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MULTIPLICITY FOR THE ELLIPTIC BOUNDARY VALUE PROBLEM VIA ORDERED BANACH SPACE TECHNIQUES

CANDIDATO :

Dr. Franic NJOKU

RELATORE :

Chiar.mo Prof. Giovanni VIDOSSICH

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TRIESTE

ISSA - SCUOLA
TERNAZIONALE
SUPERIORE
STUDI AVANZATI
TRIESTE
Strada Costiera 11

INTRODUCTION

In considering a differential equation, one of the main problems to face is the estimation of the number of solutions. Many mathematicians have contributed quite a lot in this direction for the nonlinear boundary value problems (BVP's) in the past two and a half decades. In the literature there are various methods to attack this question. We shall therefore in this thesis give a survey of some of the most important methods and results in the literature concerning the existence of multiple solutions. Particular attention will be given to i) the techniques from ordered Banach spaces (OBS), which usually provide a lower bound for the number of solutions and ii) the topological degree techniques, which in some cases provide the exact number of solutions.

The plan of the work is the following: in Chapter 0, we give a brief review of OBS and the fixed point index. It is good to remark that a nice introduction to these, full of historical and bibliographical materials is given by Amann [1].

The definitions, notations and results of this chapter are basic for the whole of the thesis. They will be constantly used, usually without further mention. It is also good to remark that some of the results which we shall see were proved for more general cases but we shall only be concerned with the Dirichlet problem:

$$(*) \qquad \begin{aligned} Lu &= f(x,u) \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

where Ω is a bounded domain of $\mathbb{R}^n$ whose boundary is an (N-1)-dimensional $C^{2+\mu}$-manifold for some $\mu \in (0,1)$. L denotes a differential operator of the form

$$L := -\sum_{j,k=1}^{N} a_{jk} D_j D_k + \sum_{j=1}^{N} a_j D_j + a$$

with symmetric coefficient matrix (a_{jk}). We suppose that a_{jk}, a_j, $a \in C^{\mu}(\Omega)$ and that $a \geq 0$. L is also assumed to be strongly uniformly elliptic. Also we shall assume that the maximum principle holds. The effect of which is to provide not only a way to invert the linear part, but also to have an inverse operator which is positive in the sense that it maps positive functions into positive functions, something very peculiar and useful. For the case when N=1, Ω is just a bounded open interval, $\Omega = (x_0, x_1)$ and L is a differential operator of the form

$$Lu := -a_{11}u'' + a_1 u' + au$$

with continuous coefficients $a_{11}, a_1, a \in C(\Omega)$ such that $a_{11}(x) > 0$ and $a(x) \geq 0 \ \forall x \in \bar{\Omega}$. For the various cases to be considered, we shall define the nonlinearity f, accordingly and f may even depend on the gradient term, Du.

In chapter one we shall prove some multiplicity results of Amann [1], [2], [3] . Here the consideration will be for the case when the nonlinearity is independent of the gradient term. The abstract tools to be used are results on linear, order-preserving maps and the fixed point index for compact map of a positive cone into itself; in chapter two, the consideration will be for the case when the nonlinearity depends on the gradient term. As we shall see, because of the presence of the gradient term in the nonlinearity, most of the known techniques are not applicable and the existence theory becomes much more intractable. The main results of this chapter are by Amann and Crandall [4]; Hofer [10]; Ambrosetti and Hess [5]. Finally in chapter three, we present some results by Berestycki [7], in which the proofs rely on simple eigenvalue comparison arguments.

CHAPTER 0

Let E be a nonempty set. A non-empty subset P of E is called a cone if P is closed, convex, invariant under multiplication by elements of $\mathbb{R}_+$ and if $P\cap(-P) = 0$. Each cone induces a partial ordering in E usually denoted by $\leq$, through the rule $y \leq z$ if $z-y \in P$. This ordering is antisymmetric, reflexive, transitive, compatible with the linear structure and with the topology. A non-empty set together with an ordering is called an ordered set. The elements in

$$\dot{P} := P \setminus \{o\}$$

are positive and P is said to be the positive cone of the ordering. Hence an ordered Banach space (OBS) with positive cone P is a Banach space together with a partial ordering which is induced by a given cone P. An OBS will be denoted by (E,P) and we shall write E if the cone is known.

Let E be an OBS with positive cone P, for every pair $x,y \in E$, the set

$$[x,y] := \{z \in E : x \leq z \leq y\} \equiv (x+P)\cap(y-P)$$

is called the order interval between x and y. Every order interval is obviously a closed convex subset of E. However, in general an order interval will not be bounded. A subset X of E is called order convex whenever $x,y \in X \Rightarrow [x,y] \subset X$. A cone is said to be total if $E = \overline{P-P}$, generating if $E = P-P$, normal if $\exists$ a $\delta > 0$: $\forall x,y \in P$,

$$\|x+y\| \leq \delta \max(\|x\|, \|y\|).$$

If P is a normal cone then every order interval is bounded.

Let E,F be OBS's with positive cones P and Q respectively (with the ordering in each set denoted by $\leq$). A linear operator $T: E \longrightarrow F$ is called positive if $T(P) = Q$. It is called strict ly positive if $T(\dot{P}) = \dot{Q}$ and it is called strongly positive if $T(\dot{P}) = \mathring{Q}$, where in the latter case, it is assumed that Q has non empty interior Q. A non-linear map $f:E \longrightarrow F$ is called increasing if $x \leq y \Rightarrow f(x) \leq f(y)$, strictly increasing if $x < y \Rightarrow f(x) < f(y)$ and strongly increasing if $\mathring{Q} \neq \emptyset$ and if $x < y \Rightarrow f(x) - f(y) \in \mathring{Q}$.

Let X be a non empty subset of E and let f be a map from X into F. Then f is called completely continuous if f is continuous and maps bounded subsets of X into compact sets. f is compact if it is continuous and if f(x) is relatively compact. The map $f:D(f) \subset E \rightarrow F$ is called bounded if it maps bounded sets into bounded sets. Clearly every compact map is completely continuous and the two notions coincide if X is bounded. In the special case that f is a linear operator f is called compact if f maps every bounded set into a relatively compact set.

Let X be a non empty subset of some ordered set Y. A fixed point x of a map $f:X \rightarrow Y$ is called minimal (resp. maximal) if every fixed point y of f in X satisfies $x \leq y$ (resp. $y \leq x$). Clearly, there is at most one minimal (resp. maximal) fixed point.

The following theorem of Amann [1] which will be needed in the next chapters contains the basic existence result based on monotone iteration schemes:

Thm 0.1 Let (E,P) be an OBS and let $[\bar{y}, \hat{y}]$ be a non empty order interval in E. Suppose that $f: [\bar{y}, \hat{y}] \rightarrow E$ is an increasing compact map such that

$$\bar{y} \leq f(\bar{y}) \text{ and } f(\hat{y}) \leq \hat{y} \quad .$$

Then f has a minimal fixed point $\bar{x}$ and a maximal fixed point $\hat{x}$. Moreover,

$$\bar{x} = \lim_{k\to\infty} f^k(\bar{y}) \text{ and } \hat{x} = \lim_{k\to\infty} f^k(\hat{y})$$

and the sequences $(f^k(\bar{y}))$ and $(f^k(\hat{y}))$ are increasing and decreasing respectively.

Let (E,P) be an OBS with positive cone P and let $e \in \mathring{P}$ be given. Then $e \in E$ is called an order unit for E if $[-e,e]$ is absorbing, that is, for every $x \in E$, $\exists$ a positive number λ: $-\lambda e \leq x \leq \lambda e$. Hence e is an order unit for E iff

$$E = \cup \{\lambda[-e,e] : \lambda \in \mathbb{R}_+\}$$

The following theorem whose proof is contained in [11], shows that order units are closely related to interior points of the positive cone.

<u>Thm. 0.2</u> Let (E,P) be an OBS. Then $\exists$ order units for E iff P has non empty interior. If $\mathring{P} \neq \emptyset$, then the set of all order units for E coincides with $\mathring{P}$.

Given two ordered normed linear spaces (ONLS) E and F with positive cones P and Q, resp. A linear operator $T:E\to F$ is called e-positive if $\exists$ an element $e \in \mathring{Q}$ such that, for every $x \in P$, there are numbers $\alpha = \alpha(x)$, $\beta = \beta(x) > 0$: $\alpha e \leq Tx \leq \beta e$. Obviously, every e-positive linear mapping is strictly positive and it is easily seen that every strongly positive linear operator is e-positive for every $e \in \mathring{Q}$. A mapping f: $D(f) \subset E \to F$ is called e-increasing if $\exists$ an $e \in \mathring{Q}$: for every $y,z \in D(f)$ with $y > z$, one can find constants $\alpha = \alpha(y,z)$, $\beta = \beta(y,z) > 0$ with

$$\alpha e \leq f(y) - f(z) \leq \beta e.$$

Let (E,P) be an ONLS with positive cone and let $e \in \dot{P}$ be given. Then we denote by E_e the linear subspace of E defined by $E_e \equiv \bigcup_{\lambda \geq 0} \lambda[-e,e]$ and we set $P_e \equiv P \cap E_e$. On E_e we introduce a new norm $||.||_e$, the e-norm, namely, we define $||.||_e$ to be the Minkowski functional of the order interval $[-e,e]$, i.e for every $x \in E_e$, $||x||_e \equiv \inf\{\lambda > 0 : -\lambda e \leq x \leq \lambda e\}$. In this norm (E_e, P_e) is an ONLS with positive cone. Moreover, P_e is normal and has non empty interior, namely $e \in \mathring{P}_e$. Finally, if P is a normal cone, it can be shown that E_e is continuously imbedded in E, i.e. the e-norm is stronger than the orginal norm of E on the subspace E_e. Hence if $(E, ||.||)$ is complete then $(E_e, ||.||)$ is an OBS with normal positive cone P_e and P_e has non empty interior. In the following, when referring to E_e, it will always be understood that E_e is considered with the e-norm. Finally, it should be remarked that the Banach spaces E and E_e are topologically isomorphic iff P is a normal cone with non empty interior and $e \in \mathring{P}$.

Let E be an OBS with normal positive cone P and let $e > 0$ be fixed. Suppose f maps some order interval $[y,z] \subset E$ into E_e. Then, if $\exists$ a fixed point of f, it necessarily belongs to $[y,z] \cap E_e$. Hence it suffices to consider f on this set. In the following, we denote by $[y,z]_e \equiv [y,z] \cap E_e$ with the topology induced by E_e. Moreover, let $j_e : E_e \to E$ denote the injection map. Then we define $f_e : [y,z]_e \to E_e$ by $f_e \equiv f \circ (j_e | [y,z]_e)$. It follows from the fact that every interior point of the positive cone is an order unit for the space, that every strongly increasing mapping is e-increasing for every $e \in \mathring{Q}$. In general, every e-increasing mapping is strongly increasing w.r.t. the cone P_e.

Next, we highlight some properties of the fixed point index (equivalent to the well-known Leray-Schauder degree). The essence of this is that since we shall consider, for instance, nonlinear maps in OBS, which we defined on (relatively) open subsets of the positive cone, if the positive cone does not have interior points, the Leray-Schauder degree is not immediately applicable. But due to the fact that the positive cone is a retract of the Banach space, it is possible to define a fixed point index for compact maps which are defined in the positive cone.

A non empty subset X of a metric (more generally, topological) space E is called a retract of E if $\exists$ a continuous map $r:E \to X$, a retraction, such that $r|X = id_X$. It is easily seen that every retract is a closed subspace of E. By an important theorem of Dugundji [9] every non empty closed convex subset of a Banach space E is a retract of E.

Thm. 0.3 Let X be a retract of some Banach space E. For every open subset U of X and every compact map $f: \bar{U} \to X$ which has no fixed point on ∂U, $\exists$ an integer $i(f,U,X)$ satisfying the following conditions:

(i) (NORMALIZATION) For every constantmap f mapping U into U, $i(f,U,X) = 1$

(ii) (ADDITIVITY) For every pair of disjoint open subsets U_1, U_2 of U such that f has no fixed points on $\bar{U} \setminus (U_1 \cup U_2)$,

$i(f,U,X) = i(f,U_1,X) + i(f,U_2,X)$

where $i(f,U_k,X) := i(f|U_k, U_k, X)$, $k = 1,2$;

(iii) (HOMOTOPY INVARIANCE) For every compact interval $I \subset \mathbb{R}$, and every compact map

$h: I \times \bar{U} \to X : h(\lambda,x) \neq x$ for $(\lambda,x) \in I \times \partial U$

$i(h(\lambda,.), U,X)$ is well defined and independent of $\lambda \in I$.

(iv) (PERMANENCE) If Y is a retract of X and $f(\bar{U}) \subset Y$, then

$i(f,U,X) = i(f,U\cap Y,Y)$ where $i(f,U\cap Y,Y) := i(f|\overline{U\cap Y}, U\cap Y, Y)$

The family $\{i(f,U,X) \mid X$ retract of E, U open in X, $f:\bar{U}\to X$ compact without fixed points on $\partial U\}$ is uniquely determined by the properties (i)-(iv) and $i(f,U,X)$ is called the fixed point index of f (over U w.r.t. X).

