Scuola Internazionale Superiore di Studi Avanzati

International School for Advanced Studies

$SO_q(N,R)$-Symmetric Harmonic Oscillator on the N-dim Real Quantum Euclidean Space R_q^N

Thesis submitted for the degree of

"Magister Philosophiæ"

CANDIDATE

Gaetano Fiore

SUPERVISOR

Prof. Loriano Bonora

March 1992

Index

to my family

Introduction

The development of Physics has been often characterized by the introduction of some more general and accurate theories as sort of deformations of already known and accepted ones. A well-known example is special relativity, which can be viewed as a deformation of Galileo's relativity; the velocity of light plays the role of deformation parameter. Another example is quantum mechanics, which can be seen as a deformation of classical mechanics, Planck constant being the deformation parameter.

Within the recent increasing interest for quantum groups the question has been raised [1] whether these fascinating mathematical objects can replace (or generalize) Lie groups in the description of the fundamental symmetries of physics, since they can be considered as continuous deformations of Lie groups themselves [2],[3]. One may ask whether the axioms of quantum mechanics are compatible with a more general description of continuous symmetries than the usual one, which is provided by Lie groups and their representations over the Hilbert spaces of physical states; many new possibilities in this direction seem to be open [4].

In particular it looks tempting to consider deformations of the symmetries of space(time) [5],[6],[7],[8]; in such a case quantum groups and/or the underlying quantum spaces [3] replace classical space(time) and represent examples of noncommutative geometries [9]. As known, such geometries look promising for describing the microscopic structure of spacetime.

To understand what is going on and to introduce some important concepts which will be heavily used in the next chapters, let us consider for instance a one-particle system in ordinary 3-dimensional space. First we consider it as a system described by classical mechanics.

Let r,π be the position and the momentum of the particle; (r,π) define the state of the system. We introduce a reference frame S and the coordinate and momentum (vector) functions $\bar{x},\bar{p}$:

$$\bar{x} : r \to \bar{x}(r) \in \mathbf{R}^3 \qquad \bar{p} : \pi \to \bar{p}(\pi) \in \mathbf{R}^3; \tag{0.1}$$

$x^i(r)$, $p^i(\pi)$ are respectively the coordinates and the momentum components of the particle in the frame S. The functions x^i, p^i commute:

$$[x^i, x^j] = 0 = [p^i, p^j] = [x^i, p^j]. \tag{0.2}$$

All other observables are functions (power series) of x^i, p^j and make up a (commutative) $\mathbf{C}^*$-algebra $Fun(\mathbf{R}^3_{\bar{x}} \times \mathbf{R}^3_{\bar{p}})$.

We denote by $Fun(G)$ the commutative algebra of functions (power series) on the group $G := SO(3, \mathbf{R})$. The basic variables $T^i_j \in Fun(G)$, $i, j = 1, 2, 3$, are defined by $T^i_j(g) = g^i_j$ ($g \in G$ and $\|g^i_j\| \in adj(G)$, where $adj(G)$ denotes the adjoint representation of G). The physical description of the system is covariant w.r.t. the group of rotations G. We can express this covariance using the language of corepresentations, i.e. introducing a (left) coaction ϕ_L

$$\phi_L : Fun(\mathbf{R}^3) \rightarrow Fun(G) \bigotimes Fun(\mathbf{R}^3) \tag{0.3}$$

defined on the basic variables x^i, p^i by

$$\phi_L(x^i) := T^i_j \bigotimes x^j \qquad\qquad \phi_L(p^i) := T^i_j \bigotimes p^j \tag{0.4}$$

(either multiplet $(x^i), (p^i)$ gives the fundamental corepresentation) and extended as an algebra homomorphism

$$\phi_L(ab) := \phi_L(a)\phi_L(b), \qquad\qquad a, b \in Fun(\mathbf{R}^3_{\bar{x}} \times \mathbf{R}^3_{\bar{p}}). \tag{0.5}$$

The coordinates $x'^i(r)$ of the particle in a new frame S' (obtained from S by a rotation $\|g^i_j\|$) will be given by

$$[\phi_L(x^i)](g, r) = [T^i_j \bigotimes x^j](g, r) := T^i_j(g)x^i(r) = g^i_j x^j(r) = x'^i(r); \tag{0.6}$$

in a similar way we get the new momenta $p'^i(\pi)$.

The transition to quantum mechanics is characterized by the following replacements. The functions $\bar{x}, \bar{p}$ are substituted by operators $\bar{X}, \bar{P}$ on a Hilbert space $\mathcal{H}$. In the socalled coordinate representation

$$\begin{array}{ccc} |u> \in \mathcal{H} & \textit{is represented by} & f(x) \in \mathcal{L}^2(\mathbf{R}^3_x) \\ \bar{X} & " & \bar{x}\cdot \\ \bar{P} & " & \frac{\hbar}{i}\partial_x \end{array} \tag{0.7}$$

This implies that the commutation relations (0.2) are to be replaced by

$$[X^i, X^j] = 0 = [P^i, P^j] \qquad\qquad [X^i, P^j] = i\hbar\delta^{ij} \neq 0. \tag{0.8}$$

X^i, P^j generate the $\mathbf{C}^*$-algebra of the observables of the system. The (left) coaction is now defined on the latter algebra replacing the basic definition (0.4) by the new one

$$\phi_L(X^i) := T^i_j \bigotimes X^j = U_T(\mathbf{1}_{Fun(G)} \bigotimes X^i)U_T^{-1}$$

$$\phi_L(P^i) := T^i_j \bigotimes P^j = U_T(\mathbf{1}_{Fun(G)} \bigotimes P^i)U_T^{-1}; \tag{0.9}$$

$\mathbf{1}_{Fun(G)}$ is the unit element of $Fun(G)$ and $U_T \in Fun(G) \otimes \mathcal{B}(\mathcal{H})$ evaluated at the point $g \in G$ gives a unitary operator $U_g := U_T(g, \cdot)$ on $\mathcal{H}$. As in the classical case ϕ_L is extended as an homomorphism through the relation (0.5). The coordinate observables X'^i in the rotated frame S' are given by

$$X'^i = [\phi_L(X^i)](g, \cdot) = g^i_j X^j = U_g X^i U_g^{-1} \tag{0.10}$$

and a similar relation gives P'^i. As shown in formula (0.8), the coordinates X^i (resp. the momenta P^i's) commute with each other, and so do the functions T^i_j's:

$$[T^i_j, T^h_k] = 0. \tag{0.11}$$

Next we ask whether we can deform the commutation relations of the coordinates (or of the momenta) preserving the definition (0.9) of ϕ_L together with the natural requirement (0.5) that it be a homomorphism. We can easily see that this is not possible, unless we deform the commutation relations (0.11), as well; the converse is also true. In fact (0.9),(0.5) imply

$$\phi_L([X^i, X^j]) = T^i_h T^j_k \bigotimes X^h X^k - T^j_k T^i_h \bigotimes X^k X^h. \tag{0.12}$$

Then

$$\{(0.12)\ and\ [X^i, X^j] = 0\} \Rightarrow 0 = \phi_L([X^i, X^j]) = (T^i_h T^j_k - T^j_k T^i_h) \bigotimes X^h X^k \Rightarrow$$

$$\Rightarrow [T^i_h, T^j_k] = 0. \tag{0.13}$$

To prove the converse implication we need to specify the kind of deformed commutation relations we wish for the coordinates. We assume that they are homogeneous:

$$\mathcal{P}_A{}^{ij}_{hk} X^h X^k = 0, \tag{0.14}$$

with a nontrivial matrix $\mathcal{P}_A$. Then

$$\{(0.12)\ and\ [T^i_h, T^j_k] = 0\} \Rightarrow 0 = \phi_L(\mathcal{P}_A{}^{ij}_{hk} X^h X^k) = \mathcal{P}_A{}^{ij}_{hk} T^h_l T^k_m \bigotimes X^l X^m \Rightarrow$$

$$\Rightarrow \mathcal{P}_{A}{}^{ij}_{hk} T^h_l T^k_m = 0 \Rightarrow \mathcal{P}_{A}{}^{ij}_{hk} \propto (\delta^i_h \delta^j_k - \delta^i_k \delta^j_h) \Rightarrow \{(0.14)\ reads\ [x^i, X^j] = 0\}; \tag{0.15}$$

the second implication in the last line is due to the fact that the relations $\mathcal{P}_{A}{}^{ij}_{hk} T^h_l T^k_m = 0$ must be equivalent to the commutation relations $[T^i_h, T^j_k] = 0$.

Now we come back to the general discussion keeping the previous example in mind and generalizing the notation used there in an obvious way. It is easy to guess that for a physical system with group of symmetry G the deformation of $Fun(G)$, and the deformation of the algebra of the observables are coupled. Quantum groups and the underlying quantum spaces are naturally conceived to provide manageable tools for nontrivial examples of such deformations.

Following the line suggested in ref. [5] and [7],[8] it is possible to define differential calculi respectively over quantum groups and over quantum spaces, in such a way that they are covariant and reduce to the classical ones in the socalled classical limit, where the deformation disappears and the generating elements (x^i, T^i_j in the previous example) of these algebras become commuting variables. If we consider deformations of space alone (leaving the time as a classical coordinate), it is natural to ask whether we can endow the space of "functions" of these (generally noncommuting) "coordinates" with a suitable notion of scalar product, so as to consider it as the Hilbert space of a finite-dimensional quantum mechanical model. In other terms, the idea is to mimic the originary construction of quantum mechanics, where we naturally endow the space of square integrable functions with the corresponding well-known Hilbert space structure. If this is possible one could build the deformed (position, momentum, hamiltonian,...) operators in terms of the deformed coordinates and derivatives and set the eigenvalue problem for the hamiltonian and the other observables. In ref. [10] the construction of a 4-dimensional harmonic oscillator with $SU_q(2)$ symmetry was performed along these lines using three generating (noncommuting) "coordinates" of the quantum group $SU_q(2)$ and an additional commuting variable $r \in \mathbf{R}^+$ (the radius).

Here we consider the harmonic oscillator in the N-dimensional real quantum euclidean space $\mathbf{R}_q^N$ ($N \geq 3$) as the deformation of the classical isotropic harmonic oscillator in $\mathbf{R}^N$. Correspondingly, the symmetry group $SO(N,\mathbf{R})$ is deformed into the quantum group $SO_q(N,\mathbf{R})$.

To understand the line of development of the present work, let us briefly review the basic mathematical tools which allow the formulation of classical (i.e. nondeformed) quantum mechanics in the coordinate representation Π (over the N-

dimensional space $\mathbf{R}^N$). We can summarize its main ingredients in the following list (with self-evident notation):

• 1) There exists a differential calculus D on $\mathbf{R}^N$ (*derivatives* $\equiv \partial^i \in D$).

• 2) There exists an antilinear involutive antihomomorphism defined on the algebra of functions of x, ∂, the socalled complex conjugation $*$.

• 3) The vectors belonging to the Hilbert space $\mathcal{H}$ are represented by

$$\Pi : |u> \in \mathcal{H} \rightarrow \psi_u(x) \in \mathcal{L}^2$$

•4) Relevant operators (observables etc.) are represented in terms of products of $x\cdot, \partial$ (and their functions). Eigenvalue equations are represented by differential equations (at least in a domain dense in $\mathcal{L}^2(\mathbf{R}^N)$).

•5) scalar products are evaluated by means of Riemann integration,

$$< u|v >= \int d^N x \ \psi_u^* \psi_v,$$

which satisfies Stoke's theorem and therefore automatically makes the momentum operators $\frac{1}{i}\partial^i$ hermitean.

•6) The Schroedinger equation for the harmonic oscillator on $\mathbf{R}^N$ admits an algebraic solution by means of the creation and destruction operators (which are also represented using $x\cdot, \partial$)

In the present work we present a q-deformed version of each of these points. The analogs of points 1), 2) were thoroughly developed in Ref. [11],[3], the analogs of the remaining points are essentially constructed here; some of the latter results are anticipated in [12]. Using these q-deformed tools, we show that a sensible q-deformed harmonic oscillator on $\mathbf{R}_q^N$ (with symmetry $SO_q(N, \mathbf{R})$) can be constructed. In other words we will show that such a model satisfies all the fundamental axioms of quantum mechanics.

The plan of the thesis is as follows.

Chapter 1 is an introduction (based essentially on Ref. [3] [11]) to the quantum group $SO_q(N, \mathbf{R})$, the quantum space $\mathbf{R}_q^N$ (Sect. 1) and the (two) differential calculi $D, \bar{D}$ on $\mathbf{R}_q^N$ (Sect. 2).

In Chapter 2, Sect. 3, the time-independent Schroedinger equation is formulated in terms of the q-deformed laplacians of $D, \bar{D}$ and it is solved using a suitable generalization of the classical creation/destruction operators. The spectrum is bounded from below, as physics requires. The eigenfunctions are the "q-deformed" Hermite functions. Then (Sect. 4) we show that any function of the type *polynomial* $\cdot$ *gaussian* can be expressed as a combination of q-deformed Hermite functions (as in the classical case), a property which enables to handle the question of completeness.

Chapter 3 deals with the definition of integration over $\mathbf{R}_q^N$. Integration is thoroughly defined using Stoke's theorem only on some functions of the type *polynomial* $\cdot$ *gaussian*; the latter will be involved in the definition of the scalar products of states of the harmonic oscillator. To define integration in full generality one should carefully delimit the domain of functions on which commutation of integration and infinite sums makes sense; this is out of the scope of this work. In Sect. 5 we analyse the desired requirements that an honest definition of integration should satisfy; among them Stoke's theorem plays a special role. In Sect. 6 and appendix A we carry out the construction of the integrals for the abovementioned relevant fucntions; at the end of that section we comment on a surprising feature regarding the behaviour of integration under dilatation of the integration variables, a sort of "quantized" scaling invariance.

In Chapter 4 we construct the Hilbert space of the harmonic oscillator. First a pre-Hilbert space $\mathcal{H}$ is introduced by representing the states in two different ways over the space $Fun(\mathbf{R}_q^N)$ (Sect. 7). The two representations $(\Pi, \bar{\Pi})$ correspond respectively to $D, \bar{D}$. It is shown that the square lenght, the square momentum and hence the hamiltonian of the harmonic oscillator are observables, i.e. hermitean operators. In Sect. 8 another observable, the square angular momentum, is found, and its spectrum and eigenfuntions are determined. With the help of these results we prove (Sect.9) the positivity of the scalar product introduced in Sect. 7. This allows the completion of $\mathcal{H}$ into a Hilbert space $[\mathcal{H}]$.

Section 10 contains the conclusions of the present work.

q-deformed harmonic oscillators have already been treated by other authors [13] starting from a purely algebraic approach, in the sense that creation/destruction operators with some prescribed commutations relation are postulated from the very beginning without any reference to a geometrical framework. Here and in [10], on the contrary, a geometrical framework is the starting point and creation/destruction operators are constructed out of the deformed "coordinates" and "derivatives".

Acknowledgements

I thank L. Bonora for the choice of the subject of this thesis, for stimulating discussions and for encouraging suggestions. I thank J. Wess for introducing the problem to me and for enlightening discussions. I am indebted with P. Nurowski: his precious suggestions helped me to improve the clearness of exposition, but above all he went on in declaring himself sincerely interested in my thesis, in spite of the huge number of misprints he had detected. I feel obliged to Simonetta A. I (successor to Pasquale P. I), since she initiated me into the Mysteries of **Macro.tex**. I am grateful to M. O'Campo, M. Schlieker and W. Weich for many useful discussions. Finally, I would like to thank M. Abud for his continuous encouragement.

Chapter 1

The Quantum Group $SO_q(N,R)$, the Quantum Space R_q^N, and its two Differential Calculi

1. The quantum group $SO_q(N,R)$ and its quantum space R_q^N

In Ref. [3] one parameter deformations of the classical simple Lie groups and Lie algebras are presented. For each Lie group G a family $Fun(G_q)$ of Hopf algebras parametrized by $q \in \mathbf{C}$ ($q \equiv$ the parameter of deformation) is given, and for $q = 1$ (which corresponds to the socalled classical limit) $Fun(G_q)$ reduces to the Hopf algebra $Fun(G)$ of functions on G. With a suggestive expression, $Fun(G_q)$ is said to be the Hopf algebra of functions on the " quantum group " G_q. If $q \neq 1$ the expression " quantum group G_q " has no intrinsic meaning: all " geometrical " properties of G_q are to be translated and understood in terms of properties of $Fun(G_q)$. We will be concerned with the deformation $SO_q(N)$ ($N \geq 3$) of $SO(N)$, more precisely with its real section $SO_q(N, \mathbf{R})$.

