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SPONTANEOUS COMPACTIFICATION IN SIX-DIMENSIONAL EINSTEIN-LANCZOS-MAXWELL THEORY

Thesis submitted for the degree of

"Magister Philosophiae"

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Supervisor:

Prof. John Strathdee

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C O N T E N T S

INTRODUCTION

In recent years, gravity theories in more than four dimensions have been extensively studied, starting from an old idea of Kaluza [1], aimed at the unification of gravitation and electromagnetism.

He considered a five-dimensional spacetime with metric g_{MN} and showed that if g_{MN} does not depend on the fifth dimension, so that it is unobservable, four dimensional equations of general relativity and electromagnetism are automatically obtained by identifying the g_{M5} components of the metric with the four-dimensional electromagnetic potential, provided that the five-dimensional lagrangian is of the Einstein-Hilbert type.

Later on Klein, looking for a wave equation in five-dimensional space, noticed that Kaluza's theory can describe an infinite tower of charged massive particles, the masses and the charges assuming discrete values [2].

Further investigations were performed by Einstein and Bergmann [3], who proposed to take seriously into account (and not as a mere mathematical artifact) the idea of the fifth dimension, allowing for the spacetime components of g_{MN} to depend periodically on the fifth dimension, with a very short period. This corresponds to taking the fifth dimension to be a circle of very small radius. This idea was generalized by Thiry and Jordan [4], who considered also the g_{55} component of the metric as a function of spacetime coordinates, corresponding to a scalar field (dilaton).

Furthermore, fermions were introduced in the theory by Pauli [5].

In more recent years, these ideas were generalized in more than five dimensions by the introduction of Yang-Mills fields instead of the simple Maxwell field. Starting from a suggestion of De Witt [6], this was first done by Rayski, Kerner [7,8] and by Cho and Freund [9] in the analogue of the Thiry-Jordan theory *.

But the most important advance was the proposal by Cremmer and Scherk of a mechanism of "spontaneous compactification" [11]. They showed that explicit solutions of the equations of motion exist which compactify the extra dimensions, if matter fields are added to the metric field.

This mechanism, if on the one hand gives a physical justification of the anisotropy between spacetime and "internal" dimensions, on the other hand loses the simplicity of the original model, which rested only on the geometry of spacetime.

Many other spontaneously compactifying solutions have been discussed afterwards, in particular based on supergravity in higher dimensions [12,13] or on quantum effects [14].

The study of the classical stability of this kind of solutions was started by Randjbar-Daemi, Salam and Strathdee [15,16] who used the formalism of harmonic expansion of fields on coset spaces [17]. Many models have been studied since then, but no general criterion of stability has been stated.

In spite of the large amount of work performed in the framework of Kaluza-Klein theories, very few attempts have been made up to now to modify the Einstein-Hilbert lagrangian as the lagrangian of the higher dimensional general relativity [18-20].

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* This was achieved by thinking of the extra coordinate as a parametrization of the gauge group manifold [8,9] or of a homogeneous space on which the group acts [10].

It turns out, however, that this lagrangian can be generalized in higher dimensional spaces, if we release the demand that second order differential equations in the metric must be obtained [21,22]. This leads to lagrangians that are quadratic, cubic, etc. in the curvature tensor.

In this work we consider the simplest non-trivial case: a quadratic lagrangian in six-dimensional spacetime ,where spontaneous compactification is triggered by a U(1) field in a monopole-like configuration. We shall study the stability of the compactified solution by employing the methods introduced by Randjbar-Daemi et al. [15,16].

As we shall see, by imposing some very natural requirements on the equations of motion, it turns out that in six dimensions there is a unique term that can be added to the Einstein-Hilbert lagrangian: it is proportional to the well-known Gauss-Bonnet four-dimensional density:

$$R_{KLMN}R^{KLMN} - 4R_{LM}R^{LM} + R^2$$

In four dimensions this is a total derivative and has no influence on the equations of motion, though it has some interesting topological properties. [23,24].

Another remarkable feature of this lagrangian is that, when expanded around a flat background, the term quadratic in the fluctuations of fields vanishes, leading to a theory free from the presence of ghosts that generally affect quadratic gravity theories in absence of torsion [25,27].

This fact has induced some authors to conjecture that the quadratic terms that should be present in the low energy expansion of string theories [30] should have this simple form [28,29].

In four dimensions, quadratic lagrangians (but obviously different from the one considered above) have already been proposed in several different contexts, such as conformal theories of gravity [31], and quantum gravity, due to their nice renormalization properties [25,32,33]. Unfortunately these properties cannot be extended to six dimensions.

Various properties of this kind of lagrangians will be discussed in chapter one and in the appendix. In chapter two we shall give a brief exposition of the foundations of Kaluza Klein theories. Finally, in chapter three we shall present and discuss our six-dimensional calculations, showing the stability and displaying the mass spectrum which stems from our extended gravitational lagrangian coupled to a Maxwell field.

I. THE LANCZOS LAGRANGIAN

I.1 Generalization of the Einstein lagrangian

The introduction of the Einstein equations in the vacuum is usually obtained by solving the following problem: to seek all the tensors A_{ij} with the properties:

a) A_{ij} is a function of the metric g_{ab} and its first and second derivatives;

b) $A_{ij} = A_{ji}$;

c) $A_{ij;j} = 0$

d) A_{ij} is linear in the second derivatives of the metric g_{ab}, so that second order differential equations for g_{ab} hold.

The field equations in the vacuum are then assumed to take the form

$$A_{ij} = 0 \tag{1}$$

With these assumptions the conclusion arises that [34]:

$$A_{ij} = a\, G_{ij} + b\, g_{ij} \tag{2}$$

where G_{ij} is the Einstein tensor.

If we release condition (d), more general terms can be added to A_{ij}, which however vanish in four dimensions, so that (2) remains the most general lagrangian in four dimensions even abandoning condition (d).

More precisely, the following theorem can be stated [21]:

In 2m (or 2m-1) dimensions, the only symmetric tensor $A_{ij}(g_{rs}, g_{rs,t}, g_{rs,tu})$ for which $A_{ij;j} = 0$ is

$$A^{i}_{j} = \sum_{1}^{m-1}{}_{p}\, a_p\, \delta^{i\, i_1 \cdots i_{2p}}_{j\, j_1 \cdots j_{2p}}\, R_{i_1 i_2}{}^{j_1 j_2} \cdots\cdots R_{i_{2p-1} i_{2p}}{}^{j_{2p-1} j_{2p}} + a_0\, \delta^{i}_{j} \tag{3}$$

where a_o and a_p are arbitrary constants and

$$\delta^{i_1\ldots i_N}_{j_1\ldots j_N} = \det \begin{vmatrix} \delta^{i_1}_{j_1} & \ldots & \delta^{i_1}_{j_N} \\ \delta^{i_N}_{j_1} & \ldots & \delta^{i_N}_{j_N} \end{vmatrix} = \varepsilon^{i_1\ldots i_N} \varepsilon_{j_1\ldots j_N} \qquad (4$$

By virtue of the fact that if $n<N$ $\delta^{i_1\ldots i_N}_{j_1\ldots j_N}=0$ identically, we can formally express (3) as an infinite series which in fact has only a finite number of terms depending on the dimension of the space. The simplest lagrangian density from which these equations of motion can be obtained is given by [21]:

$$\mathcal{L} = \sum_{0}^{m-1}{}_p \, 2a_p G_p =$$

$$= \sqrt{g}\left[\sum_{1}^{m-1}{}_p \, 2a_p \, \delta^{i_1\ldots i_{2p}}_{j_1\ldots j_{2p}} R_{i_1 i_2}{}^{j_1 j_2} \ldots R_{i_{2p-1} i_{2p}}{}^{j_{2p-1} j_{2p}} + 2a_o\right] \qquad (5$$

where $g= |\det g_{ab}|$.

If the manifold is compact, of dimension 2m, and the metric is positive definite, then we obtain [22]:

$$\int_M G_m d^{2m}x = (-1)^m \, 2^{2m} \pi^m m! \, \chi(M) \qquad (6$$

where $\chi(M)$ is the Euler characteristic, which is a topological invariant of fundamental importance in classifying the topology of the manifold M [23,24]. In fact G_m is the Euler class on M, and (6) is the well known Gauss-Bonnet theorem* [23].

In dimensions d=2m, G_m is locally a divergence, so one obtains identities from the equations of motion.

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* Another topological invariant useful in the classification of four-dimensional manifolds is the Pontrjagin number defined as

$$\tau(M) = \frac{1}{96\pi^2}\int \varepsilon^{ijkl} R_{kl}{}^{mn} R_{ij}{}^{mn} \sqrt{g}\, d^4x$$

For example, for d=2 one obtains the identity

$$G_{ij}=0 \tag{7}$$

For d=4 the well known Bach-Lanczos identity holds [35]

$$C_{iklm}C_j^{\ klm}-\frac{1}{4}g_{ij}C_{klmn}C^{klmn}=0 \tag{8}$$

where C_{ijkl} is the Weyl tensor.

For $d > 2m$, instead, G_m contributes new terms to the equations of motion. In the following we shall be interested in the six dimensional case. The most general lagrangian density is then:

$$\mathcal{L}=\sqrt{\gamma}\left[\lambda+\frac{1}{k^2}R-\frac{\beta}{2k^2}\left(R_{KLMN}R^{KLMN}-4R_{LM}R^{LM}+R^2\right)\right] \tag{9}$$

where we have redefined, for later convenience, the coupling constants.

Following Madore [19], we call the new term, quadratic in the Riemann tensor and its contractions, the Lanczos lagrangian [35] and the entire class of lagrangians of type (5), Gauss-Bonnet lagrangians.

The equations of motion following from (9) are

$$-\frac{k^2}{2}\lambda\ \gamma_{MN}+G_{MN}-\beta H_{MN}=0 \tag{10}$$

where

$$G_{MN}=R_{MN}-\frac{1}{2}\gamma_{MN}R$$

$$H_{MN}=R_{MPQR}R_N^{\ PQR}+2R_{MPNQ}R^{PQ}-2R_{MP}R^{NP}+RR_{MN}-\frac{1}{4}\left(R_{PQRS}R^{PQRS}-4R_{PQ}R^{PQ}+R^2\right)\gamma_{MN} \tag{11}$$

A more detailed discussion of the properties of the Gauss-Bonnet lagrangians can be found in the appendix.

I.2 Quadratic lagrangians

Lagrangians quadratic in the curvature tensor have been considered since the early days of relativity theory because of their scale invariance [31], and more recently have been extensively investigated in connection with quantum gravity, because of their renormalizability properties [6,32,33].

The main problem with these theories is that they give rise to negative energy states (ghosts) when one looks for the particle content of the quantum theory [25,26]. This is essentially due to the presence of p^4 terms in the propagators of quantum particles, originating from the higher derivatives of the metric present in quadratic lagrangians. In four dimensions this problem can be overcome only by introducing torsion terms in the lagrangian [27].

This problem does not arise in the case of Gauss-Bonnet lagrangians in more than four dimensions [28,29]. These lagrangians, in fact, when expanded around a constant background, give rise at first order to bilinear terms which contain only second derivatives of the fluctuations of the metric, and if the background is flat, reduce to total derivatives.

The particle content of quadratic theories has been discussed by Stelle |26|, who showed that in the general case they comprise a massless and a massive graviton plus a massive scalar.

In the case we are interested in, however, since the linearized quadratic lagrangian is a total derivative, only the massless graviton remains in the flat six-dimensional theory.

Recently, the possibility of the appearance of curvature squared terms in the low energy limit of string theories has been discussed [30,36].

Zwiebach [28] pointed out that in order to avoid ghost particles, that should not be present in string theory, the low energy string lagrangian should be of the Lanczos type.

This conjecture has been confirmed by Romans and Warner [36] who studied a Chapline-Manton [37] ten-dimensional supergravity coupled to supersymmetric Yang-Mills,modified by the introduction of the gravitational Chern-Simons term needed to cancel the anomalies. This theory is supposed to be the low energy limit of type I superstring theory.

It turns out that if supersymmetry has to be preserved, the Lanczos term appears among the new terms in the lagrangian.

Further investigations are in progress concerning the low energy limit of string theories, but no definitive result has been obtained up to now.

II. KALUZA-KLEIN THEORIES

II.1 Kaluza's theory

To give a simple example which shows the mechanism of dimensional reduction, we shall expose in the following the original Kaluza's model [1].

Kaluza started with a five dimensional Einstein theory with metric γ_{MN} $(M=1,\ldots,5)$ and signature $(-++++)$ constrained by the condition*

$$\frac{\partial \gamma_{MN}}{\partial x^5} = 0 \tag{1}$$

This corresponds to requiring that the fields do not depend on the fifth coordinate which is therefore, in some sense, unobservable. This condition was released by Einstein and Bergmann [3] who postulated that the fifth dimension was closed in a circle with radius of microscopic size, so that at ordinary energies it is not possible to observe the extra dimension, and only the average on it of physical quantities is measurable. The fields have therefore to be periodic in the fifth dimension and can be Fourier expanded [2,8], as we shall see later. In this case the weaker condition holds

$$\frac{\partial \gamma_{n5}}{\partial x^5} = 0 \tag{2}$$

A second condition that is imposed on the four-dimensional metric is

$$\frac{\partial \gamma_{55}}{\partial x^m} = 0 \tag{3}$$

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* From now on we indicate with latin letters spacetime indices, with greek letters extra-dimensional indices and with capital letters the full set of indices.

This implies that the fifth dimension has a constant radius. (If we give up this condition we obtain the Jordan-Thiry theory [4] which contains a scalar Brans-Dicke field besides the graviton and the photon).

