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APPENDIXES

INTRODUCTION

In this work the origin of the conformal anomaly which arises in the Polyakov functional quantization of string theories is discussed.

In particular the critical dimension d=26 for the bosonic string is computed and a preliminary result, which suggests the absence of conformal anomaly in the Green-Schwarz formulation of the heterotic string, is presented.

The plan of the work is the following:
in the first part we introduce the generalized Riemann zeta function as a way to regularize the determinants of certain differential operators. The connection of zeta function with heat kernel is exploited.

In sect. 1.3 formulas are derived which relate heat kernel and zeta function to the trace anomaly.

In sect. 1.4 heat kernel is used in order to establish an index theorem.
The results of the first chapter are mostly based on the

following references [1÷4]

In ch. 2 a method is presented which allows us to evaluate the trace anomaly for the operators we are interested in. It is basically a first order perturbative computation of a Schrödinger-like equation.

In ch. 3 we review the Polyakov prescription for the quantization of the bosonic string following as standard references [5÷11]

The results of ch. 1 and 2 are then used in order to derive the Riemann-Roch theorem and the critical dimension for the bosonic string.

In the final section the Green-Schwarz formulation of the heterotic string is introduced; then we compute the conformal anomaly for an operator which appears in the quantization of this theory. The relevance of the result in proving the conformal invariance of the heterotic string is discussed.

CHAPTER 1

THE HEAT KERNEL METHOD AND THE ZETA FUNCTION REGULARIZATION

1.1 Introductory remarks and notations

We will describe a method which enables us to give a precise meaning to the partition function for theories formulated on a curved space-time background or, equivalently stated, for evaluating the determinants of certain differential operators.

In order to manage with well-defined mathemathical objects we will work in the euclideanized version of the path integral. Moreover, in order to make use of the spectral properties of compact linear operators,[12] we will work with compact manifolds (since we have in mind to apply our results to the closed strings the manifolds we consider are also assumed to be boundaryless and oriented).

Let $g_{ab}(x)$ be a riemannian metric (we think of it as an external background) for a manifold M .

The real fields ϕ over M ($\phi : M \to \mathbb{R}$) belong to a Hilbert space whose scalar product is given by:

$$\langle \phi | \psi \rangle \underset{def.}{=} \int_M dx \sqrt{g(x)}\, \phi(x) \psi(x) \qquad g(x) = \det g_{ab} \tag{1.1}$$

The eigenstates of the positions are the generalized vectors $|y\rangle$ such that

$$\forall |\phi\rangle , \qquad \langle y | \phi \rangle = \phi(y)$$

Then

$$|y\rangle \equiv \psi(x) = \delta(x,y) \frac{1}{\sqrt{g(y)}} \tag{1.2}$$

The following relations hold:

$$\langle y | z \rangle = \delta(y,z) \frac{1}{\sqrt{g(z)}} \quad \text{(orthonormality condition)} \tag{1.3}$$

$$\int dx \sqrt{g(x)} |x\rangle\langle x| = \mathbb{1} \qquad \text{(completeness relation)} \tag{1.4}$$

Let Ω be an operator ($\Omega : \phi \mapsto \phi'$) which admits a complete set of proper eigenstates ϕ_n :

$$\Omega\, \phi_n = \lambda_n \phi_n$$

$$\Sigma_n |\phi_n\rangle\langle\phi_n| = \mathbb{1} \qquad\qquad \langle \phi_n | \phi_m \rangle = \delta_{nm}$$

The operator Ω will be specified by giving all its matrix elements

$$\langle \phi | \Omega | \psi \rangle \quad \forall \phi, \psi \quad \text{or} \quad \langle x | \Omega | y \rangle \quad \forall x, y .$$

Let $\quad \langle x | \Omega | \psi \rangle = \Omega(x) \psi(x)$

Then
$$\langle x | \Omega | y \rangle \underset{def}{=} \Omega(x,y) = \Omega(x)\, \delta(x,y) \frac{1}{\sqrt{g(y)}} = \Omega(x) \langle x | y \rangle \tag{1.5}$$

$Tr\,\Omega$ is defined as follows:

$$Tr\,\Omega = \Sigma_n \lambda_n = \Sigma_n \langle \phi_n | \Omega | \phi_n \rangle = \qquad (1.6)$$
$$= \int dx \sqrt{g(x)} \langle x | \Omega | x \rangle$$

A generic state $|\phi\rangle$ can be expressed through its mode expansion as $\quad |\phi\rangle = \Sigma_n c_n |\phi_n\rangle \qquad (1.7)$

Let us consider now the following partition function:

$$Z(g) = e^{-\frac{1}{\hbar}W(g)} = \int \mathcal{D}\phi(x)\, e^{-\frac{1}{\hbar}\int dx \sqrt{g(x)}\, \phi(x)\, \Omega(x)\, \phi(x)} \qquad (1.8)$$

For the moment the eigenvalues of the operator Ω are supposed to be strictly positive $\left(\Omega \phi_n = \lambda_n \phi_n \ \forall_n \lambda_n > 0 \right)$

In a D -dimensional space we can fix the mass dimension of $\phi_n(x)$ and $\Omega(x)$ to be:

$$[\phi_n(x)] = \frac{D}{2} \qquad\qquad [\Omega(x)] = 2$$

Then the mass dimension of $\phi(x),\ \lambda_n,\ c_n$ will be:

$$[\phi(x)] = \frac{D}{2} - 1 \quad , \quad [\lambda_n] = 2 \qquad\qquad [c_n] = -1$$

We define the measure $\mathcal{D}\phi(x)$ through the mode expansion of $\phi(x)$, following a procedure which is similar to the one introduced by Fujikawa [13] for computing the chiral anomaly

$$\int \mathcal{D}\phi(x) \underset{\text{def.}}{=} \Pi_n \left(\mu\, dc_n \right) \qquad (1.9)$$

An arbitrary massive factor μ must be inserted in order to have a dimensionless measure.

Naively the functional integral will be given by:

$$Z(g) = \prod_n \left(\mu \int dc_n \, e^{-\frac{\lambda_n c_n^2}{\hbar}} \right) \tag{1.10}$$

Since we know that $\displaystyle\int_{-\infty}^{+\infty} dx \, e^{-\alpha x^2} = \sqrt{\frac{\pi}{\alpha}}$ (1.11)

then

$$\begin{aligned} Z(g) &= \prod_n \left(\mu \sqrt{\frac{\pi\hbar}{\lambda_n}} \right) = \prod_n \left(\frac{\mu^2 \pi \hbar}{\lambda_n} \right)^{\frac{1}{2}} = \\ &= \left[\prod_n \frac{\lambda_n}{\mu^2 \pi \hbar} \right]^{-\frac{1}{2}} = \left(\det \frac{\Omega}{\pi \hbar \mu^2} \right)^{-\frac{1}{2}} \end{aligned} \tag{1.12}$$

So, in order to give a precise meaning to the partition function we have introduced, we need a prescription which allows us to manage with the product of an infinite number of eigenvalues. The presence of the arbitrary factor μ is unavoidable and this simply reflects the arbitrariness to the choice of the normalization constant for the path integral.

To make sense of expressions like 1.12 we will use the zeta-function technique. Before doing this we just mention another approach which is worth to know because it is widely used in literature,[6] and which turns out to be equivalent to the zeta-function technique. This is the so-called proper-time regularization.

It is based on the following representation of the logarithm function:

$$\log x = -\int_0^{+\infty} \frac{dt}{t}\left(e^{-tx} - e^{-t}\right) \qquad (1.13)$$

The determinant of a finite dimensional operator A can be expressed then as:

$$\log \det A = -\int_0^{+\infty} \frac{dt}{t}\left(e^{-tA} - e^{-t\mathbb{1}}\right) \qquad (1.14)$$

The determinant of the ∞ -dimensional operator Ω we are considering can therefore be regularized by introducing a cutoff ε as follows:

$$\log \det_\varepsilon \Omega = -\mathrm{Tr}\int_\varepsilon^{+\infty} e^{-t\Omega}\frac{dt}{t} = -\sum_i \int_\varepsilon^{+\infty} e^{-t\lambda_i}\frac{dt}{t} \qquad (1.15)$$

We point out that the presence of the cutoff ε spoils the scale invariance one naively expects from the (formally infinite) expressions like $F(x) = \int_0^{+\infty} e^{-tx}\frac{dt}{t}$

A drawback of this method is the fact that in order to remove the cutoff dependance of the results one needs to implement a renormalization prescription.

1.2 Heat equation and zeta function

In this section we will consider only non-negative, self-adjoint, elliptic operators (for a definition of elliptic operator see [14]).

To such an operator Ω we can associate the following equation (the heat equation):

$$\frac{\partial}{\partial \tau} G_\Omega(x,y,\tau) = -\Omega(x)\, G_\Omega(x,y,\tau) \tag{1.16}$$

with the boundary condition

$$\lim_{\tau \to 0^+} G_\Omega(x,y,\tau) = \delta(x,y)\frac{1}{\sqrt{g(y)}} \tag{1.17}$$

The heat kernel $G_\Omega(x,y,\tau)$ represents the diffusion in parameter time τ of a unit quantity of heat (or ink) placed at the point y at $\tau=0$.

The equation can be formally solved by writing down:

$$G_\Omega(x,y,\tau) = \sum_n e^{-\lambda_n \tau}\phi_n(x)\,\phi_n(y) = e^{-\tau\Omega(x)}\delta(x,y)\frac{1}{\sqrt{g(y)}} \tag{1.18}$$

Otherwise expressed is

$$G_\Omega(x,y,\tau) = \langle x| e^{-\tau\Omega} |y\rangle \tag{1.19}$$

since for a generic function $f(\Omega)$ is

$$\langle x| f(\Omega) |y\rangle = f(\Omega(x))\,\delta(x,y)\frac{1}{\sqrt{g(y)}} \tag{1.20}$$

It is obvious to define:

$$\mathrm{Tr}\, G_{\Omega}(x,y,\tau) \underset{\mathrm{def.}}{=} \mathrm{Tr}\, e^{-\tau\Omega} = \sum_n e^{-\tau\lambda_n} \tag{1.21}$$

It follows:

$$\mathrm{Tr}\, G_{\Omega}(x,y,\tau) = \int dx \sqrt{g(x)} \langle x | e^{-\tau\Omega} | x \rangle = \int dx \sqrt{g(x)}\, G_{\Omega}(x,x,\tau) \tag{1.22}$$

We introduce now the generalized Riemann zeta function. To the non-negative operator Ω we are considering we can associate the following function

$$\zeta_{\Omega}(s) = \sum_n{}' \frac{1}{\lambda_n^s} \tag{1.23}$$

with the sum taken over the positive eigenvalues only. The sum is convergent for large $\mathrm{Re}\, s > 0$; for other s, $\zeta_{\Omega}(s)$ must be defined by means of analytic continuation. One can prove that $\zeta_{\Omega}(s)$ is a meromorphic function and in particular analytic at $s = 0$.

The usual Riemann zeta function ζ_R is defined for the eigenvalues of the harmonic obscillator $\zeta_R = \sum_n \frac{1}{n^s}$ and it is convergent for $\mathrm{Re}\, s > 1$.

For Ω the primed determinant $\det{}'\Omega$ built up with its positive eigenvalues only, is formally given by

$$\det{}'\Omega = \prod_n{}' \lambda_n , \quad \ln \det{}'\Omega = \sum_n{}' \ln \lambda_n \tag{1.24}$$

For $\zeta_{\Omega}(s)$ we can write

$$\frac{d}{ds}\zeta_{\Omega}(s) = \frac{d}{ds}\sum_n{}' e^{-s\ln\lambda_n} = -\sum_n{}' \ln\lambda_n\, e^{-s\ln\lambda_n} \tag{1.25}$$

At a formal level we have $\zeta'_{\Omega}(0) = -\ln \det{}'\Omega$

Therefore, since the definition of ζ_{Ω} is such that at $s=0$, ζ_{Ω} is analytical, it makes sense to regularize $\det{}'\Omega$ by defining

$$\ln \det{}'\Omega \underset{def.}{=} -\frac{d}{ds}\zeta_{\Omega}(s)\Big|_{s=0} \qquad (1.26)$$

We want now to make a connection between zeta function and heat kernel by expressing $\zeta_{\Omega}(s)$ in terms of $G_{\Omega}(x,y,\tau)$ or, more specifically, of its trace.