The elements of the Hopf algebra $Fun(SO_q(N))$ are formal ordered power series in the generating elements $\{T^i_j\}$, $i,j = 1,2,...,N$. The latter satisfy the relations

$$TCT^t = \mathbf{1}_{SO_q(N)}C \tag{1.1}$$

and

$$\hat{R}(T \otimes T) = (T \otimes T)\hat{R}. \tag{1.2}$$

Here $C := ||C_{ij}||$ denotes the (q-deformed) metric matrix, $\mathbf{1}_{SO_q(N)}$ denotes the unit of the algebra, and the tensor product appearing in eq. (1.2) just means that we are tensoring indices. $\hat{R} := ||\hat{R}^{ij}_{hk}||$ is the braid matrix and satisfies the Yang-Baxter equation (in the braid version)

$$(\hat{R} \otimes \mathbf{1}_1)(\mathbf{1}_1 \otimes \hat{R})(\hat{R} \otimes \mathbf{1}_1) = (\mathbf{1}_1 \otimes \hat{R})(\hat{R} \otimes \mathbf{1}_1)(\mathbf{1}_1 \otimes \hat{R}), \tag{1.3}$$

$\mathbf{C}^N$. Eq. (1.3) itself is most commonly written in the form

$$\hat{R}_{12}\hat{R}_{23}\hat{R}_{12} = \hat{R}_{23}\hat{R}_{12}\hat{R}_{23} \tag{1.4}$$

with self-explaining notation. Eq.'s (1.2), (1.4) imply

$$f(\hat{R})(T \otimes T) = (T \otimes T)f(\hat{R}) \tag{1.5}$$

$$f(\hat{R}_{12})\hat{R}_{23}\hat{R}_{12} = \hat{R}_{23}\hat{R}_{12}f(\hat{R}_{23}) \tag{1.6}$$

for any polynomial $f(\hat{R})$ in the variable $\hat{R}$. $C, \hat{R}$ are explicitly given by

$$C_{ij} := q^{-\rho_i}\delta_{ij'}, \qquad\qquad j' := N + 1 - j \tag{1.7}$$

$$\hat{R} = q\sum_{i\neq i'} e^i_i \otimes e^i_i + \sum_{i\neq j,j',\ or\ i=j=j'} e^j_i \otimes e^i_j + q^{-1}\sum_{i\neq i'} e^{i'}_i \otimes e^i_{i'} + \tag{1.8}$$

$$+(q-q^{-1})[\sum_{i<j} e^i_i \otimes e^j_j - \sum_{i<j} q^{-\rho_i+\rho_j} e^{j'}_i \otimes e^j_{i'}], \tag{1.9}$$

where $(e^i_j)^h_k := \delta^{ih}\delta_{jk}$ and

$$(\rho_i) := \begin{cases} (\frac{N}{2}-1, \frac{N}{2}-2, ..., 0, 0, ..., 1-\frac{N}{2}) & if\ N\ even \\ (\frac{N}{2}-1, \frac{N}{2}-2, ..., \frac{1}{2}, 0, -\frac{1}{2}..., 1-\frac{N}{2}) & if\ N\ odd \end{cases}. \tag{1.10}$$

For instance for $N = 3$

$$C = \left\| \begin{matrix} & & q^{\frac{-1}{2}} \\ & 1 & \\ q^{\frac{1}{2}} & & \end{matrix} \right\| \tag{1.11}$$

In general $C^{-1} = C$, so that $C^{ij} = C_{ij}$. The fact that the matrix C is not diagonal for q=1 is due to the choice of non real coordinates for the fundamental representation of $SO(N)$. The $\hat{R}$-matrix can be decomposed using the three orthogonal projectors corresponding to its three eigenvalues $q, -q^{-1}, q^{1-N}$:

$$\hat{R} = q\mathcal{P}_S - q^{-1}\mathcal{P}_A + q^{1-N}\mathcal{P}_1, \tag{1.12}$$

and

$$\mathbf{1}_2 = \mathcal{P}_S + \mathcal{P}_A + \mathcal{P}_1 \tag{1.13}$$

Here now $\mathbf{1}_2$ denotes the unit matrix acting on $\mathrm{C^N}\otimes\mathrm{C^N}$. Therefore any polynomial in the $\hat{R}$ variable reduces to a combination of these three projectors. It can be also expressed as a combination of three other linearly independent functions of

$\hat{R}$. For instance the choice of the variables $\hat{R}, \mathbf{1}, \mathcal{P}_1$ will be convenient for many calculations, so we solve here equations (1.12), (1.13) for $\mathcal{P}_A, \mathcal{P}_S$:

$$\begin{aligned} \mathcal{P}_A &= \frac{1}{q+q^{-1}}[-\hat{R} + q\mathbf{1} - (q - q^{1-N})\mathcal{P}_1]; \\ \mathcal{P}_S &= \frac{1}{q+q^{-1}}[\hat{R} + q^{-}\mathbf{1} - (q^{-1} + q^{1-N})\mathcal{P}_1]. \end{aligned} \tag{1.14}$$

The projectors $\mathcal{P}_A, \mathcal{P}_1, \mathcal{P}_S$, have respectively dimension $\frac{N(N-1)}{2}, 1, \frac{N(N+1)}{2} - 1$. If q=1 they reduce to the projectors over the irreducible corepresentations (antisymmetric, singlet and symmetric respectively) of the tensor product $x \bigotimes x$ of the fundamental corepresentation (x) of $SO(N)$. The projector $\mathcal{P}_1$ is related to the metric matrix C by

$$\mathcal{P}_1{}^{ij}_{hk} = \frac{C^{ij}C_{hk}}{Q_N}, \qquad Q_N := C^{ij}C_{ij} \tag{1.15}$$

The matrix $\hat{R}$ is symmetric

$$\hat{R}^T = \hat{R}; \tag{1.16}$$

$\hat{R}$ and its inverse satisfy the relations

$$C_{mi}\hat{R}^{\pm 1}{}^{ij}_{hk} = \hat{R}^{\mp 1}{}^{jn}_{ml}C_{nk}. \tag{1.17}$$

As a direct consequence, they also satisfy the following one

$$[\hat{R}^{\pm 1}, P \cdot (C \otimes C)] = 0, \tag{1.18}$$

and so does any polynomial function $f(\hat{R})$ (in particular each one of the three projectors):

$$[f(\hat{R}), P \cdot (C \otimes C)] = 0, \tag{1.19}$$

$$f(\hat{R})^T = f(\hat{R}) \tag{1.20}$$

The coproduct ϕ

$$\phi : Fun(SO_q(N)) \rightarrow Fun(SO_q(N)) \bigotimes Fun(SO_q(N)) \tag{1.21}$$

and counity ε

$$\varepsilon : Fun(SO_q(N)) \rightarrow \mathbf{C} \tag{1.22}$$

are defined on the basic variables by

$$\phi(T^i_j) := T^i_k \bigotimes T^k_j, \qquad \varepsilon(T^i_j) := \delta^i_j \tag{1.23}$$

and extended to all $Fun(SO_q(N))$ as algebra homomorphisms:

$$\phi(ab) = \phi(a)\phi(b), \qquad \varepsilon(ab) = \varepsilon(a)\varepsilon(b) \qquad \forall a, b \in Fun(SO_q(N)). \tag{1.24}$$

The antipode S is defined by

$$S(T^i_j) = C^{il} T^m_l C_{mj} \tag{1.25}$$

on the basic variables and is extended as a linear antihomomorphism to all $Fun(SO_q(N))$:

$$S(ab) = S(b)S(a). \tag{1.26}$$

ϕ, ε, S have the properties

$$(\phi \bigotimes id) \circ \phi = (id \bigotimes \phi) \circ \phi \qquad (id \bigotimes \varepsilon) \circ \phi = id = (\varepsilon \bigotimes id) \circ \phi \tag{1.27}$$

$$m \circ (id \bigotimes S) \circ \phi = i \circ \varepsilon = m \circ (S \bigotimes id) \circ \phi, \tag{1.28}$$

$$(\varepsilon \bigotimes id) \circ \phi = id = (id \bigotimes \varepsilon) \circ \phi \tag{1.29}$$

$$\varepsilon \circ S = \varepsilon \tag{1.30}$$

$$\phi \circ S = \tau \circ (S \bigotimes S) \circ, \qquad \tau(a \bigotimes b) := (b \bigotimes a). \tag{1.31}$$

Here id is the identity operator on $Fun(SO_q(N))$,

$$m : Fun(SO_q(N)) \bigotimes Fun(SO_q(N)) \to Fun(SO_q(N)) \tag{1.32}$$

is the multiplication operator ($m(a \otimes b) := ab$) and $i : \mathbf{C} \to Fun(SO_q(N))$ is the injection operator defined by $i(c) := c\mathbf{1}_{SO_q(N)}$.

If $q \in \mathbf{R}$ there exists an antilinear involution $*$ on $Fun(SO_q(N))$. It is called complex conjugation, since it reduces to the ordinary complex conjugation for q=1. On the basic variables it is defined by

$$(T^i_j)^* := S(T^j_i) = C^{li} T^l_m C_{jm} \tag{1.33}$$

and extended as an antilinear antihomomorphism to all $Fun(SO_q(N))$:

$$(ab)^* := b^* a^*. \tag{1.34}$$

It is easy to show that $*$ is compatible with the relations (1.1),(1.2) defining $Fun(SO_q(N))$, namely that the relations obtained by taking the complex conjugated of (1.1),(1.2) are identically satisfied. We explicitly check the compatibility with the first relation, which we rewrite here by showing indices

$$T^i_j C^{jk} T^i_k = C^{il} \mathbf{1}_{SO_q(N)} \tag{1.35}$$

Since $\mathbf{1}^*_{SO_q(N)} = \mathbf{1}_{SO_q(N)}$ and the matrix elements of C are real, complex conjugation on the LHS gives

$$(LHS(1.35))^* = C^{il} \mathbf{1}_{SO_q(N)}; \tag{1.36}$$

as for the RHS, using formula (1.1) itself and the property $C^{-1} = C$ we find

$$\begin{aligned}(RHS(1.35))^* &= (T^l_k)^* C^{jk} (T^i_j)^* = C^{ml} T^m_n C_{kn} C^{jk} C^{pi} T^p_q C_{jq} = \\ &= C^{ml} T^m_j C_{jq} T^p_q C^{pi} = C^{ml} C_{mp} C^{pi} \mathbf{1}_{SO_q(N)} = C^{il} \mathbf{1}_{SO_q(N)};\end{aligned} \tag{1.36}$$

they are equal, as announced. Similarly one shows that (1.2) is transformed by $*$ into an identity. By multiplying relations (1.1),(1.2) by powers of T^i_j one generates new relations involving higher order polynomials in T^i_j; the latter are compatible with $*$ too, since $*$ is an antihomomorphism.

Finally it is straightforward to show that the complex conjugation " commutes " with the coproduct and the counity:

$$\phi \circ * = (* \bigotimes *) \circ \phi \tag{1.37}$$

The Hopf algebra $Fun(SO_q(N))$ equipped with $*$ is the compact real section of this Hopf algebra and will be denoted by $Fun(SO_q(N, \mathbf{R}))$. If $q = 1$ it reduces to the Hopf algebra of functions on the compact group $SO(N, \mathbf{R})$.

The algebra $O_q(N)$ (this name is due to Ref. [3]) of functions on the quantum euclidean space is introduced as the algebra of formal ordered power series in the generating elements $\{x^i\}$, $i = 1, 2, ..., N$ modulo the relations

$$\mathcal{P}_A{}^{ij}{}_{hk} x^h x^k = 0. \tag{1.38}$$

For instance for $N=3$ eq.'s (1.38) amount to the three independent relations

$$x^1x^2-qx^2x^1=0, \quad x^2x^3-qx^3x^2=0, \quad x^1x^3-x^3x^1+(q^{\frac{1}{2}}-q^{\frac{-1}{2}})(x^2)^2=0. \quad (1.39)$$

For $q=1$ and any N $\mathcal{P}_{A}{}^{ij}_{hk}=\frac{1}{2}(\delta^i_h\delta^j_k-\delta^i_k\delta^j_h)$, so that the x^i " coordinates " become commuting variables and their order in each monomial doesn't matter any more. In other terms the underlying geometry is no more noncommutative, but classical (i.e. commutative).

The left coaction $\phi_L : O_q(N) \rightarrow Fun(SO_q(N)) \otimes O_q(N)$ of the quantum group $SO_q(N)$ is defined on the basic variables by

$$\phi_L(x^i)=T^i_j \bigotimes x^j \qquad (1.40)$$

and is extended as an algebra homomorphism:

$$\phi_L(ab)=\phi_L(a)\phi_L(b) \qquad a,b \in O_q(N). \qquad (1.41)$$

The conditions (1.38) are covariant w.r.t. the quantum group $SO_q(N)$ (hence they are compatible with the coaction) because of relation (1.5) (with $f(\hat{R})=\mathcal{P}_A$):

$$\phi_L(\mathcal{P}_{A}{}^{ij}_{hk}x^hx^k)=\mathcal{P}_{A}{}^{ij}_{hk}T^h_{h'}T^k_{k'}\bigotimes x^{h'}x^{k'}=T^i_{i'}T^j_{j'}\bigotimes \mathcal{P}^{A}{}^{i'j'}_{hk}x^hx^k. \qquad (1.42)$$

Imposing condition (1.38) puts both handsides equal to zero.

ϕ_L satisfy the properties

$$(\phi \bigotimes id)\circ\phi_L=(id\bigotimes\phi_L)\circ\phi_L \qquad (\varepsilon\bigotimes id)\circ\phi_L=id_{O_q(N)} \qquad (1.43)$$

Therefore by the introduction of ϕ_L $O_q(N)$ gets a " left $SO_q(N)$-comudule ". Similarly one could define a right coaction and show that $O_q(N)$ is a " right $SO_q(N)$-comodule ".

It is easy to check that the square length $xCx := x^iC_{ij}x^j$ is central in $O_q(N)$ and is a scalar under the coaction of the quantum group:

$$\phi_L(xCx)=1\bigotimes(xCx) \qquad (1.44)$$

As for the first point, note that relation (1.12),(1.13),(1.38) imply

$$q^{-1}\hat{R}^{ij}_{hk}x^hx^k=x^ix^j+\frac{1-q^2}{\mu}q^{-N}C^{ij}(xCx) \qquad (1.45)$$

$$q\hat{R}^{-1}{}^{ij}_{hk}x^h x^k = x^i x^j + \frac{q^2-1}{\mu}C^{ij}(xCx); \tag{1.46}$$

hence, using property (1.17),

$$\begin{aligned} x^i(xCx) &= x^i x^l C_{lm}x^m = [q^{-1}\hat{R}^{il}_{hk}x^h x^k - \frac{1-q^2}{\mu}q^{-N}C^{il}(xCx)]C_{lm}x^m = \\ &= q^{-1}C_{hl}\hat{R}^{-1}{}^{li}_{km}x^h x^k x^m - \frac{1-q^2}{\mu}q^{-N}(xCx)x^i \\ &= q^{-1}\hat{C}_{hl}x^h[q^{-1}x^l x^i + q^{-1}\frac{q^2-1}{\mu}C^{li}xCx] + -\frac{1-q^2}{\mu}q^{-N}(xCx)x^i \\ &= [q^{-2}+q^{-2}\frac{q^2-1}{\mu}+q^{-N}\frac{q^2-1}{\mu}](xCx)x^i = (xCx)x^i, \end{aligned} \tag{1.47}$$

as claimed. As for the second point, relation (1.44), it is a straightforward consequence of the definition of ϕ_L and property (1.1).

As before if $q \in \mathbf{R}$ one can define an antilinear involution $*$, the complex conjugation (we will use the same symbol used for $Fun(SO_q(N))$) on $O_q(N)$ by setting

$$(x^i)^* := x^j C_{ji} \tag{1.48}$$

on the basic variables $\{x^i\}$ and extending it as an antilinear antihomomorphism. The basic conditions (1.38) defining $O_q(N)$ are compatible with complex conjugation. In fact, using properties (1.19), (1.20), we find

$$[\mathcal{P}_A{}^{ij}_{hk}x^h x^k]^* = \mathcal{P}_A{}^{ij}_{hk}x^{k'}x^{h'}C_{h'h}C_{k'k} = (\mathcal{P}_A{}^{j'i'}_{k'h'}x^{k'}x^{h'})C^{j'j}C^{i'i}; \tag{1.49}$$

Imposing eq.'s (1.38) sets both handsides of this relation equal to zero. The coaction and the complex conjugation commute:

$$\phi_L \circ * = (* \bigotimes *) \circ \phi_L. \tag{1.50}$$

It is enough to check this property on the basic variables x^i, since $\phi_L, *$ are respectively an homomorphism and an antihomomorphism. Indeed

$$\phi_L((x^i)^*) = C_{ji}\phi_L(x^j) = C_{ji}T^j_h \bigotimes x^h \tag{1.51}$$

and

$$(T^i_k)^* \bigotimes (x^k)^* = C^{ji}T^j_h C_{kh} \bigotimes x^l C_{lk} = C_{ji}T^j_h \bigotimes x^h, \tag{1.52}$$

upon use of (1.30), (1.45) and of the property $C^{-1} = C$.

It is easy to check that $(xCx)^* = xCx$, i.e. that the square lenght is real (for $q \in \mathbf{R}$.

The algebra $O_q(N)$ equipped with the involution $*$ will be suggestively called " the algebra $O_q(N, \mathbf{R})$ of functions on the N-dimensional real quantum euclidean space $\mathbf{R}_q^N$ " (again the expression $\mathbf{R}_q^N$ has a well-defined meaning only in relation with $O_q(N, \mathbf{R})$). From the above contruction it should be clear that the structure of the quantum space $\mathbf{R}_q^N$ is strictly dependent from the symmetry we endow it with, namely the symmetry w.r.t. the quantum group $SO_q(N, \mathbf{R})$. Then the construction of $\mathbf{R}_q^N$ from $SO_q(N, \mathbf{R})$ can be seen as a noncommutative realization of Felix Klein's program for geometry: the geometry of spaces should be determined by the symmetries we want them to satisfy. In this approach the quantum group is defined before the quantum space. On the contrary, in Manin's approach [6] the quantum spaces are defined first (as deformations of the commutative spaces), and the quantum groups are defined so as to be their symmetries.

