Supergravity Effective Theories
From Superstrings

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CONTENTS

INTRODUCTION

This thesis contains a brief review of some aspects of string theories and their supergravity effective models. The principal aim has been not that of exposing original contributions, which are summarized in chapter 5, but of giving a general overview and the principal results which have been obtained.

The emphasis will be on the connections between the string and supergravity theories because this problem arises as soon as one is interested in studying the macroscopic (or "phenomenological") predictions of a string theory. The interest will be focused, as much as possible, not on a particular model and its low–energy predictions, but on the general properties of an effective theory emerging from a string model.

In the late '70, theoretical physicists were faced with the problem of constructing a quantum theory describing the unification of gravity with the other forces. A first step in this direction was done with the introduction of supersymmetry. It was found that a local–supersymmetric theory (supergravity) describes in a consistent way at the **classical** level the coupling of gravity with the other forces. The purely geometric, eleven dimensional supergravity was proposed to be the unification theory; in fact, through a Kaluza–Klein compactification to four dimensions, both matter and Yang–Mills fields can arise. But many problems, like the non–renormalizability of the theory, the difficulties in getting in four dimensions chiral fermions or the spontaneous breaking of supersymmetry, were not solved.

Then, it was realized that some string theories could be good candidates as the "final" quantum unification theory, the "Theory of Everything", that is the theory which describes the unification of all interactions at very high energy. Although steps in the comprehension of the properties of string theories have been made, there are still big problems which are not solved, first of all that of the formulation of a string–field theory.

When superstring theories were proposed as unification models, it was immediately realized that supergravity appears in the classical field limit of the massless modes of a string theory at the Planck energy scale. In this thesis we will discuss ways of deriving such effective theories and some of their features. Thus, although supergravity can be consistently formulated as a classical theory of gravity, here it will be considered as an effective theory emerging from the fundamental quantum theory. This means that issues like renormalizability or supersymmetry breaking must be studied first in string theories.

To construct a string effective theory we cannot proceed as for a standard quantum field theory because strings are non–local theories and in many models the dimension of the spacetime (critical dimension) is greater than four. Then, to construct a string effective theory, one must take a "field–theory" limit and compactify the extra dimensions. In this way one obtains a classical, four–dimensional supergravity theory which is of a non–standard type. In fact this theory has an infinite number of higher–derivative terms which come both from the string non–locality and the massive Kaluza–Klein modes. The computation of the bosonic and fermionic components of the first higher derivative terms in the four–dimensional supergravity theory constitutes the original part of this thesis.

The thesis is organized as follows. In the first chapter we outline the essentials of the above program. First there is a brief introduction to string theories and the concept of background spacetime as a classical vacuum for a string is introduced. Then it is introduced the concept of "string effective" theory and its meaning is discussed.

Before entering the details of the construction of a string effective theory, in chapter 2 the main aspects of the structure of a string theory are recalled. A first quantized string is a two dimensional quantum conformal theory: the absence of conformal anomalies gives a constraint (the vanishing of the central charge of the Virasoro algebra) for the construction of consistent string models and dictates the critical dimension. Some examples of bosonic and supersymmetric string models are given in 26, 10 and 4 dimensions.

In chapters 3, 4 and 5 we discuss the construction of a D=4 N=1 supergravity as an effective theory emerging from the D=10, $E_8 \otimes E_8$ heterotic string on a flat background.

In chapter 3 it is shown how to obtain a D=10 classical field–theory effective action. The first method reviewed is the "scattering amplitude" one. It consists in computing the scattering amplitudes between the massless modes in string theory and then construct the classical action which reproduces them. The theory constructed in such a way is a non–standard D=10 N=1 supergravity model. It is made by the standard "Chapline–Manton" theory plus an infinite number of higher derivative terms; between these terms there are those necessary to the Green–Schwarz anomaly–cancelling mechanism to work. Then it is reviewed how the same results can be obtained within the non–linear sigma model approach.

It is shown how from the condition of the vanishing of the Weyl anomaly one can obtain the equations of motion for the massless modes of the string theory and from these reconstruct the classical effective action.

Chapter 4 deals with the compactification to four dimensions. This is done in a Kaluza–Klein like way assuming that the D=10 equations of motion have a solution in which the spacetime is of the form $M_{10} = M_4 \otimes K_6$ where K_6 is a compact space. The request of having one unbroken supersymmetry charge in four dimensions leads to a compactification on a "Calabi–Yau" manifold; we review the analysis of the spectrum of gauge interactions and matter fields present in four dimensions is made. The related appropriate truncation à la Witten of the D=10 N=1 action is discussed in some details. The last issue presented in this chapter is the relation between the four dimensional, ten dimensional and string coupling constants, and the phenomenological consequences of these relations.

Chapter 5 is mainly devoted to a review of the author contributions. First it is constructed the standard D=4 N=1 supergravity model which is inspired from the considerations done in the previous chapters. Then it is considered the Lorentz Chern–Simons form which in ten dimensions plays a great role in the mechanism for the cancellation of the anomalies. The supersymmetric completion of the Lorentz Chern–Simons form in four dimensions is explicitly computed, both for the bosonic and fermionic terms. It is given by the Gauss–Bonnet combination of the gravitational curvature multiplets. (Really, the supersymmetric term coming in four dimensions from the D=10 Lorentz Chern–Simons is the square of the Weyl curvature multiplet, but to avoid the propagation of ghosts with spin bigger than one, also the square of the Ricci and scalar curvature multiplets in the Gauss–Bonnet combination are introduced. This is possible because these extra terms vanish on–shell.) Knowing the explicit expression of the super Gauss–Bonnet multiplet, it is possible to construct the four–dimensional, off–shell effective supergravity lagrangian.

But the higher derivative terms present in this lagrangian create a new problem: the unwanted propagation of some auxiliary fields; these modes however decouple in a Minkowski background. Then, through a study of the scalar potential in a constant background, some issues, like the value of the cosmological constant and the spontaneous supersymmetry breaking patterns, are studied. To conclude this chapter, some more technical considerations on the contribution of the other higher derivative terms and on the superconformal formulation of the four–dimensional effective supergravity theories are done.

In the last chapter, a more general approach to the study of the relations between the world–sheet and spacetime properties of string theories is presented. Starting directly from string theories formulated in four dimensions, we discuss the general connections between the world–sheet and spacetime supersymmetry. Then it is shown how combining informations coming from the two dimensional conformal field theory and the properties of the D=4 N=1 supergravity, it is possible to formulate two "non–renormalization" theorems for the superpotential of the effective theory.

CHAPTER 1

Strings and Their Effective Theories

One of the main purposes of the theoretical physics research in the last part of this century is that of finding a unique theory which describes all the known interactions between the elementary particles, these theories are called "Unified Theories".

The fundamental forces of nature actually known are: gravity, the weak force, the electromagnetic force and the nuclear force. The electromagnetic, weak and nuclear forces, although at low energy scale appear to have different properties, are unified in the sense that they are all described by one quantum field theory called the "Standard Model" [1]. This theory is based on the gauge group $SU(3) \otimes SU(2) \otimes U(1)$ and has three independent gauge coupling constants, one for each interaction. Going towards higher energies, the values of the coupling constants become closer until they should reach the same value at a scale called Grand Unification Scale. At this scale the three interactions should become an unique force. Some models, like the $SU(5)$ G.U.T., have been proposed to describe this unification theory, but all the known ones have some problems, like a too fast proton–decay or the presence of some light particles which are not observed.

Since the G.U.T. scale can be of order 10^{15} GeV, gravitational effects can become important. Thus, it is necessary to add gravity to the unification theory. But the quantization of gravity is still an open problem. In ref. [2] it was shown that a quantum theory of a spin–two particle requires the introduction of an infinite tower of higher spin particles and it is not known how to couple consistently particles with spin bigger than two[1]. In any case, a consistent quantization of a spin two particle, whenever it will be known, needs the construction of a lagrangian with infinite terms.

To turn round the problem, one can try to renounce to some hypothesis made in the

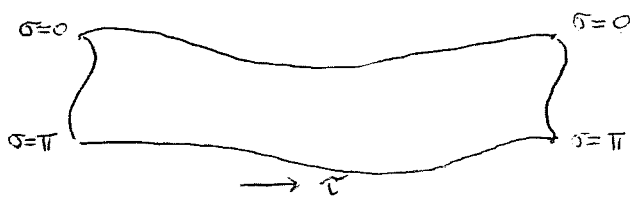
[1] Some recent attempts were made in ref. [3].

construction of a quantum theory. The starting point of the construction of a string theory is the renunciation to locality.

The usual quantum mechanics starts from the description of the motion of a material point in the space evolving with the time. Its position is described by some coordinates in the space at a fixed time τ, $x_\mu(\tau)$. Letting the time evolve, the point particle describes a trajectory in the spacetime.

Consider, instead, a one–dimensional extended object, a "string", described by $x_\mu(\sigma,\tau)$ where τ is the time ($-\infty < \tau < +\infty$) and σ parametrizes the length of the string (conventionally $0 \le \sigma \le \pi$)

Through its motion the string sweeps out a two–dimensional surface called "world–sheet"



The classical action [4] is obtained requiring that the 2d surface swept out during the string's evolution be minimal

$$\mathcal{S} = -\frac{1}{4\pi\alpha'}\int d^2\xi\sqrt{-h}h^{\alpha\beta}(\xi)\partial_\alpha X^M(\xi)\partial_\beta X^N(\xi)g_{MN} \tag{1.1}$$

where $\xi = (\sigma,\tau)$, $\alpha,\beta,\cdots$ are 2d indices, $X^M(\xi)$ are the coordinates of the string in the D–dimensional spacetime (or background space) and g_{MN} is the metric of the background space.

The first step towards the construction of a quantum field theory based on a string instead of a point, is the study of the quantum mechanics or 1^{st} quantized theory of the string. A 1^{st} quantized string theory is nothing else but a 2d quantum field theory described by the action (1.1) where the 2d quantum fields X^M are interpreted as coordinates of the background space.

The next step should be the construction of the string–field theory, i.e. a quantum field theory where the coordinates of the spacetime are 2d quantum fields. Up to now we know very little of the string–field theories, instead in the last years important results were obtained in the study of two dimensional quantum field theories.

In the next chapter we will review some of the most interesting features of string theories. Here we just recall two fundamental facts:

i) The spectrum is composed by an infinite tower of particles of all spins;

ii) The massless spectrum of some superstring models (like heterotic superstring in D=10) contains spin 2, $\frac{3}{2}$, 1, $\frac{1}{2}$ particles, i.e. describes gravity coupled to Yang–Mills systems (and, in dimensions lower than 10, also to matter).

Thus, superstring theories can realize at the same time a complete quantum description of gravity and its unification with the other forces.

The action (1.1) can be physically interpreted as describing a string moving in a flat background. In fact it describes the quantum excitations of a graviton around the flat metric g_{MN}, that is $G_{MN}(X) = g_{MN} + h_{MN}(X)$ (usually we will choose the background metric to be Minkowski $g_{MN} = \eta_{MN}$), and the quantum oscillations of the other modes around their zero configuration. An action describing a string moving in a given background is constructed by multiplying the vertex operators[2] which creates an excitation in the 2d quantum field theory, times the associated background configuration,

$$\mathcal{S} = -\frac{1}{4\pi\alpha'} \int d^2\sigma \sum_i V^i(\sigma) B_i(X(\sigma)) \tag{1.2}$$

where $V^i(\sigma)$ is the vertex operator for the i^{th} spacetime particle and $B_i(X(\sigma))$ the background configuration of the spacetime field (for example $V^g(\sigma) = \partial^\alpha X^M \partial_\alpha X_M$ and $B_g(X(\sigma)) = g_{MN}(X(\sigma))$). Notice that the background configuration is given a priori, $B_i(X)$ is a fixed function of the spacetime coordinates. Thus, in a arbitrarily given non–flat background, the string action is a 2d non–linear sigma model. Notice that the background cannot be chosen completely freely, it must be conformally invariant, as we will see in the next chapter.

The choice of the background corresponds to the choice of the (classical) vacuum state of the string theory. The condition of conformal invariance implies that the vacuum state must be a 2d conformal field theory.

In a flat background the string theory is a 2d quantum conformal field theory of free fields. This is the case which has been most studied also because there is no general idea of

[2] amputated of the momentum term $\exp(ik \cdot X)$.

how making computations in a string theory in an arbitrary background[3]. Since we don't know enough of string–field theory, we cannot say which is (or could be) the true, stable string vacuum; all what we can do is to find some "nice" vacua. An example of "nice" vacuum is the following: a consistent, anomaly–free string theory is the heterotic string in a background spacetime which has ten dimensions. A "nice" vacuum is one in which six of the ten dimensions are compactified, obtaining in this way a four dimensional spacetime string.

Once constructed a string theory, one would obtain some phenomenological predictions at the experimental energy scale. Since string theories describe gravity, their energy scale M_S is of the order of the Planck scale M_{Pl}. The simplest way to obtain phenomenological predictions at the weak scale M_W from string theories is to construct an "Effective Theory".

We will first review the definition of "Effective Theory" for a standard quantum field theory and then explain how it is possible to construct a "String Effective Theory".

§1.1 String Effective Theories

Consider a quantum field theory with action S_0 and fields $\{\varphi_i\}$, $\{\pi_i\}$ defined by the Feynman path integral

$$\mathcal{Z}[\varphi,\pi] = \int \prod_{i,j} [d\varphi_i][d\pi_j] e^{-S_0(\varphi_i,\pi_j)} \tag{1.3}$$

Suppose we are able to do the integral over the $\{\pi_i\}$ fields, the "Effective Theory" for the $\{\varphi_i\}$ fields emerging from S_0 is defined to be:

$$e^{-S_{Eff}(\varphi_i)} \equiv \int \prod_{j} [d\pi_j] e^{-S_0(\varphi_i,\pi_j)} \tag{1.4}$$

Physically the $S_{Eff}(\varphi_i)$ action describes all interactions with external states φ_i and internal interactions between the $\{\varphi_i, \pi_j\}$ fields, although in $S_{Eff}(\varphi_i)$ the $\{\pi_j\}$ fields do not appear anymore (see fig. 1).

In general, $S_{Eff}(\varphi_i)$ is a much more complicated action than $S_0(\varphi_i,\pi_j)$, but up to now we haven't done any approximation. In the usual physical situation the $\{\pi_j\}$ fields are massive and their masses are much bigger than the physical energy scale. In this case

[3] The case of a group–manifold background was studied in ref. [5].

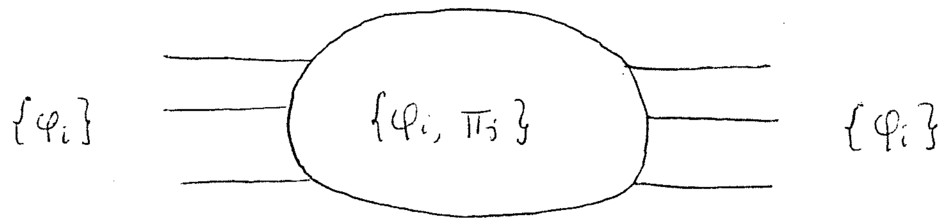


fig. 1

the $\{\pi_j\}$ fields are not observed as external (physical) states, so we can integrate them out. Moreover, the effective interactions between the $\{\varphi_i\}$ fields coming from the integration of the $\{\pi_j\}$ fields are suppressed by powers of M_{Phys}/M_π, where M_π is the mass scale of the $\{\pi_j\}$ fields and M_{Phys} is the physical energy scale. Being $M_{Phys} \ll M_\pi$ it is obvious that at most a few terms of the effective interactions between the $\{\varphi_i\}$ fields can give contributions to the physical phenomena. Thus, the effective action can be expanded in powers of M_{Phys}/M_π and the approximation consists in cutting the expansion at the preferred order.

Although the low energy spectrum can be very different from the complete one, the effective lagrangian of the light modes should have two fundamental properties:

i) it must satisfy all the symmetries which are exact symmetries of the underlying theory;

ii) it should reproduce exactly the same anomalies as the fundamental theory ('t Hooft criterium [6]).

Since the fundamental theory must be anomaly–free, ii) implies that also the effective theory must be anomaly free.

§1.1.1 *Superstrings' Effective Theories*

Superstring theories are supersymmetric 2d quantum field theories. In the spirit of constructing an effective action we would "integrate" the massive modes of the spectrum because their masses are multiple of the Planck mass.

The main problem is that (super–) string theories are two dimensional field theories but we need to know the D–dimensional background quantum action to compute the integration in (1.4) . At the moment this problem is not solved. Then it is not possible to explicitly integrate out the massive string modes in (1.4) , neither in an approximated way.

Thus, to construct a string effective theory one has first to obtain a field theory. This can be achieved computing the amplitudes between the massless modes and constructing the **classical** lagrangian which reproduces them[4], other methods (like σ–model techniques) can also be used. We will call the process to compute this classical lagrangian the "Field Theory Limit".

In the computation of the classical lagrangian two kinds of approximations are done. The first one refers to the fact that the amplitudes are computed at a finite order in the 2d perturbation theory. The perturbation expansion in string theory is equivalent to the sum over the various 2d world–sheet Riemann surfaces of different genus. The usual approximation is to consider the sphere ($g = 0$) and the torus ($g = 1$)[5].

In the amplitude approach, the (connected) interaction term in the field–theory effective lagrangian between n particles (n>1) is obtained directly from the n–point amplitude. Obviously, it is impossible to compute all the n–point amplitudes, the second approximation is done computing just the first few terms.

Notice that if we compute in string theory the scattering amplitudes between massless modes, in the intermediate states there can propagate both massless and massive modes. It turns out that the contribution to the effective action of a scattering amplitude can be expanded in powers of $(\frac{1}{M})^n$, thus each scattering amplitude gives rise to new higher order vertices between the massless modes. The approximation usually done consists in computing the 2, 3 and 4–point amplitudes to obtain the classical, field–theory effective lagrangian.

If one adopts the σ–model approach to compute the field–theory limit, since the principal objects to be computed are the β–functions, the corresponding approximation is done computing them with perturbation theory up to four loops.

§1.1.2 *Supergravity as a Superstring Effective Theory*

The field–theory effective action obtained following the criteria shown in the previous paragraphs, must be a supersymmetric theory of gravity coupled to matter, i.e. a Super-

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[4] Since the amplitudes are computed in a 2d quantum field theory, which is the only ambitus where we can make computations, the background D dimensional action obtained in this way is not a quantum action.

[5] σ–model techniques are usually applied only to tree level strings, i.e. $g = 0$ Riemann surfaces.

gravity theory. There are also some good physical reasons to have a supersymmetric theory, for example supersymmetry can help to solve the hierarchy problem. Moreover, since up to now we have only "integrated" the massive modes and taken the field theory limit, the effective theory should give a proper treatment of gravity starting from the fact that its energy scale should be the Planck scale, this is actually the case for a supergravity theory.

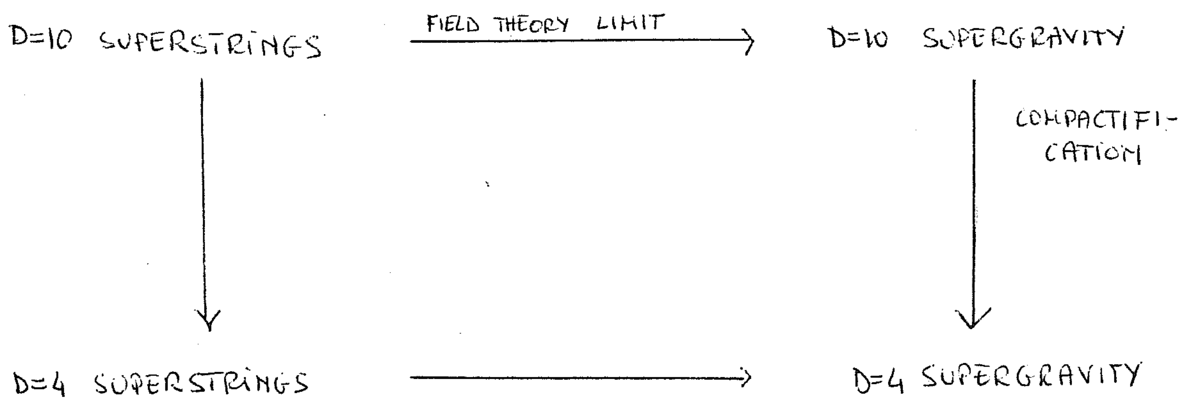


fig. 2

The D=10 heterotic superstring on a flat background with gauge group $E_8 \otimes E_8$ is a string theory which has a realistic spectrum and is anomaly free. This model will be the main subject of this thesis. Since the dimension of the background space is ten, taking the field theory limit one obtains a D=10 N=1 supergravity coupled to an $E_8 \otimes E_8$ super Yang–Mills theory (see fig. 2). The next problem is to obtain an effective theory in four dimensions; this is possible with a "Kaluza–Klein" like compactification. Compactifying a D=10 N=1 chiral supergravity theory one obtains a N=4 D=4 supergravity model (N is the number of unbroken supersymmetry charges). But in D=4 only the N=1 supergravity admits chiral fermions so that we must have at most one unbroken supersymmetry after the compactification. On the other hand, since supersymmetry helps to solve the hierarchy problem and impose severe constraints on the structure of the theory, it seems natural to preserve at least one unbroken supersymmetry charge in D=4. Thanks to the higher order terms which guarantee the cancellation of the anomalies in D=10, it is possible to make such a compactification to a D=4 N=1 supergravity theory.

The fundamental theory (i.e. superstrings) should also furnish a mechanism such that going to low energies supersymmetry should spontaneously break obtaining the "Standard Model" at the weak energy scale. Up to now it is not known any reliable mechanism of supersymmetry breaking (up to consider non–perturbative phenomena).

It is possible to construct other string theories which have a realistic spectrum. Usually they have a flat four dimensional background space, some of these models are stringy compactifications on a non–flat background of the D=10 heterotic superstring. Although D=4 superstrings are more complicated than the D=10 models, it is possible to obtain some extra informations which are lost making first the field theory limit. For example, one can study the relation between the spacetime and world–sheet supersymmetry or the non–renormalization properties of the potential of the effective theory. These issues will be considered in the last chapter.

CHAPTER 2

Strings, Superstrings and Conformal Field Theories

§2.1 Strings and Conformal Field Theory

A free bosonic string in a flat Minkowski background is a 2d quantum field theory described by the action

$$\mathcal{S} = -\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{g}g^{\alpha\beta}(\sigma)\partial_\alpha X^M\partial_\beta X_M \tag{2.1}$$

This action has two important local symmetries, reparametrization invariance and Weyl (or scale) invariance. A first consequence of Weyl invariance is that the stress energy tensor is traceless. The quantum theory (defined for example with a path integral formalism [7]) has three gauge invariances, two reparametrizations and the Weyl scaling. We can use these symmetries to fix a gauge on the 2d metric $g_{\alpha\beta}$

$$g_{\alpha\beta} = e^{\varphi}\eta_{\alpha\beta} \tag{2.2}$$

After having fixed the gauge, there survives a residual gauge symmetry called "conformal symmetry" which is given by a combination of the reparametrization and Weyl transformations. The conformal symmetry guarantees that the conformal metric factor φ cancels out from the action. It is convenient to use complex notations, defining

$$\begin{aligned}
&z = \tau + i\sigma & &\bar{z} = \tau - i\sigma \\
&\partial_z = \frac{1}{2}\left(\frac{\partial}{\partial\tau} + i\frac{\partial}{\partial\sigma}\right) & &\partial_{\bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial\tau} - i\frac{\partial}{\partial\sigma}\right) \\
&ds^2 = e^{\varphi}\left(d\sigma^2 + d\tau^2\right) = e^{\varphi}dzd\bar{z}
\end{aligned} \tag{2.3}$$

The conservations laws $\partial_{\bar{z}} T_{zz} = 0 = \partial_z T_{\bar{z}\bar{z}}$ imply that T_{zz} ($T_{\bar{z}\bar{z}}$) is an (anti–) analytic function of z ($\bar{z}$). An infinitesimal conformal transformation is an (anti–) analytic change of variables

$$z \longrightarrow z + \epsilon(z) \tag{2.4}$$

and is generated by

$$T_\epsilon = \oint_C \frac{dz}{2\pi i} \epsilon(z) T_{zz} \tag{2.5}$$

where C is a contour surrounding the origin [8]. The correct gauge fixing procedure requires the introduction of the Faddeev–Popov ghost–fields (c, $\bar{c}$, b, $\bar{b}$) [9]. Thus, the gauge–fixed action is

$$\mathcal{S} = -\frac{1}{4\pi\alpha'} \int dz d\bar{z} \partial_z X^M \partial_{\bar{z}} X_M + \frac{1}{2\pi\alpha'} \int \left(b_{zz} \partial_{\bar{z}} c^z + \bar{b}_{\bar{z}\bar{z}} \partial_z \bar{c}^{\bar{z}} \right) dz d\bar{z} \tag{2.6}$$

Thus a bosonic string theory is a 2d quantum conformal field theory. Properties of conformal field theories are very useful in studying strings. Conformal (or primary) fields are those 2d fields that under a conformal transformation

$$z \longrightarrow z'(z) \qquad \bar{z} \longrightarrow \bar{z}'(\bar{z}) \tag{2.7}$$

transform according to

$$\Phi(z, \bar{z}) \longrightarrow \left(\frac{\partial z'}{\partial z} \right)^h \left(\frac{\partial \bar{z}'}{\partial \bar{z}} \right)^{\bar{h}} \Phi(z', \bar{z}') \tag{2.8}$$

$h + \bar{h} = d$ is called the dimension or conformal weight and $h - \bar{h} = s$ the spin of Φ. Usually one considers only the analytic part of the theory because the classical equations of motion, $\Box X = 0$, admit a solution $X = X(z) + \overline{X}(\bar{z})$[6]. In this case $\bar{h} = 0$ and $s = d$.