Let $\hat{\Omega}$ be the operator obtained by restricting the action of Ω to the subspace $(\mathrm{Ker}\,\Omega)^{\perp}$

Let $\Pi(\mathrm{Ker}\,\Omega)$ be the projector over $\mathrm{Ker}\,\Omega$.

We remember that the operators we are considering are Fredholm operators, havin a finite-dimensional kernel.

It is:

$$\frac{1}{\hat{\Omega}^{s}} = \int_0^{+\infty} d\tau\, e^{-\tau\hat{\Omega}} = \sum_n{}' \frac{1}{\lambda_n^s} |\phi_n\rangle\langle\phi_n| \qquad (1.27)$$

Therefore

$$\mathrm{Tr}'\frac{1}{\hat{\Omega}^s} = \sum_n{}' \langle\phi_n|\frac{1}{\Omega^s}|\phi_n\rangle = \sum_n{}' \frac{1}{\lambda_n^s} = \zeta_{\Omega}(s) \qquad (1.28)$$

We make use of the Mellin transform

$$\int_0^{+\infty} d\tau\, \tau^{s-1} e^{-\tau\hat{\Omega}} = \frac{1}{\hat{\Omega}^s}\int_0^{+\infty} d\tau\, e^{-\tau}\tau^{s-1} \qquad (1.29)$$

(to understand this relation simply applies both sides to an eigenvector $|\phi_n\rangle$) in order to write

$$\frac{1}{\hat{\Omega}^s} = \frac{1}{\Gamma(s)}\int_0^{+\infty} d\tau\, \tau^{s-1} e^{-\tau\hat{\Omega}} \qquad (1.30)$$

$\Gamma(s)$ is here the Gamma-function which is given for $Re\, s > 0$ by:

$$\Gamma(s) = \int_0^{+\infty} d\tau \, \tau^{s-1} e^{-\tau} \tag{1.31}$$

We can write

$$\begin{aligned} \zeta_\Omega(s) &= Tr' \frac{1}{\hat{\Omega}^s} = \sum_m' \langle \phi_m | \frac{1}{\Gamma(s)} \int_0^{+\infty} d\tau \, \tau^{s-1} e^{-\tau\hat{\Omega}} | \phi_m \rangle = \\ &= \sum_n \langle \phi_m | \frac{1}{\Gamma(s)} \int_0^{+\infty} d\tau \, \tau^{s-1} \left[e^{-\tau\Omega} - \pi(Ker\,\Omega) \right] | \phi_m \rangle \end{aligned} \tag{1.32}$$

The final expression is

$$\zeta_\Omega(s) = \frac{1}{\Gamma(s)} \int dx \sqrt{g(x)} \int_0^{+\infty} d\tau \, \tau^{s-1} \left[G_\Omega(x,x,\tau) - \dim(Ker\,\Omega) \right] \tag{1.33}$$

We end this section with a remark: an operator $\Omega_{\mathbb{C}}$ acting on complex fields can be thought obtained from the "doubling" of a corresponding operator $\Omega_{\mathbb{R}}$ acting on real fields. Naively one expects $\det \Omega_{\mathbb{C}} = (\det \Omega_{\mathbb{R}})^2$ (1.34)

The zeta function regularization is consistent with this requirement.

1.3 Connection of zeta function with trace anomaly.

In this section we show the connection of the zeta-function to the trace anomaly.

If the operator Ω is associated with $\zeta_{\Omega}(s)$, the operator $\alpha\Omega$ (α is a constant >0) is associated with

$$\zeta_{\alpha\Omega}(s) = \frac{1}{\alpha^s}\zeta_{\Omega}(s) \qquad (1.35)$$

It follows

$$\ln \det{}'\alpha\Omega = -\frac{d}{ds}\left(\zeta_{\alpha\Omega}(s)\right)\Big|_{s=0} = \ln\det{}'\Omega + \ln\alpha\,\zeta_{\Omega}(0) \qquad (1.36)$$

We remark that $\zeta_{\alpha\Omega}(0) = \zeta_{\Omega}(0)$ (1.37)

This relation, as we will see, has the consequence that trace anomaly does not depend on the arbitrary parameter μ which normalizes the path integral.

The following classical action

$$S = \frac{1}{2}\int dx \sqrt{g(x)}\, g^{ab}\partial_a\phi\,\partial_b\phi \qquad (1.38)$$

(ϕ are real bosonic scalars) is invariant under diffeomorphisms and, in $D=2$ dimensions, also under Weyl transformations ($g_{ab}\mapsto e^{\alpha(x)}g_{ab}$, ϕ unchanged).

Taking the parameter of Weyl transformations be independent of space-time we get the dilatation invariance

$$0 = \delta S = \int dx\, \delta g_{ab}\frac{\delta S}{\delta g_{ab}} = \delta\alpha\int dx\, g_{ab}\frac{\delta S}{\delta g_{ab}} \qquad (1.39)$$

We define the energy-momentum tensor $T^{ab}_{(g)}$ as

$$T^{ab}_{(g)} \underset{\text{def.}}{=} \frac{2}{\sqrt{g_{(x)}}} \frac{\delta S}{\delta g_{ab}^{(x)}} \tag{1.40}$$

The relation 1.39 , since $\sqrt{g_{(x)}}>0 \ \forall x$, implies the tracelessness of T^{ab} : $\quad T^{a}{}_{a} = 0 \qquad (1.41)$

(this relation is obtained without making use of the equations of motion).

From a quantum point of view we have the partition function

$$Z_{(g)} = e^{-\frac{1}{\hbar} W_{(g)}} = \int \mathcal{D}\phi_{(x)}\, e^{-\frac{1}{2\hbar}\int dx \sqrt{g_{(x)}}\, g^{ab} \partial_a \phi \partial_b \phi} \tag{1.42}$$

With our prescription

$$Z_{(g)} = e^{-\frac{1}{2\hbar} \ln \det' \frac{\Omega}{\mu^2}}$$

Ω is the operator

$$\Omega_{(x)} = \frac{1}{\sqrt{g_{(x)}}} \partial_a \sqrt{g_{(x)}}\, g^{ab} \partial_b \tag{1.43}$$

Then

$$W_{(g)} = -\frac{\hbar}{2} \frac{d}{ds} \zeta_{\Omega/\mu^2}^{(s)} \Big|_{s=0} \tag{1.44}$$

Under an infinitesimal dilatation we get

$$\delta W_{(g)} = \frac{\hbar}{2} \delta\alpha\, \zeta_{\Omega/\mu^2}^{(0)} = \frac{\hbar}{2} \delta\alpha\, \zeta_{\Omega}^{(0)} \tag{1.45}$$

Since $\zeta_{\Omega}^{(0)}$ as we will see later is different from zero, this relation means that the dilatation invariance is broken at the quantum level.

We now define the quantum energy-momentum tensor T^{ab} as

$$T^{ab} \underset{\text{def.}}{=} \frac{2}{\sqrt{g_{(x)}}} \frac{\delta W}{\delta g_{ab}} \tag{1.46}$$

We get $\int dx \sqrt{g(x)}\, T^{a}{}_{a} = \hbar \zeta_{\Omega}(0)$ (1.47)

The energy-momentum tensor is no longer traceless.

Until now we have just performed formal manipulations, without worrying about how to compute things. The machinery which enables us to extract informations from the operators we are interested in will be developped in the next chapter. Here however it turns out to be useful to anticipate a result we will prove later: working in 2 dimensions the heat kernel $G_{\Omega}(x,x,\tau)$ can be expanded at small τ as follows

$$G_{\Omega}(x,x,\tau) = \frac{a_{-1}}{\tau} + a_0(x) + O(\sqrt{\tau}) \tag{1.48}$$

(the first coefficient a_{-1} is independent of x).

We want to show now that the value of $\zeta_{\Omega}(s)$ at $s=0$ is related to the coefficient $a_0(x)$ appearing in the expansion above, in such a way that (almost)all the informations about the trace anomaly are encoded in this coefficient.

We start by considering $\zeta_{\Omega}(s)$ as given in the equation (1.33). At small s $[\Gamma(s)]^{-1} \approx s + \gamma s^2 + O(s^3)$

To have a non-vanishing $\zeta_{\Omega}(s)$ at $s=0$ we have therefore to look for the contribution of poles $\frac{1}{s}$ arising from the integration $\int_0^{+\infty} d\tau\, \tau^{s-1}(\dots)$

The integration $\int_0^{+\infty} d\tau(\ldots)$ can be decomposed as
$\int_0^{+\infty}(\ldots) = \int_0^{\tau_0}(\ldots) + \int_{\tau_0}^{+\infty}(\ldots)$ for an arbitrary τ_0 .
The integration $\int_{\tau_0}^{+\infty} d\tau(\ldots)$ gives us an analytic function of s ,
in such a way that the only terms containing poles can arise from integrating $\int_0^{\tau_0} d\tau(\ldots)$
If τ_0 is chosen small enough we can insert the small τ expansion for $G_\Omega(x,x,\tau)$ inside the integration.
At this point it can be easily realized that a pole is present with residue $\left[\int dx\sqrt{g(x)}\,a_0(x) - \dim(\mathrm{Ker}\,\Omega)\right]$

At the end we have:

$$\zeta_\Omega(0) = \left[\int dx\sqrt{g(x)}\,a_0(x) - \dim\left(\mathrm{Ker}\,\Omega\right)\right] \tag{1.49}$$

We must observe that we are interested in full Weyl invariance, not simply in scale invariance. To extract the dependance of the determinant of the operators under Weyl transformations some extra work and an additional hypothesis is needed.

We think of an infinitesimal transformation $\Omega \mapsto \Omega + \delta\Omega$ parametrized by the infinitesimal function $\delta\sigma(x)$. Let us suppose that under the transformation the variation of the eigenvalue λ_n is given by:

$$\delta\lambda_n = k\,\lambda_n \langle\phi_n|\,\delta\hat{\sigma}\,|\phi_n\rangle \tag{1.50}$$

where K is a constant, ϕ_n are the eigenvectors corresponding to λ_n and $\delta\hat{\sigma}|x\rangle = \delta\sigma(x)|x\rangle$

Therefore $\quad \delta\Omega = K\,\delta\hat{\sigma}\,\Omega \qquad (1.51)$

The formula one gets from this assumption can be easily applied to the case of the laplacian operators acting in string theory.

We notice that with the hypothesis done

$$\mathrm{Ker}\,\Omega = \mathrm{Ker}(\Omega+\delta\Omega) \tag{1.52}$$

After performing some manipulations we arrive at the intermedial step

$$\delta\zeta_{\Omega}(s) = -\frac{s}{\Gamma(s)}\int_0^{+\infty} d\tau\,\tau^{s-1}\left[\int dx\sqrt{g(x)}\,\delta\sigma(x)\,G_{\Omega}(x,x,\tau) - \mathrm{Tr}\,\delta\hat{\sigma}\,\Pi(\mathrm{Ker}\,\Omega)\right] \tag{1.53}$$

By inserting now the small τ expansion for the heat kernel $G_{\Omega}(x,x,\tau)$ as done before, one can write the final result:

$$\delta \ln \det{}'\Omega = K\left[\int dx\sqrt{g(x)}\,\delta\sigma(x)\,a_0(x) - \mathrm{Tr}\,\delta\hat{\sigma}\,\Pi(\mathrm{Ker}\,\Omega)\right] \tag{1.54}$$

1.4 Connection of heat kernel with index theorem.

In order to fully exploit and appreciate the consequences of the heat kernel approach we have previously introduced, it is useful in this final section of the first chapter to sketch an argument which shows the connection of heat kernel with an index theorem.[15] The relation we get turns out to be useful in chapter 3, where we will be able to derive the Riemann-Roch theorem.

For our purpose we have to consider a compact, oriented, boundaryless manifold M which is thought to be the base manifold for some vector bundles (say V_+, V_-).

Let us denote as E_+ (respectively E_-) the sets of sections of our vector bundles (therefore

$$E_+ = \{\phi \mid \phi : M \to V_+\} , \quad E_- = \{\phi \mid \phi : M \to V_-\})$$

Furthermore let us suppose that E_+, E_- have the structure of Hilbert space with a well-defined, positive, scalar product (to be definite we can think of E_+, E_- as the sets of tensor fields introduced in app. 1).