Geometrically condition (1) means that there exists a Killing vector a_M for the metric and condition (2) means that its flux lines are geodesic.

A special coordinate system can now be found where

$$a_m = \gamma_{5m} \qquad\qquad a_5 = \gamma_{55} = 1 \tag{4}$$

The theory is now invariant under the following transformations of coordinates:

$$x^a \to \bar{x}^a(x^1, x^2, x^3, x^4) \qquad\qquad x^5 \to x^5 \tag{5}$$

$$x^a \to x^a \qquad\qquad x^5 \to x^5 + f(x^1, x^2, x^3, x^4) \tag{6}$$

It is easy to show that under transformations (5) γ_{mn} and a_m behave like four-dimensional tensors, while under transformations (6) they behave in the following manner:

$$a_m \to a_m - \frac{\partial f}{\partial x^m} \tag{7}$$

$$\gamma_{mn} \to \gamma_{mn} - \gamma_{m5}\frac{\partial f}{\partial x^n} - \gamma_{n5}\frac{\partial f}{\partial x^m} + \frac{\partial f}{\partial x^m}\frac{\partial f}{\partial x^n} \tag{8}$$

We can now recognize in (5) general four-dimensional coordinate transformations, and if we identify a_m with the electromagnetic potential times a constant, we can recognize in (7) gauge transformations.

Finally, if we redefine

$$g_{mn} = \gamma_{mn} - a_m a_n \tag{9}$$

we see that, in our coordinate system, g_{mn} has only the four dimensional components non-vanishing, and is invariant under the gauge transformation (6), while it is a tensor with respect to the general coordinate transformations (5).

So we have recovered electromagnetic and gravitational fields in four dimensions. Furthermore, the five dimensional generalization of the Einstein action (where $\hat{R}$ is the Ricci scalar in five dimensions $\hat{R} = \hat{R}^{K}{}_{LLK}$):

$$-\frac{1}{2\hat{\varkappa}} \int \sqrt{\gamma}\, \hat{R}\, d^5x \qquad \gamma = |\det(\gamma^{MN})| \tag{10}$$

can be written in terms of four-dimensional quantities as

$$-\frac{1}{2\varkappa} \int \sqrt{g} \left(R + \frac{\ell^2}{4} F_{\mu\nu} F^{\mu\nu}\right) d^4x \tag{11}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \qquad A_\mu = \ell^{-1} a_\mu \tag{12}$$

$$\hat{\varkappa} = 2\pi r \varkappa \qquad \ell = (2\varkappa)^{\frac{1}{2}} \sim 1{,}1 \cdot 10^{-32}\, \mathrm{cm} \tag{13}$$

In the above formulae, $\varkappa$ is the Newton constant, ℓ is the Planck length, and r is a constant with the dimension of a length. In Einstein-Bergmann theory it can be identified with the radius of the internal dimension.

(10) is exactly the four dimensional Einstein-Maxwell lagrangian, from which the Einstein and Maxwell field equations follow! (Notice that if a timelike signature for the fifth dimension had been chosen, the wrong sign for the Maxwell lagrangian would have been obtained).

Furthermore, if we project out on the 4-manifold the equation of the five-dimensional geodesic:

$$\frac{du^L}{d\tau} + \Gamma^L_{MN}\, u^M u^N = 0 \qquad\qquad u^M = \frac{dx^M}{d\tau} \tag{14}$$

we obtain, after some calculations

$$\frac{du^m}{d\tau} + \Gamma^{\ell}_{mn}\, u^m u^n + \ell Q F_{mn}\, u^n = 0 \tag{15}$$

with

$$Q = \frac{dx^5}{d\tau} = \mathrm{cost} \tag{16}$$

But this is the equation of motion for a charged particle in general relativity, if we pose $Q = \frac{e}{M\ell}$, where M is the mass of the particle. From this and from (16) we see that we can identify the electric charge with the component of the momentum along the fifth dimension, in suitable units.

So the particles follow geodesic trajectories in the fifth dimension. In four dimensions, particles of different charge follow different trajectories because their initial "momentum" in the fifth dimension is different.

Let us now consider a field in the cylindrical space obeying the Klein equation [2]:

$$\Box \varphi = 0 \qquad\qquad \Box = \gamma^{MN} \nabla_M \nabla_N \tag{17}$$

where ∇_M is the covariant derivative. Because of its periodicity, we can expand φ in Fourier series in the fifth coordinate [2,8]:

$$\varphi(x^m, x^5) = \sum_{-\infty}^{\infty}{}_{k}\; \varphi_k(x^m)\; \exp\left[\frac{ikx^5}{2\pi\ell}\right] \tag{18}$$

By substituting (18) in (17) we can deduce the four dimensional equation obeyed by the harmonics:

$$\left(\nabla_m - iA_m \frac{k\ell}{2\pi\ell}\right) \varphi_k(x^m) = 0 \tag{19}$$

from which, putting $e = \frac{\ell}{2\pi r}$, $e_k = ke$; $m = \frac{1}{2\pi r}$, $m_k = km$, we see that we have obtained the four-dimensional Klein-Gordon equation for particles with electric charge e_k and mass m_k, so that the elementary electric charge is given by the ratio between the Planck length and the length of the Klein circle*, and also the masses are quantized. The masses of the charged particles are very large (of the order of the Planck mass $\sim 10^{19}$ GeV), so that they are unobservable.

Also the graviton gives rise to a tower of massive particles. This is easily seen by linearizing the action around a flat background and then expanding in Fourier series, as in the previous case. In detail, putting

$$\gamma_{MN} = \eta_{MN} + \sqrt{\kappa}\, h_{MN} \qquad \eta_{MN} = \begin{pmatrix} -1 & & & & 0 \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ 0 & & & & 1 \end{pmatrix} \tag{20}$$

the action bilinear in h_{MN} becomes

$$A = -\int d^5x \, \sqrt{\gamma} \left(\frac{1}{4} \nabla_L h_{MN} \nabla_L h_{MN} - \frac{1}{4} \nabla_L h_{MM} \nabla_L h_{NN} + \frac{1}{2} \nabla_L h_{LM} \nabla_M h_{NN} - \frac{1}{2} \nabla_L h_{LM} \nabla_N h_{NM} \right) \tag{21}$$

and after imposing the gauge constraints $\frac{\partial \gamma_{MN}}{\partial x^N} = 0$, $\gamma_{MM} = 0$, we obtain

$$A = -\int d^5x \, \sqrt{\gamma} \left(\frac{1}{4} \nabla_L h_{MN} \nabla_L h_{MN} \right) \tag{22}$$

Expanding h_{MN} as

$$h_{MN}(x^m, x^5) = \sum_k h_{MN}^{(k)} \exp\left[\frac{i k x^5}{2\pi r} \right] , \tag{23}$$

substituting in (22) and integrating in dx^5, one obtains

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* Actually, $\frac{e}{2\pi r} = \frac{e}{\sqrt{\hbar}} \frac{1}{\sqrt{137}}$ if e is the electron charge. This fixes the radius of Klein circle.

$$\sum_k \frac{2\pi r}{n} \int d^4x \sqrt{g}\; h^{(k)}_{LM} \left(\partial^2 - \frac{k^2}{(2\pi r)^2}\right) h^{(k)}_{LM} \tag{24}$$

from which the equations of motion for the $h^{(k)}_{MN}$ fields follow

$$\left[\partial^2 - \frac{k^2}{(2\pi r)^2}\right] h^{(k)}_{MN} \; (x^m) = 0 \tag{25}$$

These correspond to a tower of massive spin two particles, with large masses $\frac{k}{2\pi r}$, that add to the massless graviton, photon and (eventually) dilaton, obtained from zero modes, corresponding to $h^{(0)}_{mn}$, $h^{(0)}_{m5}$ and $h^{(0)}_{55}$.

II.2 <u>Spontaneous compactification and stability</u>

The main problem with Kaluza-Klein theory was the origin of Kaluza's constraint (1), which breaks up the covariance of the five-dimensional theory without a physical justification.

An important progress in understanding how this can happen was the proposal by Cremmer and Scherk [11] of the mechanism of "spontaneous compactification" of the higher dimensional spacetime.

They showed that, if an SO(N+1) gauge theory is coupled to gravity in 4+N dimensions, a solution of the field equations exists corresponding to monopole-like configurations of the Yang-Mills field, such that the space assumes the configuration of the direct product of the Minkowski space M^4 and an N-sphere S^N.

This solution was generalized [10,38] and weaker conditions for the gauge field were found [39]. In particular it was shown that in order to obtain the compactification on a coset space G/H, a Yang-Mills field invariant under H is sufficient.

Other kinds of spontaneous compactification have been found, where gravity is coupled with scalar fields [40] or with the antisymmetric tensor gauge fields which arise in higher-dimensional supergravity* [12,13].

Also quantum effects have been advocated [14] to justify dimensional reduction: in this case the energy-momentum tensor responsible for compactification is supposed to arise not from topologically non-trivial configurations of the matter fields, but from one-loop quantum fluctuations of matter fields around their trivial vacuum states.

The classical stability of compactified solutions was investigated at first by Randjbar-Daemi, Salam and Strathdee [15] who used harmonic analysis of fields on coset spaces [17] to classify the excitation modes. The absence of tachyons and ghosts shows the classical stability of the theory. The result of these investigations was that for six dimensional gravity theories coupled to gauge fields that compactify to $M^4 \times S^2$ only the abelian case is stable.

Much work has been done in this context, but very little is known about general criteria of stability, except the case of gravity plus Yang-Mills compactified to $M^4 \times S^N$ [41].

Also semiclassical stability of the Kaluza-Klein theory has been discussed. In this case an instability can arise if different vacuum solutions are separated by a finite energy barrier. Witten [42] showed that the original Kaluza-Klein theory is in this sense unstable, due to the fact that it is not possible to find a positive energy theorem for such a manifold.

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* Actually, also for pure gravity with squared curvature terms in the lagrangian, compactified solutions have been obtained [18], but, as we shall see later, they do not apply to our case.

II.3 Harmonic expansion on hyperspheres

In our study on the stability of the solutions of Einstein-Lanczos-Maxwell theory in six dimensions, we shall use the method of harmonic expansion in coset spaces, introduced by Salam and Strathdee [17,43]. We give here a brief sketch of the basic features of such formalism, with particular reference to the six-dimensional case of $M^4 \times S^2$, that will be used in the following.

Coset spaces are defined in the following way: if H is a subgroup of a continuous group G, then the space of left cosets of H in G, denoted by G/H, is called a coset space, and it is invariant under the action of G. Coset spaces are a particular kind of homogeneous space , since the group G acts freely on G/H.

The relevance of coset spaces is due to their higher degree of symmetry; for example the N-dimensional hyperspheres can be defined as coset spaces:

$$S^N = \frac{SO(N+1)}{SO(N)} \quad .$$

The starting point in Kaluza-Klein compactification on coset spaces is to find a solution of the field equations with the topology $M^4 x G/H$, where M^4 is Minkowski space. This is generally obtained by imposing a monopole configuration on the gauge fields present in the theory, which are supposed to be invariant under the gauge group G_{YM}, that contains H as a subgroup.

In the following, we shall consider for simplicity the case where G_{YM}=H, which is a sufficient condition for the existence of such a solution [39].

The ground state invariance group is given in general by $P^4 \times G \times K$, where P^4 is the Poincaré group and K is a subgroup of G_{YM}: in the case G_{YM} = H, K is the whole group H.

The ground state invariance group is very useful in classifying the fluctuations around the vacuum solution. In particular, the fields associated with zero modes, which are relevant to the low energy sector, will belong to the irreducible representations of this group.

The basic idea of harmonic expansion is to expand the fields in complete sets of functions belonging to irreducible representations of the group of symmetry G of the coset space and to obtain the four dimensional lagrangian by integrating on this space.

To this end, let us study the geometry of coset spaces. We start by introducing some notation : let z^M parametrize our manifold, with $z^M = (x^m, y^\mu)$ and $x^m \in M^4$, $y^\mu \in G/H$.

The Lie algebra of G is spanned by the generators $Q_{\hat{\alpha}}$ ($\hat{\alpha} = 1,\ldots,\dim G$), which obey the commutation relations:

$$[Q_{\hat{\alpha}}, Q_{\hat{\beta}}] = c_{\hat{\alpha}\hat{\beta}}{}^{\hat{\gamma}} Q_{\hat{\gamma}} \tag{26}$$

Among the $Q_{\hat{\alpha}}$ there is a set $Q_{\bar{\alpha}}$ that spans the Lie algebra of H ($\bar{\alpha} = 1,\ldots,\dim H$). The remaining N parameters Q_α span the tangent space of G/H. Obviously $N = \dim G/H = \dim G - \dim H$.

Let us choose from each class of G/H an element L_y to represent it. Then multiplication from left by any $g \in G$ will carry L_y into another coset, with representative element L'_y , and we can write

$$g L_y = L_{y'} h \tag{27}$$

where y' and h are uniquely defined as functions of y and g.