The energy momentum tensor is

$$T(z) = T_{matter} + T_{ghosts} = -\frac{1}{2} : (\partial_z X)^2 : -2 : b(z) \partial_z c(z) : - : \partial_z b(z) \cdot c(z) : \tag{2.9}$$

and the infinitesimal conformal transformations are defined by

$$\delta_\epsilon \Phi(z) = [T_\epsilon, \Phi(z)]|_{\text{at equal time}} \tag{2.10}$$

[6] The background index M for the conformal theory is an enumerating index of the fields; when it is not necessary we will not write it.

This correlation function can be computed using the Wick theorem or the functional integration. We refer to the literature for the technical details [8]. The most important properties of a conformal field theory are encoded in the operator product expansion of the fields with the stress energy tensor:

$$T(z')\Phi(z) \equiv \frac{d}{(z'-z)^2}\Phi(z) + \frac{1}{z'-z}\partial_z\Phi(z) + \cdots \tag{2.11}$$
$$T(z')T(z) \equiv \frac{c/2}{(z'-z)^4} + \frac{2}{(z'-z)^2}T(z) + \frac{1}{z'-z}\partial_z T(z) + \cdots$$

where d is the conformal weight of Φ and c is a number called the central charge of the Virasoro algebra. The Virasoro algebra is the algebra obeyed by the fourier components of the stress energy tensor. The fourier analysis of a general tensor field of dimension d is

$$\Phi_n = \oint \frac{dz}{2\pi i} z^{n+d-1}\Phi(z) \qquad \Phi(z) = \sum_{n=-\infty}^{+\infty} \Phi_n z^{-n-d} \tag{2.12}$$

The stress energy tensor has dimension $d = 2$ so that $T(z) = \sum_{n=-\infty}^{+\infty} L_n z^{-n-2}$ and the Virasoro algebra is

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n+1)(n-1)\delta_{m+n,0} \tag{2.13}$$

The central charge c is an additive constant and for one massless free scalar field ($d = 1$) has the value $c = 1$. The conformal central charge of the $(b,\ c)$ system is $c = -26$.

Looking at (2.13) , one important thing must be noted: the quantum conformal algebra is different from the classical one unless $c = 0$. This means that we are in presence of an anomaly. In fact we can expect that an anomaly can break the scale invariance, in field theory this usually happens and is called trace anomaly. Although in some cases the trace anomaly is well accepted, in this case the conformal symmetry is a true local symmetry of the theory and we need to cancel the anomaly (i.e. $c = 0$). Polyakov has explicitly shown that if $c \neq 0$ the partition function is not any more invariant under a conformal transformation.

In our approach to conformal theories there is another reason why c must be zero, the B.R.S.T. charge is not nihilpotent unless $c = 0$ (which means exactly that the quantum theory is conformally invariant only if $c = 0$). The B.R.S.T. charge [10] is a hermitean operator which commutes with the hamiltonian, raises the ghost number by one unit and annihilates all ghosts–free, gauge invariant states. Following ref. [10], the B.R.S.T. charge for the analytic sector of the bosonic string is

$$\begin{aligned} Q_{BRS} &= \oint \frac{dz}{2\pi i} J_{BRS} \\ J_{BRS} &= c_z \left(T^X(z) + \frac{1}{2} T^{(b,c)}(z) \right) + \frac{3}{2} \partial^2 c_z \end{aligned} \tag{2.14}$$

(the last term is a total derivative so in correlation function can be dropped, but ensures that J_{BRS} is a conformal field and the conservation of Q_{BRS} on a curved world surface). The B.R.S.T. charge acts on the fields as

$$\begin{aligned} [Q_{BRS}, O(z)] &= \oint \frac{dw}{2\pi i} J_{BRS}(w) O(z) \\ [Q_{BRS}, X(z)] &= c^z \partial_z X(z) \\ \{Q_{BRS}, c^z\} &= c^z \partial_z c^z \\ \{Q_{BRS}, b_{zz}\} &= T(z) \end{aligned} \tag{2.15}$$

It is easy to check that $(Q_{BRS})^2 = 0$ if and only if $c = 0$.

Thus, a necessary condition to be satisfied in string theory is the vanishing of the central charge. This can be achieved easily, it is sufficient to notice that the central charge is additive so that the total central charge of the action (2.6) is $c = D - 26$ which implies $D = 26$. Then the quantization of a string theory induces a constraint on the dimensionality of the spacetime (we will see that this constraint is not so strong as it appears now).

In the same way one can study the case of a 2d supersymmetric conformal field theory in a flat background. It is sufficient to introduce the superfields for the scalar and ghost fields, $X(\theta, \overline{\theta}) = X + \theta\psi + \overline{\theta}\overline{\psi} + \theta\overline{\theta}F$, $B = \beta + \theta b$, $C = c + \theta\gamma$.
The total supersymmetric action is

$$\mathcal{S} = \frac{1}{4\pi\alpha'} \int d^2 d^2\theta \left(\overline{D}X DX + (B\overline{D}C + \text{ h.c.}) \right) \tag{2.16}$$

The B.R.S.T. charge is

$$\begin{aligned} Q_{BRS} &= -\oint \frac{dz}{2\pi i} \int d\theta : C \left(T^X + \frac{1}{2} T^{(B,C)} \right) + \frac{3}{4} D(C(DC)B) := \\ &= +\oint \frac{dz}{2\pi i} : \left[c(z) \left(T_B^{(X,\psi)} + \frac{1}{2} T_B^{(b,c,\beta,\gamma)} \right) - \gamma(z) \left(T_F^{(X,\psi)} + \frac{1}{2} T_F^{(b,c,\beta,\gamma)} \right) \right] : \end{aligned} \tag{2.17}$$

where $T(z,\theta) = T_F + \theta T_B$.

It is convenient to define $\widehat{c}^{matter} = \frac{2}{3}c^{(X,\psi)}$ so that the contribution to the central charge of a chiral superfield (of a N=1 2d supersymmetry) is $\widehat{c} = 1$ (i.e. $c = \frac{3}{2}$, this is obvious because a scalar field has $c = 1$ and a fermion $c = \frac{1}{2}$). Since $c^{(b,c)} + c^{(\beta,\gamma)} = -15$ or $\widehat{c}^{(b,c,\beta,\gamma)} = -10$, the vanishing of the central charge implies that $\widehat{c}^{matter} = +10$.

Let us consider now the case of a string in a non–flat background. By definition it is a 2d quantum field theory with local reparametrization and Weyl invariances. We can always choose the conformal gauge and introduce the conformal ghosts (b,c). Thus a string theory is a 2d conformal field theory, this implies that the background must be conformal. As before, the conformal invariance is guaranteed by the nihilpotence of the B.R.S.T. charge which is equivalent to the vanishing of the central charge of the Virasoro algebra; as we know, the conformal ghosts have central charge $c = -26$. We can also construct supersymmetric string theories. The superconformal gauge fixing of the local supersymmetry implies the presence of the superconformal ghosts $(\beta,\ \gamma)$ which have central charge $c = +11$.

The first requirement in the construction of a string theory is that the central charge vanishes. It is easy to compute the central charge for a 2d free field (i.e. in the case of a string in a flat background), but things are much more complicated for interacting 2d fields (i.e. strings on a non–flat background). We can fulfill the requirement of vanishing central charge by summing conformal theories such that $\sum_i c^i \equiv c^{matter} = +26$ (or $\widehat{c} = +10$ if supersymmetric). Thanks to the possibility of factorizing the string, we can choose different theories on the right and left sectors, up to the fact of preserving unitarity and modular invariance. We can also choose to have supersymmetry only in one sector, these models are called "heterotic".

Let us make some examples of string theories.

i) The simplest string theories are the non–supersymmetric ones on a flat background. The bosonic string theory described by the action (2.6) must have 26 coordinate fields to satisfy $c = 0$, thus the dimension of the spacetime is D=26. Other model can be constructed introducing 2d fermion fields, however one must be careful to construct a model where all anomalies cancel. In any case the central charge of the "matter" system must be 26. An example is the case in which there are 10 coordinate fields X^M and 32 fermions λ^I.

ii) A supersymmetric string on a flat background is described by the action (2.16) .

Since $\widehat{c} = 10$, the dimension of the spacetime is D=10.

iii) The D=10, $E_8 \otimes E_8$ heterotic superstring in a flat background is a supersymmetric string of type ii) in the right sector and a non–supersymmetric string of type i) in the left one. The dimensionality of the spacetime is D=10 and follows from the vanishing of the central charge in the right sector. In the left sector there are 10 coordinate fields, thus one has to add a $c = 16$ system. This is obtained for example introducing a system of 32 free fermions λ^I; the action in this case is

$$\mathcal{S}_{het} = \frac{1}{4\pi\alpha'} \int d^2z \, [\partial_{\bar{z}} X^M \partial_z X_M - \psi^M \partial_{\bar{z}} \psi_M - \lambda^I \partial_z \lambda^I + \\ + \text{ghosts } (b,\ c,\ \beta,\ \gamma,\ \bar{b},\ \bar{c})] \tag{2.18}$$

Obviously this system has (0,1) supersymmetry by construction.

iv) D=4 string models can be constructed summing a D=4 string on a flat background and a compact conformal theory as internal space chosen in such a way to get the vanishing of the central charge. The D=4 string can be of the types described in points i), ii), or iii). Some interesting examples of D=4 heterotic superstring were considered by Gepner [11]. He takes as internal space any solvable conformal field theory with N=2 world–sheet supersymmetry[7]. Solvable conformal field theories are models in which all correlation and partition functions can be computed exactly. Moreover, one needs also the theory to be modular invariant, this is possible if the left and right conformal structure are identical. The heterotic string theories in D=4 constructed by Gepner are given by the free D=4 coordinates sector, a $c = 9$ conformal theory both on the left (superstring–like) and right (non–supersymetric) sector (this is done to achieve modular invariance) and a system of free bosons in the right sector with $c = 13$ moving on the maximal torus of a rank–13 Lie group.

v) A more explicit example of a D=4 heterotic string can be constructed as the sum of a D=4 string on a flat background and a D=6 string on a non–flat (interacting) compact background. One starts from the flat heterotic string (2.18) and compactifies six coordinates X^m on a Calabi–Yau like manifold K_6[8]. This means to divide the action in a D=4 part and a D=6 one, and to consider the D=6 part in a Calabi–Yau like background (thus obtaining

[7] As we will see in the last chapter, N=2 world–sheet supersymmetry is necessary to have N=1 spacetime supersymmetry.

[8] See chapter 4 for a detailed description of the Calabi–Yau like compactification and the related issues.

a 2d non–linear sigma model on K_6). The model can be constructed as follows.
Consider the $E_8 \otimes E_8$ D=10 heterotic superstring (2.18) , the 32 fermions λ^I transform in the $(16,1)\oplus(1,16)$ representation of the $SO(16)\otimes SO(16)$ subgroup of $E_8\otimes E_8$. We can pick an $SO(6)$ subgroup of $SO(16)\otimes SO(16)$ such that the fields λ^i for i=7,...,32 are singlets and λ^l l=1,...,6 transform in the vector representation of $SO(6)$. Under an $SU(3)$ subgroup of $SO(6)$ these fields transform as $(3\oplus 3)$, let us denote them by λ^r, $\lambda^{\bar{r}}$. (This division of the fields is done following the ideas of the Calabi–Yau compactification (see chapter 4) where the relevant gauge group in the compact space is $SU(3)$.)
Since under the compactification the D=10 Lorentz group splits in $SO(10)\sim SO(4)\otimes SO(6)$, in an analogous way we can split the coordinate fields $(X^M,\ \psi^M)$ in D=4 coordinates and fields in the six dimensional space K_6 (for details see ref. [12]). Once splitted in this way the action (2.18) , we can introduce in the six dimensional part the background fields. The explicit expression of this model is

$$\begin{aligned}
\mathcal{S} &= \mathcal{S}_4 + \mathcal{S}_{int} \qquad (2.19)\\
\mathcal{S}_4 &= \frac{1}{4\pi\alpha'}\int d^2z\,\big[\partial_{\bar{z}}X^\mu\partial_z X_\mu - \psi^\mu\partial\bar{z}\psi_\mu - \lambda^i\partial_z\lambda^i + \\
&\quad \text{ghosts }(b,\ c,\ ,\beta,\ \gamma,\ \bar{b},\ \bar{c})\big]\\
\mathcal{S}_{int} &= \frac{1}{4\pi\alpha'}\int d^2z\,\big[g_{mn}(X_m)\partial_{\bar{z}}X^m\partial_z X_m + \varepsilon^{\alpha\beta}B_{mn}(X_m)\partial_\alpha X^m\partial_\beta X^n + \\
&\quad -2\psi^m\partial_{\bar{z}}\psi_m + \psi^m\left(\omega_p^{mn}(X_m) - H_p^{mn}(X_m)\right)\lambda^n\partial_{\bar{z}}X^p + \\
&\quad -2\left(\lambda^{\bar{r}}\partial_z\lambda^r + iA_m^\alpha(X_m)(T^\alpha)_{\bar{r}r}\partial_z X^m\lambda^{\bar{r}}\lambda^r\right) + F_{mn}^\alpha(X_p)\lambda^{\bar{p}}(T^\alpha)_{\bar{r}s}\lambda^s\psi^m\psi^n\big]
\end{aligned}$$

where m,n,p=5,...,10, g is the Calabi–Yau metric, B the antisymmetric tensor with field strength H and A_m^α the $SU(3)$ gauge fields with field strength F. The free part of the action, $\mathcal{S}_4$, is an heterotic string theory in a flat four dimensional spacetime with central charge $c=-9$ ($\hat{c}=-6$) in the right sector and $c=-9$ in the left sector. Thus $\mathcal{S}_{int}$ should describe a conformal field theory with central charge $c=+9$ on both the left and right sectors. The 2d sigma model $\mathcal{S}_{int}$ is a conformal interacting theory, thus one should verify that the central charge, which for the free theory is correctly $c=+9$, do not change because of the interactions. Although not a general proof has been given, it has been verified to four–loop order in the sigma model perturbation theory that on a Calabi–Yau manifold the central charge is $c=+9$ [13].

In the following chapters we will mainly study the effective theories obtained from the

D=10 heterotic string on a flat background (iii)). In the last chapter we will consider a D=4 heterotic superstring obtained from the D=10 one by sigma model compactification on a Calabi–Yau manifold, as in example v).

§2.2 Vertex Operators and String Spectrum

The Hilbert space of a string theory on a flat background can be easily constructed acting on the vacuum with the creation operators obtained from the Fourier analysis of the fields. The vacuum state turns out not to be the state with minimum energy. The state of minimum energy $|\Omega\rangle$ is created acting on the vacuum with a ghost oscillator (in the bosonic case) and has negative energy, thus $|\Omega\rangle$ describes a tachyon. Moreover the ground–state has a non zero ghost charge. This is a consequence of the fact that the ghost number current $J_z = bc$ is anomalous.

The string states can be constructed applying to the ground–state some composite operators called Vertex operators, $|\psi\rangle = V(z)|\Omega\rangle$. B.R.S.T. invariance of the physical states implies that the vertex operators must be conformal primary fields of dimension 1. For example, the following operators are vertex operators for the D=26 bosonic string

$$\zeta^M \partial_z X_M e^{ip\cdot X}, \quad \zeta^{MN} \partial_z X_M \partial_z X_N e^{ip\cdot X}, \quad \zeta^M \partial_z^2 X_M e^{ip\cdot X}, \quad \cdots \tag{2.20}$$

We are working in the RNS formalism in which the 2d Lorentz group is linearly realized. The operators that we have introduced up to now are bosonic or fermionic operators from a two dimensional point of view, but on the spacetime they are all bosons. This means that we can construct only bosonic spacetime states. Notice however that half of these states from a 2d point of view are fermions; hence there is a conflict of statistics because these states obey a Bose–Einstein statistics on the spacetime and a Fermi–Dirac statistics on the world–sheet.

To construct spacetime spinor states one introduces the spin–fields S_α[9]. The spin–field S_α is antiperiodic on circles around the origin, which means that it has a square–root branch point in the origin. It can be constructed using the bosonized version of the 2d fermion fields (ghosts included). The spin field is singular on the world–sheet but transforms in the spinor

[9] For a detailed description of the spin–fields see refs. [14, 8, 15].

representation of the spacetime Lorentz group. Acting on the vacuum with the spin field we can construct spacetime spinor states. As for the spacetime bosons, these states are spacetime fermions but on the world–sheet they can be both fermions or bosons.

It is possible to make a consistent truncation of the spectrum, called "G.S.O." projection, which preserve only the states with the correct quantum statistics. It is done by means of a quantum operator which is odd both for spacetime and world–sheet spinors, even otherwise; it is in some sense a generalization of the γ_5 operator in D=4 quantum field theory. Requiring that a physical state has eigenvalue +1 of the G.S.O. projection operator, or equivalently that a physical vertex operator satisfies $Q_{GSO} V Q_{GSO}^{-1} = V$, only the states with consistent statistics survive.

The effect of the G.S.O. projection on the spectrum of the supersymmetric string is that of eliminate the tachyon state, in this way the lightest physical states are the massless ones. Moreover the massless states are a vector and a spinor which, on shell in ten dimensions, have the same number of degrees of freedom. This is not an accident because the physical spectrum, after the G.S.O. projection, is supersymmetric in spacetime. It is easy, in fact, to define a spinor operator which plays the role of an on–shell spacetime supersymmetry charge

$$Q_\alpha \sim \oint \frac{dz}{2\pi i} S_\alpha(z) \tag{2.21}$$

Let us analyze in some more details the spectrum of the D=10 heterotic string. The D=10 heterotic string can be thought as the direct product of a superstring and a bosonic string in which part of the fields are fermionized. The condition of B.R.S.T. invariance implies that the direct product of the states must be taken between states with the same mass. Let us suppose that the 32 λ^I fermions of the non–supersymmetric sector transform in the fundamental representation of $SO(32)$ ($I=$ index in the fundamental representation of $SO(32)$). Because the G.S.O. projection keeps as the lowest physical state of the superstring sector the massless states, thanks to the constraint on the masses of the left and right sectors, the bosonic tachyon is cancelled from the physical spectrum. The massless states of the non–supersymmetric sector are given by an $SO(10)$ vector constructed with $\partial_z X^M$, $|M\rangle$, and another bosonic state, Lorentz singlet, which transforms in the antisymmetric second–rank–tensor representation of $SO(32)$, $|I,J\rangle$ (which indeed is the adjoint representation of $SO(32)$). This state is constructed by the product $\lambda^I \lambda^J$ which explains the antisymmetry of the indices. Notice that, through the equal mass constraint, the G.S.O. projection acts also on the non–supersymmetric sector eliminating not only the tachyonic state but also the states created with an odd number of λ^I which have wrong statistics and mass.

Thus, the ground–massless states of the heterotic string are

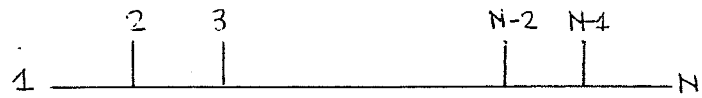
$$(|M\rangle \oplus |\alpha\rangle) \otimes (|N\rangle \oplus |I,J\rangle) = \qquad (2.22)$$

$$(|M\rangle \otimes |N\rangle \ \oplus \ |\alpha\rangle \otimes |N\rangle) \oplus (|M\rangle \otimes |I,J\rangle \ \oplus \ |\alpha\rangle \otimes |I,J\rangle)$$

Obviously the spectrum is completely spacetime supersymmetric and is composed by the D=10 super Yang–Mills multiplet of $SO(32)$ ($|M\rangle \otimes |I,J\rangle = A_M^{I,J}$ are the gauge vectors, $|\alpha\rangle \otimes |I,J\rangle = \chi_\alpha^{I,J}$ are the gauge fermions) and the D=10 supergravity multiplet (the symmetric part of $|M\rangle \otimes |N\rangle$ is the metric tensor g_{MN}, the trace part is the dilaton D, the antisymmetric part is the antismmetric tensor B_{MN} and $|\alpha\rangle \otimes |N\rangle$ the related fermions).

The most common heterotic string is the one in which the 32 fermions λ^I are divided in two groups of 16 each transforming in the fundamental representation of an $SO(16) \otimes SO(16)$ subgroup of $SO(32)$. It turns out that the massless vector bosons are 496 states and generate a $E_8 \otimes E_8$ gauge symmetry. The explicit construction of the extra conserved currents of $E_8 \otimes E_8$ is a little tricky and we will not review it.

Since L_0 is the string equivalent of the hamiltonian of the theory, knowing the propagator ($\sim 1/L_0$) and the vertex operators for the emission of the physical particles, it is not difficult, in principle, to compute the scattering amplitudes between the string physical states[10]. For example, the scattering between N particles described by



has a tree scattering amplitude which schematically is

$$\langle V_1 | V_2 \Pi V_3 \Pi \quad \cdots \quad \Pi V_{N-2} \Pi V_{N-1} | V_N \rangle \qquad (2.23)$$

where V_I is the vertex emission operator for the I^{th} particle and Π is the string propagator.

10 One must be careful to use the opportune vertex operators to compensate the ghost charge of the vacuum.

CHAPTER 3

The 10-D Field Theory Limit

In the first chapter we have already discussed the meaning of a low energy effective action. The energy scale of string theories (M_S) should be of the order of the Planck scale (M_P) because string theories realize a fully consistent quantum approach to gravity. As already discussed, the massive particles have masses proportional to the string scale, thus in the low energy limit only the massless modes are considered. The string effective theory for the massless modes looks terribly complicated because of the appearance of an infinite number of higher derivative interactions. Following the two criteria given in the first chapter and the approximations there explained, we will now review how one can construct a field–theory string effective theory.

The first observation is that the massless modes of the D=10 heterotic superstring correspond exactly to the field content of the N=1 D=10 super Yang–Mills coupled to supergravity theory, which we will call "Chapline–Manton" theory[11] [16]. Once fixed the gauge group to be $E_8 \otimes E_8$, there is no room for arbitrariness in this theory.

The coupling constants of the Chapline–Manton lagrangian are the gravitational constant k_{10} and the gauge coupling constant g_{10} and in the next paragraph we will show how they are related to the string parameters. First, we have to discuss which are the string parameters. Until now we have introduced only one string parameter, the Regge slope α' (or its inverse T, the string tension). It has dimensions M^{-2} and in the previous chapter has been fixed to $\alpha' = \frac{1}{2}$.

Just on dimensional ground, being α' the only dimensionful parameter, we can reintroduce α' in the mass formula obtaining $m^2 \propto \frac{1}{\alpha'}$ from which $M_S \propto (\alpha')^{-\frac{1}{2}}$ (for definiteness,

[11] Notice that we are not saying that the Chapline–Manton theory is the field theory limit of the heterotic string, but only a good point from which start with.

M_S can be thought to be the mass of the first excited state). The direct relation between M_S and the Planck scale will be seen after the compactification to 4d, because is k_4 which is directly related to M_P. Thus the effect of integrating the massive modes is that of introducing an infinite set of terms in powers of α'.

As it is defined, string theory has a natural perturbation expansion where the loop diagrams are higher genus world–sheet Riemannian manifolds but does not have a free dimensionless parameter to be interpreted as the string coupling constant [17]. This follows from the fact that the coupling constant which measures the strength of the interaction between strings can be absorbed into a redefinition of the massless dilaton field (trace part of the spacetime metric). Thus, the dilaton expectation value plays the role of the string coupling constant [18].

A hint to this fact can be given within the Polyakov functional integral approach to the string. Consider the 2d action in presence of background fields; for a bosonic closed string

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \left(\sqrt{\gamma} R^2 D(X) + \sqrt{\gamma}\gamma^{ab}\partial_a X^\mu \partial_b X^\nu g_{\mu\nu}(X) + \ldots \right) \tag{3.1}$$

The partition function is

$$\mathcal{Z} = \sum_{loops} \int [d\gamma][dX] e^{-S[\gamma, X \ldots]} \tag{3.2}$$

Making a constant shift of the dilaton field $D(X) \to D(X) + a$ and remembering that

$$a\frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{\gamma} R^2 = \frac{a}{\alpha'} 2(1-L) \tag{3.3}$$

(L is the number of genus or string loops) one obtains

$$\begin{aligned} \mathcal{Z} &= \sum_{L=Loops} e^{\frac{2a}{\alpha'}(L-1)} \int [d\gamma][dX] \ldots e^{-S[\gamma, X, \ldots]} \\ &= \sum_L g_{st}^{2(L-1)} \int [d\gamma][dX] \ldots e^{-S[\gamma, X, \ldots]} \end{aligned} \tag{3.4}$$

where $g_{st} = e^{a/\alpha'}$ is the string coupling constant. In the following we will ridefine the dilaton field so that it has zero expectation value.