Corollary 0.1

The fixed point index has these further properties.

(v) (EXCISION) For every open subset $V \subset U$: f has no fixed point in $\bar{U} \setminus V$,

$$i(f,U,X) = i(f,V,X)$$

(vi) (SOLUTION PROPERTY). If $i(f,U,X) \neq 0$, then f has at least one fixed point in U.

We shall also require a generalization of the classical fixed point index to strict set contractions (s s c) defined on certain metric absolute neighbourhood retracts as given by Nussbaum [14]. Let E be a metric space and let X be a bounded subset of E. Then the measure of non compactness of X, $\gamma(X)$ is defined by $\gamma(X) := \inf\{\delta > 0 \mid X$ can be covered by finitely many subsets of E of diameter less or equal to $\delta\}$. Clearly, X is totally bound iff $\gamma(X)=0$.

If E and F are metric spaces, A mapping $f:E\to F$ is called an α-set contraction if it is continuous and for every bounded subset $X \subset E$, $\gamma_2(f(X)) \leq \alpha\gamma_1(X)$ where γ_1, γ_2 are the measures of non-compactness in E and F resp. The mapping f is called a strict set contraction (s s c) if f is an α-set contraction with $\alpha < 1$. Hence, every completely continuous map is a s s c. For our purpose, it will suffice to consider ssc's defined on closed convex subsets of Banach spaces. In the following proposition we state the main properties of the fixed point index for the case when f is a ssc. For the proof we refer to Nussbaum [14].

Propostion 0.1 Let X be a closed convex subset of a Banach space and let U be a bounded open subset of X. Let $f:\bar{U}\to X$ be a ssc which has no fixed points on ∂U. Then one can define an integer valued function $i_X(f,U)$, the fixed point index of f, which has the following properties:

(1) If $i_X(f,U)\neq 0$ then f has a fixed point in U.

(2) If U_1 and U_2 are disjoint open subsets of U containing all the fixed points of f then

$$i_X(f,U) = i_X(f,U_1) + i_X(f,U_2).$$

In particular, if f has no fixed points, then $i_X(f,U) = 0$ (Additivity property).

(3) Let $F:\bar{U}\times[0,1]\to X$ be a continuous map such that, for each $s\in[0,1]$ and each $x\in\partial U$, $F_s(x)\equiv F(x,s)\neq x$. Suppose that each F_s is an α-set contraction with $\alpha<1$ and α independent of s. Finally suppose that $F(X,.):[0,1]\to X$ is uniformly continuous w.r.t. $x\in\bar{U}$. Then

$$i_X(F_o,U) = i_X(F_1,U)$$

(Homotopy property)

(iv) Let U_i be bounded open subsets of closed convex subsets X_i of Banach spaces E_i, i=1,2 resp. Suppose $f_1:\bar{U}_1\to X_2$ and $f_2:\bar{U}_2\to X_1$ are α_1 and α_2 - set contractions resp. with $\alpha_1\alpha_2<1$. Finally, suppose that $f_1\circ f_2$ has no fixed points on $\partial(f_2^{-1}(U_1))$. Then $f_2\circ f_1$ has no fixed points on $\partial(f_1^{-1}(U_2))$ and

$$i_{X_1}(f_2\circ f_1,f_1^{-1}(U_2)) = i_{X_2}(f_1\circ f_2, f_2^{-1}(U_1))$$

(Commutativity property)

(5) Suppose X is compact. Then,

$$i_X(f,X) = \Lambda(f),$$

where Λ denotes the Lefschetz number. (Normalization property).

We shall recall two simple consequences of these properties. For proof, see Amann [2].

Lemma 0.1 Let X and X_1 be closed convex subsets of a Banach space E and $X \subset X_1$. Let U be a bounded open subset of X_1 and let $f:\bar{U} \to X$ be a ssc with no fixed points on ∂U. Then

$$i_X(f, U \cap X) = i_{X_1}(f,U)$$

Lemma 0.2 Suppose X and U are as in Proposition 0.1 and suppose $f:\bar{U} \to X$ is a constant map with $f(\bar{U}) \in U$. Then $i_X(f,U)=1$.

After these preparations, we are now in a position to state a fixed point theorem for a ssc mapping a closed bounded convex subset of a Banach space into itself.

Theorem 0.4 Let X be a closed bounded convex subset of a Banach space E. Let $f:X \to X$ be a ssc. Then

$$i_X(f,X) = 1$$

and f has a fixed point.

Now we look at a slight gene ralization of a ssc. Let E be a metric space with measure of noncompactness γ. A mapping $f:E \to E$ is called condensing if it is continuous and, for every bounded subset $X \subset E$ with $\gamma(X) > 0$, we have $\gamma(f(X)) < \gamma(X)$. Obviously, every ssc is condensing but Nussbaum [14] has shown that there are condensing maps which are not ssc's.

Let E,F be normed vector spaces (nvs). Then we denote by $L(E,F)$ the nvs of all continuous linear operators $f:E\to F$. For every $f\in L(E) := L(E,E)$ we denote by $r(f)$ its spectral radius, that is, $r(f) := \lim_{k\to\infty} ||f^k||^{1/k}$. Let P be a total cone in E. A map $f:P\to F$ is said to be differentiable at $x\in P$ along P if $f'(x)\in L(E,F)$:

$$\lim_{\substack{h\in P\\ h\to 0}} \frac{||f(x+h)-f(x)-f'(x)h||}{||h||} = 0$$

$f'(x)$ is uniquely defined. The mapping f is said to be asymptotically linear along P if $\exists\ f'(\infty)\in L(E,F)$:

$$\lim_{\substack{x\in P\\ ||x||\to\infty}} \frac{||f(x)-f'(\infty)x||}{||x||} = 0$$

Again, $f'(\infty)$ is uniquely determined and called the derivative at infinity along P. So, $f'(\infty)$ does not have a positive eigenfunction belonging to an eigenvalue greater or equal to 1 if $r(f'(\infty)) < 1$. We shall define by

$L_1^+(E) := \{f\in L(E)\,|\,f$ has a positive eigenfunction belonging to eigenvalue > 1 and no positive eigenfunction with eigenvalue $= 1\}$

and

$L_1^-(E) := \{f\in L(E)\,|\,f$ does not have a positive eigenfunction belonging to an eigenvalue $\geq 1\}$.

The condition for $f\in L(E)$ to belong to $L_1^+(E)$ simplifies if it is known that f has at most one eigenvalue having a positive eigenfunction. For example, this is the case if f is e-positive

We now state a lemma which generalizes Krasnose l'skii's result [13] for completely continuous maps - if f is completely continuous, E complete and P is generating, the derivative along P at x (resp. at ∞) is a completely continuous linear operator. This result is based on the fact that in case (E,P) is OBS with P generating $\exists \beta \geq 1$: for every $y,z \in P$ with $x=y-z$ and $\max (\|y\|,\|z\|) \leq \|x\|$. However, in the general case, where $f:P \to F$ is supposed to be an α-set contraction, this implies only that $f'(x)$ (resp. $f'(\infty)$) is a $2\text{-}\alpha\beta$-set contraction where the value of β is not known in general.

<u>Lemma 0.3.</u> Let E and F be nvs and let P be a total cone in E. Let $f:P \to F$ be an α-set contraction which is asymptotically linear along P and for some $x \in P$, differentiable at x along P. Then $f'(\infty)|P$ and $f'(x)|P$ are α-set contractions.

Proof: See Amman [3] .

CHAPTER ONE

MULTIPLICITY RESULTS - TECHNIQUES FROM OBS

In this chapter, we shall prove some multiplicity results by Amann [1], [2], [3], using techniques from OBS. In the first section, we prove a theorem using the extension of the theory of topological degree to functions defined on cones of OBS. The second section establishes criteria which ensure multiple fixed points of a nonlinear mapping of a cone into itself. The abstract tools used here are results on linear, orderpreserving maps and the fixed point index for condensing maps of a cone into itself. In the last section, asymptotically linear completely continuous maps which leaves invariant a cone in a Banach space to ssc are considered. The proof is also based on the fixed point index. As we shall see all these have applications to second order elliptic BVP where the nonlinearity is independent of the gradient term.

Section one

In this section, we consider increasing maps, using some topological tools, we shall prove the existence of multiple solutions under appropriate conditions. In the 1-dimensional case, if f: $[y,z] \to \mathbb{R}$ be a continuous function on some nontrivial interval $[y,z] \subset \mathbb{R}$: $(f(y)-y)\ (z-f(z)) > 0$. Then the intermediate value theorem implies the existence of a fixed point of f in (y,z). This has been generalized by replacing the interval $[y,z]$ by an order interval. As in Theorem 0.1, which was for increasing compact self maps of a given order interval, the existence of a fixed point - (as well as that of a minimal and a maximal fixed point) was deduced by means of an iteration method. We shall here be concerned with the conditions placed on an operator (continuous) mapping some bounded order interval $[y,z] \subset E$ into itself and the condition on the ordering, to ensure the existence of two or more fixed points. Indeed, if $f: [y,z] \to E$ is increasing and compact, then Theorem 0.1 (or Schauder's theorem) guarantees the existence of a fixed point provided $y \leq f(y)$ and $f(z) \geq z$. We now state a multiplicity result due to Amann [1] for the case when P is positive, $\overset{\circ}{P} \neq \emptyset$, f compact and strongly increasing.

Thm. 1.1. (Amann) Let (E.P) be an OBS whose positive cone has non empty interior. Suppose that there are four points y_k, $z_k \in E$, k=1,2 with

$$y_1 < z_1 < y_2 < z_2$$

and a compact strongly increasing map $f: [y_1, z_2] \to E$ such that

$$y_1 \leq f(y_1), \quad f(z_1) < z_1, \quad y_2 < f(y_2), \quad f(z_2) \leq z_2$$

Then f has at least three distinct fixed points,

$$x, x_1, x_2 \; : \; y_1 \leq x_1 \ll z_1 \quad , \; y_2 \ll x_2 \leq z_2 \; , \text{ and } y_2 \nleq x \nleq z_1 .$$

Remark 1.1 (Application to Dirichlet problem) - Applying Theorem 1.1 to nonlinear elliptic BVP of the type

$$(1.1) \qquad \begin{aligned} Lu &= f(x,u) && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega \end{aligned}$$

We obtain the following multiplicity result.

Thm. 1.2 (Amann) Let $f: \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ be of class C^1. Suppose $\exists$ a subsolution Φ_1, a strict supersolution Ψ_1, a strict subsolution Φ_2 and a supersolution Ψ_2 for the BVP (1.1) :

$$\Phi_1 < \Psi_1 < \Phi_2 < \Psi_2 .$$

Then the BVP (1.1) has at least three distinct solutions u_k, $k=1,2,3$: $\Phi_1 \le u_1 < u_2 < u_3 \le \Psi_2$.

Remark 1.2 The proof of Thm. 1.1 is based on the following simple lemma.

Lemma 1.1 (Amann) Let X be a retract of some Banach space and let $f: X \to X$ be a compact map. Suppose that X_1 and X_2 are disjoint retracts of X and let U_k, $k=1,2$ be open subsets of $X: U_k \subset X_k$, $k=1,2$. Moreover, suppose that $f(X_k) \subset X_k$ and that f has no fixed points on $X_k \setminus U_k$, $k=1,2$. Then f has at least three distinct fixed points, x, x_1, x_2 with $x_k \in X_k$, $k=1,2$ and $x \in X \setminus (X_1 \cup X_2)$.

Proof (Sketch) We know if $f: \bar{U} \subseteq X \to X$ is a given compact function, X some retract of a Banach space E, U open in X and $r: E \to X$ a retraction, that the fixed point index

$$i(f, U, X) = \deg(f \circ r, r^{-1}(U), 0)$$

is well defined. By the additivity property

$$i(f, X \setminus \overline{(U_1 \cup U_2)}, X) = i(f, X, X) - \sum_{k=1}^{2} i(f, U_k, X)$$

By the permanence property

$$i(f, U_k, X) = i(f, U_k, X_k)$$

and the excision property gives

$$i(f, U_k, X_k) = i(f, X_k, X_k).$$

Consequently,

$$i(f, X \setminus (\overline{U_1 \cup U_2}), X) = i(f, X, X) - \sum_{k=1}^{2} i(f, X_k, X_k) .$$

But we have (from the proof of Schauder's fixed point theorem) that

$$i(f,C,C) = 1$$

for every compact self-map f on an arbitrary retract C. Hence

$$i(f,X,X) = i(f,X_1,X_1) = i(f,X_2,X_2) = 1$$

and so

$$i(f,X \setminus \overline{(U_1 \cup U_2)}, X) = -1$$

This implies the existence of a fixed point of f on $X \setminus \overline{(U_1 \cup U_2)}$, which is different from x_1 and x_2. q.e.d.