2. The differential calculi $D, \bar{D}$ on R_q^N

Following the successful lines suggested in Ref. [5] and [7],[8], the differential calculus on the quantum space $\mathbf{R}_q^N$ (see Ref. [11]) can be defined by a few essential requirements: its $SO_q(N, \mathbf{R})$-covariance; Leibniz rule and nilpotency for the exterior derivative d whatever q; homogeneous commutation relations between 1-forms and 0-forms (i.e. functions). It turns out that the differential calculus reduces to the classical one on $\mathbf{R}^N$ for $q = 1$. The steps are the following. One first introduces the basic 1-forms ξ^i by applying the exterior derivative to x^i's. Then one looks at sensible homogeneous commutation relations between x^i's and ξ^j. Finally one introduces " derivatives " ∂_i's " w.r.t. the coordinates " by setting $d := \xi^i \partial_i$. We will see that two independent differential calculi are admissible. They are mapped one into the other by the complex conjugation.

The exterior derivative of the differential calculus D is denoted by d. It is nilpotent, namely

$$d^2 = 0. \tag{2.1}$$

Denoting by $\Lambda_q^p(O^N)$ the space of p-forms over $\mathbf{R}_q^N$, we define

$$d\alpha_p| := d\alpha_p - (-1)^p \alpha_p d \tag{2.2}$$

Leibniz rule amounts to the statement

$$\alpha_p \in \Lambda_q^p \quad \Rightarrow \qquad d\alpha_p| \in \Lambda_q^{p+1} \tag{2.3}$$

In fact it implies

$$d\alpha_p\beta| = d\alpha_p|\beta + (-1)^p \alpha_p d\beta| \tag{2.4}$$

for any form β. In the common notation $d\alpha_p|$ would be denoted by $(d\alpha_p)$, so that Leibniz rule (2.4) would take the usual form $d(\alpha_p\beta_q) = (d\alpha_p)\beta_q + (-1)^p\alpha_p(d\beta_q)$. We prefer to use the new symbol $d\alpha_p|$ to keep in mind that this form can be written as the difference (2.2); this will enable us to define the complex conjugation $*$ in an explicit antihomomorphic form on all arguments (also on d, see below) and to avoid many notation ambiguities.

In particular if $f \in O_q^N \equiv \Lambda_q^0$, $df|$ is a 1-form. We denote by $\xi^i := dx^i|$ the exterior derivatives of the basic coordinates x^i; $\{\xi^i\}$ is a " basis " of Λ_q^1, the space of 1-forms. The latter can be obtained by arbitary combinations of formal products $f(x)\xi^i g(x)$, $f, g \in O_q^N(\mathbf{R})$. One should be able to reduce all such combinations to combinations of terms either of the type $f\xi^i$ or of the type $\xi^i g$, and to this end one needs to prescribe commutations relations between the x^i's and the ξ^j's. In the classical case these relations are homogeneous; therefore, as already anticipated, we look for homogeneous ones for any q:

$$x^i\xi^j = M^{ij}_{hk}\xi^h x^k, \qquad M \textit{ invertible.} \tag{2.5}$$

M is fixed by the requirement of covariance, consistency with the q-space relations (1.38) for x^i's and consistency with Leibniz rule. By covariance we mean that the coaction should be naturally extended as an homomorphism to all Λ_q^p by the fundamental requirement that the exterior derivative " commutes " with the coaction:

$$\phi_L \circ d = (id_{SO_q(N)} \bigotimes d) \circ \phi_L \tag{2.6}$$

As an immediate consequence of (2.6) and the definition $\xi^i := dx^i|$ we get for instance

$$\phi_L(\xi^i) = T^i_j \bigotimes \xi^j. \tag{2.7}$$

Applying ϕ_L to both sides of (2.5) we find that M must satisfy the condition

$$T^i_{i'}T^j_{j'} \bigotimes x^{i'}x^{j'} = M^{ij}_{hk}T^h_{h'}T^k_{k'} \bigotimes \xi^{h'}x^{k'} = M^{ij}_{hk}T^h_{h'}T^k_{k'}M^{-1}{}_{i'j'}^{h'k'} \bigotimes x^{i'}\xi^{j'}, \quad (2.8)$$

in other words

$$M(T \otimes T) = (T \otimes T)M. \quad (2.9)$$

Therefore M must be a function $f(\hat{R})$ of $\hat{R}$ (see formula (1.5)), hence a combination of three linearly independent functions of $\hat{R}$, let us take $\hat{R}, \hat{R}^{-1}, \mathcal{P}_1$:

$$M = a\hat{R} + b\hat{R}^{-1} + c\mathcal{P}_1. \quad (2.10)$$

Let us now consider the matrix $B(a,b,c)$ appearing in the commutation relations

$$x^i x^j \xi^k = B^{ijk}_{lmn}\xi^l x^m x^n \quad (2.11)$$

and obtained by applying two times relations (2.5) with M as given in (2.10). Consistency with the relations (1.38) means that contracting both sides of (2.11) with $\mathcal{P}_A{}^{hk}_{ij}$ we should get zero, namely

$$(\mathcal{P}_A)_{12}B(\mathcal{P}_\Omega)_{23} = 0, \qquad \Omega = S, 1 \quad (2.12)$$

A little lenghty calculation shows that this gives two equations in the unknowns a, b, c; they have two solutions:

$$\textit{either} \qquad b = c = 0, \qquad\qquad \textit{or} \qquad a = c = 0 \quad (2.13)$$

The remaining constant is fixed by the requirement of consistency of Leibniz rule with the relations (1.38). Upon use of the (1.12),(1.15) (2.10) we find for the two solutions (2.13) respectively the identities

$$d\mathcal{P}_A{}^{ij}_{hk}x^h x^k| = \mathcal{P}_A{}^{ij}_{hk}\xi^h x^k + \mathcal{P}_A{}^{ij}_{hk}x^h\xi^k = \mathcal{P}_A{}^{ij}_{hk}\xi^h x^k \cdot \begin{cases} (1 - aq^{-1}) \\ (1 - bq) \end{cases}; \quad (2.14)$$

the LHS vanishes because of (1.38), (2,2) so we conclude that it must be $a = q$ and $b = q^{-1}$ respectively. We denote the 1-forms and the exterior derivative corresponding to the first and second solution by ξ^i, d and $\bar{\xi}^i, \bar{d}$ respectively. Summing up, $\xi^i =: dx^i|$, $\bar{\xi}^i := \bar{d}x^i|$ and

$$x^i\xi^j = q\hat{R}^{ij}_{hk}\xi^h x^k \quad (2.15)$$

$$x^i\bar{\xi}^j = q^{-1}\hat{R}^{-1}{}^{ij}_{hk}\bar{\xi}^h x^k. \tag{2.16}$$

The wedge product of two 1-forms is essentially determined by the requirement of nilpotency of the exterior derivative. In fact, using relation (2.16) and the decomposition (1.12) of $\hat{R}$ we find

$$\begin{aligned} d^2x^ix^j| &= d(\xi^ix^j + x^i\xi^j)| = d[(\mathbf{1} + q\hat{R})^{ij}_{hk}\xi^h x^k]| = \\ &= -[(1+q^2)\mathcal{P}_S + (1+q^{2-N})\mathcal{P}_1]^{ij}_{hk}\xi^h\xi^k; \end{aligned} \tag{2.17}$$

consistency of this relation with $d^2 = 0$ requires that

$$\mathcal{P}_S(\xi \otimes \xi) = 0 = \mathcal{P}_1(\xi \otimes \xi). \tag{2.18}$$

Therefore the wedge product $\wedge$ of 1-forms is to be defined as the tensor product $\otimes$ modulo the relations (2.18). In other words:

$$\mathcal{P}_S(\xi \wedge \xi) = 0 = \mathcal{P}_1(\xi \wedge \xi). \tag{2.19}$$

In this way one defines Λ_q^2. Higher degree forms are to be defined in a similar way, namely as tensor products of 1-forms modulo relations (2.18) for all neighbouring tensor factors. In a similar way one shows that $\bar{d}^2 = 0$ implies

$$\mathcal{P}_S(\bar{\xi} \wedge \bar{\xi}) = 0 = \mathcal{P}_1(\bar{\xi} \wedge \bar{\xi}) \tag{2.20}$$

for the barred 1-forms. This enables us to define the space $\bar{\Lambda}_q^p$ of barred p-forms. As a direct consequence of eq.'s (2.19), (2.20)

$$\mathcal{P}_A(\xi \wedge \xi) = (\xi \wedge \xi), \qquad \mathcal{P}_A(\bar{\xi} \wedge \bar{\xi}) = (\bar{\xi} \wedge \bar{\xi}). \tag{2.21}$$

In the sequel we will drop the symbol $\wedge$.

The decompositions $d = \xi^i\partial_i$, $\bar{d} = \bar{\xi}^i\bar{\partial}_i$ define the derivatives ∂_i, $\bar{\partial}_i$ corresponding to each coordinate x^i. In general, indices are raised and lowered by the metric matrix C (which is its own inverse), for instance

$$\partial_i = C_{ij}\partial^j, \qquad \partial^i = C^{ij}\partial_j. \tag{2.22}$$

The coaction should be extended in a natural way as an homomorphism to the larger algebras generated by x^i, ξ^i, ∂^i and $x^i, \bar{\xi}^i, \bar{\partial}^i$ respectively. The requirement that the exterior derivative be invariant, namely

$$\phi_L(d) = \mathbf{1}_{SO_q(N)} \bigotimes d, \qquad \phi_L(\bar{d}) = \mathbf{1}_{SO_q(N)} \bigotimes \bar{d} \tag{2.23}$$

implies that the coaction must act on the derivatives in the following way:

$$\phi_L(\partial^i) := T^i_j \bigotimes \partial^j \qquad \phi_L(\bar{\partial}^i) := T^i_j \bigotimes \bar{\partial}^j. \tag{2.24}$$

(we have used the orthogonality relations (1.1) of $SO_q(N)$).

The commutation relations involving the derivatives are already fixed by the previous constraints, let us determine them. First, notice that dx^i can be written in two ways

$$dx^i = \xi^j \partial_j x^i, \qquad dx^i = \xi^i + x^i d = \xi^i + x^i \xi^h \partial_h = \xi^i + q\hat{R}^{ih}_{jk} \xi^j x^k \partial_h, \tag{2.25}$$

whence, by comparison,

$$\partial_j x^i = \delta^i_j + q\hat{R}^{ih}_{jk} x^k \partial_h, \tag{2.26}$$

or, equivalently,

$$\partial^i x^j = C^{ij} + q\hat{R}^{-1}{}^{ij}_{hk} x^h \partial^k. \tag{2.27}$$

These are the " commutation relations " of the derivatives with the coordinates; notice that the Leibniz rule for the derivatives ∂ holds only for $q = 1$ (in fact for $q = 1$ $\hat{R}^{ij}_{hk} = \delta^i_k \delta^j_h$). Similarly one can show that

$$\bar{\partial}^i x^j = C^{ij} + q^{-1}\hat{R}^{ij}_{hk} x^h \bar{\partial}^k. \tag{2.28}$$

To find the commutation relations between two derivatives note that

$$d^2 = d\xi^i \partial_i = -\xi^i d\partial_i = -\xi^i \xi^j \partial_j \partial_i. \tag{2.29}$$

Using properties (2.19) (2.21) of the wedge product and formulas (2.1), (1.19) we derive that the commutation relations between the derivatives must be of the form

$$(\mathcal{P}_A + a\mathcal{P}_S + b\mathcal{P}_1)^{ij}_{hk} \partial^h \partial^k = 0. \tag{2.30}$$

Apply both sides of the preceding relation to x^m and use twice the derivation rule (2.27). It is easy to check that the constants a, b have to vanish to get again zero at both sides. An analogous argument applies to the $\bar{\partial}^i$ derivatives. Summing up:

$$\mathcal{P}_A{}^{ij}_{hk} \partial^h \partial^k = 0 \tag{2.31}$$

$$\mathcal{P}_A{}^{ij}_{hk} \bar{\partial}^h \bar{\partial}^k = 0 \tag{2.32}$$

It remains to find out the commutation relations between the derivatives and the 1-forms. They can be determined by the requirement of covariance and consistency with the commutation relations (2.15), (2.27). The calculations are straightforward and one finds:

$$\partial^i \xi^j = q^{-1} \hat{R}^{ij}_{hk} \xi^h \partial^k \tag{2.33}$$

$$\bar{\partial}^i \bar{\xi}^j = q \hat{R}^{-1}{}^{ij}_{hk} \bar{\xi}^h \bar{\partial}^k. \tag{2.34}$$

By the above discussion we have shown that the forementioned consistency requirements are satisfied for any product $\eta^i \eta^j \eta^k$ of three basic elements η ($\eta^l = x^l, \xi^l, \partial^l$). Then consistency will be satisfied for any product of any arbitrary number n of elements. One can easily show that this statement is a consequence of two facts: first, ϕ (resp. $*$) is a homomorphism (resp. a antihomomorphism); second, the matrices $\hat{R}_{i,i+1}$, $i = 1, 2, ..., n$ provide a representation of the braid group (with generating elements $\sigma_{i,i+1}, \sigma^{-1}_{i,i+1}$).

One can easily show that no combination of the 1-forms $\{\xi^i\}$ can reproduce the $\{\bar{\xi}^j\}$ satisfying the above relations, in other terms barred and non barred forms are linearly independent objects. This is also the case for the $\{\partial^i\}$ and $\{\bar{\partial}^j\}$ derivatives.

The above relations define the differential calculi $D = \{d,\ \xi^i,\ \partial^i\}$ and $\bar{D} = \{\bar{d},\ \bar{\xi}^i, \bar{\partial}^i\}$. Note that all commutation relations inside $\bar{D}$ can be obtained from the corresponding ones of D by replacing $d, \xi^i, \partial^i, \Delta, \Lambda_q, q, \hat{R}_q$ by $\bar{d}, \bar{\xi}^i, \bar{\partial}^i, \bar{\Delta}, \bar{\Lambda}_q, q^{-1}, \hat{R}_q^{-1}$. We omit to look for the commutation relations between objects in D and objects in $\bar{D}$ [14], since we won't need them.

It is natural to define the laplacians $\Delta, \bar{\Delta}$ by

$$\Delta := \partial^i \partial_i = \partial^i C_{ij} \partial^j \qquad \bar{\Delta} := \bar{\partial}^i \bar{\partial}_i = \bar{\partial}^i C_{ij} \bar{\partial}^j, \tag{2.35}$$

Just in the same way as for the square lenght xCx, one can prove that they are central elements respectively in the algebra of the ∂ and $\bar{\partial}$ derivatives; and that they are scalars under the coaction of the quantum group:

$$\partial^i \Delta = \Delta \partial^i \qquad \bar{\partial}^i \bar{\Delta} = \bar{\Delta} \bar{\partial}^i, \tag{2.36}$$

$$\phi_L(\Delta) = \mathbf{1} \bigotimes \Delta, \qquad \phi_L(\bar{\Delta}) = \mathbf{1} \bigotimes \bar{\Delta}, \tag{2.37}$$

Now let us consider the effect of the application of the complex conjugation on both handsides of all the commutation relations of this section. For $q \in \mathbf{R}$ we define the complex conjugation $*$ as an involutive antilinear antihomomorphism acting on the algebras generated by D and $\bar{D}$ respectively:

$$(AB)^* = B^* A^* \tag{2.38}$$

One can show that no new relations are introduced by the application of $*$ to the upper relations; more precisely, the relations involving the D calculus are transformed into relations equivalent to those involving the $\bar{D}$ calculus, and viceversa. Let us consider for instance relation (2.15). It can be rewritten in the form

$$\xi^l x^m = q^{-1} \hat{R}^{-1}{}_{ij}^{lm} x^i \xi^j. \tag{2.39}$$

By taking the complex conjugation of both handsides and using the definition (1.45) we get

$$x^{m'} C_{m'm} (\xi^l)^* = q^{-1} \hat{R}^{-1}{}_{ij}^{lm} x^{i'} (\xi^j)^* C_{i'i} \tag{2.40}$$

It is straightforward to check that, performing the replacement $(\xi^i)^* \rightarrow \bar{\xi}^j C_{ji}$ and using property (1.17) of the $\hat{R}$-matrix, the previous relation becomes exactly the commutation relation (2.16) between the coordinates x^i's and the barred 1-forms $\bar{\xi}^j$, so one can identify $(\xi^i)^*$ and $\bar{\xi}^j C_{ji}$. Looking at the explicit definition $\xi^i = dx^i - x^i d$, $\bar{\xi}^i = \bar{d}x^i - x^i \bar{d}$ we see that d^* behaves as $-\bar{d}$, so they can be identified,too. Finally, by means of the commutation relations (2.33) between derivatives and 1-forms and the decompositions $d = \xi^i \partial_i$, $\bar{d} = \bar{\xi}^i \bar{\partial}_i$ of the exterior derivatives, we are led to the identification of $\bar{\partial}^i$ with $-q^N (\partial^j)^* C_{ji}$; in fact one can check that the latter is consistent with all other relations (2.28),(2.34) etc. Summarizing, one can say that $*$ maps $x^i, \xi^i, \partial^i, d$ into a combination of $\bar{x}^i, \bar{\xi}^i, \bar{\partial}^i, \bar{d}$ respectively, in the following way

$$(x^i)^* = x^j C_{ji}, \qquad (\xi^i)^* = \bar{\xi}^j C_{ji}, \qquad (\partial^i)^* = -q^{-N} \bar{\partial}^j C_{ji}, \qquad d^* = -\bar{d}. \tag{2.41}$$

For q=1 the two calculi D, $\bar{D}$ are the same and these relations become the usual ones characterizing the classical calculus.