This argument can seem to be quite arbitrary because a priori it is not evident that a shift of the dilaton field is allowed. This is justified by the fact (as noted in ref. [19]) that string amplitudes have a well defined (loop dependent) scaling property under the rescaling of the dilaton vertex. This property, which will be important in the construction of the D=4 effective lagrangian, is also shared by the Chapline–Manton lagrangian. The Chapline–Manton lagrangian is

$$\frac{1}{e}\mathcal{L}^{CM} = -\frac{1}{2k_{10}^2}R^{10} - \frac{1}{4g_{10}^2\phi}tr(F_{MN}^2) - \frac{1}{k_{10}^2}\left(\frac{\partial_M\phi}{\phi}\right)^2 - \frac{3k_{10}^2}{8g_{10}^4\phi^2}H_{MNP}^2 + \text{fermions} \tag{3.5}$$

(k_{10} has dimensions (length)$^{-4}$, g_{10} (length)$^{-3}$) H_{MNP} is the field strength of B_{MN} and ϕ is the dilaton field. With a suitable Weyl rescaling of the metric g_{MN}, it becomes

$$\frac{1}{e}\mathcal{L}^{CM} = \frac{1}{\phi^2}\left[-\frac{1}{2}R^{10} - \frac{1}{4}tr(F_{MN}^2) + 2\left(\frac{\partial_M\phi}{\phi}\right)^2 - \frac{3}{4}H_{MNP}^2 + \dots\right] \tag{3.6}$$

which obviously scales as $\mathcal{L}^{CM} \to \lambda^{-2}\mathcal{L}^{CM}$ when $\phi \to \lambda\phi$ (in the previous formula under $g_{MN} \to t^{-2}g_{MN}$, $\psi \to t^{\frac{1}{2}}\psi$ and $\phi \to t^2\phi$, $\mathcal{L}^{CM} \to t^{-8}\mathcal{L}^{CM}$). This scaling property can be seen to correspond to the scaling property of the tree string amplitude. Notice that this scaling law is not a quantum symmetry because at quantum level the normalization of the action is not at all irrelevant.

Thus, g_{st} is the (dimensionless) string coupling constant. g_{st} should appear in each vertex operator because it creates a particle state which interacts with the string; moreover each loop contributes to a scattering amplitude with g_{st}^{2L-2}, then for a generic scattering amplitude of M particles at the L loop of string holds

$$A_L(M) = g_{st}^{2L-2+M}T_L(M) \tag{3.7}$$

This indeed is the explicit result already obtained from dual models long time ago [20].

String theory has thus an infinite set of (classical) vacua labelled by the expectation value of the dilaton field. Quantum mechanically this vacuum degeneracy should be removed leaving a well defined ground state, so that the expectation value of the dilaton would be dynamically (non–pertubatively) determined. In what follows, mainly tree string loop level will be considered, in this ambitus we will see that the scaling property of the dilaton field will have far reaching consequences.

Before entering the details of the computation of the "field–theory limit", we want to recall some important facts on D=10 supergravity theories. The D=10 N=1 supergravity formulated by the authors of ref. [16] which we generically call "Chapline–Manton" theory (3.5) , is a classical, standard (i.e. with terms containing up to two derivatives) local supersymmetric and local gauge invariant action. In view of a possible quantization, it is meaningful to ask if this theory is anomaly–free. The theory described by the action (3.5) suffers both from Yang–Mills and gravitational anomalies. Green and Schwarz discovered how one has to modify this action to cancel the anomalies. First one must redefine the field strength of B, H. In the Chapline–Manton lagrangian (3.5) the field strength of B is $H = dB - \omega_{3Y}$ where ω_{3Y} is the Yang–Mills Chern–Simons three form; it becomes $H = db - \omega_{3Y} + \omega_{3L}$ where ω_{3L} is the Lorentz Chern–Simons three form. At the same time the field B must transform under a local Lorentz transformation.

The introduction of the Lorentz Chern–Simons form is of fundamental importance. First because it is an higher derivative term and then the action is not any more a "standard" one. Second because it is a bosonic term. The problem of its supersymmetrization in D=10 is very difficult and it has been not completely solved up to now. We know that the supersymmetrization of ω_{3L} exists and requires the introduction of an infinite number of higher derivative terms [21]. The situation is completely different in four dimensions where, as we will see in chapter 5, it is possible to explicitly compute the supersymmetrization of the Lorentz Chern–Simons form.

§3.1 Effective Action from Scattering Amplitudes

Our aim is to construct a (classical) field theory that is an effective action for the massless modes of the D=10, flat background, $E_8 \otimes E_8$ heterotic string[12].The simplest approach to this problem [22] is that of computing the string scattering amplitudes of the massless particles and then construct the effective classical lagrangian which reproduces the S–matrix. This can be done in a "perturbative" fashion. One first constructs the effective lagrangian $\mathcal{L}_{2pt}$ that describes the massless free particles of the theory; one then adds the cubic terms which describe their three–point couplings, as given by the string vertices, thus obtaining $\mathcal{L}_{3pt}$ (effective lagrangian up to three–point interactions).

Eventually one considers the four–point scattering amplitudes. Unitarity guarantees that the massless poles will be those generated by the tree graphs of $\mathcal{L}_{3pt}$; In fact the amplitude can be divided in two parts: the first one is made by the massless pole contribution

[12] We are considering only the case of the tree string loop now.

and the other given by the rest. Simbolically

$$A_4 \sim \frac{1}{k^2} + f(k^2) \tag{3.8}$$

$f(k^2)$ has no singularities for small external momenta and is due to the massive particle exchange. One can easily convince himself of this fact considering the contribution to the scattering amplitude of a graph with a massless particle as intermediate state and one with a massive particle. Thus, $f(k^2)$ can be expanded in a power series of k^2. Each term in this expansion can be reproduced by some local vertex, V_{4pt}, which starts out quartic in the massless fields; from these objects one can easily compute $\mathcal{L}_{4pt}$. This procedure can be repeated for the higher–order–point scattering amplitudes, thereby yielding, in principle, the effective lagrangian to all orders.

The effective lagrangian so constructed will not be unique since a redefinition of the fields will not affect the scattering amplitudes. In fact, if $\mathcal{L}[\phi_i]$ yields the S–matrix for particles described by fields ϕ_i and the transformation $\phi_i \to \phi_i'(\phi_i)$ is non singular, then the lagrangian $\mathcal{L}'[\phi_i'] \equiv \mathcal{L}[\phi_i'(\phi_i)]$ gives the same S–matrix. $\mathcal{L}'$ gives the same equations of motion since the extrema of $\mathcal{L}$ and $\mathcal{L}'$ coincide as long as the Jacobian $\frac{\partial \phi_i'}{\partial \phi_j}$ is non singular

$$\frac{\partial \mathcal{L}[\phi]}{\partial \phi_i} = \frac{\partial \mathcal{L}'[\phi'(\phi)]}{\partial \phi_i} = \sum_i \frac{\partial \mathcal{L}'}{\partial \phi_j'} \frac{\partial \phi_j'}{\partial \phi_i} \tag{3.9}$$

§3.1.1 *Two–Point Effective Action*

The bosonic massless modes of the heterotic string are the graviton multiplet $g = (h_{ab}, B_{mn}, D, \ldots)$ and the Yang Mills fields A^I_m with field strength F^I_{mn}. The field strength of the antisymmetric field B_{mn} is $H_{abc} = \partial_{[c} B_{ab]} = \partial_c B_{ab} + \partial_a B_{cb} + \partial_b B_{ca}$. Let's start considering the two point amplitudes between these fields [13]; since they are the propagators of the fields, $\mathcal{L}_{2pt}$ is nothing else but the part of the lagrangian containing the kinetic terms[14]

$$\frac{1}{\sqrt{-g}} \mathcal{L}_{2pt} = -\frac{1}{2k_{10}^2} R - \frac{e^{-2cD}}{6} H_{mnp} H^{mnp} - \frac{1}{2} (\nabla_m D)(\nabla^m D) - \frac{e^{cD}}{4} F^I_{mn} F^I_{mn} \tag{3.10}$$

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[13] see [22,20, 23].

[14] We are using here a rather different normalization of the fields but at the end of the paragraph by means of a field redefinition we will go back to the usual normalization.

Notice that this lagrangian contains also three–point couplings (they must be there, otherwise gauge invariance would be lost). For example the three–point on–shell expansion of $\sqrt{-g}R$ is

$$-\frac{\sqrt{-g}}{2k_{10}^2}R|_{3pt} = -k_{10}\left(h^{ab}h^{cd}\partial_a\partial_b h_{cd} + 2\partial^d h_{ab}\partial^a h^{bc}h_{cd}\right) \tag{3.11}$$

where $g_{mn} = \eta_{mn} + 2k_{10}h_{mn}$.

§3.1.2 *Three–Point Effective Action*

All the possible three-point amplitudes can be computed using the vertex operators ([22, 20, 23]); using short–hands notations the result is

$$\begin{aligned} A_{ggg} &= g_{st}\left\{(k_2\theta_1 k_2)tr(\theta_2\theta_3^T) + (k_3\theta_2\theta_3^T\theta_1 k_2)+ \right.\\ &\left.+(k_1\theta_3\theta_2^T\theta_1 k_2) + \frac{1}{4}(k_2\theta_1 k_2)(k_3\theta_2\theta_3^T k_1)\right\} + \text{cyclic permutations} \\ A_{AAg} &= -\frac{1}{2}g_{st}\left\{tr(A_1\dot{A} - 2)(k_1\theta_3 k_1)+\right.\\ &\left.+tr(A_1^a A_2^b)(k_{2a}k_1^c\theta_{3cb} + k_{3b}k_1^c\theta_{3ca})\right\} \\ A_{AAA} &= -g_{st}\left\{tr\left([A_1^m, A_2^m]A_3^n\right)k_{1n} + \text{cyclic permutations}\right\} \end{aligned} \tag{3.12}$$

where θ_{ab} is the polarization tensor of an element of the gravitational multiplet $g = (h_{ab}, B_{mn}, D)$; the symmetric traceless part of θ_{ab} refers to the graviton g_{ab}, the antisymmetric to B_{mn} and the trace part to the dilaton D.

The three point–amplitude contain two– and four–derivatives terms[15]. The two derivatives terms must be invariant under the symmetries of the theory and this implies that they must have the same expression of the two–derivative terms already present in $\mathcal{L}_{2pt}$. In other words, they renormalize $\mathcal{L}_{2pt}$ and from the two–derivative terms of the three–point amplitudes we can compute the relations between the coupling constant k_{10}, c, g_{10} and the string coupling constant g_{st} and α'.

In fact, consider the three gravitons scattering whose amplitude A_{hhh} is obtained choosing $\theta_{ab} = h_{ab}$ (symmetric and traceless). This contribution can be obtained from the lagrangian

$$\mathcal{L}_{hhh} = -\frac{1}{2}g_{st}\left(h^{ab}h^{cd}\partial_a\partial_b h_{cd} + 2\partial^d h_{ab}\partial^a h^{bc}\cdot h_{cd}\right) \tag{3.13}$$

[15] Remeber that $k \leftrightarrow i\partial$.

Comparing this lagrangian with (3.11) one immediately obtains

$$2k_{10} = g_{st}(2\alpha')^2 \tag{3.14}$$

(remember that $[\alpha'] = m^{-2}$ and $\alpha' = \frac{1}{2}$).

In the same way from the BBD term and the hAA amplitude one obtains

$$\begin{aligned} c &= \sqrt{\frac{1}{2}}k_{10} \\ g_{st} &= g_{10}(2\alpha')^{-\frac{3}{2}} \end{aligned} \tag{3.15}$$

from which follows the famous relation

$$\frac{(k_{10})^2}{(g_{10})^2} = \frac{\alpha'}{2} \tag{3.16}$$

There is one important exception to the two–point rule we have just stated. From the AAB amplitude one obtains a term which cannot be eliminated by field redefinitions and is not included in $\mathcal{L}_{2pt}$. The inclusion of this term [23] leads to a redefinition of H_{abc} as

$$\begin{aligned} H_{abc} &= \partial_{[c}B_{ab]} - \frac{k_{10}}{4}\left(\Omega_{3y}\right)_{abc} \\ \left(\Omega_{3y}\right)_{abc} &= tr\left(A_{[c}F_{ab]} - \frac{g_{10}}{3}A_{[a}A_{[b}A_{c]]}\right) \end{aligned} \tag{3.17}$$

Indeed it is well known that to couple super Yang–Mills to supergravity in D=10 is necessary to modify in this way the H_{abc} tensor, obtaining the Chapline–Manton lagrangian (to maintain covariance we must also require that B_{mn} transforms under gauge transformations as $B \rightarrow B + \frac{k_{10}}{4}Tr(A \wedge \Lambda)$). Thus the Chapline–Manton theory can be considered to be a first approximation of the field theory limit of the heterotic string.

Consider now the four–derivative terms in the three–point amplitude; they can be written as

$$\begin{aligned} A^{4\theta}_{ggg} = g_{st}\, \big[&(k_2\theta_1 k_2)(k_3\theta_2\theta_3^T k_1) + (k_3\theta_2 k_3)(k_1\theta_3\theta_1^T k_2) + \\ &+ (k_1\theta_3 k_1)(k_2\theta_1\theta_2^T k_3)\big] \end{aligned} \tag{3.18}$$

As shown in [22], the hBB amplitude leads to another field redefinition of H_{abc}

$$\begin{aligned} H_{abc} &= \partial_{[c}B_{ab]} - \frac{k_{10}}{4}\left(\Omega_{3y}\right)_{abc} + \frac{\alpha'}{8k_{10}}\left(\Omega_{3L}\right)_{abc} \\ \left(\Omega_{3L}\right)_{abc} &= tr\left(\omega_{[c}R_{ab]} - \frac{1}{3}\omega_{[a}\omega_{[b}\omega_{c]]}\right) \end{aligned} \tag{3.19}$$

The introduction of the Lorentz Chern–Simons form Ω_{3L} and the modification of the transformation law of B under gauge and Lorentz transformations in $B \to B + \frac{k_{10}}{4} tr(A \wedge \Lambda) - \frac{k_{10}}{4} tr(\omega \wedge \Omega)$, is what is needed for the Green–Schwarz anomaly–cancelling mechanism to work [24].

The hhh and the hhD contribution to $A^{4\theta}_{ggg}$ are generated by the following term in the lagrangian

$$\frac{\alpha'}{16k_{10}^2} e^{-\frac{k_{10}D}{\sqrt{2}}} R_{abcd}R^{abcd} \tag{3.20}$$

Terms quadratic in the Riemann tensor are dangerous because they modify the graviton propagator and can lead to the propagation of negative norm states (ghosts) [25]. To avoid this we can use the arbitrariness of making a field redefinition to construct the Gauss–Bonnet combination of the curvature tensor. In fact the $R_{ab}R^{ab}$ and R^2 terms can be generated by a field redefinition of $\mathcal{L}_{2pt}$, the resulting Gauss–Bonnet combination $R_{abcd}R^{abcd} - 4R_{ab}R^{ab} + R^2$ is a total divergence so that any correction to the graviton propagator vanishes explicitly [25]. It is not known if these terms appear in the supersymmetrization of the Lorentz Chern–Simons form and if it is possible to introduce them freely also in the supersymmetric theory.

Finally the hBB contribution of $A^{4\theta}_{ggg}$ to the effective lagrangian produces terms of the form RH^2 or $(\nabla H)^2$.

Thus, the three–point effective action is

$$\begin{aligned} \frac{1}{\sqrt{g}}\mathcal{L}_{3pt} = & -\frac{1}{2k_{10}^2}R - \frac{e^{-2\sqrt{2}k_{10}D}}{6}H_{abc}H^{abc} - \frac{1}{2}(\nabla_c D)(\nabla_c D) + \\ & - \frac{e^{-k_{10}D/\sqrt{2}}}{8}Tr(F_{mn}F^{mn}) + \frac{\alpha' e^{-k_{10}D/\sqrt{2}}}{16(k_{10})^2}\left(R_{abcd}R^{abcd} - 4R_{ab}R^{ab} + R^2\right) + \\ & + \frac{\alpha' e^{-3k_{10}D/\sqrt{2}}}{8}\left(\nabla^a H^{bcd}\nabla_c H_{dab} + \nabla_a H^{acd}\nabla^b H_{bcd}\right) + \\ & + \text{terms generated by field redefinitions} \end{aligned} \tag{3.21}$$

We can regain the standard supergravity notations defining

$$\phi = e^{k_{10}D/\sqrt{2}} \qquad \text{which implies} \qquad -\frac{1}{2}(\nabla_c D)^2 = -\frac{1}{(k_{10})^2}\left(\frac{\partial_c \phi}{\phi}\right)^2$$

so that

$$\frac{1}{\sqrt{g}}\mathcal{L}_{3pt} = -\frac{1}{2k_{10}^2}R - \frac{1}{6\phi^2}H_{abc}H^{abc} - \frac{1}{k_{10}^2}\left(\frac{\partial_c \phi}{\phi}\right)^2 - \frac{1}{8\phi}Tr(F_{mn}F^{mn}) + \tag{3.22}$$

$$+\frac{\alpha'}{16k_{10}^2\phi}\left(R_{abcd}R^{abcd}-4R_{ab}R^{ab}+R^2\right)+$$
$$+\frac{\alpha'}{8\phi^3}\left(\nabla^a H^{bcd}\nabla_c H_{dab}+\nabla_a H^{acd}\nabla^b H_{bcd}\right)+$$
$$+\text{ terms generated by field redefinitions}$$

Now rescaling $A_m \to \frac{1}{g_{10}}A_m$, $F_{mn} \to \frac{1}{g_{10}}F_{mn}$, $\Omega_{3Y} \to \frac{1}{g_{10}^2}\Omega_{3Y}$, $B \to \frac{k_{10}}{g_{10}^2}B$ and $H \to \frac{3k_{10}}{2g_{10}^2}H$ (which means $F = dA + [A,A]$, $\Omega_{3Y} = Tr(FA - \frac{1}{3}A^3)$ and $H = \frac{1}{3}[dB - \Omega_{3Y} + \Omega_{3L}]$) we obtain exactly the usual Chapline–Manton lagrangian (3.5) plus the Lorentz Chern–Simons form and other corrections.

Notice that eliminating α' through $\alpha'/2 = (k_{10}/g_{10})^2$ we obtain

$$\begin{aligned}-\frac{1}{8\phi g_{10}^2}Tr(F_{mn}F^{mn})+\frac{\alpha'}{16k_{10}^2\phi}\left(R_{abcd}R^{abcd}-4R_{ab}R^{ab}+R^2\right)&=\\ =-\frac{1}{8\phi g_{10}^2}\left[Tr(F_{mn}F^{mn})-(R_{abcd}R^{abcd}-4R_{ab}R^{ab}+R^2)\right]&\end{aligned} \tag{3.23}$$

This is exactly the same combination of gauge and gravitational tensors as the one we will find in D=4 as a consequence of the supersymmetrization of the Lorentz Chern–Simons form.

§3.1.3 *Four–Point Effective Action*

What has been done for the three–point scattering amplitude can be repeated for the four–scattering amplitude and any higher N–point amplitude. An explicit expression for $\mathcal{L}_{4pt}$ is given in ref. [22]. Most of the new four– (and higher derivative) terms seem to make part of the supersymmetrization of the Lorentz Chern–Simons form. However, there is at least one term which hardly can belong to the supersymmetrization of the Lorentz Chern–Simons form; it is

$$\frac{\zeta(3)}{32^7k_{10}^2}e^{3k_{10}D/\sqrt{2}}t^{abcdefgh}t^{mnpqrsuv}\bar{R}_{abmn}\bar{R}_{cdpq}\bar{R}_{efrs}\bar{R}_{ghuv} \tag{3.24}$$

where $\zeta(3)$ is the ζ–Riemann function, $t_{abcdefgh}$ is a combinatorial factor, $\bar{R}_{abmn} = R_{abmn} + k_{10}e^{-k_{10}D/\sqrt{2}}\nabla_{[a}H_{b]mn} - \frac{1}{\sqrt{8}}k_{10}\eta_{[a}^{[m}\nabla_{b]}\nabla^{n]}D$ and η_b^a is a combination of the polarization tensors.

This term should not belong to the supersymmetrization of the Lorentz Chern–Simons because it does not seem possible to generate the ζ–Riemann function by supersymmetry unless this supersymmetrization has some free parameters which can be chosen to be $\zeta(3)$. It is more probable that at the four scattering amplitude level an entire set of new terms, related by supersymmetry and having $\zeta(3)$ as numerical coefficient, appears in the massless–modes field–theory effective lagrangian of the heterotic superstring.

§3.1.4 *One–Loop String Amplitudes*

The scattering amplitudes have been computed (although still not completely) also at one–loop (i.e. on the torus) [26]. The one loop corrections to the two– and three–point functions vanish as well as the low order terms in the four–point functions. This means that the Chapline–Manton supergravity action (3.5) (i.e. the part of the effective action with at most two–derivative terms) is not renormalized by one–loop string corrections.

Instead the fourth–order terms in the field curvatures get renormalized by the one–loop string amplitudes. These corrections are of phenomenological significance for compactifications schemes, as they modify the classical equations that should be satisfied by any proposed manifold of compactification. Notice, however, that these are higher order corrections (starting from $O(\alpha'^2)$) and they will not modify the lowest order "Calabi–Yau" like compactifications (see chapter 4).

§3.2 Effective Action from Sigma Model

The string propagation in a flat Minkowski background is described by the action

$$S_0 = -\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{h}h^{\alpha\beta}\partial_\alpha X^M\partial_\beta X^N\eta_{MN} \tag{3.25}$$

where η_{MN} is the Minkowski metric in the spacetime which we usually didn't displayed. We have already said that the propagation of a string in an arbitrary background (for the gravitational part) is given by

$$S = -\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{h}h^{\alpha\beta}\partial_\alpha X^M\partial_\beta X^N g_{MN}(X) \tag{3.26}$$

where $g_{MN}(X)$ is the spacetime metric tensor of the background which is an arbitrary given function of X. The action (3.26) to describe a string theory must be quantum conformally invariant, this gives a constraint on the acceptable backgrounds.

A generalization of this theory is to consider $g_{MN}(X)$ as a undetermined function of X. We can argue that in this case the action (3.26) describes the dynamics of the string graviton. In fact, suppose that $g_{MN}(X) = \eta_{MN} + f_{MN}(X)$, where $f_{MN}(X)$ is treated as a perturbation. Consider the world–sheet path integral of the action (3.26)

$$\begin{aligned} \mathcal{Z} &= \int [dX_M][dh_{\alpha\beta}]e^{-S} = \\ &= \int [dX_M][dh_{\alpha\beta}]e^{-S_0}\Big(1 + \frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{h}h^{\alpha\beta}\partial_\alpha X^M\partial_\beta X^N f_{MN}(X) + \\ &\quad + \frac{1}{2}[\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{h}h^{\alpha\beta}\partial_\alpha X^M\partial_\beta X^N f_{MN}(X)]^2 + \ldots\Big) \\ &= \int [dX_M][dh_{\alpha\beta}]e^{-S_0}e^V \end{aligned} \tag{3.27}$$

where

$$V(X,h) = \frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{h}h^{\alpha\beta}\partial_\alpha X^M\partial_\beta X^N f_{MN}(X) \tag{3.28}$$

$V(X,h)$ is the vertex operator for the emission of a graviton of wave function $f_{MN}(X)$[16]. The insertion of e^V in the usual path integral (which leads to (3.27)) describes the interaction of the string with a coherent state of gravitons. Also in this case action (3.26) must be quantum conformally invariant. This condition now translates in an equation that $g_{MN}(X)$ must satisfy. This equation can be interpreted as the spacetime equation of motion of the graviton[17]. We will now study in more detail this issue.

§3.2.1 *The Bosonic 2d Non–Linear Sigma Model*

The massless states of the bosonic closed string are the graviton, the antisymmetric tensor and the dilaton. It is easy to construct the background action for the graviton and the antisymmetric tensor

$$S = -\frac{1}{4\pi\alpha'}\int d^2\sigma\left[\sqrt{h}h^{\alpha\beta}\partial_\alpha X^M\partial\beta X^N g_{MN}(X) + \varepsilon^{\alpha\beta}\partial_\alpha X^M\partial_\beta X^N B_{MN}(X)\right] \tag{3.29}$$

[16] Until now we have considered gravitons whose wave function is a plane wave, but there is no reason not to consider instead a wave function which is a general superposition of plane waves.

[17] These considerations apply equally to all the other string modes.

where $\varepsilon^{\alpha\beta}$ is the antisymmetric 2d tensor.