Proof of Theorem 1.1 (Sketch) Let $X: = [y_1, z_2]$ and $X_k := [y_k, z_k]$, $k=1,2$.

Then X, X_1 and X_2 are retracts of E with $X_k \subset X$ and $X_1 \cap X_2 = \emptyset$. Hence X_k, $k=1,2$ is a retract of X. Moreover, since f is increasing, the hypotheses imply that $f(X) \subset X$ and $f(X_k) \subset X_k$, $k=1,2$. Since f is strongly increasing and $f(z_1) < z_1$, it follows from Theorem 0.1 that f has a maximal fixed point x_1 in X_1 and $x_1 \ll z_1$. Consequently, X_1 has nonempty interior U_1 in X and f has no fixed point on the boundary $X_1 \setminus U_1$ of X_1 in X. Similarly, the existence of a minimal fixed point of f in X_2 and the fact that f is strongly increasing imply that X_2 has non empty interior U_2 in X and that f has no fixed point on $X_2 \setminus U_2$. Hence lemma 1.1 is applicable and the assertion follows. q.e.d.

Remark 1.3 Suppose it is only known that the compact strongly increasing map f maps one order interval into itself. Then, by Thm. 0.1, f has a minimal and a maximal fixed point $\bar{x}$ and $\hat{x}$ resp. Suppose, in addition that $\bar{x} < \hat{x}$. Is it true that f has a third fixed point in this situation. The following theorem gives an answer to this question.

THEOREM 1.3 (AMANN). Let (E,P) be an OBS whose positive cone has nonempty interior. Suppose that $y < z$ and let $f:[y,z] \longrightarrow E$ be a strongly increasing compact map such that $y < f(y)$ and $f(z) \ll z$. Suppose in addition that the minimal fixed point $\bar{x}$ and the maximal fixed point $\hat{x}$ are distinct, and that f has strongly positive derivatives at $\bar{x}$ and $\hat{x}$ respectively. Then f possesses at least three distinct fixed points, provided

$$r(f'(\bar{x})) \neq 1 \quad \text{and} \quad r(f'(\hat{x})) \neq 1.$$

REMARK 1.4. The condition of Theorem 1.1 are somewhat restrictive. It can be relaxed to more general situations. For example, it suffices to assume that f is increasing (instead of being strongly increasing) if it is known that the maximal fixed point x_1 of f in $[y_1,z_1]$ satisfies $x_1 \ll z_1$ and the minimal fixed point x_2 in $[y_2,z_2]$ satisfies $y_2 \ll x_2$. Moreover, the hypotheses that f be increasing can be completely dropped if it is known that f maps each of the order intervals X, X_1 and X_2 into itself such that f has no fixed points on the boundaries of X_k, $k=1,2$. For example, this can be guarateed if the existence of strongly increasing majorants and minorants are presupposed as given in the following Section.

SECTION TWO

THEOREM 1.4 (AMANN). Let E be an OBS with normal positive cone P. Let $[y,z] \subset E$ be an order interval and suppose, for some $e > o$, $f : [y,z] \longrightarrow E_e$ is such that f_e is a ssc. Suppose there exist e-increasing condensing maps $\bar{f}$, $\hat{f}$: $[y,z] \rightarrow E$ with $R(\bar{f})$, $R(\hat{f})$ $\subset E_e$, such that, for all $x \in [y,z]$

$$\bar{f}(x) \leq f(x) \leq \hat{f}(x)$$

Suppose there exist $y_j, z_j, j = 1, \ldots, m$ with

$$y = y_1 < z_1 < y_2 < \ldots < z_{m-1} < y_m < z_m = z$$

such that

$$y \leq \bar{f}(y) \quad , \quad \hat{f}(z) \leq z$$

and

$$y_j < \bar{f}(y_j), \quad j=2, \ldots, m \; ; \; \hat{f}(z_j) < z_j \quad , \quad j=1, \ldots, m-1$$

Then f has at least 2m-1 distinct fixed points $x_1^*, \ldots, x_{2m-1}^*$ with $y_j \leq x_{2j-1}^* \leq z_j$, $j=1, \ldots, m$ and $y_{j+1} \nleq x_{2j}^* \nleq z_j$, $j=1, \ldots, m-1$.

REMARK 1-5

Theorem 1-4 establishes criteria (applicable to the BVP (1.1)) which ensure multiple fixed points of a nonlinear mapping of a cone into itself. Just as in Theorem 1-1, the following which is also due to Amann [2] is an application of Theorem 1-4.
Let $f: \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ be C^1. Suppose that $v_1, \ldots, v_m$ are strict subsolutions of the BVP and $w_1, \ldots, w_m$ are strict supersolutions such that

$$v_1 < w_1 < v_2 < \ldots < w_{m-1} < v_m < w_m$$

Then the BVP (1) has at least 2m-1 distinct solutions

$$u_1^* < \ldots < u_{2m-1}^*$$

such that

$$v_j < u_{2j-1}^* < w_j, \; j=1, \ldots, m \quad \text{and} \quad u_{j+1} \nleq u_{2j}^* \nleq u_j, \; j=1, \ldots, m-1$$

PROOF OF THEOREM 1-4 (sketch).

This depends on the following two lemmas due to Amann [2], which we state without proof:

LEMMA 1-2

Let E be an OBS and suppose $[y,z]$ is a bounded order interval. Suppose also that f: $[y,z] \longrightarrow E$ is an increasing condensing map which satisfies

$$y \leq f(y) \quad , \quad f(z) \leq z .$$

Then f has at least one fixed point. Moreover, there is a minimal and a maximal fixed point x_1 and x_2 resp. in the sense that every fixed point x^* satisfies $x_1 \leq x^* \leq x_2$. Finally, the iteration method

$$x_{k+1} = f(x_k) \ , \ k= 0, 1, \dots$$

converges to x_1 from below if $x_o=y$ and it converges to x_2 from above if $x_o=z$.

LEMMA 1-3

Let E be an OBS with normal positive cone. Let f be a ssc mapping an order interval $[y,z]$ into itself. Suppose there exist

$$y = y_1 < z_1 < y_2 < \dots < z_{m-1} < y_m < z_m = z$$

such that, for every j=1, ... m

$$f([y_j, z_j]) \subset [y_j, z_j]$$

and

$$f([y_1, z_j]) \subset [y_1, z_j]$$

Moreover, suppose there exist open subsets $U_1, \dots U_m; U_1^1, \dots U_m^1$ of $[y,z]$ such that f has no fixed points on

$$[y_j, z_j] \setminus U_j \text{ and } [y, z_j] \setminus U_j^1 \ , \ j=1, \dots, m.$$

Then f has at least 2m-1 distinct fixed points $x_1^*, \dots, x_{2m-1}^*$ with

$$y_j \leq x_{2j-1}^* \leq z_j \ , \ j=1, \dots, m \quad \text{and} \quad y_{j+1} \nleq x_{2j}^* \nleq z_j, \ j=1, \dots, m-1$$

We are now in a position to prove Theorem 1-3. The idea is to use Lemma 1-3.

Set $\bar{e}_j \equiv \bar{f}(y_j)$ and $\hat{e}_j \equiv \hat{f}(z_j)$, j=1, ..., m. f_e maps each of $[\bar{e}_1, \hat{e}_j]$, $[\bar{e}_j, \hat{e}_j] \subset E_e$, j=1, ..., m into itself and

$\bar{e}_1 < \hat{e}_1 < \bar{e}_2 < \dots < \hat{e}_{m-1} < \bar{e}_m < \hat{e}_m.$

By Lemma 1-3, to complete the proof, it suffices to prove the existence of open subsets of that Lemma. We note that

$$\hat{f}[\bar{e}_k, \hat{e}_j] \subset [\bar{e}_k, \hat{e}_j] \subset E \ , \ k \leq j$$

and so $\hat{f}$ has a maximal fixed point $\hat{x}_{j,k}$ in each of $[\bar{e}_k, \hat{e}_k]$ which is of course unique for all k=1, ..., m; i.e.

$$\hat{x}_{j,1} = \hat{x}_{j,2} = \dots = \hat{x}_{j,j} = \hat{x}_j \ .$$

We claim that every fixed point x^* of f in $[\bar{e}_k, \hat{e}_j]$ satisfies $x^* \leq \hat{x}_j$. For, if $x^* \in [\bar{e}_k, \hat{e}_j]$ and $f(x^*) = x^*$. Then

$$x^* \leq \hat{f}(x^*), \ \hat{e}_j \geq \hat{f}(\hat{e}_j)$$

and f increasing, $\hat{f}([x^*,\hat{e}_j]) \subset [x^*,\hat{e}_j]$. Again f has a maximal fixed point in $[x^*, \hat{e}_j]$ which coincides with $\hat{x}_j$. Hence $x^* \leq \hat{x}_j$.

Similarly, in $[\bar{e}_j, \hat{e}_j]$, $\bar{f}$ has a minimal fixed point $\bar{x}_j$ and every fixed point x^* of f in $[\bar{e}_j, \hat{e}_j]$ satisfies $x^* \geq \bar{x}_j$.

By the fact that $\hat{f}$ is e-increasing, we have that for j=1, ..., m-1, there exist $\delta_j > o$:

$$\hat{e}_j - \hat{x}_j = \hat{f}(z_j) - \hat{f}(\hat{x}) \geq \delta_j e \ , \ j=1, \dots, m-1$$

(Since obviously $z_j > \hat{x}_j$, j=1, ..., m-1 . Similarly there exist $\delta_j^1 : \hat{x}_j - \bar{e}_j \geq \delta_j^1 e$, j=2, ..., m.

At this juncture we are at the following situation. We have a ssc operator f_e that maps each of $[\bar{e}_1, \hat{e}_j]$, $[\bar{e}_j, \hat{e}_j] \subset E_e$ into itself, j=1, ..., m, where

$$\bar{e}_1 < \hat{e}_1 < \bar{e}_2 < \dots < \hat{e}_{m-1} < \bar{e}_m < \hat{e}_m \ .$$

Also, there exist $\bar{x}_j$, $\hat{x}_j$ with $\bar{x}_j \leq \hat{x}_j$ such that every fixed point of f_e in $[\bar{e}_j, \hat{e}_j]$ and $[\bar{e}_1, \hat{e}_j]$ are already in $[\bar{x}_j, \hat{x}_j]$ and $[\bar{e}_1, \hat{x}_j]$ resp.

But the above inequalities show that, for j=1, ..., m

$$([\bar{e}_j, \hat{x}_j] + \epsilon[-e, e]) \cap [\bar{e}_1, \hat{e}_m] \subset [\bar{e}_1, \hat{e}_j]$$

and

$$([\bar{x}_j, \hat{x}_j] + \epsilon[-e, e]) \cap [\bar{e}_1, \hat{e}_m] \subset [\bar{e}_j, \hat{e}_j] .$$

Hence since $[-e, e]$ is the unit ball of E_e, it follows that there exist open subsets

$$U_j , U'_j , j=1, \ldots, m \text{ of } [\bar{e}_1, \hat{e}_j]$$

such that

$$U_j \subset [\bar{e}_j , \hat{e}_j] \quad , \quad U'_j \subset [\bar{e}_1, \hat{e}_j]$$

and such that f_e has no fixed points in $[\bar{e}_j, \hat{e}_j] \setminus U_j$ and $[\bar{e}_1, \hat{e}_j] \setminus U'_j$, $j=1, \ldots, m$.

Finally P being normal implies that E_e is an OBS with normal positive cone and the statement follows from Lemma 1-3.

SECTION THREE

Here we shall consider the case of asymptotically linear maps in OBS. The following result is due to Amann [3].

THEOREM 1-4

Let (E,P) be an OBS with total positive cone and let $f:P \longrightarrow E$ be a ssc which is asymptotically linear along P. Suppose there exists a fixed point $\bar{x}$ of f such that, for every $x \geq \bar{x}$, $f(x) \geq \bar{x}$ and such that f is differentiable at $\bar{x}$ along P. Then P has a fixed point $x^* > \bar{x}$ provided one of the following conditions is satisfied:

i/ $f'(\bar{x}) \in L_1^-(E)$ and $f'(\infty) \in L_1^+(E)$

ii/ $f'(\bar{x}) \in L_1^+(E)$ and $f'(\infty) \in L_1^-(E)$.

REMARK 1-6

We consider an application when the map f depends on a parameter whose proof follows directly from Theorem 1-4.

