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The solution set for nonconvex differential inclusions

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Section 1. Introduction.

Purpose of this thesis is to present an account of results concerning the solution set of differential inclusion Cauchy problems, of the type:

$$(F) \qquad x' \in F(t,x) \qquad x(0) = \xi$$

,where $F: I \times X \to Y$ is a measurable multifunction with compact values contained in $Y = R^n$, and verifying some extra regularity assumptions, namely with respect to the variable x. Here $I = [0,T]$ is a compact interval in R, X is a compact subset of R^n. With the word "multifunction" we mean that F is a correspondence associating to each point (t,x) in $I \times X$ some nonempty subset of Y, called the value of F at (t,x); and "solution of (F) " will designate any continuous function $x : I \to X$ possessing a (Lebesgue) integrable derivative on I, and verifying $x'(t) \in F(t, x(t))$ a.e. on I. Saying that F is measurable means here $\mathcal{L} x \mathcal{B}$- measurable (for precise definitions see Section 2). We shall assume moreover that F is uniformly integrable, in the sense that there exists a (Lebesgue) integrable mapping $M : I \to R$ such that:

$$y \in F(t,x) \quad \Rightarrow \quad |y| \leq M(t), \qquad \forall (t,x) \in I \times X$$

(here $|y|$ denotes the usual Euclidian norm in R^n). Let M_1 be larger than $\int_I M(t)\,dt$; we shall suppose that X contains a ball around Ξ, of radius M_1, where Ξ is some compact convex set in R^n, in which all initial data ξ lives. This will force all the action to take place inside X. Note that the compactness assumption on X is not a restriction , but a byproduct of the compactness assumed on I and on Ξ.

Besides problem (F), we shall study also a closely related one, namely:

$$(F\text{-}A) \qquad x' \in F(t,x) - Ax \qquad x(0) = \xi$$

, where F is as above and $A: D(A) < R^n \to R^n$ is a maximal monotone operator. This implies in particular that the closure of $D(A)$ is convex, and all the values Ax of A are convex closed in R^n. In this case we cannot take anymore M_1 just larger than $\int_I M(t)\,dt$ as before, but we shall nevertheless be able to get another a priori estimate for the values of all possible solutions of (F-A), based on a well-known estimate for solutions of maximal monotone differential inclusions. Naturally we shall suppose that now the initial data set Ξ is contained also in the closure of **D(A)** . We shall say (following [3],[11],[13]) that **x** is a solution of **(F - A)**$_\xi$ if there exists a selection **v** from **F(., x(.))** s.t. **x'(t)** $\in$ **v(t) - A x(t)**, a.e. , **x(0)**=ξ.We shall see that the maximal monotone perturbation does not affect essentially the results that can be obtained for the unperturbed problem **(F)** : the results for **(F-A)** are obtained via a straightforward derivation from the results for **(F)**. This fact allows us to include here both types of problems without compromising the unity of exposition. In many cases we shall speak only about problem **(F-A)** ; it is clear that problem **(F)** is just a particular case, obtained by setting **A = 0.**

For more informations about multifunctions (also called set-valued or multivalued maps, or also correspondences, in the literature) and for a complete review of known results on differential inclusions (also called generalized or multivalued differential equations), see the monograph by Aubin - Cellina [2].

The regularity assumptions we consider on **F** depend naturally on the particular result to be presented, and are as follows. We shall present results in which **F** is allowed to be:

-lower semicontinuous **(lsc)** in **x**

-continuous in **x**

-Lipschitz continuous in **x**

-continuous in **(t,x)**.

The expression "solution set of **(F-A)**" is used to denote the set of all the solutions of **(F-A)**, equipped with the metric of the supremum on **I**. We shall also say something about the set of derivatives of solutions, with an appropriate (weak) topology ; and about the attainable set, i.e. the set of points **b** in **X** such that there is a solution **x** with **x (T) = b.**

In this exposition we consider only the finite dimensional case, since in spaces of infinite dimension almost nothing is known about the solution set of differential inclusions, apart from the fact that it is nonempty in some special cases. And to treat nonemptiness would mean to treat existence theory - and this would constitute, on its own, a topic for a thesis of this kind. In infinite dimensional spaces, the only known results that give something besides nonemptiness are those based on Baire category arguments (see below some further comments on this method).

We now describe the contents of each section. In Section 3 we summarize some classical results on the solution set for convex problems, of the type:

$$\textbf{(coF)} \qquad x' \in \text{co } F(t,x) \qquad x(0) = \xi$$

with **F** as above, continuous (or even upper semicontinuous (**usc**)) in **(t,x)**, due to Filippov [16], [17] and to Cellina [7],[8],[9]. Namely it is stated that the solution set (and the attainable set) is nonempty compact connected, has an **usc** dependence on initial data and parameters, and enjoys some selection properties; and that the set of derivatives of solutions is (weakly) compact and has an **usc** dependence on initial conditions. Moreover if the attainable set corresponding to a compact convex set Ξ of initial conditions is contained in Ξ, then there exists a fixed point, i.e. a solution **x** with **x (0) = x (T)**.

We present also in this section the classical result of Filippov-Waszewski [29],[17] , stating the density of the solution set of **(F)** in the solution set of **(coF)**, in the Lipschitz case. This section is presented without proofs,since we wish to focus our attention on the nonconvex case.

In section 4 we present a series of results for the nonconvex case, obtained through the technique of constructing a continuous selection from a **lsc** multifunction **G** associated to **F.** Each value of **G** is the family of all the integrable selections from the Nemyitski (multivalued) operator associated to **F**, hence it is decomposable [we will call **G** the selection multifunction associated to **F**]. This kind of technique originated in the work of Antosiewicz-Cellina [1], and was further developped by Bressan [4], Fryszkowski [19], and Bressan - Colombo [5]. It has been showing itself to be a very powerfull technique to handle nonconvex differential inclusions. It is in fact the only general tool that works constructively in the nonconvex case ; the only other general tool known to date is the Baire category approach, and this is obviously non-constructive. The importance of having constructive techniques appears clearly on reading Section 4: there it is shown how to construct a solution exactly where we need it, and several usefull applications of this technique are given.

Pianigiani, in a paper [28] that we examine here in detail,applied the technique of Antosiewicz - Cellina to obtain a generalization of the density theorem of Filippov - Waszewski, for **F** satisfying hypothesis of Kamke type; to obtain two partial density results for the continuous case (allowing measurability in **t**); and to prove upper semicontinuity of a subset of the solution set (also in the continuous case with measurability in **t**). We show in this section how a modification of Bressan's proof [4], following the ideas of Pianigiani, can be used to extend the three last results of Pianigiani to the case of problems of the type **(F)** and **(F-A)** as described above. We also show how the more abstract methods of Fryszkowski [19] (see also Bressan - Colombo [5] for a further generalization of Fryszkowski's methods) can be used to obtain the same results with much less effort. Moreover, we prove that any solution of **(coF - A)** can be approximated by solutions of approximating problems $(\mathbf{F_n} - \mathbf{A})$, with $\mathbf{F_n}$ converging uniformly to **F**. (The regularity of the approximants $\mathbf{F_n}$ depends on the regularity of the given **F**). These are new results that will be published in [27]. To deal with the maximal monotone perturbation we use a technique that was first developped in a paper by Cellina - Marchi [11]. Subsequently this same technique was used in [13] to obtain an extension of the results in [11] and [4].

A couple of words is due here to the Baire category approach. This technique was first used by Cellina first used by Cellina [10] , in the 1-dimensional case, to prove existence of solutions to differential inclusions ; and then by DeBlasi - Pianigiani [14],[15], in the finite and infinite dimensional cases. More precisely ,they proved that the solution set of **(F)** , considered as a subset of the solution set of **(coF)** , is a dense countable intersection of open sets, in the case of **F** (Hausdorff) continuous (besides another technical condition) and having interior of **co F (t,x)** nonempty for all **t,x**. The Baire category technique has the disadvantage of beeing nonconstructive; but as a counterpart it has the advantage of giving an enourmous quantity of solutions to problem **(F)** , not only relative to the solution set of **(coF)** as explained above, but also relative to the space of continuous functions (where it is again a countable intersection of open sets), and also in itself (since it has the cardinality **c** of the continuum, and is topologically complete).

Moreover, while other techniques, like the one explained in Section 4, use at some point a compactness argument (or possibly a contraction mapping argument) and therefore impose restrictions on **F** in order to obtain this compactness (or to obtain regularity on **F** , respectively), the Baire category approach seems to work well exactly in the opposite situation i.e. with **F** having values as large as possible. In this respect the two techniques are complementary, and give disjoint results. It was our original intention to include in this thesis also a section on the Baire category method , but lack of space does not allow us to .

In the Appendix we present some technical definitions and some technical lemmas, related to decomposable essentially bounded sets, that are used in Section 4. These are adapted from similar statements by Bressan - Colombo [5] (see also Fryszkowski [19], to whom the basic ideas of these statements are due) , but modified to suit our need. The appendix contains also some results that are the abstract counterpart of similar statements in section 4.

Section 2: Notation ,definitions and some useful results on multifunctions.

(2.1): Let I= **[0,T]** be a compact interval in the real line **R** , and **X** be a compact subset of the n-dimensional Euclidian space $\mathbf{Y} = \mathbf{R}^n$. Denote by $\mathbf{L^1} = \mathbf{L^1}(\mathrm{I}, \mathbf{R})$ the Banach space of (equivalence classes of) real-valued (Lebesgue) integrable functions on I ; by $\mathbf{L^1}_\mathbf{Y} = \mathbf{L^1}(\mathrm{I},\mathbf{Y})$ [respectively $\mathbf{L}^\infty{}_\mathbf{Y}=\mathbf{L}^\infty(\mathrm{I}, \mathbf{Y})$] the Banach space of (equivalence classes of) $\mathbf{R}^n$-valued (Lebesgue) integrable [respectively essentially bounded] functions on I. The (usual) norms in $\mathbf{L}^1{}_\mathbf{Y}$, $\mathbf{L}^1{}_\mathbf{Y}$ will be denoted , respectively , by $|.|_1$, $|.|_\infty$.
Ξ denotes a compact convex subset of **Y** , where all initial data ξ is supposed to lie.

Let $\mathcal{L}$ be the σ-algebra of (Lebesgue) measurable subsets of I , μ the (Lebesgue) measure on I , $\mathcal{B}$ the σ-algebra of Borel subsets of **X** , $\mathcal{B}_\mathbf{Y}$ the σ-algebra of Borel subsets of **Y**. Given $\mathbf{A} \in \mathcal{L}$, $\mathrm{X}(\mathbf{A})$ denotes the characteristic function of the set **A** .

(2.2) :(see [5])

A set $\mathbf{K} < \mathbf{L^1}_\mathbf{Y}$ is said decomposable if :

$$u, v \in K , \quad A \,\varepsilon\, \mathcal{L} \quad \Rightarrow \quad u\, \mathrm{X}(A) + v\, \mathrm{X}(\mathrm{I} \setminus A) \in K.$$

We now recall some concepts from the theory of multifunctions (see for example [2],[21],[20]).

(2.3) If **Z,W** are topological spaces, we say that $\mathbf{H : Z \to W}$ is a multifunction if **H** is a correspondence associating to each point $\mathbf{z} \in \mathbf{Z}$ some nonempty subset **H(z)** of **W** , called the value of **H** at **z** . Given a subset **S** of **W** , we set :

$\mathbf{H^-(S)} := \{z \in Z : H(z) \quad S \neq \varnothing\}$ $\qquad \mathbf{H^+(S)} := \{ z \in Z : H(z) < S \}$

We say **H** is lower semicontinuous (**lsc**) if:

S open in **W** $\Rightarrow$ **H⁻(S)** open in **Z** (or equivalently, if : **S** closed in **W** $\Rightarrow$ $\mathbf{H^+(S)}$ closed in **Z**); and we say **H** is upper semicontinuous (**usc**) if:

S closed in **W** $\Rightarrow$ $\mathbf{H^-(S)}$ closed in **Z** (or equivalently, if: **S** open in **W** $\Rightarrow$ $\mathbf{H^+(S)}$ open in **Z**)

and naturally, **H** is said continuous if it is **usc** and **lsc**.