The introduction of the dilaton is a bit more complicated. One way is to consider (3.29) as a 2d non–linear sigma model [27]. Let us assume that this theory is renormalizable. The vertex operators are simply those composite operators whose anomalous dimension, in the sense of the renormalization group, is two [28]. As usual the renormalization will mix the operators of the same naive dimension. String theories are necessarily defined on a curved world–sheet so that the two–dimensional curvature is not zero and can contribute to form a composite operator. In our case the graviton and the antisymmetric tensor vertices are operators of naive dimension two, to complete them we must add the identity operator times the 2d curvature which also has naive dimension two. Using again the prescription given to construct the non–linear sigma model, the identity vertex becomes a scalar field $F(X)$ (in 2d a scalar field has dimension zero). Then the renormalization group forces us to add to the action the term

$$S_{dil} = -\frac{1}{4\pi\alpha'}\int d^2\sigma\alpha'\sqrt{h}R^{(2)}(\sigma)F(X) \tag{3.30}$$

(In the same way one can introduce massive vertex operators considering composite operators of naive dimension four and anomalous dimension two. Roughly speaking, one can think that the difference (naive dimension) – (anomalous dimension=2) is proportional to the mass of the state described by the vertex operator.)

$F(X)$ is the dilaton background field. Notice that S_{dil} is a renormalizable action which (on dimensional ground) is of the first order in α'. α' plays the role of the coupling constant of the non–linear sigma model, so that S_{dil} is in fact a one–loop term in the non–linear sigma model perturbation theory. Moreover, S_{dil} is **not** Weyl invariant (i.e. invariant under a local rescaling of the 2d metric $h^{\alpha\beta} \longrightarrow \Lambda(\sigma)h^{\alpha\beta}$) but is needed to cancel the one loop anomaly. In fact in this theory Weyl invariance is explicitly broken by S_{dil} and implicitly by anomalies, so we can ask that the two contributes cancel one each other.

If we want this non–linear sigma model to describe the dynamic of the massless modes of the string, it must have the same symmetries of a string theory; in particular it must be conformally invariant. We start considering the Weyl invariance and we will show that it is sufficient to guarantee the full quantum conformal invariance.

The Weyl anomaly or anomaly of the trace of the energy momentum tensor was computed in [29], it is

$$4\pi\alpha'\sqrt{h}T^{\alpha}_{\alpha} = \sqrt{h}h^{\alpha\beta}\partial_\alpha X^M\partial_\beta X^N B^g_{MN}(X) + \varepsilon^{\alpha\beta}\partial_\alpha X^M\partial_\beta X^N B^B_{MN}(X) + \tag{3.31}$$

$$+\alpha'\sqrt{h}R^{(2)}B^D(X)$$

where

$$\begin{aligned}
B^g_{MN}(X) &= \beta^g_{MN}(X) + \nabla_{(M}v_{N)} \\
B^B_{MN}(X) &= \beta^B_{MN}(X) + H^P_{MN}v_P + \nabla_{[M}W_{N]} \\
B^D(X) &= \beta^D(X) + v^M\nabla_M D(X)
\end{aligned}$$

$\beta^g_{MN}(X)$, $\beta^B_{MN}(X)$ and $\beta^D(X)$ are the three beta functions and v_m, W_N are some functions constructed from the D–dimensional Riemann tensor R_{MNPQ}, the field strength of the B_{MN} field H_{MNP} , $\partial_M D$ and their derivatives and which can be calculated perturbatively [29, 30]. It has been noticed that both the beta functions and the v_M, W_N functions are renormalization scheme dependent but the combination occurring in the B functions depends only on the subtraction prescription and not on the nature of the background or of the wave function renormalization. Further, the dependence of the B functions on the subtraction scheme can be absorbed into a redefinition of the spacetime fields.

If B^B and B^g vanish [27, 31], then the trace free parts of the energy momentum tensor, T_{zz} and $T_{\bar{z},\bar{z}}$, generate a Virasoro algebra

$$[T_{zz}, T_{ww}] = \frac{1}{2}(T_{zz} + T_{ww})\delta'(z-w) + \frac{1}{12}c\delta'''(z-w) \tag{3.32}$$

(and the same for the anti–analytic sector). The central charge, c, in this algebra is identically equal to the function B^D (from the Bianchi identities follows that the covariant derivative of B^D on the two dimensional manifold is zero, which means that B^D is a $\mathbb{C}$–number on M and the equation $c = B^D$ is consistent). Then the vanishing of the Weyl anomaly guarantees that the 2d non–linear sigma model is conformal invariant.

At the lowest order the B functions are

$$\begin{aligned}
B^g_{MN}(X) &= -\alpha'\left[R_{MN}(X) - H_{MPQ}(X)H_N^{\ PQ}(X) + \nabla_M\nabla_N D(X)\right] + O(\alpha'^2) \qquad (3.33)\\
B^B_{MN}(X) &= -\alpha'\left[-\nabla^P H_{MNP}(X) + H^P_{MN}(X)\nabla_P D(X)\right] + O(\alpha'^2) \\
B^D(X) &= \frac{1}{3}(D-26) - \alpha'\left[-\frac{1}{2}\nabla^2 D(X) + \frac{1}{2}(\nabla D(X))^2 + \right. \\
&\left. -\frac{1}{3}H_{MNP}(X)H^{MNP}(X)\right] + O(\alpha'^2)
\end{aligned}$$

where $H_{MNP} = \frac{3}{2}\partial_{[M}B_{NP]}$ and in B^D the -26 comes from the reparametrization ghosts contribution. The vanishing of the B functions can be interpreted as the equations of motion for the background fields. So, in this indirect way, one can construct the field theory effective action for the massless modes of the string. In fact the $B = 0$ equations can be obtained from the following classical lagrangian in 26 dimensions

$$\frac{1}{\sqrt{g}}\mathcal{L}_{26} = e^{-D}\left[R - \frac{1}{3}(H_{MNP})^2 + (\nabla D)^2\right] \tag{3.34}$$

which indeed corresponds (up to the normalization of the fields) to the bosonic part of the lagrangian previously obtained. Also this lagrangian, coming from the equations of motion, is fixed up to the redefinitions of the fields which don't change the equations of motion. It is exactly the same situation found in the S–matrix approach to the effective action. In fact it has been suggested [22] that the equations of motion obtained from the S–matrix lagrangian and the B functions should coincide up to a field redefinition, which means that there should exists a "metric" $K_{ij}(\phi)$ in the field space such that

$$B_i(\phi) = K_{ij}(\phi)\frac{\delta\mathcal{L}_{S-mat}}{\delta\phi_j} \tag{3.35}$$

Although a general proof of this relation has not been given, this equation is certainly true to low orders in perturbation theory where it is possible a direct construction of the K_{ij} metric.

An important fact that we should stress, is the relation between the B functions and the β functions. At the lowest order it is

$$\begin{aligned} B^g_{MN} &= \beta^g_{MN} - \alpha'\nabla_M\nabla_N D \\ B^B_{MN} &= \beta^B_{MN} - \alpha' H^P_{MN}\nabla_P D \\ B^D &= \beta^D - \frac{1}{2}\alpha'(\nabla D)^2 \end{aligned} \tag{3.36}$$

(at higher orders these relations become more complicated).

The beta functions appear as coefficients of the global scale anomaly ($h^{\alpha\beta} \rightarrow \Lambda h^{\alpha\beta}$, Λ independent of σ)

$$\begin{aligned} 4\pi\alpha'\int d^2\sigma\sqrt{h}T^\alpha_\alpha = \int d^2\sigma \Big[&\sqrt{h}h^{\alpha\beta}\partial_\alpha X^M\partial_\beta X^N\beta^g_{MN}(X) + \\ &+\varepsilon^{\alpha\beta}\partial_\alpha X^M\partial_\beta X^N\beta^B_{MN}(X) + \alpha'\sqrt{h}R^{(2)}\beta^D(X)\Big] \end{aligned} \tag{3.37}$$

It seems that for the bosonic non–linear sigma model the vanishing of the beta functions does not imply that the model is conformal invariant but only that it is globally scale invariant. On the other hand it has been shown that at two loop order, using field redefinitions and integration by parts, the effective action can be written as

$$S_{eff} = \frac{1}{\alpha'} \int d^{26}X \sqrt{g} e^{-D(X)} \left[g^{MN}(X) B^g_{MN}(X) - 2B^D(X) \right] + O(\alpha'^2) = \tag{3.38}$$
$$= \frac{1}{\alpha'} \int d^{26}X \sqrt{g} e^{-D(X)} \left[g^{MN}(X) \beta^g_{MN}(X) - 2\beta^D(X) \right] + O(\alpha'^2)$$

Moreover Tseytlin, generalizing the C–theorem of Zamolodchikov [31], has argued [32] that the effective action should have this form to all orders in perturbation theory. It is not clear if up to field redefinitions and terms vanishing on shell, the zeros of the beta functions coincide with those of the B functions so that the 2d non–linear sigma model at the fixed point of the beta functions actually are conformally invariant. The form (3.38) of the action indeed suggests that this is the case.

All these facts hold on the basis of two conjectures:

i) the 2d non–linear sigma model is renormalizable

ii) the condition of Weyl invariance of the 2d non–linear sigma model is equivalent to the tree level Bose string equations of motion.

§3.2.2 *B.R.S. Invariance in the 2d Non–Linear Sigma Model*

There is another way of introducing the dilaton in the sigma model action which physically is more clear. As it was done for the graviton and the antisymmetric tensor, one can start from the dilaton vertex operator. The dilaton vertex operator is the unique vertex which depends on the ghosts. In fact [33]

$$V_D(z,\bar{z}) = \int d^2z \left[\partial_z X^M \partial_{\bar{z}} X_M + \frac{1}{3}\partial_z \left(\bar{c}(\bar{z})\bar{b}(\bar{z}) \right) + \frac{1}{3}\partial_{\bar{z}} \left(c(z)b(z) \right) \right] e^{ik\cdot X} \tag{3.39}$$
$$= \int d^2z \left[\partial_z X^M \partial_{\bar{z}} X_M - \frac{1}{3}\partial_z \bar{J}^{(b,c)}(\bar{z}) - \frac{1}{3}\partial_{\bar{z}} J^{(b,c)}(z) \right] e^{ik\cdot X}$$

where $J^{(b,c)}(z) = b(z)c(z)$ is the anomalous (b,c) ghost number current. Using the anomalous equation of conservation of the ghost number current, one can obtain the coupling of the dilaton to the scalar curvature $R^{(2)}$. On the other side, from equation (3.39) , one can construct a coupling of the dilaton field to the ghost number current. The sigma model action is [34]

$$S = \frac{1}{4\pi\alpha'} \int d^2 z \left[g_{MN}(X) \partial_z X^M \partial_{\bar{z}} X^N + B_{MN}(X) \partial_\alpha X^M \partial_\beta X^N \varepsilon^{\alpha\beta} + \right. \tag{3.40}$$
$$\left. + 2b(z)\partial_{\bar{z}} c(z) + 2\bar{b}(\bar{z})\partial_z \bar{c}(\bar{z}) + \frac{4}{3}\partial_{\bar{z}} D(X) J^{(b,c)}(z) + \frac{4}{3}\partial_z D(X) \bar{J}^{(b,c)}(z) \right]$$

This action is actually B.R.S.T. invariant. Thus, one can compute Q_{BRS} and impose $(Q_{BRS})^2 = 0$. Q_{BRS} is a 2d B.R.S.T. charge and its nihilpotence is equivalent to the vanishing of the central charge. The central charge is not a number but it is a function of the background fields, so that from the nihilpotence condition of the B.R.S. charge one obtains the equations of motion of the background fields. The explicit computation of $(Q_{BRS})^2 = 0$ (or of the O.P.E. of the energy tensor with itself) can be done using 2d non–linear sigma model perturbation techniques. The computation is done in an easier way observing that the action (3.40) , its B.R.S.T. transformations and the B.R.S.T. charge can be obtained with the substitution $c \to e^{4D/3} c$, $b \to e^{-4D/3} b$ from the same quantities obtained from the action

$$S = \frac{1}{4\pi\alpha'} \int d^2 z \left[\partial_z X^M \partial_{\bar{z}} X^N g_{MN}(X) + \varepsilon^{\alpha\beta} \partial_\alpha X^M \partial_\beta X^N B_{MN}(X) + \right. \tag{3.41}$$
$$\left. + 2b(z)\partial_{\bar{z}} c(z) + 2\bar{b}(\bar{z})\partial_z \bar{c}(\bar{z}) \right]$$

But the transformation which relates the action (3.41) to the action (3.40) is generated by the ghost number current which is anomalous; then making this transformation one must be very careful about this anomaly. Since the energy momentum tensor of action (3.41) is very simple (it does not have exponential coupling of the dilaton), one can try to use it to compute $(Q_{BRS})^2 = 0$. But in the transformation from the energy momentum tensor of action (3.40) to that of action (3.41) there is also an anomalous contribution to the energy momentum tensor from the ghost number current so that the equivalent expression is

$$T_{action(3.40)} \approx T_{action(3.41)} + \frac{1}{2}\partial_z^2 D(X) = T' \tag{3.42}$$

Using this expression of the energy momentum tensor, one can easily compute $(Q_{BRS})^2 = 0$ or the Virasoro algebra of T'. At the first loop order one finds some equations of motion for the background fields which are exactly those obtained imposing the vanishing of the B functions. Thus the vanishing of the B functions or the $(Q_{BRS})^2 = 0$ condition give rise to the same effective action.

§3.2.3 *The Supersymmetric Non–Linear Sigma Model*

Using superspace background field methods, in ref.[35] the graviton beta function for the supersymmetric 2d non–linear sigma model was computed up to four loops. As in the bosonic case [30] the one–loop contribution is just the Ricci tensor, but the two and three loops contributions to the β^g function vanish (in the bosonic case at two–loop there is an $R_{MPQR}R_N^{PQR}$ term). So up to three-loop the type II superstring has as gravitational background a Ricci flat space (solution of the graviton equation of motion). We will see the consequences of this fact on the possibilities of compactification to four dimensions from ten dimensions. But at four–loop arises the $\zeta(3)(R_{MNPQ})^4$ term which we have already met considering the four point scattering amplitude results. So at four loop a Ricci flat space is not anymore a solution of the graviton equation of motion.

For the heterotic superstring only a few results are known up to two loops [27]. The main features are the modification of the B_{MN} field strength H_{MNP} with the introduction of the Lorentz and Yang–Mills Chern–Simons forms and the presence (at two–loop order) in the graviton beta function of the term $R_{MPQR}R_N^{PQR} - trF^2_{MN}$ (remember that the term quadratic in the Ricci tensor and scalar curvature can be introduced by field redefinitions). These terms should lead to the same effective action found by the scattering amplitude method.

These results were obtained considering a string theory at tree level. It is not known how higher string loop contributions can modify the situation. Moreover the sigma model approach seems to be technically more difficult than the scattering amplitude one to compute the effective theory; it is also plagued with some serious problems, like the inclusion of the spacetime fermion fields[18], and the explicit expression of the Weyl anomaly in the case of supersymmetric sigma models is not known, although the approach proposed in ref. [34] with the B.R.S. charge gives an independent way for obtaining the same results.

We end summarizing the known results for the field theory effective action of the heterotic superstring.

The effective action is constructed from the tree string contributions (strings propa-

[18] Probably, within the Green–Schwarz formulation of the superstring it will be possible to include also the background fermion fields.

gating on a genus zero Riemann surface). It is a classical action expanded in powers of a dimensionful parameter α', it is a local supersymmetric (supergravity) action with N=1 in D=10 and $E_8 \otimes E_8$ as Yang–Mills gauge group. Anomalies are cancelled by means of the Green–Schwarz mechanism.

Up to α'^2 the effective action contains also terms quartic in the Riemann tensor besides all those of the Chapline–Manton Green–Schwarz N=1 D=10 supergravity theory.

§3.2.4 *Sigma Model Perturbation Theory*

In this paragraph we will recall some features of the 2d non–linear sigma model theory which will be useful in future. In a 2d quantum field theory the action for a bosonic field is

$$I = \int d^2\sigma \partial_\alpha X(\sigma) \partial_\beta X(\sigma) h^{\alpha\beta} \tag{3.43}$$

Thus in d=2 a scalar field is dimensionless[19].

A string theory is a D=2 quantum field theory where the 2d quantum fields X_M are also the coordinates of a background D–dimensional space; then X_M has dimension -1. To have a dimensionless 2d action for X_M one has to introduce the Regge slope α' which has mass dimension +2. Then

$$I_{strings} = \int d^2\sigma \frac{1}{4\pi\alpha'} \partial_\alpha X_M(\sigma) \partial_\beta X_N(\sigma) h^{\alpha\beta} \sqrt{h}\ \eta^{MN} \tag{3.44}$$

The background string action for the graviton is just a 2d sigma model where $g_{MN}(X(\sigma))/4\pi\alpha'$ is the 2d coupling constant

$$I_{b.g.} = \int d^2\sigma \frac{g^{MN}(X(\sigma))}{4\pi\alpha'} \partial_\alpha X_M(\sigma) \partial_\beta X_N(\sigma) h^{\alpha\beta} \sqrt{h} \tag{3.45}$$

and α', since it plays the role of $\hbar$, is the 2d loop–counting parameter.

The 2d sigma model quantum theory must be considered as a perturbation around a classical solution of the equations of motion $X^0_M(\sigma)$ such that

$$X_M(\sigma) = X^0_M(\sigma) + \xi_M(\sigma) \tag{3.46}$$

[19] The metric tensor is always dimensionless.

where $\xi_M(\sigma)$ are the quantum fields[20]. Thus one can expand the action in a Taylor series in powers of $\xi_M(\sigma)$. We do not review these techniques which are rather complicated and require the introduction of a normal–coordinates expansion [36], but we just give a simple example. Expand in orthonormal coordinates the action and make $\xi_M(\sigma)$ dimensionless as any good 2d quantum field by $\xi_M(\sigma) \to \sqrt{2\pi\alpha'}\xi_M(\sigma)$, one obtains

$$I_{b.g.} = \int d^2\sigma\sqrt{h}h^{\alpha\beta}\left(\frac{1}{2}g^{MN}(X^0(\sigma))\partial_\alpha\xi_M(\sigma)\partial_\beta\xi_N(\sigma)+ \right. \\ \left. -\frac{1}{6}(2\pi\alpha')R^{MNPQ}(X^0(\sigma))\xi_M(\sigma)\xi_N(\sigma)\partial_\alpha\xi_P(\sigma)\partial_\beta\xi_Q(\sigma)+\cdots\right) \tag{3.47}$$

The 2d sigma model action is then expressed as an expansion in powers of α'.

Thus α' plays a triple role in the 2d sigma model: it appears in the coupling constant, it is the loop–counting parameter and the massive modes (i.e. higher derivatives) expansion parameter.

[20] D. Friedan has shown [36] that the choice of X^0_M is arbitrary, since changing X^0_M is equivalent to make a field redefinition on the 2d quantum theory.

CHAPTER 4

Compactifications to Four Dimensions

In the previous chapter we have seen that a N=1 D=10 non–standard supergravity is the effective field theory for the massless modes of the heterotic superstring. We want now to construct a four dimensional theory. This can be done looking for solutions of the equations of motion of the D=10 supergravity in which 6 of the 10 dimensions are compactified. It is not known why the theory should choose this solution of its equations of motion, but it is already of great importance the fact that the theory admits such a solution. In this case, being the space $M_{10} = M_4 \otimes K_6$ where K_6 is compact, one can in principle solve the equations of motion on K_6 so to obtain the equations of motion for the four dimensional fields. As we will see in details, this procedure gives rise to a D=4 spectrum composed by some massless modes and an infinite tower of massive modes. The D=4 massive modes have masses of the order of $1/r$ where r is the radius of K_6 ($M_{K_6} \sim 1/r$). As usual the D=4 effective theory is constructed only with the massless modes.

In principle, one should be able to take care also of the effective interactions due to the K_6 (or Kaluza–Klein) massive modes in the construction of the D=4 effective action. In this sense the D=4 effective action should appear as a series expansion in powers of r; moreover we already know that there is a power expansion in the string loops and in α'.

In the following, as we have done until now, we will consider just the tree string loop and first order in α' contributions and we will study the D=4 massless spectrum and its symmetries. These informations are sufficient to specify the D=4 effective supergravity theory (up to two derivative terms, which means neglecting the massive modes and α' contributions) as we will discuss in the next chapter.

§4.1 "Calabi Yau" like compactifications

We are interested in studying a solution of the equations of motion of the D=10 supergravity theory emerging as an effective theory from superstring with the following properties [37, 38]

a) the equation of motion for the D=10 graviton g_{MN} has a solution in which the vacuum spacetime is of the form $M_4 \otimes K_6$ where M_4 is a maximally symmetric space and K_6 a compact space. We will call the coordinates of M_4 x_μ ($\mu = 1, \ldots, 4$), of K_6 y_m ($m = 5, \ldots, 10$) and of M_{10} x_M ($M = 1, \ldots, 10$).

b) there exists one (unbroken) local supersymmetry acting on the D=4 fields on M_4.

c) the gauge group and fermion spectrum should be realistic.

The consequence of the c) condition is to consider $E_8 \otimes E_8$ as the D=10 gauge group.
The most important and far reaching condition is b), so we will study first its consequences.

§4.1.1 *Unbroken Supersymmetry in D=4*

An unbroken supersymmetry Q is simply a conserved supercharge that annihilates the vacuum state $|\Omega\rangle$, this is equivalent to say that for all operators U, $\langle\Omega|\{Q,U\}|\Omega\rangle = 0$. This certainly holds if U is a bosonic operator, if instead U is fermionic $\{Q,U\} = \delta_Q U$ so that the condition for an unbroken supersymmetry is $\langle\Omega|\delta_Q(\epsilon)U|\Omega\rangle = 0$[21] for any fermion field U. Obviously to have an unbroken supersymmetry in D=4 we must start with an unbroken supersymmetry in D=10.

The D=10 fermi fields are the gravitino ψ_M, the dilatino λ and the gauginos χ^α. Their supersymmetry variations are

$$\begin{aligned}
\delta_Q(\eta)\psi_M &= \frac{1}{k_{10}} D_M\eta + \frac{3k_{10}}{32g_{10}^2\phi}\left(\Gamma_M^{\ NPQ} - 9\delta_M^N\Gamma^{PQ}\right)\eta H_{NPQ} + (\text{fermi})^2 \qquad (4.1)\\
\delta_Q(\eta)\chi^\alpha &= -\frac{1}{4g_{10}\sqrt{\phi}}\Gamma^{MN}F^\alpha_{MN}\eta + (\text{fermi})^2\\
\delta_Q(\eta)\lambda &= -\frac{1}{\sqrt{2}\phi}(\Gamma\cdot\partial\phi)\eta + \frac{k_{10}}{8\sqrt{2}g_{10}^2\phi}\Gamma^{MNP}\eta H_{MNP} + (\text{fermi})^2\\
H &= dB - \omega_{3YM} + \omega_{3L}
\end{aligned}$$

[21] Notice that at the classical (tree) level, as the one we have to consider, $\delta_Q U$ and $\langle\delta_Q U\rangle$ coincide.

These transformations laws are at the lowest order in α' except for the presence of the Lorentz Chern–Simons form ω_{3L}. The inclusion of this term will play an important role in the discussion.

To simplify the study one assumes that $\langle H\rangle = \langle d\phi\rangle = 0$, the generalization for the arbitrary case will be considered later. The important point is that the solution we will find of $\langle\delta_Q U\rangle = 0$ will turn out to be a solution of the equations of motion of the D=10 theory at the first order in α'.

The simplifying hypothesis implies that $\langle\delta_Q\lambda\rangle = 0$ is identically satisfied. Then it remains to study

$$\begin{aligned}
&\langle\delta_Q(\eta)\psi_M\rangle = \langle D_M\eta\rangle = 0 \\
&\langle\delta_Q(\eta)\chi^\alpha\rangle = \langle\Gamma^{MN}F^\alpha_{MN}\eta\rangle = 0 \\
&dH = trR\wedge R - \frac{1}{30}TrF\wedge F
\end{aligned} \tag{4.2}$$

The Bianchi identities for the field strength of the antisymmetric tensor field B_{MN} are consistent with the position $\langle H\rangle = 0$ because it means that $trR\wedge R$ and $TrF\wedge F$ are in the same cohomology class (or that $trR\wedge R - \frac{1}{30}TrF\wedge F$ is in the zero cohomology class). Notice moreover that the B_{MN} field, due to its gauge invariance under $B_{MN}\rightarrow B_{MN} + \partial_M\Lambda_N - \partial_N\Lambda_M$, can appear in the lagrangian essentially only through its field strength H_{MNP}.

