_{phys\ z,\chi} = K'^{1}_{z,\chi} \ / \ \{\chi\} \tag{3.49}$$

$$K_{phys\ z,v_{+}} = K'^{1}_{z,v_{+}} \tag{3.50}$$

The complete spaces are then obtained by a fock procedure.

It appears clear that the physical spaces of the theorem 3.1 are not invariant under the action of the Poincare' group; this may be seen by considerig the fact that the subsidiary condition (3.42) is not leaved invariant by the time translations:

$$D(\exp(iak_{o})\hat{f}(k))\big|_{C_{+}} = (z + iak_{o})\hat{f}(k) \tag{3.51}$$

To the same conclusions one arrives by observing that the subsidiary condition (3.42) is not in accordance with the conditions (3.23).

However the (3.51) itself may be interpreted as a subsidiary condition identifying another physical space and this because of the Poincare' invariance of the intrinsic inner product. One has indeed the following mapping

$$U(a)\ \mathcal{D}'_{z} = \mathcal{D}'_{z+iak} \tag{3.52}$$

Thus $\mathcal{D}'_{z}$ is leaved invariant only by the spatial translations. The subgroup of the time translations is spontaneously broken in all the spaces of theorem (3.1). This result goes in the same direction of the results obtaines by Thirring with other tecniques.

Using the formula (A1.34) it is possible to show that the Lorentz transformed of a function obeying the subsidiary condition (3.42) must satisfy the following subsidiary condition:

$$\hat{D}f(\Lambda^{-1}k)\big|_{C_+} = [\,k_o^2(\Lambda^{-1}k)_o^2\; z\; f(\Lambda^{-1}k)]\big|_{C_+} + (k_o\Lambda_o{}^i\partial_i'\; f(\Lambda^{-1}k)\big|)\big|_{C_+} \tag{3.53}$$

where ∂' means derivative with respect to the transformed variables $\Lambda^{-1}k$. When $\Lambda = 1$ the (3.53) reduces to the (3.42).

If one denotes $\mathcal{D}'_{z,\Lambda}$ the set of functions satisfying (3.53) then

$$U(\Lambda)\; \mathcal{D}'_z = \mathcal{D}'_{z,\Lambda} \tag{3.54}$$

The condition (3.53) implies that purely spatial rotations map each physical space onto itself while the subgroup of the boosts is spontaneosly broken.

The spontaneous breaking of the time translations and of the boosts relies on the choice that has been made of expressing the subsidiary conditions in terms of the operator D.

It is however possible to rephrase the entire formulation of the theory in terms of the operators D_c of appendix 1.B .

In particular making use of $D_{c=o} = k_i\; \partial/\partial k_i$ one may obtain physical spaces in which there is a spontaneous breaking of the spatial translations and rotations . This alternative possible formulation has never been noticed in past.

However the question of the possibility of constructing a

Poincare' invariant nontrivial physical space is still unanswered and may be the first continuation of this work.

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