(2.4): Given a σ-algebra $\mathcal{A}$ of subsets of **Z**, we say **H** is $\mathcal{A}$ - measurable if::

S closed in **W** $\Rightarrow \mathbf{H^-(S)} \in \mathbf{A}$; and we say **H** is $\mathcal{A}$-weakly measurable if :

S open in **W** $\Rightarrow \mathbf{H^-(S)} \in \mathbf{A}$; and we say **H** is $\mathcal{A}\mathrm{x}\mathcal{B}(\mathrm{W})$-measurable if:

$\mathrm{graph}(\mathbf{H}) := \{(z,w) \in ZxW : w \in H(z) \} \in \mathcal{A}\mathrm{x}\mathcal{B}(\mathrm{W})$ [the σ-algebra generated by the sets of the form **AxB** , where $\mathbf{A}\, \varepsilon \mathcal{A}$, $\mathbf{B} \in \mathcal{B}(\mathbf{W})$ i.e., **B** is a Borel subset of **W**].

(2.5): Suppose now W is a complete separable metric space with metric d. For $a,b \in W$, A,B nonempty bounded subsets of W, we set :

$d(a,B) := \inf \{d(a,B) : b \in B\}$, $d^*(A,B) = \sup \{d(a,B) : b \in B\}$

$d(A,B) := \max \{ d^*(A,B), d^*(B,A) \}$

$B(A, \varepsilon) := \{b \in W : d(b,A) < \varepsilon \}$; $\underline{B}(A, \varepsilon) := \{b \in W : d(b,A) \leq \varepsilon\}$.

Then clearly $A < \underline{B}(B, D^*(A,B) < \underline{B}(B, d(A,B))$.

We say H is Lipschitz with constant L if:

$$d(H(z_1), H(z_2)) \leq L \, d(z_1,z_2), \qquad \forall z_1,z_2 \in Z$$

We say H is Hausdorff lower semicontinuous (H-lsc) if:

$\forall z_0 \in Z, \forall \varepsilon>0 \quad \exists N(z0,\varepsilon)$ nbd of z_0 in Z s.t.

$$z \in N(z_0,\varepsilon) \Rightarrow H(z_0) < B(H(z),\varepsilon).$$

We say H is Hausdorff upper semicontinuous (H-usc) if :

$\forall z_0 \in Z \quad \forall \varepsilon>0 \quad \exists N(z_0,\varepsilon)$ nbd of z_0 in Z s.t.

$$z \in N(z_0, \varepsilon) \Rightarrow H(z) < B(H(z_0), \varepsilon).$$

Naturally, H is said Hausdorff continuous if it is **H-usc** and **H-lsc.**

In general, the following is true : H usc $\Rightarrow$ H H-usc ; H H-lsc $\Rightarrow$ H lsc.
If moreover H is compact-valued , then the reverse implications also hold.
If moreover Z is a metric space and H is usc with closed values then graph (H) is closed in ZxW , equipped with the product topology ; while if graph(H) is compact then H is usc.

(2.6): Let now (Z,A,ν) be a positive measure space, W as in (2.5) , and let $\mathcal{A}_\nu$ be the completion of the σ-algebra $\mathcal{A}$ relative to ν, i.e. A_ν is the collection of all the sets of the form $A \; E$ where $A \in \mathcal{A}$ and $\nu^*(E) = 0$ [ν^* is the outer measure associated with ν] . Denote by $\mathcal{A}_\nu x\mathcal{B}_W$ the σ-algebra generated by all sets of the form AxB , where $A \in \mathcal{A}\nu$ and B is a Borel subset of W. Then we have:

$$M \in \mathcal{A}_\nu x \, \mathcal{B}_W \Rightarrow pr_Z (M) = \{ z \in Z : (z,w) \in M \text{ for some } w \in W \} \in \mathcal{A}_\nu.$$

If moreover H has closed values, then :

H is $\mathcal{A}\nu$-measurable $\Leftrightarrow$ H is $\mathcal{A}\nu$-weakly measurable $\Leftrightarrow$ H is $\mathcal{A}_\nu x \, \mathcal{B}_W$-measurable.
In such a situation , we shall say simply that H is ν-measurable to indicate all these properties, or simply " measurable " if ν is the Lebesgue measure μ on the interval I.

(2.6): Consider now a multifunction $F: I \times X \to Y$, with I, X, Y as in (2.1) ; we shall say F is measurable if it is $\mathcal{L} \chi \mathcal{B}$-measurable. If F is measurable with closed values and if the multifunction $F(t, .) : X \to Y$ is **lsc** for each $t \in I$ (we shall say simply that F is **lsc** in **x**), then a Scorza-Dragoni type property holds , namely :
(see [22] Corollary 5 and Remark 1)) $\forall \varepsilon > 0 \quad \exists I_\varepsilon < I$, I_ε compact , $\mu(I \setminus I_\varepsilon) < \varepsilon$, s.t. $F|_{I_\varepsilon \times X}$ is **lsc**.

If moreover $F(t, .)$ is continuous then the same conclusion holds , with $F|_{I_\varepsilon \times X}$ continuous.
If F is measurable with compact values then **coF** is a measurable multifunction.
If $F(t,.)$ is **lsc** then $coF(t, .)$ is **lsc**. If $F(t, .)$ is **usc** with compact values then $coF(t, .)$ is **usc**.
If F is H-continuous then $co\, F(t, .)$ is H-continuous.

(2.7): If F is as in (2.6) , then we say F is uniformly integrable if it is measurable and there exists a map $M \in L^1$ s.t. $y \in F(t,x) \Rightarrow |y| \leq M(t)$ a.e. $\forall x \in X$.
[i.e. $F(. , x(.))$ is integrably bounded, uniformly in $x: I \to X$] .
If F is uniformly integrable and $F_1 : I \times X \to Y$ is a multifunction with
$clco\ F(t,x(t)) = clco\ F1(t,x(t)), \quad \forall x \in X$ then :

$$\int F(t,x(t))\, dt := \{ \int v(t)\, dt : v \text{ is a } L^1\text{-selection from } F(. , x(.)) \} =$$
$$= \int F^1(t,x(t))\, dt.$$

If $w \in L^1_Y$ and $x \in C^0(I, R^n)$, $x(t) \in X \ \forall t$, then there exists a $v \in L^1_Y$ s.t.
$v(t) \in F(t,x(t))$ a.e. and $|w(t) - v(t)| = d(w(t) , F(t,x(t)))$ a.e..

(2.8): (see [6],[3],[11],[2],[13] for example)
Let $A: D(A) < Y \to Y$ be a maximal monotone operator ,
$v \in L^1_Y$, $\xi \in clD(A)$; then the problem :

$$(v - A)_\xi \qquad x' \in -Ax + v(t) \qquad x(0) = \xi$$

has a unique solution, denoted $x=i(v,\xi)$, and $x(t) \in D(A)$, $\forall t>0$. Moreover :

(a) $|i(v,\xi)(t) - \xi_0| \leq |\xi - \xi_0| + \int_0^t |v(s) - \eta_0|\, ds$
$\forall t \in I, \quad \forall \xi_0 \in D(A), \quad \forall \eta_0 \in A\xi_0$

(b) $|i(v,\xi)(t) - i(v_1, \xi_1)(t)| \leq |\xi - \xi_1| +$
$+ \int_0^t |v(s) - v_1(s)|\, ds \quad \forall t \in I$

(c) $|i(v,x)'|_1 \leq C_A [(1 + T + |v|_1) (1 + |i(v,x)|_\infty) + |\xi|^2]$.

Note that (a) gives an estimate for $i(v,\xi)$ in L^{∞}_Y, hence (c) gives an estimate for $i(v,\xi)'$ in L^1_Y. Moreover, (b) tells us that the map $i: L^1_Y x X \to C^0(I,Y)$ is continuous. Then we have $: |i(v,x)|_{\infty} \le 3|\Xi| + T|\eta_0| + |v|_1, \ \forall\ t \in I, \forall v \in L^1_Y$ where $|\Xi|$ denotes the maximum of $|\xi|$ for in Ξ, and η_0 is any point in $A\,\Xi$, and

$$| i(v,x)' |_1 \le C_A [(1+T+|v|_1)(1+3|\Xi|+T|\eta_0|+|v|_1) + |\xi|^2] \le$$

$$\le 2\, C_A [1+T(1+|\eta_0|)+3|\Xi|+|v|_1]^2.$$

Set $C = 2\, C_A [1 + T(1+|\eta_0|) + 3|\Xi| + M_1]^2$; and consider the problem:

$$(F - A)_\xi \qquad x' \in F(t,x) - A x \qquad x(0) = \xi$$

,where F is as in (2.7). We shall say (following [3], [11],[13]) that x is a solution of $(F - A)_\xi$ if there exists a L^1 selection v from $F(., x(.))$ s.t. x is a solution of problem $(v - A)_\xi$. Suppose $x \in L^1_Y$, $g(x) \in L^1_Y$ are given, s.t. $g(x)(t) \in F(t,x(t))$ a.e.

; then we have :

$$| i(g(x_1),\xi_1)(t) - i(g(x),\xi)(t) | \le | \xi - \xi | + \int_0^t | g(x_1) - g(x) | dt$$

hence:

$$| i(g(x),\xi) - i(g(x_1),\xi_1) |_{\infty} \le | \xi_1 - \xi_2 | + | g(x_1) - g(x_2) |_1 .$$

The above inequalities give then:

$$| g(x)(t) | \le M(t), \ |g(x)|_1 \le M_1, \ | i(g(x),\xi) |_{\infty} \le |\xi| + M_1 < C, \ | i(g(x),\xi)' |_1 < C.$$

If we set

$K_1 = \{ x \in C^0(I,R^n) : x(0) \in X, \ x(t) \varepsilon X \ \ \forall t, \ x' \in L^1_Y, |x'|_1 \le C \}$ with the topology of L^1_Y,

$K'_1 = \{x' : x \in K_1 \}$ with the weak topology of L^1_Y,

$K_\infty = \{ x \in K_1 : |x'(t)| \le M(t) \text{ a.e. } \}$ with the topology of $C^0(I,R^n)$

$K'_\infty = \{ x' : x \in K_\infty \}$ with the weak topology of L^1_Y,

then K_1, K_∞, K'_∞ are convex, and they are compact metric spaces. In particular any sequence (x_k) in K_∞ has a subsequence (x_{k_j}) converging to some x in K_∞, and the derivatives (x'_{k_j}) converge to x' in K'_∞. Therefore if $g_k: K_1 \to L^1_Y$ is a sequence of continuous maps s.t.

$|g_k(x)(t)| \le M(t)$ a.e. and we set $h_k: K_1 \to K_\infty$, $h_k(x)(t) = \xi + \int_0^t g_k(x)(s)\, ds$, for some $\xi \in \Xi$, then the sequence $(h_k(x))$ has a subsequence converging to some $h(x)$, while $(h_k(x)') = (g_k(x))$ converges to $g(x) = h(x)'$ weakly. If in particular we know that

$\int_o^t g_k(x)(s)\ ds \to \int_o^t g(x)(s)\ ds$ equiuniformly [i.e. equi in $x \in K_1$, uniformly in $t \in \mathbf{I}$]

then $h_k \to h$ uniformly . Also ,by Schauder fixpoint Theorem h_k, h will have fixpoints in K_∞, and clearly if $g_k(x)(t) \in F(t,x(t))$ a.e. $\forall\ k \in \mathbf{N}$, then the fixpoints of h_k will be solutions of problem **(F)**.