THEOREM 1-5 (AMANN)

Let (E,P) be an OBS with total positive cone and let $f: P \times \mathbb{R}_+ \to P$ be a map such that $f(o,.) = o$ and such that for every $\lambda \in \mathbb{R}_+$, $f(.,\lambda)$ is a ssc. Suppose that there exist u_o, $u_\infty \in L(E)$ such that for $m=0, \infty$ and every $\lambda \in \mathbb{R}_+$

$$f(x,\lambda) = \lambda u_m(x) + r_m(x,\lambda)$$

with $r_m(x,\lambda) = o(\|x\|)$ as $\|x\| \to m$ in P. Moreover, suppose that there exists a unique non negative eigenvalue μ_m of u_m having a positive eigenfunction. Then, if $\mu_o \neq \mu_\infty$, for every $\lambda \in \left(\frac{1}{\mu_o}, \frac{1}{\mu_\infty}\right)$ there exists a positive fixed point of $f(\cdot,\lambda)$.

REMARK 1-7

In proving Theorem 2-4, the main tool to be used is also the fixed point index $i(f,U,P) := i(f,U)$ for a ssc f defined on the closure of the relatively open subset U of the cone P with values in P. The proof follows immediately from this Lemma.

LEMMA 1-4

Let (E,P) be an OBS with P positive and total, $f: P \to P$ be a ssc and B the open unit ball in E.

(1) Suppose that f is asymptotically linear along P. Then there exist $\rho_\infty > o$: for all $\rho \geq \rho_\infty$

(a) $i(f, \rho B \cap P) = 1$ if $f'(\infty) \in L_1^-(E)$;

(b) $i(f, \rho B \cap P) = 0$ if $f'(\infty) \in L_1^+(E)$.

(2) Suppose that $f(\theta) = 0$ and that f is differentiable at o along P. Then there exist $\rho_o > o$ such that for $\rho \in (o, \rho_o)$

(c) $i(f, \rho B \cap P) = 1$ if $f'(o) \in L_1^-(E)$;

(d) $i(f, \rho B \cap P) = 0$ if $f'(o) \in L_1^+(E)$.

SKETCH OF PROOF.

Let $u \in L_1^-(E) \cup L_1^+(E)$ be given. Suppose $u|P$ is a ssc. Then the closeness of $(id-u)|P$ on bounded sets implies that $(id-u)(S \cap P)$ is closed. Hence since $o \notin (id-u)(S \cap P) \Longrightarrow$ there exist $\alpha > o$:

$$\|x - u(x)\| \geq \alpha \|x\| .$$

Under hypotheses of the lemma, for $m = o, \infty$, there exist $\alpha_m > 0$: for all $x \in P$,

$$\|x - f'(m)x\| \geq \alpha_m \|x\| .$$

Moreover, $f'(m)$ are positive. Choose $\rho_m > o$ such that for all $x \in P$,

$$\|f(x) - f'(x)x\| < \left(\frac{\alpha_m}{2}\right)\|x\| \quad (\|x\| \geq \rho_\infty,\ \|x\| \leq \rho_o)$$

CLAIM:

For every $\rho \geq \rho_\infty$ (resp. $\rho \leq \rho_o$) every $y \in P$ with $\|y\|/\rho < \alpha_m/2$, and every $\lambda \in [o,1]$,

$$(1-\lambda)\ (f'(m) + y) + \lambda f$$

has no fixed point on $\rho S \cap P$. Indeed, for $x \in \rho S \cap P$,

$$\begin{aligned} \|x - (1-\lambda)(f'(m)\,x + y) - \lambda f(x)\| &\geq \|x - f'(m)\,x\| \\ &\quad - \|f(x) - f'(m)x\| - \|y\| \\ &\geq \rho\left(\alpha_m - \frac{\alpha_m}{2} - \frac{\|y\|}{\rho}\right) \\ &> 0 . \end{aligned}$$

Hence, for every $y \in P$ with $\|y\| < \rho\alpha_m/2$

$$i(f, \rho B \cap P) = i(f'(m) + y, \rho B \cap P)$$

CASES (a) AND (c).

Set $y = o$ and observe that $f'(m) \in L_1^-(E)$ imply that for every

$\lambda \in [0,1]$, $x - \lambda f'(m)x = o$ does not have a positive solution. Hence

$$i(f, \rho B \cap P) = i(f'(m), \rho B \cap P) = i(o, \rho B \cap P) = 1$$

CASES (b) AND (d).

For this denote by h_m a positive eigenfunction of $f'(m)$ belonging to an eigenvalue $\lambda_m > 1$. Then, for every $\alpha > o$, the equation $x - f'(m)x = \alpha h_m$ does not have a solution in P. Indeed, suppose $x_m > o$ and there exist $\beta_m \geq o$ such that $x_m \geq \beta_m h_m$ and for every $\beta > \beta_m$, $x_m \not\geq \beta h_m$. But then

$$x_m = f'(m)x_m + \alpha h_m \geq f'(m)(\beta_m h_m) + \alpha h_m > (\beta_m + \alpha) h_m$$

which contradicts the maximality of β_m. Now setting $y = \alpha h_m$ with $\alpha > o$ sufficiently small, we find

$$i(f, \rho B \cap P) = i(f'(m) + \alpha h_m, \rho B \cap P) = 0$$

since $f'(m) + \alpha h_m$ has no fixed point on $\rho B \cap P$.

PROOF OF THEOREM 1-5 (sketch).

Define $g: P \to P$ by

$$g(x) := f(x + \bar{x}) - f(\bar{x}) = f(x + \bar{x}) - \bar{x}.$$

Then g is a ssc which is asymptotically linear along P with $g'(\infty) = f'(\infty)$; and g is differentiable at o along P with $g'(o) = f'(\bar{x})$. Hence, without loss of generality, we may assume $\bar{x} = o$. By the above lemma, there exist $\rho_o > o$ and $\rho_\infty > o$ with $\rho_o < \rho_\infty$ such that $i(f, \rho_m B \cap P)$, $m = o, \infty$ is well defined. Hence

$$\begin{aligned} i(f, (\rho_\infty B \setminus \rho_o \bar{B}) \cap P) &= i(f, \rho_\infty B \cap P) - i(f, \rho_o B \cap p) \\ &= -1 \quad \text{(in case (i))} \\ &= 1 \quad \text{(in case (ii))} \end{aligned}$$

Therefore in each case, f has a fixed point in $\rho_\infty B \setminus \rho_o \bar{B}$, that is, a nonzero fixed point. q.e.d.

SKETCH OF PROOF OF THEOREM 1-6.

It suffices to observe that the stated inequalities for λ imply that $f(.,\lambda)$ satisfies the hypothes es of Theorem 1-5, q.e.d.

CHAPTER TWO

MULTIPLICITY RESULTS

CASE INVOLVING THE GRADIENT-DEPENDENT NONLINEARITY

We present in this chapter three main theorems due to Amann and Crandall; Hofer; Ambrosetti and Hess respectively. All of these deal with the situation where the nonlinearity depends on the gradient term. In the first section, Amann and Crandall [4] proved a multiplicity result using the monotone iterative techniques. In fact this is an extension of the first result (by Amann) of the last chapter. By assuming (i) a growth condition on the nonlinearity and (ii) the existence of a lower and an upper solution it is proved that there is a least and a greatest solution. In the second section, we see a result due to Hofer [10] in which the multiplicity result depends on a parameter. His line of proof is very similar to that of Amann and Crandall. Lastly, in this chapter, a result of Ambrosetti and Hess [5] is proved. Here the existence of at least two solutions is proved by assuming also a growth condition for the nonlinear part and certain properties of two upper and lower solutions. Also use is made of a connected set of lower solutions and this is useful in applying the Lyapunov-Schmidt method.

SECTION ONE

We consider some multiplicity results by Amann and Crandall [4] concerning the semi-linear elliptic BVP

$$Lu = f(x,u,Du) \text{ in } \Omega \quad \ldots (2.1)$$
$$u = o \qquad \text{on } \partial\Omega$$

provided that suitable sub- and supersolutions $\bar{v}$ and $\hat{v}$ resp. are known. Because of the presence of the gradient term in f, most of the known techniques are not directly applicable here. The essential step in the technique of proof is to associate solutions of (2.1) with fixed points of order-preserving compact mappings with the same sub- and supersolutions as (2.1). To achieve this goal, instead of (2.1), they solved an auxiliary equation of the form

$$\left.\begin{array}{ll} u + \lambda(Lu - f(x,u,h(Du))) = g & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{array}\right\} \cdots (2.2)$$

for u, when g in $[\bar{v}, \hat{v}]$ is given, where $\lambda > o$ and $h \in C^1(\mathbb{R}^N; \mathbb{R}^N)$. The associated mapping is denoted by $u = T(g)$. The "modulo technicalities" is to show that $\lambda > o$ and h can be chosen so that T is well-defined, compact, order-preserving and the fixed points of T are solutions of (2.1). This allows result for (2.1) to be deduced from standard abstract principles applied to T.Roughly, speaking, they used the monotone iterative technique.
At this juncture, we remark that if f(x,s,t) grows at most quadratically in t, the existence of an ordered pair of sub- and supersolutions implies the existence of a solution.

<u>THEOREM 2-1.</u>

Let $f: \bar{\Omega} \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ be a continuous function such that $\partial f/\partial s$ and $\partial f/\partial t$ exist and are continuous where (x,s,t) denotes a generic point of $\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^N$. Assume moreover that there is an increasing function

$$c : \mathbb{R}_+ \to \mathbb{R}_+$$

such that

$$|f(x,s,t)| \leq c(|s|)(1 + |t|^2) \qquad \ldots \quad (2.3)$$

for

$$(x,s,t) \in \bar{\Omega} \times \mathbb{R} \times \mathbb{R}^N$$

Let $\bar{v}$ and $\hat{v}$ be a sub- and a supersolution resp. of (2.1) such that $\bar{v} \leq \hat{v}$. Then (2.1) has a least and a greatest solution $\bar{u}$ and $\hat{u}$ resp. in the order internal $[\bar{v}, \hat{v}]$.

THEOREM 2-2.

Let the hypotheses of Theorem 2-1 hold. Suppose that $\bar{v}_j$ is a subsolution and $\hat{v}_j$ is a supersolution for j=1,2 such that $\bar{v}_1 < \hat{v}_1 < \bar{v}_2 < \hat{v}_2$. Assume, moreover that $\hat{v}_1$ and $\hat{v}_2$ are strict. Then (2.1) has at least three solutions u_j such that

$$\bar{v}_1 \leq u_1 < u_3 < u_2 \leq \hat{v}_2 \quad \text{and} \quad u_j \in [\bar{v}_j, \hat{v}_j] \text{ for } j=1,2.$$

THEOREM 2-3.

Let the hypotheses of Theorem 2-1 hold. Let $\bar{v}$ and $\hat{v}$ be a strict sub- and a strict supersolution resp. such that $\bar{v} < \hat{v}$. If $u_1 = \bar{u}$ and $u_2 = \hat{u}$ are the least and greatest solutions of (2.1) in the interval $[\bar{v}, \hat{v}]$, $u_1 \neq u_2$ and the BVP.

$$\begin{aligned} Lh - \sum_{j=1}^{N} f_{t_j}(s,u_i(x),Du_i(x))D_j h - f_s(x,u_i(x),Du_i(x))h &= 0 \quad \text{in } \Omega \\ h &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

does not have a positive solution for i=1 or i=2, then (2.1) has at least three distinct solutions in $[\bar{v}, \hat{v}]$.

REMARK 2-1.

Theorems 2-1; 2-2; and 2-3 follow at once from the following proposition and known abstract tools [1].

PROPOSITION 2-1.

Let the hypothesis of Theorem 2-1 hold. Let $\bar{v}$ be a subsolution and $\hat{v}$ be a supersolution of (2.1). Then there exist $h \in C^1(\mathbb{R}^N; \mathbb{R}^N)$ and $\lambda > o$ such that

(a) for every $g \in [\bar{v}, \hat{v}]$ the problem

$$u + \lambda(Lu - f(x,u, h(Du))) = g$$
$$u = 0$$

has exactly one solution u satisfying $u \in [\bar{v}, \hat{v}]$.
This solution is denoted by $u = T(g)$.

(b) A function $u \in [\bar{v}, \hat{v}]$ is a solution of (2.1) iff $u = T(u)$.

(c) Let $C^1_B(\bar{\Omega}) = \{u \in C^1(\bar{\Omega}): u = 0 \quad \text{on } \partial\Omega\}$. Then $T : [\bar{v},\hat{v}]_{L^p(\Omega)} \rightarrow [\bar{v}, \hat{v}]_{C^1_B(\Omega)}$ is continuous, compact and strongly increasing. (Note if X is a Banach space of real-valued functions on $\bar{\Omega}$, set

$$[\bar{v}, \hat{v}]_X : = [\bar{v}, \hat{v}] \cap X$$

and regard $[\bar{v}, \hat{v}]_X$ as having the relative topology from X).

(d) If $w \in [\bar{v}, \hat{v}]$ is a strict subsolution (resp. strict supersolution) of (2.1), then $w < T(w)$ (resp. $T(w) < w$).