To define a covariant basis, let us consider the one-form:

$$e(y) = L_y^{-1}\, dL_y \tag{28}$$

which belongs to the Lie algebra of G and can therefore be expanded as a linear combination of its generators:

$$L_y^{-1} dL_y = e^{\alpha}(y)\, Q_{\alpha} + e^{\bar{\alpha}}(y)\, Q_{\bar{\alpha}} \tag{29}$$

Under left translations e(y) behaves as

$$e'(y') = h\, e(y)\, h^{-1} + h\, dh^{-1} + h\, L_y^{-1} g^{-1} dg\, L_y h^{-1} \tag{30}$$

or, introducing the adjoint representation of G, $g^{-1} Q_{\hat{\alpha}}\, g = D_{\hat{\alpha}}{}^{\hat{\beta}}(g) Q_{\hat{\beta}}$:

$$e^{\hat{\alpha}}(y') = e^{\hat{\beta}}(y)\, D_{\hat{\beta}}{}^{\hat{\alpha}}(h^{-1}) + (g\, dg^{-1})^{\hat{\beta}} D_{\hat{\beta}}{}^{\hat{\alpha}}(L_y h^{-1}) + (h\, dh^{-1})^{\hat{\alpha}} \tag{31}$$

From this formula it is easy to obtain the transformation properties of e^{α} and $e^{\bar{\alpha}}$. They indicate that e^{α} behaves as an invariant vielbein on G/H, while $e^{\bar{\alpha}}$ can be considered as a connection one-form. In fact, from (29) we can write

$$dL_y^{-1} + e^{\bar{\alpha}}(y)\, Q_{\bar{\alpha}}\, L_y^{-1} = -e^{\alpha} Q_{\alpha}\, L_y^{-1} \tag{32}$$

so that we can define the covariant derivative ∇_{α} by

$$d + e^{\bar{\alpha}}(y)\, Q_{\bar{\alpha}} = a\, e^{\alpha} \nabla_{\alpha} \tag{33}$$

where a is a scale parameter related to the radius of the coset space.

To perform the harmonic expansion on a coset space G/H it is useful to consider the matrices of the unitary representations of the group G. These provide a complete set for representing functions on G and can be suitably restricted for functions on G/H.

It is well known that, given a function ϕ on a group G, it can be expanded as

$$\phi(g) = \sum_{n} \sum_{p,q} \sqrt{d_n}\; D^{n}_{pq}(g)\, \phi^{n}_{pq} \tag{34}$$

where D^n_{pq} are unitary matrices of dimension d_n and the sum is performed on all the matrix elements of the unitary irreducible representations $g \to D^n(g)$.

The coefficients ϕ^n_{pq} can be obtained by integration on the group:

$$\phi^{n}_{pq} = \frac{\sqrt{d_n}}{V_G} \int_G d\mu \; D^{n}_{pq}(g^{-1})\, \phi(g) \tag{35}$$

where $d\mu$ is the invariant measure normalized to volume V_G.

To expand functions on a coset space G/H, we must consider the representations D^n of G constrained by the requirement that they should contain a representation $\mathcal{D}_{ij}$ of the group H. In formulae:

$$D^{n}(hg) = \mathcal{D}(h)\, D^{n}(g) \tag{36}$$

Every function belonging to an irreducible representation of H, $\mathcal{D}(h)$ of dimension $d_{\mathcal{D}}$, can be expanded as

$$\phi_i(g) = \sum_{n} \sum_{\zeta,q} \sqrt{\frac{d_n}{d_{\mathcal{D}}}}\; D^{n}_{i\zeta,q}(g)\, \phi^{n}_{q\zeta} \tag{37}$$

where ζ is needed if $\mathcal{D}(h)$ is contained more than once in D^n. Then

$$\phi^{n}_{q\zeta} = \frac{1}{V_H} \sqrt{\frac{d_n}{d_{\mathcal{D}}}} \int_{G/H} d\mu\; D^{n}_{q,i\zeta}(L_y)\, \phi_i(y) \tag{38}$$

A generic function $\phi_i(x,y)$ on $M^4 \times G/H$, belonging to an irreducible representation of H labeled by i, can thus be written as

$$\phi_i(x,y) = \sum_{n} \sum_{\zeta,q} \sqrt{\frac{d_n}{d_{\mathcal{D}}}}\; D^{n}_{i\zeta,q}(L_y^{-1})\, \phi^{n}_{q\zeta}(x) \tag{39}$$

The orthogonality conditions satisfied by the D^n are

$$\frac{1}{V_{G/H}} \int_{G/H} d\mu \; D^{n}_{q,i\zeta}(L_y)\, D^{n'}_{i\zeta',q'}(L_y^{-1}) = \frac{d_{\mathbb{D}}}{d_n}\, \delta_{nn'}\, \delta_{qq'}\, \delta_{\zeta\zeta'} \qquad (40$$

Another useful property of the $D^n(L_y^{-1})$ is their simple behaviour under covariant differentiation. In fact (32) and (33) yield

$$\nabla_\alpha L_y^{-1} = -\frac{1}{a} Q_\alpha L_y^{-1} \qquad (41$$

and therefore

$$\nabla_\alpha D^{n}_{i\zeta,q}(L_y^{-1}) = -\frac{1}{a} D^{n}_{i\zeta,q}(Q_\alpha L_y^{-1}) \qquad (42$$

so that all ~~the~~ differential operations on G/H can be reduced to algebraic manipulations. For example, the action of the laplacian $\nabla^2 = \nabla_\alpha \nabla_\alpha$ is given by:

$$\nabla^2 D^{n}_{i\zeta,q}(L_y^{-1}) = \frac{1}{a^2} D^{n}_{i\zeta,q}(Q_\alpha Q_\alpha L_y^{-1}) = \frac{1}{a^2}\left[C_G(D^n) - C_H(\mathbb{D})\right] D^{n}_{i\zeta,q}(L_y^{-1}) \qquad (43$$

where $C_G(D^n)$ and $C_H(\mathbb{D})$ are the values taken by the quadratic Casimir operators of G and H in the representations D^n and $\mathbb{D}$.

Example

The two-sphere S^2 can be described as a coset space $\frac{SO(3)}{SO(2)} \sim \frac{SU(2)}{U(1)}$. It can be parametrized by the two angles ϑ, φ, but two patches are needed. In terms of the SU(2) generators Q_1, Q_2, Q_3 we choose the boosts:

$$L_{\vartheta\varphi} = e^{-\varphi Q_3}\, e^{-\vartheta Q_2}\, e^{\varphi Q_3} \qquad 0 \le \vartheta \le \frac{\pi}{2} \qquad 0 \le \varphi \le 2\pi$$

$$L'_{\vartheta\varphi} = e^{-\varphi Q_3}\, e^{-\vartheta Q_2}\, e^{-\varphi Q_3} \qquad \frac{\pi}{2} \le \vartheta \le \pi \qquad 0 \le \varphi \le 2\pi \qquad (44$$

On $\vartheta = \frac{\pi}{2}$ $L' = L\, e^{-2\varphi Q_3}$.

The action of ~~a~~ an SU(2) element g is given by

$$g L_{\theta\varphi} = L_{\theta'\varphi'} h \tag{45}$$

with h= exp ζQ_3 belonging to h and depending on θ, φ and g.

The 1-form $L^{-1} dL$ belongs to the algebra of SU(2) and contains the covariant basis $E^{\pm}$ and the U(1) connection e^3:

$$L^{-1} dL = \frac{1}{a} E^{+} \frac{Q_1 - iQ_2}{\sqrt{2}} + \frac{1}{a} E^{-} \frac{Q_1 + iQ_2}{\sqrt{2}} + e^3 Q_3 \tag{46}$$

where

$$E^{\pm} = \pm \frac{a}{i\sqrt{2}} e^{\pm i\varphi} (d\theta \mp i \sin\theta\, d\varphi) \tag{47}$$

$$e^3 = -d\varphi\ (\cos\theta - 1)$$

According to (45), the action of SU(2) induces a U(1) rotation on S^2, so that the vielbein transform according to

$$\begin{aligned} E^{\pm}(y) &\to E^{\pm}(y') = E^{\pm}(y)\, e^{\pm i\zeta} \\ e^3(y) &\to e^3(y') = e^3(y) - d\zeta \end{aligned} \tag{48}$$

Referred to the frame $E^{\pm}$, tensors on S^2 are automatically decomposed into irreducible representations of SO(2)$\sim$U(1), and the component φ_λ carrying the U(1) quantum number λ can be represented by the equation

$$\varphi_\lambda(x,\theta,\varphi) = \sum_{\ell \geq |\lambda|} \sum_{-\ell}^{\ell}{}_m \sqrt{2\ell+1}\ D^{\ell}_{\lambda m}(L^{-1}_{\theta\varphi})\, \phi^{\ell}_{\lambda m}(x) \tag{49}$$

where $D^{\ell}_{\lambda m}$ belongs to the $(2\ell+1)$ dimensional representation of SU(2):

$$D^{\ell}_{\lambda m}(L^{-1}_{\theta\varphi}) = e^{\mp i\lambda\varphi}\, d^{\ell}_{\lambda m}\, e^{im\varphi} \tag{50}$$

The covariant derivative is defined by

$$\nabla L^{-1} = dL^{-1} + e^3 Q_3 L^{-1} = E^{+}\nabla_{+} L^{-1} + E^{-}\nabla_{-} L^{-1} \tag{51}$$

from which, by using (46), can be easily obtained the relation

$$\nabla_{\pm} D^{\ell}_{\lambda m}(L^{-1}) = -\frac{1}{a\sqrt{2}} D^{\ell}_{\lambda m}(Q_{\pm} L^{-1}) = \frac{i}{a}\sqrt{\frac{(\ell\mp\lambda)(\ell\pm\lambda+1)}{2}}\ D^{\ell}_{\lambda\pm1,m} \tag{52}$$

III. SIX-DIMENSIONAL MODELS

III.1 Einstein-Maxwell theory in six dimensions

To introduce the study of the Lanczos action in six dimensions, we summarize what happens in the simpler case of the six-dimensional Einstein-Maxwell theory [15], characterized by the action

$$S_{EM} = -\frac{1}{V}\int d^6z \left[\frac{1}{\kappa^2}R + \frac{1}{4}F_{MN}F^{MN} + \lambda\right] \tag{1}$$

where κ is a constant with the dimension of a length, $g = |\det g_{MN}|$, and V will be taken to be the volume of the compact space. We put

$$\begin{aligned} R &= g^{LM}R_{LM} = g^{KN}g^{LM}R_{KLMN} \\ R^K{}_{LMN} &= \partial_M\Gamma^K_{NL} - \partial_N\Gamma^K_{ML} + \Gamma^K_{AN}\Gamma^A_{ML} - \Gamma^K_{AM}\Gamma^A_{NL} \\ F_{MN} &= \partial_M A_N - \partial_N A_M \end{aligned} \tag{2}$$

The field equations arising from the variation of the action are

$$\begin{aligned} R_{MN} - \frac{1}{2}g_{MN}R &= -\frac{\kappa^2}{2}T_{MN} + \frac{\lambda\kappa^2}{2}g_{MN} \\ \nabla_M F_{MN} &= 0 \\ T_{MN} &= (F_{ML}F_N{}^L - \frac{1}{4}g_{MN}F^2) \end{aligned} \tag{3}$$

We look for a solution of the type $M^4 \times S^2$. Thus we make the ansatz *:

$$\begin{aligned} g_{MN}\,dz^M dz^N &= g_{mn}(x)\,dx^m dx^n + g_{\mu\nu}(y)\,dy^\mu dy^\nu \\ A_M\,dz^M &= A_\mu(y)\,dy^\mu \end{aligned} \tag{4}$$

---

* We remind the conventions we are using: $z^M = (x^m, y^\mu)$, where m = 0,1,2,3; μ = 5,6 so that x refers to spacetime and y to internal dimensions.

where g_{mn} is an orthogonal metric of a maximally symmetric space (later on we shall restrict it to be the Minkowski space, which is the case of interest for us), and $g_{\mu\nu}$ are polar coordinates on the 2-sphere:

$$g_{\mu\nu}\, dy^\mu dy^\nu = a^2 (d\vartheta^2 + \operatorname{sen}^2\vartheta\, d\varphi^2) \tag{5}$$

where a is the radius of the sphere to be determined in terms of the parameters of the theory.

The 1-form $A_M dy^M$ is required to be invariant (up to gauge transformations) under rotations of the 2-sphere. This leads us to consider the monopole-like configuration

$$A_\mu(y)\, dy^\mu = \frac{n}{2e} (\cos\vartheta \pm 1) \tag{6}$$

where n is a positive or negative integer and e is the U(1) gauge coupling constant, the two signs corresponding to the two patches ($0 \le \vartheta \le \frac{\pi}{2}, \frac{\pi}{2} \le \vartheta \le \pi \quad 0 \le \varphi < 2\pi$) necessary to specify the configuration. The field strength corresponding to (6) is given by the form

$$F = dA = -\frac{n}{2ea^2} \quad a\, d\vartheta \wedge a \sin\vartheta\, d\varphi \tag{7}$$

Corresponding to that, the components of the energy-momentum tensor in our basis are

$$T_{ab} = -\frac{n^2}{8e^2a^4} g_{ab} \qquad T_{a\beta} = T_{\alpha b} = 0$$
$$T_{\alpha\beta} = \frac{n^2}{8e^2a^4} g_{\alpha\beta} \tag{8}$$

In the same basis the non-vanishing part of the Riemann tensor is given by

$$R_{abcd} = \frac{K^2\Lambda}{6} (g_{ac}g_{bd} - g_{ad}g_{bc})$$
$$R_{\alpha\beta\gamma\delta} = \frac{1}{a^2} (g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}) \tag{9}$$

where the four-dimensional cosmological constant Λ and the radius a of the compact space are determined by substituting in the Einstein equations (3). In particular, we are interested in the flat case where $\Lambda = 0$.