Finally , if we set $h_k : K_1 \to K^1$, $h_k(x) = i(g_k(x),\xi)$, for some $\xi \in \Xi$, then h_k is continuous [and depends continuously on the parameter ξ], and , again by Schauder fixpoint Theorem , there exist fixpoints x_k of h_k, and they verify $x_k = i(g(x_k),\xi)$ i.e. , they are the unique solutions of : $x'_k \in -A\,x_k + g_k(x_k)$, $x(o)=\xi$, i.e. of $x'_k(t) \in F(t,x_k(t)) - A\,x_k(t)$ a.e. , $x(0) = \xi$, and this means x is a solution of $(F - A)_\xi$.

We define now the <u>solution set map</u> as the multifunction:

$\mathcal{M}_{(F-A)} : \Xi \to K_1$, $\mathcal{M}_{(F-A)}(\xi) = \{ x \in K_1 :\ x \text{ is solution of } (F-A)_\xi \}$;

and the <u>derivative of solution set map</u> as the multifunction:

$\mathcal{M}'_{(F-A)} : \Xi \to K'_1$, $\mathcal{M}'_{(F-A)}(\xi) = \{ x' : x \in \mathcal{M}_{(F-A)}(\xi) \}$;

and the <u>attainable set map</u> as the multifunction :

$\mathcal{A}_{(F-A)} : \Xi \to Y$, $\mathcal{A}_{(F-A)}(\xi) = \{ x(T) :\ x \in \mathcal{M}_{(F-A)}(\xi) \}$.

In the special case when A=0 , we have:

$\mathcal{M}_{(F)} : \Xi \to K_\infty$, $\mathcal{M}_{(F)}(\xi) = \{ x \in K_\infty : x \text{ is solution of } (F)_\xi \}$

$\mathcal{M}'_{(F)}(\xi) = \{x' :\ x \in \mathcal{M}_{(F)}(\xi) \}$.

Section 3. Results obtained by construction of a suitable continuous selection from the associated selection multifunction.

(3.1) Remark.

We use in this section the notations and assumptions described in Section 2 , namely relative to $I = [0,T]$; $X, \Xi, Y = R^n$; $M \in L^1$; $A: D(A) < Y \to Y$ maximal monotone operator ; $F: I \times X \to Y$ measurable uniformly integrable multifunction with compact values , lsc in x ; $K_\infty < C^0(I, R^n)$, $K_1 < L^1_X$; $K'_\infty < L^1_Y$, with weak topology ; $\mathcal{M}(F)$, $\mathcal{M}(F\text{-}A)$, $\mathcal{M}(\text{coF-}A)$ solution set maps , from Ξ to K_∞ and to K_1 ; $\mathcal{M}'(F)$, $\mathcal{M}'(F\text{-}A)$, $\mathcal{M}'(\text{coF-}A)$ derivative of solution set maps , from Ξ to K'_∞ .

We begin by some lemmas needed to prove Theorem (3.5) , but not to prove the other results in Section 3 (only Lemma (3.3) is needed after Theorem (3.5)).

(3.2) Lemma.

Let $F : I \times x \to Y$ be a measurable multifunction , **lsc** in x , with compact values . Then , for any $u_o \in L^1_X$ and any $\varepsilon_1 > 0$,

$\exists E = E(u_0, \epsilon_1)$ compact contained in I , $\mu(I \setminus E) < \varepsilon_1$ s.t.

$\forall \varepsilon_2 > 0 \quad \exists \delta = \delta(u_o, \varepsilon_1, \varepsilon_2)$:

$$u \in L^1_X , \quad \sup\{|u(t) - u_o(t)| : t \in E\} < \delta \quad \Rightarrow$$

$$\Rightarrow \quad F(t, u_o(t)) < B(F(t, u(t)), \varepsilon_2), \ \forall t \in E .$$

Proof:

The first part of this lemma is a generalization of [4 , Proposition 1] , and we follow more or less his steps . For each $u_o : I \to X$ measurable, $\Psi = F(. , uo(.))$ is a measurable multifunction (see Lemma (3.3) (i)) , with closed values , hence by the Scorza - Dragoni property of (2. 11) we have :

$\exists$ Eo = Eo (uo , ε1) < I , $\mu(I \setminus Eo) < \varepsilon 1/4$ s.t. $\psi|$ Eo is H-lsc , hence

$\forall$(to , xo) εE1xX , $\exists \rho = \rho(\varepsilon 1, to, xo, \sigma) < \sigma$: $\forall$ (t,x) εE1xX ,

$$d((t, x), (to, xo)) < \rho \quad \Rightarrow \quad F(to, xo) < B(F(t,x), \varepsilon 2/2).$$

Set E = E (uo , ε1) = Eo E1 , E2 = E2 (u0 , ε1) = { (t,u(t)) : t ∈ E} ;

then clearly $\mu(I \setminus E) < \varepsilon 1/2$, and since E2 is compact , we can cover it with balls $B_i = B(c_i, \rho_i)$, $c_i = (t_i, u_o(t_i)) \in E_2$, $\rho_i < \sigma$, $i = 1,...,m$, so that :

$$(t,x) \in E \times X, \quad d((t,x), (t_i, u_o(t_i))) < \rho_i \quad \Rightarrow \quad F(t_i, u_o(t_i)) < B(F(t,x), \varepsilon_2/2)$$

Set A = B_i and $\delta = \min\{\rho_i , i=1,...,m\}$; then clearly $B(E_2, \delta) < A$. Therefore if $(t,x) \in E \times X$ and $|x - u_o(t)| < \delta$ then $(t,x) \in A$, hence $(t,x) \in B_i$ for some i , hence $d((t,x),(t_i,u_o(t_i))) < \rho_i$, and this implies $F(t_i,u_o(t_i)) < B\,(F(t,x)\ ,\ \varepsilon_2/2)$. Now if $|u(t) - u_o(t)| < \delta \ \ \forall t \in E$ then $t \in E \Rightarrow (t,u(t)) \in A \Rightarrow d(t,u(t)),(t_i,u_o(t_i))) < \rho_i$, for some i $\Rightarrow d(t,t_i) < \sigma \Rightarrow F(t,u_o(t)) < B\,(\,F(t_i,u_o(t_i)),\ \varepsilon_2/2) < B(\,F(t,u(t)),\ \varepsilon_2)$, and this proves the lemma.

(3.3) Lemma.

Let $\mathbf{F : I{\times}X \to Y}$ be a measurable multifunction with compact values. Then :

(i) for each $\mathbf{u : I \to X}$ measurable , the multifunction $\psi_u : \mathbf{I \to Y}$, $\psi_u(t) = \mathbf{F(t,u(t))}$ is measurable ;

(ii) for each $\mathbf{u} \in \mathbf{L^1{}_X}$ s.t. $\mathbf{u(t) \in X}$ a.e. , the set

$$\mathbf{G(u)} = \{\ \mathbf{v} \in \mathbf{L^1{}_Y} : \mathbf{v(t)} \in \mathbf{F(t,u(t))} \quad \text{a.e.}\ \}$$

is nonempty closed decomposable ;

(iii) if moreover **F** is **lsc** in **x** and uniformly integrable then the multifunction

$$\mathbf{G : K_1 \to L^1{}_Y}$$

, **G(u)** as in (ii), is **H-lsc** and uniformly integrable.

Proof.

(i) Let $u \in L^1{}_X$; then Ψ_u is measurable iff for each open $O < X$, $\Psi_u^-(O) \in \mathcal{L}$. But
$\Psi_u^-(O) = \{t \in I : F(t,u(t))\ \ O \neq \varnothing\} = \{\ t \in I\ : \forall\, x \in X$ with $F(t,x)\ \ O \neq \varnothing$ and $(t,x) \in$ graph (u)
$= pr_I\ \{\ (t,x) : (t,x) \in F^-(O)$ graph (u) $\}$
, and since F is measurable and $u \in L^1{}_X$, $F^-(O)$ and graph (u) are measurable and , by (2.), the projection on I of their intersection is in $\mathcal{L}$.

(ii) Let $u \in L^1{}_X$; then Ψ_u is measurable with closed nonempty values , hence by Kuratowski - Ryll Nardzewski's Theorem there exists a measurable selection v from Ψ_u , and $v \in G(u)$. Suppose now (w_k) is a sequence in $L^1{}_Y$; then a subsequence $w_{k_j(t)} \to w(t)$ a.e. , and since $\Psi_u(t)$ is closed , $w(t) \in \Psi_u(t)$ a. e. , that is $w \in G(u)$.

(iii) To show that G is H-lsc , fix u in K and let $(u_k) \to u$ in K_1 ; then each subsequence of (u_k) as a subsequence converging a.e. ; denote by : $d_k(t) = d^*(F(t,u(t)),F(t,u_k(t)))$ the corresponding sequence of distances ; since it is integrably bounded , to prove that it goes to zero in L^1 is enough to prove that it consists of measurable functions and that each of its subsequences has a subsequence going to zero a.e.. The later property follows from the fact that whenever (u_j) is a subsequence of (u_k) going to zero a.e. , also (d_j) goes to zero a.e. , by H-lsc of F in the second variable. To obtain the former note that since G has closed decomposable uniformly integrable values , by corollary (A6) (c), $D^*(G(u),G(u_j))(t) = d^*(G(u)(t),G(u_j)(t)) = d^*(F(t,u(t)),F(t,u_j(t))) = d_j(t)$ a.e.,hence d_j is measurable. This means d_k goes to zero in $L^1{}_Y$, i.e. $d^*(G(u),G(u_k))$ goes to zero , hence G is H-lsc.

An alternative proof can be given as follows (cf. [13]). First using Theorem 3.5 (e) , Theorem 6.5 and Theorem 6.6 in [20] , one shows that d_i is a measurable function. As before d_i goes to zero in L^1 , and , for each $v \in G(u)$, by (2.), there exists $v_i(v) \in G(u_i)$ s.t. $d(v_i(v)(t), v(t)) \leq d_i(t)$ a.e., hence $d(v_i(v),v) \leq \int d_i(t)\, dt$. This means $d(v_i(v), G(u_i)) \leq |d_i|_1$ $\forall$ $v \in G(u)$. Therefore $d^*(G(u),G(u_i)) \leq |d_i|_1 \to 0$ as $i \to \infty$.

(3.4) Lemma.

Let $F : I\mathrm{x}X \to Y$ be a measurable uniformly bounded multifunction , **lsc** in **x** with compact values. Let $f_*: I\mathrm{x}X \to Y$ be a Caratheodory selection from **coF** . Fix any $\varepsilon > 0$ and any positive decreasing sequence (ε_k) s.t. $\varepsilon_0 + \varepsilon_1 + ... < \varepsilon/(4M+2T)$.

Set $G : K_1 \to L^1{}_Y$, $G(u) = \{ v \in L^1{}_Y : v(t) \in F(t,u(t))$ a.e. $\}$, where K_1 is as in Remark (3.1) , and $g_*: K_1 \to L^1{}_Y$, $g_*(u)(t) = f_*(t,u(t))$ a.e. .