Let P be an operator connecting E_+ with E_- :

$P : E_+ \to E_-$ (P^+ is its adjoint, $P^+ : E_- \to E_+$).

We define as $\Omega_{(+)}, \Omega_{(-)}$ the operators:

$$\Omega_{(+)} \underset{\text{def.}}{=} P^+P \qquad \left(\Omega_{(+)} : E_+ \to E_+ \right) \tag{1.55}$$

$$\Omega_{(-)} \underset{\text{def.}}{=} PP^+ \qquad \left(\Omega_{(-)} : E_- \to E_- \right) \tag{1.56}$$

It turns out that $\Omega_{(\pm)}$ are self-adjoint operators $\left((\Omega_\pm)^+ = \Omega_\pm \right)$ We suppose them to be elliptic in order (1.57) to guarantee the completeness of their eigenvectors.

The operators which satisfy the hypothesis above have some important properties. A first property is expressed by the relations:

$$Ker\,\Omega_{(+)} = Ker\,P \qquad Ker\,\Omega_{(-)} = Ker\,P^+ \tag{1.58}$$

which follow from

$$P\phi = 0 \Rightarrow \quad \Omega_{(+)}\phi = 0 \qquad \text{for } \phi \in E_+ \tag{1.59}$$

$$\Omega_+\phi = 0 \Rightarrow \quad \langle \phi | \Omega_{(+)} | \phi \rangle = 0 \Rightarrow \langle P\phi, P\phi \rangle = 0 \Rightarrow \; P\phi = 0 \tag{1.60}$$

Another important property is the isomorphism between the subspaces $(Ker\,\Omega_+)^\perp \subset E_+$ and $(Ker\,\Omega_-)^\perp \subset E_-$ (1.61)

The isomorphism can be seen as follows:

let $\phi_m \in E_+$ be an eigenvector of $\Omega_{(+)}$ with positive eigenvalue $(\Omega_+\phi_m = \lambda_m \phi_m,\ \lambda_m > 0)$ If we put $\quad \Psi'_m = P\phi_m \quad (\Psi'_m \in E_-)$ (1.62)

it then follows from the relation

$$P\,\Omega_{(+)} = P(P^+P) = (PP^+)P = \Omega_{(-)}P \tag{1.63}$$

that Ψ'_m is an eigenvector of $\Omega_{(-)}$ with positive eigenvalue λ_m.

As a further consequence, this not only means an isomorphism between $(Ker\,\Omega_+)^\perp$ and $(Ker\,\Omega_-)^\perp$ and a one-to-one correspondance between the positive eigenvalues of Ω_+ and Ω_- but it also implies that Ω_+ has the same spectrum of positive eigenvalues of Ω_- .

If the eigenvectors ϕ_n , labelled by n , are chosen in such a way to form an orthonormal basis for the subspace $(Ker\,\Omega_+)^\perp$, we can also see from the relation $\langle\phi_n|P^+P|\phi_m\rangle = \lambda_n\delta_{nm}$ that the eigenvectors ψ_n of $\Omega_{(-)}$, $\psi_n = \frac{1}{\sqrt{\lambda_n}}\psi'_n$ (1.64) form an orthonormal basis for $(Ker\,\Omega_-)^\perp$

At this point we remember that for our operators the kernel is a finite dimensional subspace and therefore it makes sense to define an analytic index $I_A(P)$ as follows:

$$I_A(P) \underset{def.}{=} \dim(Ker\,P) - \dim(Ker\,P^+) \qquad (1.65)$$

The index theorem connects the analytical properties (which are expressed by the analytic index $I_A(P)$) of the operators P, P^+ with topological properties which are expressed by the so called topological index, by stating that the two indices are equal.

Now we want to show how the machinery developped in the previous sections will enable us to compute the analytic index $I_A(P)$.

From the discussion above it is clear that we can write:

$$I_A(P) = \sum_n e^{-\lambda_n \tau} - \sum_m e^{-\lambda_m \tau} \qquad (1.66)$$

(the parameters n, m label the eigenvalues respectively of $\Omega_{(+)}$ and $\Omega_{(-)}$). This relation can also be expressed as

$$I_A(P) = \mathrm{Tr}\, G_{\Omega_+}(x,x,\tau) - \mathrm{Tr}\, G_{\Omega_-}(x,x,\tau) \qquad (1.67)$$

($G_{\Omega_\pm}(x,y,\tau)$ is the heat kernel for $\Omega_\pm$).

This equality holds for any τ.

In particular we can choose τ to be small enough so that we can insert in the r.h.s. the small τ expansion for $G_{\Omega_\pm}(x,x,\tau)$. Therefore, in order to compute $I_A(P)$ one has simply to compute the coefficient $a_0(x)$ (working in 2 dimensions) in the small τ expansion for $G_{\Omega_+}(x,x,\tau)$ and $G_{\Omega_-}(x,x,\tau)$ This will be done, as already explained, for the laplacian operators introduced in app. 1. We will see then that the index so computed is a topological invariant quantity.

We close this section pointing out that the discussion about index theorem we have done could be repeated in the framework of Witten's approach to the index of supersymmetric theories.[16]

In this supersymmetric context we can think of a Hilbert space $\mathcal{H}$ given by $\mathcal{H} = E_+ \oplus E_-$

It is possible to introduce a hermitian operator Q acting on $\mathcal{H}$, as a"supersymmetry operator", by defining:

$$Q \underset{\text{def.}}{=} \begin{pmatrix} 0 & P^+ \\ P & 0 \end{pmatrix} \quad (1.68)$$

The operator $(-)^F$, given by

$$(-)^F = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (1.69)$$

plays the role of "fermion number operator" and H,

$H = Q^2$ plays the role of supersymmetric hamiltonian. (1.70)

The following commutation relations are satisfied:

$$[H, Q] = [H, (-)^F] = 0 \quad (1.71)$$

$$\{Q, (-)^F\} = 0 \quad (1.72)$$

It is easy to check that for each positive eigenvalue of H there is a pairing of a "bosonic" state with a "fermionic" one. The correspondance is broken for states having "zero energy". The analytic index $I_A(P)$ can therefore be expressed as $I_A(P) = \mathrm{Tr}(-)^F$ (1.73)

This viewpoint is at the basis of an alternative way to look at the heat kernel expansion, by exploiting its connection with supersymmetric quantum mechanical systems.[17] [18]

CHAPTER 2

COMPUTATION OF TRACE ANOMALY

2.1 Explanation of the method used.

We have mentioned in the previous chapter that in order to get the trace anomaly one need not know the full solution to the heat equation; actually it is sufficient to know the behaviour of $G_\Omega(x,y,\tau)$ for $x=y$ at small τ.

In this chapter we present the method which allows us to compute the coefficient $a_0(x)$ in the expansion 1.48 for a certain class of elliptic operators acting on functions which are sections over a 2-dimensional manifold. The method proposed is worked out in detail since it is interesting in itself and it is worth to know because it can be easily generalized, along the same lines, to get informations for operators acting on objects defined over higher dimensional manifolds.

We use the perturbative approach: therefore if $\Omega(x)$ is the operator we are considering, we can think of it as splitted into two pieces:

$$\Omega_{(x)} = \Omega_{0}(x) + V(x) \tag{2.1}$$

where $\Omega_{0}(x)$ is an operator for which we suppose to know the exact solution of the heat equation.

We want to express $e^{-\tau\Omega(x)}$ as:

$$e^{-\tau\Omega(x)} = \left[e^{-\tau\Omega(x)} e^{\tau\Omega_0(x)} \right] e^{-\tau\Omega_0(x)} \tag{2.2}$$

and make use of the Campbell-Hausdorff formula, which reads as follows

$$e^{A} \cdot e^{B} = e^{C(A,B)} \tag{2.3}$$

with $\quad C(A,B) = A+B+\frac{1}{2}[A,B]+\frac{1}{12}\left[A,[A,B]\right]+\frac{1}{12}\left[B,[B,A]\right]+\dots$

$$e^{A+B} = \left[e^{A+B}e^{-B}\right]e^{B} = e^{C(A+B,-B)}e^{B} \tag{2.4}$$

Identifying $\quad A = -\tau\Omega_0 \,, \quad B = -\tau V$

and applying both sides of 2.2 to $\delta(x,y)\frac{1}{\sqrt{g(y)}}$

we can write the formula:

$$G_{\Omega}(x,y,\tau) = \left\{ e^{-\tau V - \frac{1}{2}[\tau V, \tau\Omega_0] + \frac{1}{12}[\tau V,[\tau V, \tau\Omega_0]] - \frac{1}{12}[\tau\Omega_0[\tau\Omega_0, \tau V]] + \dots} \right\} G_{\Omega_0}(x,y,\tau)\frac{1}{\sqrt{g(y)}} \tag{2.5}$$

with $\quad G_{\Omega_0}(x,y,\tau) = e^{-\tau\Omega_0(x)}\delta(x,y) \qquad (2.6)$

If we work in 2 dimensions we can take as Ω_0 the flat laplacian

$$\Omega_0 = -\alpha\left(\partial^2_{x_1} + \partial^2_{x_2}\right) + m^2 \tag{2.7}$$

It is an easy exercise to check that the function $G_0(x,y,\tau)$ which solves the equation

$$\frac{\partial}{\partial\tau} G_0(x,y,\tau) = -\Omega_0\, G_0(x,y,\tau) \tag{2.8}$$

with the boundary condition

$$\lim_{\tau\to 0^+} G_0(x,y,\tau) = \delta(x,y) \tag{2.9}$$

is precisely given by:

$$G_0(x,y,\tau) = \frac{1}{4\pi\alpha\tau}\, e^{-\frac{(x-y)^2}{4\alpha\tau} - m^2\tau} \tag{2.10}$$

Before going ahead we want to stress that the elliptic hermitian operators under consideration are the laplacians which play a role in string theory and which are introduced in app. 2.

It turns out that, when expressed in real coordinates these laplacians are of the following kind:

$$\Omega(x) = -\left[A(x)\left(\partial^2_{x_1} + \partial^2_{x_2}\right) + B(x)\left(\partial_{x_1} + i\partial_{x_2}\right) + C(x)\right] \tag{2.11}$$

In this chapter we find convenient to work with real coordinates. The final formula we get will be translated into the complex notation we use for the strings.

The theories we are considering are invariant under diffeomorphisms. This has the consequence that we can fix $y=0$, compute $G_\Omega(x,0,\tau)$ at $x=0$ and work in a particular frame of reference. The particular frame given by the normal coordinates expansion [19] around $x=0$ implies that for our laplacian operators we can put

$$A(0)=1 \qquad \partial_{x_{1,2}} A(x)\Big|_{x=0}=0 \tag{2.12}$$

Since from our definition the heat kernel $G_\Omega(x,y,\tau)$ is a scalar both in x and in y the results we obtain, whose form depends on the particular choice we have done, can be reexpressed at the end in a manifestly covariant form.

We are interested in computing the corrections of $G_0(0,0,\tau)$ at small τ . We notice that the application of the operator $\partial^2=\partial^2_{x_1}+\partial^2_{x_2}$ at $G_0(x,0,\tau)$ has the effect of multiplying $G_0(x,0,\tau)$ by the factor

$$\left[\frac{-1}{\alpha\tau}+\frac{x^2}{4\alpha^2\tau^2}\right]$$

The piece $\frac{x^2}{4\alpha^2\tau^2}$ vanishes when we evaluate the trace ($x\equiv 0$). It is then possible to introduce a sort of τ -dimensionality of the operators: a derivative ∂ has a τ -dim.: $[\partial]=-\frac{1}{2}$ (2.13)

A $\varkappa$ factor has a τ -dim. $[\varkappa] = \frac{1}{2}$ (2.14)

since, in order to have a non-vanishing value when we compute $G_\Omega(x,0,\tau)$ at $x=0$ it must be "eaten" by a derivative.

We can now expand the functions $A(x)$, $B(x)$, $C(x)$ in Taylor series around $x=0$.