One finds the relations:

$$\Lambda = \frac{-n^2}{16e^2a^4} + \frac{\ell}{2}$$

$$\frac{2}{\kappa^2a^2} = \frac{3n^2}{16e^2a^4} + \frac{\ell}{2} \tag{10}$$

so that $\Lambda = 0$ if

$$\ell = \frac{n^2}{8e^2a^4} = \frac{8e^2n^2}{\kappa^4} \tag{11}$$

In this case the radius takes the value

$$a^2 = \frac{n^2}{8}\frac{\kappa^2}{e^2} = \frac{1}{\ell\kappa^2} \tag{12}$$

The vacuum state is invariant under the local group P_4 x SU(2) x U(1), where SU(2) corresponds to the x-dependent rotations of the 2-sphere, U(1) corresponds to the four dimensional part of the Maxwell gauge group and P_4 is the Poincaré group of the Minkowski space.

The spectrum of masses of the theory is obtained by expanding the action functional around the ground state background and by taking the terms bilinear in the fluctuations h_{MN} and V_M: namely, we must substitute in (1) the expansions:

$$g_{MN} = g^{(0)}_{MN} + \kappa\, h_{MN}$$

$$A_M = A^{(0)}_M + V_M \tag{13}$$

where $g^{(0)}_{MN}$ and $A^{(0)}_M$ are the ground state solutions obtained above.

In an orthonormal basis the bilinear terms reduce to

$$\begin{aligned} S_{EM}^{(2)} = \int d^6 z \sqrt{g^{(0)}} \Big[& -\frac{1}{4} h_{AA} h_{BB;CC} + \frac{1}{4} h_{AB} h_{AB;CC} - \frac{1}{2} h_{CA} h_{BA;BC} + \frac{1}{2} h_{AB} h_{CC;AB} \\ & + \frac{1}{2} V_A V_{A;BB} - \frac{1}{2} V_A V_{B;BA} + \kappa \left(h_{AB} F_{BC} - \frac{1}{4} h_{BB} F_{AC} \right) \left(V_{C;A} - V_{A;C} \right) \\ & - \frac{1}{2} R_{BC} \left(h_{AB} h_{AC} - h_{AA} h_{BC} \right) + \frac{1}{2} R_{ABCD} h_{AC} h_{BD} + \frac{1}{2} R_{AB} V_A V_B \\ & - \frac{\kappa^2}{2} F_{BD} F_{CD} \left(h_{AB} h_{AC} - \frac{1}{2} h_{AA} h_{BC} \right) - \frac{\kappa^2}{4} F_{AB} F_{CD} h_{AC} h_{BD} \\ & + \left(R + \frac{\kappa^2}{4} F_{AB} F^{AB} + \lambda \right) \left(\frac{1}{4} h_{AB} h_{AB} - \frac{1}{8} h_{AA} h_{BB} \right) \Big] \end{aligned} \qquad (14$$

where R_{ABCD} , R_{AB} , R , F_{AB} are the background values given by (7) and (9) and the semicolon denotes the covariant derivative with respect to the background connection. Summations are performed by means of the background metric.

This expression conserves the invariance under general coordinate transformations and U(1) gauge transformations of the original one, where the infinitesimal transformations are now

$$\begin{aligned} \delta h_{AB} &= \frac{1}{\kappa} \left(\xi_{A;B} + \xi_{B;A} \right) \\ \delta V_A &= \omega_{;A} - F_{AB} \xi_B \end{aligned} \qquad (15$$

Thus we must fix the gauge to perform calculations. We choose the light-cone gauge, for which the gauge conditions are

$$\begin{aligned} n_M V_M &= 0 \\ n_M h_{MN} &= 0 \end{aligned} \qquad (16$$

where n_M is a light-like vector, for example (1,0,0,1,0,0). In light-cone components $x_{\gtrless} = \frac{1}{\sqrt{2}}(x_0 \pm x_3)$ it can be written as

$$\begin{aligned} V_< &= 0 \\ h_{A<} &= 0 \end{aligned} \qquad (17$$

Writing down the equations of motion for $V_>$, $h_{A>}$ and $h_{>>}$, and substituting them in the action, it turns out that all terms containing $V_>$, $h_{A>}$ and $h_{>>}$ cancel out, so that the action is obtained (where latin indices take now the values 1,2):

$$
\begin{aligned}
S^{(2)}_{EM} = \int \sqrt{g^{(0)}}\Big[\tfrac{1}{4} h_{AB} h_{AB;CC} &- \tfrac{1}{2} R_{BC}(h_{CA} h_{BA} - h_{AA} h_{BC}) \\
&+\tfrac{1}{2} V_A V_{A;BB} - \kappa h_{BA} F_{BC} V_{A;C} + \tfrac{1}{2} R_{AB} V_A V_B \\
&-\tfrac{\kappa^2}{2} F_{BD} F_{CD} h_{AC} h_{AB} - \tfrac{\kappa^2}{4} F_{AC} F_{BD} h_{AB} h_{CD} - \tfrac{\kappa^2}{2} F_{AB} F_{AC} V_B V_C\Big]
\end{aligned} \tag{18}
$$

Some simplifications have been obtained by using the background solution.

At this point we must decompose the fields into irreducible pieces under SO(2). They are listed below:

$$
\begin{array}{lll}
h_{\pm\pm} = \tfrac{1}{2}(h_{55} - h_{66} \mp 2i h_{56}) & \lambda = \pm 2 & s = 0 \\
h_{a\pm} = \tfrac{1}{\sqrt{2}}(h_{a5} \mp i h_{a6}) & \lambda = \pm 1 & s = 1 \\
h_{+-} = \tfrac{1}{2}(h_{55} + h_{66}) & \lambda = 0 & s = 0 \\
V_{\pm} = \tfrac{1}{\sqrt{2}}(V_5 \mp i V_6) & \lambda = \pm 1 & s = 0 \\
\overset{b}{h}_{ab} = h_{ab} - \tfrac{1}{2} g_{ab} h_{cc} & \lambda = 0 & s = 2 \\
h_{cc} & \lambda = 0 & s = 0 \\
V_a & \lambda = 0 & s = 1
\end{array} \tag{19}
$$

The fields quoted above are also irreducible under representations of the rotation group in M^4 and their SO(2) charge λ and their spin s are indicated.

Notice that light-cone gauge imposes a constraint on the above fields, coming from the $h_{>>}$ equation of motion:

$$
h_{+-} = -\tfrac{1}{2} h_{aa} \tag{20}
$$

By substituting into the action and by using the background values of the curvature tensors and of the electromagnetic field, one obtains the bilinear action in terms of the physical fields

$$S_{EH}^{(2)} = \int d^6 z \sqrt{g^{(0)}} \Big[\frac{1}{4} h^t_{ab} (\partial^2+\nabla^2) h^t_{ab} + \frac{1}{4} h_{aa} (\partial^2+\nabla^2) h_{bb} + \frac{1}{2} h_{++} (\partial^2+\nabla^2) h_{--} + h_{a+} (\partial^2+\nabla^2) h_{a-}$$

$$+ \frac{1}{2} V_a (\partial^2+\nabla^2) V_a + V_+ (\partial^2+\nabla^2) V_-$$

$$- \frac{i\sqrt{2}}{a} \Big[\frac{1}{2} h_{aa} (\nabla_- V_+ - \nabla_+ V_-) + h_{--} \nabla_+ V_+ + h_{++} \nabla_- V_- + h_{a-} \nabla_+ V_a - h_{a+} \nabla_- V_a \Big] \qquad (21$$

$$- \frac{1}{a^2} (h_{aa} h_{bb} + h_{a+} h_{a-} + h_{++} h_{--} + 3 V_+ V_-) \Big]$$

where ∂^2 is the spacetime laplacian and $\nabla^2 = (\nabla_+ \nabla_- + \nabla_- \nabla_+) = (\nabla_5^2 + \nabla_6^2)$, ∇ being the covariant derivative with respect to the background metric. The positivity of the coefficients of the spacetime derivatives is a sufficient condition for the absence of ghost states, provided that there are no tachyons.

Introducing harmonic expansin for the fields in (21) and using (II.52) to evaluate the covariant derivatives,one obtains the equations for the harmonic components which are reported below, with the SU(2) indices omitted. Notice that in the light-cone gauge the equations for fields carrying different spin automatically decouple.

$$\Big[\partial^2 - \frac{\ell(\ell+1)}{a^2} \Big] h^t_{ab} = 0$$

$$\Big[\partial^2 - \frac{\ell(\ell+1)}{a^2} \Big] h_{a\pm} \pm \frac{\sqrt{\ell(\ell+1)}}{a^2} V_a = 0$$

$$\Big[\partial^2 - \frac{\ell(\ell+1)}{a^2} \Big] V_a + \frac{\sqrt{\ell(\ell+1)}}{a^2} (h_{a+} - h_{a-}) = 0$$

$$\Big[\partial^2 - \frac{\ell(\ell+1)}{a^2} \Big] h_{aa} + \frac{\sqrt{\ell(\ell+1)}}{a^2} (V_+ - V_-) = 0 \qquad (22$$

$$\Big[\partial^2 - \frac{\ell(\ell+1)-2}{a^2} \Big] h_{\pm\pm} \pm 2 \frac{\sqrt{(\ell-1)(\ell+2)}}{a^2} V_\pm = 0$$

$$\Big[\partial^2 - \frac{\ell(\ell+1)+2}{a^2} \Big] V_\pm \pm \frac{\sqrt{(\ell-1)(\ell+2)}}{a^2} h_{\pm\pm} \pm \frac{\sqrt{\ell(\ell+1)}}{a^2} h_{aa} = 0$$

The solution of this system of equations yields the spectrum of masses of the theory

The massive states have the following masses:

$$M_2^2 = \frac{\ell(\ell+1)}{a^2} \qquad \ell \geq 0$$

$$M_{1\pm}^2 = \frac{\ell(\ell+1) \pm \sqrt{2\ell(\ell+1)}}{a^2} \qquad \ell \geq 1 \tag{23}$$

$$M_{0\pm}^2 = \frac{2\ell(\ell+1)+1 \pm \sqrt{1+12\ell(\ell+1)}}{2a^2} \qquad \ell \geq 0$$

(the subscript indicates the spin of the particle).

The massless states are given by one spin two particle (graviton) and four spin one particles (photon plus three Yang-Mills bosons), corresponding to the local invariance of the ground state under $P_4 \times SU(2) \times U(1)$. No massless scalar is present.

Moreover all masses squared are positive and since,as we have seen, also ghosts are absent, the ground state is at least perturbatively stable.

III.2 Einstein-Lanczos-Maxwell theory: ground state and fluctuations

In the present and in the following sections, we study in some detail a gravity theory coupled to an abelian gauge field in six dimensions, with the gravity lagrangian given by the Einstein-Hilbert plus the Lanczos lagrangians.

The action has the following form:

$$S_{ELM} = -\frac{1}{V}\int \sqrt{g}\, d^6z \left\{ \lambda + \frac{1}{\kappa^2}R - \frac{\beta}{2\kappa^2}\left(R_{KLMN}R^{KLMN} - 4R_{LM}R^{LM} + R^2\right) + \frac{F_{LM}F^{LM}}{4} \right\} \tag{24}$$

where κ, β, λ are dimensionful parameters, $g = |\det g_{MN}|$ and V is the volume of the compact space.

The field equations stemming from this action are [19,6]:

$$G_{MN} - \beta H_{MN} = -\frac{\kappa}{2}\left(T_{MN} - \lambda g_{MN}\right)$$
$$\nabla_M F_{MN} = 0 \tag{25}$$

with

$$G_{MN} = R_{MN} - \frac{1}{2} g_{MN} R$$

$$H_{MN} = R_{MPQR} R_N{}^{PQR} + 2 R_{MPNQ} R^{PQ} - 2 R_{MP} R_N{}^P + R R_{MN} - \frac{1}{4}(R_{PQRS} R^{PQRS} - 4 R_{PQ} R^{PQ} + R^2) g_{MN} \quad (26$$

$$T_{MN} = F_{ML} F_N{}^L - \frac{1}{4} F_{LP} F^{LP} g_{MN}$$

As in the case examined in the previous section, we shall look for a solution of the type $M^4 \times S^2$, where M^4 is Minkowski space.

This leads to the ansatz *:

$$g_{MN} dz^M dz^N = g_{mn}(x) dx^m dx^n + g_{\mu\nu}(y) dy^\mu dy^\nu$$
$$A_M dz^M = A_\mu(y) dy^\mu \quad (27$$

with

$$g_{mn} dx^m dx^n = \eta_{mn}(x) dx^m dx^n$$
$$g_{\mu\nu} dy^\mu dy^\nu = a^2 (d\theta^2 + \sin^2\theta \, d\varphi^2) \quad (28$$

For the time being, we shall take η_{mn} to be an orthogonal metric of a maximally symmetric space (so that spacetime can be de Sitter, anti-de Sitter or Minkowski depending on the parameter Λ). These assumptions yield the following expressions for the curvature tensor:

$$R_{abcd} = \frac{\kappa^2 \Lambda}{6} (\eta_{ac}\eta_{bd} - \eta_{ad}\eta_{bc})$$
$$R_{\alpha\beta\gamma\delta} = \frac{1}{a^2} (g_{\alpha\gamma} g_{\beta\delta} - g_{\alpha\delta} g_{\beta\gamma}) \quad (29$$

where Λ and a have to be determined from the equations of motion.