Then there exists a sequence of mappings $g_k: K \to L^1{}_Y$, that approximate both G and g_* in the sense that , for each $k \geq 0$:

$(a)_k$ g_k is continuous ;

$(b)_k$ $\forall$ $u \in K$, $d(g_k(u), G(u)) \leq \varepsilon_k (M+2T)/4$ and $\mu(W_k(u)) \leq \varepsilon_k/8$, where $W_k(u) = \{t \in I : d(g_k(u)(t), G(u)(t)) \geq \varepsilon_k/2 \}$;

$(c)_k$ $\forall u \in K$, $|\int_0^t (g_k(u) - g_*(u))\, ds\, | \leq \varepsilon$ $\forall t \in I$;
and , if $k>0$, $d(g_k(u),g_{k-1}(u)) \leq \varepsilon_{k-1}(T+2M)$;
and $\mu(Z_k(u)) \leq \varepsilon_{k-1}$, where
$Z_k(u) = \{t \in I : d(g_k(u)(t), g_{k-1}(u)(t)) \geq \varepsilon_{k-1} \}$;

$(d)_k$ $\exists$ $\delta_k > 0$ (with $\delta_k < \delta_{k-1}$ when $k>0$) s.t.
for each finite set V contained in K with **diam** $V < \delta_k$, $\mu(W_k(V)) < \varepsilon_k/4$,where $W_k(V) = \{ t \in I : d(g_k(u)(t),G(u)(t)) \geq \varepsilon_k/2$ **for some** $u \in V \}$
and $\mu(X_k(V)) < \varepsilon_k/8$,where
$X_k(V) = \{ t \in I : g_k(v)(t) \neq g_k(w)(t)$ **for some** $w,v \in V \}$.

Proof:

The proof is a merging of the proofs of [4, Theorem 1] and [28, Theorem 2]. It has two main parts , the first beeing the construction of g_0 and the second the construction of g_k given g_{k-1} , for n > 0. The second part is equal to the proof in [4] (he has got no first part, since he just sets g_0=0) ; the only difference is that instead of 2^{-k} we use the numbers ε_k , chosen with an adequate sum. As to the first part it is a carefull adaptation of the first part in [28] ; it is more involved because of the dependence of the sets $E(u_0)$ on u_0 in Lemma (3.2) (consequence of passing from the continuous case to the lsc one). The part in common with the first part in [28] is comparatively small , and since this is a new result , published here for the first time, we present all the details.

Step 1: Definition of g_o.

By Lemma (3.2) , $\forall\, u_o \in K$ $\exists \rho_o = \rho_o(u_o) < \delta$ and a compact $E_o = E_o(u_o) < I$, with $\mu(I \backslash E_o) < \min\{\varepsilon_o/16, \varepsilon/16M\}$, s.t. $u \in K$, $|u - u_o|_\infty < \rho_o \Rightarrow F(t,u(t)) < B(F(t,u(t)), \varepsilon_o/2)$, and $|g_*(u)(t) - g_*(u_o)(t)| \leq \varepsilon/8T$, $\forall\, t \in E_o(u_o)$.

(this is true because we can restrict a little the set E_o given by the lemma so that we have also $f_*|E_o \times X$ uniformly continuous , by the Scorza-Dragoni property) . Let E_o be equal to I^o_i , except for a set of measure less than $\varepsilon/16M$. Since K is compact , it may be covered with balls $U^o_i = B(u^o_i, \rho_o(u^o_i))$, $i=1,\ldots,m_o$, and there exists a subordinated continuous partition of unity ,

$$p^o = (p^o_1, \ldots, p^o_{mo}) : K \to [0,1]^{mo}.$$

If we set $F_i(t) := F(t, u^o_i(t)) = G(u^o_i)(t)$, and $f_i(t) = g_*(u^o_i)(t)$, then F_i , hence coF_i, is measurable ; and f_i is a measurable selection from coF_i. Liapunov convexity theorem tells us that for each measurable $J < I$, we have :

$\int_J F_i(t)\,dt := \{\int_J \varphi(t)\,dt : \varphi \in L^1_Y , \varphi'(t) \in F_i(t) \text{ a.e.}\} = \int_J coF_i(t)\,dt$,

hence $\int_J f_i \in \int_J coF_i = \int_J F_i$, and this means $\int_J f_i = \int_J v^J_i$ for some v^J_i , a measurable selection from F_i , $i=1,\ldots,m_o$. Now we divide I into disjoint intervals $J_1,\ldots,J_{ko}$, of lenght less than $\varepsilon/(32\, M\, m_o k_o)$, and define $v^o_i : I \to Y$ by $v^o_i|_{Jk} = v^{Jk}_i$, $i=1,\ldots,m_o$, a.e. $t \in I$.Then it is clear that $v^o_i(t) \in F(t, u^o_i(t)) = G(u^o_i(t)(t)$ a.e. , that is , $v^o_i \in G(u^o_i)$, and for any interval $J < I$, $|\int_J (v^o_i - f_i)| \leq \varepsilon/8 m_o k_o$, $i=1,\ldots,m_o$.

We will now construct , for each $u \in K$, a partition of our interval I in Borel subsets $J^o_i(u)$, $i=1,\ldots,m_o$, based on the partition of unity $p^o(u)$. Finally, denoting by $\chi(J^o_i(u))$ the characteristic function of the set $J^o_i(u)$, g_o will be defined by

$$g_o = \Sigma\ \chi(J^o_i(u))\, v^o_i.$$

The construction of the sets $J_i(u)$ follows now. Set $A^o_i{}^+ := E_o(u^o_i)$, $A^o_i{}^- := I \backslash A^o_i{}^+$; then $\mu(A^o_i{}^+) \geq T - \varepsilon_o/8$ and $\mu(A^o_i{}^-) \leq \varepsilon_o/8$. Define recursively the sets $Y^o_i(u)$, $Y^o_i{}^+(u)$, $Y^o_i{}^-(u)$, $J^o_i(u)$, and the map $\varphi o_{i,u}$ for $i=1,\ldots,m_o$, as follows:

$Y^o_i(u) = I \backslash \quad J_k(u)$, $Y^o_i{}^+(u) = Y^o_i(u) \quad A^o_i{}^+$, $Y^o_i{}^-(u) = Y^o_i(u) \quad A^o_i{}^-$, $\varphi o_{i,u} : Y^o_i(u) \to \mathbf{R}$

$\varphi o_{i,u}(t) = \mu([0,T] \quad Y^o_i{}^+(u))\ \chi(Y^o_i{}^+(u) + \{\mu(Y^o_i{}^+(u)) + \mu([0,T] \quad Y^o_i{}^-(u))\}\ \chi(Y^o_i{}^-(u))$

and finally $J^o_i(u) = (\varphi^o_{i,u})^{-1}([0, Tp^o_i(u)])$, $i=1,\ldots,m_o$.

It is clearly seen that the maps $\varphi^o_{i,u}$ are Borel measure preserving transformations , and that the sets $J^o_i(u)$ form a Borel disjoint cover of I , and $\mu(J^o_i(u)) = Tp^o_i(u)$. It is also clear that

$g_o(u) \in L^1 \ \forall\ u \in K$, and this implies that the sets $W_o(u), Z_o(u)$, and $X_o(V)$, $W_o(V)$ are measurable. Also a little reflection shows that :

$\mu(J^o{}_i(u) \quad A^o{}_i{}^+) = \inf \{\ Tp^o{}_i(u),\ \mu(Y^o{}_i{}^+(u))\} \geq \ \inf \{\ Tp^o{}_i(u),\ \mu(\ Y^o{}_i(u))\text{-}\varepsilon_o/8\ \}$

, and that the construction of the sets was so carefull that not only $\mu(\ A^o{}_i{}^+) \geq T\text{-}\varepsilon_o/8$ for each $i=1,\dots,m_o$, but , more precisely $\mu(\quad (J^o{}_i(u) \quad A^o{}_i{}^+)) \geq T - \varepsilon_o/8$. Clearly, this implies :

$\mu(\quad (J^o{}_i(u) \quad A^o{}_i{}^-)) \leq \varepsilon_o/8$.

It is also clear that , except for a set of measure less than $\varepsilon_o/16M$, we can say that each set $J^o{}_i(u)$ consists of the union of at most k_o intervals ; we shall call $J^o{}_{i*}(u)$ the union of these intervals, so that : $J^o{}_i(u) \quad J^o{}_{i*}(u) \quad J^o{}_\varepsilon(u)$, where $J^o{}_{i*}(u) = \quad I^o{}_{ij}(u)$, $\mu(\ J^o{}_\varepsilon(u)) < \varepsilon/16\ M$, $i=1,\dots,m_o$, where $J^o{}_\varepsilon(u)$ does not depend on i, and $I^o{}_{ij}(u)$ is an interval. Clearly also $I= J^o{}_\varepsilon(u) \quad (\quad J^o{}_{i*}(u)) = J^o{}_\varepsilon(u) \quad J^o{}_*(u))$.

Step 2: Verification of $(a)_o,(b)_o,(c)_o,(d)_o$.

$(b)_o$: $\forall\ u \in K\ ,\ \mu(W_o(u)) \ \leq\ \varepsilon_o/8$ and $d(g_o(u),G(u)) \ \leq\ \varepsilon_o\ (M+2T)/4$.

To prove this we observe first that if $u \in K$ and $t \in J^o{}_i(u) \quad A^o{}_i{}^+$ for some i , then $p_i(u) \neq 0$, hence $u \in U_i$ and $d(\ u(t),u^o{}_i(t)) \ \leq\ \rho_o(u^o{}_i)$ and this gives

$F(t,u^o{}_i(t)) \ <\ B(\ F(t,u(t)),\ \varepsilon_o/2\ )$. Therefore

$d(\ g_o(u)(t),\ G(u)(t)) \leq d(\ g_o(u)(t),G(u^o{}_i)(t)) + \varepsilon_o/2 = \ d(\ v^o{}_i(t),G(u^o{}_i)(t)) + \varepsilon_o/2 = \varepsilon_o/2$.

Therefore $t \notin W_o(u)$ and $W_o(u) < \quad (J^o{}_i(u) \quad A^o{}_i{}^-)$, and this implies $\mu(\ W_o(u)) \leq \varepsilon_o/8$.

†o prove the second inequality , note that $t \notin W_o(u) \ \Rightarrow\ d(g_o(u)(t),G(u)(t)) < \varepsilon_o/2$, hence there exists a L^1 selection v(.) from G(u)(.) [i.e. , $v \in G(u)$] verifying :

$d(g_o(u)(t),v(t)) \ <\ \varepsilon_o/2 \quad \forall\ t \notin\ W_o(u) \quad , \quad d(g_o(u)(t),v(t)) \leq 2M \quad \forall\ t \in W_o(u)$.

Therefore $d(g_o(u),G(u)) := \ \inf \{\ \int_I d(g_o(u)(t),v(t))\ dt :\ \ v \in G(u)\ \} \ \leq T\ \varepsilon_o/2 + 2M\ \varepsilon_o/8 = \varepsilon_o(M+2T)/4$, and this proves $(b)_o$.

$(c)_o \quad \forall\ u\ \in K\ , \quad \forall\ t \in I\ , \quad |\int_o{}^t(g_o(u)\ -\ g_*\ (u))\ (s)\ ds| \leq\ \varepsilon$.