A direct consequence of (2.41) is the relation

$$\Delta^* = q^{-2N} \bar{\Delta}. \tag{2.42}$$

From (2.27) it is easy to derive the following useful formulas:

$$\Delta x^i = \mu \partial^i + q^2 x^i \Delta \qquad \partial^i (xCx) = \mu x^i + q^2 (xCx) \partial^i \tag{2.43}$$

$$\bar{\Delta} x^i = \bar{\mu}\bar{\partial}^i + q^{-2} x^i \bar{\Delta} \qquad \bar{\partial}^i (xCx) = \bar{\mu} x^i + q^{-2}(xCx)\bar{\partial}^i \tag{2.44}$$

where $\mu := 1 + q^{2-N}, \bar{\mu} := 1 + q^{N-2}$.

For any function $f(x) \in O_q^N$, $\partial^i f$ can be expressed in the form

$$\partial^i f = \hat{f}^i + \tilde{f}^i_j \partial^j \qquad \hat{f}^i, \tilde{f}^i_j \in O_q^N \tag{2.45}$$

upon using (2.27) to move step by step the derivatives to the right of each x^i variable of each term of the power expansion of f, as far as the extreme right. Similarly to what has been done in formula (2.8), we denote $\hat{f}^i$ by

$$\partial^i f| := \partial^i f - \tilde{f}^i_j \partial^j (= \hat{f}^i). \tag{2.46}$$

In an analogous way we can define $\bar{\partial}^i f|$.

The q-exponential function is introduced by

$$exp_q[Z] := \sum_{n=0}^{\infty} \frac{Z^n}{(n)_q!}, \qquad (n)_q := \frac{q^n - 1}{q-1}; \tag{2.47}$$

its usefulness lies essentially in the relations

$$\partial^i \{exp_{q^2}[\frac{\alpha(xCx)}{\mu}]\} = \alpha x^i exp_{q^2}[\frac{\alpha(xCx)}{\mu}] + exp_{q^2}[\frac{q^2\alpha(xCx)}{\mu}] \tag{2.48}$$

$$\bar{\partial}^i \{exp_{q^{-2}}[\frac{\alpha(xCx)}{\bar{\mu}}]\} = \alpha x^i exp_{q^{-2}}[\frac{\alpha(xCx)}{\bar{\mu}}] + exp_{q^{-2}}[\frac{q^{-2}\alpha(xCx)}{\bar{\mu}}], \tag{2.49}$$

which imply $\partial^i exp_{q^2}[\frac{\alpha(xCx)}{\mu}]| \propto x^i exp_{q^2}[\frac{\alpha(xCx)}{\mu}]$, $\bar{\partial}^i exp_{q^{-2}}[\frac{\alpha(xCx)}{\mu}]| \propto x^i exp_{q^{-2}}[\frac{\alpha(xCx)}{\mu}]$.

From the definition (2.47) it is easy to check the following q-derivative property for the exponentials

$$\frac{exp_{q^2}[q^2 a(xCx)] - exp_{q^2}[a(xCx)]}{q^2-1} = a(xCx) exp_{q^2}[a(xCx)] \tag{2.50}$$

$$\frac{exp_{q^{-2}}[q^{-2} a(xCx)] - exp_{q^{-2}}[a(xCx)]}{q^{-2}-1} = a(xCx) exp_{q^{-2}}[a(xCx)]. \tag{2.51}$$

Finally we write down a formula which will be often used in the sequel

$$\Delta (xCx)^h = \frac{\mu^3 q^{N+2h-2}}{q^2-1} h_{q^2} (xCx)^{h-1} B - \frac{\mu^2(q^2+1)}{q^2-1} h_{q^4} (xCx)^{h-1} + q^{4h} (xCx)^h \Delta, \tag{2.52}$$

where the operator B is defined by

$$B := 1 + \frac{q^2-1}{\mu} x^i \partial_i \tag{2.53}$$

and satisfies the properties

$$B(xCx) = q^2(xCx)B \qquad B\Delta = q^{-2}\Delta B. \tag{2.54}$$

Chapter 2

The Schroedinger Equation of the Harmonic Oscillator on $\mathbf{R}_q^N$

3. The q-Deformed Harmonic Oscillator on R_q^N and its Schroedinger Equation

In this section we consider the Schroedinger equation for the harmonic oscillator in $\mathbf{R}_q^N$ with characteristic constant ω and symmetry $SO_q(N, \mathbf{R})$. As we have the two calculi $D, \bar{D}$ at our disposal, we introduce the corresponding two versions of the equation. Then we recursively determine eigenvalues and eigenfuntions by using a suitable generalization of the creation/destruction operators of the classical case. In section 4 we will see that no other eigenfunctions (at least in the form "*gaussian* $\cdot$ *polynomial* ") are possible.

Let

$$h_\omega := \frac{1}{2}(-q^N \Delta + \omega^2 xCx) \qquad \bar{h}_\omega = \frac{1}{2}(-q^{-N}\bar{\Delta} + \omega^2(xCx)) \tag{3.1}$$

be the hamiltonians corresponding to the calculi $D, \bar{D}$. Both coincide with the classical one for q=1. By the above choice q^N, q^{-N} of the factors preceding the laplacians $\Delta, \bar{\Delta}$ the eigenvalues of $h_\omega, \bar{h}_\omega$ will coincide and

$$h_\omega^* = \bar{h}_\omega. \tag{3.2}$$

From (2.24), (2.25) and for any $\alpha, \bar{\alpha} \in R$ we get respectively

$$2h_\omega(x^i + \alpha\partial^i) =$$

$$= x^i(xCx)\omega^2 - \alpha q^N \partial^i \Delta - q^N(\mu\partial^i + q^2 x^i \Delta) + \omega^2 \alpha q^{-2}[\partial^i(xCx) - \mu x^i] =$$

$$= x^i[q^2 2h_{\omega q^{-1}} - \mu\alpha\omega^2 q^{-2}] + \partial^i[\alpha 2h_{\omega q^{-1}} - \mu q^N] \tag{3.3}$$

$$2\bar{h}_\omega(x^i+\bar{\alpha}\bar{\partial}^i)=$$
$$=x^i(xCx)\omega^2-\bar{\alpha}q^{-N}\bar{\partial}^i\bar{\Delta}-q^{-N}(\bar{\mu}\bar{\partial}^i+q^{-2}x^i\bar{\Delta})+\omega^2\bar{\alpha}q^2[\bar{\partial}^i(xCx)-\bar{\mu}x^i]=$$
$$=x^i[q^{-2}2\bar{h}_{\omega q}-\bar{\mu}\bar{\alpha}\omega^2q^2]+\bar{\partial}^i[\bar{\alpha}2\bar{h}_{\omega q}-\bar{\mu}q^{-N}]. \tag{3.4}$$

Note that in both square brackets on the RHS of (3.3) (respectively (3.4)) the same operator $h_{\omega q^{-1}}$ (respectively $\bar{h}_{\omega q}$) appears.

Now assume that $\psi(x,\omega)$ (respectively $\bar{\psi}(x,\omega)$) is an eigenvector of h_ω ($\bar{h}_\omega$) with eigenvalue E ($\bar{E}$):

$$h_\omega\psi|=E\psi, \qquad\qquad \bar{h}_\omega\bar{\psi}|=\bar{E}\bar{\psi}; \tag{3.5}$$

then $\psi':=\psi(x,\omega q^{-1})$ (respectively $\bar{\psi}':=\bar{\psi}(x,\omega q)$) will be an eigenvector of $h_{\omega q^{-1}}$ ($\bar{h}_{\omega q}$) with eigenvalue Eq^{-1} (Eq) (in fact $E,\bar{E}\propto\omega$ for dimensional reasons). I look for α ($\bar{\alpha}$) such that $(x^i+\alpha\partial^i)\psi'$ ($(x^i+\bar{\alpha}\bar{\partial}^i)\bar{\psi}'$) be an eigenvector of h_ω ($\bar{h}_\omega$). Let E' ($\bar{E}'$) be the corresponding eigenvalue. Then the relations

$$2E'(x^i+\alpha\partial^i)\psi'|=2h_\omega(x^i+\alpha\partial^i)\psi'|=x^i[q2E-\mu\alpha\omega^2q^{-2}]\psi'+\partial^i[\alpha q^{-1}2E-\mu q^N]\psi'| \tag{3.6}$$

$$2\bar{E}'(x^i+\bar{\alpha}\bar{\partial}^i)\bar{\psi}'|=\bar{2}h_\omega(x^i+\bar{\alpha}\bar{\partial}^i)\bar{\psi}'|=x^i[q^{-1}2\bar{E}-\bar{\mu}\bar{\alpha}\omega^2q^2]\bar{\psi}'+\bar{\partial}^i[\bar{\alpha}q2\bar{E}-\bar{\mu}q^{-N}]\psi'| \tag{3.7}$$

must hold, namely

$$\begin{cases} 2E'\alpha=\alpha 2Eq^{-1}-\mu q^N \\ 2E'=qE-\mu\alpha\omega^2q^{-2} \end{cases} \tag{3.8}$$

$$\begin{cases} 2\bar{E}'\bar{\alpha}=\bar{\alpha}2\bar{E}q-\bar{\mu}q^{-N} \\ 2\bar{E}'=q2\bar{E}-\bar{\mu}\bar{\alpha}\bar{\omega}^2q^2. \end{cases} \tag{3.9}$$

These systems of equations have the following solutions

$$\begin{cases} \alpha=\frac{E(q-q^{-1})\pm\sqrt{E^2(q-q^{-1})^2+\mu^2\omega^2q^{N-2}}}{\mu\omega^2q^{-2}} \\ E'=Eq^{-1}-\frac{\mu}{2\alpha}q^N, \end{cases} \tag{3.10}$$

$$\begin{cases} \bar{\alpha}=\frac{\bar{E}(q^{-1}-q)\pm\sqrt{\bar{E}^2(q^{-1}-q)^2+\bar{\mu}^2\omega^2q^{2-N}}}{\bar{\mu}\omega^2q^2} \\ \bar{E}'=\bar{E}q-\frac{\bar{\mu}}{2\bar{\alpha}}q^{-N} \end{cases} \tag{3.11}$$

Formulas (3.10),(3.11) can be used to find by induction spectra and eigenfunctions of h_ω,$\bar{h}_\omega$ starting from known eigenfunctions ψ_0,$\bar{\psi}_0$ respectively. In analogy with the classical case, we try eigenfunctions of the form $\psi_0=exp_{q^2}[-\frac{AxCx}{\mu}]$, $\bar{\psi}_0=$

$exp_{q^{-2}}[-\frac{BxCx}{\bar{\mu}}]$, which depend only on (xCx) [1]; the factors A, B are fixed by the requirement that ψ_0, $\bar{\psi}_0$ are eigenfunctions of h_ω,$\bar{h}_\omega$, and one finds $A = \pm\omega q^{-N}$, $B = \pm\omega q^N$. We choose the solutions $A, B > 0$ as they correspond to the classical normalizable ground state with positive energy. Finally:

$$\psi_0 = exp_{q^2}[-\frac{\omega q^{-N} xCx}{\mu}], \qquad E_0 = \frac{Q_N \omega}{2} \tag{3.12}$$

$$\bar{\psi}_0 = exp_{q^{-2}}[-\frac{\omega q^{N} xCx}{\bar{\mu}}], \qquad \bar{E}_0 = \frac{Q_N \omega}{2} \tag{3.13}$$

where

$$Q_N := C^{ij}C_{ij} = \frac{(1-q^N)\mu}{(1-q^2)} = (q^{\frac{N}{2}-1} + q^{1-\frac{N}{2}})\frac{q^{\frac{N}{2}} - q^{-\frac{N}{2}}}{q - q^{-1}}. \tag{3.14}$$

We see that $\bar{E}_0 = E_0$ as we wanted. Let us apply formulas (3.10), (3.11) to this eigenvalue; we obtain respectively

$$\alpha = \begin{cases} \frac{q^{N+1}}{\omega} := \alpha'_1 \\ -\frac{q}{\omega} := \alpha_1 \end{cases}, \tag{3.15}$$

$$\bar{\alpha} = \begin{cases} \frac{q^{-N-1}}{\omega} := \bar{\alpha}'_1 \\ -\frac{q^{-1}}{\omega} := \bar{\alpha}_1 \end{cases}. \tag{3.16}$$

Here the primed (non-primed) solutions correspond to the choice of the plus (minus) sign in formula (3.10) (formula (3.11)). The solutions α'_1,$\bar{\alpha}'_1$ can be discarded as they yield trivial functions: $(x^i + \alpha'_1\partial^i)\psi_0(\omega q^{-1})| = 0$, $(x^i + \bar{\alpha}'_1\bar{\partial}^i)\bar{\psi}_0(\omega q^{-1})| = 0$ (see (2.29),(2.30)).

Now we devote our attention to the sequence of solutions that are determined from ψ_0 ($\bar{\psi}_0$) by a recursive application of formula (3.10) (formula (3.11)) with the choice of the minus sign. The coefficient and eigenvalue obtained after n steps will be called α_n, E_n ($\bar{\alpha}_n, \bar{E}_n$).

Proposition 1:

$$\alpha_n = -\frac{q^{2-n}}{\omega}, \quad n \geq 1; \qquad E_n = \frac{1}{2}\omega(q^{\frac{N}{2}-1} + q^{1-\frac{N}{2}})[\frac{N}{2} + n]_q \qquad n \geq 0 \tag{3.17}$$

$$\bar{\alpha}_n = -\frac{q^{n-2}}{\omega}, \quad n \geq 1; \qquad \bar{E}_n = \frac{1}{2}\omega(q^{\frac{N}{2}-1} + q^{1-\frac{N}{2}})[\frac{N}{2} + n]_q = E_n \qquad n \geq 0 \tag{3.18}$$

where

$$[n]_q := \frac{q^n - q^{-n}}{q - q^{-1}}, \tag{3.19}$$

are the q-deformed integers: $[n]_q \overset{q\to 1}{\longrightarrow} n$. The proof of this proposition is straightforward: (3.18),(3.19) yield for $E_0, \bar{E}_0$ the values of formulas (3.12),(3.13), and it is trivial algebra to show that if the proposition is true for $n = m$ then formulas (3.8), (3.9) imply that it is true also for $n = m + 1$.

As a consequence of formulas (3.17), (3.18) and the very construction of the eigenfunctions, at level n the latter are of the form

$$\psi_n^{i_n i_{n-1} \dots i_1} =$$

$$= (x^{i_n} + \alpha_n \partial^{i_n})(x^{i_{n-1}} + q\alpha_{n-1}\partial^{i_{n-1}}) \dots .(x^{i_1} + q^{n-1}\alpha_1\partial^{i_1}) exp_{q^2}[-\frac{\omega q^{-n-N} xCx}{\mu}]| =$$

$$= (x^{i_n} - \frac{q^{2-n}}{\omega}\partial^{i_n})(x^{i_{n-1}} - \frac{q^{4-n}}{\omega}\partial^{i_{n-1}}) \dots .(x^{i_1} - \frac{q^n}{\omega}\partial^{i_1}) exp_{q^2}[-\frac{\omega q^{-n-N} xCx}{\mu}]| \tag{3.20}$$

$$\bar{\psi}_n^{i_n i_{n-1} \dots i_1} =$$

$$= (x^{i_n} + \bar{\alpha}_n \bar{\partial}^{i_n})(x^{i_{n-1}} + q^{-1}\bar{\alpha}_{n-1}\bar{\partial}^{i_{n-1}}) \dots .(x^{i_1} + q^{1-n}\bar{\alpha}_1\bar{\partial}^{i_1}) exp_{q^2}[-\frac{\omega q^{+n+N} xCx}{\mu}]|$$

$$= (x^{i_n} - \frac{q^{n-2}}{\omega}\bar{\partial}^{i_n})(x^{i_{n-1}} - \frac{q^{n-4}}{\omega}\bar{\partial}^{i_{n-1}}) \dots .(x^{i_1} - \frac{q^{-n}}{\omega}\bar{\partial}^{i_1}) exp_{q^2}[-\frac{\omega q^{+n+N} xCx}{\mu}]| \tag{3.21}$$

(in fact the replacement $\omega \to \omega q^{\pm 1}$ is equivalent to the replacement $\alpha \to q^{\mp 1}$). The eigenfunctions (3.20), (3.21) are the "q-deformed Hermite functions ", since they (both) reduce to the classical Hermite functions for $q = 1$.