In this basis some useful relations hold:

$$\Gamma^L_{MN}(z) = \Gamma^L_{MN}(x) + \Gamma^L_{MN}(y)$$
$$R^K{}_{LMN}(z) = R^K{}_{LMN}(x) + R^K{}_{LMN}(y) \quad (30$$

---

* For the conventions see the footnote on page 23.

from which it follows that:

$$\mathcal{L}_L(z) = \mathcal{L}_L(x) + \mathcal{L}_L(y) - R(x)R(y)$$

$$H_{LM}(z) = H_{LM}(x) + H_{LM}(y) - R(x)G_{LM}(y) - R(y)G_{LM}(x) - \frac{1}{2}\mathcal{L}_L(x)g_{LM}(y) - \frac{1}{2}\mathcal{L}_L(y)g_{LM}(x) \qquad (31$$

where $\mathcal{L}_L$ is the Lanczos lagrangian and the variables in the parentheses indicate in what space the quantities are evaluated. (For example R(y) is the Ricci scalar of the internal space).

As discussed in the previous section, the appropriate ansatz for the Maxwell field is

$$A_\mu(y)\,dy^\mu = \frac{n}{2e}\,(\cos\theta \pm 1)\,d\varphi \qquad (32$$

Substituting (29) and (32) in (26) one obtains the equations

$$\frac{K^2\Lambda}{2} + \frac{1}{a^2} + \frac{\beta K^2\Lambda}{a^2} = -\frac{K^2 n^2}{16e^2a^4} + \frac{\lambda K^2}{2}$$

$$K^2\Lambda + \frac{\beta K^4\Lambda^2}{6} = -\frac{K^2 n^2}{16e^2a^4} + \frac{\lambda K^2}{2} \qquad (33$$

This is a fourth degree system of algebraic equations in Λ and $\frac{1}{a^2}$ and its explicit solution is not very useful.

We are, however, interested in the particular case where $\Lambda = 0$ (Minkowski space). If so, we obtain the solution

$$\frac{1}{a^2} = \frac{8e^2}{n^2K^2} \qquad (34$$

with the constraint

$$\lambda = \frac{8e^2}{n^2K^4} \qquad (35$$

which is exactly the same ground state solution we obtained in absence of the Lanczos term. This is a consequence of our choice for the ground state to be a flat spacetime times a 2-sphere. If we let Λ to be different from zero, the presence of the Lanczos term changes the ground state of the theory. In this case compactifying solutions exist also in absence of the Maxwell field, but the curvature of the compact space is comparable with that of the spacetime and the solution is devoid of physical interest*.

Obviously the solution described above admits the ground state invariance group $P_4 \times SU(2) \times U(1)$, as explained before.

Next step in our calculations is the linearization of the action through perturbative expansion around the background solution $\overset{o}{g}_{MN}$, $\overset{o}{A}_M$. If we put

$$g_{MN} = g_{MN}^{(0)} + k\, h_{MN}$$
$$A_M = A_M^{(0)} + V_M \qquad (36$$

we can express the quantities appearing in the lagrangian as a series in h_{MN} and V_M. For example:

$$g^{MN} = g^{MN(0)} + g^{MN(1)} + g^{MN(2)} + \dots$$

$$g^{MN(1)} = -h_{MN} \qquad (37$$
$$g^{MN(2)} = h^{MP} h_P{}^N$$

where the index $^{(0)}$ indicates the background value.

We give here this expansion up to second order [32]:

$$(\sqrt{g})^{(1)} = k\, (\sqrt{g})^{(0)}\, h_{AA}$$
$$(\sqrt{g})^{(2)} = k^2\, (\sqrt{g})^{(0)} \left(\tfrac{1}{8} h_{AA} h_{BB} - \tfrac{1}{4} h_{AB} h_{AB}\right) \qquad (38$$

* That is not so in more than 8 dimensions. In this case solutions of the type $M^4 \times S^N$ are available also for pure gravity if the Lanczos term is present [18,20].

$$\Gamma^{L\,(1)}_{MN} = \frac{\kappa}{2}\, g^{LP(0)} \left(h_{PM;N} + h_{PN;M} - h_{MN;P}\right)$$
$$\Gamma^{L\,(2)}_{MN} = -\frac{\kappa^2}{2}\, h^{LP} \left(h_{PM;N} + h_{PN;M} - h_{MN;P}\right) \tag{39}$$

$$R^{(1)}_{KLMN} = \frac{\kappa}{2}\left[h_{KN;LM} - h_{LN;KM} - h_{KM;LN} + h_{LM;KN} + R_{KAMN}h_{AL} - R_{LAMN}h_{AK}\right]$$
$$R^{(2)}_{KLMN} = \frac{\kappa^2}{4}\left[\left(h_{AK;N} + h_{AN;K} - h_{KN;A}\right)\left(h_{AL;M} + h_{AM;L} - h_{LM;A}\right) - (M \leftrightarrow N)\right] + \text{tot. der.} \tag{40}$$

$$R^{(1)}_{LM} = \frac{\kappa}{2}\left[h_{AA;LM} - h_{AM;LA} + h_{LM;AA} - h_{AL;MA}\right]$$
$$R^{(2)}_{LM} = \frac{\kappa^2}{4}\left[h_{BB;A}\left(h_{AL;M} + h_{AM;L} - h_{LM;A}\right) + \frac{1}{4}\left(h_{BA;M} + h_{BM;A} - h_{AM;B}\right)\left(h_{AL;B} + h_{AB;L} - h_{LB;A}\right)\right] + \text{tot. der.} \tag{41}$$

$$R^{(1)} = \kappa\left(h_{AA;BB} - h_{AB;BA} - h_{AB}R_{AB}\right)$$
$$R^{(2)} = \frac{\kappa^2}{4}\left(h_{AA;C}h_{BB;C} + h_{AB;C}h_{AB;C} - 2h_{CA;B}h_{BA;C}\right) \tag{42}$$

$$F^{(1)}_{MN} = V_{B;A} - V_{A;B}$$
$$F^{(2)}_{MN} = 0 \tag{43}$$

In the formulae above the semicolon indicates covariant derivatives with respect to the background connection and the summations are performed using the background metric.

The linear part of the action vanishes because of the equations of motion. The bilinear part of the Lanczos action can be expanded in the following way (the expansion of the other terms in the action is trivial):

$$\mathcal{L}^{(1)}_L = 2R^{(1)}_{KLMN}R^{(0)}_{KLMN} - 4R^{(1)}_{LM}R^{(0)}_{LM} - h_{KP}\left(4R^{(0)}_{KLMN}R^{(0)}_{PLMN} - 8R^{(0)}_{KM}R^{(0)}_{PM}\right) \tag{44}$$

$$\begin{aligned}\mathcal{L}^{(2)}_L = {} & \left(R^{(1)}_{KLMN}R^{(1)}_{KLMN} - 4R^{(1)}_{LM}R^{(1)}_{LM} + R^{(1)}R^{(1)}\right) + 2\left(R^{(2)}_{KLMN}R^{(0)}_{KLMN} - 4R^{(2)}_{LM}R^{(0)}_{LM} + R^{(2)}R^{(0)}\right) \\ & -2h_{KP}\left(4R^{(1)}_{KLMN}R^{(0)}_{PLMN} - 8R^{(1)}_{PM}R^{(0)}_{LM}\right) + h_{KP}h_{LQ}\left(2R^{(0)}_{KLAB}R^{(0)}_{PQAB} + 4R^{(0)}_{KALB}R^{(0)}_{PQAB}\right. \\ & \left.-4R^{(0)}_{KL}R^{(0)}_{PQ}\right) + h_{LE}h_{EA}\left(4R^{(0)}_{ABCD}R^{(0)}_{LBCD} - 8R^{(0)}_{CD}R^{(0)}_{AD}\right)\end{aligned} \tag{45}$$

Finally

$$(\sqrt{g}\,\mathcal{L}_L)^{(2)} = (\sqrt{g})^{(0)}\,\mathcal{L}_L^{(2)} + (\sqrt{g})^{(1)}\,\mathcal{L}_L^{(1)} + (\sqrt{g})^{(2)}\,\mathcal{L}_L^{(0)} \tag{46}$$

We give below the complete expression for the bilinear part of the whole action (24), comprising the Einstein-Hilbert and the Maxwell terms*

$$\begin{aligned}
S_{ELM}^{(2)} = \int d^6 z \sqrt{g} \Big\{ & -\tfrac{1}{4} h_{AA} h_{BB;CC} + \tfrac{1}{4} h_{AB} h_{AB;CC} - \tfrac{1}{2} h_{CA} h_{BA;BC} + \tfrac{1}{2} h_{AB} h_{CC;AB} \\
& + \tfrac{1}{2} V_A V_{A;BB} - \tfrac{1}{2} V_A V_{B;BA} + \kappa \left(h_{AB} F_{BC} - \tfrac{1}{4} h_{BB} F_{AC}\right)\left(V_{C;A} - V_{A;C}\right) \\
& - \tfrac{1}{2} R_{BC}\left(h_{AB} h_{AC} - h_{AA} h_{BC}\right) + \tfrac{1}{2} R_{ABCD} h_{AC} h_{BD} + \tfrac{1}{2} R_{AB} V_A V_B \\
& - \tfrac{\kappa^2}{2} F_{BD} F_{CD}\left(h_{AB} h_{AC} - \tfrac{1}{2} h_{AA} h_{BC}\right) - \tfrac{\kappa^2}{4} F_{AB} F_{CD} h_{AC} h_{BD} \\
& + \tfrac{\beta}{2} \Big[R\left(\tfrac{1}{2} h_{AA} h_{BB;CC} - \tfrac{1}{2} h_{AB} h_{AB;CC} + h_{AC} h_{AB;BC} - h_{AB} h_{CC;AB}\right) \\
& \quad + R_{CD}\big[h_{AA}\left(4 h_{BC;BD} - h_{BB;CD} - 2 h_{CD;BB}\right) + h_{AB}\left(h_{AB;CD} - 4 h_{AC;DB} + 2 h_{CD;AB}\right) \\
& \qquad + h_{AC}\left(-2 h_{BD;BA} + 2 h_{AB;BD}\right)\big] + R_{ABCD}\big[h_{BC}\left(h_{AD;EE} + 2 h_{EE;AD}\right) \\
& \quad + h_{AE}\left(4 h_{BD;CE} + 2 h_{EC;BD}\right)\big] + R_{ALMN} R_{BLMN}\left(h_{AC} h_{BC} - h_{AB} h_{CC}\right) - 2 R_{ABMN} R_{CDMN} h_{AC} h_{BD} \\
& \quad + R_{AMDN} R_{CMBN}\left(4 h_{AC} h_{BD} - 4 h_{AD} h_{BC} - 2 h_{AB} h_{CD}\right) - 4 R_{ABMN} R_{CMDN} h_{AD} h_{BC} \\
& \quad + R_{AB} R_{CD} h_{AB} h_{CD} + 4 R_{AB} R_{AC} h_{BC} h_{DD} + R R_{AB}\left(h_{AC} h_{BC} - h_{AB} h_{CC}\right) \\
& \quad + R R_{ABCD} h_{AD} h_{BC} - 6 R_{DE} R_{ABCD} h_{AE} h_{CB} + 4 R_{BD} R_{ABCD} h_{AE} h_{CE} \Big] \\
& + \left(\tfrac{1}{8} h_{AA} h_{BB} - \tfrac{1}{4} h_{AB} h_{AB}\right)\left[R - \tfrac{\beta}{2}\left(R_{KLMN} R^{KLMN} - 4 R_{LM} R^{LM} + R^2\right) + \tfrac{\kappa^2 F^2}{4} + \Lambda \kappa^2 \right] \Big\}
\end{aligned} \tag{47}$$

In deriving (47) use has been made of the constant curvature of the background, which implies

$$\nabla_A R_{KLMN}^{(0)} = 0 \tag{48}$$

To obtain the final expression, integrations by parts have been performed

---

* From now on the superscript $^{(0)}$ indicating background values will be omitted.

and the commutation relation between covariant derivatives has been largely used*:

$$T_{A;BC} - T_{A;CB} = R_{ADCB} T_D \tag{49}$$

Notice that fourth order derivatives appearing in the partial calculations cancel out only in the particular combination of the Riemann tensor and its contractions given by the Lanczos lagrangian. For example:

$$(\sqrt{g}R^2)^{(2)} = h_{AA} h_{BB;CCDD} + h_{AB} h_{CD;DCAB} - 2h_{AA;CDBB} + \dots \tag{50}$$

The expression (47) for the bilinear part of the action looks very complicated, but if we substitute in it the background values of the curvature tensors, then a miraculous simplification occurs: the Lanczos term gives rise to the expression

$$\begin{aligned} &\frac{\beta}{2a^2}\left(h_{ab}h_{ab;cc} - h_{aa}h_{bb;cc} + 2h_{cc}h_{ab;ab} - 2h_{ab}h_{ca;cb}\right) \\ &+ \frac{\beta}{2a^2}\left(h_{\alpha\beta}h_{\alpha\beta;\gamma\gamma} - h_{\alpha\alpha}h_{\beta\beta;\gamma\gamma} + 2h_{\gamma\gamma}h_{\alpha\beta;\alpha\beta} - 2h_{\alpha\beta}h_{\gamma\alpha;\gamma\beta} + h_{\alpha\alpha}h_{\beta\beta} - 2h_{\alpha\beta}h_{\alpha\beta}\right) \end{aligned} \tag{51}$$

The first term is proportional to the bilinear part of the Einstein lagrangian density in the four dimensional Minkowski space, while the second is proportional to the same object evaluated on the 2-sphere. But in two-dimensional