§4.1.2 *The Calabi–Yau Space*

The Killing equation $\langle D_M\eta\rangle = 0$ has a strong integrability condition $\langle[D_M, D_N]\eta\rangle = 0$ or

$$R_{MNPQ}\Gamma^{PQ}\eta = 0 \tag{4.3}$$

where R_{MNPQ} is the Riemann tensor. Consider now this equation on M_4, which is a maximally symmetric space, in this case $R_{\mu\nu\rho\sigma} = \frac{R}{12}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})$ from which $R = 0$; this means that the space M_4, as solution of the equations of motion of the graviton, must be the flat Minkowski space. On the Minkowski background the equation $D_\mu\eta(x_\mu, y_m) = 0$ implies that η is independent of the four uncompactified coordinates. Then $D_M\eta = 0$

reduces to the statement that on K_6 it exists a covariantly constant spinor field (Killing spinor). Before studying the consequences of this, let us recall some general facts.

Let $\Phi(X)$ be a field on M_{10}, on $M_4 \otimes K_6$ its decomposition is taken (symbolically) to be $\Phi(X) = \sum_i \Phi_i(x_\mu) \otimes \Phi_i(y_m)$. The Lorentz group $SO(10)$ of M_{10} decomposes, going to $M_4 \otimes K_6$, in $SO(4) \otimes SO(6)$, accordingly the Lorentz representation (spin) of the field must be decomposed in the product of those on M_4 and on K_6. For example, the spinor irreducible representation of $SO(10)$ is the 16 which under $SO(10) \to SO(6) \otimes SO(4)$ becomes $16 = (4,2) \oplus (\bar{4}, 2')$. Considering the dependence of a spinor χ on x_μ (i.e. on M_4), χ is a Majorana spinor (in the $2 \oplus 2'$ of $SO(4)$) with an external index in the 4 of $SU(4) \simeq SO(6)$; on the other side, as spinor on K_6, χ is the sum of a spinor of positive chirality and one with negative chirality with an external index in the 2 of $SO(4)$.

The next important concept which is useful is the holonomy group[22]. As in any gauge theory, a covariantly constant field η always returns to its original value upon parallel transport around a contractible closed curve. Thus, K_6 admits a covariantly constant spinor only if its holonomy group leaves η invariant, $U\eta = \eta$. The holonomy group is then a subgroup of $SO(6) \simeq SU(4)$ under which the decomposition of the spinor representation of $SU(4)$ has a singlet. The spinor representations of $SU(4)$ are the 4 (positive chirality) and the $\bar{4}$ (negative chirality). Assume that η is a positive chirality spinor, then $SU(4) \supset SU(3)$ and $4 = 3 \oplus 1$. Then given $H = SU(3)$ there exists a spinor which is invariant under the action of the holonomy group.

A compact manifold K_6 whose holonomy group is precisely $SU(3)$ (and not a subgroup) admits exactly one covariantly constant spinor field η of positive chirality. The complex conjugate $\overline{\eta}$ of η ($\overline{\eta}$ belongs to $\bar{4}$ of $SU(4)$) is then the unique covariantly constant spinor field of negative chirality. Looking at these two spinor fields from the D=4 point of view, remembering that the 16 of $SO(10)$ decomposes in $SO(4) \otimes SU(4)$ as $(2,4) \oplus (2', \bar{4})$ and under $SO(4) \otimes SU(3)$ as $(2,1) \oplus (2', \bar{1})$[23], they form a single real four component Majorana spinor of $SO(4)$ (the $2'$ is the complex conjugate of the 2 since the 16 we started with is a real representation). Then if we choose a compact manifold K_6 of $SU(3)$ holonomy, the resulting D=4 theory has exactly (on–shell) one unbroken local supersymmetry.

By means of η, we can construct other covariantly constant tensors which specify the

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[22] For a definition see ref. [39]
[23] Really $SO(4) \simeq SU(2) \otimes SU(2)$ and $2 = (2,1)$, $2' = (1,2)$.

characteristics of the K_6 manifold. They are

Kähler form	$K_{ij} = \bar{\eta}\Gamma_{ij}\eta$
Complex structure	$J^i_j = g^{ik}K_{kj}$
Holomorphic volume form	$\omega_{ijk} = \eta^T\Gamma_{ijk}\eta$

The existence of these covariantly constant tensors implies that K_6 is a complex, Kählerian, $SU(3)$ holonomy and Ricci flat manifold[24]. These manifolds are called Calabi–Yau manifolds since their existence was first conjectured by E. Calabi and then proven by S. T. Yau [40]. Although these manifolds are known to exist, none of their metrics is known explicitly. Then it is impossible to integrate explicitly on K_6, all we can do is to study general properties of these manifolds.

§4.1.3 *The Embedding of the Spin–Connection in the Gauge Group*

The third equation of (4.2) projected on K_6 becomes

$$\frac{1}{30}TrF \wedge F - trR \wedge R = 0 \tag{4.4}$$

(Tr is the trace on the adjoint representation, tr on the vector representation). We expect that it could be satisfied only in a trivial way, i.e. there should be some relation between R and F. For example, discarding for the moment the fact that ω is in the vector representation and the factor $\frac{1}{30}$ in (4.4) , if the gauge group on K_6 is $SU(3)$, since we can think of ω as an $SU(3)$ gauge field with R as field strength, imposing the identification $\omega_m \equiv A_m$ the equation (4.4) is trivially satisfied.

In the case at hand the gauge group is $E_8 \otimes E_8$. Then we can make an **ansatz**: a solution of the equation of motion breaks the gauge group to its maximal subgroup $(SU(3) \otimes E_6) \otimes E_8$, the $SU(3)$ charged gauge fields develop on K_6 an expectation value related to that of ω, then the unbroken gauge group in D=4 is $E_6 \otimes E_8$. The adjoint representation of $E_8 \otimes E_8$ is the $248 \oplus 248$, the second E_8 doesn't contribute to (4.4) and we limit ourselves to study the first one. The vector representation 10 of $SO(10)$ on $SU(2) \otimes SU(2) \otimes SU(3)$ decomposes as

$$10 = (2,2,1) \oplus (1,1,3) \oplus (1,1,\bar{3}) \tag{4.5}$$

[24] Ricci flatness is a trivial consequence of $SU(N)$ holonomy.

The $(2,2,1)$ part describes a vector on M_4 and a singlet on K_6, i.e. ω_μ. The $(1,1,3)\oplus(1,1,\overline{3})$ part describes the K_6 spin connection ω_m. Then on K_6 the spin connection is in the $3 \oplus \overline{3}$ representation of $SU(3)$. In the same way one can decompose the Lorentz index of $A^\alpha_M \rightarrow A^\alpha_\mu \oplus A^\alpha_m$.

From the ansatz $E_8 \rightarrow SU(3) \otimes E_6$ the 248 of E_8 decomposes as

$$248 = (3,27) \oplus (\overline{3},\overline{27}) \oplus (8,1) \oplus (1,78) \tag{4.6}$$

The $(1,78)$ part represents the gauge fields in the adjoint of E_6 and singlet under $SU(3)$ and are the gauge fields corresponding to the unbroken symmetries. The $(3,27) \oplus (\overline{3},\overline{27}) \oplus (8,1)$ are the $SU(3)$ charged gauge fields. Since on K_6 we are not interested in E_6, we identify the 27 copies of 3 and $\overline{3}$, leaving 27 copies of $SU(3)$ gauge fields transforming in the $3 \oplus \overline{3}$. It easy to show that the trace of the square of an $SU(3)$ generator in the adjoint representation (which occurs in $TrF \wedge F$ for the (8,1)) is three times the value of the same trace in the $3 \oplus \overline{3}$. The total result is that we have 30 copies of the $trF \wedge F$ where the gauge field is in the vector representation of $SU(3)$.

"Embedding the spin connection in the gauge group", which means identifying the spin connection with the $SU(3)$ gauge field, we can satisfy (trivially) the equation (4.4) .

Summarizing, the effect of the ansatz which solves the (4.4) equation, is that of breaking the gauge group to $E_6 \otimes E_8$. To obtain this result the inclusion of the Lorentz Chern–Simons form was crucial. In fact, the same analysis carried on for the pure Chapline–Manton theory leads to the conclusion that F_{mn} vanishes [41]. This implies that there aren't non–trivial gauge compactifications, fact which precludes the possibility of having chiral fermions in D=4.

§4.1.4 *The Four Dimensional Spectrum*

Let consider a ten dimensional field $\Phi(X_M)$. We can think of it also as a field on $M_4 \otimes K_6$ and $\Phi = \Phi(x_\mu, y_m)$. In four dimension Φ is an infinite set of fields parametrized by y_m where y_m is a point in the K_6 space, $\Phi = \Phi_{y_m}(x_\mu)$. In particular, consider a Dirac spinor field $\psi(x_M)$ in D=10, it obeys the following equation of motion

$$0 = i\Gamma^M D_M \psi(x_M) \tag{4.7}$$

This equation can be rewritten as

$$0 = i\sum_{i=1}^{4}\Gamma_i D_i\psi(x_M) + i\sum_{i=5}^{10}\Gamma_i D_i\psi(x_M) \tag{4.8}$$

Now we have to look for a solution of this equation on $M_4 \otimes K_6$. We know that the spinor representation 16 decomposes as $(2,4)\oplus(2',\bar{4})$ so that we can make the following ansatz

$$\psi_{\alpha_{10}}(x_M) \sim \psi_{\alpha_4,\alpha_6}(x_\mu, y_m) \sim \sum_i \psi^i_{\alpha_4}(x_\mu)\otimes\phi^i_{\alpha_6}(y_m) \tag{4.9}$$

In the same way we can decompose the gamma matrices as

$$\Gamma^{10}_M \sim (\Gamma^4_\mu \otimes \mathbb{I}, \mathbb{I}\otimes\Gamma^6_m) \tag{4.10}$$

To solve the equation (4.8) by separation of variables the two differential operators must commute, but in (4.8) rather they anticommute. This problem can be solved multiplying the equation (4.8) by $\Gamma^{(4)} \equiv i\Gamma_1\Gamma_2\Gamma_3\Gamma_4$ ($\Gamma^{(4)}$ is the chirality operator), then

$$0 = i\left(\widetilde{\not{D}}_4 + \widetilde{\not{D}}_K\right)\psi_{10} \tag{4.11}$$

where $\widetilde{\not{D}}_4 = \sum_{i=1}^{4}\widetilde{\Gamma}^i D_i$, $\widetilde{\not{D}}_K = \sum_{i=5}^{10}\widetilde{\Gamma}^i D_i$ and $\widetilde{\Gamma}_M = \Gamma^{(4)}\Gamma_M$. It is important to note that $\widetilde{\Gamma}_i$ and $\widetilde{\Gamma}_m$ obey, respectively, the proper anticommutation relations of gamma matrices of M_4 and K_6.

Now one can introduce a complete set of normalized solutions of the eigenvalue problem

$$i\widetilde{\not{D}}_K\phi_i(y_m) = \lambda_i\phi_i(y_m) \tag{4.12}$$

so that the Dirac equation for the four dimensional spinor field becomes

$$0 = \left(i\widetilde{\not{D}}_4 + \lambda_i\right)\psi_i(x_\mu) \tag{4.13}$$

Thus, each ψ_i is observed in D=4 as a fermion of mass λ_i. The masses λ_i turn out to be of order $1/r$ where r is the "radius" of the compact space K_6 [42] (this terminology comes from the seven–sphere Kaluza–Klein techniques).

Now we should compute the interaction terms emerging from the integration of the massive Kaluza–Klein modes. We are not able to do this computation in details. What we can do is to compute the spectrum of the massless modes and to study some properties of these fields. We know also that the effective D=4 theory which describes the massless degrees of freedom, besides the standard (i.e. not higher order) terms, contains also two series of higher orders interaction terms: one is an expansion in α', the other comes from the compactification and depends on r.

The important fact that we have learned is that the D=4 spectrum of the massless modes is given by the zero modes of the kinetic operators on K_6[25].

The study of the zero modes of the Dirac and Laplacian operator is easily done using algebraic geometry techniques. Without entering in any detail, we will just state the principal outcomings [37].

On K_6 one can define the de Rham cohomology groups $H^p(K_6)$ whose dimensions b^p are called the Betti numbers. It turns out that a p–antisymmetric tensor B in D=10 gives rise to p–n forms on K_6 (and a n–forms on M_4) for all n $\leq$ max(4,p); moreover the number of zero modes of B on K_6 is given by b^{p-n}.

Since K_6 is also a complex, Kählerian manifold, one can define the Dolbeault cohomology groups $H^{(p,q)}(K_6)$[26], the dimensions of these groups are the Hodge numbers $h^{(p,q)}$ which are related to the Betti numbers by

$$b^n = \sum_{p+q=n} h^{(p,q)} \tag{4.14}$$

On a $SU(d)$–holonomy, Kählerian manifold of real dimension 2d it is possible to show that chiral spinors are in one–to–one correspondence with (0,q) forms, i.e. a spinor field is the same as a collection of (0,q) forms, q=0,...,d. Moreover, the number of zero modes of a (0,q) form on K_6 is exactly $h^{(0,q)}$.

For K_6, which is an $SU(3)$–holonomy Kähler manifold, it holds

$$\begin{aligned} &h^{(0,0)} = h^{(3,3)} = h^{(0,3)} = h^{(3,0)} = 1 \\ &h^{(0,1)} = h^{(0,2)} = h^{(1,0)} = h^{(2,0)} = 0 \\ &h^{(1,1)} = h^{(2,2)} \qquad h^{(1,2)} = h^{(2,1)} \\ &b^0 = 1 \qquad\qquad\quad b^2 = h^{(1,1)} \end{aligned} \tag{4.15}$$

[25] Obviously, the considerations that we have done for the fermions can be extended to the other fields.
[26] $H^{(p,q)}(K_6)$ is the cohomology group of the p–holomorphic and q–antiholomorphic forms.

The gluino fields[27] χ^x_α have a spinor index α and a gauge index x in the adjoint representation of E_8 (we are not considering the second E_8). Under $E_8 \supset SU(3) \otimes E_6$ we know that $248 = (3, 27) \oplus (\overline{3}, \overline{27}) \oplus (8, 1) \oplus (1, 78)$. On K_6 the E_6 indices are enumerating indices of the fermions, while the $SU(3)$ are gauge indices.

The $SU(3)$ singlet modes are equivalent to (0,q) forms on K_6 without gauge indices. Since $h^{(0,1)} = 0$, $h^{(0,0)} = 1 = h^{(0,3)}$, each $SU(3)$ singlet has precisely one zero mode of positive chirality and one of negative chirality. These modes give rise in D=4 to 78 Majorana fermions in the adjoint representation of E_6, i.e. the gluino E_6 fields in D=4.

Modes that transform in the 3 of $SU(3)$, $\psi^x_{\bar{b}}$, are equivalent to (0,1) forms with an holomorphic index $\psi^a_{\bar{b}}$ on a vector bundle on K_6. Since the spin connection is embedded in the gauge group, we can equivalently relate ψ to a (2,1) form on K_6 $\widetilde{\psi}_{a_1,a_2,\bar{b}} = \omega_{a_1,a_2,a_3}\psi^{a_3}_{\bar{b}}$. Thus, from the (3,27) part one obtains in D=4 $h^{(2,1)}$ massless modes in the 27 of E_6 and the number of left–handed massless 27–families in D=4 is $N_{27} = h^{(2,1)}$. In the same way the fields in the $\overline{3}$ can be related to (1,1) forms on K_6, so that the number of massless left–handed $\overline{27}$ families in D=4 is $N_{\overline{27}} = h^{(1,1)}$. The number of generations is $N_{gen} = |N_{27} - N_{\overline{27}}| = |h^{(2,1)} - h^{(1,1)}| = \frac{1}{2}\chi(K_6)$ (χ is the Euler characteristic of K_6).

Finally, consider the modes which transform in the adjoint of $SU(3)$. These modes are E_6 singlets, so they do not carry any currently known gauge interaction. However they can play an important role in phenomenology. There is not a general formula to compute their number.

We will consider now the gravitational sector; for simplicity we will study the compactification of the bosonic D=10 modes Φ, g_{MN}, B_{MN}. First of all, notice that the equations that determine the structure of K_6 do not fix it completely, i.e. the solution of the equations introduces some integration constants ϕ_i (usually called moduli); these integration constants (moduli) will always manifest themselves in the effective D=4 theory as massless spin zero particles.

The D=10 graviton g_{MN} gives rise to three sets of fields under compactification, $g_{\mu\nu}$, $g_{\mu m}$, g_{mn}. The only massless mode which comes from $g_{\mu\nu}$ is the D=4 graviton. The massless modes coming from $g_{\mu m}$ are gauge bosons in D=4. It can be proven that they are related to the continuous symmetries of K_6. However, a manifold whose holonomy is $SU(3)$ never has continuous symmetries, then from $g_{\mu m}$ does not come any massless mode.

The massless modes of g_{mn} on K_6 represent the degeneracy of the vacuum state and

[27] By supersymmetry it is sufficient to study just the bosonic or the fermionic part of the spectrum.

are in correspondence with the moduli of K_6. Expanding $g_{mn} = g^0_{mn} + h_{mn}$ where g^0 is a Ricci flat background and h is a metric disturbance, the equation for h is $\triangle h = 0$ where $\triangle$ is a certain differential operator (Lichnerowicz Laplacian). In the case of $SU(3)$ holonomy, the independent components of this equation are $h_{a\bar{b}}$ and $h_{\bar{a}\bar{b}}$. Zero modes of $h_{\bar{a}\bar{b}}$ are complex and gives rise to complex scalar fields in D=4. If the spin connection is embedded in the gauge group, these massless scalars are in one–to–one correspondence with the massless matter fermions (in the 27 and $\overline{27}$) obtained from the Yang–Mills sector; since supersymmetry is unbroken, these fields pair up to form the matter supermultiplets. The zero modes of $h_{a\bar{b}}$ are harmonic (1,1) forms. They correspond to the moduli of K_6; harmonic (1,1) forms are naturally real fields so that to fill out a complex supermultiplet they must have pseudoscalar partners; as we will see, these partners are provided by B_{MN}. Since these fields correspond to the integration constants of K_6, their value on K_6 is not fixed (remember that $\phi(x_M) \sim \sum_i \left(\phi^i_4(x_\mu) \otimes \phi^i_K(y_m)\right)$ but in this case $\phi^i_K(y_m) = \phi^i$ are constants). Then the corresponding D=4 massless fields must satisfy a well defined scaling property because the (classical) theory should be invariant under a change of the moduli[28]. This scaling law will be very important in the construction of the D=4 effective theory.

The B_{MN} field can be seen as a 2–form in D=10, then it gives rise to 0, 1 and 2 forms on K_6 which are $B_{\mu\nu}$, $B_{\mu m}$, B_{mn}. Since $b^0 = 1$, $B_{\mu\nu}$ describes one pseudoscalar massless mode in D=4 which pairs up with the dilaton Φ to make the first component of a complex multiplet called S. This supermultiplet has some interesting properties. It satisfies a scaling law since the dilaton has this property both in string theories and in the D=10 effective supergravity theory. Moreover, B has a gauge invariant field strength which is $H = dB - \omega_{3YM} + \omega_{3L}$. In the D=10 lagrangian there is the term $\frac{1}{2}\langle H, H\rangle_{10}$ which under compactification becomes $\frac{1}{2}\langle H, H\rangle_4$. But in D=4 an antisymmetric 3–form field strength has only one on–shell degree of freedom[29], so that $y_\mu = \varepsilon^{\nu\rho\sigma}_\mu H_{\nu\rho\sigma}$. Substituting one of the two H with y, we obtain the equation of motion for y, $dy = 0$. This gives $y = d\alpha$ for some pseudoscalar field α. Since $dH = trR \wedge R - trF \wedge F$, from the definition of y we get

$$\Box\alpha = trR \wedge \widetilde{R} - trF \wedge \widetilde{F} \tag{4.16}$$

This is the standard coupling of an axion, α; then α, which is the massless pseudoscalar degree of freedom described by $B_{\mu\nu}$, satisfies a Peccei–Quinn symmetry, $\alpha \to \alpha +$ constant.

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[28] In a classical theory the normalization of the lagrangian is not important, so in D=4 this scaling law appears as $\phi^i_4 \to \lambda\phi^i_4$, $\mathcal{L} \to \lambda^\alpha \mathcal{L}$.

[29] We will see in detail this fact in the next chapter.

Since for an $SU(3)$ holonomy manifold $b^1 = 0$, there are not massless modes coming from $B_{\mu m}$.

Consider now B_{mn}. For an SU(3) holonomy manifold $b^2 = h^{(1,1)} + h^{(2,0)} + h^{(0,2)}$ but $h^{(2,0)} = h^{(0,2)} = 0$, so that there are $h^{(1,1)}$ harmonic (1,1) forms that give rise to pseudo–real massless scalars in four dimensions. These modes pair up with the $h_{a\bar{b}}$ fields to form $h^{(1,1)}$ complex chiral supermultiplets $T_{(\alpha)}$ in D=4. As for S, the real part of $T_{(\alpha)}$ satisfies a scaling law which follows from the moduli, and the pseudo–real part satisfies a Peccei–Quinn symmetry and can have an axion coupling which come from the higher order terms needed to cancel the anomalies in D=10:

$$\triangle S = \int_M B \wedge trF^2 \wedge trF^2 \tag{4.17}$$

At this point, up to specify $h^{(1,1)}$ and $h^{(2,1)}$ and the number of E_6 singlet matter fields, we have completely determined the D=4 massless spectrum.

§4.1.5 *Higher Order Corrections*

In the study of the supersymmetry transformations of the spinor fields in the last section we considered the equations to the lowest order in α' except that for the Lorentz Chern–Simons form. As we know, there is an infinite number of corrections to these equations of higher order in α'. Moreover we considered a solution of these equations in which $\langle H \rangle = 0 = \langle d\Phi \rangle$. The Calabi–Yau metric is a solution of the gravity equation of motion on the compact space K_6 and also all the other fields satisfy their equations of motion at the lowest order in α'.

From the 4–point scattering amplitude or equivalently from the the 4–loop sigma–model beta function, comes a term which modifies the Einstein equation such that they do not admit anymore a Ricci flat metric as a solution.

Thus, the Calabi–Yau compactification must be seen as a first order approximation of the to–all–orders solution of the equation of motion. Since the expression of the fields that we have found in the last paragraph, with $H = 0 = d\Phi$, are a solution of the lowest order equations of motion, we can think to compute the to-all-order solutions perturbatively starting from it. On K_6 it exists a natural parameter that we can use in the perturbative expansion; it is the "radius" of K_6, r. One can expand in powers of $1/r$[30] the ten dimensional equations, treating spacetime derivatives and gauge fields as being of order $1/r$.

To the lowest order the Calabi–Yau solution is a solution of the equations of motion, but at order $1/r^4$ there begins to be corrections. For example at $1/r^4$ order, the equations are compatible with having H of order $1/r^3$ and not any more zero. In the same way one must modify the Ricci flat Kähler metric adding non Ricci flat terms. In ref.[43] it was shown that there always exists a Kähler metric which satisfies these new equations and which is constructed perturbatively starting from the Ricci flat one.

§4.2 Dimensional Reductions: a General Truncation

It is possible to do some dimensional reductions from the D=10 supergravity to the D=4 one so to obtain results very similar to those which one would reach making a Calabi–Yau compactification. They main tool is to keep in D=4 only those modes which are invariant under some symmetries, this would correspond to say that on the chosen K_6 space only the invariant modes are zero modes and give rise to massless particles in D=4.

The simplest choice of K_6 is the six dimensional torus T_6, but if $K_6 = T_6$ the resulting D=4 supergravity has four unbroken supersymmetries. To obtain a N=1 theory we have to do something similar to have an $SU(3)$ holonomy group. On K_6 there are two $SU(3)$ groups, one is embedded in the Lorentz group $SO(6) \simeq SU(4) \supset SU(3)_L$ and the other in the gauge group, $G \supset SU(3)_G$. Let $SU(3)_D$ be the direct sum of these two groups, $SU(3)_D = SU(3)_L \oplus SU(3)_G$ and Z_3 the center of $SU(3)_D$. A choice of K_6 which leads to a N=1 D=4 supergravity is the orbifolds $K_6 = T_6/Z_3$ [44].

Here, we are not interested in the study of the orbifold compactification but of the (consistent) truncation which represents the contribution of the untwisted modes on the orbifold.