Let us come back now to the choice of the plus sign in formulas (3.10), (3.11). If we set $E = E_{n-1}$ ($\bar{E} = E_{n-1}$) we find solutions

$$\alpha' = \frac{q^{n+N}}{\omega} := \alpha'_n \qquad E' = E_{n-2} \tag{3.22}$$

$$\bar{\alpha}' = \frac{q^{-n-N}}{\omega} := \bar{\alpha}'_n \qquad \bar{E}' = E_{n-2} \tag{3.23}$$

We see that no new eigenvalue is introduced in this way. Actually in next section we are going to show that all independent eigenfunctions of h_ω (respectively $\bar{h}_\omega$) of level n are represented in formula (3.20) (respectively (3.21)), hence the eigenfunctions corresponding to (3.22), (3.23) must be combinations of them. Therefore the operators

$$\begin{cases} (x^i + \alpha'_n \partial^i)G_q \\ (x^i + \bar{\alpha}'_n \bar{\partial}^i)G_{q^{-1}} \end{cases}, \qquad G_q f(x, \partial) := f(q^{-\frac{1}{2}}x, q^{\frac{1}{2}}\partial) \tag{3.24}$$

act as destruction operators when applied to eigenfunctions of level $(n-1)$ $\psi_{n-1}, \bar{\psi}_{n-1}$ respectively, whereas the operators

$$\begin{cases} (x^i + \alpha_n \partial^i) G_q \\ (x^i + \bar{\alpha}_n \bar{\partial}^i) G_{q^{-1}} \end{cases} \tag{3.25}$$

act as creation operators.

We conclude this section with the following remarks:

1) From the basic formulas of section 2 and the whole construction of this section it has become apparent that the barred and the non barred scheme can be obtained one from the other by simple replacements:

$$\begin{array}{ccc} D & & \bar{D} \\ h_\omega & & \bar{h}_\omega \\ \psi_n & & \bar{\psi}_n \\ \alpha_n & \underset{q^{-1}\to q,\, \bar{\partial}\to\partial}{\overset{q\to q^{-1},\, \partial\to\bar{\partial}}{\rightleftarrows}} & \bar{\alpha}_n \\ \alpha'_n & & \bar{\alpha}'_n \\ \hat{R}, \hat{R}^{-1} & & \hat{R}^{-1}, \hat{R} \\ \dots & & \dots \end{array} \tag{3.26}$$

2) The energies are invariant under the replacement $q \to q^{-1}$, hence they coincide in the barred and non-barred scheme. In Chapter 4 we will show that these two schemes can be seen as two different "representations " of the same Hilbert space.

3) The spectrum is bounded from below and increasing with n for any $q \in \mathbf{R}^+$; energy levels are not equidistant as in the classical case ($q = 1$) and the difference between neighbouring energy levels diverges with n (as it was found in [13]), so it would yield to a macroscopic energy gap for great n [(2)]; the energy levels have the same degeneracy as in the classical case. As we will show in next section, these results hold at least if we look for eigenfunctions in the form $P(x) exp_{q^2}[-\frac{\alpha(xCx)}{\mu}]$, where $P(x)$ is a polynomial.

4) In the case $q = 1$ both schemes reduce to the classical ones, and the eigenfunctions (3.20), (3.21) become the classical Hermite functions.

4. The Linear Span of the q-Deformed Hermite functions

In the classical case any function of the type $P(x) exp[-\frac{\omega(xCx)}{2}]$ ($P(x)$ being a polynomial) can be expressed as a combination of particular functions of this type, the well-known Hermite functions with characteristic constant ω. Moreover, any eigenfunction of the corresponding harmonic oscillator hamiltonian is a combination of Hermite functions of one (an the same) level. In this section we want to understand whether analogous statements hold in the q-deformed case. The answer will be affirmative, provided we suitably adjust their formulations to the present case. We explicitly consider only the non barred scheme, since all results valid in this case will hold also in the barred one after their translation according to the rules (3.26).

What seems to complicate the analysis in the q-deformed case is the fact that the exponents in the exponentials of formulas (3.20),(3.21) have a n-dependent q-power; consequently an enormous proliferation of functions of the type *polynomial · exponential* for the same characteristic constant ω seems to take place. Nevertheless, upon iterative use of relation (2.31) we easily realize that one function of this type can be expressed in infinitely many equivalent ways,

$$P_n(x)exp_{q^2}[-\frac{\omega q^{-N-m}(xCx)}{\mu}] =$$

$$= P_n(x)[1-\omega q^{-N-m-2}\frac{q^2-1}{\mu}(xCx)]exp_{q^2}[-\frac{\omega q^{-N-m-2}(xCx)}{\mu}] =$$

$$\equiv P_{n+2}(x)exp_{q^2}[-\frac{\omega q^{-N-m-2}(xCx)}{\mu}] = \ldots$$

$$\equiv P_{n+2h}(x)exp_{q^2}[-\frac{\omega q^{-N-m-2h}(xCx)}{\mu}], \qquad h \geq 0, \tag{4.1}$$

where by $P_n(x)$ we mean a polynomial of degree n in x; so this proliferation is (to a great extent) only apparent.

To simplify the analysis we can consider polynomials containing only either even or odd powers of x, since both the enforcement of relation (2.31) in (4.1) and the application of the operators Δ, xCx appearing in h_ω to a function of the type *monomial · exponential* change the degree of the monomial only by ± 2. Therefore from now on

$$P_n(x) := a\ polynomial\ in\ x\ containing\ only\ powers\ of\ degree\ p = n(mod\ 2). \tag{4.2}$$

We introduce the following notation:

$$\Psi_n := linear\ span\ of\ all\ the\ \psi_n's\ of\ formula\ (3.20)$$

$$\tilde{\Psi}_{2m} := \bigoplus_{h=0}^{m} \Psi_{2h} \qquad\qquad \tilde{\Psi}_{2m+1} := \bigoplus_{h=0}^{m} \Psi_{2h+1}$$

$$V_n := linear\ span\ of\ all\ functions\ of\ the\ form\ P_n(x)exp_{q^2}[-\frac{\omega q^{-n-N}(xCx)}{\mu}].$$

$$M_n := space\ of\ homogeneous\ polynomials\ of\ degree\ n \tag{4.3}$$

Below we prove the following

Proposition 2:

$$V_n = \tilde{\Psi}_n; \qquad dim(\Psi_n) = dim(M_n) = \binom{N+n-1}{N-1}, \qquad n \in N. \tag{4.4}$$

Now we define the following spaces,

$$\begin{aligned} V &:= \sum_{n=0}^{\infty} V_n = \\ &= linear\ span\ of\ all\ functions\ of\ the\ form \\ &P_n exp_{q^2}[-\frac{\omega q^{-n-N}(xCx)}{\mu}]\ (\forall\ n \geq 0) = \\ &= linear\ span\ of\ all\ functions\ of\ the\ form \\ &P_n exp_{q^2}[-\frac{\omega q^{-n-N-2h}(xCx)}{\mu}]\ (\forall\ n, h\ \geq 0) \end{aligned} \tag{4.5}$$

(the first equality holds by the definition of V_n, the second does since P_n is a particular polynomial of the type P_{n+2h}) and

$$\begin{aligned} S &:= linear\ span\ of\ all\ eigenfunctions\ of\ h_\omega\ of\ the\ form\ \psi \\ &= P(x)exp_{q^2}[-\frac{\alpha(xCx)}{\mu}]. \end{aligned} \tag{4.6}$$

Then the following chain of inclusion relations holds:

$$V \supset S \supset \bigoplus_{n=0}^{\infty} \Psi_n = \sum_{n=0}^{\infty} V_n = V. \tag{4.7}$$

The first equality holds by proposition 2; the second inclusion relation is trivial since the eigenfunctions (3.20) are eigenfunctions of the form *polynomial* $\cdot$ *gaussian*. The first inclusion relation is true as we need choosing $\alpha = \omega q^{-n-N}$ in (4.6) in order that ψ be an eigenvector of h_ω; in fact this is the condition which must be satisfied to annihilate the coefficient of the term of degree $n+2$ in the LHS of the eigenvalue equation $h_\omega\psi| = E\psi$ (as one can easily check from formulas (2.24), (2.29)). Since the two extrema of the chain of inclusions (4.5) coincide, all the inclusions can be converted into equalities, and we get a multiple characterization of V:

$$V := \sum_{n=0}^{\infty} V_n = S = \bigoplus_{n=0}^{\infty} \Psi_n. \tag{4.8}$$

We find also the

Corollary: No eigenfunctions of h_ω other than those belonging to Ψ_n (for some n) can be found in $V = S$. Correspondingly, no eigenvalue other than those belonging to $\{E_n\}_{n\in N}$ (for some n).

Now we come to the

Proof of Proposition 2:

The proposition is trivially true for $n = 0, 1$ (see (formulas (3.20),(2.29)). The general proof is by induction. Assume that the proposition is true for $n = m-2, m-1$.

We first prove the statement

$$dim\Psi_n = dimM_n \qquad n = m, m+1 \tag{4.9}$$

which is a direct consequence of the

Lemma:

$$\begin{pmatrix} A_{i_1\ldots i_n} \text{ are such that} \\ A_{i_1\ldots i_n} x^{i_1}\ldots x^{i_n} = 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} A_{i_1\ldots i_n} \text{ are such that} \\ A_{i_1\ldots i_n} \psi_n^{i_1\ldots i_n} = 0 \end{pmatrix}. \tag{4.10}$$

To prove the $\Rightarrow$ implication in (4.10) we note that

$$A_{i_1\ldots i_n} x^{i_1}\ldots x^{i_n} = 0 \tag{4.11}$$

with nontrivial coefficients $A_{i_1\ldots i_n}$ can only occurr if at least one of the relations

$$A_{i_1\ldots i_n} = \mathcal{P}_A{}^{r_j r_{j+1}}_{i_j i_{j+1}} A_{i_1\ldots i_{j-1} r_j r_{j+1} i_{j+2}\ldots i_n} \qquad 1 \le j \le n-1 \tag{4.12}$$

is satisfied (recall that $\mathcal{P}_A{}^{ij}_{hk} x^h x^k = 0$). But if this is the case then also the expression $A_{i_1\ldots i_n}\psi_n^{i_1\ldots i_n}$ vanishes because of the relation

$$\mathcal{P}_A{}^{ij}_{hk}(x^h + \beta\partial^h)(x^k + \beta q^2 \partial^k) = 0 \tag{4.13}$$

(which can be easily checked on the basis of relations (2.3),(2.8),(2.9),(2.2)).

To prove the $\Leftarrow$ implication note that by use of the relation

$$(x^i+\beta\partial^i)(x^j+\gamma\partial^j)exp_{q^2}[\frac{\alpha xCx}{\mu}]|$$

$$= ax^ix^jexp_{q^2}[\frac{\alpha xCx}{\mu}]+bexp_{q^2}[\frac{\alpha q^2xCx}{\mu}]\partial^ix^j| \qquad (a=cost\neq 0) \tag{4.14}$$

we find

$$\psi_n^{i_n\ldots i_1}=(x^{i_n}-\frac{q^{2-n}}{\omega}\partial^{i_n})\ldots(x^{i_3}-\frac{q^{n-4}}{\omega}\partial^{i_3})\cdot$$

$$\cdot\{ax^{i_2}x^{i_1}exp_{q^2}[-\frac{\omega q^{-n-N}xCx}{\mu}]+bexp_{q^2}[-\frac{\omega q^{2-n-N}xCx}{\mu}]\partial^{i_2}x^{i_1}\}|; \tag{4.15}$$

the second term in the braces can only give a term of the type $P_{n-2}^{i_n\ldots i_1}exp_{q^2}[-\frac{\omega q^{2-n-N}xCx}{\mu}]$ when the operator standing to its left acts on it, therefore

$$\psi_n^{i_n\ldots i_1}=a(x^{i_n}-\frac{q^{2-n}}{\omega}\partial^{i_n})\ldots(x^{i_3}-\frac{q^{n-4}}{\omega}\partial^{i_3})exp_{q^2}[-\frac{\omega q^{-n-N}xCx}{\mu}]x^{i_2}x^{i_1}|+$$

$$+P_{n-2}^{i_n\ldots i_1}(x)exp_{q^2}[-\frac{\omega q^{2-n-N}xCx}{\mu}]. \tag{4.16}$$

By applying the same argument to the first term in the RHS, and then again and again, we end up with

$$\psi_n^{i_n\ldots i_1}=\tilde{a}x^{i_n}\ldots x^{i_1}exp_{q^2}[-\frac{\omega q^{-n-N}xCx}{\mu}]+$$

$$+\tilde{P}_{n-2}^{i_n\ldots i_1}(x)exp_{q^2}[-\frac{\omega q^{2-n-N}xCx}{\mu}], \qquad \tilde{a}\neq 0. \tag{4.17}$$

Let us consider

$$A_{i_n\ldots i_1}\psi_n^{i_n\ldots i_1}=\tilde{a}A_{i_n\ldots i_1}x^{i_n}\ldots x^{i_1}exp_{q^2}[-\frac{\omega q^{-n-N}xCx}{\mu}]|+$$

$$+A_{i_n\ldots i_1}\tilde{P}_{n-2}^{i_n\ldots i_1}(x)exp_{q^2}[-\frac{\omega q^{2-n-N}xCx}{\mu}]. \tag{4.18}$$

According to the induction hypothesis the second term in the RHS belongs to $\tilde{\Psi}_{n-2}$; but the first term cannot, since the n-th level eigenfunctions do not belong to $\tilde{\Psi}_{n-2}$. Consequently the vanishing of the LHS implies the vanishing of the first term, i.e. the $\Leftarrow$ implication of (4.10). So the proof of the lemma is completed.

Now the proof of the claim $V_n = \Psi_n$ for $n = m, m+1$ is straightforward, since clearly $V_n \supset \tilde{\Psi}_n$ (to check this inclusion relation use formula (4.1) to reduce the exponents to $[-\frac{\omega q^{-n-N} xCx}{\mu}]$ when necessary), and

$$dimV_n = dim(space\ of\ polyn.\ P_n\,'s) = dimM_n + dim(space\ of\ polyn.\ P_{n-2}\,'s) =$$

$$= dim\Psi_n + dimV_{n-2} = dim\Psi_n + dim\tilde{\Psi}_{n-2} = dim\tilde{\Psi}_n, \qquad n = m, m+1. \tag{4.19}$$

Here the first two equalities are trivial; statement (4.9) has been used to justify the third equality, whereas the fourth and the fifth hold because of the induction hypothesis and the defintion of $\tilde{\Psi}_n$, respectively.

It remains to show that $dimM_n = \binom{N+n-1}{N-1}$ as in the case $q = 1$. In the classical case (i.e. for $q = 1$) $\binom{N+n-1}{N-1}$ is the number of sets $\{r_1, r_2, ..., r_N\}$ satisfying the condition $\sum_{i=1}^{N} r_i = n$, or, equivalently, the number of independent ordered monomials $x^{i_1}...x^{i_n}$ modulo the relations (2.3)

$$\mathcal{P}_{A}{}^{ij}{}_{hk} x^h x^k = 0$$

where $\mathcal{P}_{A}{}^{ij}{}_{hk} = \frac{1}{2}(\delta^i_h \delta^j_k - \delta^i_k \delta^j_k)$ (for $q = 1$). The antisymmetric projector $\mathcal{P}_A$ is deformed for $q \neq 1$, but the number of relations (2.3) remains the same; consequently also the number of independent monomials. The proof of Proposition 2 is so completed $\diamondsuit$.

Whatever $q > 0$ relations (2.3) are sufficient to order any monomial according with a prescribed order relation for the indices, for instance according to increasing order; hence a basis in M_n is

$$\{x^{i_1} x^{i_2} ... x^{i_n},\ i_1 \leq i_2 \leq ... \leq i_n\} \tag{4.20}$$

and a basis in Ψ_n (because of lemma (4.10)) is provided by

$$\{\psi^{i_1 i_2 ... i_n},\ i_1 \leq i_2 \leq ... \leq i_n\}. \tag{4.21}$$

As we have already noticed, all the results of this section hold for the barred "representation " after the replacements (3.26).

Notes

(1) Other power solutions $f(xCx) = \sum_n a_n(xCx)^n$ which cannot be written in the form $P(xCx)exp_{q^2}[AxCx]$ (P being a polynomial) can be formally found solving the Schroedinger equation (the a_n are found recursively). In the classical limit they correspond to functions that are not normalizable. Postponing the discussion of integrability to a forthcoming paper, we leave them out here.

Chapter 3

Integration over R_q^N

5. Integration: formal requirements

In Ref. [7] the authors propose a definition of integration over the quantum hyperplane essentially based on the requirements of linearity and of validity of Stoke's theorem (of course in such an approach the latter is no more a "theorem "). Denoting by $< f >$, $\int \omega_n$ respectively the integral of a function f and of an n-form ω_n over the n-dimensional hyperplane (as usual they are related by definition by the identity $< f >:= \int dVf$, where dV denotes the volume form), Stoke's theorem takes respectively the forms

$$< \partial^i f| >= 0 \quad i = 1, 2, ..., n; \qquad\qquad \int d\omega_{n-1}| = 0. \tag{5.1}$$

In the classical case, if $f = P_n(x)exp[-a|x|^2]$ (P_n denotes a polynomial of degree n in x and $|x|^2$ the square lenght), then

$$\partial^i P_n(x)exp[-a|x|^2]| = P_{n-1}(x)exp[-a|x|^2] + P_{n+1}exp[-a|x|^2]; \tag{5.2}$$

relations (5.2), (5.3) imply

$$< P_{n-1}(x)exp[-a|x|^2] > + < P_{n+1}exp[-a|x|^2] >= 0. \tag{5.3}$$

Relation (5.3) allows to recursively define the integral $< f >$ (for any function f of the same kind) in terms of $< exp[-a|x|^2] >$ (which fixes the normalization of the integration). The same holds in the q-deformed case, provided one has defined the generalization of the exponential (the socalled q-exponential).