We get the following expressions:

$$\Omega = \Omega_1 + \Omega_2 + \Omega_3$$

$$\Omega_1 = -\left[a + x^a f_a + \frac{1}{2} x^a x^b e_{ab} + \dots\right]\left(\partial^2_{x_1} + \partial^2_{x_2}\right)$$

$$\Omega_2 = -\left[b + x^a d_a + \dots\right]\left(\partial_{x_1} + i\partial_{x_2}\right) \tag{2.15}$$

$$\Omega_3 = -\left[c + \dots\right]$$

where

$$a = A(0), \quad f_a = \frac{\partial}{\partial x^a} A(x)\Big|_{x=0}, \quad e_{ab} = \frac{\partial^2}{\partial x^a \partial x^b} A(x)\Big|_{x=0}$$

$$b = B(0), \quad d_a = \frac{\partial}{\partial x^a} B(x)\Big|_{x=0} \tag{2.16}$$

$$c = C(0)$$

The flat laplacian Ω_0 is given by

$$\Omega_0 = -\left[a\left(\partial^2_{x_1} + \partial^2_{x_2}\right) + c\right] \tag{2.17}$$

As told before we can put $a=1$, $f_a=0$ (2.18)

It is clear that the only operators in τV which give contribution to the lowest order in τ are those whose τ -dimensionality is $\leqslant 1$

This implies that the only operators which play a role are:

$$\tau V_1 = -\frac{1}{2} x^a x^b e_{ab} \tau \left(\partial^2_{x_1} + \partial^2_{x_2} \right) \qquad (\tau\text{-dim} = 1)$$

$$\tau V_2 = -b\tau \left(\partial_{x_1} + i\partial_{x_2} \right) \qquad (\tau\text{-dim} = \tfrac{1}{2}) \qquad (2.19)$$

$$\tau V_3 = -x^a d_a \tau \left(\partial_{x_1} + i\partial_{x_2} \right) \qquad (\tau\text{-dim} = 1)$$

We now evaluate the commutators of these operators with $\tau\Omega_0$; we get

$$[\tau\Omega_0, \tau V_1] = \tau^2 \left[\partial^2, \frac{1}{2} x^a x^b e_{ab} \partial^2 \right] =$$

$$= \tau^2 \left(e_c{}^c \partial^2 + 2 x^a e_{ac} \partial^2 \partial^c \right) \qquad (\tau\text{-dim} = 1) \qquad (2.20)$$

$$[\tau\Omega_0, \tau V_2] = 0$$

$$[\tau\Omega_0, \tau V_3] = \tau^2 \left[\partial^2, x^a d_a (\partial_{x_1} + i\partial_{x_2}) \right] = 2\alpha d_c \partial^c \tau^2 (\partial_{x_1} + i\partial_{x_2}) \qquad (\tau\text{-dim} = 1)$$

When we evaluate the commutators of 3 operators we have:

$$[\tau\Omega_0, [\tau\Omega_0, \tau V]] = [\tau\Omega_0, [\tau\Omega_0, \tau V_1]] =$$
$$= [-\tau\partial^2, 2\tau^2 x^a \varrho_{ac} \partial^2 \partial^c] = -4\tau^3 \varrho_{ac} \partial^a \partial^c \partial^2 \qquad (\tau \cdot dim = 1) \tag{2.21}$$

In $[\tau V, [\tau V, \tau\Omega_0]]$ the only operator which can give contribution for the dimensionality argument is V_2, but its commutator with Ω_0 is zero.

Clearly all the remaining commutators give no contribution at the lowest order in τ.

At this point we can apply the formula 2.5

The exponential will be expanded in power series, we get:

$$G_\Omega(0,0,\tau) = \left\{ 1 - \tau V - \frac{1}{2}[\tau V, \tau\Omega_0] + \frac{1}{12}[\tau V, [\tau V, \tau\Omega_0]] - \frac{1}{12}[\tau\Omega_0 [\tau\Omega_0, \tau V]] + \right.$$
$$\left. + \frac{1}{2!}(\tau V_2)^2 \right\} G_0(x,0,\tau)\Big|_{x=0} \cdot \frac{1}{\sqrt{g(0)}} + O(\sqrt{\tau}) \tag{2.22}$$

Taking the derivative $\partial^2_{x_1} \partial^2_{x_2}$ of $G_0(x,0,\tau)$ and putting $x=0$ has the same effect as multiplying $G_0(0,0,\tau)$ by a certain factor:

$$\partial^2_{x_1} \equiv \partial^2_{x_2} \equiv -\frac{1}{2\tau} \qquad \partial^2 \equiv -\frac{1}{\tau} \qquad \partial^2\partial^2 \equiv \frac{2}{\tau^2}$$
$$\frac{1}{2!}\tau^2 V_2^2 \equiv \frac{1}{2} b^2 \tau^2 (\partial^2_{x_1} - \partial^2_{x_2}) \equiv 0 \tag{2.23}$$

We get then the final result:

$$G_\Omega(0,0,\tau) = \left[1 + \frac{1}{6}\tau \varrho_c{}^c - \frac{1}{2}\tau(d_{x_1} + i d_{x_2}) + \tau c\right] \frac{1}{4\pi\tau} + O(\sqrt{\tau}) \tag{2.24}$$

2.2 Final result.

It is convenient to reexpress the final result in a compact form, both in real and complex notation (for complex notation see app. 1 and 2).

Let
$$\Omega(x) = -\left[A(x)(\partial_{x_1}^2+\partial_{x_2}^2) + B(x)(\partial_{x_1}+i\partial_{x_2}) + C(x)\right] \qquad (2.25)$$

$$A(0) = 1 \qquad \partial_{x_{1,2}} A(x)\Big|_{x=0} = 0 \qquad (2.26)$$

Then

$$G_\Omega(0,0,\tau) = \frac{1}{4\pi\tau} + \frac{1}{4\pi}\left[\frac{1}{6}(\partial_{x_1}^2+\partial_{x_2}^2)A\Big|_{x=0} - \frac{1}{2}(\partial_{x_1}+i\partial_{x_2})B\Big|_{x=0} + C(0)\right] + O(\sqrt{\tau})$$
$$(2.27)$$

In complex notation

$$\Omega_{\mathbb{C}} = -\left[4A\,\partial_z\partial_{\bar z} + 2B\,\partial_{\bar z} + C\right] \qquad (2.28)$$

Then

$$G_{\Omega_{\mathbb{C}}}(0,0,\tau) = \frac{1}{2\pi\tau} + \frac{1}{2\pi}\left[\frac{2}{3}\partial_z\partial_{\bar z}A\Big|_0 - \partial_{\bar z}B\Big|_0 + C(0)\right] + O(\sqrt{\tau}) \qquad (2.29)$$

In this formula a factor 2 has been inserted w.r.t. 2.27 for taking into account the fact that $\Omega_{\mathbb{C}}$ is assumed to act on complex fields.

CHAPTER 3

THE STRING THEORY

3.1 Introduction to string theory.

In this chapter we will review the Polyakov functional approach to the quantization of the closed, oriented, bosonic string and we will show how, in this approach, the critical dimension d=26 is singled out (different approaches to the quantization of the bosonic string, like the canonical quantization or the BRST quantization, turn out to be equivalent to the Polyakov formulation; in these different approaches the critical dimension d=26 is obtained from different structures[20]: for instance in the operatorial formalism the critical dimension is related to the central extension of the Virasoro algebra).

In the final section of this chapter we will discuss the conformal anomaly for the Green-Schwarz heterotic superstring.

In this introduction we will briefly discuss the basic ingredients of the classical string theory we need to know.

A string theory[21] [22] is charachterized by the fact that its fundamental objects are not point-like but one-dimensional: it is a theory of a curve whose evolution sweeps out a 2-dimensional surface, the so called world-sheet, in space-time.

At a classical level the equations of motion are determined by the Nambu-Goto action, which is given by the geometric area of the surface. Such an action is highly non-linear in the coordinates and therefore difficult to quantize.

According to the Polyakov prescription[23] we have to take as starting point an equivalent classical action which depends on an additional intrinsic metric g_{ab}. This new action is givan by:

$$I_0(X,g) = \frac{1}{2}\int_M d^2x \sqrt{g}\, g^{ab}\partial_a X^\mu \partial_b X^\mu \tag{3.1}$$

M is a compact 2-dimensional, oriented surface; g is a riemannian (positive definite) metric on M. X^μ is an embedding of M into a d -dimensional space-time: $X: M \to \mathbb{R}^d$. The space-time is thought to be flat and euclidean (we have assumed that a Wick rotation has been performed both on the 2-dimensional world-sheet and in the tangent space-time where the string moves).

When we use the variational principle to get the equations of motion we are free to vary both X^μ and g_{ab} independently. The metric g is non-dynamical and may be solved for, so that we are led again for X^μ to the equations of motion deriving from the Nambu-Goto action.

The string action I_0 admits 3 invariances at a classical level:

i) invariance under diffeomorphisms of M :

$$x \mapsto x' = x'(x)$$
$$g_{ab}(x) \mapsto \frac{\partial x'^c}{\partial x^a}\frac{\partial x'^d}{\partial x^b}\, g_{cd} \tag{3.2}$$

ii) conformal (Weyl) invariance under local rescaling of the metric:

$$g_{ab}(x) \mapsto e^{2\sigma(x)} g_{ab}(x) \tag{3.3}$$

iii) invariance under global rotations and translations of $\mathbb{R}^d$. (3.4)

As we have already seen in ch. 1, the conformal invariance will in general not be preserved at a quantum level. As a consequence, this means that when we regularize the partition function with a scheme like the proper-time regularization, the criterion of renormalizability forces us to start with the most general action having couplings of non-negative dimensions and consistent with the symmetries i) and iii) (the zeta function technique we use, however, allows us to bypass this point since there is no need to compute explicitely the counterterms).

For boundaryless manifolds the most general action is given by

$$I(X, g) = \frac{1}{2} A \int_M d^2x \sqrt{g}\, g^{ab} \partial_a X^\mu \partial_b X^\mu + B \frac{1}{4\pi} \int d^2x \sqrt{g} R + C \int_M d^2x \sqrt{g} \tag{3.5}$$

The second term is a topological invariant quantity which gives no contribution to the dynamics while, at a classical level, the equations of motion require

$$C = 0 . \tag{3.6}$$

3.2 Quantization of the bosonic string.

The Polyakov functional approach to the quantization of the bosonic string is essentially a perturbative approach which postulates a partition function Z given by:

$$Z = N \sum_h \lambda^h \int_{\text{metrics}} \mathcal{D}g \int_{\text{embeddings}} \mathcal{D}X^\mu \, e^{-I_0(X,g)} \tag{3.7}$$

Here h is an integer, the number of handles of the world-sheet surface. The sum over h is a sum over the different topologies of the world-sheet surface and takes into account the effects due to the string interactions.

λ is a coupling constant; N is the usual normalization factor.

If one wanted to compute S-matrix amplitudes one would have to include the insertion of vertex operators.

To be definite we have of course to specify the functional measure.

It turns out that a natural measure $\mathcal{D}X^\mu$ for the configuration space C of strings ($C = \{X : M \to \mathbb{R}^d\}$) is the one corresponding to the metric

$$\| \delta X^\mu \|^2 = \int_M d^2x \sqrt{g}\, \delta X^\mu \delta X^\mu \tag{3.8}$$

between two nearby maps X^μ, $X'^\mu = X^\mu + \delta X^\mu$

(we remember that in an ordinary Riemann space of finite dimension N with metric $ds^2 = g_{ij} dx^i dx^j$ the volume element dV is given by $dV = d^N x \sqrt{g}$). (3.9)

The word natural is used for the following reason: our metric is the simplest expression invariant under global translations of X^μ and under diffeomorphisms which does not involve a derivative of g or a derivative of X^μ We point out that the measure $\mathcal{D}X^\mu$ is not conformally invariant.

The same procedure allows us to determine the measure $\mathcal{D}g$: let $\mathcal{M}$ be the space of the riemannian metrics g on the world-sheet manifold. $\mathcal{M}$ turns out to ba a convex, non-compact space. A natural metric (natural has the same meaning as before) for $\mathcal{M}$ is expressed by the relation:

$$\| \delta g \|^2 = \int_M d^2x \sqrt{g} \left(G^{abcd} + u g^{ab} g^{cd} \right) \delta g_{ab} \delta g_{cd} \tag{3.10}$$

u is an arbitrary constant > 0.