After having decomposed the D=10 Lorentz representations in the products of the 4 and 6 dimensional ones, one keeps only the modes which are singlets under Z_3. Since the theory is supersymmetric, it is sufficient to consider the bosonic modes. The D=4 modes, coming from the ten dimensional gravity sector g_{MN}, B_{MN}, Φ, which are singlets under Z_3

[30] More precisely, the dimensionless expanding parameter is α'/r^2.

are

$$g_{\mu\nu}, \quad B_{\mu\nu} \sim H_{\mu\nu\rho} \sim D, \quad \Phi, g_{i\bar{j}}, \quad B_{i\bar{j}} \tag{4.18}$$

$g_{\mu\nu}$ is the D=4 graviton, (D, Φ) give rise to the bosonic part of the axion–dilaton multiplet. (D, Φ) are present in every compactification (this is related to the fact that $b^0 = 1$ for a Calabi–Yau manifold), for that reason they form the so called "universal sector". The scalar fields in a D=4 N=1 supergravity theory are coordinates of a Kähler manifold, for the universal sector the Kähler manifold is

$$\frac{SU(1,1)}{U(1)} \tag{4.19}$$

The fields coming instead from $(g_{i\bar{j}}, B_{i\bar{j}})$ are model dependent; as we have seen their number is given by $h^{(1,1)}$. For T_6/Z_3, $h^{(1,1)} = 36 = 9$ untwisted + 27 twisted states. The 9 untwisted states come directly from $(g_{i\bar{j}}, B_{i\bar{j}})$; since they transform under $SU(3)_D$ as $1 \oplus 8$, they describe in D=4 N=1 supergravity the Kähler manifold

$$\frac{SU(3,3)}{SU(3) \otimes SU(3) \otimes U(1)} \tag{4.20}$$

It is possible to do explicitly a truncation of the Chapline–Manton lagrangian obtaining the D=4 supergravity theory which describes these modes. Since D=4 N=1 supergravity is completely fixed up to two functions of the scalar fields (in the next chapter we will give more details about that), this means that from the truncation is possible to obtain the expressions of these two functions [19]. In fact, if S is the complex field arising from (D, Φ) and $T_{i\bar{j}}$ those coming from $(g_{i\bar{j}}, B_{i\bar{j}})$, the Kähler potential turns out to be $J = -\log(S + \overline{S}) - \log(\det(T_{i\bar{j}} + \overline{T}_{i\bar{j}}))$.

This truncation can be realized starting from compactifying on T_6 and then taking the Z_3 invariant modes. Thus, the N=1 theory can be obtained with a truncation from the N=4 one. But in N=4 supergravity the manifold of the scalar fields is restricted to be

$$\frac{SO(6,n)}{SO(6) \otimes SO(n)} \otimes \frac{SU(1,1)}{U(1)} \tag{4.21}$$

in our case $n = 6$. Looking for a Z_3 invariant submanifold one finds exactly the Kähler manifold of the N=1 supergravity model previously stated.

The explicit procedure for the truncation is the following [19,45]: we start from the D=10 Chapline–Manton lagrangian and we write down the N=4 D=4 theory that contains all of the fragments of the D=10 fields. We obtain this action by imposing $\theta(x_\mu, y_m) = \theta(x_\mu)$ where θ is a generic component of a generic field. The integration over the y coordinates is then carried out giving the T_6 volume that appears as a factor renormalizing the coupling constants and some of the wave functions. Then, to get the N=1 theory, we cancel from the action all the pieces containing at least one field that is not a singlet under the action of Z_3. This prescription provides an approximation to the full effective action, because the effect of the massive Klauza–Klein fields is not taken into account and the resulting action does not contain higher derivative terms.

Consider now the D=10 gauge sector; from A_M^α one obtains in D=4 the gauge fields of $SU(3) \otimes E_6 \otimes E_8$, A_μ^α. The decomposition of the fields with the Lorentz index tangent to K_6 is

$$A_m^\alpha \longrightarrow C_3^{(\bar{3},\overline{27})} \oplus C_3^{(3,27)} \oplus C_3^{(1,78)} \oplus C_3^{(8,1)} + \text{ h.c.} \tag{4.22}$$

The diagonal Z_3 singlet are the complex fields

$$C_i^{(n,a)} \quad \text{with} \quad a\epsilon 27 \text{ of } E_6, \quad n\epsilon 3 \text{ of } SU(3)_G, \quad i\epsilon 3 \text{ of } SU(3)_L \tag{4.23}$$

They give rise to 9 families in the 27 of E_6 which are triplets under the $SU(3)$ part of the gauge group.

This model, containing chiral fermions in the 3 of $SU(3)$ gauge group, is anomalous. The additional chiral fermions needed to cancel the anomaly come from the twisted sector of the orbifold compactification. In fact [46], from the twisted sector come 27 copies of fields in the $(3,1)$ of $SU(3) \otimes E_6$ which cancel the anomaly, and also 27 copies of singlets under $SU(3)$ transforming as 27 under E_6. This is consistent with the fact that for T_6/Z_3 $h^{(1,1)} = 36$, $h^{(2,1)} = 0$.

Taking in account the gravitational, matter and gauge sectors, the D=4 supergravity lagrangian is specified by the following functions of the scalar fields

$$\mathcal{G} = J(z,\bar{z}) - \log \frac{1}{4}|g| \tag{4.24}$$

$$\text{Y.M. coupling function} \qquad f_{\alpha\beta}(z) = S\delta_{\alpha\beta} \qquad \alpha,\beta \,\epsilon\, SU(3)\otimes E_6 \otimes E_8$$

where

$$\text{Kähler potential} \qquad J(z,\bar{z}) = -\log(S+\bar{S}) - \log\det(T_{i\bar{j}} + \overline{T}_{i\bar{j}} - 2C_i^{ma}\overline{C}_{\bar{j}}^{\,ma})$$

$$\text{superpotential} \qquad g(z) = d_{abc}\varepsilon_{mnl}\varepsilon^{ijk}C_i^{ma}C_j^{nb}C_k^{lc}$$

and ε_{mnl} is the antisymmetric tensor of $SU(3)$ and d_{abc} the symmetric tensor of the 27 of E_6. The maximal 9 family Kähler manifold defined by this Kähler potential is

$$\frac{SU(1,1)}{U(1)} \otimes \frac{SU(3,3+3n)}{SU(3) \otimes SU(3+3n) \otimes U(1)} \qquad n = 27 \tag{4.25}$$

From this model it is possible to obtain other models with a lower number of families considering conpactifications on T_6/Z_6, T_6/Z_{12} or $T_6/SU(3)_D$. Compactifications on $T_6/SU(3)_D$ gives rise to the minimal one–family model first obtained in ref. [19]. In this case only the singlets under $SU(3)_D$ survive, then $T_{i\bar{j}} \sim \mathbb{I}T$; since $\det \mathbb{I} = 3$, the Kähler potential becomes

$$\begin{aligned} &J(z,\bar{z}) = -\log(S+\overline{S}) - 3\log(T+\overline{T} - 2C^a\overline{C}^a) \\ &g(z) = d_{abc}C^aC^bC^c \qquad f_{\alpha\beta}(z) = S\delta_{\alpha\beta} \end{aligned} \tag{4.26}$$

This is exactly the result of ref.[19]. In this case the D=4 gauge group is $E_6 \otimes E_8$.

In the following we will mainly consider the one–family model as a toy model to study the properties of the D=4 N=1 effective supergravity theories.

§4.3 Four dimensional Coupling Constants

In the D=4 N=1 standard supergravity theory, as in the D=10 case, there are two independent coupling constants, besides those appearing in the three arbitrary functions. They are the gravitational k_4 and gauge g_4 coupling constants. k_4 is related to the Planck mass by $M_{Pl} \sim 1/k_4$.

With very simple arguments, mainly on dimensional ground, one can find the relations between the string, the D=10 and the D=4 coupling constants.

Strings have one dimensionful parameter α' which has dimension −2, and a dimensionless one, the string coupling constant which is the same as the vacuum expectation value of the dilaton field. Since the D=10 gauge coupling g_{10} has dimension −3 and the gravitational coupling k_{10} has dimension −4, it follows

$$g_{10} \sim \rho(\alpha')^{\frac{3}{2}} \qquad k_{10} \sim \rho(\alpha')^2 \tag{4.27}$$

and

$$\alpha' \sim \frac{(k_{10})^2}{(g_{10})^2} \tag{4.28}$$

(this is the result that we have already obtained by explicit computation of the field theory limit in the previous chapter).

The D=10 coupling constants are related to the D=4 ones by

$$g_4 \sim g_{10}/\sqrt{V} \qquad\qquad k_4 \sim k_{10}/\sqrt{V} \tag{4.29}$$

where V is the volume of the internal compact space K_6. This relation can be easily understood from the fact that $b^0 = 0$ and that the unique zero form on K_6 is a constant. Then, for example, going from ten to four dimensions the graviton $g_{MN}(X)$ gives rise to the D=4 graviton $g_{\mu\nu}(x_\mu, y_m) \simeq g_{\mu\nu}(x_\mu) \cdot ($ constant on $K_6)$. It follows that [45]

$$\int d^{10}X \left(-\frac{1}{2k_{10}^2} R^{(10)}\right) \longrightarrow \int d^4X \left(-\frac{1}{2k_{10}^2} R^{(4)}\right) \cdot V \tag{4.30}$$

which means $k_{10}^2/V \sim k_4^2$.

Since the D=4 coupling constants are related to the D=10 ones by the same factor $\sqrt{V}$, it follows that α' is determined by the four dimensional Newton's constant and Yang–Mills coupling as in ten dimensions, i.e.

$$\alpha' \sim \frac{k_4^2}{g_4^2} \tag{4.31}$$

In terms of the string mass scale $M_s \sim 1/\sqrt{\alpha'}$ and the Planck scale $M_{Pl} \sim 1/k_4$, this relation takes the form

$$\frac{M_s}{M_{Pl}} \sim g_4 \tag{4.32}$$

If the gauge group is to correspond to a physical gauge group in four dimensions, then the gauge coupling g_4 should be taken roughly of order unity, so the string scale is necessarily of order of the Planck mass M_{Pl}. M_s generally characterizes the scale at which intrinsically stringy effects become directly observable. The relation (4.32) thus constitutes one of the fundamental obstacles to finding observable consequences of a string theory.

In a lower–dimensional theory that does not correspond to a compactification of some higher dimensional theory (for example coming from D=4 superstrings), we might imagine a different relation emerging between the gauge and gravitational coupling g_4 and k_4. It has been shown [47] that for spacetime gauge symmetries that are realized as affine Kac–Moody algebras on the world–sheet this does not happen; generically it turns out that $M_s/M_{Pl} \geq g_4$.

Since during the compactification the gauge group is broken to a Grand Unification group[31], the G.U.T. scale coincides with the energy scale of the compact space $M_{GUT} = M_6$ ($M_6 = 1/r \sim V^{-6}$). Using again the relations between the string and field theory coupling constants one finds [48]

$$(g_4)^2 \sim (g_{st})^2 \cdot \left(\frac{M_6}{M_s}\right)^6 \tag{4.33}$$

Thus, if the Kaluza–Klein scale is just one order of magnitude below the string scale, it will be make the four dimensional coupling unphysically small (less than 10^{-6}) unless the string coupling is very large (but in this case all our discussion is not valid). This implies that for a weakly coupled string, reasonable phenomenology can be obtained only for

$$M_{GUT} = M_6 \sim M_s \tag{4.34}$$

Moreover, the scale at which spontaneous soft supersymmetry breaking occurs should have a value comparable with the weak scale, i.e. $M_W \lesssim M_{SSB}$. This leads to a "Grand Desert" scenario, that is nothing interesting happens between 10^{18}GeV, which is the energy scale of gravity, string and G.U.T., and 10^2GeV, which is the weak and supersymmetry breaking scale. Obviously, this situation is not happiling from a phenomenological point of view.

Although lots of different models have been proposed (see for example [48]), the main patterns that we have discussed here are very difficult to avoid.

31 We have seen just the case in which E_8 is broken to E_6, but on K_6 one can break E_6 by means of Wilson lines to more common G.U. groups.

CHAPTER 5

Four Dimensional Effective Supergravity Theories

In this chapter, making use of all the informations obtained previously, we will construct the N=1 D=4 effective supergravity theory which emerges from the heterotic superstring. We will start considering the standard, not higher derivative supergravity theory. This corresponds to consider the D=10 Chapline–Manton theory and to neglect the contributions of the string and Calabi–Yau massive modes. Then we will study how the higher order terms can modify the theory.

First we recall the structure of the not–higher–derivative (which means with at most two–derivative terms in the lagrangian) N=1 supergravity. The lagrangian is completely fixed up to the choice of the gauge group and of two functions of the scalars fields, $\mathcal{G}(z,\overline{z})$ and the Yang–Mills coupling functions $f_{\alpha\beta}(z)$. It is usually convenient to introduce the kähler potential $J(z,\overline{z})$ and the superpotential $g(z)$ defined by $\mathcal{G} = J - \log\frac{1}{4}|g|$. Using complex notations for the bosonic fields of the scalar multiplets, the superpotential and the Yang–Mills coupling function turn out to be analytic functions. The bosonic part of the lagrangian is [49]

$$\begin{aligned}\frac{1}{e}\mathcal{L}_B = e^{-\mathcal{G}}\left(3+\mathcal{G}^{l}_{k}\mathcal{G}^{\prime\prime -1k}_{l}\mathcal{G}^{\prime l}\right) + \frac{g_4^2}{2}\Re e f^{-1}_{\alpha\beta}\left(\mathcal{G}^{\prime i}T^{\alpha j}_{i}z_j\right)\left(\mathcal{G}^{\prime k}T^{\beta l}_{k}z_l\right) + \\ -\frac{1}{2}R + \mathcal{G}^{\prime\prime i}_{j}D_\mu z_i D^\mu \overline{z}^j - \frac{1}{4}\Re e f_{\alpha\beta}F^{\alpha}_{\mu\nu}F^{\beta}_{\mu\nu} + \frac{i}{4}\Im m f_{\alpha\beta}F^{\alpha}_{\mu\nu}\widetilde{F}^{\beta}_{\mu\nu}\end{aligned} \tag{5.1}$$

where z_i are the scalar matter fields, T^α the generators of the gauge group and g_4 the gauge coupling constant. Then it is sufficient to determine the $\mathcal{G}$, g and $f_{\alpha\beta}$ functions to get the expression of the D=4 string effective supergravity action.

§5.1 Symmetry Considerations and the General Structure of D=4 Supergravity

The relevant gauge group in D=4 is E_6 and the matter fields transform in the 27 of it except for some fields which are singlets. From the Calabi-Yau compactification we expect $\chi(K_6)/2$ families of fields in the 27 of E_6. Moreover there are $h^{(1,1)}$ singlets fields coming from the gravitational D=10 sector. We will call generically z_i all the scalar bosonic fields, where $z_1 = S$ is the axion–dilaton field, $z_{i+1} = T_i$ (i=1,...,r) are the fields coming from $(g_{i\bar{j}}, B_{i\bar{j}})$, $z_{i+r+1} = C_i$ (i=1,...) are the matter fields.

The scalar fields must satisfy some symmetries and scaling rules besides the gauge symmetries, as we have seen in the previous chapters. Imposing these constraints we will be able to almost fix the expression of the lagrangian. The relevant symmetries are [50]

i) "axion–type" symmetries. Under a shift of $\Im m S$ and $\Im m T_i$ the action should be invariant

ii) "scale invariance". The fact that the dilaton vacuum expectation value is the string coupling constant implies a scaling law in the string scattering amplitudes. This scaling law is a natural property also of the D=10 supergravity lagrangian and in D=4 becomes

$$\begin{aligned} g_{\mu\nu} &\longrightarrow t^4 g_{\mu\nu} \\ S &\longrightarrow t^4 S \\ T &\longrightarrow T \\ \mathcal{L} &\longrightarrow t^4 \mathcal{L} \end{aligned} \tag{5.2}$$

iii) "rescaling property". A similar scaling law follows from the fact that the moduli of the D=6 compact space are not fixed in the compactification, thus

$$\begin{aligned} S &\longrightarrow r^{-\frac{1}{2}} S \\ T &\longrightarrow r^{\frac{1}{2}} T \\ C &\longrightarrow r^{\frac{1}{4}} C \\ \mathcal{L} &\longrightarrow r^{-\frac{1}{2}} \mathcal{L} \end{aligned} \tag{5.3}$$

The requirement that the N=1 D=4 supergravity lagrangian satisfies these symmetries almost fixes the three functions J, g and $f_{\alpha\beta}$. For example, consider the terms

$$e \Re e f_{\alpha\beta} F^{\alpha}_{\mu\nu} F^{\beta}_{\mu\nu} + e \Im m f_{\alpha\beta} F^{\alpha}_{\mu\nu} \widetilde{F}^{\beta}_{\mu\nu} \tag{5.4}$$

to satisfy the symmetries it must be $\Re e f_{\alpha\beta} \to t^4 r^{-1/2} \Re e f_{\alpha\beta}$ which implies

$$f_{\alpha\beta} = S\delta_{\alpha\beta} \tag{5.5}$$

For definiteness, we will consider in the following the case in which there is only one T field, this means only one family because $h^{(1,1)} = 1$ and $h^{(2,1)} = 0$. The superpotential and the Kähler potential are

$$\begin{aligned} J &= -\log(S+\overline{S}) - 3\log(T+\overline{T}) + \mathcal{F}\left(\frac{C_i}{\sqrt{T+\overline{T}}}, \frac{\overline{C}_i}{\sqrt{T+\overline{T}}}\right) \\ g &= g(C_i) \end{aligned} \tag{5.6}$$

where $\mathcal{F}$ is an arbitrary function which depends on the details of the compactification. Since g comes from the trilinear part of the D=10 Chern–Simons term and is a homogeneous analytic function, it must be a polynomial of degree 3 in the fields. This means

$$g = d_{ijk} C^i C^j C^k \tag{5.7}$$

The structure of (5.5) , (5.6) and (5.7) is consistent with the explicit truncations, for example the Witten minimal model [19] is given by

$$\mathcal{F} = -3\log\left(1 - 2\frac{C_i\overline{C}^i}{T+\overline{T}}\right) \tag{5.8}$$

This is the expression of the D=4 supergravity theory coming from the string–tree level amplitudes of the heterotic superstring neglecting the contributions of the string and compactification massive modes.

In the same way one can consider the case of the string on a n–genus Riemann surface (n–loop quantum correction). The classical D=4 lagrangian should scale as

$$\mathcal{L}_{\text{n-loop}} \longrightarrow t^{4(\text{n}-1)} r^{\text{n}-\frac{1}{2}} \mathcal{L}_{\text{n-loop}} \tag{5.9}$$

which would imply a loop parameter $\sim TS^{-1}$. The corresponding functions are[32]

$$\begin{aligned} J &= -\log(S+\overline{S}) - 3\log(T+\overline{T}) + \mathcal{F}_1\left(\frac{T+\overline{T}}{S+\overline{S}}, \frac{C_i}{\sqrt{T+\overline{T}}}, \frac{\overline{C}_i}{\sqrt{T+\overline{T}}}\right) \\ f_{\alpha\beta} &= \left[S - \varepsilon(aT + bC_i\overline{C}^i)\right]\delta_{\alpha\beta} \end{aligned} \tag{5.10}$$

[32] From the explicit one string loop computations it seems that $f_{\alpha\beta}$ does not receive corrections, so that to all orders $f_{\alpha\beta} = S\delta_{\alpha\beta}$.

where ε is a small parameter and a and b are arbitrary constants. The superpotential g doesn't change.

Notice that $f_{\alpha\beta}$ could receive only a one string loop contribution which mainly says that T could have an axionic coupling. We have already seen that this term could come only from the D=10 anomaly cancelling terms and is a one loop effect (however see footnote 32).

It may seem wrong to consider the higher string loop contribution before considering the effects of the terms emerging from the massive modes contribution already at tree level. Because of the technical difficulties, we will consider separately the two corrections to the minimal standard supergravity effective action. In the rest of this chapter we will study the string tree level, massless and massive modes' contributions to the effective action. In the next chapter we will reanalyse within the contest of a more general approach the possible contributions of the string loops to the standard (not higher derivative) D=4 effective theory.

§5.2 Higher Derivative Terms Emerging from the D=10 Lorentz Chern–Simons Form

Between the infinite number of terms which come from the α' expansion in the D=10 effective theory, there is one of particular importance, the Chern–Simons form. This term plays a great role in the cancellation of the anomalies of the D=10 supergravity theory and in the compactification to four dimensions. For these reasons, we start considering its contributions to the D=4 effective action and then we will try to generalize the results to the other higher order terms.

The Lorentz Chern–Simons form appears in the D=10 lagrangian through the redefinition of the field strength of B_{MN}, $H = dB - \omega_{3YM} + \omega_{3L}$. As we have already noticed, the D=10 lagrangian is not anymore supersymmetric when the Lorentz Chern–Simons form is introduced in H. We want to stress that, contrary to the case of D=10 where the supersymmetrization of the Lorentz Chern–Simons term is not known and requires the introduction of an infinite number of terms [21],the four dimensions the knowledge of the off–shell tensor calculus easily permits to compute the supersymmetrization of ω_{3L}. Thus we can start from the lowest order in α' and consider just the H^2 term in the lagrangian, after the compactification of this term we will reconstruct in D=4 its supersymmetry partners.

The contribution in D=4 coming from ω_{3L} through H^2 can be easily found. Consider $(H_{MNP})^2$ with Lorentz indices tangent to M_4, $(H_{\mu\nu\rho})^2$. The field B in the effective lagrangian couples through the field strength H. H is invariant under the gauge transformation $B \rightarrow B + d\Lambda - \delta_v\omega_{3YM} + \delta_v\omega_{3L}$. In D=4 this gauge invariance reduces the number of degrees of freedom of $B_{\mu\nu}$ from 6 to 1. (We know already that $B_{\mu\nu}$ coupled to the dilaton gives rise in D=4 to the axion–dilaton multiplet, now we verify it explicitly.) We can show this by means of a duality transformation. In the D=4 action the term H^2 appears multiplied by a field C which essentially is the dilaton related to the radius of K_6, CH^2. We can introduce an auxiliary one form field f and a lagrange multiplier a and rewrite in an equivalent way the same lagrangian

$$CH^2 \longrightarrow Cf^2 + a(d^*f + F\widetilde{F} - R\widetilde{R}) \tag{5.11}$$

Remembering that $dH = F\widetilde{F} - R\widetilde{R}$, the equation of motion for a says that $f = H^*$, so that eliminating the lagrange multiplier a we obtain the equation we started with. On the other hand, we can integrate by parts ad^*f and then make the gaussian integral over f. In this way the effective action becomes [51]

$$\frac{1}{2C}\partial_\mu a\partial^\mu a + a(F\widetilde{F} - R\widetilde{R}) \tag{5.12}$$

Thus, a describes the unique physical component of B, which previously we called $\Im m S$. Notice that we have reobtained the axion coupling $aF\widetilde{F}$ and the expression of the $f_{\alpha\beta}$ function of the previous paragraph. But from the term $-aR\widetilde{R} = -\Im m S R\widetilde{R}$ we know also which is the D=4 expression for the Lorentz Chern–Simons term and how it couples to the matter fields, it has the same axionic coupling as $F\widetilde{F}$.

This term, being a higher derivative term, is not present in the lagrangian considered in the last paragraph; since we want to construct a D=4 supersymmetric theory, we must look for the supersymmetric completion of $R\widetilde{R}$. In D=4 there exists a very simple supersymmetric completion of $R\widetilde{R}$ which does not requires an infinite number of terms. Before studying this, it is better if we recall some general features of the N=1 D=4 supergravity.

§5.2.1 *Minimal Formulations and Auxiliary Fields of N=1 D=4 Supergravity*

The supersymmetry transformations close on the physical fields only on–shell; one must introduce some auxiliary fields in the supersymmetric multiplets to gain the off–shell closure of the algebra. In fact, usually there are less off-shell bosonic degrees of freedom than fermionic ones. For example the gravity multiplet (e^a_μ, ψ_μ) has 10 bosonic and 16 fermionic off-shell degrees of freedom; since by supersymmetry they must be equal in number, one has to introduce 6 auxiliary bosonic degrees of freedom. There are two minimal choices (which means without introducing coupled bosonic–fermionic auxiliary fields), the "Old" and the "New" minimal ones. In the "Old" minimal formulation the auxiliary fields are a complex scalar u and a vector A_μ; in the "New" minimal, an antisymmetric tensor $a_{\mu\nu}$ and a $U(1)$ gauge field B_μ. In the not–higher derivative theories the auxiliary fields do not propagate and have linear equations of motion. In this case the two formulations are equivalent (up to the $U(1)$–R– symmetry always present in the "New" minimal formulation and which is not requested, but permitted, in the "Old" minimal one) and through a Legendre transformation one can pass from one to the other[33].

Notice that having an off-shell formulation it is easy to add to a lagrangian a new term; it is sufficient that this term be off-shell supersymmetric. Obviously the equation of motion of the auxiliary fields are modified by this new term.

Of the two minimal formulations, the better known and widely used is the "Old" minimal one. Thus we have added the D=4 Lorentz Chern–Simons term within this formulation. Since the auxiliary fields will play an important role in the following, notice that it exists a formulation of D=4 N=1 supergravity independent of them, the conformal supergravity. The conformal group contains as a subgroup the Poincarè group; then, once constructed a superconformal theory, one has to break the extra symmetries to get the Poincarè theory, this can be done imposing a gauge fixing. Doing that automatically the auxiliary field arise. So the different formulations of the Poincarè supergravity can be seen to emerge from different gauge fixing of the superconformal theory. On the other side, conformal supergravity requires the introduction of a non–physical matter multiplet, called "compensator". It can be a chiral, linear or general vector multiplet and it gives the possibility of making the gauge fixing without changing the physical contents of the theory[34].

Since the higher–order terms coming from the string should depend on the formulation, superstrings should specify also which kind of compensator and which gauge fixing is realized.

[33] All the models we will consider have this $U(1)$–R–symmetry
[34] In other words, it "compensates" for the extra symmetries of the superconformal group.