---

* The part of (47) quadratic in the curvature tensors is not uniquely defined because many non-trivial relations like $R_{KLPM}R_{ML} + R_{KM}R_{PM}$ are valid for constant curvature and also Bianchi identities can be used to rearrange the terms. Many equivalent expressions can thus be given, but they obviously reduce to the same form when the explicit values of $\overset{o}{R}_{KLMN}$ are substituted.

spaces that is a total derivative (see chapter 1) and so we can drop it out, remaining with*

$$\frac{\beta}{2a^2}\left(h_{ab}\,\partial^2 h_{ab} - h_{aa}\,\partial^2 h_{bb} + 2h_{ab}\nabla_{bc}h_{ac} - 2h_{ab}\nabla_{bc}h_{ca}\right) \tag{52}$$

The bilinear part of the action (24) finally reduces to*

$$\begin{aligned} S^{(2)}_{ELM} = \int d^6 z\,\sqrt{g}\,\Big[\, & \tfrac{1}{4} h_{ab}(\gamma\partial^2+\nabla^2)h_{ab} - \tfrac{1}{4} h_{aa}(\gamma\partial^2+\nabla^2)h_{bb} \\ & + \tfrac{1}{4} h_{\alpha\beta}(\partial^2+\nabla^2)h_{\alpha\beta} - \tfrac{1}{4} h_{\alpha\alpha}(\partial^2+\nabla^2)h_{\beta\beta} \\ & + \tfrac{1}{2} h_{a\alpha}(\partial^2+\nabla^2)h_{a\alpha} - \tfrac{1}{4} h_{aa}(\partial^2+\nabla^2)h_{\alpha\alpha} \\ & + \tfrac{1}{2} h_{\alpha\beta}\nabla_{\alpha\beta}(h_{aa}+h_{\gamma\gamma}) + \tfrac{1}{2} h_{ab}\nabla_{ab}(\gamma h_{cc}+h_{\alpha\alpha}) + h_{a\alpha}\nabla_{a\alpha}(h_{aa}+h_{\gamma\gamma}) \\ & - \tfrac{1}{2} h_{a\alpha}\nabla_{\alpha\beta}h_{a\beta} - \tfrac{1}{2} h_{a\alpha}\nabla_{ab}h_{b\alpha} - h_{a\alpha}\nabla_{a\beta}h_{\alpha\beta} - h_{a\alpha}\nabla_{\alpha\beta}h_{a\beta} - \tfrac{1}{2} h_{\alpha\gamma}\nabla_{\gamma\beta}h_{\alpha\beta} - \tfrac{\gamma}{2} h_{ac}\nabla_{ab}h_{ba} \\ & - \tfrac{1}{2} V_a(\partial^2+\nabla^2)V_a + \tfrac{1}{2} V_\alpha(\partial^2+\nabla^2)V_\alpha - \tfrac{1}{2} V_a\nabla_{ab}V_b - \tfrac{1}{2} V_\alpha\nabla_{\alpha\beta}V_\beta - V_a\nabla_{a\alpha}V_\alpha - \tfrac{1}{2a^2} V_\alpha V_\alpha \\ & + \kappa F_{\alpha\beta}h_{\alpha\gamma}(\nabla_\gamma V_\beta - \nabla_\beta V_\gamma) + \kappa F_{\alpha\beta}h_{a\alpha}(\nabla_a V_\beta - \nabla_\beta V_a) - \tfrac{\kappa}{2}(h_{aa}+h_{\gamma\gamma})F_{\alpha\beta}\nabla_\alpha V_\beta \\ & - \tfrac{\kappa^2}{4} F_{\alpha\beta}F_{\gamma\delta}h_{\alpha\gamma}h_{\beta\delta} - \tfrac{\kappa^2}{2} F_{\alpha\beta}F_{\alpha\gamma}\big[h_{\beta\delta}h_{\gamma\delta} + h_{\beta a}h_{\gamma a} - \tfrac{1}{2} h_{\beta\gamma}(h_{aa}+h_{\delta\delta})\big] \\ & + \tfrac{1}{2a^2}(h_{a\alpha}h_{a\alpha} - h_{aa}h_{\alpha\alpha}) \end{aligned} \tag{53}$$

where

$$\gamma = 1 + \frac{2\beta}{a^2} \tag{54}$$

(The background solution has been used to simplify the expression).

III.3 Einstein-Lanczos-Maxwell theory: spectrum of masses and stability

The expression (47) for the bilinear part of the action is invariant under general coordinate and U(1) gauge transformations:

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* The notations are the following: ∇_A denotes the covariant derivative with background connection, $\nabla_{AB} = \nabla_A\nabla_B$, $\partial^2 = \nabla_a\nabla_a$, $\nabla^2 = \nabla_\alpha\nabla_\alpha$.

$$\delta h_{AB} = \frac{1}{\kappa}\left(\xi_{A;B} + \xi_{B;A}\right)$$
$$\delta V_A = \omega_{;A} + F_{AB}\xi_B \tag{55}$$

This is a consequence of the invariance of the original action (24).

A gauge must then be chosen in order to perform calculations: we shall use also in this case the light cone gauge, as explained in the Einstein-Maxwell case. As we have seen, the most convenient feature of this gauge is that the equations of motion for fields carrying different spin automatically decouple, so that a great simplification arises in calculations.

Let us use the light-cone coordinates

$$x_{\gtrless} = x_0 \pm x_3 \tag{56}$$

so that $x_A y_A = x_a y_a + x_\alpha y_\alpha - x_> y_< - x_< y_>$, where a takes the values 1,2.

Let us impose the gauge condition

$$V_< = h_{A<} = 0 \tag{57}$$

The following equations of motion are obtained from (53) once the gauge conditions have been imposed:

$$\begin{aligned}
h_{>>}:&\quad \gamma h_{cc} + h_{\alpha\alpha} = 0\\
h_{a>}:&\quad \partial_- h_{a+} = \nabla_b h_{ab} + \frac{1}{\gamma}\nabla_\alpha h_{a\alpha}\\
h_{\alpha>}:&\quad \partial_- h_{\alpha+} = \nabla_a h_{a\alpha} + \nabla_\beta h_{\beta\alpha} - (1-\gamma)\nabla_\alpha h_{aa} - \kappa F_{\alpha\beta}V_\beta\\
V_>:&\quad \partial_- V_+ = \nabla_a V_a + \nabla_\alpha V_\alpha
\end{aligned} \tag{58}$$

After substitution in (53) of these equations, the terms containing $V_>$ and $h_{A>}$ cancel out and the expansion is obtained

$$\begin{aligned}
S^{(2)}_{ELM} = \int d^6 x \sqrt{g}\Big\{ & \frac{1}{4}h_{ab}(\gamma\partial^2+\nabla^2)h_{ab} + \frac{1}{4}(1-\gamma)h_{aa}\left[\gamma\partial^2+(1-\gamma)\nabla^2\right]h_{bb}\\
&+\frac{1}{4}h_{\alpha\beta}(\partial^2+\nabla^2)h_{\alpha\beta} + \frac{1}{2}h_{a\alpha}(\partial^2+\nabla^2)h_{a\alpha} - \frac{1}{2}\left(1-\frac{1}{\gamma}\right)h_{a\alpha}\nabla_{\alpha\beta}h_{\beta a} - \frac{1}{2}(1-\gamma)h_{\alpha\beta}\nabla_{\alpha\beta}h_{aa}\\
&+\frac{1}{2a^2}(h_{a\alpha}h_{a\alpha} - \gamma h_{aa}h_{bb}) - \frac{\kappa^2}{4}F_{\alpha\beta}F_{\gamma\delta}h_{\alpha\gamma}h_{\beta\delta} - \frac{\kappa^2}{2}F_{\alpha\beta}F_{\alpha\gamma}\left[h_{\beta\delta}h_{\gamma\delta} + h_{\beta a}h_{\gamma a} - \frac{1}{2}(1-\gamma)h_{\beta\gamma}h_{aa}\right]\\
&+\frac{1}{2}V_a(\partial^2+\nabla^2)V_a + \frac{1}{2}V_\alpha(\partial^2+\nabla^2)V_\alpha - \frac{\kappa^2}{2}F_{\alpha\beta}F_{\alpha\gamma}V_\beta V_\gamma - \frac{1}{2a^2}V_\alpha V_\alpha\\
&-\kappa F_{\alpha\beta}h_{\alpha\gamma}\nabla_\beta V_\gamma - \kappa F_{\alpha\beta}h_{a\alpha}\nabla_\beta V_a + (1-\gamma)\frac{\kappa}{2}F_{\alpha\beta}h_{aa}\nabla_\alpha V_\beta\Big\}
\end{aligned} \tag{59}$$

(Latin indices run now from 1 to 2).

Finally we must decompose the fields into irreducible SO(2) representations. The physical fields are the same ones that are listed in (19):

$$
\begin{aligned}
&h_{++} = \frac{1}{2}\,(h_{55}-h_{66}-2ih_{56}) && h_{--} = \frac{1}{2}\,(h_{55}-h_{66}+2ih_{56}) \\
&h_{a+} = \frac{1}{2}\,(h_{a5}-ih_{a6}) && h_{a-} = \frac{1}{2}\,(h_{a5}+ih_{a6}) \\
&h_{+-} = h_{-+} = \frac{1}{2}\,(h_{55}+h_{66}) && \\
&h^{t}_{ab} = h_{ab} - \frac{1}{2}\,g_{ab}h_{cc} && h_{aa} \\
&V_{+} = \frac{1}{2}\,(V_5-iV_6) && V_{-} = \frac{1}{2}\,(V_5+iV_6) \\
&V_{a} &&
\end{aligned}
\qquad (60
$$

Notice that from the first equation in (58)^{ic} follows that h_{+-} and h_{aa} are linearly dependent:

$$
h_{+-} = -\frac{\gamma}{2}\,h_{aa} \qquad (61
$$

In the basis implied by (60) the components of $F_{\alpha\beta}$ are

$$
\begin{aligned}
&F_{+-} = -F_{-+} = \frac{i\sqrt{2}}{ak} \\
&F_{++} = F_{--} = 0
\end{aligned}
\qquad (62
$$

If we substitute (60), (61) and (62) in (59), we obtain

$$
\begin{aligned}
S^{(2)}_{ELM} = \int d^6 x\sqrt{g}\,\Big\{ & \frac{1}{4}\,h^{t}_{ab}\,(\gamma\partial^2+\nabla^2)\,h^{t}_{ab} + \frac{1}{8}\,h_{aa}\left[(3\gamma-\gamma^2)\,\partial^2-(3-2\gamma+\gamma^2)\nabla^2\right]h_{bb} \\
& + \frac{1}{2}\,h_{++}(\partial^2+\nabla^2)\,h_{--} + h_{a+}\left[\partial^2+\nabla^2-(1-\tfrac{1}{\gamma})\nabla_{-+}\right]h_{a-} \\
& -\frac{1}{2}\,(1-\tfrac{1}{\gamma})\left[h_{a+}\nabla_{--}h_{a+}+h_{a-}\nabla_{++}h_{a-}\right] - \frac{1}{2}\,(1-\gamma)\,(h_{++}\nabla_{--}+h_{--}\nabla_{++})\,h_{aa} \\
& -\frac{1}{a^2}\left(\frac{\gamma^2}{4}\,h_{aa}h_{bb}+h_{++}h_{--}+h_{a+}h_{a-}\right) \\
& +\frac{1}{2}\,V_a\,(\partial^2+\nabla^2)\,V_a + V_+\,(\partial^2+\nabla^2)\,V_- - \frac{3}{a^2}\,V_+V_- \\
& +\frac{i\sqrt{2}}{a}\Big[(h_{a+}\nabla_- - h_{a-}\nabla_+)\,V_a + h_{++}\nabla_-V_- - h_{--}\nabla_+V_+ \\
& \qquad -\frac{1}{2}\,(1-2\gamma)\,h_{aa}\,(\nabla_+V_- - \nabla_-V_+)\Big]
\end{aligned}
\qquad (63
$$

The equations of motion for the fluctuation fields can now be written down:

spin 2

$$\frac{1}{2}\left(\gamma\partial^2+\nabla^2\right)h^t_{ab}=0 \tag{64a}$$

spin 1

$$\begin{aligned}
&(\partial^2+\nabla^2)V_a-\frac{i\sqrt{2}}{a}\left(\nabla_-h_{a+}-\nabla_+h_{a-}\right)=0\\
&\left[\partial^2+\nabla^2-\left(1-\tfrac{1}{\gamma}\right)\nabla_{\pm\mp}\right]h_{a\pm}-\frac{1}{a^2}h_{a\pm}-\left(1-\tfrac{1}{\gamma}\right)\nabla_{\mp\mp}h_{a\mp}\pm\frac{i\sqrt{2}}{a}\nabla_\pm V_a=0
\end{aligned} \tag{64b}$$

spin 0

$$\begin{aligned}
&\frac{1}{4}\left[(3\gamma-\gamma^2)\partial^2+(3-2\gamma+\gamma^2)\nabla^2\right]h_{aa}-\frac{1}{2}(1-\gamma)\left(\nabla_{--}h_{++}+\nabla_{++}h_{--}\right)-\frac{\gamma^2}{4a^2}h_{aa}-\frac{1}{2}(1-2\gamma)\left(\nabla_+V_--\nabla_-V_+\right)=0\\
&(\partial^2+\nabla^2)V_\pm-\frac{3}{a^2}V_\pm\mp\frac{i\sqrt{2}}{a}\nabla_-h_{++}\pm\frac{i}{\sqrt{2}a}(1-2\gamma)\nabla_\pm h_{aa}=0\\
&\frac{1}{2}(\partial^2+\nabla^2)h_{\pm\pm}-\frac{1}{a^2}h_{\pm\pm}-\frac{1}{2}\nabla_{\pm\pm}h_{aa}-\frac{i\sqrt{2}}{a}\nabla_\pm V_\pm=0
\end{aligned} \tag{64c}$$