G^{abcd} is a projector onto the space of symmetric, traceless tensors:

$$G_{ab}{}^{cd} = \frac{1}{2}\left(\delta_a{}^c\delta_b{}^d + \delta_a{}^d\delta_b{}^c - g_{ab}g^{cd}\right) \tag{3.11}$$

$\mathcal{D}g$ is of course the measure associated with this metric and it is, just like $\mathcal{D}X^\mu$, not conformally invariant. We will see that for $d=26$ the conformal anomaly associated with $\mathcal{D}g$ precisely cancels the conformal anomaly associated with $\mathcal{D}X^\mu$.

Both the action and the measure are covariantly defined so that, at least formally, the partition function is invariant under a reparametrization of the world-sheet. Physically equivalent configurations are counted manyfold: the configuration space over which one is integrating is the product space $C \times \mathcal{M}$, while the space of physical equivalent configurations is the quotient space $(C \times \mathcal{M})/\mathrm{Diff.}(M)$

The functional integral contains an overall infinite factor which has to be removed by restricting the integral to a gauge slice, i.e. to a subspace of metrics which meets each orbit of local gauge group exactly once.

The full diffeomorphism group $\mathrm{Diff.}(M)$ may be thought of as a discrete transformation in combination

with an element of $Diff_0(M)$, the subgroup of diffeomorphisms connected with the identity.
It turns out that the deiscrete diffeomorphisms may be anomalous. Here we will use the Fadeev-Popov technique in order to factor out the integration over $Diff_0(M)$.

To apply the Fadeev-Popov technique we have to decompose the most general variation δg_{ab} as follows:

$$\delta g_{ab} = \delta h_{ab} + 2 g_{ab} \delta \tau \qquad (3.12)$$

with $\quad \delta h_{ab} = G_{ab}{}^{cd} \delta g_{cd}$

(δh_{ab} is a symmetric traceless tensor).
Inserting this decomposition in $\|\delta g\|^2$ we can write:

$$\|\delta g\|^2 = \int_M d^2x \sqrt{g}\, G^{abcd} \delta h_{ab} \delta h_{cd} + 16u \int_M d^2x \sqrt{g} (\delta\tau)^2 \qquad (3.13)$$

which implies the following relation

$$\mathcal{D}g = (\mathcal{D}h)(\mathcal{D}\tau) \qquad (3.14)$$

(we have to warn that 3.12 is a decomposition of a tangent vector in an unspecified orthonormal frame which is in general not integrable and does not lead to a coordinate system labelled by h and τ).

The infinitesimal variation δg_{ab} is specified by 3 arbitrary functions: 2 for the traceless tensor δh_{ab} and 1 for $\delta\tau$. It is useful at this point to express the generic variation δg_{ab} in terms of an infinitesimal diffeomorphism connected with the identity (such a diffeomorphism involves 2 arbitrary functions) and of an infinitesimal Weyl transformation which involves 1 arbitrary function (it must be said that not every δg_{ab} can be expressed in this way: we will come back on this point later).

Our infinitesimal diffeomorphism is specified by the infinitesimal vector field $\delta V^a(x)$. The corresponding variation of g_{ab} is given by

$$\delta_D g_{ab} = \nabla_a \delta V_b + \nabla_b \delta V_a \tag{3.15}$$

(∇_a is the covariant derivative).

Under an infinitesimal Weyl transformation we have:

$$\delta_W g_{ab} = 2\delta\sigma g_{ab} \tag{3.16}$$

Therefore
$$\delta g_{ab} = \delta_D g_{ab} + \delta_W g_{ab} \tag{3.17}$$

It follows

$$\delta h_{ab} = G_{ab}{}^{cd}\,\delta g_{cd} = 2G_{ab}{}^{cd}\,\nabla_c \delta V_d = (P\delta V)_{ab} \tag{3.18}$$

$$2g_{ab}\delta\tau = \frac{1}{2} g_{ab} g^{cd} \delta g_{cd} = g_{ab}\left[2\delta\sigma + g^{cd}\nabla_c \delta V_d\right]$$

$$\Downarrow$$

$$\delta\tau = \delta\sigma + \frac{1}{2} g^{cd}\nabla_c \delta V_d \tag{3.19}$$

The operator P maps vectors into traceless tensors.

Changing our variables from h_{ab}, τ to V_a, σ we can write the measure as:

$$(\mathcal{D}h)(\mathcal{D}\tau) = (\mathcal{D}\sigma)(\mathcal{D}V)\,J \qquad (3.20)$$

J is the jacobian

$$J = \left| \frac{\partial(\tau, h)}{\partial(\sigma, V)} \right| = \left| \det \begin{pmatrix} 1 & * \\ 0 & P \end{pmatrix} \right| = \left(\det P^{+}P\right)^{\frac{1}{2}} \qquad (3.21)$$

(the term denoted as $*$ gives no contribution to the determinant because the matrix is triangular).

P^{+} is the adjoint of P and maps traceless tensors into vectors.

There are two crucial observations; the first one concerns the zero-modes of the operator P : the vectors δV which satisfy $P\delta V = 0$ (3.22)
belong to $\mathrm{Ker}\,P$ and are called conformal Killing vectors. A diffeomorphism generated by such a vector is equivalent to a change in the conformal factor and must be omitted since the deformation of the metric has to be counted only once. The correct jacobian is therefore given by the primed determinant $\left(\det' P^{+}P\right)^{\frac{1}{2}}$ which is built with the positive eigenvalues only.

The second observation concerns the fact that not every deformation δh_{ab} can be expressed as $P\delta V$.

It is very simple to realize this if we go back to the discussion of sec. 1.4. There is a one-to-one correspondence between the subspaces of vector fields given by $(\mathrm{Ker}\, P)^{\perp}$ and the subspace of traceless tensor fields given by $(\mathrm{Ker}\, P^{+})^{\perp}$

The procedure used to isolate the volume of gauge group is incomplete and we are still required to integrate over the deformations which belong to the finite-dimensional subspace $\mathrm{Ker}\, P^{+}$. These are the so called Teichmüller deformations.

We are thus led to write our partition function as

$$Z = N \sum_{\text{topologies}} \int \mathcal{D}\sigma\, \mathcal{D}t\, \Omega^{\perp}_{\text{diffeo}}\, \mathcal{D}X^{\mu} (\det{}'\, P^{+}P)^{\frac{1}{2}}\, e^{-I_0(X, \hat{g}_t e^{2\sigma})} \tag{3.23}$$

$\Omega^{\perp}_{\text{diffeo}}$ is the volume of diffeomorphisms which are perpendicular to the conformal Killing vectors.

t denotes the Teichmüller deformations ($\mathcal{D}t$ is the integration over the Teichmüller deformations).

$\hat{g}$ is a given reference metric.

Since the action I_0 is quadratic in X^{μ} the integration over $\mathcal{D}X^{\mu}$ can be performed in the usual way:

Let $\hat{X}^\mu$ be the classical solution of the equations of motion in presence of a background metric $\hat{g}$.

We have

$$\int \mathcal{D}X^\mu e^{-I_0(X,\hat{g})} = e^{-I_0(\hat{X},\hat{g})} \left(\det{}'\Delta_0\right)^{-\frac{d}{2}} \int_{0\text{-modes}} \mathcal{D}X_0^\mu \qquad (3.24)$$

Δ_0 is the laplacian operator 1.43 associated with the action I_0 (see sec. 1.3).

With the symbol $\int_{0\text{-modes}} \mathcal{D}X_0^\mu$ we denote the eventual integration over the zero-modes for the laplacian operator.

Since our classical action is conformally invariant we have

$$I_0(\hat{X},\hat{g}) = I_0(\hat{X},\hat{g}e^{2\sigma}) \qquad (3.25)$$

3.3 The conformal anomaly for the bosonic string.

In this section we will compute the conformal anomaly for the bosonic string, that is to say we will compute the dependence of the partition function Z on the conformal factor $\sigma(x)$.

If in Z the integrand is not affected by a change of $\sigma(x)$ ($\sigma(x)\mapsto\sigma(x)+\delta\sigma(x)$), then the measure $\mathcal{D}\sigma(x)$ can be absorbed in the normalization factor N and the theory is conformally invariant.

To simplify the formulas in our derivation we do not worry about the complications introduced by Teichmüller deformations (for instance we can assume to work in the sector $h=0$ where our base manifold M is equivalent to a sphere which admits no Teichmuller deformations).

The partition function 3.23 can be therefore rewritten as

$$Z = N\int \mathcal{D}\sigma \, \frac{\Omega^{\perp}_{diff_0}}{\Omega_{diff_0}} \, \Omega_{diff_0} \left(\det{}' P^{+}P\right)^{\frac{1}{2}} \left(\det{}'\Delta_0\right)^{-\frac{d}{2}} \int_{0\text{-modes}} \mathcal{D}X_0^{\mu} \qquad (3.26)$$

where the ratio $\Omega_{diff_0}/\Omega^{\perp}_{diff_0}$ is essentially Ω_{CKV} , the volume of the conformal Killing vectors.

We must observe at this point that the operator P which maps vectors (specified by 2 real components) into traceless tensors (also specified by 2 real components) is expressed in the complex formalism introduced in app. 2 by the covariant derivative ∇_1^2 which maps tensors $\in \mathcal{T}^1$ (2 real components) into tensors $\in \mathcal{T}^2$ (2 real components too).

Therefore the operator P^+P coincides with the laplacian Δ_1^+ of app. 2.

In the same way Δ_0 is related to Δ_0^+ (the difference is that the latter acts on complex objects, while we have assumed that Δ_0 acts on real fields).

To work out the conformal anomaly we have therefore simply to apply the method shown in ch. 1 and 2 to the operators $\Delta_n^{(+)}, \Delta_m^{(-)}$ of app. 2.

The first ting to do is to compute the small τ expansion for the heat kernel $G_{\Delta_n^\pm}(x,x,\tau)$

In order to do this we have to specialize the formula (2.29) to our operators $\Delta_n^\pm$,insetring the proper coefficients.

We remember that the invariance under diffeomorphisms allows us to fix at $x=0$

$$\sigma(0) = 0 \quad , \quad \partial_a \sigma(x)\Big|_{x=0} = 0 \tag{3.27}$$

with this choice we can write:

$$G_{\Delta_n^+}(0,0,\tau) = \frac{1}{2\pi\tau} + \frac{3n+1}{12\pi}\left(-2\partial^2\sigma\right)\Big|_{x=0} + O(\sqrt{\tau}) \tag{3.28}$$

$$G_{\Delta_n^-}(0,0,\tau) = \frac{1}{2\pi\tau} - \frac{3n-1}{12\pi}\left(-2\partial^2\sigma\right)\Big|_{x=0} + O(\sqrt{\tau}) \tag{3.29}$$

In order to express $G_{\Delta_n^\pm}(x,x,\tau)$ in a manifestly covariant form we have to notice that, with the position 3.27 we have $R(0) = -2\partial^2\sigma\Big|_{x=0}$ (3.30)

Therefore the traces are given by

$$\mathrm{Tr}\, G_{\Delta_n^+}(x,x,\tau) = \frac{1}{2\pi\tau}\int d^2x\sqrt{g(x)} + \frac{3n+1}{12\pi}\int d^2x\sqrt{g(x)}\,R(x) + O(\sqrt{\tau}) \tag{3.31}$$

$$\mathrm{Tr}\, G_{\Delta_n^-}(x,x,\tau) = \frac{1}{2\pi\tau}\int d^2x\sqrt{g(x)} - \frac{3n-1}{12\pi}\int d^2x\sqrt{g(x)}\,R(x) + O(\sqrt{\tau}) \tag{3.32}$$

The Riemann-Roch theorem is at this point easily derived: the analytic index $I_A(\nabla_n^z)$ is given by

$$I_A(\nabla_n^z) = \dim(\mathrm{Ker}\,\nabla_n^z) - \dim(\mathrm{Ker}(\nabla_n^z)^+) = \tag{3.33}$$

$$= \mathrm{Tr}\, G_{\Delta_n^+}(x,x,\tau) - \mathrm{Tr}\, G_{\Delta_{n+1}^-}(x,x,\tau) \tag{3.34}$$

Therefore

$$I_A(\nabla_n^z) = (2n+1)\frac{1}{4\pi}\int d^2x\sqrt{g}\,R = (2n+1)\,\chi(M) \tag{3.35}$$

(we point out that in our formula the dimension of the kernels are given counting the number of real components).