§5.3 D=4 Supersymmetric Completion of the Lorentz Chern–Simons Form

We must look for a multiplet which contains $R\tilde{R}$. The gravitational Bianchi identities (in the "Old" minimal formulation) can be solved in terms of three basic multiplets

$$\begin{array}{ll} \text{Weyl curvature} & W_{\mu\nu\rho} \\ \text{Scalar curvature} & R \\ \text{Ricci curvature} & E_{\mu\dot{\mu}} \end{array} \tag{5.13}$$

which contain the related Poincarè tensor. From these, three chiral multiplets quadratic in the curvature can be constructed

$$\begin{array}{ll} W_{\mu\nu\rho}W^{\mu\nu\rho} & \text{(gravitational analogue of } W^{\alpha}_{YM}W^{\alpha}_{YM}) \\ \Sigma(E_{\mu\dot{\mu}}E^{\mu\dot{\mu}}) & \\ RT(R) & \end{array} \tag{5.14}$$

where Σ denotes the operation of projecting a vector multiplet in a chiral one and T is the kinetic operator. In ref. [52] we have explicitly computed the bosonic and fermionic components of the three gravitational multiplets. We will give their expression in the next paragraph. Since $R\tilde{R}$ appears in $W_{\alpha\beta\gamma}W^{\alpha\beta\gamma}$, the supersymmetrization of $\Im m S(F\tilde{F} - R\tilde{R})$ is

$$\left[f(S)\left(\delta_{\alpha\beta}W^{\alpha}_{YM}W^{\beta}_{YM} - W_{\mu\nu\rho}W^{\mu\nu\rho}\right)\right]_F \tag{5.15}$$

where $f_{\alpha\beta} = f\cdot\delta_{\alpha\beta}$ and $[\ ]_F$ is the F–density action for a chiral multiplet. As we have already observed, as long as tree–level S–matrix elements for massless particles are concerned, the coefficients of R^2 and $R^2_{\mu\nu}$ are completely arbitrary, only the square of the Weyl tensor $W^2_{\mu\nu\rho\sigma}$ being fixed by the on–shell amplitudes. Thus, we have the freedom to add to (5.15) a linear combination of $\Sigma(E_{\alpha\dot{\alpha}}E^{\alpha\dot{\alpha}})$ and $RT(R)$. In general a theory containing terms quadratic in the gravitational curvatures propagates – besides the usual massless states – massive particles of spin ≤ 2. It results that, whereas the new massive particles with spin $\leq \frac{1}{2}$ are physically acceptable positive norm states, the ones with spin ≥ 1 are bound to be "Poltergeists" (i.e. negative norm states). Of course superstring should be poltergeists–free. There is only one combination[35] which is poltergeists–free, it is the Gauss-Bonnet combination of curvatures

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2 \tag{5.16}$$

The Gauss–Bonnet theorem states that (5.16) is a total derivative proportional to the density of the Euler characteristic of M_4. As we have seen, this term enters the lagrangian with a field dependent coefficient which is given by the axion–dilaton S (otherwise it would drop from the lagrangian).

The Gauss–Bonnet combination of the curvature multiplets is [53,52]

$$(W_{GB})^2 = RT(R) + \Sigma(E_a E^a) + \frac{9}{16} W_{\mu\nu\rho} W^{\mu\nu\rho} \tag{5.17}$$

The super–Gauss–Bonnet theorem states that

$$\begin{aligned} \frac{8}{9}\frac{1}{32\pi^2}\Re e\left[(W_{GB})^2\right]_F &= \frac{1}{32\pi^2}\sqrt{g}\left[(R_{\mu\nu\rho\sigma})^2 - 4(R_{\mu\nu})^2 + R^2\right] \\ &\quad + \text{a total derivative of a globally defined, local 3–form} \\ \frac{8}{9}\frac{1}{32\pi^2}\Im m\left[(W_{GB})^2\right]_F &= \frac{1}{32\pi^2}\sqrt{g}R_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma} \\ &\quad + \text{a total derivative of a globally defined, local 3–form} \end{aligned} \tag{5.18}$$

In this way, the real part of the super–Gauss–Bonnet combination is a natural representative of the Euler class and the imaginary part of the Pontryagin class.

Then, we **assume** that the supersymmetric completion of the Lorentz Chern–Simons form in D=4 is

$$\left[f(S)\left(\delta_{\alpha\beta}W^{\alpha}_{YM}W^{\beta}_{YM} - (W_{GB})^2\right)\right]_F \tag{5.19}$$

The complete lagrangian is

$$\mathcal{L} = -\frac{1}{2}\left[\Phi(z,\bar{z})\right]_D + \left[g(z)\right]_F - \left[f(S)\left(\delta_{\alpha\beta}W^{\alpha}_{YM}W^{\beta}_{YM} - (W_{GB})^2\right)\right]_F \tag{5.20}$$

where $J = 3\log(-\Phi/3)$.

[35] Actually one can add αR^2 to the Gauss-Bonnet combination, but $\alpha \neq 0$ requires the propagation of an additional scalar which doesn't come from strings.

§5.3.1 *Propagation of the Auxiliary Fields*

Although thanks to the super–Gauss–Bonnet combination, there aren't gravitational negative norm states, the analysis must be supplemented with a study of the auxiliary fields sector. As we have explicitly computed (see ref. [52]), the equations of motion of the auxiliary fields get drastically modified by the Gauss–Bonnet terms; they become a coupled system of non–linear, differential equations (this is a general feature of higher derivative supergravity). In fact most of the new massive degrees of freedom are just auxiliary fields which get promoted to physical propagating ones. Moreover, if the quadratic terms in the gravitational curvatures form the poltergeists–free Gauss-Bonnet combination, also the auxiliary fields do not propagate at the linearized level around a **supersymmetric background**.

Since in higher derivative supergravity the auxiliary field equations are not longer linear, in general there are more than one solution even for translational invariant backgrounds. In the situation of interest here, the auxiliary field equations have two solutions in a constant background. The "non–perturbative" one (which is not continuously connected with the standard result when we let the higher derivative terms vanish) can be seen as a "vacuum" in which the auxiliary fields get non trivial v.e.v., then breaking supersymmetry, and become propagating with a finite mass of order 1 Planck mass units. However, since they are poltergeists, this solution cannot be taken too seriously, but it is just a spurious effect due to the fact that our lagrangian is not a good approximation to the real string dynamics at Planck energies.

This discussion becomes non trivial when one considers the fields in a non–supersymmetric background. In ref. [52] we studied in details the kinetic matrix of the gravitational auxiliary fields in the case of the Witten minimal model. We found that there is a whole propagating poltergeist multiplet. But these states get an infinite mass in a Minkowski background so that they decouple when we perturbe around a flat vacuum. Thus, the new terms in the auxiliary fields equations amount just to new derivative couplings for the standard physical fields.

Similarly, no ghost states appear perturbatively in the gravitational sector, when expanding around a flat space, since the quadratic piece of the Riemann terms just cancel because of the Gauss–Bonnet theorem.

§5.4 Scalar Potential and Spontaneous Superymmetry Breaking

One interesting issue is the study of the vacuum configuration of this supergravity model. This can be done examining the scalar potential for the physical fields, i.e. after having eliminated the auxiliary fields through their equations of motion. From the analysis of the scalar potential and of the vacuum configurations, one can discuss the possible symmetry breakings, both of supersymmetry and of the gauge group.

We will first consider the standard (i.e. not higher–derivative) case and then we will show which are the modifications, which come through the elimination of the auxiliary fields, in the case with the Lorentz Chern–Simons form.

§5.4.1 *The Scalar Potential of the Standard Effective Theory*

Let $\mathcal{F} = \mathcal{F}\left(C_i\overline{C}^i/(T+\overline{T})\right)$, then the scalar potential of the standard theory in the notation of the first paragraph can be written as

$$V = \frac{e^{\mathcal{F}}}{(S+\overline{S})(T+\overline{T})}\left[|g|^2 + \frac{T+\overline{T}}{\mathcal{F}'}\left|\frac{\partial g}{\partial C_i}\right|^2 + \frac{\triangle}{\alpha\beta\mathcal{F}'}\left(u\mathcal{F}'W + C^i\frac{\partial}{\partial C_i}W\right)^2\right] \qquad (5.21)$$
$$+ \frac{1}{(S+\overline{S})}\left(\frac{\mathcal{F}'}{T+\overline{T}}\right)^2 d^\alpha d^\alpha$$

where

$$d^\alpha = C_i(T^\alpha)^i_j C^j$$
$$\triangle = (\mathcal{F}')^2 - 3\mathcal{F}''$$
$$\alpha = 3 + u\mathcal{F}'$$
$$\beta = \mathcal{F}' + u\mathcal{F}''$$
$$u = \frac{C_i\overline{C}^i}{T+\overline{T}}$$

Since $J_j^{\prime\prime i}$ $(i,j = S, T, C_i)$ provides the kinetic lagrangian of the chiral scalars and fermions, it has to be positive definite; this implies

$$\beta > 0$$

$$\mathcal{F}' > 0$$
$$T + \overline{T} > 0$$

For $\triangle \geq 0$, the scalar potential is positive definite with a global minimum at $g = 0$, $\frac{\partial}{\partial C_i} g = 0$ which is supersymmetric ($V = 0$). When $\triangle \geq 0$ is not satisfied, the potential is not positive; however it has an extremum at $g = 0 = \frac{\partial}{\partial C_i} g$ which is a supersymmetric minimum with vanishing cosmological constant. So in any case, it exists a minimum with zero cosmological constant and unbroken supersymmetry.

Notice that the critical case, $\triangle = 0$, corresponds to the Witten minimal model

$$\mathcal{F} = -3 \log(1 - 2u) \tag{5.22}$$

This study can be repeated also in the general case when string loops contributions are taken into account. In this case, all what can be said is that the origin, where $g = 0 = \frac{\partial}{\partial C_i} g$, is a local supersymmetric minimum with vanishing cosmological constant and with flat valleys in the S and T directions.

Thus, in this approximation string theories don't provide a tool for supersymmetry breaking. Also if we consider the string loop corrections to the not–higher order D=4 supergravity effective theory, supersymmetry remains unbroken. The only possibility is to impose that supersymmetry is broken by nonperturbative phenomena, for example by gaugino condensation in the hidden sector or by a non vanishing vacuum expectation value of the field strength of the antisymmetric tensor B_{MN} in the compact directions. For the simplest expressions of the $\mathcal{F}$ function and $\triangle \geq 0$, it turns out [50] that the goldstino field is the superpartner of the T field; in the T direction the potential is flat and in the S direction it has a minimum giving a mass to the S field. Moreover the scalar potential has a vanishing minimum where $\langle S + \overline{S} \rangle$ and $\langle T + \overline{T} \rangle$ do not vanish but $\langle C_i \rangle = 0$. Therefore we conclude that for a large class of $\mathcal{F}$ functions the relevant features of the Kähler potential are preserved, i.e. we get supersymmetry breaking with vanishing cosmological constant, absence of soft–breaking terms in the observable sector and existence of flat directions which imply that the gauge group remains unbroken.

§**5.4.2** *Scalar Potential of The Higher Derivative Theory*

In the higher derivative case we limit ourselves to discuss the Witten minimal model. In a constant background[36] we were able to solve the equation of motion of the auxiliary fields and to compute the scalar potential of the physical fields. We have considered the solution of the equations of motion of the auxiliary fields connected to the standard one, in this case the scalar potential is[37] [52]

$$
\begin{aligned}
V &= \frac{\beta^2 + \frac{1}{2}e^{\mathcal{G}} h(\frac{3}{2} - h)}{1 + 3\lambda h^2} \\
h &= \frac{3}{2} - \frac{2}{\sqrt{27\lambda}} \left[1 + \frac{27}{4}\lambda + \frac{4}{9}(S + \overline{S})\beta^2\right]^{\frac{1}{2}} \sin\vartheta \\
\vartheta &= \frac{1}{3}\arcsin\left[\frac{\sqrt{27\lambda}\left(1 + \frac{27}{4}\lambda + \frac{2}{3}(S + \overline{S})\beta^2\right)}{2\left(1 + \frac{27}{4}\lambda + \frac{4}{9}(S + \overline{S})\beta^2\right)^{\frac{3}{2}}}\right] \\
\lambda &= \frac{2}{27}\exp\left[-3\log(T + \overline{T} - 2C_i\overline{C}^i) + \log|\hat{g}|^2\right] \\
\beta^2 &= \frac{1}{S + \overline{S}}\left(\mathcal{G}'^i T_i^{\alpha j} C_j\right)^2 + \frac{9}{24}\frac{|g_i|^2}{(S + \overline{S})^{\frac{1}{3}}\phi^2} \\
\mathcal{G} &= J + \log|\hat{g}|^2 = -3\log\left(-\frac{\phi}{3}\right) + \log|\hat{g}|^2 = \\
&= -\log(S + \overline{S}) - 3\log(T + \overline{T} - 2C_i\overline{C}^i) + \log|\hat{g}|^2 \\
\hat{g} &= g + \frac{1}{2}(S + \overline{S})\overline{\lambda}_L^\alpha \lambda_L^\alpha
\end{aligned}
\tag{5.23}
$$

where

$$
\begin{aligned}
0 \leq \vartheta \leq \frac{\pi}{6} \\
0 < h < \frac{3}{2}
\end{aligned}
$$

The potential is positive semi–definite, then as in the standard case the cosmological constant vanishes. At the minimum $\beta^2 = 0$ and $|\hat{g}| = 0$ which implies $\langle C_i \rangle = 0$; thus the gauge group is unbroken. Moreover, if there are not gauge fermions condensates ($\left\langle \overline{\lambda}_L^\alpha \lambda_L^\alpha \right\rangle = 0$) supersymmetry is unbroken, otherwise supersymmetry is broken. The Goldstone fermion is again the superpartner of the T field since $\left\langle \delta\chi_L^T \right\rangle = -\frac{1}{8}\left\langle (S + \overline{S})^{2/3} \right\rangle \cdot \left\langle \overline{\lambda}_R^\alpha \lambda_R^\alpha \right\rangle \varepsilon_L$.

[36] By supersymmetry it must be Minkowski or anti–de–Sitter.

[37] We add also the possible contribution of gluino condensates $\overline{\lambda}_L^\alpha \lambda_L^\alpha$.

Thus the higher order gravitational corrections do not change the vacuum configuration; in particular the cosmological constant remains exactly zero. Classically, supersymmetry is unbroken but it can be broken by the same quantum effects as suggested in the framework of the standard theory, that is by gluino condensation. In this last case too, the vacuum energy vanishes even when the higher derivative corrections are taken in account.

§5.5 Higher Derivative Supergravity

§5.5.1 *The "Non–Renormalization" of the Vacuum*

The fact that the vacuum is not "renormalized" by the addition of the Gauss–Bonnet interactions (though the potential and the auxiliary field equations are dramatically modified) is just an explicit example of a general situation.

The addition of higher derivative terms to the standard supergravity theory can spoil the basic geometric properties of N=1 D=4 supergravity. One of them is the Kählerian structure which in the standard (i.e. not higher–derivative) theory is a consequence of supersymmetry.

The Kähler structure of the D=4 N=1 supergravity is roughly the fact that the kinetic metric of the scalar fields $g_{i,\overline{j}}(z,\overline{z})\partial_\mu z_i \partial^\mu \overline{z}^{\overline{j}}$ can be written as the second derivative of a function $\Phi(z,\overline{z})$ called Kähler potential $g_{i,\overline{j}}(z,\overline{z}) = \partial_i \partial_{\overline{j}} \Phi(z,\overline{z})$. Obviously this metric is invariant under the Kähler transformation $\Phi(z,\overline{z}) \rightarrow \Phi(z,\overline{z}) + \Lambda(z) + \overline{\Lambda}(\overline{z})$. Because of supersymmetry this invariance extends to a super–Kähler invariance, where Φ and Λ are chiral multiplets. One can study [54] which restrictions the super–Kähler invariance imposes on the possible higher curvature terms (of all orders). The main properties which the higher derivative terms must have to satisfy the super–Kähler invariance, are certain scaling properties among the Kähler potential and polynomials in the Lorentz and Yang–Mills curvatures of a given degree.

Interesting, these scaling properties, in the case of superstrings, turn out to be a consequence of the global scale covariance of the two dimensional action under a shift of the dilaton $\phi \rightarrow \phi + c$. Thus, it seems that the dilaton scaling law, which is exact order by order in string perturbation theory, guarantees that the D=4 N=1 effective theory is super–Kähler invariant.

One can also study the full scalar potential of a higher order N=1 D=4 supergravity theory. This is possible because the scalar potential can be written in a model–independent form as follows [55]

$$V(z,\overline{z}) = 2C^I(z,\overline{z})Z_I^J(z,\overline{z})C_J(z,\overline{z}) - 3M^2(z) \tag{5.24}$$

where C^I, Z_J^I and M are defined by

$$\begin{aligned} \delta_Q \chi_L^I &= C^I(z,\overline{z})\varepsilon_L \\ \delta_Q \psi_{\mu R} &= \tfrac{1}{2}\gamma_m u M(z,\overline{z})\varepsilon_L \\ \mathcal{L}_{KF} &= -\frac{1}{2} Z_J^I \overline{\chi}^J \partial\!\!\!/ \chi_I + \text{h.c.} \end{aligned}$$

and the Einstein and Rarita–Schwinger terms in the lagrangian have canonical form and there are no mixed spin $\frac{1}{2}$–spin $\frac{3}{2}$ kinetic terms[38]. Since this expression of the scalar potential is model independent, it applies equally to the standard case and to the higher derivative one. Let C_{0I}, Z_{0J}^I and M_0 be the expression of the functions in the standard case, it has been proven that the C, M and Z functions have a formal power series expansion in the variables C_{0I}, Z_{0J}^I and M_0. Suppose that there exists a configuration for which $C_{0I} = M_0 = 0$ (this is mainly the condition of unbroken supersymmetry and gauge group), then the overall scalar potential turns out to be

$$V(z,\overline{z}) = V_0(z,\overline{z}) + \delta V(C_0, M_0, Z_0) \tag{5.25}$$

with $\delta V(C_0, M_0, Z_0)$ at least cubic in C_0 and M_0. It follows that $\frac{\partial V}{\partial z} = 0$, $V = 0$ is still solved by $C_0 = M_0 = 0$. Moreover since $\frac{\partial^2 \delta V}{\partial z^\alpha \partial \overline{z}^\beta} = 0$, all the particle masses are the same as in the standard theory. This is so because $Z_I^J = Z_{0I}^J$ when $C_0 = M_0 = 0$.

Applying these results to the superstring in the case of unbroken supersymmetry, it follows that the flat directions of the potential, the gauge symmetry breaking patterns and the particles masses computed in the naive approximation remain exact to all orders in α'; in a word, the vacuum is not renormalized even if higher curvature corrections are considered. In the case of spontaneously broken supersymmetry, nothing can be said in general.

Thus, though the higher order terms dramatically modify the theory, the general features of the D=4 effective supergravity theory are completely described by its standard formulation.

[38] These terms can always be removed by an appropriate redefinition of the gravitino field.

§5.5.2 *The Off–Shell Structure of the Theory*

The results discussed until now were obtained in the ambitus of the "Old" minimal formulation of N=1 D=4 supergravity. Although the minimal formulations for the not–higher derivative supergravity theories are equivalent[39], when one considers higher derivative supergravity the situation can change.

In fact, in this case the equivalence should be understood in a broad sense. Consider for example a lagrangian in the "New" minimal formulation made by the standard one plus a quadratic combination of the gravitational multiplets. After the appropriate Legendre transformation, one obtains an "Old" minimal lagrangian but, a priori, with a **different** quadratic combination of the gravitational multiplets. Starting with the Gauss–Bonnet combination in the "New" minimal, one could end with a different combination in the "Old" minimal.

Thus it is interesting to reformulate supergravity as much as possible in a way independent from the gravitational auxiliary fields, and to see if strings say something about the off–shell structure of supergravity. Up to now a partial answer has been given to both these problems.

The conformal formulation of supergravity, besides the fact to be technically more simple, it is also independent from the choice of the gravitational auxiliary fields. In this formulation the gravitational auxiliary fields appear only after the gauge–fixing choice which breaks the superconformal group to the super–Poincarè. To have the possibility of making the gauge choice without changing the matter content of the theory, an extra (matter) multiplet called compensator is introduced; the gauge fixing can be realized fixing the value of the components of the compensator or of every other multiplet. Not all the component of the multiplet chosen to break the superconformal algebra are fixed; the components which are not constrained become part of the auxiliary fields of the super–Poincarè gravity multiplet. Thus a different gauge choice leads to different (but physically equivalent) formulations of Poincarè supergravity.

Superstring should tell us not only which is the gauge fixing to be chosen, but also if

[39] Only if the "Old" minimal formulation has the $U(1)$–R–symmetry characteristic of the "New" minimal one.

the compensator is a chiral, linear or vector multiplet.

We must also study how the Poincarè gravitational curvature multiplets are described in conformal supergravity. Two of the three gravitational curvature multiplets appear in conformal supergravity as matter multiplets [56], and precisely they are build out of the multiplet on which the gauge is fixed, so that after the gauge–fixing they reduce to purely gravitational multiplets. Thus, from the knowledge of the gravitational curvature multiplets one should be able to find on which multiplet the gauge is fixed in superstring effective theories.

From the structure of the gravitational vertices in heterotic superstring one can argue that the gravitational curvature multiplets satisfy some Bianchi identities which are those obtained constructing the curvatures and then fixing the gauge on a linear multiplet[40]. These superconformal Bianchi identities are [56]

$$
\begin{aligned}
&\overline{D}^{(L)}_{\dot\alpha} W_\alpha = D^{(L)}_\alpha \overline{W}_{\dot\alpha} = 0 \\
&\overline{D}^{(L)}_{\dot\alpha} W_{\alpha\beta\gamma} = D^{(L)}_\alpha \overline{W}_{\dot\alpha\dot\beta\dot\gamma} = 0 \\
&D^{(L)\alpha} W_\alpha = \overline{D}^{(L)}_{\dot\alpha} W^{\dot\alpha} \\
&D^{(L)\gamma} W_{\alpha\beta\gamma} = \sum_{\alpha\beta} \left(\frac{1}{2} i D^{(L)}{}_{\alpha}{}^{\dot\gamma} E_{\beta\dot\gamma} - \frac{1}{3} D^{(L)}_\alpha W_\beta\right) \\
&\overline{D}^{(L)\dot\gamma} \overline{W}_{\dot\alpha\dot\beta\dot\gamma} = \sum_{\dot\alpha\dot\beta} \left(\frac{1}{2} i D^{(L)\gamma}{}_{\dot\alpha} E_{\gamma\dot\beta} - \frac{1}{3} \overline{D}^{(L)}_{\dot\alpha} \overline{W}_{\dot\beta}\right) \\
&D^{(L)\alpha} E_{\alpha\dot\alpha} = -2\overline{W}_{\dot\alpha} \\
&\overline{D}^{(L)\dot\alpha} E_{\alpha\dot\alpha} = -2W_\alpha
\end{aligned}
\tag{5.26}
$$

where $D^{(L)}$ is the L–associated superconformal covariant derivative defined in ref. [57]. In ref. [56] we have computed the explicit expressions of the gravitational curvature multiplets which satisfy the Bianchi identities (5.26) and are constructed from L.

The square of the Weyl curvature is

$$
\begin{aligned}
W_{\alpha\beta\gamma} W^{\alpha\beta\gamma} = \frac{1}{4} \Big[& R_{ab}(Q) P_R \overline{R_{ab}(Q)}, \\
& \left(2\sigma_{cd} W^{cov}_{ab}(M^{cd}) - 3i R_{ab}(A)\right) P_R \overline{R_{ab}(Q)}, \\
& \frac{1}{2} W^{cov}_{ab}(M^{cd}) W^{cov}_{ab}(M^{cd}) - \frac{1}{4} \varepsilon^{cdef} W^{cov}_{ab}(M_{cd}) W^{cov}_{ab}(M_{ef}) + \\
& + 4 R_{ab}(Q) P_R \overline{R^{cov}_{ab}(S)} - \frac{3}{4} R_{ab}(A) R_{ab}(A) + \frac{3}{4} R_{ab}(A) \widetilde{R}_{ab}(A) \Big]
\end{aligned}
\tag{5.27}
$$

[40] These Bianchi identities can be called "New" minimal Bianchi identities since the gauge group is fixed on a linear multiplet.

where:

$$
\begin{aligned}
W^{cov}_{mn}(M^{ab}) &= \text{Weyl tensor of } R^{cov}_{mn}(M^{ab}) \\
&= R^{cov}_{mn}(M^{ab}) + \frac{i}{4}[\delta_{ma}\widetilde{R}_{nb}(A) - \delta_{mb}\widetilde{R}_{na}(A) + \\
&\quad - \delta_{na}\widetilde{R}_{mb}(A) + \delta_{nb}\widetilde{R}_{ma}(A)] \\
\widetilde{R}_{ab}(A) &= \frac{1}{2}\varepsilon_{abcd}R_{cd}(A)
\end{aligned}
$$

$R_{ab}(Q)$, $R_{ab}(A)$, $R^{cov}_{mn}(M^{ab})$, $R^{cov}_{ab}(S)$ are the improved gauge curvature associated to the Q, A, M^{ab}, S generators of the superconformal algebra (for the explicit expressions of these curvatures see ref [58]).