The integral over the hyperplane defined according to (5.1), (5.2)has the following properties: a) it is covariant w.r.t. $GL_q(n)$ (in the sense that will defined below); b) it coincides with the classical Riemann integral for q=1 (by a suitable choice of the normalization factor); c) it satisfies the reality condition

$$< f >^* = < f^* > \tag{5.4}$$

for any q ($\in \mathbf{R}^+$). Therefore the relation

$$< f^* f > \geq 0, \qquad < f^* f > = 0 \Leftrightarrow f = 0 \tag{5.5}$$

(positivity condition) holds, at least in a (f-dependent) neighbourhood of q=1, since it holds for q=1. If there exists a neighbourhood $U \subset \mathbf{R}^+$ of q=1 such that positivity holds $\forall q \in U$, a scalar product can be introduced through the definition

$$(f, g) := < f^* g >, \tag{5.6}$$

and one can convert into a Hilbert space a suitable subspace of the algebra of functions on the quantum hyperplane.

In the case of the real quantum euclidean space the situation is complicated by the fact that there exist two sets of linearly independent derivatives belonging respectively to the differential calculi $D, \bar{D}$, hence potentially two kinds of integrations $< \ >, \ll \ \gg$ and two versions of Stoke's theorem:

$$< \partial^i f| > = 0 \quad i = 1, 2, ..., N; \qquad \int d\omega_{n-1}| = 0.$$

$$<< \bar{\partial}^i f| >> = 0 \quad i = 1, 2, ..., N; \qquad \bar{\int} \bar{d}\bar{\omega}_{n-1}| = 0. \tag{5.7}$$

It is not difficult to guess that reality condition (5.4) for each of the two integrations $< \ >, \ll \ \gg$ is no more guaranteed by Stoke's theorems (5.7) because $*$ maps derivatives $\partial \in D$ into derivatives $\bar{\partial} \in \bar{D}$, and viceversa. Therefore we should be prepared to abandon or to modify some of the formal requirements that we wish integration to satisfy.

First, we list these requirements; then we ask whether they are compatible. If they are not we should investigate how they can be modified to become such. Even though here we are considering $\mathbf{R}_q^N$, the following analysis should be valid for any quantum space. Through the relation

$$< f > = \int dV \ f \tag{5.8}$$

statements regarding integral of functions can be translated into ones regarding integrals of N-forms, and viceversa, so often they will be written only in one of the two versions.

We would like an integration $< \ >$ to be defined on a not too poor subspace $\mathcal{V}$ of $O_q^N(\mathbf{R})$ and to satisfy:

1) linearity;
2) covariance;
3) correspondence principle for $q \to 1$;
4) reality;
5) positivity.

Of course linearity means

$$< \alpha f + \beta g >= \alpha < f > + \beta < g >, \qquad \alpha, \beta \in \mathbf{C} \quad f, g \in \mathcal{V} \tag{5.9}$$

and one has to check that if f vanishes because of relations (2.), then so does $< f >$, in other terms

$$f(x) = A_{ij} \mathcal{P}_A{}^{ij}_{hk} x^h x^k \cdot g(x) \qquad \Rightarrow \quad < f >= 0. \tag{5.10}$$

By covariance we mean

$$\mathbf{1}_{SO_q(N)} < f >= (id_{SO_q(N)} \bigotimes < \; >) \circ \phi_L(f), \tag{5.11}$$

where $\mathbf{1}_{SO_q(N)}$ and $id_{SO_q(N)}$ denote respectively the unit element and the identity operator on $Fun(SO_q(N, \mathbf{R}))$, and ϕ_L is the left coaction of $SO_q(N, \mathbf{R})$ on $O_q^N(R)$. More explicitly, if $f^{i_1 i_2 \dots i_k} := x^{i_1} x^{i_2} \dots x^{i_k} g(xCx)$, then covariance means

$$\mathbf{1}_{SO_q(N)} < f^{i_1 i_2 \dots i_k} >= T^{i_1}_{j_1} T^{i_2}_{j_2} \dots T^{i_k}_{j_k} < f^{j_1 j_2 \dots j_k} >, \tag{5.12}$$

in other words the numbers $< f^{i_1 i_2 \dots i_k} >$, $i_j = 1, 2, \dots, N$, are the components of an "isotropic " tensor; in the classical case this corresponds to the well-known property of tensors such as

$$\int d^N x \; g(|x|^2) x^i = 0, \qquad \int d^N x \; g(|x|^2) x^i x^j \propto \delta^{ij},$$

$$\int d^N x \; g(|x|^2) x^i x^j x^k x^l \propto (\delta^{ij} \delta^{kl} + \delta^{ik} \delta^{jl} + \delta^{il} \delta^{jk}), \dots \tag{5.13}$$

namely the property that the latter are invariant under an orthogonal transformation of the coordinates $x^i \to x'^i := g^i_j x^j$. The simplest nontrivial example of a tensor satisfying (5.12) is for $k = 2$, $< f^{ij} > \propto C^{ij}$. In general tensors satisfying (5.12) involve matrix products among $\hat{R}$-matrices (or, equivalently, $\hat{R}^{-1}$-matrices)) and contractions with metric matrices C: the former reorder indices by means of the RTT relations (2.), whereas the latter transform a couple of neighbouring

T-matrices into a commuting number.Therefore an integral $< x^{i_1}...x^{i_k}g(xCx) >$ should be factorizable as a product

$$< x^{i_1}...x^{i_k}g(xCx) >= S^{i_1...i_k}\alpha_g, \tag{5.14}$$

and $S^{i_1...i_k} = 0$ for k odd; the g-dependence of the RHS of (5.14) is concentrated in the constant α_g, which essentially is a (yet unspecified) integral along the "" radial " direction. Explicit solutions $S^{i_1...i_k}, \bar{S}^{i_1...i_k}$ satisfying (5.12) will be found in setion 4.

Point 3) means that we require a q-deformed integral to reduce to a classical one (with some integration measure $\rho(x)d^N x$) when q=1. Maybe it is timely to recall the fact that the x^i coordinates are not real (even for q=1), but are complex combinations of the usual real cartesian coordinates; the latter can be used to perform the integral when q=1.

The reality and positivity conditions 4), 5) in the form (5.4),(5.5) or in some other form should guarantee that the definition (5.6) or what takes its place introduces an honest scalar product (,) in a suitable subspace of $O_q(N, \mathbf{R})$

$$(f, g)^* = (g, f); \qquad (f, f) \geq 0, \ \ (f, f) = 0 \Leftrightarrow f = 0 \qquad f, g \in \mathcal{V}, \tag{5.15}$$

to convert this subspace into a Hilbert space.

To the five previous points we add a requirement characterizing the specific problem we are dealing with here, namely that the hamiltonian (or the position/momentum operators) of the harmonic oscillator be hermitean operators w.r.t. (,). As it will be clear in the sequel, we are led to ask for the validity of

6) Stoke's theorem

in the form (5.7). As in the classical case, point 6) involves a definite choice of the "radial " part of the integration (whereas the latter is left unspecified by 2) alone). As already noticed, Stoke's theorem is a formidable tool to define (up to a normalization factor) the corresponding integration.

Now we briefly discuss compatibility of requirements 1) - 6).

It is straightforward to check that linearity is compatible with covariance because of property (1.5) (taking $f(\hat{R}) = \mathcal{P}_A$). Requirement 3) is obviously compatible with 1),2) since classical integration is linear and there exist $SO(N, \mathbf{R})$

invariant integration measures. As for reality, we consider two possible formulations, (5.4) and

$$< f >^* = \ll f^* \gg, \tag{5.16}$$

where we consider a couple of integrations $< \ >, \ll \ \gg$ (virtually the ones satisfying Stoke's theorems $(5.7)_a$ and $(5.7)_b$). It is straightforward to prove that both formulations are compatible with linearity (because of relation (1.19), where again we take $f(\hat{R}) = \mathcal{P}_A$), with covariance (apply $*$ to eq. (5.11) and use property (1.34)) and with the correspondence principle (classical real integrations satisfy the reality condition). Positivity in the form (5.5) is clearly compatible with requirements 1),3) and with reality in either form (5.4) or (5.16). At this stage is not easy to understand if it is compatible with covariance. Using the results which will be presented in Sect.'s 6.,7. one could prove that this is the case (decomposing f in a combination of eigenfunctions of the square angular momentum L^2, and using orthogonality of the eigenfunctions corresponding to different eigenvalues of L^2). The question is not strictly relevant for the solution of problem (5.15) in the specific case of the harmonic oscillatos, since the scalar product $(\ , \ )$ that we are going to introduce in Sect. 5 is not of the form (5.6). In fact, we will prove that the latter *is* positive definite.

Let us analyse now the compatibility of Stoke's theorems (5.7) with points 1) - 5). It is straightforward to check that both $(5.7)_a$ and $(5.7)_b$ are compatible with linearity: in fact this compatibility is reduced to the consistency of both differential calculi $D, \bar{D}$ with the q-space relations (1.35) and the commutation relations (2.15),(2.16) respectively. Similarly, the compatibility with covariance is guaranteed by the commutation relation (2.6) of the exterior derivative with the coaction. As for the correspondence principle, compatibility is ensured by the fact that in the limit $q \to 1$ both D and $\bar{D}$ go to the classical differential calculus, and Riemann integration (on smooth functions) satisfies Stoke's theorem. It is hard to say whether Stoke's theorems are compatible with reality in the form (5.4), and we are led to think that this is not the case, but it is surely compatible with reality in the form (5.16). To understand this point, let us consider the spaces of formal relations

$$\mathcal{F} = \{\partial^i f - \partial^i f| - f^i_j \partial^j = 0\}, \qquad i,j = 1,..,N \quad f \in \mathcal{V}\} \tag{5.17}$$

$$\bar{\mathcal{F}} := \{\bar{\partial}^i f - \bar{\partial}^i f| - \bar{f}^i_j \bar{\partial}^j = 0\}, \qquad i,j = 1,..,N \quad f \in \mathcal{V}\}, \tag{5.18}$$

where: 1) $\partial^i f|, f^i_j, \bar{\partial}^i f|, \bar{f}^i_j$ are the functions introduced in formula (2.43); 2) $\mathcal{V}$ is some subspace of $O_q^N(\mathbf{R})$ closed under complex conjugation and containing also the functions $\partial^i f|, f^i_j, \bar{\partial}^i f|, \bar{f}^i_j$ for any $f \in \mathcal{V}$. In the classical case the space V_{cl} of functions of the type $P(x)exp[-a|x|^2]$ (P being a polynomial) is an example of such a subspace $\mathcal{V}$, and we will see that an analogous space will be available in the q-deformed case, too. Under these assumptions it is immediate to recognize that the two sets (5.17),(5.18) are mapped into each other by $*$, since $* : D \rightarrow \bar{D}$ and $* : \bar{D} \rightarrow D$. In other terms $\mathcal{F}^* = \bar{\mathcal{F}}$. If we define subspaces $\mathcal{A}, \bar{\mathcal{A}} \subset \mathcal{V}$ as the linear spans of functions $\partial^i f|$ and $\bar{\partial}^i f|$ respectively, the previous remark implies

$$\mathcal{A}^* = \bar{\mathcal{A}} \tag{5.19}$$

For each $a \in \mathcal{A}$ let $\bar{a} \in \bar{a}$ be the function such that $a^* = \bar{A}$. Stoke's theorems respectively imply

$$(5.7)_a \quad \Rightarrow \quad < a >= 0 =< a >^* \quad \forall a \in \mathcal{A} \tag{5.20}$$

$$(5.7)_b \quad \Rightarrow \quad \ll \bar{a} \gg = 0 = \ll \bar{a} \gg^* \quad \forall \bar{a} \in \bar{\mathcal{A}}, \tag{5.21}$$

hence reality in both forms (5.4) and (5.16) is trivially satisfied for the integrals $< a >, \ll \bar{a} \gg$. If q=1 and we take $\mathcal{V} = V_{cl}$ one easily realizes that any $f \in \mathcal{V}$ can be expressed in the form

$$f = a + c_f f_0, \quad a \in \mathcal{A}, \; c_f \in \mathbf{C} \tag{5.22}$$

(as anticipated at the beginning of this section), where f_0 is defined by $f_0 := exp[-a|x|^2]$. Consequently

$$< f >= c_f < f_0 > . \tag{5.23}$$

For self-evident reasons we call f_0 the refernce function of the integral. In next sections we will see that the a similar situation occurs also in the q-deformed case, for instance by taking $\mathcal{V} = V$ (V was defined in Sect. 4.) and $f_0 := exp_{q^2}[-a\frac{(xCx)}{\mu}]$. In any case f_0 should be a real function not belonging to $\mathcal{A}$ and should go to a smooth rapidly decreasing classical function in the limit $q \rightarrow 1$. Taking the complex conjugate of eq. (5.22) we get

$$f^* = \bar{a} + c^*_f f_0, \quad \bar{a} \in \bar{\mathcal{A}}, \; c_f \in \mathbf{C}, \tag{5.24}$$

which implies

$$\ll f^* \gg = c_f^* \ll f_0 \gg \tag{5.25}$$

We are free to fix $< f_0 >, \ll f_0 \gg$ as we like. If we impose the reality condition in the form (5.24) on the reference function we see that it is transfered to all functions belonging to $\mathcal{V}$, as claimed. Since $f_0^* = f_0$, the reality condition (5.16) on the reference function reads $< f_0 > = \ll f_0 \gg^*$. In the sequel we will take $< f_0 > \in \mathbf{R}^+$.

Finally the compatibility of Stoke's theorem with the positivity condition in the form (5.5) is left as an open question, but again is not relevant for our specific problem; whereas we will see in Sect. 7 that the scalar product in the Hilbert space of the harmonic oscillator, defined using the integrals $< \ >, \ll \ \gg$ is positive defined.

6. Integration: construction

In this section we use Stoke's theorem (in its two versions (5.7)) as a tool for constructing the integrations. The systematic enforcement of Stoke's theorems generates a set of formal relations between integrals of different functions. We determine these relations in two steps. First, we find out the isotropic tensors $S^{i_1...i_k}, \bar{S}^{i_1...i_k}$: hence, according to (5.14) , the integrals $< f >$, $\ll f \gg$ of a non scalar function f will be expressed in terms of integrals of a scalar one. Second, we determine the equations relating integrals of different scalar functions; in this way we will be able to express integrals of scalar functions in terms of the integrals $< f_0 >$, $\ll f_0 \gg$ of a particular one, what we call the reference function f_0. $< f_0 >$, $\ll f_0 \gg$ are normalization constants and can be fixed quite arbitrarily. So to say, the second step amounts to integration over the radial coordinate. As an example we will explicitly consider in this section the reference function $f_0 = exp_{q^2}[\frac{-\alpha x C x}{\mu}]$; in section 5. we will take an other reference function which is more suitable for defining the scalar products of states of the harmonic oscillator. In this way the integrals can be defined for infinitely many independent functions $\{f_i\}_{i \in N}$ and therefore for finite combinations of them. This is enough for the scopes of this work, since it will enable us to define a positive definite scalar product inside the span of states of the harmonic oscillator (see section 5.); then the completion of this pre-Hilbert space will be done w.r.t. the corresponding norm. Nevertheless, to further enlarge the domain of definition of the integrals one could consider series expansions in the $\{f_i\}$, and we will briefly address this problem at the end of this section.

The preliminary discussion of the previous section has shown that the two basic integrations $<\ >$, $\ll\ \gg$ are linear, covariant and coincide with the classical Riemann integration for q=1. Therefore the explicit recursive application of the two Stoke's theorems will determine (up to a factor) isotropic tensors $S^{i_1...i_k}$, $\bar{S}^{i_1...i_k}$ (see (5.12)). As we are going to see, up to a factor these tensors do not depend on the choice of the function $g(xCx)$ in formula (5.a).