Each field can now be expanded in harmonics of S^2 and formulae (II.52) can be used to compute covariant derivatives. For example

$$\begin{aligned}
&h_{aa}(x,\vartheta,\varphi)=\sum_{\ell\geq 0}\sqrt{2\ell+1}\sum_m D^\ell_{0m}(\vartheta,\varphi)\,[h_{aa}]^\ell_m(x)\\
&\nabla_+\left[D^\ell_{0m}\,[h_{aa}]^\ell_m\right]=\frac{i}{a}\sqrt{\frac{\ell(\ell+1)}{2}}\,D^\ell_{1m}\,[h_{aa}]^\ell_m
\end{aligned} \tag{65}$$

The resulting equations of motion for the fluctuation fields are then: (SU(2) indices have been omitted)

spin 2

$$\left[\gamma\partial^2-\frac{\ell(\ell+1)}{a^2}\right]h^t_{ab}=0 \tag{66a}$$

spin 1

$$\begin{aligned}
&\left[\partial^2-\frac{\ell(\ell+1)}{2a^2}-\frac{\ell(\ell+1)}{2\gamma a^2}\right]h_{a\pm}\pm\frac{\sqrt{\ell(\ell+1)}}{a^2}V_a=0\\
&\left[\partial^2-\frac{\ell(\ell+1)}{a^2}\right]V_a+\frac{\sqrt{\ell(\ell+1)}}{a^2}\left(h_{a+}-h_{a-}\right)=0
\end{aligned} \tag{66b}$$

spin 0

$$\left[(3\gamma-\gamma^2)\partial^2-\frac{(3-2\gamma+\gamma^2)\ell(\ell+1)+2\gamma^2}{a^2}\right]h_{aa}+(1-\gamma)\frac{\sqrt{(\ell-1)\ell(\ell+1)(\ell+2)}}{a^2}(h_{++}+h_{--})+$$
$$+2(2\gamma-1)\frac{\sqrt{\ell(\ell+1)}}{a^2}(V_+-V_-)=0$$
$$\left[\partial^2-\frac{\ell(\ell+1)-2}{a^2}\right]h_{\pm\pm}+\frac{1}{2}(1-\gamma)\frac{\sqrt{(\ell-1)\ell(\ell+1)(\ell+2)}}{a^2}h_{aa}\pm 2\frac{\sqrt{(\ell-1)(\ell+2)}}{a^2}V_\pm=0 \qquad (66c$$
$$\left[\partial^2-\frac{\ell(\ell+1)+2}{a^2}\right]V_\pm\pm\frac{\sqrt{(\ell-1)(\ell+2)}}{a^2}h_{\pm\pm}+\frac{1}{2}(2\gamma-1)\frac{\sqrt{\ell(\ell+1)}}{a^2}h_{aa}=0$$

The spectrum of masses can then be easily evaluated by diagonalizing the equations. It turns out that ten massless states are present, corresponding to a graviton (spin 2, $\ell = 0$), a photon (spin 1, $\ell = 0$) and a triplet of Yang Mills bosons* (spin 1, $\ell = 1$), corresponding to the background local invariance under $P_4 \times SU(2) \times U(1)$.

The massive states have the following masses

$$M_2^2=\frac{\ell(\ell+1)}{a^2} \qquad \ell\geq 0$$
$$M_{1\pm}^2=\frac{\ell(\ell+1)\pm\sqrt{2\ell(\ell+1)}}{a^2} \qquad \ell\geq 1 \qquad (67$$
$$M_{0\pm}^2=\frac{1}{(6-2\gamma)a^2}\left[(5-\gamma)\ell(\ell+1)+2\gamma\pm\sqrt{[(1-\gamma)\ell(\ell+1)]^2-4(\gamma^2-\gamma-12)+4\gamma^2}\right] \qquad \ell\geq 0$$

In the formulae above the zero values must be discarded and the subscript indicates the spin of the particle.

We see that the Lanczos term does not influence the mass of the spin 1 particles, and multiply by a factor $\gamma^{-\frac{1}{2}}=\left(1+\frac{2\beta}{a^2}\right)^{-\frac{1}{2}}$ that of the spin 2 ones. Effects on scalar particles are more substantial, but in the limit $\beta\to 0$ we recover the previous results.

It is easily seen that masses squared are positive for $0<\gamma\leq 3$, i.e. $-\frac{a^2}{2}<\beta\leq a^2$

The physical significance of these limits will be discussed in the next

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* We remember that ℓ labels the SU(2) representations.

section.

It can be shown that in our gauge a sufficient condition for the absence of ghosts is that (if tachyons are absent) all coefficients of the space-time derivatives have the same sign. (This is equivalent to the requirement that all the residua of the propagators at the poles have the same sign). This condition is fulfilled in the interval $0<\gamma<3$, so that one can conclude that the ground state is classically stable under perturbations.

The limit values $\gamma = 0$ and $\gamma = 3$ are of some interest. In the first case no propagating graviton exists. In the second case the number of scalar degrees of freedom is halved, but the ground state remains stable.

III.4 Einstein-Lanczos-Maxwell theory: the zero-mode ansatz

In this section we derive an effective four dimensional lagrangian for the long range sector of the six-dimensional system. That is to say, we shall suppose that the radius a tends to zero, so that all harmonic components of mass of order $\frac{1}{a}$ are discarded from the theory.

This problem has been studied in the case where $\beta = 0$ in ref. [32]. We shall use the results of that paper and shall limit ourselves to the dimensional reduction of the Lanczos lagrangian. This is a quite involved calculation on its own, so we shall report only the main steps.

The "zero mode ansatz" consists in writing down the components of the vielbein and of the Maxwell potential one-form in terms of the four-dimensional massless fields of the theory. As we have seen in the previous section, these

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* We use here the formalism of forms.

are given by the graviton $E_m^a(x)$, the Maxwell field $V_m(x)$ and a Yang-Mills triplet, which is a mixture of metric and Maxwell fields. In order to take into account this mixing, we start by introducing two independent $\ell=1$ four-vectors $W_m^{\hat\alpha}(x)$ and $U_m^{\hat\alpha}(x)$ into the ansatz. One combination of them will reveal itself as massive and will be discarded. Thus, we put

$$\begin{aligned} E^a(x,y) &= dx^m e_m{}^a(x) \\ E^\alpha(x,y) &= dy^\mu e_\mu{}^\alpha(y) - \kappa\, dx^m W_m{}^{\hat\beta} D_{\hat\beta}{}^\alpha \\ A(x,y) &= dx^m V_m(x) + \frac{n}{2ea}\left[dy^\mu e_\mu{}^3 - \kappa\, dx^m U_m{}^{\hat\beta} D_{\hat\beta}{}^3\right. \end{aligned} \qquad (68$$

where $a=0,1,2,3$; $\alpha=+,-$; $\hat\alpha=1,2,3$; $y^\mu=(a\theta,a\varphi)$ and $D_{\hat\alpha}{}^{\hat\beta}$ are matrices belonging to the adjoint representation of G. The field strengths corresponding to W_m, U_m, and V_m will be respectively denoted by W_{mn}, U_{mn} and V_{mn}.

With this ansatz, the metric can be written in terms of the Killing vectors $K_{\hat\alpha}{}^\mu = D_{\hat\alpha}{}^\beta e_\beta{}^\mu$ as

$$\begin{pmatrix} g_{mn} + \kappa^2 K_{\hat\alpha}{}^\rho K_{\hat\beta\rho} W_m^{\hat\alpha} W_n^{\hat\beta} & -\kappa W_m{}^{\hat\alpha} K_{\hat\alpha\nu} \\ -\kappa\, W_n{}^{\hat\alpha} K_{\hat\alpha\mu} & g_{\mu\nu} \end{pmatrix} \qquad (69$$

The fields transform under SU(2) according to

$$\begin{aligned} E^\alpha(x,y) &\to E'^\alpha(x,y') = E^\beta(x,y)\, D_\beta{}^\alpha(h^{-1}) \\ A(x,y) &\to A'(x',y') = A(x,y) - \frac{n}{2e}\, d\zeta \end{aligned} \qquad (70$$

where $h = \exp[\zeta(y,g)\, Q_3]$. It follows that, in order to mantain the form of the ansatz under such transformations, U and V must transform as SU(2) gauge fields, while V(x) must be invariant:

$$\begin{aligned} V'^{\hat\alpha}(x) &= U^{\hat\beta}(x)\, D_{\hat\beta}{}^{\hat\alpha}(g^{-1}) - \frac{a}{\kappa}\,(g\,dg^{-1})^{\hat\alpha} \\ W'^{\hat\alpha}(x) &= W^{\hat\beta}(x)\, D_{\hat\beta}{}^{\hat\alpha}(g^{-1}) - \frac{a}{\kappa}\,(g\,dg^{-1})^{\hat\alpha} \end{aligned} \qquad (71$$

With our ansatz, the components of the six-dimensional Riemann tensor are given by * [13]:

$$
\begin{aligned}
\hat{R}_{abcd} &= R_{abcd} - \frac{\kappa^2}{4} K_{\hat{\rho}\alpha} K_{\hat{\sigma}}{}^{\alpha} \left(2 W_{ab}^{\hat{\rho}} W_{cd}^{\hat{\sigma}} + W_{ac}^{\hat{\rho}} W_{bd}^{\hat{\sigma}} - W_{ad}^{\hat{\rho}} W_{bc}^{\hat{\sigma}}\right) \\
\hat{R}_{\alpha\beta\gamma\delta} &= R_{\alpha\beta\gamma\delta} \\
\hat{R}_{\alpha b\gamma d} &= \frac{\kappa}{2} (\nabla_{\gamma} K_{\hat{\rho}\alpha}) W_{bd}^{\hat{\rho}} + \frac{\kappa^2}{4} K_{\hat{\rho}\alpha} K_{\hat{\sigma}}{}^{\alpha} W_{ad}^{\hat{\rho}} W_{ab}^{\hat{\sigma}} \\
\hat{R}_{ab\gamma\delta} &= \kappa (\nabla_{\delta} K_{\hat{\rho}\gamma}) W_{ab}^{\hat{\rho}} + \frac{\kappa^2}{4} (K_{\hat{\rho}\delta} K_{\hat{\sigma}\gamma} - K_{\hat{\rho}\gamma} K_{\hat{\sigma}\delta}) W_{ca}^{\hat{\rho}} W_{cb}^{\hat{\sigma}} \\
\hat{R}_{\alpha bcd} &= -\frac{\kappa}{2} D_b W_{cd}^{\hat{\rho}} K_{\hat{\rho}\alpha} \\
\hat{R}_{a\beta\gamma\delta} &= 0
\end{aligned}
\tag{72}
$$

In the formulae above, ∇_{α} denotes the covariant derivative on S^2 and D_a is the gauge covariant derivative defined as

$$
D_a W_{bc}^{\hat{\rho}} = \nabla_a W_{bc}^{\hat{\rho}} - \frac{\kappa}{a} \varepsilon_{\hat{\rho}\hat{\sigma}\hat{\tau}} A_b^{\hat{\sigma}} W_{bc}^{\hat{\tau}} \tag{73}
$$

The six-dimensional quantities appearing in the action can now be written in terms of the four-dimensional ones in the following way

$$
\hat{R} = R + R_{S^2} + \frac{\kappa^2}{4} W_{ab}^{\hat{\rho}} W_{ab}^{\hat{\sigma}} K_{\hat{\rho}\alpha} K_{\hat{\sigma}}{}^{\alpha} \tag{74}
$$

$$
\begin{aligned}
\hat{\mathcal{L}}_L = {} & R^2 - 4 R_{ab} R_{ab} + R_{abcd} R_{abcd} \\
& + R_{S^2}^2 - 4 R_{\alpha\beta} R_{\alpha\beta} + R_{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} + 2 R_{S^2} R \\
& + \kappa^2 \left[\frac{1}{2} R W_{ab}^{\hat{\rho}} W_{ab}^{\hat{\sigma}} - 4 R_{ab} W_{ca}^{\hat{\rho}} W_{cb}^{\hat{\sigma}} - R_{abcd} (W_{ab}^{\hat{\rho}} W_{cd}^{\hat{\sigma}} - W_{ad}^{\hat{\rho}} W_{bc}^{\hat{\sigma}}) \right] K_{\hat{\rho}\alpha} K_{\hat{\sigma}}{}^{\alpha} \\
& + \kappa^2 \left[\frac{1}{2} R_{S^2} W_{ab}^{\hat{\rho}} W_{ab}^{\hat{\sigma}} g_{\alpha\beta} + 2 R_{\alpha\beta} W_{ab}^{\hat{\rho}} W_{ab}^{\hat{\sigma}} \right] K_{\hat{\rho}}^{\alpha} K_{\hat{\sigma}}^{\beta} + 3\kappa^2 (\nabla_{\delta} K_{\hat{\rho}\gamma} \nabla_{\delta} K_{\hat{\sigma}\gamma}) W_{ab}^{\hat{\rho}} W_{ab}^{\hat{\sigma}} \\
& + \kappa^2 \left[(D_a W_{bc}^{\hat{\rho}})(D_a W_{bc}^{\hat{\sigma}}) - 2 (D_a W_{ab}^{\hat{\rho}})(D_c W_{cb}^{\hat{\sigma}}) \right] K_{\hat{\rho}\alpha} K_{\hat{\sigma}}{}^{\alpha} + 3\kappa^3 (K_{\hat{\rho}\alpha} \nabla_{\alpha} K_{\hat{\tau}\beta} K_{\hat{\sigma}\beta}) W_{ab}^{\hat{\rho}} W_{bc}^{\hat{\sigma}} W_{ca}^{\hat{\tau}} \\
& + \kappa^4 f(W^4)
\end{aligned}
\tag{75}
$$

where R_{S^2} denotes the Ricci scalar of the internal space.