To start with we have , setting $f_i(t) = g_*(u^o{}_i)(t)$:

$|\int_o{}^t \chi(\ I^o{}_{ij}(u))\ (v^o{}_i\text{-}\ f_i)| \leq\ \varepsilon/(8\ m_o k_o)$, $i=1,\dots,m_o$, $j = 1,\dots,k_o$, since $I^o{}_{ij}(u) \quad [0,T]$ is an interval . Therefore

$\Sigma\ |\int_o{}^t\ \chi(\ J^o{}_{i*}\ (u)) \ \ (v^o{}_i - f_i)\ (s)\ ds\ | \leq \Sigma\,\Sigma \ \ |\int_o{}^t\ \chi\ (\ I^o{}_{ij}(u))\ \ (v^o{}_i\text{-}\ f_i\ )\ | \leq\ \ m_o k_o \varepsilon/(8 m_o k_o) =\varepsilon/8$. Finally : $|\int_o{}^t(g_o(u)(s)\text{-}g_*(u)(s)\ )\ \ ds\ | \ \leq |\ \int \chi(\ J^o{}_*(u)) \ \ (g_o(u)\ \text{-}g_*(u))(s)\ \ ds\ | + + |\ \int_o{}^t \chi(\ J^o{}_\varepsilon(u)) \ \ (g_o(u)\ -\ g_*(u))\ (s)\ ds\ | \ \leq |\ \Sigma\ \int_o{}^t\ \chi\,(J^o{}_{i*}(u)) \ \ (g_o(u)\ -\ g_*(u))\ (s)\ ds\ |+ + 2M\ \ \varepsilon/16M \ \ \leq\ \Sigma\ \ |\int_o{}^t\ \ \chi\,(J^o{}_{i*}(u))\ \ (\ v^o{}_i\ -\ f_i\ )\ (s)\ ds\ \ | + |\Sigma\ \int_o{}^t\ \ \chi\,(J^o{}_{i*}(u)) \ \ \ (g_*(u)\ - g_*(u^o{}_i))\ (s)\ ds\ | + \varepsilon/8 \ \leq\ \varepsilon/8 + \varepsilon/8 \ + |\ \int_o{}^t\ \Sigma\ \chi\,(J^o{}_{i*}(u))\ (s)\ \ (f(s,u(s)) \ \ -\ f(s,u^o{}_i(s)))\ \ ds| \leq$

$\leq \varepsilon/4 + \varepsilon/8T \mid \int_{I\varepsilon} \Sigma \chi (J^o{}_{i^*}(u))(s)\ ds \mid + 2M\ \varepsilon/16M \leq \varepsilon/4 + \varepsilon/8 + \varepsilon/8T\ \mu(I_\varepsilon) \leq \varepsilon/2.$

$(d)_o$: $\exists\ \delta_o > 0$ s.t. for each finite set **$V < K$** with **diam $V \leq \delta_o$,** $\mu(\ W_o(V)) \leq \varepsilon_o/4$ and $\mu(\ X_o(V)) \leq \varepsilon_o/8$.

To prove this, choose $\delta_o > 0$ s.t. $v, w \in K, |v-w|_\infty \leq \delta_o \Rightarrow |Tp^o(v) - Tp^o(w)| \leq \varepsilon_o/(8m_o(m_o+1))$. Set $\Lambda^o{}_k(V) = \{ t \in J^o{}_k(v) \setminus J^o{}_k(w) : v,w \in V \}$ and $\Lambda^o(V) = \quad \Lambda^o{}_k(V)$, for each finite set $V < K$ with diam $V \leq \delta_o$. Then it is easy to see by induction on k that $\mu(\Lambda^o{}_k(V)) \leq 2k\,\varepsilon_o/(8m_o(m_o+1))$, $k=1,\ldots,m_o$, using the fact that $v,w \in V \Rightarrow$ $|\ T\,p^o{}_k(v) - T\,p^o{}_k(w) \mid \leq \varepsilon_o/(8m_o(m_o+1))$. Therefore $\mu(\Lambda^o(V)) \leq \varepsilon_o/8$.
Note that $\Lambda^o(V)$ is the set of points in I which are not always in the same set $J_i(V)$ of the partition of I as u varies in V; hence $t \notin \Lambda_o(V) \Rightarrow \exists\, i \leq m_o : t \in J_i(u)\ \ \forall u \in V$. Now it is clear that if $t \in J^o{}_i(u)$ for any $u \in V$, then $g_o(u)(t) = v^o{}_i(t)$ for any $u \in V$, and this means that t is not in $X_o(V)$; hence $X_o(V) \leq \Lambda^o(V)$, and $\mu(X_o(V)) \leq \varepsilon_o/8$.
To prove the second inequality, recall that $\mu(\quad (J^o{}_i(u) \quad A^o{}_i{}^-)) \leq \varepsilon_o/8$, and that $W_o(V) = \{ t \in I : d(g_o(u)(t), F(t,u(t))) \geq \varepsilon_o/2$, for some u in $V \} < \quad (J^o{}_i(u) \quad A^o{}_i{}^-)$. This implies that, fixing any u in V, we have $\mu(W_o(V)) \leq \mu(\quad (J^o{}_i(u) \quad A^o{}_i{}^-)) +$ $+\ \varepsilon_o/8 + \varepsilon_o/8 = \varepsilon_o/4$, since for each $t \notin \Lambda^o(V)$ there exists some i s.t. $t \in J^o{}_i(u)$ for any u in V. This proves (d).

$(a)_o$ $g_o : K \to L^1{}_Y$ is continuous.

To prove this we remark that using a reasoning like in the first part of the proof of $(d)_o$, we obtain : $\forall\, \varepsilon_o' > 0\ \exists\, \delta_o' : |u - v|_\infty < \delta_o' \Rightarrow \mu(\{t \in I : g_o(u)(t) \neq g_o(v)(t)\}) < \varepsilon_o'/2M$ $\Rightarrow d(g_o(u),g_o(v)) = \int_I |g_o(u) - g_o(v)| \leq 2M\,\varepsilon_o'/2M = \varepsilon_o'$. This proves $(a)_o$.

Step 3 : Definition of g_k given g_{k-1}.

Suppose now g_{k-1} is constructed and δ_{k-1} was chosen so that $(d)_{k-1}$ holds. By Lemma (3.2), $\forall u \in K\ \exists\, \rho_k(u) < \delta_{k-1}/2$ and a compact $E_k(u) < I$ with $\mu(I \setminus E_k) < \varepsilon_k/8$ s.t. $d(x,u(t)) < \rho_k \Rightarrow F(t,u(t)) < B(F(t,x), \varepsilon_k/2)\ \ \forall t \in E_k$.
K can be covered with balls $U^k{}_i = B(u^k{}_i, \rho_k(u^k{}_i))$, $i = 1,\ldots,m_n$, and we can find a continuous partition of unity $p^k : K \to [0,1]^{m_k}$. It is possible to choose $v^k{}_i \in G(u^k{}_i)$ s.t. $d(v^k{}_i(t), g_{k-1}(u^k{}_i)(t)) < \varepsilon_{k-1}/2$, $\forall t \notin W_{k-1}(u^k{}_i)$. Finally, define $A^k{}_i{}^+$, $A^k{}_i{}^-$, $Y^k{}_i(u)$, $Y^K{}_i{}^+(u)$, $Y^k{}_i{}^-(u)$, $\varphi^k{}_{i,u}$, $J^k{}_i(u)$ as for the case $n=0$, and define

$$g_k(u)(t) = \Sigma\ \chi(J^k{}_i(u))(t)\ \ v^k{}_i(t).$$

Step 4 : Verification of $(a)_K$, $(b)_k$, $(c)_k$, $(d)_k$.

$(b)_k$, $(d)_k$, $(a)_k$ are verified just as in the case n=0.

$(c)_k$ $\forall\ u \in K$, $\mu(Z_k(u)) \leq \varepsilon_{k-1}$, $d(g_k(u), g_{k-1}(u)) \leq \varepsilon_{k-1}$ (T+2M) and $|\int_o^t (g_k(u)\ (s)\ ds\ -\ g_*(u)(s)\ )\ ds\ | \leq \varepsilon$.

To prove this , set for each u in K , $V_k(u) = \{u\}\ \ \{\ u^k_i : p^k_i(u) > 0\ \}$. Since $\rho_k(u^k_i) < \delta_{k-1}/2$, we must have $|u^k_i - u|_\infty < \delta_{k-1}/2$ whenever $u_i \in V_k(u)$, hence diam ($V_k(u)) \leq \delta_{k-1}$. Aplying $(d)_{k-1}$ to $V = V_k(u)$ we obtain two sets , $W^*_{k-1}(u) := W_{k-1}(V_k(u)) = \{\ t \in I\ :\ d(g_{k-1}(v)(t)\ ,\ F(t,v(t))) \geq \varepsilon_{k-1}/2$ for some $v \in V_k(u)\ \}$ and $X^*_{k-1}(u) := X_{k-1}(V_k u) = \{\ t \in I : g_{k-1}(v)(t) \neq g_{k-1}(w)(t)$ for some $v,w \in V_k(u)\ \}$ with $\mu(\ W^*_{k-1}(u)\ ) \leq \varepsilon_{k-1}/4$ and $\mu\ (\ X^*_{k-1}(u)) \leq \varepsilon_{k-1}/8$. Let $t \notin W^*_{k-1}(u)\ \ X^*_{k-1}(u)$; then $g_{k-1}(u)(t) = g_{k-1}(u^k_i)(t)\ \ \forall\ u^k_i \in V_k(u)$ and $d\ (g_{k-1}(u^k_i)(t)\ ,\ v^k_i(t)) < \varepsilon_{k-1}/2\ \ \forall\ u^k_i \in V_k(u)$, by definition of v^k_i . Therefore if t is also in $J^k_{io}(u)$ then $u^k_{io} \in V_k(u)$, $g_k(u)(t) = v^k_{io}(t)$, hence $d(g_k(u)(t), g_{k-1}(u)(t)) = d\ (v^k_{io}(t), g_{k-1}(u^k_{io})(t)) < \varepsilon_{k-1}/2$. Therefore $t \notin Z_k(u)$, and this means $Z_k(u) < W^*_{k-1}(u)\ \ X^*_{k-1}(u)$, and $\mu\ (Z_k(u)) \leq \varepsilon_{k-1}/4 + \varepsilon_{k-1}/8 < \varepsilon_{K-1}$. To prove the second inequality , $\forall\ u \in K$, $d(\ g_k(u), g_{k-1}(u)) = \int_I d(g_k(u)(t)\ ,\ g_{k-1}(u)(t))\ dt \leq \int_{I\setminus Zk(u)} d(\ ) + \int_{Zk-1(u)} d\ (\ ) < \varepsilon_{k-1}T + 2M\ \varepsilon_{k-1}$. Finally , $|\int_o^t (g_k(u)(s) - g_*(u)(s))\ ds\ | \leq |\int_o^t (g_o(u)(s) - g_*(u)(s))\ ds\ | + \Sigma \int_o^t |\ g_{i+1}(u)(s) - g_i(u)(s)\ |\ ds \leq \varepsilon/2 + \Sigma\ d\ (\ g_{i+1}(u), g_i(u)) \leq \varepsilon/2 + \Sigma\ \varepsilon_i\ (T+2M) \leq \varepsilon/2 + \varepsilon/2\ \ \forall\ t \in I$. this completes the proof of Lemma (3.4).

(3.5) Theorem.

Let $\mathbf{F} : \mathbf{I} \times \mathbf{X} \to \mathbf{Y}$ be a measurable uniformly bounded multifunction , **lsc** in **x** , with compact values , and let $\mathbf{f}_*$ be a Caratheodory selection from **coF.**

Then there exists a sequence $(\mathbf{x}_k)$ of solutions of (F) which converges uniformly to a solution of $(\mathbf{f}_*)$.