The choice $g = exp_{q^2}[\frac{-\alpha xCx}{\mu}]$ (or, alternatively, we could take $g = exp_{q^{-2}}[\frac{-\alpha xCx}{\bar{\mu}}]$) is particularly convenient for this goal. Using relation (2. we find

$$\partial^{i_1}x^{i_2}...x^{i_k}exp_{q^2}[\frac{-\alpha xCx}{\mu}]| =$$

$$-\alpha x^{i_1}x^{i_2}...x^{i_k}exp_{q^2}[\frac{-\alpha xCx}{\mu}] + exp_{q^2}[\frac{-q^2\alpha xCx}{\mu}]\partial^{i_1}x^{i_2}...x^{i_k}| =$$

$$= -\alpha x^{i_1}x^{i_2}...x^{i_k}exp_{q^2}[\frac{-\alpha xCx}{\mu}] + exp_{q^2}[\frac{-q^2\alpha xCx}{\mu}]M_{k,j_3...j_k}^{\ \ i_1...i_k}x^{j_3}...x^{j_k}| \qquad (6.1)$$

and

$$\bar{\partial}^{i_1}x^{i_2}...x^{i_k}exp_{q^2}[\frac{-\alpha xCx}{\mu}]| =$$

$$-q^{N-2}\alpha x^{i_1}x^{i_2}...x^{i_k}exp_{q^2}[\frac{-q^{-2}\alpha xCx}{\mu}] + exp_{q^2}[\frac{-q^{-2}\alpha xCx}{\mu}]\partial^{i_1}x^{i_2}...x^{i_k}| =$$

$$= -q^{N-2}\alpha x^{i_1}x^{i_2}...x^{i_k}exp_{q^2}[\frac{-q^{-2}\alpha xCx}{\mu}] + exp_{q^2}[\frac{-q^{-2}\alpha xCx}{\mu}]\bar{M}_{k,j_3...j_k}^{\ \ i_1...i_k}x^{j_3}...x^{j_k}, \qquad (6.2)$$

where the tensors $M_{k,j_3...j_k}^{\ \ i_1...i_k}$, $\bar{M}_{k,j_3...j_k}^{\ \ i_1...i_k}$ are introduced together with the ones $N_{k,j_1...j_k}^{\ \ i_1...i_k}$, $\bar{N}_{k,j_1...j_k}^{\ \ i_1...i_k}$ by the defining relations

$$\partial^{i_1}x^{i_2}...x^{i_k} = M_{k,j_3...j_k}^{\ \ i_1...i_k}x^{j_3}...x^{j_k} + N_{k,j_1...j_k}^{\ \ i_1...i_k}x^{j_1}...x^{j_{k-1}}\partial^{j_k} \qquad (6.3)$$

$$\bar{\partial}^{i_1}x^{i_2}...x^{i_k} = \bar{M}_{k,j_3...j_k}^{\ \ i_1...i_k}x^{j_3}...x^{j_k} + \bar{N}_{k,j_1...j_k}^{\ \ i_1...i_k}x^{j_1}...x^{j_{k-1}}\partial^{j_k}. \qquad (6.4)$$

Taking the integrals $<\ >$, $\ll\ \gg$ respectively of (6.1), (6.2) and applying Stoke's theorems we find

$$< x^{i_1}...x^{i_k}exp_{q^2}[-\frac{\alpha xCx}{\mu}] >= \frac{1}{\alpha}M_{k,j_3...j_k}^{\ \ i_1...i_k} < x^{j_3}...x^{j_k}exp_{q^2}[-\frac{q^2\alpha xCx}{\mu}] > \qquad (6.5)$$

$$\ll x^{i_1}...x^{i_k} exp_{q^2}[-\frac{\alpha x C x}{\mu}] \gg = \frac{q^{-N}}{\alpha} \bar{M}_{k,j_3...j_k}^{\ \ i_1...i_k} \ll x^{j_3}...x^{j_k} exp_{q^2}[-\frac{\alpha x C x}{\mu}] \gg . \tag{6.6}$$

Starting from $k = 0, 1$ and noting that Stoke's theorem (or, equivalently, covariance) imply $< x^i exp_{q^2}[-\frac{\alpha x C x}{\mu}] >= 0$, we see that the recursive application of relations (6.5), (6.6) determines tensors $S_k^{i_1...i_k}$, $\bar{S}_k^{i_1...i_k}$ satisfying (5.12), in the form

$$S_k^{i_1...i_k} := 0 := \bar{S}_k^{i_1...i_k} \quad if\ k\ is\ odd \tag{6.7}$$

$$S_{2n} := M_{2n} \cdot M_{2(n-1)} \cdot ...M_2 \tag{6.8}$$

$$\bar{S}_{2n} := \bar{M}_{2n} \cdot \bar{M}_{2(n-1)} \cdot ...\bar{M}_2, \tag{6.9}$$

where we have used the shorthand notation

$$(M_{2k} \cdot M_{2(k-1)})_{j_5 j_6...j_{2k}}^{i_1 i_2...i_{2k}} = M_{2k,l_3 l_4...l_{2k}}^{\ \ i_1 i_2...i_{2k}} M_{2(k-1),j_5 j_6...j_{2k}}^{\ \ l_3 l_4...l_{2k}} \tag{6.10}$$

(and similarly for the $\bar{M}$ tensors). To give an idea of what these tensors $M_k, \bar{M}_k$ look like, we draw the explicit expression for $M_2, \bar{M}_2, M_4, \bar{M}_4$ using the derivation rules (2. :

$$M_2^{ij} = C^{ij} = \bar{M}_{ij}$$

$$M_{4,j_3 j_4}^{\ \ i_1 i_2 i_3 i_4} = C^{i_1 i_2} \mathbf{1}_{j_3 j_4}^{i_3 i_4} + q\hat{R}^{-1}\,{}_{j_3 s}^{i_1 i_2} C^{s i_3} \mathbf{1}_{j_4}^{i_4} + q^2 \hat{R}^{-1}\,{}_{j_3 s}^{i_1 i_2} \hat{R}^{-1}\,{}_{j_4 v}^{s i_3} C^{v i_4}$$

$$\bar{M}_{4,j_3 j_4}^{\ \ i_1 i_2 i_3 i_4} = C^{i_1 i_2} \mathbf{1}_{j_3 j_4}^{i_3 i_4} + q^{-1} \hat{R}_{j_3 s}^{i_1 i_2} C^{s i_3} \mathbf{1}_{j_4}^{i_4} + q^{-2} \hat{R}_{j_3 s}^{i_1 i_2} \hat{R}_{j_4 v}^{s i_3} C^{v i_4}. \tag{6.11}$$

Now it is easy to realize that

$$< x^{i_1} x^{i_2} ... x^{i_{2n}} g > \propto S_{2n}^{i_1 i_2...i_{2n}} \qquad \ll x^{i_1} x^{i_2} ... x^{i_{2n}} g \gg \propto \bar{S}_{2n}^{i_1 i_2...i_{2n}} \tag{6.12}$$

also for a different choice of the reference function $g(xCx)$. In fact, looking at the power series defining g one immediately finds that $\partial^i g(xCx)| = \tilde{g}(xCx)x^i$, $\bar{\partial}^i g(xCx)| = \bar{\tilde{g}}(xCx)x^i$ with some functions $\tilde{g}, \bar{\tilde{g}} \in O_q(N)$. Then, applying both sides of (6.3), (resp. (6.4)) to g and taking the integral $< \ >$ (respectively $\ll \ \gg$) we find

$$0 =< \partial^{i_1} x^{i_2} ... x^{i_{2n}} g| >= M_{2n,j_3...j_{2n}}^{\ \ i_1 i_2...i_{2n}} < x^{j_3} ... x^{j_{2n}} g > + N_{2n,j_1...j_{2n}}^{\ \ i_1...i_{2n}} < x^{j_1} ... x^{j_{2n}} \tilde{g} > \tag{6.13}$$

$$0 =\ll \bar{\partial}^{i_1} x^{i_2} ... x^{i_{2n}} g| \gg = \bar{M}_{2n,j_3...j_{2n}}^{\ \ i_1 i_2...i_{2n}} \ll x^{j_3} ... x^{j_{2n}} g \gg +$$

$$+ \bar{N}_{2n,j_1...j_{2n}}^{\ \ i_1...i_{2n}} \ll x^{j_1} ... x^{j_{2n}} \bar{\tilde{g}} \gg . \tag{6.14}$$

This result holds for any function g, in particular for the previous choice $g = exp_{q^2}[\frac{-\alpha xCx}{\mu}]$; by comparison with (6.5), (6.6), (6.8), (6.9) we infer the invertibility of the matrices N_{2n}, $\bar{N}_{2n}$, the relations

$$N_{2n}^{-1} \cdot S_{2n} \propto S_{2n} \qquad \bar{N}_{2n}^{-1} \cdot \bar{S}_{2n} \propto \bar{S}_{2n} \tag{6.15}$$

and hence the relations

$$< x^{i_1}...x^{i_{2n}}\tilde{g} >= c_{n,g}S_{2n}^{i_1...i_{2n}} \tag{6.16}$$

$$\ll x^{i_1}...x^{i_{2n}}\tilde{g} \gg= \bar{c}_{n,g}\bar{S}_{2n}^{i_1...i_{2n}}, \tag{6.17}$$

for any function $\tilde{g}(xCx)$. By contracting the free indices $i_1, i_2, ..., i_{2n}$ with $C_{i_1i_2},...,C_{i_{2n-1}i_{2n}}$ we reduce the determination of the constants $c_{n,\tilde{g}}, \bar{c}_{n,\tilde{g}}$ to the evaluation of integrals of purely scalar functions:

$$< x^{i_1}...x^{i_{2n}}\tilde{g} >= S_{2n}^{i_1...i_{2n}}\frac{< (xCx)^n\tilde{g} >}{S_{2n}} \tag{6.18}$$

$$\ll x^{i_1}...x^{i_{2n}}\tilde{g} \gg= \bar{S}_{2n}^{i_1...i_{2n}}\frac{\ll (xCx)^n\tilde{g} \gg}{\bar{S}_{2n}}; \tag{6.19}$$

here

$$\begin{aligned} S_{2n} &:= C_{i_1i_2}...C_{i_{2n-1}i_{2n}}S_{2n}{}^{i_1,i_2,...,i_{2n}}, \\ \bar{S}_{2n} &:= C_{i_1i_2}...C_{i_{2n-1}i_{2n}}\bar{S}_{2n}{}^{i_1,i_2,...,i_{2n}}, \end{aligned} \tag{6.20}$$

For later use we derive here the following very useful formulas:

$$\Delta^n x^{i_1}x^{i_2}...x^{i_{2n}}| = (\mu)^n n_{q^2}!S_{2n}^{i_1i_2...i_{2n}} \tag{6.21}$$

$$\bar{\Delta}^n x^{i_1}x^{i_2}...x^{i_{2n}}| = (\bar{\mu})^n n_{q^{-2}}!\bar{S}_{2n}^{i_1i_2...i_{2n}}. \tag{6.22}$$

Their proof is by induction. For $n = 1$ (6.21), (6.22) are true, since $\Delta x^{i_1}x^{i_2}| = \mu\partial^{i_1}x^{i_2} = \mu C^{i_1i_2}$, $\bar{\Delta}x^{i_1}x^{i_2}| = \bar{\mu}\bar{\partial}^{i_1}x^{i_2} = \bar{\mu}C^{i_1i_2}$. Now assume that they are true for $n = m-1$. Then

$$\Delta^m x^{i_1}x^{i_2}...x^{i_{2m}}| = \mu\Delta^{m-1}\partial^{i_1}x^{i_2}...x^{i_{2m}}| + q^2\Delta^{m-1}x^{i_1}\Delta x^{i_2}...x^{i_{2m}}| =$$

$$= \mu(1+q^2)\Delta^{m-1}\partial^{i_1}x^{i_2}...x^{i_{2m}}| + q^4\Delta^{m-2}x^{i_1}\Delta^2 x^{i_2}...x^{i_{2m}}| =$$

$$= .... =$$

$$= \mu m_{q^2} \Delta^{m-1} \partial^{i_1} x^{i_2} ... x^{i_{2m}} | + q^{2m} x^{i_1} \Delta^m x^{i_2} ... x^{i_{2m}} |. \tag{6.23}$$

The second term in the last expression is zero, since the $2m$ derivatives contained in Δ^m act on $(2m-1)$ coordinates x; using the definition (6.3) of the tensor M_{2m}, the induction hypothesis and the definition (6.8) of S_{2m} we are able to rewrite the first term as

$$\mu m_{q^2} M^{i_1 i_2 ... i_{2m}}_{2m, j_3 ... j_{2m}} \Delta^{m-1} x^{j_3 ... j_{2m}} | = (\mu)^m m_{q^2}! M_{2m, j_3 ... j_{2m}}^{i_1 i_2 ... i_{2m}} S^{j_3 ... j_{2m}}_{2(m-1)} =$$

$$= (\mu)^m m_{q^2}! S^{i_1 i_2 ... i_{2m}}_{2m}, \tag{6.24}$$

which shows that (6.21) is true also for $n = m$. In a similar way one proves (6.22). It is not difficult to evaluate the constants $\mathcal{S}_{2n}, \bar{\mathcal{S}}_{2n}$. From the definition and formula (2.52) we derive

$$\mathcal{S}_{2n} = \Delta^n (xCx)^n | = \mu^2 n_{q^2} (\frac{N}{2} + n - 1)_{q^2} \Delta^{n-1} (xCx)^{n-1} | =$$

$$= n_{q^2}! (\frac{N}{2} + n - 1)_{q^2} ... (\frac{N}{2})_{q^2} \mu^{2n} > 0; \tag{6.25}$$

similarly

$$\bar{\mathcal{S}}_{2n} = \Delta^n (xCx)^n | = n_{q^{-2}}! (\frac{N}{2} + n - 1)_{q^{-2}} ... (\frac{N}{2})_{q^{-2}} \bar{\mu}^{2n} > 0. \tag{6.26}$$

Let us analyze the "radial " dependence of the two integrals $< \;>$, $\ll \;\gg$. We introduce the operators

$$B := 1 + \frac{q^2 - 1}{\mu} x^i \partial_i = q^{-N} (1 + \frac{q^2 - 1}{\mu} \partial^i x_i)$$

$$\bar{B} := 1 + \frac{q^{-2} - 1}{\bar{\mu}} x^i \bar{\partial}_i = q^N (1 + \frac{q^{-2} - 1}{\bar{\mu}} \bar{\partial}^i x_i); \tag{6.27}$$

it is straightforward to check that $B(xCx) = q^2 (xCx) B$, $\bar{B}(xCx) = q^{-2} (xCx) \bar{B}$ and therefore

$$B f(xCx) = f(q^2 xCx) B, \qquad \bar{B} f(xCx) = f(q^{-2} xCx) \bar{B}, \tag{6.28}$$

for any $f \in O_q(N)$ depending only on (xCx); hence

$$q^{-N} (f + \frac{q^2 - 1}{\mu} \partial^i x_i f |) = f(q^2 xCx), \qquad q^N (f + \frac{q^{-2} - 1}{\bar{\mu}} \bar{\partial}^i x_i f |) = f(q^{-2} xCx). \tag{6.29}$$

By taking the integrals $<\ >$, $\ll\ \gg$ respectively of $(6.29)_a$, $(6.29)_b$ and by applying Stoke's theorems (5.7) we find the formal relations

$$< f(q^2xCx) > q^N =< f(xCx) >, \qquad \ll f(q^2xCx) \gg q^N =\ll f(xCx) \gg; \tag{6.30}$$

but the integrals $< f >$, $\ll f \gg$ of any $f \in O_q(N)$, if they exist, are reduced to combinations of integrals of radial functions by means of (6.18), (6.19), therefore property (6.30) can be generalized as follows

$$< f(qx) > q^N =< f(x) >, \qquad \ll f(qx) \gg q^N =\ll f(x) \gg . \tag{6.31}$$

This fundamental relation characterizes both integrations $<\ >$, $\ll\ \gg$ defined by means of Stoke's theorem and will be called "scaling property " of such integrations, for reasons which will become clear at the end of this section.

So far we have not specified the domain of functions $f \in O_q^N(\mathbf{R})$ for which the integrals $< f >$, $\ll f \gg$ can be defined. Therefore all the previous relations were purely formal. Now we pick up a particular reference function $f_0(xCx)$. We ask what are the functions f such that the corresponding integrals $< f >$, $\ll f \gg$ can be reduced to the ones $< f_0 >$, $\ll f_0 \gg$ by means of iterated application of Stoke's theorems and of linearity, and turn out to be finite. Of course we wish to include in this space of "integrable " functions as many $f \in O_q^N(\mathbf{R})$ as possible.

As an example we take $f_0 = exp_{q^2}[-\frac{\alpha xCx}{\mu}]$, $\alpha > 0$, which for q=1 reduces to a well known smooth rapidly decreasing classical function, the gaussian. First we consider functions f of the type $f(xCx) = P(xCx)exp_{q^2}[-\frac{\alpha xCx}{\mu}]$, P being an arbitrary polynomial. Using property (6.31) and the q-derivative property (2. of the exponential we show that

$$< exp_{q^2}[-\frac{\alpha xCx}{\mu}](xCx)^h >= \left(\frac{\mu}{\alpha}\right)^h \left(h-1+\frac{N}{2}\right)_{q^2} \cdots \left(\frac{N}{2}\right)_{q^2} \cdot$$
$$\cdot q^{-h(N+h-1)} < exp_{q^2}[-\frac{\alpha xCx}{\mu}] > \tag{6.32}$$

and, in the same way,

$$\ll exp_{q^2}[-\frac{\alpha xCx}{\mu}](xCx)^h \gg=$$
$$= \left(\frac{\mu}{\alpha}\right)^h \left(h-1+\frac{N}{2}\right)_{q^2} \cdots \left(\frac{N}{2}\right)_{q^2} \cdot q^{-h(N+h-1)} \ll exp_{q^2}[-\frac{\alpha xCx}{\mu}] \gg . \tag{6.33}$$

In fact

$$< exp_{q^2}[-\frac{\alpha xCx}{\mu}](xCx)^{k-1} >= q^{N+2(k-1)} < exp_{q^2}[-\frac{q^2\alpha xCx}{\mu}](xCx)^{k-1} >=$$

$$= q^{N+2(k-1)} < exp_{q^2}[-\frac{\alpha xCx}{\mu}](xCx)^{k-1} > +$$

$$-\frac{\alpha}{\mu}(q^2-1)q^{N+2(k-1)} < exp_{q^2}[-\frac{\alpha xCx}{\mu}](xCx)^{k} >, \tag{6.34}$$

whence

$$\begin{aligned} < exp_{q^2}[-\frac{\alpha xCx}{\mu}](xCx)^{k} > = (\frac{\mu}{\alpha})(k-1+\frac{N}{2})_{q^2}q^{-N-2(k-1)} \\ < exp_{q^2}[-\frac{\alpha xCx}{\mu}](xCx)^{k-1} >; \end{aligned} \tag{6.35}$$

applying h times formula (6.35) for $k = h, h-1, ..., 1$ we find (6.32).