The latter equality makes use of the Gauss-Bonnet theorem and shows that the analytic index is a topological invariant quantity (by the way, if one would not know the content of Gauss-Bonnet theorem, our analysis could prove that $\frac{1}{4\pi}\int_M \sqrt{g}\, R$ is an integer).

In particular for $n=1$ we get the relation:

$$\#\,\text{conf. Kill. vect.} - \#\,\text{Teichm. deform.} = 6-6h \tag{3.36}$$

The variation of the operator $\Delta_n^+(\sigma)$ under an infinitesimal Weyl transformation $\sigma \mapsto \sigma+\delta\sigma$ is given by:

$$\begin{aligned} \delta\Delta_n^+(\sigma) &= \Delta_n^+(\sigma+\delta\sigma) - \Delta_n^+(\sigma) = \\ &= -2(n+1)\,\delta\sigma\,\Delta_n^+(\sigma) + 4n\,(\nabla_m^z)^+\,\delta\sigma\,\nabla_m^z \end{aligned} \tag{3.37}$$

The corresponding variation of a positive eigenvalue λ_i of $\Delta_n^+(\sigma)$ is:

$$\delta\lambda_i = \langle\varphi_i|\,\delta\Delta_n^+(\sigma)\,|\varphi_i\rangle \tag{3.38}$$

$|\varphi_i\rangle$ is the normalized eigenvector of $\Delta_n^+(\sigma)$ which corresponds to the eigenvalue λ_i (therefore $|\varphi_i\rangle \in [\mathrm{Ker}\,\Delta_n^+(\sigma)]^\perp$ and it is a n-rank tensor).

It follows

$$\begin{aligned} \delta\lambda_i &= -2(n+1)\langle\varphi_i|\,\delta\sigma\,\Delta_n^+(\sigma)|\varphi_i\rangle + 4n\langle\nabla_m^z\varphi_i|\delta\sigma|\nabla_m^z\varphi_i\rangle = \\ &= -2(n+1)\lambda_i\langle\varphi_i|\delta\sigma|\varphi_i\rangle + 2n\lambda_i\langle\psi_i|\delta\sigma|\psi_i\rangle \end{aligned} \tag{3.39}$$

where $|\psi_i\rangle = \sqrt{\frac{2}{\lambda_i}}\,\nabla_m^z\varphi_i$ belongs to $\mathcal{T}^{n+1}$ and is a normalized eigenvector of $\Delta^{(-)}_{n+1}(\sigma)$ ($\Delta^{-}_{n+1}(\sigma)|\psi_i\rangle = \lambda_i|\psi_i\rangle$ which means $|\psi_i\rangle \in (\mathrm{Ker}\,\Delta^{(-)}_{n+1})^{\perp}$).

We define $\ln \det{}' \Delta^{+}_{n}(\sigma)$ by making use of the generalized zeta function. It is clear that in order to compute the variation $\delta \ln \det{}' \Delta^{+}_{n}(\sigma)$ we have simply to repeat, with only slight modifications, the procedure explained at the end of sec. 1.3. We get the final result

$$\delta \ln \det{}' \Delta^{+}_{n}(\sigma) = -\frac{6n^2+6n+1}{6\pi}\int d^2x \sqrt{g(x)}\, R(x)\, \delta\sigma(x) + 2(n+1)\,\mathrm{Tr}(\delta\sigma\, \pi_{\mathrm{Ker}\, \nabla^{z}_{m}}) - 2n\,\mathrm{Tr}(\delta\sigma\, \pi_{\mathrm{Ker}(\nabla^{z}_{m})^{+}}) \qquad (3.40)$$

The expression above is a differential equation which can be integrated. In order to do this we have to express the quantities g, R in terms of a reference metric $\hat{g}$: then we have $g = \hat{g} \exp 2\sigma$ (3.41)

The first term in the r.h.s. can be easily integrated, while the terms containing a trace can be reexpressed as

$$(2n+1)\,\mathrm{Tr}\,\delta\sigma\, \pi_{\mathrm{Ker}\,\Delta^{+}_{n}} = \delta \ln \det H(\Delta^{+}_{n}) \qquad (3.42)$$

$$-2n\,\mathrm{Tr}\,\delta\sigma\, \pi_{\mathrm{Ker}\,\Delta^{-}_{n+1}} = \delta \ln \det H(\Delta^{-}_{n+1}) \qquad (3.43)$$

Here $H(\Delta^{+}_{n})$ is a finite-dimensional matrix, defined as $H_{rs} \underset{\mathrm{def}}{=} \langle \phi^{o}_{r} | \phi^{o}_{s} \rangle$ (3.44) where ϕ^o_r, ϕ^o_s span a basis for $\mathrm{Ker}(\Delta^+_n)$ and are taken to be independent of the conformal factor. A similar position holds for $H(\Delta^-_{n+1})$ in terms of $\mathrm{Ker}\,\Delta^-_{n+1}$. At the end we get

$$\ln \det{}' \Delta^+_n(\sigma) = -\frac{6n^2+6n+1}{12\pi}\int_M d^2x\,\sqrt{\hat{g}}\left[\hat{g}^{ab}\partial_a\sigma\partial_b\sigma + \hat{R}\sigma\right] + \ln\det H(\Delta^+_n) + \ln\det H(\Delta^-_{n+1}) + F(\hat{g}) \tag{3.45}$$

$F(\hat{g})$ is a term which is independent of the conformal factor.

We are now ready to compute the conformal anomaly. A careful analysis conducted in [24], [9] and which takes into account the contribution of conformal Killing vectors and Teichmüller deformations shows that the correct Fadeev-Popov determinant is given by:

$$J = \left[\frac{\det{}'\Delta^+_1}{\det H(\Delta^+_n)\cdot\det H(\Delta^-_{n+1})}\right]^{\frac{1}{2}} \tag{3.46}$$

The contribution of the jacobian to the conformal anomaly is therefore contained in the piece

$$-\frac{13}{12\pi}\int_M d^2x\,\sqrt{\hat{g}}\left[\hat{g}^{ab}\partial_a\sigma\partial_b\sigma + \hat{R}\sigma\right] \tag{3.47}$$

The last thing we have to compute is $(\det{}'\Delta_0)^{-\frac{d}{2}}\int_{0\text{-modes}}\mathcal{D}X_0^\mu$

For what concerns $\det{}'\Delta_0$, it is obtained from 3.45 by putting $n=0$ and an overall factor $\frac{1}{2}$ which is due to the fact that Δ_0 acts on real fields.

We get then

$$\delta \ln\det{}'\Delta_0^+ = -\frac{2}{12\pi}\int\sqrt{g}\,R\delta\sigma + 2\,\mathrm{Tr}\left(\delta\sigma\,\Pi\,\mathrm{Ker}\,\Delta_0^+\right) \tag{3.48}$$

$$\delta \ln\det\Delta_0 = -\frac{1}{12\pi}\int\sqrt{g}\,R\delta\sigma + 2\,\mathrm{Tr}\left(\delta\sigma\,\Pi\,\mathrm{Ker}\,\Delta_0\right) \tag{3.49}$$

The second term in the r.h.s. and which depends on $\mathrm{Ker}\,\Delta_0$ is compensated by the variation $\delta\int_{0\text{-modes}}\mathcal{D}X_0^\mu$ of the integration over the zero-modes of the laplacian as the following analysis proves:

we expand a function $\phi\in\tau^0$ in an orthonormal basis as done in sec. 1.1. We get at $\sigma(x)$ and $\sigma'(x)=\sigma(x)+\delta\sigma(x)$

$$\phi = \Sigma_n c_n|\varphi_n\rangle_\sigma \qquad \langle\varphi_n|\varphi_m\rangle_\sigma = \delta_{nm}$$

$$\phi = \Sigma_n c'_n|\varphi'_n\rangle_{\sigma'} \qquad \langle\varphi'_n|\varphi'_m\rangle_{\sigma'} = \delta_{nm}$$

We have to take into account only the zero-modes.

The eigenvectors which correspond to a zero eigenvalue for the laplacian operator Δ_0^+ are independent of $\sigma(x)$.

Therefore, if $|\varphi_0\rangle$, $|\varphi'_0\rangle$ are zero-modes at σ,σ' respectively, it follows: $|\varphi'_0\rangle = (1+\delta\lambda)|\varphi_0\rangle$

$\delta\lambda$ is a constant.

Since

$$\int d^2x \sqrt{\hat{g}}\,(1+2\delta\sigma_{\omega})(1+\delta\lambda)\varphi_0^*(1+\delta\lambda)\varphi_0 = \int d^2x \sqrt{\hat{g}}\,\varphi_0^*\varphi_0 = 1 \tag{3.50}$$

we have
$$\delta\lambda = -\frac{\int d^2x \sqrt{\hat{g}}\,\delta\sigma\,\phi_0^*\phi_0}{\int d^2x \sqrt{\hat{g}}\,\phi_0^*\phi_0} \tag{3.51}$$

It is $c_0' = c_0(1-\delta\lambda)$ and then follows

$$\int_{\substack{0.\,\text{modes}\\ \text{at}\ \sigma'=\sigma+\delta\sigma}} - \int_{\substack{0.\,\text{modes}\\ \text{at}\ \sigma}} = \int dc_0' - \int dc_0 = -\delta\lambda \int dc_0 \tag{3.52}$$

$\int dc_0$ is a constant factor which appears in the normalization of the partition function; the variation inside the partition function is then precisely given by $Tr(\delta\sigma\,Ker\,\Delta_0^+)$ Since in our case the laplacian operator Δ_0 acts on d fields we get at the end $\int_{\substack{0.\,\text{modes}\\ \text{at}\ \sigma'}} - \int_{\substack{0.\,\text{modes}\\ \text{at}\ \sigma}} = d\,Tr(\delta\sigma\,\pi\,Ker\,\Delta_0)$ (3.53) which precisely cancels the corresponding term appearing in $(det'\Delta_0)^{-\frac{d}{2}}$

The final result is that in d=26 dimensions the contribution of the laplacian operator precisely cancels the contribution of the Fadeev-Popov determinant and the partition function turns out to be conformally invariant.

In dimension d different from the critical one, we get that the partition function Z contains a term which is the Liouville model, given by

$$\int \mathcal{D}\sigma(x)\, e^{+\frac{d-26}{24\pi}\int d^2x \sqrt{\hat{g}}\left[\hat{g}^{ab}\partial_a\sigma\partial_b\sigma + \hat{R}\sigma + Ae^{\sigma}\right]} \qquad (3.54)$$

The term proportional to the constant A has been inserted in order to have the most general action invariant under diffeomorphisms (remember the discussion at the end of sec. 3.1).

3.4 The heterotic string.

The last application of the method we have described concerns the computation of the conformal anomaly for the Green-Schwarz formulation of the heterotic string.

There are two ways to describe superstrings. The first one is due to Neveu, Schwarz and Ramond [25] [26] and makes the 2-dimensional world-sheet supersymmetric, while the embedding is an ordinary space-time.
The second, due to Green and Schwarz, [27] [28] is a manifestly supersymmetric formulation in space-time, which appears to be equivalent to the supersymmetric truncation of N.S.R. model. [29] For G.S. the string world-sheet is an ordinary 2-dimensional surface, but the embedding is a superspace of 10 commuting and 16 or 32 anticommuting coordinates. [30] [31]

The N.S.R. formulation has the advantage that the gauge-fixing procedure is straightforward, but the 10-dimensional supersymmetry is not manifest.

In contrast the G.S. action is manifestly supersymmetric but difficult to quantize for the presence of a complicated local supersymmetry: calculations may be performed in the light-cone gauge formulation.

A general gauge-fixing procedure has been suggested in [32] [33] in the framework of Batalin-Vilkovisky [34] [35] method. In this framework is interesting to verify the absence of conformal anomaly because it can be an indirect proof of the correctness of the quantization (in the light-cone gauge the absence of conformal anomaly is related with the absence of Lorentz anomaly).