Proof

Fix $\varepsilon>0$. In Lemma (3.4) we proved the existence of a sequence of continuous approximate selections of the selection multifunction G associated to F , in the sense that properties $(a)_k$, $(b)_k$,$(c)_k$,$(d)_k$ hold. Clearly , $(c)_k$ implies that $(g_k(u))$ is a Cauchy sequence , uniformly in $u \in K$; hence (g_k) is a Cauchy sequence and converges uniformly to some continuous g_ε: $K \to L^1{}_Y$. As $\varepsilon_k \to 0$, $(b)_k$ implies that $g_\varepsilon(u) \in G(u)$, $\forall u \in K$,since G has closed values . Using $(a)_k$, one sees that g_ε is a continuous selection from G . Moreover , for any $t \in I$, $|\int_0^t (g_\varepsilon(u)(s) - g_*(u)(s))\, ds | \le |\int_0^t (g_\varepsilon(u)(s) - g_k(u)(s))\, ds | + |\int_0^t (g_k(u)(s) - g_*(u)(s))\, ds| \le |g_\varepsilon(u) - g_k(u)|_1 + \varepsilon$, and letting $k \to \infty$, we obtain :
$|\int (g_\varepsilon(u)(s) - f_*(s,u(s)))\, ds | \le \varepsilon$, $\forall u \in K$ $\forall t \in I$. Consider now $h_\varepsilon(u)(t) = \int_0^t g_\varepsilon(u)(s)\, ds$; then $h_\varepsilon : K_\infty \to K_\infty$ is well-defined and is continuous , and since K_∞ is compact convex , by Schauder fixpoint theorem , There exists at least a fixpoint $u_\varepsilon = h_\varepsilon(u_\varepsilon)$; and this means
$u'_\varepsilon = g_\varepsilon(u_\varepsilon) \in G(u_\varepsilon)$,i.e. , $u'_\varepsilon(t) \in F(t,u(t))$ a.e.. Moreover,

$$|\int_0^t (u'_\varepsilon(s) - f_*(s,u_\varepsilon(s)))\, ds | = |\int_0^t (g_\varepsilon(u_\varepsilon)(s) - g_*(u_\varepsilon)(s))\, ds | \le \varepsilon.$$

Set now $u_k = u_{\varepsilon_k}$, $k \in N$. Then $u_k \in K_\infty$, $u'_k(s) \in F(s,u_k(s))$ a.e. and $|\int_0^t (u'_k(s) - f(s,u_K(s)))\, ds| < \varepsilon_k$ $\forall t \in I$. But K'_∞ is sequentially compact , hence a subsequence (u_i) of (u_k) converges to some $u \in K_\infty$, while the derivatives (u'_i) converge weakly to u' , hence we have :
$|\int_0^t (u'(s) - f_*(s,u(s)))\, ds | \le |\int_0^t (u'(s) - u'_i(s))\, ds | + |\int_0^t (u'_i(s) - f(s,u(s)))\, ds | +$
$|\int_0^t (f(s,u_i(s)) - f(s,u(s)))\, ds | \le |\int_0^t (u'(s) - u'_i(s))\, ds | + \varepsilon_i + |\int_0^t (f_*(s,u_i(s)) - f(s,u(s)))\, ds |$,
and letting $i \to \infty$, we obtain $\int_0^t u'(s)\, ds = \int_0^t f(s,u(s))\, ds$ $\forall t \in I$ hence in particular $u'(t) = f(t,u(t))$ a.e. Therefore we have a sequence (u_i) of solutions of (F) converging uniformly to a solution u of (f_*) , and the theorem is proved.

We now present a much stronger theorem , obtained with the help of the abstract results in the Appendix . The results that follow do not depend on (3.2) , (3.4) , or (3.5).

(3.6) Theorem.

Let $\mathbf{F}: \mathbf{I}\times\mathbf{X} \to \mathbf{Y}$ be a unifomly integrable multifunction , **lsc** in **x**, with closed values , and let f_* be a Caratheodory selection from **coF**. Let $\mathbf{A}: \mathbf{D(A)} < \mathbf{Y} \to \mathbf{Y}$ be a maximal monotone operator.

Then there exists a sequence $(\mathbf{x}_k)$ of solutions of $(\mathbf{F}-\mathbf{A})_\xi$ converging uniformly to a solution $\underline{\mathbf{x}}$ of $(f_*-\mathbf{A})_\xi$.

If moreover A=0 then the sequence $(\mathbf{x}'_k)$ converges weakly to $\underline{\mathbf{x}}'$.

Proof:

By Lemma (3.3) , the selection multifunction $G:K_1\to L^1_Y$ associated with F is well-defined and is lsc and uniformly integrable . If we set , for each $u \in K_1$,

$G_*(u) = \{ v \in L^1_Y : v(t) \in \text{co } F(t,u(t)) \text{ a.e. } \}$

, then the hypothesis of Remark (A.13) are satisfied , with $g_*(u)(t) = f_*(t,u(t))$ (the continuity of g_* follows from the fact that $g_*(u) \in L^1_Y$ for each $u \in K_1$, and f_* is Caratheodory) by the Lyapunov theorem on the range of a vector measure. By Theorem (A.14), there exists a sequence (g_k) of continuous selections from G s.t. $\int_0^t g_k(u)\,ds \to \int_0^t g_*(u)\,ds$ equiuniformly (i.e. uniformly in t , equi in u). Therefore , setting

$h^\xi_k , h^\xi_*: K_1\to K_1$, $h^\xi_k(u) = i(g_k(u),\xi)$, $h^\xi_k(u)=i(g_*(u),\xi)$

, we obtain a fixpoint $u_k=h^\xi_k(u_k)$ i.e. $u_k(0)= \xi$ and $u'_k \in -A\, u_k +g(u_k)$, hence $u'_k(t) \in F(t,u_k(t))- A\, u_k(t)$ a.e. . Since (u_k) is a sequence in the compact K_1 , we may suppose $u_k \to \underline{u}$ uniformly, and we have:

$| u_* - h^\xi_*(\underline{u})|_1 \le | \underline{u} - u_k |_1 + | u_k - h^\xi_k(u_k) |_1 + | h^\xi_k(u_k) - h^\xi_*(u_k) |_1 + | h^\xi_*(u_k) -$
$- h^\xi_k(\underline{u}) |_1$; but $u_k \to \underline{u}$ in L^1 , $u_k= h^\xi_k (u_k)$, $h^\xi_*(u_k) \to h^\xi_*(\underline{u})$ in L^1 ; therefore we only need to show that if $\gamma_k(t) = h^\xi_k (u)(t) - h^\xi_*(u)(t) = i(g_k(u),\xi)\,(t) - i(g_*(u),\xi)(t)$, then $|\gamma_k|_1 \to 0$ uniformly in $u \in K_1$. But we have (see [6, Lemme 3.1 , formula (28) , theoreme 3.4 in p.65 ; proposition 3.8 in p.82]):

$|\gamma_k(t)|^2 \le 2 \int_0^t < g_k(u)(s) - g_*(u)(s),\gamma_k(s) > ds$; hence setting $\beta_k(t) = g_K(u)(t) -g_*(u)(t)$, and $\alpha_k(t) = |\int_0^t \beta_k(s)\,ds|$ and $\varphi_k(t) = | \gamma_k(t) |$ we have :

$\varphi_k^2(t) \le 2 <\int_0^t \beta_k(s)\,ds , \int_0^t \gamma_k(s)\,ds > \le 2\, \alpha_k(t) \,|\int_0^t \gamma_k(s)\,ds| \le 2\,\alpha_k(t)\int_0^t \varphi_k(s)\,ds.$

But $\alpha_k(t) \le \varepsilon_k$ independently of t in I and u in K_1, and $\varphi_k(t) \le 2C$ since $h^\xi_k: K_1\to K_1$; therefore $\varphi_k^2(t) \le 2\,\varepsilon_k\, T\, 2\, C$ hence $| \varphi_k |_1 \le 2\, T\, (T\, C\, \varepsilon_k)^{1/2} \to 0$ as $k \to \infty$, uniformly for u in K. We have thus shown that $h^\xi_k \to h^\xi_*$, uniformly , and $\underline{u} = h^\xi_k(\underline{u}) = i(g(\underline{u}),\xi)$, i.e. $\underline{u}(0) = \xi$, $\underline{u}' \in -A\, \underline{u} + g_*(\underline{u})$, $\underline{u}'(t) \in -A\, \underline{u}(t) + f_*(t,\underline{u}(t))$ a.e. .

This means $\underline{u}$ is a solution of $(f_* -A)$ and the theorem is proved .

;

(3.7) Theorem.

Let $F : I\times X \to Y$ be a uniformly integrable multifunction , **lsc** in **x** , with compact values . Let $A : D(A) < Y \to Y$ be a maximal monotone operator.

Let x_* be a solution of **(coF - A)**$_\xi$ .Then there exists :

- a Caratheodory selection f_* from **co F** s.t. x_* is a solution of $(f_* - A)_\xi$
- a sequence (x_k) of solutions of $(F - A)_\xi$ which converges uniformly to a solution $\underline{x}$ of $(f_* - A)_\xi$.

If moreover $A = 0$ then the sequence (x'_k) converges weakly to $\underline{x}'$.

Proof:

Since x_* is a solution of (co F - A) , by definition there exists a v_* in $L^1{}_Y$ s.t. x_* is a solution of $(v_* - A)_\xi$, This means that $x'_*(t) \in -Ax(t) + v_*(t)$ a.e. , $x_*(0) = \xi$.

Define the multifunction $F_* : I\times X \to R^n$,

$$F_*(t,x) = \begin{cases} v_*(t) & \text{if } x = x_*(t) \text{ , i.e. } (t,x) \in \text{graph } x^* \\ \text{co } F(t,x) & \text{otherwise.} \end{cases}$$

Let $a(t) = (t,x_*(t))$ for each $t \in I$. Then for each open subset O of R^n , we have :

$F_*^-(O) = \{ (t,x) : F_*(t,x) \quad O \neq \varnothing \} = \{(t,x) \in \text{graph } x_* : v_*(t) \in O\}$

$\{ (t,x) \in \text{graph}^c\, x_* : \text{coF}(t,x) \quad O \neq \varnothing \} = \{a(t) : t \in v_*^{-1}(O)\}$

$(\text{graph}^c\, x_* \quad \text{coF}^-(O)) \qquad a(v_*^{-1}(O)) \qquad (\text{graph}^c\, x_* \quad \text{coF}^-(O))$.

Since coF is measurable , and x_* is measurable , $\text{graph}^c(x_*) \quad \text{coF}^-(O)$ is measurable ; and since v_* is measurable, $v_*^{-1}(O)$ is measurable and since a is a measurable function , $a(v_*^{-1}(O))$is measurable (in fact, $a(v_*^{-1}(O))$ is the graph of the restriction of x_* to the measurable set $v_*^{-1}(O)$). Therefore , F_* is a measurable multifunction . Now, for each fixed $t=t_0$, and each closed subset c of R^n , it is easy to see that :

$t_0 \notin v_*^{-1}(C) \Rightarrow F_*(t_0, .)^+(C) = \text{co } F(t_0, .)^+(C)$

(i.e. if $v_*(t_0) \notin C$ then $F_*(t_0,x) \in C \Rightarrow x \neq x_*(t_0) \Rightarrow F_*(t_0,x) = \text{co } F(t_0,x)$)

$t_0 \in v_*^{-1}(C) \Rightarrow F_*(t_0,.)^+(C) = \{x_*(t_0)\} \quad \text{coF}(t_0,.)^+(C)$,

and in both cases $F_*(t_0, .)^+(C)$ is closed.

Therefore , F_* is measurable and F_* is lsc for each t in I , and F_* has convex closed values , hence all the hypothesis of Fryszkowski's theorem are satisfied , therefore there exists a Caratheodory selection f_* from F_* ; clearly f_* is a Caratheodory selection from co F , and moreover : $f_*(t,x_*(t)) = v_*(t)$ a.e. in I . Therefore x_* is a solution of $(f_* - A)$, i.e. $x'_*(t) \in f_*(t,x_*(t)) - A\, x_*(t)$ a.e.

The rest of the statement of the theorem follows from Theorem (3.6).

(3.8) Theorem.

Let $\mathbf{F}: I \times X \to Y$ be a uniformly integrable multifunction , **lsc** in $\mathbf{x}$, with compact values. Let $\mathbf{A}: D(A) < Y \to Y$ be a maximal monotone operator. Let x_* be a solution of $(\mathbf{co}\ \mathbf{F} - \mathbf{A})_\xi$.