The components of the "scalar" curvature multiplet are

$$
W_{\alpha} = -\frac{3i}{2}[(P_R\eta)_{\alpha},\ iP_{R,\hat{\beta}\hat{\gamma}}((P_R\mathbb{C}^{-1})_{\hat{\gamma}\alpha}N + (\sigma_{ab}P_R\mathbb{C}^{-1})_{\hat{\gamma}\alpha}P_{ab}),\ (P_R\not{D}\eta)_{\alpha}] \tag{5.28}
$$

where

$$
\begin{aligned}
\eta &= -\frac{1}{C}\not{D}\zeta + \frac{1}{C}(\not{B} - \not{D}Ci\gamma_5)\lambda - 4i\gamma_5\lambda(\bar{\lambda}\lambda) \\
N &= \frac{1}{2C^2}(D_aC)^2 - \frac{1}{C}\Box C + \frac{1}{2C^2}(B_a)^2 + (\bar{\lambda}i\gamma_5\frac{1}{C}\not{D}\zeta) + \\
&\quad - 2(\bar{\lambda}i\gamma_5\frac{1}{C}\not{B}\lambda) - 6(\bar{\lambda}\lambda)(\bar{\lambda}\lambda) \\
P_{ab} &= D_a[\frac{i}{C}B_b - i(\bar{\lambda}i\gamma_5\gamma_b\lambda) + \frac{1}{C}D_bC] - (a \leftrightarrow b) \\
\mathbb{C} &= \text{charge conjugation matrix} \\
\lambda &= \frac{i\gamma_5}{2C}\zeta
\end{aligned}
$$

$\hat{\beta}$ is a four-component-spinor index and the components of the linear multiplet are

$$
L = (C, \zeta, 0, 0, B_m, -\not{D}\zeta, -\Box C)
$$

The Einstein curvature multiplet is

$$
E_m = [C^E_m, \zeta^E_m, H^E_m, K^E_m, B^E_{m,n}, \Lambda^E_m, D^E_m] \tag{5.29}
$$

where

$$C_m^E = \frac{1}{C}B_m - \frac{3}{2}(\bar\lambda i\gamma_5\gamma_m\lambda)$$

$$\zeta_m^E = i\gamma_5\frac{1}{C}(D_m\zeta - \gamma_m \not{D}\zeta) + \frac{2}{C}B_m i\gamma_5\lambda - \frac{3}{2C}\not{D}C\gamma_m\lambda - 6(\bar\lambda\lambda)\gamma_m\lambda - \frac{3}{2C}\not{B}\gamma_m i\gamma_5\lambda$$

$$H_m^E = 3(\bar\lambda i\gamma_5\lambda)\frac{1}{C}D_mC + (\bar\lambda\frac{1}{C}D_m\zeta) + \frac{1}{2}(\bar\lambda\gamma_m\frac{1}{C}\not{D}\zeta) - (\bar\lambda\lambda)\frac{1}{C}B_m$$

$$K_m^E = 3(\bar\lambda\lambda)\frac{1}{C}D_mC - 2(\bar\lambda\frac{i\gamma_5}{2C}D_m\zeta) + (\bar\lambda\gamma_m\frac{i\gamma_5}{2C}\not{D}\zeta) + (\bar\lambda i\gamma_5\lambda)\frac{1}{C}B_m$$

$$\begin{aligned}
B_{m,n}^E = {} & 3\frac{1}{C}D_pC(\bar\lambda\gamma_n\gamma_p\gamma_m\lambda) - 2(\bar\lambda\gamma_n\frac{i\gamma_5}{2C}D_m\zeta) + 3(\bar\lambda\gamma_m\frac{i\gamma_5}{2C}D_n\zeta) + \\
& - 2(\bar\lambda\gamma_n\gamma_m\frac{i\gamma_5}{2C}\not{D}\zeta) - \frac{i}{C}\varepsilon_{mnpq}D_pB_q - \frac{1}{C}D_mD_nC - \frac{1}{C}B_m(\bar\lambda i\gamma_5\gamma_n\lambda) + \\
& + \frac{1}{2C^2}B_mB_n + \frac{3}{2C^2}D_nCD_mC + \frac{3i}{2}\varepsilon_{amcn}\frac{1}{C^2}D_cCB_a + 9(\bar\lambda\lambda)(\bar\lambda\lambda)\delta_{mn} + \\
& + \frac{1}{C}\Box C\delta_{mn} - \frac{3}{4C^2}(D_aC)^2\delta_{mn} - \frac{3}{4C^2}(B_a)^2\delta_{mn} - 3(\bar\lambda\frac{1}{C}\not{B}i\gamma_5\lambda)\delta_{mn} + \\
& - 2i\widetilde{R}_{mn}(A) + 2(R_{mn}(Q)\lambda)
\end{aligned}$$

$$\begin{aligned}
\Lambda_m^E = {} & (\bar\lambda\lambda)[18\lambda\frac{1}{C}D_mC - \frac{18}{C}\not{D}C\gamma_m\lambda + \frac{i\gamma_5}{C}D_m\zeta - \frac{i\gamma_5\gamma_m}{C}\not{D}\zeta - \frac{6}{C}B_m i\gamma_5\lambda] + \\
& + (\bar\lambda i\gamma_5\lambda)[-\frac{1}{C}D_m\zeta + \gamma_m\frac{1}{C}\not{D}\zeta] + (\bar\lambda i\gamma_5\gamma_p\lambda)[-2\gamma_p\frac{1}{C}D_m\zeta + 2\gamma_p\gamma_m\frac{1}{C}\not{D}\zeta + \\
& + 3\gamma_m\frac{1}{C}D_p\zeta - \frac{3}{C}\not{D}\zeta\delta_{mp}] - \frac{5}{C}B_m i\gamma_5\frac{1}{C}\not{D}C\lambda + \frac{3}{C}D_mCi\gamma_5\frac{1}{C}\not{B}\lambda + \\
& + \frac{3i\gamma_5}{C^2}\not{D}C\not{B}\gamma_m\lambda + \frac{1}{C}\not{B}\frac{1}{C}D_m\zeta - \frac{1}{C}\not{B}\gamma_m\frac{1}{C}\not{D}\zeta - \frac{3}{2C}B_a\gamma_m\frac{1}{C}D_a\zeta + \\
& + \frac{1}{C}B_m\frac{1}{C}\not{B}\lambda + \frac{1}{C}B_m\frac{1}{C}\not{D}\zeta + \frac{3}{C}D_mC\frac{1}{C}\not{D}C\lambda - \frac{3}{C^2}(D_aC)^2\gamma_m\lambda + \\
& + \frac{i\gamma_5}{C}\not{D}C\frac{1}{C}D_m\zeta + \frac{i\gamma_5}{2C}\not{D}C\gamma_m\frac{1}{C}\not{D}\zeta - \frac{3}{2}i\gamma_5\gamma_m\frac{1}{C}D_aCD_a\zeta + \frac{1}{C}\Box C\gamma_m\lambda + \\
& - \frac{1}{C}D_m\not{D}C\lambda + [(R_{ab}(Q)\lambda) - i\widetilde{R}_{ab}(A)]\gamma_a\gamma_b\gamma_m\lambda - \frac{i\gamma_5}{C}D_m\not{B}\lambda + \\
& + \frac{i\gamma_5}{C}D_aB_b\gamma_a\gamma_m\gamma_b\lambda + i\gamma_5\gamma_n\lambda R_{nm}(A) + i\gamma_n\lambda\widetilde{R}_{nm}(A) - \frac{2i\gamma_5}{C}D_m\not{D}\zeta + \\
& + \frac{i\gamma_5}{C}\gamma_m\not{D}\not{D}\zeta + \frac{i\gamma_5}{C}\not{D}D_m\zeta
\end{aligned}$$

$$\begin{aligned}
D_m^E = {} & -\frac{12}{C}B_m(\bar\lambda\lambda)(\bar\lambda\lambda) + \frac{1}{2C^3}(B_a)^2B_m - \frac{3}{C^2}B_m(\bar\lambda i\gamma_5\not{B}\lambda) + \\
& + \frac{9}{C^2}(D_aC)^2(\bar\lambda i\gamma_5\gamma_m\lambda) + -\frac{5}{2C^3}B_m(D_aC)^2 \\
& + \frac{3}{C^3}D_mC(B_aD_aC) - \frac{9}{C}D_mC(\bar\lambda i\gamma_5\frac{1}{C}\not{D}C\lambda) + \frac{5}{2C^2}D_aCD_aB_m + \\
& - \frac{3}{2C^2}D_aCD_mB_a - (\bar\lambda i\gamma_5\gamma_a\lambda)i\varepsilon_{cdma}\frac{1}{C}D_cB_d - \frac{i}{2}\varepsilon_{abcm}\frac{1}{C^2}B_aD_bB_c + \\
& + 3(\bar\lambda\frac{1}{C}\not{D}\zeta)\frac{1}{C}D_mC - 6(\bar\lambda\gamma_m\frac{1}{C}D_a\zeta)\frac{1}{C}D_aC - 6(\bar\lambda\lambda)(\bar\lambda\frac{1}{C}[D_m\zeta - \gamma_m\not{D}\zeta]) +
\end{aligned}$$

$$
\begin{aligned}
&+3(\overline{\lambda}\frac{1}{C}\not{D}C\frac{1}{C}D_m\zeta)+\frac{1}{C}B_m(\overline{\lambda}\frac{i\gamma_5}{C}\not{D}\zeta)-(\overline{\lambda}\frac{i\gamma_5}{C^2}\not{B}[D_m\zeta-\gamma_m\not{D}\zeta])+\\
&-\frac{1}{4C^2}(\overline{\not{D}\zeta}i\gamma_5\gamma_m\not{D}\zeta)+\frac{3}{4C^2}(\overline{D_a\zeta}i\gamma_5\gamma_m D_a\zeta)-\frac{1}{2C^2}(\overline{D_m\zeta}i\gamma_5\not{D}\zeta)+\\
&-\frac{1}{C}\Box B_m+(\overline{\lambda}[\gamma_m\delta_{ab}-\gamma_b\delta_{am}]\frac{1}{C}D_aD_b\zeta)-\frac{1}{8}(\overline{\lambda}\gamma_m\gamma_a\gamma_b\frac{1}{C}[D_a,D_b]\zeta)+\\
&+[-2\delta_{pm}\delta_{ab}+3\delta_{am}\delta_{pb}-\delta_{bm}\delta_{ap}](\overline{\lambda}i\gamma_5\gamma_p\lambda)\frac{1}{C}D_aD_bC+\\
&+[-\frac{3}{2}\delta_{bm}\delta_{ac}+\frac{1}{2}\delta_{am}\delta_{bc}+\frac{1}{2}\delta_{ab}\delta_{cm}]\frac{1}{C}B_a\frac{1}{C}D_bD_cC-\frac{i}{C}B_a\widetilde{R}_{am}(A)+\\
&+\frac{3}{2C}D_aCR_{am}(A)-\frac{1}{2}(R^{cov}_{bm}(S)i\gamma_5\gamma_b\lambda)+2i(\overline{\lambda}i\gamma_5\gamma_a\lambda)\widetilde{R}_{am}(A)+\\
&-\frac{7}{2}(R_{am}(Q)i\gamma_5\lambda)\frac{1}{C}D_aC+(R_{am}(Q)\lambda)\frac{1}{C}B_a-\frac{1}{2}(R_{am}(Q)\frac{1}{C}D_a\zeta)
\end{aligned}
$$

The next step is to identify the multiplet L between those present in the lagrangian. There are two possibilities:

i) L is the compensator

ii) L is a matter field.

At a first sight it seems that the second choice is not possible in the stringy case because there aren't linear multiplets; instead there is a linear multiplet, it is the axion–dilaton multiplet. Indeed we have defined the S field by means of a dualization of $H_{\mu\nu\rho}$, but in supergravity the dual of a chiral multiplet is a linear multiplet so that, naturally, the axion–dilaton multiplet is the only one linear multiplet in a string effective supergravity theory.

From considerations on string field theories, W. Siegel [59] has argued that the compensator is a chiral multiplet, so that only the second possibility is left. In this case, if we construct the gravitational curvatures with the axion–dilaton multiplet and then we fix the gauge on it, the physical degrees of freedom of this multiplet flow into the compensator components. Now, to recast the Poincarè supergravity theory so obtained in the canonical form, we would have to do some non trivial field redefinitions.

The important fact is that this theory, in which all matter multiplets are chiral, is in the "New" minimal formulation, i.e. the auxiliary fields of the gravity multiplet are a vector B_μ and an antisymmetric tensor $a_{\mu\nu}$. Moreover, the $U(1)$–R–symmetry is present because it is related to the axion–dilaton stringy scaling law, and the "New" minimal formulation describes this symmetry in a more natural way. Thus, it seems that string theories prefer the "New" minimal formulation of N=1 D=4 supergravity.

These considerations must be handled with caution because just a few of them have been checked explicitly[41]. It is still possible that the true "off–shell" structure of the effective supergravity theory emerging from superstrings is much more complicated and still not discovered.

[41] See for example refs. [56,59].

CHAPTER 6

Relations Between World–Sheet and Spacetime Properties of String Theories

In this chapter we want to introduce a more general approach to the study of the background properties of string theories. The main idea is to find the relations between the conformal properties of the two dimensional theory and the symmetries and invariances of the spacetime modes. We consider D=4 superstrings with N=1 supersymmetry constructed, as explained in chapter two, in such a way that the total central charge is zero, unitarity and modular invariance are preserved [11,60,61]. These models can be thought to be the direct product of a free (heterotic) superstring theory on a D=4 flat Minkowski background times a compact [61], $c = 9$ internal conformal field theory.

These conformal theories, with at least (0,1) superconformal invariance, are **classical** vacuum states for the heterotic superstring; the desirable phenomenological properties should be formulated as constraints on the two dimensional superconformal field theories. The most interesting results have been obtained using arguments based on the properties of the effective lagrangians [62,63,64]. The effective theory of the heterotic superstring is a N=1 D=4 supergravity theory with some peculiar properties. It is interesting to discover how these properties are realized in the 2d conformal theory; notice that spacetime arguments apply only to the massless modes of the string while the conformal ones to the whole string theory. In this way one should be able to promote the "perturbative" effective results to "non–perturbative" ones, i.e. one should discover the "true" classical vacuum of the string theory.

§6.1 Spacetime and World–Sheet Supersymmetry

We have already stressed the importance of preserving one spacetime supersymmetry in a classical superstring ground–state. It is of great interest to have a simple criterion for checking whether a 2d conformal field theory has this property. A necessary and sufficient criterion is the existence of a N=2 supersymmetry current algebra on the world–sheet, plus a charge quantization condition on the $U(1)$ current contained in this algebra [61,12].

The heterotic string is given by the product of a supersymmetric string on the right sector and a non–supersymmetric string on the left one. The supersymmetry charges are constructed only from the right sector, then we will consider only the superconformal part of the heterotic string.

Witten and Hull [65] showed that in any spacetime supersymmetric classical vacuum of a D=4 heterotic string described by a non–linear sigma model, the local N=1 superconformal invariance of the two dimensional (world–sheet) field theory extends to a **global** N=2 superconformal invariance.

We will not prove that (0,2) world–sheet supersymmetry ensures the existence of N=1 spacetime supersymmetry (see refs. [12,61]), but we now review the construction of the conserved spacetime supersymmetry charge.

The construction is independent of the details of the theory, it is sufficient to have an heterotic string which, on the right sector, is made by a D=4 N=1 superstring theory plus an N=2 superconformal theory with $\widehat{c} = 6$. All what is needed to know of the internal sector is the N=2 superconformal algebra.

The N=2 superconformal algebra is [66]

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{1}{4}\widetilde{c}(m^3 - m)\delta_{m,-n} \qquad (6.1)\\
[L_m, G^i_n] &= \left(\frac{1}{2}m - n\right) G^i_{m+n}\\
[L_m, T_n] &= -nT_{m+n}\\
[T_m, T_n] &= \widetilde{c}m\delta_{m,-n}\\
[T_m, G^i_n] &= i\epsilon^{ij}G^j_{m+n}
\end{aligned}$$

$$[G^i_m, G^j_n] = 2\delta^{ij}L_{m+n} + i\epsilon^{ij}(m-n)T_{m+n} + \tilde{c}\left(m^2 - \frac{1}{4}\right)\delta^{ij}\delta_{m,-n}$$

where the Virasoro generators L_m are the Fourier coefficients of the traceless stress energy tensor, T_m are the coefficients of the $U(1)$ current algebra $J(z) = \sum_m T_m z^{-m-1}$, and G^i_m (i=1,2) are the coefficients of the fermionic partner fields $G^i(z) = \sum_m G^i_m z^{-m-\frac{3}{2}}$ which complete the N=2 super stress–energy tensor ($\tilde{c} = c/3 = \hat{c}/2$).

The next requirement is that the $U(1)$ current $J(z)$ can be expressed as $J(z) = i\sqrt{3}\partial_z H(z)$, where $H(z)$ is a canonically normalized free scalar field, i.e. $H(z)H(w) = -\log(z-w)$ (this requirement is equivalent to a charge quantization condition on the $U(1)$ current).

Now define $\Sigma(z) = e^{i\sqrt{3}H/2}$ and $\Sigma^\dagger(z) = e^{-i\sqrt{3}H/2}$, the N=1 spacetime supersymmetry currents are

$$\begin{aligned} V^\alpha(z) &= e^{-\varphi/2}\mathcal{S}_\alpha\Sigma(z) \\ V^{\dot\alpha}(z) &= e^{-\varphi/2}\mathcal{S}_{\dot\alpha}\Sigma(z) \end{aligned} \tag{6.2}$$

where $e^{-\varphi/2}$ is a spin field for the (β,γ) superconformal ghosts system, $\mathcal{S}_\alpha$ and $\mathcal{S}_{\dot\alpha}$ are the spin fields for the (free) world–sheet fermions ψ^μ with four dimensional Minkowsky indices. Then, the spacetime supersymmetry charges are

$$\begin{aligned} Q_\alpha &= \oint dz V^\alpha(z) \\ Q_{\dot\alpha} &= \oint dz V^{\dot\alpha}(z) \, . \end{aligned} \tag{6.3}$$

Following the same line of reasoning , it has been shown [61] that N=2 spacetime supersymmetry implies N=4 world–sheet supersymmetry and the internal algebra splits into a piece with $\hat{c} = 4$ (and N=4 superconformal invariance), and a piece with $\hat{c} = 2$ constructed from two free dimension–$\frac{1}{2}$ superfields. N=4 spacetime supersymmetry requires that the entire $\hat{c} = 6$ be represented by six free superfields.

§6.2 Non–Renormalization Theorems

Since it is very difficult to obtain informations about the background properties of a string theory studying directly its microscopic, 2d conformal field theory, and, as we have seen in the last chapter, on the other side the semiclassical effective theory approach is very powerful, one can study the properties of the background effective actions and try to relate them to the 2d theory. In a second time, one should obtain the same results only in the 2d conformal field theory ambitus.

The generalization from the background to the world–sheet is not so obvious (besides the technical difficulties) because the results stated with the effective theory formalism are true only for the massless modes of the string and in a perturbative contest. Instead the results obtained in a microscopic 2d approach should be true in the whole string theory (consider for example the case of the spacetime supersymmetry discussed in the previous paragraph).

We consider now a N=1 D=4 supergravity theory obtained as the effective theory of a D=4 superstring model constructed as a (0,2) supersymmetry, sigma–model–compactification conformal theory (see for example (2.19)). A priori, the D=4 supergravity theory contains an infinite number of higher derivative terms coming from the string loop contributions, the α' expansion and the integration of the massive modes on the compact, six–dimensional space.

But, as we have argued in the last chapter, these higher derivative terms **should not** modify the general properties of the standard supergravity theory. Thus one can start considering the standard supergravity formulation, i.e. the effective field theory for the string massless modes at zero string–loop, at the lowest order in the expansion in α' and in the sigma model perturbation theory. The sigma model coupling constant is $1/r$ (r is the "radius" of the compact K_6 manifold), we assume that r is large and we look for (0,2) superconformal sigma models in an expansion in $1/r$. We can try, then, to compute the exact supergravity effective theory perturbatively in these three parameters.

Let us first consider the sigma model perturbation theory. The Calabi–Yau solution is a solution at the leading order for every r. In fact, we have seen that r shows up in D=4 as a massless scalar whose vacuum expectation value describes the size of the compact space; we have called this field $\Re e T$. Its vertex operator in the NSR formalism is

$$V_r(k) = \int d^2\sigma g_{i\bar{i}}(X)\bar{\partial}X^{\bar{i}}\left[\partial X^i + i(k\cdot\psi)\psi^i\right]e^{ik\cdot y} + \ i \leftrightarrow \bar{i} \tag{6.4}$$

where $i,\ \bar{i} = 1,\ldots,3$, X labels the six compact dimensions, y labels the four non–compact dimensions and ψ's are the ten right moving NRS fermions. The supersymmetry partner of r is a massless pseudoscalar b which originates from B_{MN}, previously we have called it $\Im mT$. Its vertex operator is

$$V_b(k) = \int d^2\sigma b_{i\bar{i}}(X)\bar{\partial}X^{\bar{i}}\left[\partial X^i + i(k\cdot\psi)\psi^i\right]e^{ik\cdot y} - i \leftrightarrow \bar{i} \tag{6.5}$$

These two fields form the first component of the chiral superfield T, $T = r + ib$. Since the (1,1) form $b_{i\bar{i}}$ is closed, at zero momentum the b vertex operator is a total derivative. However, since this form is not exact, this term does not always vanish. It vanishes whenever the world–sheet is mapped to a topologically trivial two–surface in the compact space. All configurations in the sigma model perturbation theory are topologically trivial and therefore the zero momentum mode of b decouples from all correlation functions to all orders in sigma model perturbation theory. Hence, we have obtained from the properties of the conformal theory the result that the four dimensional effective lagrangian is invariant under the Peccei–Quinn symmetry $b \to b +$ const. [64].

This symmetry guarantees that only derivatives of the b field appear in the effective lagrangian. Therefore the spacetime superpotential g is independent of $b = \Im mT$. Since the superpotential is an **analytic** function of the superfields, it is also independent of the whole superfield T.

The fact that g is independent of T means that it does not depend on r. But r is the coupling constant of the sigma model (and the Peccei–Quinn symmetry is true to all orders in the sigma–model perturbation theory), therefore g is given by the leading order result. In other words, since g is independent of the sigma–model coupling constant, it is not renormalized by the sigma–model loops. This is the first non–renormalization theorem [62,64,62].

The spacetime superpotential plays an important role in supersymmetric theories and the fact that it is not renormalized has many important consequences:

i) if supersymmetry is unbroken at the leading order, it is unbroken to all orders in this expansion (up to the presence of Fayet–Iliopulos D–terms);

ii) the cosmological constant is zero to all orders;

iii) the states which are massless at the classical level remain massless to all orders. This includes both the gauge bosons and matter fields. Clearly, the low–energy gauge group cannot be broken by radiative corrections.

These results are true for the massless modes of the string if one assumes that the field theory effective lagrangian is given by a N=1 D=4 **standard** supergravity. We **assume** that the higher order terms do not give corrections to these results.

The last expansion is in the string loops. Also in this case a non–renormalization theorem for the spacetime superpotential can be formulated. The proof is very similar to the previous one [67,62] but applied to the axion–dilaton multiplet. In fact, the real part of S is the dilaton, and we know that it plays the role of the string coupling constant. The axion $a \sim B_{\mu\nu}$ is the imaginary part of S. Since the axion exactly decouples at zero momentum, it satisfies a Peccei–Quinn symmetry to all orders in string perturbation theory. This means that the superpotential g does not depend on S and then is not renormalized in string perturbation theory.

This non–renormalization theorem obviously has the same consequences on the cosmological constant, supersymmetry, masses and gauge group as the previous one. It is worth to notice that both these two Peccei–Quinn symmetries can be broken by world–sheet non perturbative phenomena. Consider for example the case of the first theorem. Non–perturbatively the world–sheet can be mapped in a topologically non–trivial two–surface in the compact space. Then the b field does not decouple at zero momentum and the Peccei–Quinn symmetry is explicitly broken [64,68]. Holomorphic world–sheet istantons do renormalize the superpotential and lead to a dependence of g on T of the form $g = e^{-\mathrm{const}\cdot T}$ [64,62]. Such a superpotential can trigger spontaneous supersymmetry breaking and destabilize the vacuum. In ref. [52] we showed that the consequences for the effective supergravity theory of the holomorphic istantons renormalization of the superpotential are quite the same as assuming the existence of a gluino condensate.

The next steps in the study of the properties of the string background structures should be of obtaining these results directly with the two dimensional conformal theory approach. Some work in this direction has already been made, see for example ref. [61,69,70].

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