In its original form the covariant G.S. action described the embedding of a 2-dim. surface with minkowskyan metric into a minkowskyan flat superspace. The embedding must be minkowskyan because Majorana-Weyl spinors in 10-dimension riemannian manifolds do not exist. It is convenient however, as done before, to take a riemannian world-sheet (we also assume it to be boundaryless).

The Green-Schwarz lagrangian for the heterotic string is given by: [31]

$$\mathcal{L} = -\frac{1}{2}\sqrt{g}\, g^{ab}\, \Pi_a^{\mu}\Pi_{\mu b} + \varepsilon^{ab}\partial_a X^{\mu}(\bar{\theta}\gamma_{\mu}\partial_b\theta) \tag{3.55}$$

with $\Pi_a^{\mu} = \partial_a X^{\mu} - i\bar{\theta}\gamma^{\mu}\partial_a\theta$

θ is an anticommuting Majorana-Weyl spinor (having 16 real components).
An extra-term which contains the sector of left-moving "matter-fields" must be added to the lagrangian.

Besides global super-Poincaré invariance, the Green-Schwarz action admits a local supersymmetry given by

$$\begin{aligned} \delta\theta &= 2i\,\frac{M_{\bar z} M_z \kappa}{\Pi^2} \\ \delta X^\mu &= i\,\bar\theta \gamma^\mu \delta\theta \\ \delta e^a_z &= -8\left(e^b_z \partial_b \bar\theta\, \frac{M_z \kappa}{\Pi^2}\right) e^a_{\bar z} \end{aligned} \qquad (3.56)$$

κ is the gauge paramete; $\quad M_{z,\bar z} = e^a_{z,\bar z} \Pi_a^\mu \gamma_\mu$

$e^a_z, e^a_{\bar z}$ are "zweibeins".

The local supersymmetry reduces at a half (from 16 to 8) the number of fermionic physical degrees of freedom, in such a way that they match the number of bosonic physical degrees of freedom as it is required from global supersymmetry.

The local supersymmetry closes only on-shell: this fact is at the basis of the difficulties in the quantization procedure (just like those supergravity theories for which no set of auxiliary fields which closes the algebra is known).

As a gauge choice we can take the light-cone gauge condition

$$\gamma^+\theta = \frac{1}{\sqrt{2}}\left(\gamma^0+\gamma^9\right)\theta = 0 \qquad (3.57)$$

In order to impose it near the gauge slice we have to choose a gauge parameter

$$\kappa = \frac{-1}{4i}\left(e^a_{\bar z}\,\Pi^+_a\right)^{-1} M_{\bar z}\,\gamma^+\theta \qquad (3.58)$$

We must notice that this expression is singular at zeros of $e^a_{\bar z}\Pi^+_a$ (this fact is probably related with the problem we will discuss later).

However the jacobian of the change of variable from $\gamma^+\theta$ to κ is independent of $e^a_{\bar z}\Pi^+$ and this change of variable should be allowed.

With this gauge-condition our lagrangian simplifies to:

$$\begin{aligned}\mathcal{L} = &-\frac{1}{2}\sqrt{g}\,g^{ab}\left(\partial_a X^i\partial_b X^i - 2\partial_a X^-\partial_b X^+\right) + \\ &+ \partial_a X^+\left[-i\sqrt{g}\,g^{ab}\bar\theta\gamma^-\partial_b\theta - \varepsilon^{ab}\bar\theta\gamma^-\partial_b\theta\right]\end{aligned} \qquad (3.59)$$

Introducing the projectors $P^{ab}_\pm = \frac{1}{2}\left(g^{ab} \pm i\varepsilon^{ab}/\sqrt{g}\right)$ (3.60)

the last term can be rewritten as

$$(-2i\sqrt{g})\,\partial_a X^+ P^{ab}_-\,\bar\theta\gamma^-\partial_b\theta$$

To understand the passage from 3.55 to 3.59 we have to take into account the properties of γ matrices, like the hermiticity condition

$$(\gamma^+)^+ = -\gamma^- \qquad \gamma^{i+} = \gamma^i$$
$$i = 1,2,\ldots,8 \qquad \gamma^- = \frac{1}{\sqrt{2}}(\gamma^0 - \gamma^9) \tag{3.61}$$

and the commutation relations

$$\gamma^+\gamma^i = -\gamma^i\gamma^+$$
$$\gamma^+\gamma^0 = \gamma^0\gamma^+ \tag{3.62}$$

which imply that terms like $\bar{\theta}\gamma^i\partial\theta$ are vanishing in our gauge.

It is an aesy exercise to check that the classical equations of motion for X^μ are given by

$$\Delta X^i = 0$$
$$\Delta X^+ = 0 \tag{3.63}$$
$$\Delta X^- = \frac{2i}{\sqrt{g}}\partial_a\left(P_-^{ab}\bar{\theta}\gamma^-\partial_b\theta\sqrt{g}\right)$$

In order to quantize our theory we have to split the X^μ fields into a classical background $\hat{X}^\mu$ which solves the equations of motion plus a quantum correction:

$$X^\mu = \hat{X}^\mu + X^\mu_Q \tag{3.64}$$

The terms which in our lagrangian contain the quantum fields X^μ_Q, θ are therefore given by

$$-2iP_-^{ab}\partial_a\hat{X}^+(\bar{\theta}\gamma^-\partial_b\theta)\sqrt{g} \underbrace{- 2iP_-^{ab}\partial_a X^+_Q(\bar{\theta}\gamma^-\partial_b\theta)\sqrt{g}}_{I} +$$

$$-\frac{1}{2}\sqrt{g}g^{ab}\Big[\partial_a X^i_Q \partial_b X^i_Q + \underbrace{2\partial_a \hat{X}^i \partial_b X^i_Q}_{\mathrm{II}} - \underbrace{2\partial_a \hat{X}^+ \partial_b X^-_Q}_{\mathrm{III}} - \underbrace{2\partial_a X^+_Q \partial_b \hat{X}^-}_{\mathrm{IV}} - 2\partial_a X^+_Q \partial_b X^-_Q\Big] \tag{3.65}$$

If we integrate by parts II and III and take into account, respectively, the equations of motion of $\hat{X}^i$ and $\hat{X}^+$, our lagrangian simplifies.

Another simplification is obtained by integrating by parts I and IV and taking into account the equations of motion of $\hat{X}^-$.

We have also to notice (as we can see by integrating by parts $\sqrt{g}g^{ab}\partial_a\hat{X}^+\partial_b\hat{X}^-$) that in our lagrangian there is no dependence on $\hat{X}^-$.

The only term in the lagrangian which contains the Majorana-Weyl θ fields is therefore

$$-2iP_-^{ab}\partial_a\hat{X}^+(\bar{\theta}\gamma^-\partial_b\theta)\sqrt{g}$$

with $\hat{X}^+$ such that $\Delta_0\hat{X}^+=0$

The integration over θ leads us to $(\det D)^4$ where D is the operator

$$D = P_-^{ab}\,\partial_a\hat{X}^+\partial_b \tag{3.66}$$

expressed in local complex coordinatesas:

$$D = g^{z\bar{z}}\partial_{\bar{z}}\hat{X}^+\partial_z \tag{3.67}$$

The integration over X^{μ}_{Q} leads to $\left(\det \Delta_0\right)^{-\frac{10}{2}}$

To look at the conformal anomaly of our theory we have therefore to compute the conformal anomaly for the operator D .

It is easily proven that for an operator $D = g^{z\bar{z}} U_{\bar{z}} \partial_z$

D^+ , given by $D^+ = g^{z\bar{z}} U_z \partial_{\bar{z}}$ (3.68)

is its adjoint if and only if $\partial_z U_{\bar{z}} = 0$ (3.69)

It is well known [36] that for a boundaryless manifold M with h handles this equation has h solutions, each of one with $2(h-1)$ zeros ($U_{\bar{z}}$ is antianalytical because 3.69 is a Cauchy-Riemann equation).

However, and this fact is of difficult interpretation for our case, the Liouville theorem requires that no solution can be expressed as $U_{\bar{z}} = \partial_{\bar{z}} \phi$.

If external sources are present $\partial_{\bar{z}} \hat{X}^+$ is a meromorphic function with n poles and $2(h-1)+n$ zeros.

What we have done is to apply the machinery of ch. 1 and 2 to compute the conformal anomaly for the operator

$$\Omega = D^+ D = -g^{z\bar{z}} U_{\bar{z}} \partial_z g^{z\bar{z}} U_z \partial_z = -(g^{z\bar{z}})^2 U_z U_{\bar{z}} \partial_z \partial_{\bar{z}} - g^{z\bar{z}} U_{\bar{z}} \left[\partial_z \left(g^{z\bar{z}} U_z\right)\right] \partial_{\bar{z}} \quad (3.70)$$

with the condition $\partial_z U_{\bar{z}} = 0$ (3.71)

In our case $g^{z\bar{z}}\sqrt{|U_z|^2}$ plays the role of the metric.

It must be said that since U_z is not nowhere vanishing (the only exception is the unique solution to $\partial_z U_{\bar{z}\bar{z}}=0$ on a torus $h=1$), the metric is singular at certain points and the application of the method presented in ch. 2 has to be done in a more careful way.

We have to insert the coefficients

$$A = \frac{1}{4}(g^{z\bar{z}})^2 U_z U_{\bar{z}} \tag{3.72}$$

$$B = \frac{1}{2} g^{z\bar{z}} U_{\bar{z}} \partial_z (g^{z\bar{z}} U_z) \tag{3.73}$$

in the formula (2.29)

It is $g^{z\bar{z}} = 2e^{-2\sigma}$

We define $e^{-4\sigma'} = e^{-4\sigma}|U|^2$

The condition analogous to 3.27 is now

$$\sigma'(0)=0 \qquad \partial_{z,\bar{z}}\sigma'\big|_0 = 0$$

It is easily shown that

$$\partial_{\bar{z}}B\big|_0 = \frac{1}{2}\partial_{\bar{z}}\left[g^{z\bar{z}}U_{\bar{z}}\,\partial_z(g^{z\bar{z}}U_z)\right]\Big|_0 = \frac{1}{2}\partial_{\bar{z}}\left[4(g^{z\bar{z}})^{-1}U_z^{-1}\partial_z(g^{z\bar{z}}U_z)\right]\Big|_0$$

which implies

$$\partial_{\bar{z}}B\big|_0 = 2\partial_z\partial_{\bar{z}}\left[\ln g^{z\bar{z}} + \ln U_z\right]$$

In the formula 2.29 a term proportional to

$$\partial_z\partial_{\bar{z}}\sigma'\big|_0 = \partial_z\partial_{\bar{z}}\sigma\big|_0 - \frac{1}{4}\partial_z\partial_{\bar{z}}\ln|U|^2\big|_0$$

is also present.

The final result (expressed in a manifestly covariant form) is

$$G_{\Omega}(0,0,\tau) = \frac{1}{2\pi\tau} - \frac{1}{12\pi} R(0) - \frac{1}{12\pi} \frac{2}{\sqrt{g}} \partial_z \partial_{\bar{z}} \ln U_z \Big|_0 + O(\sqrt{\tau}) \qquad (3.74)$$

The last term gives contribution only in presence of zeros and poles of U_z .

It is useful to compare the formula above with the formula for the laplacian operator Δ_0^+ :

$$G_{\Delta_0^+}(0,0,\tau) = \frac{1}{2\pi\tau} + \frac{1}{12\pi} R(0) + \ldots \qquad (3.75)$$

Under an infinitesimal Weyl transformation $\sigma \mapsto \sigma + \delta\sigma$ the variation of Ω is given by

$$\Omega \mapsto \Omega' = e^{-2\delta\sigma} D^+ e^{-2\delta\sigma} D = \Omega - 2\delta\sigma D^+ D - 2 D^+ \delta\sigma D \qquad (3.76)$$

If we repeat the analysis done in the previous section for the $\Delta_n^{(+)}$ laplacians we get that the conformal anomaly for D^+D is given by:

$$\delta \ln \det(D^+D) = \frac{4}{12\pi} \int \sqrt{g}_{(x)}\, \delta\sigma_{(x)} \left[R_{(x)} + \frac{2}{\sqrt{g}} \partial_z \partial_{\bar{z}} \ln U_z \Big|_0 \right] \qquad (3.77)$$

The corresponding term for the laplacian Δ_0^+ is

$$\delta \ln \det(\Delta_0^+) = \frac{-2}{12\pi} \int \sqrt{g}_{(x)}\, \delta\sigma_{(x)} R_{(x)} \qquad (3.78)$$

The presence of the extra-term in 3.77 has not yet been interpreted.