Suppose **F** verifies the following one-sided Lipschitz type condition:

$\exists\ L = L(x_*)$ s.t.

$$d^*(F(t,x_*(t)),F(t,x(t))) \leq L(x_*)\ |x_*(t) - x(t)|, \quad \forall t \in I .$$

Then there exists a sequence (x_k) of solutions of $(\mathbf{F} - \mathbf{A})_\xi$ converging uniformly to x_* . In particular the solution set of $(\mathbf{F} - \mathbf{A})_\xi$ is dense in the solution set of $(\mathbf{co}\ \mathbf{F} - \mathbf{A})_\xi$ if for each boundary solution x_* of $(\mathbf{co}\ \mathbf{F} - \mathbf{A})_\xi$ there corresponds one such Lipschitz constant $\mathbf{L}(x_*)$. If moreover $\mathbf{A}=0$ then (x'_k) converges weakly to x_* ; i.e. , the derivative of solution set of $(\mathbf{F})_\xi$ is dense in the derivative of solution set of $(\mathbf{co}\ \mathbf{F})_\xi$.

Proof:

By Corollary (A.6) (c) , we have :

$$\int_J d^*(F(t,x_*(t)),F(t,x(t)))\, dt = \int_J D^*(G(x_*)(t),G(x)(t))\, dt = \int_J D^*(G(x_*),G(x))\,(t)\, dt \leq$$
$$\leq L(x_*) \int_J |x_* - x|\, ds$$

As in the proof of Theorem (A16) (b) , consider a positive decreasing sequence (ε_k) , and find a sequence (g_k) of continuous selections fro G , verifying :

$$\left|\int_0^t (g_k(x)(s) - v_*(s)\, ds\right| \leq \varepsilon_k + L \int_0^t |x^* - x|\, ds , \quad \forall x \in K .$$

Suppose x^* is a solution of $x'_* \in - A x_* + v_*$, and set

$h_k: K_1 \to K_*$, $h_k(u) = i(g_k(u), \xi)$; then , by Schauder fixpoint Theorem , h_k has a fixpoint $x_k = h_k(x_k)$, i.e. $x'_k \in - A x_k + g_k(x_k)$, , hence x_k is aa solution of $(F - A)_\xi$.

Set $v_k = g_k(x_k)$. Then the above inequality gives :

$\left|\int_0^t (v_k - v_*)(s)\, ds\right| \leq \varepsilon_k + L \int_0^t |x_*(s) - x_k(s)|\, ds$, or , setting

$\alpha_k(t) = \left|\int_0^t (v_k(s) - v_*(s))\, ds\right|$, $\quad \varphi_k(t) = |x_k(s) - x_*(s)| = |i(v_k,\xi)(s) - i(v_*,\xi)|$,

$\alpha_k(t) \leq \varepsilon_k + L \int_0^t \varphi_k(s)ds$;

while the estimate computed in Theorem (3.6) gives : $\quad \varphi_k^2(t) \leq 2\,\alpha_k(t) \int_0^t \varphi_k(s)\, ds$.

Combining the two , we obtain :

$\varphi_k^2(t) \leq 2\,(\varepsilon_k + L \int_0^t \varphi_k(s)\, ds) \int_0^t \varphi_k(s)\, ds = 2\,\varepsilon_k \int_0^t \varphi_k(s)\, ds + 2\,L\,(\int_0^t \varphi_k(s)\, ds)^2$,

and $\varphi_k(0) = 0$; since we want to obtain an upper bound for $\varphi_k(t)$ in I, the worst possible situation occurs when we have the equality sign . Differentiating we obtain :

$$2\varphi_k(t)\varphi'_k(t) = 2\varepsilon_k\varphi_k(t) + 4L\,\varphi_k(t) \int_0^t \varphi_k(s)\, ds$$

, and supposing $\varphi_k(t) \neq 0$ we have : $\varphi'_k(t) \leq 2\,\varepsilon_k + 4\,L \int_0^t \varphi_k(s)\, ds$, $\quad \varphi'_k(0) = 2\varepsilon_k$.

The unique solution of $\varphi''_k(t) = 4\,L\,\varphi_k(t)$, $\varphi'_k(0) = 2\,\varepsilon_k$, $\varphi_k(0) = 0$ is

$\varphi_k(t) = \varepsilon_k\ \mathrm{sh}(2\,L^{.5}\,t)/L^{.5} \leq \varepsilon_k\ \exp(2\,L^{.5}\,T)/L^{.5} \leq 1/k$ if we take

$\varepsilon_k = L(x_*)^{.5}\ \exp(-2\,L(x^*)^{.5}\,T)/k$. Since this represents the maximum possible growth for $\varphi_k(t)$, we have : $|x_k - x_*|_\infty \leq 1/k$, hence x_k converges uniformly to x_*.

Finally a boundary solution is a solution x_* s.t. the corresponding $v_*(t)$ is a.e. on the boundary of $coF(t,x_*(t))$; the assertion that only boundary solutions need to be checked is justified by Theorem (3.9).[The last assertion in the statement follows directly from Theorem (A16)(b)].

(3.9) Theorem.

Let $\mathbf{F} : I x X \to Y$ be a uniformly integrable multifunction with compact values , Lipschitz in x [with Lipschitz constant $L(.) \in L^1$] . Let $A: D(A) < Y \to Y$ be a maximal monotone operator. Let x_* be a solution of $(coF - A)_\xi$.

Then there exists :

- a Lipschitz - Caratheodory selection f_* from **coF** [with Lipschitz constant $4\ n\ L(.)$] , s.t. x_* is a solution of $(f_* - A)_\xi$;

– a sequence (x_k) of solutions of $(F - A)_\xi$ which converges uniformly to x_*.

In particular , the solution set of $(F - A)_\xi$ is dense in the solution set of $(coF - A)_\xi$.
If moreover $A=0$ then the sequence (x'_k) converges weakly to x'_* ; hence the derivative of solution set of $(F)_\xi$ is dense in the derivative of solution set of $(co\ F)_\xi$.

Proof:

The theorem of Lojasiewicz jr ([23]) gives a Lipschitz-Caratheodory selection f_* from coF , with constant $4\ n\ L(.)$, verifying $f_*(t,x_*(t)) = v_*(t)$ a.e. in I , where v_* is such that $x'_*(t) \in -A\ x_*(t) + v_*(t)$ a.e.. The rest of the statement follows from Theorem (3.6) , and from the uniqueness of solutions of $(f_* - A)_\xi$. To prove this uniqueness , set :
$w(t)= |\underline{x}(t) - x_*(t)|^2$, where x_* , $\underline{x}$ are two solutions of $(f_* - A)_\xi$; and
$\underline{v}(t) = f_*(t, \underline{x}(t)) - \underline{x}'(t)$, $v_*(t) = f_*(t, x_*(t)) - x'_*(t)$ $\forall\ t \in I$.
Then $\underline{v}$ is a measurable selection from $A\ \underline{x}$, and v_* is a measurable selection from $A\ x_*$, hence $\underline{v}(t) \in A\ \underline{x}(t)$, $v_*(t) \in A\ x_*(t)$ a.e. and , by definition of monotone operator ,

$$< \underline{v}(t) - v_*(t) , \underline{x}(t) - x_*(t) > \geq 0 \text{ a.e. hence } < \underline{x}'(t) - x'_*(t) , \underline{x}(t) - x_*(t) > =$$
$$= < f_*(t,x(t)) - v(t) - f_*(t,x_*(t)) + v_*(t) , x(t) - x_*(t) > =$$
$$= < f_*(t,\underline{x}(t)) - f_*(t,x_*(t)) , \underline{x}(t) - x_*(t) > - < \underline{v}(t) - v_*(t) , \underline{x}(t) - x_*(t) > \leq$$
$$\leq | f_*(t,\underline{x}(t)) - f_*(t,x_*(t)) , \underline{x}(t) - x_*(t) > + 0 \leq L(t) | \underline{x}(t) - x_*(t) |^2 = L(t)\ w(t) .$$

This means $w'(t) \leq 2\ L(t)\ w(t)$, $w(0) = 0$, and since the Cauchy problem $w'(t) = 2\ L(t)\ w(t)$, $w(0) = 0$, as unicity of solution , we have $w(t) \equiv 0$, i.e. $\underline{x} = x_*$, and the theorem is proved .

(3.10) Theorem .

Let $F : I\times X \to Y$ be a uniformly integrable multifunction , **lsc** in **x** , with compact values. Let $A : D(A) < Y \to Y$ be a maximal monotone operator .

Let x_* be a solution of $(coF - A)_\xi$ s.t. x_* is a solution of $(v_* - A)_\xi$ with $v_*(t) \in$ **int co** $F(t,x_*(t))$ in a set of positive measure.

Then there exists a sequence (x_k) of solutions of $(F - A)_\xi$ which converges uniformly to x_*.[And the two last assertions in the statement of Theorem (3.9) hold .]

Proof:

Define the selection multifunction G_* associated with co F as usual , and note that $v_* \in \text{int } G_* (x_*)$, and G_* is H-lsc , by Lemma (3.3) (iv) , and is uniformly integrable . Therefore the proof of Theorem (A 16)(a) gives us a continuous selection g_* from G_* verifying $g^*(x) = v_*$, $\forall x \in B_*$, where $B_* = cl\, B(x_*, \delta)$, $\delta > 0$, is a nbd of x_*. Like in Theorem (A 14) we can find a sequence (gk) of continuous selections from G (the selection multifunction associated with G) s.t.

$$| \int_0^t (g^*(x) - g_k(x)) \, ds | \le \varepsilon_k , \quad \text{where } \varepsilon_k \to 0 \text{ as } k \to \infty.$$

Set $h_* : B_* \to B_*$, $h_* (x) (t) = \int_0^t g_*(x)(s) \, ds = \int v_*(s) \, ds$,

$$h_k : B_* \to K_\infty \;, \; h_k(x)(t) = \int_0^t g_k(x)(s) \, ds.$$

Then $|\int h_k(x)(t) - h_*(x)(t) | = | h_k(x)(t) - \int_0^t v_*(s) \, ds| = |\int_0^t (g_k(x) - g_*(x)) \, ds | \le \varepsilon_k$.
Therefore we can find k_0 large enough so that $k \ge k_0 \Rightarrow | h_k(x) - \int_0^t v_* |_\infty \le \delta$, since $\varepsilon_k \to 0$.
This means also that $\int_0^t g_k(x) \, ds \to \int_0^t g_*(x) ds$, equiuniformly (in $x \in B_*$, $t \in I$).
As in the proof of Theorem (3.6) , we set :
$h_k : K_1 \to K_1$, $h_k(x)(t) = i(g_k(x), \xi)$; $h_* : K_1 \to K_1$, $h_k(x)(t) = i(g_*(x),\xi)$.
Like in the proof of Theorem (3.6) , we find fixed points $x_k = h_k(x_k) \to \underline{x} = h_*(\underline{x})$ uniformly.
Therefore $x_k = i(g_k(x_k),\xi)$ i.e. $x_k(0) = x$ and $x'_k \in - A x_k + g_k(x_k)$, hence
$x'_k(t) \in - A x_k(t) + F(t,x_k(t))$, i.e. x_k is a solution of $(F - A)_\xi$, and similarly x_* is a solution of $(f_* - A)_\xi$, and the proof of the theorem is complete.

(3.11) Corollary.

Let $F: I\times X \to Y$ be a uniformly integrable multifunction , lsc in **x** , with compact values.Let x_* be a solution of $(co\ F)_\xi$ s.t. $x'_*(t) \in$ **int co** $F (t,x_*(t))$ in a set of positive measure.

Then there exists a sequence (x_k) of solutions of $(F)_\xi$ which converges uniformly to x_*, and the sequence (x'_k) of derivatives converges weakly to x'_*.
In particular if $F(t,x) = \partial coF(t,x)$, $\forall t,x$, then the solution set and the derivative of solution set of $(F)_\xi$ are dense in the solution set , derivative of solution set , of $(coF)_\xi$, respectively .