(e) If $\bar{v}$ and $\hat{v}$ are strict sub- and supersolutions, then $[\bar{v}, \hat{v}]_{C^1_B(\bar{\Omega})}$ has non-empty interior in $C^1_B(\bar{\Omega})$. Moreover, as a self-mapping of $[\bar{v}, \hat{v}]_{C^1_B(\bar{\Omega})}$ T has a strongly positive Frechét derivative $T'(u)$ for u in the interior of $[\bar{v}, \hat{v}]_{C^1_B(\Omega)}$. Finally for each fixed point u of T in $[\bar{v}, \hat{v}]$, $T'(u)h = h$ for $h \in C^1_B(\Omega)$ exactly when h is a solution of the linear BVP.

$$Lh - \sum_{j=1}^{N} f_{t_j}(s,u(x),Du(x))\, D_j h - f_s(s,u(x),Du(x))\, h = 0 \quad \text{in } \Omega$$
$$h = 0 \quad \text{on } \partial\Omega$$

Proposition 2-1 is proved by maximum principle and continuation arguments in conjunction with the following a priori estimate:

PROPOSITION 2-2.

Let f satisfy (2.3). Then there is an increasing function $\beta : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that if u is a solution of (2.1) then

$$\|u\|_{W^{2,p}(\Omega)} \leq \beta(\|u\|_{C(\bar{\Omega})})$$

Moreover, β depends only on L, Ω, N, p and C.

We now show how these Theorems follow from proposition 2-1. Then proposition 2-1 is established assuming proposition 2-2. Finally proposition 2-2 is proved.

PROOF OF THEOREM 2-1.

If T is the mapping of proposition 2-1, it follows immediately from (a), (b) and (c) that $\hat{u}_n = T^n(\hat{v})$ decreases to an element $\hat{u}$ of $[\bar{v}, \hat{v}]$ as $n \to \infty$ and $\hat{u}$ is the maximal solution of (2.1) in $[\bar{v}, \hat{v}]$. Similarly, $\bar{u}_n = T^n(\bar{v})$ increases to the minimal solution $\bar{u}$ (or see Theorem 0-1)

PROOF OF THEOREM 2-2.

This result follows at once from proposition 2-1 and Theorem 1-1 applied to T.

PROOF OF THEOREM 2-3.

The assertions of Theorem 2-3 follow from proposition 2-1 and Theorem 1-3 applied to T provided that it can be shown that the Frechét derivative T'(u) has a spectral radius different from 1 if u is u_1 or u_2. Since $T'(u_i)$ is a strongly positive compact endomorphism of $C^1_B(\bar{\Omega})$ by proposition 2-1 (e), the spectral radius is an eigenvalue and it is the only eigenvalue with a positive eignevector. These assertions follow from the Krein-Rutman theorem.

Consequently, if $T'(u_i)h \neq h$ for every $h \in C^1_B(\bar{\Omega})$ with $h > 0$, then the spectral radius of $T'(u_i)$ is not 1. But, according to proposition 2-1 (e), this holds under the assumptions of Theorem 2-3.

PROOF OF PROPOSITION 2-1 (sketch).

Let $\bar{v}$, $\hat{v} \in W^{2,P}(\Omega)$ be a sub- and supersolution of (2.1) with $\bar{v} \leq \hat{v}$. Let

$$m = \max(\|\bar{v}\|_{C^1(\bar{\Omega})}, \|\hat{v}\|_{C^1(\bar{\Omega})}) + 1 \qquad \ldots (2.4)$$

Let $H = \{h \in C^1(\mathbb{R}^N; \mathbb{R}^N) : |h(t)| \leq 2|t| \text{ for } t \in \mathbb{R}^N\}$

It follows from proposition 2-2 and the imbedding theorems that there is a constant M with the following properties:

$$\left.\begin{array}{l} \text{(i)} \quad m \leq M \\ \text{(ii) If } h \in H \text{ and } u \in [\bar{v}, \hat{v}] \text{ is a solution of} \\ \qquad Lu = f(x,u,h(Du)) \text{ in } \Omega \text{ and } u = 0 \text{ on } \partial\Omega, \\ \qquad\qquad \text{then } \|u\|_{C^1(\bar{\Omega})} < M \end{array}\right\} \qquad \ldots (2.5)$$

Choose $h \in H$ which satisfies

$$h(t) = t \quad \text{for } |t| < M \quad \text{and } h(\mathbb{R}^N) \qquad (2.6)$$

is bounded and consider the problem

$$\begin{array}{ll} u + \lambda(Lu - f(x,u,h(Du))) = g & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{array} \qquad (2.7)$$

where $\lambda > 0$ and $g \in [\bar{v}, \hat{v}]$. If (2.7) has a solution u and u = g, then

$$\|u\|_{C^1(\Omega)} < M$$

by (2.5). By (2.6) we thus have $h(Du) = Du$ and u is a solution of (2.1). Conversely, any solution u of (2.1) with $u \in [\bar{v}, \hat{v}]$ is a solution of (2.7) with g=u. We next show that (2.7) in fact has a unique solution $u \in [\bar{v}, \hat{v}]$ for every $g \in [\bar{v}, \hat{v}]$ and the mapping $g \longmapsto u = T(g)$ so defined has the properties of proposition 2.1 provided that λ is chosen suitably small.

Set $k(x, s,t) = f(x, s,h(t))$ and let $\lambda > 0$ satisfy

1- $\lambda k_s(x, s,t,) > 0$ for $x \in \bar{\Omega}$, $|s| \leq m$, $|t| \in R^N$ ...(2.8)

It is possible to choose such a λ since $h\ (R^N)$ is bounded and $k_s\ (x, s, t\ ) = f_s(x, s, h(t))$. Fix λ as above and define

$$G\ :\quad C^1(\bar{\Omega}) \rightarrow C\ (\Omega)$$

by

$$G\ (u)\ (x) \doteq k\ (x, u(x), Du(x)) \qquad \ldots (2.9)$$

So (2.7) can be abbreviated to

$$\begin{aligned} u + \lambda(Lu - G(u)) &= g \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned} \qquad \ldots (2.10)$$

We make use of the following form of the maximum principle to prove the existence and uniqueness of solutions of (2.10).

LEMMA 2-1

Suppose $a_j \in L^\infty(\Omega)$ for $j = 0,\ldots,$ N and $a_0 > 0$. Let $u \in W^{2,P}(\Omega)$ sqtisfy the inequalities

$$\begin{aligned} Lu + \sum_{j=1}^{N} a_j D_j u + a_0 u &\geq 0 \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

Then $u \geq 0$. Moreover, if $u=0$ then $u\ (x) > 0$ for every $x \in \Omega$. If $u \neq 0$ and $u\ (x_0) = 0$ for some $x_0 \in \partial\Omega$ then $\left(\frac{\partial u}{\partial \alpha}\right)(x) < 0$ where α is an arbitrary outwqrd pointing vector at x which is not tangential to $\partial\Omega$.

PROOF

The assertion follows from Bony's maximum principle [8] by means of standard arguments as given, for example in [15].

Lemma 2 - 1 may be used to prove the following comparsion result:

LEMMA 2 - 2

Suppose that $u, v \in W^{2,p}(\Omega)$ satisfy

$$\|u\|_{C(\bar{\Omega})}, \|v\|_{C(\bar{\Omega})} \leq m$$

and the inequalities

$$\begin{aligned} u + \lambda(Lu - G(u)) &\geq v + \lambda(Lv - LG(v)) && \text{in } \Omega \\ u &\geq v && \text{on } \partial\Omega \end{aligned}$$

Then $u \geq v$. Moreover if $u \not\equiv v$ then $u(x) > v(x)$ for $x \in \Omega$. If $u \not\equiv v$ and $u(x_0) = v(x_0)$ for some $x_0 \in \partial\Omega$, then $\left(\frac{\partial u}{\partial \alpha}\right)(x_0) - \left(\frac{\partial v}{\partial \alpha}\right)(x_0)$ where α is an arbitrary outwqrd pointing vector at x which is not tangential on $\partial\Omega$.

PROOF

Set $w = u - v$. Then the hypotheses imply the inequalities

$$\begin{aligned} Lw + \sum_{j=1}^{N} a_j D_j w + a_0 w &\geq 0 && \text{in } \Omega \\ w &= 0 && \text{on } \partial\Omega \end{aligned}$$

where $a_j(x) = -\int_0^1 k_{t_j}(x, v(x) + \beta w(x), Dv(x) + \beta Dw(x))\, d\beta$ for $j = 1, \ldots, N$ and

$a_0(x) = \lambda^{-1} \int_0^1 (1 - \lambda k_s(x, v(x) + \beta w(x), Dv(x) + \beta Dw(x))\, d\beta$

We have $a_0 > 0$ by (2.8) and hence lemma 2 - 1 implies the desired hypotheses.

LEMMA 2 - 3

Let $g \in [\bar{v}, \hat{v}]$. Then the problem (2. 10) has a unique solution $u \in [\bar{v}, \hat{v}]$.

PROOF

The uniqueness assertion follows at once from lemma 2 - 2.
Let $g \in [\bar{v}, \hat{v}]$ be fixed and set

$$\hat{\psi} = \hat{v} + \lambda(L\hat{v} - G(\hat{v})) - g .$$

Since $\hat{v}$ is a supersolution and $\hat{v} - g \geq 0$ we have $\hat{\psi} \geq 0$.
From the L^p - theory of elliptic BVP's and lemma 2 - 1 it

follows that for every $q \in L^p(\Omega)$ the problem

$$(L+1)\,w = q \quad \text{in } \Omega$$
$$w = o \quad \text{on } \partial\Omega$$

has a unique solution $w = Kq \in W^{2,p}(\Omega)$. Moreover, $K : L^p(\Omega) \to W^{2,p}(\Omega)$ is continuous. Set $\hat{z} = -K(L+1)\hat{v}$ and $\hat{w} = \hat{z} + \hat{v}$, Then $\hat{w} \in W^{2,p}(\Omega)$, $(L+1)\hat{w} = o$, $\hat{w} = \hat{v}$. Define

$$H : C^1(\bar{\Omega}) \times R \longrightarrow C^1(\bar{\Omega})$$

by

$$H(u,\beta) = \lambda(u - \beta\hat{w}) + K\left[(1-\lambda)u - \lambda G(u) - g - \beta(\hat{\psi} - \lambda\hat{w})\right] \quad \dots (2.12)$$

Clearly, H is continuously differentiable and

$$\left.\begin{array}{l} H(u,\beta) = K\left[u + \lambda(Lu - G(u)) - g - \beta\hat{\psi}\right] \\ \text{provided } u - \beta\hat{w} \in W^{2,p}(\Omega) \text{ and } (u - \beta\hat{w}) = o \end{array}\right\} \quad \dots 2.13$$

Hence $H(\hat{v}, 1) = o$ and a solution u of $H(u, o) = o$ is a solution of (2.10). To show how to continue the solution $u = v$, $\beta = 1$ of $H = o$ to a solution at $\beta = o$. Let $D_1H(u,\beta)$ denote the Frechét derivative of the mapping $u \to H(u,\beta)$. Clearly, $D_1H(u,\beta) : C^1(\bar{\Omega}) \to C^1(\bar{\Omega})$ is given by

$$D_1H(u,\beta)\,h = \lambda h + K\left[(1-\lambda)\,h - \lambda G'(u)h\right]$$

where

$$G'(u)h = \sum_{j=1}^{N} K_{t_j}(x,u,Du)\,D_jh + K_s(x,u,Du)\,h$$

A function $h \in C^1(\bar{\Omega})$ satisfies $D_1H(u,\beta)h = o$ iff $h \in W^{2,p}(\Omega)$, $h = o$ on $\partial\Omega$ and $h + \lambda(L - G'(u))h = o$ in Ω. Lemma 2 - 1 and (2 - 8) imply therefore that $D_1H(u,\beta)$ is injective if $\|u\|_{C(\bar{\Omega})} \leq m$. Since K is bounded from $L^p(\Omega)$ to $W^{2,p}(\Omega)$, it is compact from $L^p(\Omega)$ to $C^1(\bar{\Omega})$ and it follows that $D_1H(u,\beta)$ is an automorphism of $C^1(\bar{\Omega})$ for $(u,\beta) \in C^1(\bar{\Omega}) \times \mathbb{R}$

with $\|u\|_{C(\bar{\Omega})} \leq m$. Therefore by the implicit function theorem there is a neighbourhood $V \times W$ of $(v, 1)$ in $C^1(\bar{\Omega}) \times R$ such that $H(u,\beta) = 0$ has a unique solution $u = u(\beta)$ for $\beta \in W$ with $u(\beta) \in V$. In particular, $u(1) = \hat{v}$ since $\|\hat{v}\|_{C(\bar{\Omega})} < m$ and $\beta \to u(\beta)$ is continuous, we can assume $\|u(\beta)\|_{C(\bar{\Omega})} < m$ by choosing W sufficiently small. From $H(u(\beta), \beta) = o$ and (2. 13) we deduce that $u(\beta) \in W^{2,p}(\Omega)$ and

$$u(\beta) + \lambda\left[Lu(\beta) - G(u(\beta))\right] = g + \beta\hat{\Psi}$$
$$u(\beta) = \beta\hat{\omega} = \beta\hat{v} = \hat{v}$$

for $\beta \in W \cap [0, 1]$. Since also

$$g + \beta\hat{\Psi} \leq g + \hat{\Psi} = \hat{v} + \lambda(L\hat{v} - G(\hat{v}))$$

for such β, lemma 2 - 1 implies $u(\beta) \leq \hat{v}$. In a similar way, we see that $u(\beta) \geq \bar{v}$ for $\beta \in W \cap [0,1]$. Using in addition (2.6), (2.9) and the defn. of h, it follows that $G(u(\beta))$ remains bounded in $L^{\infty}(\Omega)$. Then (2. 12) and $H(u(\beta),\beta) = o$ show that $\{u(\beta) : \beta \in W \cap [0,1]\}$ is bounded in $W^{2,p}(\Omega)$ and hence precompact in $C^1(\bar{\Omega})$.