For a better understanding one can think to delete a small area from the world-sheet surface around each of the zeros of U_z and to compute the conformal anomaly. To do this a modification of the machinery presented in ch. 2 is required, in order to treat with manifolds with boundary.[37]

If we discard for the moment the extra terms we can see that the contribution to the conformal anomaly of $\eth^{+}\eth$ is such that the heterotic string is conformally invariant for left and right movers separately (this also means, as Alvarez has pointed out,[38] that the Lorentz anomaly vanishes as well: the Lorentz anomaly has in fact the same absolute value as the conformal one; in the ordinary bosonic string is not present because left and right movers give opposite contribution).

To see the cancellation of the conformal anomaly we have to look at the right moving sector only because the left moving sector is already equivalent to the standard 26-dimensional bosonic string.

The determinants involved in the computation are the laplacian Δ_0 acting on 10 bosonic coordinates X^μ, the Fadeev-Popov determinant for the diffeomorphisms $(\det \Delta_1^+)^{\frac{1}{2}}$ and the contribution of $(\det D^+D)^{\frac{1}{2}}$ which acts on the Majorana-Weyl fields. One has to take into account the fact that the laplacian and the Fadeev-Popov determinant have to be restricted to the right movers sector. The computation which shows the cancellation of the conformal anomaly is sketched in app. 3.

APPENDIX 1

REAL AND COMPLEX NOTATION FOR 2-DIMENSIONAL ORIENTED MANIFOLDS

Every oriented 2-dimensional real manifold is a complex manifold of complex dimension 1. If such a manifold is also connected, it is then called a Riemann surface.

Let x_1, x_2 be the real coordinates which, in a given chart, specify a point of our surface. The connection between real and complex notation is given by the following relations:

$$z = x_1 + i x_2 \qquad \bar{z} = x_1 - i x_2 \tag{A.1}$$

$$\partial_z = \frac{1}{2}\left(\partial_{x_1} - i\partial_{x_2}\right) \qquad \partial_{\bar{z}} = \frac{1}{2}\left(\partial_{x_1} + i\partial_{x_2}\right) \tag{A.2}$$

$$4\partial_z\partial_{\bar{z}} = \partial_{x_1}^2 + \partial_{x_2}^2 = \Delta_0 \tag{A.3}$$

The condition $\partial_{\bar{z}} f = 0$ (or respectively $\partial_z f = 0$) (A.4)
is the Cauchy-Riemann equation which implies the analyticity (or the antianalyticity) of f

A basic fact in 2 dimensions is the following: for any given riemannian metric $g_{ab}(x_1, x_2)$ there is always a reparametrization $x \mapsto x' = x'(x)$ (a change of chart) which at least locally, i.e. in a given chart, makes the metric conformally euclidean. We can therefore make use of the reparametrization invariance to put ourselves locally in the so-called "conformal coordinate system", with the metric given by:

$$ds^2 = e^{-2\sigma(x_1,x_2)}(dx_1^2 + dx_2^2) = e^{-2\sigma} dz\, d\bar{z} \tag{A.5}$$

In our system we can write the metric ds^2 in terms of complex coordinates as

$$ds^2 = g_{z\bar{z}}\, dz\, d\bar{z} + g_{\bar{z}z}\, d\bar{z}\, dz$$

with metric tensor components given by

$$\begin{aligned} g_{zz} &= g_{\bar{z}\bar{z}} = 0 \\ g_{z\bar{z}} &= g_{\bar{z}z} = \frac{1}{2} e^{2\sigma} \end{aligned} \tag{A.6}$$

We remark that an analytic change of coordinates $z \mapsto z' = f(z)$, $\partial_{\bar{z}} f = 0$ is the most general coordinates transformation which preserves the conformal nature of our local coordinate system. Under such an analytic change of coordinates the metric component $g_{z\bar{z}}$ transforms as a tensor so that $g_{z\bar{z}}\, dz\, d\bar{z}$ is a scalar.

In general a n-rank tensor field T (for n integer) is a function of $z, \bar{z}$ which under the analytic reparametrization $z \mapsto z' = z'(z)$ transforms as follows:

$$T \mapsto T' = \left(\frac{\partial z'}{\partial z}\right)^n T \tag{A.7}$$

If $n \geqslant 0$, such a tensor will be denoted as $T^{z\ldots z}$ with n upper indices; if $n<0$ it will be denoted with n lower indices as $T_{z\ldots z}$

It is sufficient to consider tensors which have only z indices because the metric $g_{z\bar{z}}$ can be used to trade a $\bar{z}$ index for a z index.

The space of n-rank tensor fields $T^{z\ldots z}$ will be denoted as $\mathcal{T}^{n}$.

$\mathcal{T}^{n}$ has the structure of Hilbert space with an inner product given by:

$$\text{for } S, T \in \mathcal{T}^{n} \qquad \langle S | T \rangle \underset{\text{def.}}{=} \int d^2z \sqrt{g}\, (g_{z\bar{z}})^{n}\, S^{*} T \tag{A.8}$$

(if S is a n-rank tensor in z , S^{*} is a n-rank tensor in $\bar{z}$)

Such a definition makes our inner product an invariant quantity under reparametrization.

We specify that the conventions we have used are the following:

$$\begin{aligned} \sqrt{\det g_{ab}(x_1,x_2)} &= \sqrt{g(x_1,x_2)} = e^{2\sigma} \\ \sqrt{\det g(z,\bar{z})} &= \sqrt{g(z,\bar{z})} = g_{z\bar{z}} = \tfrac{1}{2} e^{2\sigma} \\ g^{z\bar{z}} &= (g_{z\bar{z}})^{-1} = 2 e^{-2\sigma} \\ d^2 z &\equiv 2\, dx_1 dx_2 \end{aligned} \tag{A.9}$$

APPENDIX 2

COVARIANT DERIVATIVES

In app. 1 we have introduced the space $\mathcal{T}^n$ of n-rank tensor fields. Here we will introduce the covariant derivatives which transform a tensor field into a tensor field.

It is easy to verify that the operator ∇_n^z , expressed in local coordinates by $\nabla_n^z = g^{z\bar{z}}\partial_{\bar{z}}$ (A.10)

has the following property $\nabla_n^z : \mathcal{T}^n \to \mathcal{T}^{n+1}$

which makes it a "raising" operator.

Conversely, the operator ∇_z^n , expressed by

$$\nabla_z^n = (g^{z\bar{z}})^n \partial_z (g_{\bar{z}z})^n = \partial_z + 2n\,\partial_z\sigma \qquad (A.11)$$

in the local conformal system, satisfies

$$\nabla_z^n : \mathcal{T}^n \to \mathcal{T}^{n-1}$$

(∇_z^n is a " lowering " operator).

It is easily checked that, with our scalar product, raising and lowering operators are adjoint of each other:

$$(\nabla_n^z)^+ = -\nabla_z^{n+1} \qquad (A.12)$$

With our raising and lowering operators we can build up two different kinds of self-adjoint elliptic operators

$\Delta_n^{\pm}$, such that $\Delta_n^{\pm} : \tau^n \to \tau^n$

They are defined respectively by:

$$\Delta_n^{(+)} = -2\nabla_z^{n+1}\nabla_n^z \qquad (A.13)$$

$$\Delta_n^{(-)} = -2\nabla_{n-1}^z\nabla_z^n \qquad (A.14)$$

In the conformal coordinates system they are given by:

$$\Delta_n^{+} = -4e^{-2\sigma}\left[\partial_z\partial_{\bar z} + 2n(\partial_z\sigma)\partial_{\bar z}\right] \qquad (A.15)$$

$$\Delta_n^{-} = -4e^{-2\sigma}\left[\partial_z\partial_{\bar z} + 2n(\partial_z\sigma)\partial_{\bar z} + 2n(\partial_z\partial_{\bar z}\sigma)\right] \qquad (A.16)$$

In real coordinates we have:

$$\Delta_n^{+} = -e^{-2\sigma}\left[\Delta_0 + 2n\left((\partial_{x_1}-i\partial_{x_2})\sigma\right)(\partial_{x_1}+i\partial_{x_2})\right] \qquad (A.17)$$

$$\Delta_n^{-} = -e^{-2\sigma}\left[\Delta_0 + 2n\left((\partial_{x_1}-i\partial_{x_2})\sigma\right)(\partial_{x_1}+i\partial_{x_2}) + 2n\Delta\sigma\right] \qquad (A.18)$$

$\Delta_n^{\pm}$ are called generalized laplacians because in the limit of flat metric they coincide with the usual flat laplacian (this also explains the choice of 2 in A.13/14 as normalizing factor).

We close this appendix by introducing the curvature R , which is expressed by the following relation:

$$\nabla_z^{n+1}\nabla_n^z - \nabla_{n-1}^z\nabla_z^n = \frac{n}{2}R \qquad (A.19)$$

Despite of the form of the l.h.s., R is a function, not an operator.

R turns out to be a scalar object which is given, in the local conformal system, by

$$R = -2\, e^{-2\sigma} \Delta_0 \sigma \tag{A.20}$$

The Gauss-Bonnet theorem ensures us that the quantity $\frac{1}{4\pi}\int_M d^2x \sqrt{g}\, R$ is a topological invariant quantity which precisely equals the Euler charachteristic $\chi(M)$ of the manifold M (we remember that a compact, boundaryless, orientable, real, 2-dimensional manifold M is topologically equivalent to a sphere with some handles; for such a manifold $\chi(M)$ is given by $\chi(M) = 2-2h$ (A.21) where h is the number of handles).

APPENDIX 3

CANCELLATION OF THE CONFORMAL ANOMALY IN THE BOSONIC AND IN THE HETEROTIC STRING

We give here the final computation which shows the cancellation of the conformal anomaly in the bosonic and in the heterotic string.

The notation used is a bit sloppy (for explanations and details see the text).

The operators we are interested in are Δ_0^+, Δ_1^+ and D^+D.

They satisfy

$$\delta \ln \det \Delta_0^+ = -\frac{2}{12\pi}\int \sqrt{g}\, R\, \delta\sigma \qquad (A.22)$$

$$\delta \ln \det \Delta_1^+ = -\frac{26}{12\pi}\int \sqrt{g}\, R\, \delta\sigma \qquad (A.23)$$

$$\delta \ln \det D^+D = +\frac{4}{12\pi}\int \sqrt{g}\, R\, \delta\sigma \qquad (A.24)$$

We have $\ln\det\Delta_0 = \frac{1}{2}\ln\det\Delta_0^+$ and $\ln\det D^+D_{\mathbb{R}} = \frac{1}{2}\ln\det D^+D$ (A.25)

In the bosonic case

$$\left(\ln\det\Delta_1^+\right)^{\frac{1}{2}} + \left(\ln\det\Delta_0\right)^{-\frac{d}{2}} \Rightarrow$$

$$\Rightarrow \quad \frac{1}{2}\left(\frac{-26}{12\pi}\right) - \frac{d}{2}\left(\frac{-1}{12\pi}\right) = \frac{d-26}{24\pi} \qquad (A.26)$$

which implies that the critical dimension is 26.

In the heterotic case we have:

$$\frac{1}{2}\left(\ln\det\Delta_1^+\right)^{\frac{1}{2}} + \frac{1}{2}\left(\ln\det\Delta_0\right)^{-\frac{10}{2}} + \left[\left(\ln\det D^+D_{\mathbb{R}}\right)^{\frac{1}{2}}\right]^{\frac{8}{2}} \Rightarrow$$

right movers

$$\Rightarrow \quad -\frac{26}{48\pi} \quad -\frac{10}{4}\left(\frac{-1}{12\pi}\right) \quad +\frac{8}{4}\left(\frac{2}{12\pi}\right) = 0 \tag{A.27}$$

which implies the cancellation of the conformal anomaly.

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