Proof:

Set $A = 0$ and $v_* = x'_*$ in Theorem (3.9).

(3.12) Theorem.

Let $F : I\mathrm{x}X \to Y$ be a uniformly integrable continuous multifunction with compact values. Let $A : D(A) < Y \to Y$ be a maximal monotone operator.

Then there exists a sequence (F_k) of Lipschitz multifunctions with compact values, $F_k : I\mathrm{x}X \to Y$, s.t. $F_k \to F$ uniformly in the Hausdorff metric, and :

$$\mathcal{M}_{(coF-A)}(\xi) < \mathcal{M}_{(Fk-A)}(\xi).$$

Proof:

We show that there exists a sequence (F_k) of multifunctions as stated, verifying: $F(t,x) < F_k(t,x) < \underline{B}(F(t,x), 1/k)$;
and if $x_* \in \mathcal{M}_{(coF-A)}(\xi)$ i.e., x_* is a solution of $(coF-A)_\xi$, then there exists a sequence $(x_k) \to x_*$ uniformly, with $x_k \in \mathrm{M}_{(Fk-A)}(\xi)$ i.e., x_k is a solution of $(F_k - A)_\xi$.
Fix $\varepsilon > 0$. Since F, considered as a map from the compact metric space $I\mathrm{x}X$ to the metric space of compact nonempty subsets of Y with Hausdorff metric, is continuous, there exists a Lipschitz compact-valued multifunction H_ε s.t.

$d_\bullet(H_\varepsilon, F) := \sup\{ d(H_\varepsilon(t,x), F(t,x)) : (t,x) \in I\mathrm{x}X \} \leq \varepsilon/2$.

Set $Ge(t,x) := \underline{B}(H_\varepsilon(t,x), e/2)$ $\forall$ t,x ; then G_ε is a compact valued multifunction, and:
$F(t,x) < G_\varepsilon(t,x) < B(F(t,x), \varepsilon)$, $\forall$ t,x.
Therefore $x_* \in \mathcal{M}_{(coF-A)}(\xi) < \mathcal{M}_{(coG\varepsilon-A)}(\xi)$; and since G_e is Lipschitz, by Theorem (3.8), there exists a sequence (u_m) in $\mathcal{M}_{(G\varepsilon-A)}$, $u_m \to x_*$ uniformly. Choose m_ε s.t.
$| u_{m_\varepsilon} - x_* |_\infty < \varepsilon$; choose a sequence (ε_k), $0 < \varepsilon_k < 1/k$, and set $F_k = G\varepsilon_k$; $x_k = u_{m_{\varepsilon k}}$; then $x_k \in \mathcal{M}_{(Fk-A)}$, $| x_k - x_* |_\infty < 1/k$, $d_\infty(F_k,F) < 1/k$, hence $F_k \to F$, $x_k \to x_*$, and this proves the theorem.

(3.13) Proposition:

Let $\mathbf{F} : I\times X \to Y$ be a uniformly integrable multifunction , **lsc** in **x** , with compact values . Let $\mathbf{A} : D(A) < Y \to Y$ be a maximal monotone operator.

Then there exists a sequence (F_k) of multifunctions with the same properties as F , s.t. $F_k \to F$ in the Hausdorff metric , and :

$$\mathcal{M}_{(coF-A)}(\xi) < \mathcal{M}_{(Fk-A)}(\xi).$$

Proof:

Let x* Œ M(coF-A)(x) . Consider the multifunctions $F_k : I\times X \to Y$, $F_k(t,x) = B(F(t,x), 1/k)$. Then F_k is uniformly integrable , lsc in x, with compact values, and $F_k \to F$ uniformly in the Hausdorff metric . Let $x'_* \in -A x_* + v_*$; then $v_*(t) \in \operatorname{int} coF_k(t,x_*(t))$ a.e. , hence by Theorem (3.9) there exists a sequence $(u^i{}_k)_i$ of solutions of $(F_k - A)_\xi$, $u^i{}_k \to x_*$, uniformly, as $i \to \infty$. Choose i_k s.t. $| u^{i_k}{}_k - x_* | < 1/k$, and set $x_k = u^{i_k}{}_k$; then x_k is a solution of $(F_k - A)_x$, $| x_k - x^* |_\infty < 1/k$, hence $x_k \to x_*$ uniformly , and the result is proved.

(3.14) Remark.

Let $\mathcal{G}$ be the space of continuous selections from **G** ,endowed with the uniform topology. Let $\mathcal{H}$ be the space of primitives of elements of $\mathcal{G}$, i.e. ,

$\mathcal{H}$ = { **h**: $K_1 \to K_1$ s.t. $h(u) = i(g(u), \xi)$ for some $g \in \mathcal{G}$ and some $\xi \in \Xi$ } ,

with the uniform topology. Clearly , to each $h \in \mathcal{H}$ there corresponds at least one solution of **(F-A)**, a fixed point of **h**. Now , if x_* is a solution of **(F-A)** , $x'_* \in -A x_* + v_*$, then Corollary (A.12') , with **G** in place of G_* gives us a continuous selection g_* from **G** s.t. $g_*(x_*) = v_*$. Then $x'_* \in -A x_* + g_*(x_*)$ i.e., $x_* = i(g_*(x_*),\xi) = h(x_*)$ for some $h \in \mathcal{H}$. This shows that conversely , any solution x_* of **(F-A)** is a fixpoint of at least one $h \in \mathcal{H}$. In other words , the solution set that can be obtained as the set of fixpoints of elements of $\mathcal{H}$ is the whole solution set of **(F - A)**.

In the following theorem we shall consider a compact subset $\mathcal{H}_0$ of $\mathcal{H}$, and we shall denote by $\mathcal{M}_0$, $\mathcal{M}'_0$ the subsets of the solution set and of the derivative of solution set , respectively , corresponding to solutions of **(F-A)** obtained as fixpoints of elements **h** of $\mathcal{H}_0$.

(3.15) Theorem.

Let $F : I \times X \to Y$ be a uniformly integrable multifunction , **lsc** in **x**, with compact values. Let $A: D(A) < Y \to Y$ be a maximal monotone operator.

Let $\mathcal{H}$ be the space of primitives of continuous selections of the selection multifunction **G** associated with **F** , as in Remark (3.14) , and suppose $\mathcal{H}_0$ is a compact subset of $\mathcal{H}$. Let $\mathcal{M}_0(\xi)$, $\mathcal{M}'_0(\xi)$ be the subsets of the solution set and derivative of solution set ,respectively, of $(F - A)_\xi$ corresponding to solutions which are fixpoints of maps in $\mathcal{H}_0$.

Then the multifunctions $\mathcal{M}_0$, $\mathcal{M}'_0$ have closed graph , and in particular $\mathcal{M}_0$ is **usc**. If moreover **A=0** then also $\mathcal{M}'_0$ is **usc**.

Proof:

We have $\mathcal{M}_0 : \Xi \to K_1$, $M' : \Xi \to K'_1$, and if moreover A=0 then $\mathcal{M}'_0 : \Xi \to K'_\infty$; since K_1, K'_∞ are compact the usc property is justified ,and it is enough to prove that the graphs are closed. Let $((\xi_k,w_k))$ be a sequence in graph$(\mathcal{M}'_0)$, $\xi_k \to \xi$ and $w_k \to w$. Set $u_k(t) = \xi_k + \int_0^t w_k(s)\, ds$, $u(t) = \xi + \int_0^t w(s)\, ds$; by hypothesis , $u_k=h_k(u_k)$, $h_k \in \mathcal{H}_0$, for k=1,2,... ; and since $\mathcal{H}_0$ is compact relative to the uniform topology in $\mathcal{H}$, we may suppose $h_k \to h \in \mathcal{H}_0$.Also by hypothesis , $u_k=h_k(u_k) = i(g_k(u_k), \xi_k)$, $h(u)= i(g(u), \xi)$,where the g_k , g are continuous selections from the selection multifunction G associated to F. If we show that $u=h(u)$ then $u = i(g(u),\xi)$, and this means $(\xi,u) \in$ graph $(\mathcal{M}'_0)$, hence this graph is closed. But the weak convergence of (w_k) to w gives the uniform convergence of (u_k) to u ;while (h_k) converges uniformly to h and h is continuous , hence:

$|u-h(u)| \leq |u-u_k| + |u_k-h_k(u_k)| + |h_k(u_k)-h(u_k)| + |h(u_k) - h(u)| \to 0$ as $k \to \infty$.

Similarly if $((\xi_k,u_k))$ is a sequence in graph $(\mathcal{M}_0)$, with $\xi_k \to \xi$, $u_k \to u$ uniformly , then $(\xi,u) \in$ graph $(\mathcal{M}_0)$, and this proves the theorem.

(3.16) Theorem.

Let $F : I \times X \to Y$ be a uniformly integrable multifunction, continuous in **x** , and let $A : D(A) < Y \to Y$ be a maximal monotone operator.

Suppose that the solution set of **(F - A)** is dense in the solution set of **(coF - A)** [see (3.8),(3.9),(3.11),(3.12) for sufficient conditions] , then the solution set map of **(F - A)**is **H-usc**.

Suppose that the derivative of solution set of **(F)** is dense in the derivative of solution set map of **(co F)** ; then the derivative of solution set map of **(F)** is **H-usc**.

Proof:

We prove first that the solution set map of (co F - A) is usc. Let $((\xi_k,w_k))$ be a sequence in graph($\mathcal{M}'_{(coF-A)}$) , as in Theorem (3.15) .By Remark (3.14) , $u_k=h_k(u_k) = i(g_k(u_k),\xi_k)$ for some $h_k \in \mathcal{H}$, k=1,2,... ; setting $v_k=g_k(u_k)$, we get a sequence (v_k) in the compact K'_∞ , hence we may suppose that $v_k \to v$ weakly. Since $v \in$ clco {vk: k=1,2,... } and this set is weakly compact and closed in L1 we may suppose that $u_k \to u$ uniformly ; and since

$$v_k(t) = g_k(u_k)(t) \in F(t,u_k(t)) < \text{co } F(t,u_k(t)) \quad \text{a.e.}$$

,we may apply the convergence theorem and conclude that $v(t) \in \text{co } F(t,u(t))$ a.e. , i.e. $v \in G_*(u)$. But the map $(x,u) \mapsto i(g(u),\xi)$ is continuous from $X \times K_1$ to K_1 , hence: $u_k=i(g(u_k),\xi_k) \in i(g(u),\xi)=u$, and this means that u is a solution of (coF - A) , i.e., $(\xi,u) \in$ graph ($\mathcal{M}'_{(coF-A)}$) , and this graph is closed. Since the solution set is in the compact K_1 , the solution set map of (coF-A) is usc , and in particular it is H-usc.

By H-usc of $\mathcal{M}_{(coF-A)}$, for each ξ_0 fixed in Ξ , and each $\delta>0$ we can find a $\varepsilon>0$ s.t. $|\xi - \underline{\xi_0}| < \varepsilon \Rightarrow \mathcal{M}_{(coF-A)}(\xi) < B\,[\mathcal{M}_{(coF-A)}(\xi_0), \delta/2]$;

and by density , $\mathcal{M}_{(coF-A)}(\xi_0) < B\,[\mathcal{M}_{(F-A)}(\xi_0), \delta/2]$; therefore ,

$$|\xi-\xi_0| < \varepsilon \Rightarrow \mathcal{M}_{(F-A)}(\xi) < \mathcal{M}_{(coF-A)}(\xi) < B\,[\mathcal{M}_{(coF-A)}(\xi_0), \delta/2] < B\,[\mathcal{M}_{(F-A)}(\xi_0), \delta].$$

This proves the first part of the theorem. The second part has a similar proof , and we omit it.

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