Calculations are performed with the Killing metric $g^{\alpha\beta} = K_{\hat{\rho}}^{\alpha} K_{\hat{\rho}}^{\beta}$ on the

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* From now on we denote with a caret the six-dimensional quantities.

2-sphere, that yields the relation

$$\nabla_\alpha K_\beta{}^{\hat\rho} + \nabla_\beta K_\alpha{}^{\hat\rho} = 0 \tag{76}$$

Another useful relation coming from the definition of Killing vectors is:

$$K_{\hat\rho}{}^{\alpha}\,\partial_\alpha K_{\hat\sigma}{}^{\beta} - K_{\hat\sigma}{}^{\alpha}\,\partial_\alpha K_{\hat\rho}{}^{\beta} = -c_{\hat\rho\hat\sigma}{}^{\hat\tau}\,K_{\hat\tau}{}^{\beta} \tag{77}$$

where $c_{\hat\rho\hat\sigma}{}^{\hat\tau}$ are the structure constants of the group G.

After some rearrangement performed upon use of (76) and (77), the integration over the 2-sphere can be carried out by means of the general relations:

$$\int K_{\hat\rho\alpha} K_{\hat\sigma}{}^{\alpha}\, d\mu = V\,\frac{(\dim \frac{G}{H})}{(\dim G)}\,\delta_{\hat\rho\hat\sigma}$$

$$\int \nabla_\alpha K_{\hat\rho\beta}\nabla^\alpha K_{\hat\sigma}{}^{\beta} = -\int \nabla^2 K_{\hat\rho\beta} K_{\hat\sigma}{}^{\beta} d\mu = -\left[C_2(G)-C_2(H)\right]\frac{V}{a^2}\,\frac{(\dim \frac{G}{H})}{(\dim G)}\,\delta_{\hat\rho\hat\sigma} \tag{78}$$

where $C_2(G)$ and $C_2(H)$ are the quadratic Casimir invariants of the adjoint representation of the groups G and H.(Compare with II.43).

The final result is

$$\hat R = R + \frac{\kappa^2}{6} W_{ab}^2 + \frac{1}{a^2} \tag{79}$$

$$\begin{aligned}\hat{\mathcal{L}}_L = \mathcal{L}_L - \frac{4R}{a^2} &+ \frac{2}{3}\kappa^2\left[\frac{1}{2} R W_{ab}^2 - 4 R_{ab} W_{ac}\cdot W_{bc} - R_{abcd}(W_{ab}\cdot W_{cd} + W_{ac}\cdot W_{bd})\right]\\ &+ \frac{2}{3}\kappa^2\left[(D_a W_{bc})^2 - 2(D_a W_{ab})^2\right] + i\frac{\kappa^3}{a}\,\varepsilon_{\hat\rho\hat\sigma\hat\tau}\, W_{ab}^{\hat\rho} W_{bc}^{\hat\sigma} W_{ca}^{\hat\tau}\\ &+ \frac{4}{9}\kappa^4\left[\frac{1}{16}(W_{ab}\cdot W_{ab})^2 + \frac{1}{8}(W_{ab}\cdot W_{cd})^2 + \frac{1}{8}(W_{ab}\cdot W_{cd}\, W_{ac}\cdot W_{bd}) - \frac{1}{2}(W_{ab}\cdot W_{ac})^2\right]\end{aligned} \tag{80}$$

where the dot indicates the summation over SU(2) indices.

Dimensional reduction of F^2 yields [15,45]

$$\frac{1}{4}\hat F^2 = \frac{1}{4} V_{ab}^2 + \frac{1}{6}\left[W_{ab} + D_a(V-W)_b - D_b(V-W)_a\right]^2 + \frac{2}{3a^2}(V_a - W_a)^2 + \frac{1}{a^4 \kappa^4} \tag{81}$$

The unusual terms containing the interaction between R and W, and the W^3 and W^4 terms appearing in (80) are depressed by a factor β* and we shall not

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* We remind that β must be of the order of a^2 to ensure stability.

consider them in the following.

The other terms give rise to the effective lagrangian

$$\mathcal{L}_{eff} = -\Big\{\frac{1}{\kappa^2}\left(1+\frac{2\beta}{a^2}\right)R - \frac{\beta}{2\kappa^2}\mathcal{L}_L + \frac{1}{6}W_{ab}^2 + \frac{1}{4}V_{ab}^2 + \frac{1}{6}\left[W_{ab} + D_a(V-W)_b - D_b(V-W)_a\right]^2 + \frac{1}{6}\frac{4}{a^2}(V_a - W_a)^2\Big\} \quad (82$$

By defining

$$A = \frac{1}{\sqrt{3}}(V+W) \qquad\qquad X = \frac{1}{\sqrt{2}}(V-W) \quad (83$$

(82) can be rearranged to read:

$$-\frac{\gamma}{\kappa^2}R + \frac{\beta}{2\kappa^2}\mathcal{L}_L - \frac{1}{4}V_{ab}^2 - \frac{1}{4}A_{ab}^2 - \frac{1}{4}(D_aX_b - D_bX_a) - \frac{1}{2}\frac{4}{a^2}X_a^2 + \dots \quad (84$$

where now

$$D_a X_b^{\gamma} = \nabla_a X_b^{\gamma} - \frac{\sqrt{3}}{2}\frac{\kappa}{a}\varepsilon_{\hat{\alpha}\hat{\beta}}{}^{\hat{\gamma}} A_a^{\hat{\alpha}} X_b^{\hat{\beta}} \quad (85$$

On physical grounds A_m corresponds then to the massless SU(2) gauge field, while X_m is a massive SU(2) triplet of bosons, whose mass is given by $M_{1+} = \frac{4}{a^2}$

The physical coupling constants are given by

$$\kappa_4 = \kappa\left(1+\frac{\beta}{2a^2}\right)^{-\frac{1}{2}} = \frac{\kappa}{\sqrt{\gamma}}$$

$$e = \frac{m}{2\sqrt{2}}\frac{\kappa}{a} \quad (86$$

$$f = \frac{\sqrt{3}}{2}\frac{\kappa}{a} = \frac{\sqrt{6}}{m}e$$

where e is the U(1) coupling constant, f is the SU(2) coupling constant and κ_4 is related to the Newton constant G by

$$\kappa_4 = \sqrt{16\pi G} \quad (87$$

Notice that the condition $\gamma > 0$ obtained in the study of the stability of the theory corresponds to demanding the correct sign for the gravitational coupling constant (i.e. positivity of the gravitational energy).

Compared with standard models, this one admits a greater freedom in the choice of the Klein radius, that might be even much smaller than usually supposed whenever γ approaches zero. This is due to the presence of the new free parameter β, which is not fixed by the equations of motion.

APPENDIX

Some properties of the Gauss-Bonnet lagrangians

In this appendix we study some perturbative properties of the Gauss-Bonnet lagrangians [29]. For this purpose, it is useful to introduce some basic notions of differential geometry.

We shall need the vielbein 1-forms e^a and the connection 1-forms $\omega_a{}^b$

$$e^a = e^a{}_m(x)\, dx^m \tag{1a}$$

$$\omega_a{}^b = \omega_a{}^b{}_m(x)\, dx^m \tag{1b}$$

Torsion T and curvature R are defined as

$$T^a = De^a \equiv de^a + \omega^a{}_b \wedge e^b \tag{2a}$$

$$R_a{}^b = d\omega_a{}^b + \omega_a{}^c \wedge \omega_c{}^b \tag{2b}$$

where D is the covariant derivative. They satisfy the Bianchi identities:

$$(DR)_a{}^b = 0 \tag{3a}$$

$$(DT)^a = R^a{}_b \wedge e^b \tag{3b}$$

We consider only the case of Riemannian geometry for which the torsion vanishes:

$$T^a = 0 \tag{4}$$

With the notation given above, the lagrangians (I.5) in an n-dimensional space can be written as

$$L_{k,n} = R_{a_1a_2} \wedge R_{a_3a_4} \wedge \ldots \wedge R_{a_{2k-1}a_{2k}} \wedge e_{a_{2k+1}} \wedge \ldots e_{a_n}\, \varepsilon^{a_1\ldots a_n} \tag{5}$$

Clearly, the number k of curvature 2-forms present in the lagrangian must be $\leqslant \frac{n}{2}$.

For example, in six dimensions we have the four possibilities:

$$\begin{aligned} L_{0,6} &= e_a \wedge e_b \wedge e_c \wedge e_d \wedge e_f \wedge e_g\ \varepsilon^{abcdfg} \\ L_{1,6} &= R_{ab} \wedge e_c \wedge e_d \wedge e_f \wedge e_g\ \varepsilon^{abcdfg} \\ L_{2,6} &= R_{ab} \wedge R_{cd} \wedge e_f \wedge e_g\ \varepsilon^{abcdfg} \\ L_{3,6} &= R_{ab} \wedge R_{cd} \wedge R_{fg}\ \varepsilon^{abcdfg} \end{aligned} \tag{6}$$

The first expression is proportional to the cosmological constant, the second to the Einstein-Hilbert lagrangian, the third to the Lanczos lagrangian and the last to the six-dimensional Euler characteristic, which is a total divergence, as it is shown in the note at the end of the appendix.

Let us consider now the variation of the Lanczos lagrangian $L_{2,6}$ around a constant background described by the vielbein $\eta^a{}_m$

$$\ell^a{}_m = \eta^a{}_m + h^a{}_m \tag{7}$$

Variation of the vielbein and connection forms yields

$$\begin{aligned}\delta L_{2,6} = {} & 2\,\delta R_{ab} \wedge R_{cd} \wedge \ell_f \wedge \ell_g\, \varepsilon^{abcdfg} \\ & + 2\, R_{ab} \wedge R_{cd} \wedge \ell_f \wedge \delta\ell_g\, \varepsilon^{abcdfg}\end{aligned} \tag{8}$$

But from (3b) we have

$$(\delta R)_{ab} = (D\,\delta\omega)_{ab} \tag{9}$$

so that the first term on the right hand side of (9) becomes

$$2\,(D\,\delta\omega)_{ab} \wedge R_{cd} \wedge \ell_f \wedge \ell_g\, \varepsilon^{abcdfg} \tag{10}$$

On the other hand, from (4a) and (5) we have

$$\begin{aligned} & 2d(\delta\omega_{ab} \wedge R_{cd} \wedge \ell_f \wedge \ell_g)\, \varepsilon^{abcdfg} = \\ & = 2D(\delta\omega_{ab} \wedge R_{cd} \wedge \ell_f \wedge \ell_g)\, \varepsilon^{abcdfg} = \\ & = 2\,(D\,\delta\omega_{ab} \wedge R_{cd} \wedge \ell_f \wedge \ell_g + 4\,\delta\omega_{ab} \wedge R_{cd} \wedge \ell_f \wedge T_g)\, \varepsilon^{abcdfg}\end{aligned} \tag{11}$$

So, if the torsion vanishes, (9) can be written as

$$\begin{aligned}\delta L_{2,6} = {} & 2d(\delta\omega_{ab} \wedge R_{cd} \wedge \ell_f \wedge \ell_g\, \varepsilon^{abcdfg}) + \\ & + 2\, R_{ab} \wedge R_{cd} \wedge \ell_f \wedge \delta\ell_g\, \varepsilon^{abcdfg}\end{aligned} \tag{12}$$

Discarding the total derivative, and expanding R in powers of the variation of the vielbein $h^a{}_m$ (that is possible if the torsion is vanishing):

$$R_{ab} = \overset{0}{R}_{ab} + R_{ab}(h) + R_{ab}(h^2) + \dots \tag{13}$$

the part of the action bilinear in h can be obtained

$$(4 R^0_{ab} \wedge R_{cd}(h) \wedge e_f \wedge h_g + 2 R^0_{ab} \wedge R^0_{cd} \wedge h_f \wedge h_g)\, \varepsilon^{abcdfg} \quad (14$$

The conclusions arising from (14) are the following: if we expand around flat spacetime, R^o_{ab} vanishes, and then there are no terms bilinear in h in the expansion of the lagrangian, which starts from $o(h^3)$. If, on the other hand, we have a more general background, we see from (14) that no term of the type $R_{ab}(h) \wedge R_{cd}(h)$ is present in the bilinear lagrangian, so that second derivatives of h arising from $R_{ab}(h)$ are present only to the first power, avoiding in this way the problems that could arise from higher derivative terms, leading to the appearence of ghosts [7].

This argument can be generalized to all lagrangians of the Gauss-Bonnet type in every dimension [44]. In particular, if the lagrangian contains k curvature forms, its expansion around flat space will start from terms of the order k+1. Analogously, second derivatives of h will be always present only linearly in the bilinear part of the expansion around a non-flat background.

Note

By applying the above procedure to $L_{3,6}$ it is easy to show that it is a total derivative:

$$\delta L_{3,6} = 3 d\, (\delta\omega_{ab} \wedge R_{cd} \wedge R_{fg}\, \varepsilon^{abcdfg}) \quad (15$$

corresponding to the fact that its integral is a topological invariant. This is obviously true for all lagrangians of the type $L_{\frac{n}{2},n}$ in spaces of even dimension n.

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