A standard continuation argument now establishes the existence of a continuous mapping $u : [0,1] \to C^1(\bar{\Omega})$ such that $H(u(\beta),\beta) = o$ and $\bar{v} \leq u(\beta) \leq \hat{v}$ for $\beta \in [0,1]$. Hence lemma 2-3 is proved.

According to Lemma 2 - 3 we now have defined a mapping $g \in [\bar{v},\hat{v}] \to u = T(g)$ where u is the unique solution of (2.10) in $[\bar{v},\hat{v}]$. From (2.5) and (2;6) the fixed points of T are precisely the solutions of (2.1) which lie in the interval $[\bar{v},\hat{v}]$.

Thus (a) and (b) of Proposition 1 hold. Moreover, as above, we see that $T([\bar{v}, \hat{v}])$ is bounded in $W^{2,p}(\Omega)$ and hence precompact in $C^1_\beta(\bar{\Omega})$. To see that T is continuous as a mapping $T: [\bar{v},\hat{v}]_{L^p(\Omega)} \to [\bar{v},\hat{v}]_{C^1_\beta(\bar{\Omega})}$ we then need only check that it has closed graph, which follows at once from the uniqueness. The fact that T is strongly increasing follows at once from Lemma 2-2. Thus (c) of Proposition 1 holds. Next let w be a subsolution of (2.1) with $w \in [\bar{v}, \hat{v}]$. The interval $[\bar{v},\hat{v}]$ may be replaced by $[w, \hat{v}]$ in the above proof (w satisfies the same assumptions as $\bar{v}$ and $\bar{v} \leq w \leq \hat{v}$ implies the choices of λ, m and h work for w in place of $\bar{v}$), so $T([w,\hat{v}]) \subseteq [w,\hat{v}]$ and so $w \leq Tw$. If w is not a solution, then $w \neq Tw$ so $w < Tw$. Similarly, supersolutions w of (2.1) satisfy $T(w) \leq w$ with strict inequality for strict supersolutions. This establishes (d). We turn now to (e) If $\bar{v}$ and $\hat{v}$ are strict we have $\hat{v} > \bar{v}$. By (c), there is an $\epsilon > 0$ such that if B_ϵ is the ball of radius ϵ about the origin in $C^1_\beta(\bar{\Omega})$ then

$$T(\hat{v}) - T(\bar{v}) + B_{2\epsilon} \subseteq \{u \in C^1_\beta(\bar{\Omega}) : u \geq 0\}.$$

But then

$$[\bar{v}, \hat{v}] \supseteq [T\bar{v}, T\hat{v}] \supset \frac{T(\bar{v}) + T(\hat{v})}{2} + B_\epsilon ,$$

which shows that $[\bar{v},\hat{v}]$ has nonempty interior. To see that T is continuously differentiable as a mapping from the interior of $[\bar{v}, \hat{v}]_{C^1_\beta(\bar{\Omega})}$ (a set which clearly contains the fixed points of T if $\bar{v}$ and $\hat{v}$ are strict) to $C^1(\bar{\Omega})$, we need only recall that T was obtained by applying the implicit function theorem. Indeed, let us write $H(u, \beta, g)$ to indicate the dependence of H on g in (2. 13). Since H is of class C^1, $h(T(g), 0, g) = 0$ and $D_1H(T(g), 0, g)$ is an automorphism of $C^1(\bar{\Omega})$, $g \to T(g)$ is continuously differentiable. Calculating the derivative shows that $T'(g)h = w$ is equivalent to $w \in W^{2,p}(\Omega)$ and

$$w + \lambda(Aw - G'(T(g))w) = h \text{ in } \Omega$$
$$w = 0 \text{ on } \partial\Omega$$

Hence Lemma 2 - 1 implies that $T'(g)$ is a strongly positive linear operator. Finally, if $u = g$ is a fixed point of T, the above characterization of T establishes the last assertion of (e).

Now the proof of Proposition 2 - 2 follows at once from the next Lemma whose proof will be omitted.

LEMMA 2 - 4

For every $b \in L(\Omega)$ there is exactly one solution $u \in W^{2,p}(\Omega)$ of the problem:

$$(A+1)u = b(1+|Du|^2) \quad \text{in } \Omega$$
$$u = o \quad \text{on } \partial\Omega$$

Moreover, there is an increasing function $c_o : R_+ \to R_+$ such that $\|u\|_{W^{2,p}(\Omega)} \leq c_o(\|b\|_{L^\infty(\Omega)})$. The function depends only on A, Ω, p and N.

REMARK.

Indeed, if u is a solution of (2.1), it is also a solution of $(A+1)u = b(1+|Du|^2)$, $u = o$, where $b = (f(x, u, Du)+u)/(1+|Du|^2)$. Thus if we set $\beta(r)$ ì $c_o(c(r) + r)$, the assertion of Proposition 2 - 2 follow from Lemma 2 - 4.

SECTION II

Next in this chapter we see another multiplicity result by Hofer [10] . In fact, here the existence and multiplicity of solutions is proved in dependence on a parameter. The BVP under consideration is

$$(P_t) \qquad Lu = G((x,u,\ Du) + tr \text{ in } \Omega$$
$$u = o \qquad \text{on } \partial\Omega$$

where t is a real parameter value. L is the usual strongly uniformly elliptic differential operator and adding αI, $\alpha > 0$ to both sides of (P_t), $a_o(x) \geq \varepsilon > o \ \forall\ x \in \Omega$. Moreover,

(1)... $r \in C(\bar{\Omega}) \setminus \{o\}$ satisfies $r(x) \geq o$ in Ω

If λ_1 denotes the principal eigenvalue of the linear BVP $Lu = \lambda u$ in Ω, $u = o$ on $\partial\Omega$, since $a_o \geq \varepsilon$ we have $\lambda_1 > o$.

For the nonlinearity we assune:

(GI) $G : \Delta = \bar{\Omega} \times R \times R^N \to R$ is continuous, continuously differentiable with respect to s and t and satisfies the growth conditions

(i) $G(x,\ s, \xi) \leq c_1(|s|)\ (1 + |\xi|^2)$ for all $(x,\ s, \xi) \in \Delta$

(ii) $G(x,s,\xi) \leq c_2(s^+)\ (1 + |s| + |\xi|)$

for all $(x,\ s, \xi) \in \Omega^o \times R \times R^N$

where $C_i : R^+ \to R^+$ are suitable increasing functions and Ω^o denotes the set $\{x \in \Omega \mid r(x) = o\}$ and $s^+ = s$ if $s \geq 0$ and $s^+ = o$ otherwise.

(G 2) There exist a continuous function

$G^* : \bar{\Omega} \times R \to R$ such that

$G(x,s,t) \geq G^*(x,s,)$ for all $(x,\ s,\ t) \in \Delta$ and

(i) $\limsup_{s \to -\infty} \dfrac{G^*(x,s)}{s} < \lambda_1$

(ii) $\limsup_{s \to +\infty} \dfrac{G^*(x,s)}{s} > \lambda_1$

uniformly for $x \in \bar{\Omega}$.

The main results are :

THEOREM 2 - 4 (HOFER)

Assume r and G satisfy (1) and (G1), (G2) resp. Then there exists a $t_0 \in R$ such that (P_t^+) is solvable for $t < t_0$ and not solvable for $t > t_0$.

THEOREM 2 - 5

Assume the hypotheses of THM. 2 - 4 are satisfied and in addition that there exists for all bounded intervals $\Gamma \subset R$ a constant $M = M(\Gamma)$ such that $u_t(x) \leq M$ for all $x \in \bar{\Omega}$, where u is an arbrtrary solution of (P_t), $t \in \Gamma$. Then there is a $t_0 \in R$ such that (P_t) possesses at least two solutions for $t < t_0$, at least one solution for $t = t_0$ and no solution for $t > t_0$.

REMARK

The proof of the above results are achieved by constructing a global strongly increasing fixed point operator in a suitable function space and observing that the fixed point operator has some properties similar to a strongly positive linear endomorphism. This has an advantage in studying multiplicity over the method of Amann and Crandall. The following proposition and Lemma whose proofs are contained in [10] (and will be omitted here) will be needed in the proof of the main results.

PROPOSITION

Assume $\gamma : C^1(\bar{\Omega}) \longrightarrow L^\infty(\Omega)$ is a continuous operator which satisfies $|\gamma(u)(x)| \leq c(|u(x)|)(1 + |Du(x)|^2)$ for all $x \in \bar{\Omega}$, where $c : R^+ \longrightarrow R^+$ is a suitable increasing function and $\gamma(o) = o$. Moreover assume γ admits for all $u, v \in C^1(\bar{\Omega})$ the representation

$$\gamma(u) - \gamma(v) = \sum_{i=1}^{N} b_i D_i (u-v) + b_o (u-v)$$

with $b_i = b_i(u, Du, v, Dv) \in L^\infty(\Omega)$, $i = o, \ldots, N$ and $b_o \geq 0$. Then we have

(a) The problem

$$(*)\quad Lu + \gamma(u) = f \text{ in } \Omega\,,\quad u = o \text{ on } \partial\Omega$$

is for all $f \in L^{\infty}(\Omega)$ uniquely solvable in $W^{2,p}(\Omega)$ and the solution operator $H : L^{\infty}(\Omega) \longrightarrow C^{1}_{B}(\bar{\Omega})$ is strongly increasing and compact where the function spaces are equipped with the natural order stuructures. If $S \subset L^{\infty}(\Omega)$ is a L^{∞}- bounded set, equipped with the induced L^{p} - topology given by the imbedding $L^{\infty}(\Omega) \longrightarrow L^{p}(\Omega)$, then the operator $H_{S} : S \longrightarrow C^{1}_{B}(\bar{\Omega}) : f \longrightarrow Hf$ is continuous.

(b) If there exist an open subset $\Omega^{*} \subset \Omega$ and $\forall\, v \in C^{1}(\bar{\Omega})$ a constant $M(v) \geq o$ such that

$|\gamma(w+v)(x) - \gamma(v)(x)| \leq M(v)\,(|w(x)| + |Dw(x)|)$ for $x \in \Omega^{*}$ and $w \leq o$, and if r satisfies (1) with $r(x) > o$ for all $x \in \Omega \setminus \Omega^{*}$, then for all $a \in C(\Omega)$ and all $\phi \in C^{1}_{B}(\bar{\Omega})$ we find $T = T(a, \phi, r) \in R$ such that $H(a + tr) \leq \phi$ for all $t \leq T$.

LEMMA

For every $b \in L^{\infty}(\Omega)$ there is exactly one solution $u \in W^{2,p}(\Omega)$ of the problem $Lu = b\,(1 + |Du|^{2})$ in Ω, $u = o$ on $\partial\Omega$. Moreover there is an increasing function $d : R^{+} \to R^{+}$ such that $\|u\|_{2,p} \leq d(\|b\|_{\infty})$. The function d is depending only on L, Ω, p and n.

PROOFS OF THE MAIN RESULTS

Proof of Theorem 1. Define $\bar{\gamma} : \bar{\Omega} \times R \times R^{n} \to R$ by ($G_{s} = \partial G/\partial s$)

$$\bar{\gamma}(x, s, \xi) = \int_{o}^{s} (\operatorname{sign}(G_{s}(x,u,\xi)) - |G_{s}(x, u, \xi)|)\, du + s$$

Then the maps $s \longrightarrow \bar{\gamma}(x, s, \xi)$ and $s \to \bar{\gamma}(x,s,\xi) + G(x,s,\xi)$ are strictly increasing for all fixed $(x,\xi) \in \Omega \times R^{n}$. Consider

$$Lu + \bar{\gamma}(x, u, Du) = G(x, v, Du) + \bar{\gamma}(x, v, Du) + tr \text{ in } \Omega$$

$$u = o \quad \text{on } \partial\Omega$$

where v is a given function in $C(\bar{\Omega})$. We will show that (2) is for all v uniquely solvable and that the solution operator $T : R \times C(\bar{\Omega}) \to C^1_B(\bar{\Omega}) : (t,v) \to u(t,v)$ is strongly increasing in both arguments and compact. Define $\bar{\gamma}_v : \bar{\Omega} \times R \times R^n \to R$ by

$$\bar{\gamma}_v(x, s, \xi) = \bar{\gamma}(x, s, \xi) - G(x, v(x), \xi) - \bar{\gamma}(x, v(x), \xi)$$
$$+ G(x, v(x), 0) + \bar{\gamma}(x, v(x), 0)$$

Then $\bar{\gamma}_v$ is locally uniformly lipschitz continuous in s and ξ, satisfies $|\bar{\gamma}_v(x,s,\xi)| \leq c_v(|s|)(1+|\xi|^2)$ for a suitable increasing function $c_v : R^+ \to R^+$ depending on v, and is increasing in s. Moreover we have

$$\bar{\gamma}_v(x, s, \xi) - \bar{\gamma}_v(x, s', \xi') = \bar{\gamma}_v(x, s, \xi) - \bar{\gamma}_v(x, s', \xi) + \bar{\gamma}_v(x, s', \xi)$$
$$- \bar{\gamma}_v(x, s', \xi')$$
$$= b_0(s-s') + \sum_{i=1}^{n} b_i(\xi_i - \xi'_i)$$

where for i=1...n

$$b_i(x) = \begin{cases} (\bar{\gamma}_v(x, s', \xi) - \bar{\gamma}_v(x, s', \xi'))(\xi_i - \xi'_i)^{-1}, & \xi_i \neq \xi'_i \\ 0 & \text{otherwise} \end{cases}$$

and

$$b_0(x) = \begin{cases} (\bar{\gamma}_v(x, s, \xi) - \bar{\gamma}_v(x, s', \xi))(s-s')^{-1} & s \neq s' \\ 0 & \text{otherwise} \end{cases}$$

Since $\bar{\gamma}_v$ is strictly increasing in s we have $b_0(x) > 0$ and by the lipschitz condition for i = 0....n: $b_i \in L^\infty(\Omega)$. Let $\Omega^* = \Omega^0$.

Since the mapping $(s, \xi) \to \bar{\gamma}_v(x, s, \xi)$ is locally uniformly lipschitz continuous and satisfies for given $t \in R$ $|\bar{\gamma}_v(x, s, \xi)| \leq c_2(t)\ (1+|s|+|\xi|)$ for all $(x, s, \xi) \in \Omega^* \times R \times R^n$ with $s \leq n$, we find for all given $w \in C^1(\bar{\Omega})$ a constant M (w) such that

$$|\bar{\gamma}_v(x, s+w(x), \xi+Dw(x)) - \bar{\gamma}_v(x, w(x), Dw(x))| \leq M(w)(|s|+|\xi|)$$

for all $(x, s, \xi) \in \Omega^* \times R \times R^n$. ... (3)

Let us define the operator $\gamma_v : C^1(\bar{\Omega}) \to L^\infty(\Omega)$ by $\gamma_v(u) = \bar{\gamma}_v(\,.,\, u,\, Du\,)$. By (3) we can find for all $w \in C^1(\bar{\Omega})$ a constant $M(w) > 0$ such that

$$|(\gamma_v(x+w) - \gamma_v(w))(x)| \leq M\,(w)(|u(x)| + |Du(x)|\,)$$

for all $x \in \Omega^*$ and all $u \in C^1(\bar{\Omega})$ with $u \leq 0$.

The preceding discussion of $\bar{\gamma}_v$ implies that γ_v satisfied the hypotheses of the proposition.

This implies that (2) is uniquely solvable because it is equivalent to

$$Lu + \gamma_v(u) = G(\,.\,,v,0) + \bar{\gamma}_v(\,\cdot\,,\, v,\, 0) + tr \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega \quad (4)$$

Let $T : R \times C(\bar{\Omega}) \to C^1_B(\bar{\Omega}) : (t,v) \to u(t,v)$ be the solution operator of (4). Using Lemma 2 and proceeding as in the proof of proposition, part (a), we deduce that T is strongly increasing and compact. The solution u of

$$Lu + \gamma_0(u) = G(\,.\,,0,0) + \bar{\gamma}_0(\,.\,,0,0) + tr \quad \text{in } \Omega,\ u = 0 \text{ on } \partial\Omega$$

is by definition $T(t,0)$. Applying now the propostion, we conclude for $t^* \leq 0$ small enough $T(t^*,0) \ll 0$. From results in [12] (Proposition 2.13) we obtain because of the asymptotic behaviour of G as $s \to -\infty$, that there exists a strict subsolution $\bar{u} \leq 0$ of (P_{t^*}) which implies $T\,(t^*,\bar{u}) \gg \bar{u}$ in $C^1_B(\bar{\Omega})$. Since $T\,(t^*,.)$ is strongly increasing, it maps $V \equiv [\bar{u},\, 0]_{C^1_B(\bar{\Omega})} \to$ int(V). Moreover V is bounded in $C\,(\bar{\Omega})$ which implies that $T(t^*,\, V)$ is bounded in $C^1_B(\bar{\Omega})$. Hence, by the compactness of $T(t^*,.) : C(\bar{\Omega}) \to C\,(\bar{\Omega})$, we find a fixed point in int (V) by Schauder's fixed point theorem. Let $t_0 \equiv \sup\{t \in R \mid (P_t) \text{ is solvable}\}$. If (P_t) has a solution u_t for some t we infer that u_t is a strict supersolution for all $t' < t$. As above we find a strict subsolution $\bar{u} \leq u_t$ which implies the existence of a solution $u_{t'} \in \mathrm{int}([\bar{u}, u_t]_{C^1_B(\bar{\Omega})})$ of $(P_{t'})$.

By the asymptotic behaviour of G as $s \to +\infty$, one can show that $t_0 < +\infty$ (see [12], ~oof of Theorem 3.4, P 636).

Proof of Theorem 2. Let T be as in Theorem 1 and choose $t < t_0$. As we have seen in the proof above there exist strict sub- and supersolutions of (P_t) : $\bar{u} \leq \hat{u}$. Denote the order interval $[\bar{u},\hat{u}]_{C^1_B(\bar{\Omega})}$ by A. Since T = T(t,.) maps $A \to \mathrm{int}\,(A)$ we find that the fixed points of T in A are in int(A).We may assume there is only one fixed point $u \in A$ (otherwise we are done).Finally we find for some $\epsilon > 0$ such that $u + \epsilon B \subset \mathrm{int}\,(A)$, Where B denotes the open unit ball in $C^1_B(\Omega)$, making use of the standard properties of the Leray-Schauder-Degree and the fixed point index

$$\deg(I-T,u+\epsilon B,0) = i(T,u+\epsilon B,\ C^1_B(\Omega)) = i(T,u+\epsilon B,A)$$
$$= i(T,A,A) = 1$$

For a solution u of $T(\bar{t},u)=u$, $\bar{t} \in [t,t_0+1] \equiv \Gamma$ we have $u(x) \leq M(\Gamma)$. On the other hand solutions are bounded from below by

$$Lu = G(x,u,Du)+tr$$
$$\geq G^*(x,u)+tr$$
$$\geq (\lambda_1-\bar{\epsilon})u+h \text{ in } \Omega$$
$$u=0 \text{ on } \partial\Omega$$

for some $\bar{\epsilon} > 0$ small enough and a suitable constant h depending only on $\bar{\epsilon}$ and Γ. Hence $u \geq K(h)$, where K denotes the positive solution operator of

$$Lu-(\lambda_1-\bar{\epsilon})u=g \quad \text{in } \Omega, \qquad u = 0 \text{ on } \partial\Omega .$$

Since the solutions of $(P_{\bar{t}})$, $\bar{t} \in \Gamma$, are contained in a $C(\bar{\Omega})$-bounded set, they must be bounded in $C^1_B(\bar{\Omega})$ by a constant k-1. We deduce, since we may assume $u+\epsilon B \subset kB$, making use of the additiviy, homotopy and excision properties of the Leray-Schauder-Degree

$$\deg(I-T,\ kB \setminus u+\epsilon B,0) = \deg(I-T,Kb,o)-1$$

$=\deg(I-\mathbb{T}(t_0+1\ldots),\ kB,0)-1$

$=-1$ which implies the existence of a second solution. To study (P_{t_0}) choose a sequence (t_n), $t_n<t_0, t_n \to t_0$. One can show as before that the sequence (u_n) of solutions u_n (P_{t_n}) is relatively compact in $C^1_B(\bar{\Omega})$. Hence we may assume(for a subsequence) $u_n \to u$ strongly in $C^1_B(\bar{\Omega})$. Taking the limit for $T(t_n, u_n)=u$ we find $T(t_0,u)=u$ which completes the proof.

SECTION III

In this section, we see a result of Ambrosetti and Hess[5] where the existence of pairs of solutions for a certain class of nonlinear elliptic equations is proved provided the nonlinear part grows quadratically with respect to the first derivatives and existence and certain properties of two supersolutions and two subsolutions are assumed.

THEOREM (AMBROSETTI AND HESS)

Given the BVP

$$\begin{aligned} Lu &= f(x,u,Du) \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned} \qquad \ldots (1)$$

where $f \in C^1(\bar{\Omega} \times R \times R^N; R)$. Suppose the following is satisfied:

(H_1) $|f(x,s,t)| = c(1+|t|^2)$ for all $(x,s,t) \in \bar{\Omega} x R x R^N$

Moreover assume

(H_2) (1) has two supersolutions $\bar{V}_1, \bar{V}_2$ with $\bar{V}_1(x) < \bar{V}_2(x)$ for all $x \in \bar{\Omega}$;

(H3) (1) has a set S of subsolutions which is connected in the space $E := C(\bar{\Omega})$;

(H4) there exist $\underline{V}_1$ and $\underline{V}_2$ in S such that:

$$\underline{V}_1 \leq \bar{V}_1 \quad \text{in } \bar{\Omega}$$

$$\underline{V}_2 > \overline{V}_2(x_0) \qquad \text{for some } x_0 \in \Omega$$

Then (1) admits at least two distinct solutions.

REMARK

It is known (Amann and Crandall [4]) that if (H1) is satisfied and if (1) has a subsolution $\underline{V}$ and a supersolution $\overline{V}$ with $\underline{V} \leq \overline{V}$, then (1) has a solution u such that $\underline{V} \leq u \leq \overline{V}$ in Ω.

PROOF OF THEOREM (SKETCH)

By the above Remark, (1) possesses a solution u with $\underline{V}_1 \leq u \leq \overline{V}_1$.

To obtain a second solution, set

$$S' = \{\underline{V} \in S : \underline{V}(x) < \overline{V}_2(x) \text{ for all } x \in \overline{\Omega}\}$$

$$S'' = \{\underline{V} \in S ; \text{ there exist } x_0 \in \Omega \text{ with } \underline{V}(x_0) > \overline{V}(x_0)$$

By (H_4) and the fact that $\overline{V}_1(x) < \overline{V}_2(x)$, for all $x \in \Omega$, it follows that S' and S'' are not empty. Since S is equipped with the topology of uniform convergence, it is easy to see that S' and S'' are open in S . Moreover, $S' \cap S'' = \emptyset$ and hence, S being connected,

$$S \setminus (S' \cup S'') \neq 0$$

Let $z \in S \setminus (S' \cup S'')$. Since $z \in S \setminus S''$

$$z \leq \overline{V}_2 \quad \text{in } \Omega \qquad --- \quad (2)$$

Further, $z \notin S'$ implies the existence of $\hat{x} \in \overline{\Omega}$:

$$z(\hat{x}) = \overline{V}_2(\hat{x}) \qquad --- (3)$$

From (2) it follows that (1) has a further solution u_2 with $z \leq u_2 \leq \overline{V}_2$ in Ω. By (3), $u_2(\hat{x}) = \overline{V}_2(x)$ and hence

$$V_1(\hat{x}) \leq u_1(\hat{x}) \leq \overline{V}_1(\hat{x}) < \overline{V}_2(\hat{x}) = u_2(x)$$

Thus the two solutions u_1 and u_2 are distinct solutions.

CHAPTER THREE

MULTIPLICITY RESULTS - CASE WHICH RELIES ON SIMPLE EIGENVALUE COMPARSION ARGUMENTS

Here we shall present two results of Berestycki [7] where the principal eigenvalue plays a central role. The proofs are based principally on the systemical aid of the arguments of the comparsions of the eigenvalues of certain linear problems combined with the positivity properties and the nodal characterization of the corresponding eigenfunctions. The first result (which deals with the Dirichlet problem $Lu = g(x,u) + h(x)$ $(x \in \Omega)$, $u = o$ on $\partial\Omega$) gives an elementary and more general proof of the theorem of Ambrosetti and Prodi [6], which sta tes under what conditions the problem has 0, 1, or 2 solutions. In the second section of the chapter we see a result for the nonlinear eignevalue problem $Lu+f(u)=\lambda u$ in Ω $u = o$ on $\partial\Omega$ which determines the number of solutions for various values of λ.

SECTION ONE

In this section, we see some complements to a classic result of Ambrosetti and Prodi which concerns the problem of the type

$$(1) \ldots \quad \begin{aligned} Lu &= g(x,u) + h(x) \text{ in } \Omega \\ u &= o \quad \text{on } \partial\Omega \end{aligned}$$

where $h(x)$ is a given function in $C^{0,\alpha}(\Omega)$.

THEOREM I (BERESTYCKI)

Given the BVP (1) where $g : \bar{\Omega} \times R \to R$ is of class C^1 and satisfies the following conditions:

(2) $\forall x \in \Omega$, $s \to g(x,s)$ is a strictly convex and increasing

function;

(3) $\lim\limits_{s\to\infty} \frac{g(x,s)}{s} \leq \lambda_2$, uniformly w.r.t. $x \in \bar{\Omega}$

where λ_i denotes the ith eigenvalue of the Dirichlet problem.

$$(4) \quad L\varphi = \lambda\varphi \text{ in } \Omega, \ \varphi = 0 \text{ on } \partial\Omega$$

Then (1) admits at most 2 solutions, independent of $h \in C^{0,\alpha}(\bar{\Omega})$.

Furthermore, if (1) possesses exactly two solutions, u and v, then these are ordered.

Proof (Sketch)

Let a be a continuous function in $\bar{\Omega}$: $a \geq 0$, $a \not\equiv 0$ in Ω, denote by $\mu_i(a)$ the ith eigenvalue of the Dirichlet problem.

$$(5) \quad L\varphi = \mu a\varphi \text{ in } \Omega, \ \varphi = 0 \quad \text{on } \partial\Omega$$

Then

$$\mu_i(1) = \lambda_i; \quad \mu_i(\lambda_i) = 1; \quad \mu_i(\lambda_j) < 1 \text{ if } i<j \text{ and}$$
$$\mu_i(\lambda_j) > 1 \text{ if } i>j.$$

Also if,

$$a \leq b, \ a \not\equiv b \text{ in } \Omega, \text{ then } \mu_i(a) > \mu_i(b) \ \forall i .$$

Suppose that (1) admits two distinct solutions, u and v in $C^{2,\alpha}(\Omega)$. Then $w = u - v \not\equiv 0$ satisfies the equation

$$(6) \quad Lw = \rho(x)w \quad \text{in } \Omega, \ w = 0 \quad \text{on } \partial\Omega$$

where

$$\rho(x) = \frac{g(x,u(x)) - g(x, v(x))}{u(x) - v(x)} \quad \text{when } u(x) \not\equiv v(x)$$

and

$$\rho(x) = g'(x, u(x)) \text{ for } u(x) = v(x)$$

Since $w \not\equiv 0$, (6) means that $1 = \mu_i(\rho)$ for a certain integer i.

Now conditions (2), (3) imply $0 < \rho(x) < \lambda_2 \quad \forall x \in \Omega$

So

$$1 = \mu_i(\rho) > \mu_i(\lambda_2) = \frac{\lambda_i}{\lambda_2}$$

Therefore this implies that $i < 2$, meaning that $i=1$: $1= \mu_i(\rho)$. It is then classical that w is considered as an eigenfunction of (5) (where $a= \rho$) associated with the first eigenvalue, is of constant sign. So, the two distinct solutions of (1) are necessarily ordered.

We show next that (1) cannot possess three distinct solutions. If (1) has three distinct solutions, u,v and z, one can suppose without loss of generality, from the preceeding, that for example $u < v < z$ in Ω. So $w=v-u$ and $\hat{w} = z-v$ satisfy

$$(7) \quad \begin{array}{llll} Lw = \rho w & \text{in } \Omega, & w=0 & \text{on } \partial\Omega \\ L\hat{w} = \hat{\rho}\hat{w} & \text{in } \Omega, & w=0 & \text{on } \partial\Omega \end{array}$$

where $$\rho(x) = \frac{g(x, v(x)) - g(x, u(x))}{v(x) - u(x)}$$

and $$\hat{\rho}(x) = \frac{g(x, z(x)) - g(x, v(x))}{z(x) - v(x)}$$

Since $0 < \rho, \hat{\rho} < \lambda_2 \; \forall \; x \in \Omega$, we obtain from (7) that

$$1 = \mu_i(\rho) = \mu_i(\hat{\rho})$$

which is impossible. For, by using the convexity hypotheses, one gets $\rho(x) < \hat{\rho}(x)$, $x \in \Omega$. (for example, because $\rho(x) < g'_v(x, v(x)) < \hat{\rho}(x)$) and this implies that $\mu_1(\rho) > \mu_1(\hat{\rho})$ which is a contradiction q.e.d.

SECTION TWO

In this section, we consider the problem

$$Lu + f(x,u) = \lambda au \text{ in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega \qquad \ldots \quad (2.1)$$

We suppose that $a \in C^0(\bar{\Omega})$, $a > 0$ in $\bar{\Omega}$ and that $f: \bar{\Omega} \times R^+ \to R$ is continuous and satisfies

$$f(x,0) = 0; \quad f(x,s) = o(s) \text{ in the n.b.d. of } s = 0 \text{ uniformly w.r.t. } x \qquad (2.2)$$

$$s \to \frac{f(x,s)}{s} \text{ (defined as 0 for } s = 0) \text{ is a strictly increasing function on } R^+ \ \forall \ x \in \Omega \quad (2.3)$$

$$\lim_{s \to +\infty} \frac{f(x,s)}{s} = +\infty, \text{ uniformly w.r.t. } x \in \Omega \ldots \quad (2.4)$$

As usual, let λ_i denote the ith-eigenvalue of the problem

$$L\phi = \lambda a\phi \text{ in } \Omega, \ \phi = 0 \text{ on } \partial\Omega \qquad (2.5)$$

THEOREM (BERESTYCKI)

Under the hypotheses (2.2), (2.3) and (2.4), for every $\lambda > \lambda_1$, there is a unique positive solution, $u_\lambda \in W^{2,P}(\Omega) \ \forall \ p$ of problem (2.1). The map $\lambda \longrightarrow u_\lambda$ is continuous for $(\lambda_1, +\infty)$ in $C^{1+\alpha}(\bar{\Omega})$ and the branch $\{(\lambda, u_\lambda), \lambda \in (\lambda_1, +\infty)\}$ bifurcates starting from the straight line of the trivial solution on $(\lambda_1, 0)$. Moreover, u_λ is strictly increasing with λ: if $\lambda < \mu$, then $u_\lambda < u_\mu$ in Ω and $\frac{\partial u_\lambda}{\partial n} > \frac{\partial u_\mu}{\partial n}$ on $\partial\Omega$. Lastly, when $\lambda \longrightarrow +\infty$, we have $u_\lambda(x) \longrightarrow +\infty$, $\forall \ x \in \Omega$, the convergence of u_λ being uniform on every compact set in Ω.

Proof.

For $\rho \in C^0(\bar{\Omega})$, denote by $v_i(\rho)$ the ith eigenvalue of the problem

$$L\phi + \rho\phi = va\phi \text{ in } \Omega, \quad \phi = 0 \text{ on } \partial\Omega \qquad \ldots (2.6)$$

So, $\lambda_i = v_i(0)$ and if $\rho \leq \hat{\rho}$, $\rho \not\equiv \hat{\rho}$ in Ω, then $v_i(\rho) < v_i(\hat{\rho})$ $\forall$ $i \in N$. Observe, that (2.1) does not admit any non trivial solution if $\lambda \leq \lambda_1$. Indeed, if $u \geqslant 0$ is a solution of (2.1), u satisfies (2.6) with

$$\rho(x) = \frac{f(x,u(x))}{u(x)} \text{ if } \quad u(x) > 0$$

$$= \quad 0 \qquad \text{if} \quad u(x) = 0 \quad \text{and } \lambda = v.$$

We then have $\lambda = v_1(\rho) > v_1(0) = \lambda_1$ since $\rho \geqslant 0$, $\rho \not\equiv 0$. Remark that the fact that $\lambda = v_1(\rho)$ $\forall$ non trivial solution of (2.1) implies in particular that $u > 0$ in Ω and $\frac{\partial u}{\partial n} < 0$ on $\partial\Omega$.

One obtains the existence (well known) of a positive solution of (2.1) for $\lambda > \lambda_1$ then by observing that (2.1) admits a lower and an upper solution which are ordered. In fact, if ϕ_1 is the first eigenfunction of (2.5) associated to λ_1 we can immediately verify by using (2.2) and $\lambda > \lambda_1$ that $\varepsilon\phi_1$, for $\varepsilon > 0$ very small, is a lower solution of (2.1). Furthermore, (2.4) implies that there is a constant $M > 0$ sufficiently large, $(:\lambda aM - f(x,M) \leq 0 \;\; \forall \;\; x \in \Omega)$ is an upper solution of (2.1). We then have a positive solution of (2.1) lying between $\varepsilon\phi_1$ and M.

More precisely, we remark that $M = M_\lambda$ is such that

$\lambda a(x)s - f(x,s) < 0$, $\forall$ $x \in \Omega$, $\forall$ $s \geqslant M_\lambda$.

It is easy to verify, with the use of the maximum principle, that every solution $u \geq 0$ of (2.1) satisfies u M. Since M is an upper solution, it means that for every $\lambda > \lambda_1$, (2.1) possesses a positive maximum solution u_λ : $0 < u_\lambda < M_\lambda$. In fact, u_λ is the unique (non trivial) positive solution of (2.1). Indeed, if (2.1) admits one other positive solution u, we would have $u < u_\lambda$. On the other hand, taking into account of (2.3) this contradicts

$\lambda = v_1\left[\frac{f(\cdot,u)}{u}\right] = v_1\left[\frac{f(\cdot,u_\lambda)}{u_\lambda}\right]$ (since $\frac{f(\cdot,u)}{u} < \frac{f(\cdot,u_\lambda)}{u_\lambda}$

Then, u_λ is the unique solution of (2.1).

Let $\lambda_1 < \lambda < \mu$. Since u_λ is a lower solution of problem (2.1) corresponding to μ, we have $u_\lambda < u_\mu$ in Ω and $\frac{\partial u_\lambda}{\partial n} > \frac{\partial u_\mu}{\partial n}$ on $\partial\Omega$; u_λ increases (strictly) with λ. We deduce , by a classical reasoning, the uniqueness of the solution of (2.1) and the *a priori* estimate is obtained trivially from M_λ; that $\lambda \to u_\lambda$ is continuous, for example on $(\lambda_1, +\infty)$ in $C^{1,\alpha}(\bar{\Omega})$ where $\alpha \in (0,1)$. Since for $\lambda = \lambda_1$, the unique solution of (2.1) is $u \equiv 0$, the branch

$\{(\lambda,u_\lambda) : \lambda \in (\lambda_1,+\infty)$ bifurcates in $(\lambda_1,0)$ starting from the straight line of trivial solutions $R \times \{0\}$ in $R \times C^{1,\alpha}(\Omega)$.

We now show that $u_\lambda(x) \to +\infty$ as $\lambda \to +\infty$, for $x \in \Omega$. Define, for $\gamma > 0$

$$\Lambda(\gamma) = \lambda\|a\|_{C^0(\bar{\Omega})} + \max_{x\in\Omega} \frac{f(x,\gamma)}{\gamma}$$

for every $\gamma > 0$, $\Lambda(\gamma) < +\infty$; Λ is an increasing function of $\gamma > 0$. Let ϕ_1 be the first eigenfunction of (2.5) (associated with λ_1) satisfying $\|\phi_1\|_{L^\infty} = 1$ $\forall$ $\lambda \geq \Lambda(\gamma)$, $\gamma\phi_1$ is a lower solution of (2.1). Indeed, (2.3) implies that $f(x,\gamma\phi_1) \leq (f(x,\gamma)/\gamma)\ \gamma\phi_1$. We then have

$$L(\gamma\phi_1) + f(x,\gamma\phi_1) \leq \left\{\lambda_1 a + \frac{f(x,\gamma)}{\gamma}\right\}\gamma\phi_1 \leq \Lambda(\gamma)\gamma\phi_1$$

Let, for $\Lambda(\gamma)$

$$L(\gamma\phi_1) + f(x,\gamma\phi_1) \leq \lambda\gamma\phi_1 \text{ in } \Omega, \quad \gamma\phi_1 = 0 \text{ on } \partial\Omega$$

$\gamma\phi_1$ being a lower solution of (2.1), we have $\gamma\phi_1 \leq u_\lambda$. In summary, we have shown that $\forall\ \gamma > 0$, there exists $\Lambda(\gamma) > 0$: $\lambda \geq \Lambda(\lambda) \Rightarrow \gamma\phi_1 \leq u_\lambda$ in Ω. Since ϕ_1 is positive in Ω, we deduce that in Ω $u_\lambda(x) \to +\infty$ when $\lambda \to +\infty$ with u_λ converging uniformly on compact sets of Ω. This completes the proof. q.e.d.

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