If we want the reality condition (5.16) to be satisfied, then we can normalize the two integrations $< \ >$, $\ll \ \gg$ by setting

$$< exp_{q^2}[-\frac{\alpha xCx}{\mu}] >=\ll exp_{q^2}[-\frac{\alpha xCx}{\mu}] \gg= c \in \mathbf{R}^+; \tag{6.36}$$

then, as a consequence of (6.30), (6.31), $< f >=\ll f \gg$ for purely radial functions $f = f(xCx)$. The same result holds for any other choice of the reference function f_0, since the scaling property (6.31) has the same form for both integrations.

Relations (6.18), (6.19), (6.32), (6.33), (6.36) allow to define the integrations $< \ >$, $\ll \ \gg$ on all functions of the type $f = P(x)f_0$, where $P(x)$ is an arbitary polynomial in x and $f_0 := exp_{q^2}[-\frac{\alpha xCx}{\mu}]$. We could enlarge the domain of definition of the integrations by admitting functions $P(x)$ in the form of power series $P(x) = \sum\limits_{n=0}^{\infty}\sum\limits_{i_1,i_2,...,i_n} A_{i_1i_2...i_n}x^{i_1}x^{i_2}...x^{i_n}$ such that the series

$$\sum_{n=0}^{\infty}\sum_{i_1,i_2,...,i_n} A_{i_1i_2...i_n} < x^{i_1}x^{i_2}...x^{i_n}f_0 >$$

$$\sum_{n=0}^{\infty}\sum_{i_1,i_2,...,i_n} A_{i_1i_2...i_n} \ll x^{i_1}x^{i_2}...x^{i_n}f_0 \gg \tag{6.37}$$

converge; the integrals $< f >$, $\ll f \gg$ would then be defined as the limit $(6.37)_a$ and the limit $(6.37)_b$ respectively. A further step towards the enlargement of the domain of definition of the integrations could be done along the following lines. In the previous formula we could take $P = P(xCx) = \sum_{n=0}^{\infty} a_n(xCx)^n$ with coefficients $a_n \in \mathbf{R}$ and such that $< P(xCx)exp_{q^2}[-\frac{\alpha xCx}{\mu}] >:= c'$ is finite. Then we could define a new reference function by letting $f_0 := P(xCx)exp_{q^2}[-\frac{\alpha xCx}{\mu}]$: by means of formulas (6.18), (6.19), (6.30) we should be able to evaluate $< P(x)f_0 >$, $\ll P(x)f_0 \gg$ in terms of c' for all polynomials $P(x)$. Thus one could include in the domain of integrable functions also functions f susceptible of a decomposition $f = P(x)f_0$, $P(x) = \sum_{n=0}^{\infty} \sum_{i_1,i_2,\ldots,i_n} A_{i_1 i_2 \ldots i_n} x^{i_1} x^{i_2} \ldots x^{i_n}$ such that the series (6.35) with this new f_0 converge. It is natural to figure that to the new choice of the reference function there should correspond an actual enlargement of the domain of integrable functions. This operation could be iterated in a sort of continuation of the functionals $< \ >$, $\ll \ \gg$, so as to enlarge to the maximum possible size the space of integrable functions. It is out of the scope of this work to face this problem by analysing which conditions the coefficients $\{A_{i_1 i_2 \ldots i_n}\}$ of an expansion of the type $f = f_0 \sum_{n=0}^{\infty} \sum_{i_1,i_2,\ldots,i_n} A_{i_1 i_2 \ldots i_n} x^{i_1} x^{i_2} \ldots x^{i_n}$ should satisfy in order that f be integrable[(1)].

We just briefly note that, having defined the integrations $< \ >$, $\ll \ \gg$ using Stoke's theorems (5.7), one could define new integrations $< \ >_\rho$, $\ll \ \gg_\rho$ satisfying (at least) requirements 1) - 3) of the preceding section, by setting

$$< f >_\rho := < f \cdot \rho > \qquad \ll f \gg_\rho := \ll f \cdot \rho \gg; \tag{6.38}$$

the "weight " ρ should be a real scalar function.

Now let us come back to property (6.31). By its iterative application we find

$$< f(q^n x) > q^{nN} = < f(x) >, \qquad \ll f(q^n x) \gg q^{nN} = \ll f(x) \gg \qquad n \in \mathbf{Z}, \tag{6.39}$$

or, equivalently, in differential form notation

$$\begin{aligned} \int dV\ q^{nN} f(q^n x) >&= \int dV\ f(x), \\ \bar{\int} d\bar{V}\ q^{nN} f(q^n x) &= \bar{\int} d\bar{V}\ f(x). \qquad n \in \mathbf{Z}. \end{aligned} \tag{6.40}$$

Relation (6.40) states that under the change of integration variables $x \to ax$ with $a = q^n$ the integrals $\int, \bar{\int}$ are invariant if we let dV transform according to

$dV \to a^N dV$, namely like $d^N x$. This explains the name "scaling property " for relations (6.30),(6.31),(6.39), (6.40). In both the classical and the q-deformed case this property characterizes the integrals satisfying Stoke's theorem; for q=1 the latter reduce to the usual Riemann integral, which has a "homogeneous " (i.e. translation invariant) measure.

One can now ask if the scaling property holds even if the dilatation parameter $a \notin Q := \{q^n,\ n \in \mathbf{Z}\}$. One can easily check that this is not the case (take for instance $f_0 = exp_{q^2}[-b(xCx)]$ and define $f = exp_{q^2}[-ab(xCx)]$ to be the function which we want to integrate by choosing f_0 as reference function). In other terms, the function

$$F(a) := < f(ax) > a^N \tag{6.41}$$

is periodic in the variable $b = ln(a)$ with period $ln(q)$, but is not identically zero. To be specific, assume for instance $q > 1$. If $a \in \mathbf{R}^+$ and $q^n < a < q^{n+1}$ the function F fluctuates around the value $F(1) = < f(x) >$, the width of the fluctuation being the same $\forall n \in \mathbf{Z}$, therefore also around large a. But if we take the deformation parameter q very close to 1, then Q is, so to say, "almost dense " in $\mathbf{R}^+$, i.e. a can be approximated quite well by an element of Q. In other terms, at a macroscopic scale (i.e. for $a \in \mathbf{R}^+$ such that $|\frac{ln(a)}{\ln(q)}| \gg 1$) deviations from the classical scaling property would not be detectable, even though they would be relevant at microscopic ones (i.e. for $a \in \mathbf{R}^+$, $|\frac{ln(a)}{\ln(q)}| \sim 1$). This surprising feature might be considered as a very interesting indication of the occurrence of a dishomogeneity of the observable properties of space when the usual euclidean commutative space is replaced by the corresponding quantum space.

Chapter 4

The Hilbert Space of the Harmonic Oscillator on $\mathbf{R}_q^N$

7. The pre-Hilbert space of the harmonic oscillator and the observables R^2, P^2, H_ω

We introduce the pre-Hilbert space $\mathcal{H}$ of the $SO_q(N)$-symmetric (isotropic) harmonic oscillator with characteristic constant ω in the following way. Let $|0>$ be the ground state with the energy E_0 given in formula (3.12). We introduce a direct (Π, V) and a barred $(\bar{\Pi}, \bar{V})$ representation by first assuming

$$\begin{array}{ccl} & & exp_{q^2}[-\frac{\omega q^{-N}(xCx)}{\mu}] \in V \\ & \overset{\Pi}{\nearrow} & \\ |0> \in \mathcal{H} & & \\ & \overset{\bar{\Pi}}{\searrow} & \\ & & exp_{q^{-2}}[-\frac{\omega q^{N}(xCx)}{\bar{\mu}}] \in \bar{V}. \end{array} \tag{7.1}$$

Up to a normalization factor, creation and destruction operators $A^{i\dagger}, A^i$ are to be represented respectively by

$$\begin{array}{ccl} & & (x^i + \alpha_n \partial^i) G_q \equiv a_n^{i\dagger} \\ & \overset{\Pi}{\nearrow} & \\ A^{i\dagger} & & \\ & \overset{\bar{\Pi}}{\searrow} & \\ & & (x^i + \bar{\alpha}_n \bar{\partial}^i) G_{q^{-1}} \equiv \bar{a}_n^{i\dagger} \end{array} \tag{7.2}$$

when acting on states of level $(n-1)$ (to give states of level n), and by

$$\begin{array}{ccl} & & (x^i + \alpha'_n \partial^i) G_q \\ & \overset{\Pi}{\nearrow} & \\ A^{i} & & \\ & \overset{\bar{\Pi}}{\searrow} & \\ & & (x^i + \bar{\alpha}'_n \bar{\partial}^i) G_{q^{-1}} \end{array} \tag{7.3}$$

when acting on states of level $(n-1)$ (to give states of level $(n-2)$); here

$$\alpha_n = -\frac{q^{2-n}}{\omega} \qquad \bar{\alpha}_n = -\frac{q^{n-2}}{\omega}$$

$$\alpha'_n = \frac{q^{N+n}}{\omega} \qquad \bar{\alpha}'_n = \frac{q^{-n-N}}{\omega}, \tag{7.4}$$

and the operator G_q was defined in formula (3.24). The space $\mathcal{H}_n$ of states of level n will be introduced as linear span of the vectors

$$|i_n, i_{n-1}, ...i_1> := A^{i_n\dagger} A^{i_{n-1}\dagger} ... A^{i_1\dagger}|0> . \tag{7.5}$$

The vector $|i_n, ...i_1>$ can be assigned the $SO_q(N, \mathbf{R})$ transformation law

$$\phi_l(|i_n, ...i_1>) = T^{i_n}_{j_n} ... T^{i_1}_{j_1} \otimes |j_n, ...j_1> \tag{7.6}$$

since both $\psi_n^{i_n...i_1}$ and $\bar{\psi}_n^{i_n...i_1}$ have transformation laws of this kind. Any $|u> \in \mathcal{H}_n$ is an eigenvector with eigenvalue

$$E_n = \omega \frac{1}{2}(q^{\frac{N}{2}-1} + q^{1-\frac{N}{2}})[\frac{N}{2} + n]_q, \qquad n \geq 0 \tag{7.7}$$

of the hamiltonian H_ω, which is represented by

$$\begin{array}{ccl} & & h_\omega = \frac{1}{2}(-q^N \Delta + \omega^2(xCx)) \\ & \overset{\Pi}{\nearrow} & \\ H_\omega & & \\ & \overset{\bar{\Pi}}{\searrow} & \\ & & h_\omega = \frac{1}{2}(-q^{-N}\bar{\Delta} + \omega^2(xCx)); \end{array} \tag{7.8}$$

$\mathcal{H}$ itself is defined as

$$\mathcal{H} := \bigoplus_{n=1}^{\infty} \mathcal{H}_n. \tag{7.9}$$

By the above construction any vector $|u> \in \mathcal{H}$ will be represented both by a vector $\psi_u \in V$ and by a vector $\bar{\psi}_u \in \bar{V}$ (see Sect. 2.):

$$\begin{array}{ccl} & & \psi_u \\ & \overset{\Pi}{\nearrow} & \\ |u> & & \\ & \overset{\bar{\Pi}}{\searrow} & \\ & & \bar{\psi}_u. \end{array} \tag{7.10}$$

With reference to the notation of Sect. 2., we know that any function of the type $\psi_u = P_n(x)exp_{q^2}[-\frac{\omega q^{-n-N-2m}xCx}{\mu}]$ belongs to V. From the above construction the corresponding $\bar{\psi}_u := \bar{\Pi}\Pi^{-1}\psi_u \in \bar{V}$ will be of the form $\bar{\psi}_u = \bar{P}_n(x)exp_{q^{-2}}[-\frac{\omega q^{+n+N+2m}xCx}{\bar{\mu}}]$ where the polynomial $\bar{P}_n(x)$ is obtained from $P_n(x)$ by the following steps: 1) writing $P_n(x)exp_{q^2}[-\frac{\omega q^{-n-N-2m}xCx}{\mu}]$ as a combinations of ψ_m's; 2) replacing ψ_m's by $\bar{\psi}_m$'s. If we consider the explicit form of $\psi_m, \bar{\psi}_m$ involving only the coordinates (without derivatives) the second step amounts to the substitutions $q \leftrightarrow q^{-1}$, $\hat{R} \leftrightarrow \hat{R}^{-1}$; in particular if the $\hat{R}, \hat{R}^{-1}$ matrices are written in terms of the projectors $\mathcal{P}_S, \mathcal{P}_A, \mathcal{P}_1$ alone, then we only need to interchange q with q^{-1}. Let us consider for instance $\psi_{u^i} := x^i exp_{q^2}[-\frac{\omega q^{-1-N}xCx}{\mu}]$, $\psi_{u^{ij}} := x^i x^j exp_{q^2}[-\frac{\omega q^{-2-N}xCx}{\mu}] \equiv P_2^{ij}(x)exp_{q^2}[-\frac{\omega q^{-2-N}xCx}{\mu}]$. The reader can easily verify that

$$\bar{\psi}_{u^i} = q^N exp_{q^{-2}}[-\frac{\omega q^{1+N}xCx}{\bar{\mu}}]$$

$$\bar{\psi}_{u^{ij}} \equiv \bar{P}_2^{ij}(x)exp_{q^{-2}}[-\frac{\omega q^{2+N}xCx}{\bar{\mu}}] = exp_{q^{-2}}[-\frac{\omega q^{2+N}xCx}{\bar{\mu}}]\{q^{2N+3}x^i x^j +$$

$$+C^{ij}[\frac{1}{\omega}(1-q^{N+1}+q^{-1}\frac{q^N-1}{q^{-N}-1})+xCx\frac{q_2-1}{\bar{\mu}}(q^{2N+3}-q^{N+2}+\frac{q^{-4}-1}{q^{-N}+1})]\}; \quad (7.11)$$

notice that the fact that $P_2^{ij}(x)$ contains no term of zero order in x doesn't imply that also in $\bar{P}_2^{ij}(x)$ no such a term is present (except when q=1).

From the above correspondence rule we immediately realize that if $\bar{\psi}_u = \bar{\Pi}\Pi^{-1}\psi_u$, then $xCx\bar{\psi}_u = \bar{\Pi}\Pi^{-1}(xCx\psi_u)$. This means that the square lenght operator R^2 can be defined over $\mathcal{H}$ and represented by

$$R^2 \quad \begin{array}{l} \overset{\Pi}{\nearrow} \quad xCx \\ \\ \underset{\bar{\Pi}}{\searrow} \quad xCx. \end{array} \qquad (7.12)$$

From (7.7), (7.11) we see that $P^2 := H_\omega - \omega^2 R^2$ is a well defined operator $\mathcal{H}$ represented by

$$P^2 \quad \begin{array}{l} \overset{\Pi}{\nearrow} \quad -q^N \Delta \\ \\ \underset{\bar{\Pi}}{\searrow} \quad -q^{-N}\bar{\Delta} \end{array} \quad ; \qquad (7.13)$$

it will obviously be called the square momentum, since it reduces to the classical square momentum for q=1.

We define the scalar product of two vectors $|v>, |u> \in \mathcal{H}$ by

$$(u,v) := < \bar{\psi}_u^* \psi_v > + \ll \psi_u^* \bar{\psi}_v \gg . \tag{7.14}$$

Indeed (,) is manifestly sesquilinear and

$$(v,u)^* = < \bar{\psi}_v^* \psi_u >^* + \ll \psi_v^* \bar{\psi}_u \gg^* = \ll \psi_u^* \bar{\psi}_v \gg + < \bar{\psi}_u^* \psi_v > = (u,v) \tag{7.15}$$

as required (see also relation (5.15)). Relation (7.15) implies that $(u,u) \in \mathbf{R}$; its positivity (i.e. $(u,u) \geq 0$ and $(u,u) = 0 \Leftrightarrow u = 0$) $\forall q \in \mathbf{R}^+$ will be proved in Sect. 7. Here we just note that it must hold at least in a ($|u>$-dependent) neighbourhood of q=1, as it holds for q=1 and (u,u) is a continuous function of q. The abstract definition of the hermitean conjugate $T^\dagger$ of an operator T is the usual one

$$(u, Tv) = (T^\dagger u, v) \tag{7.16}$$

We have chosen for the scalar product the (apparently cumbersome) form (7.15) to make the operators R^2, P^2 (and therefore H_ω itself) hermitean. It is trivial to check that R^2 is hemitean, so let us check P^2 is. Using the notation introduced in formula (2.46)

$$\Delta f = f'(x) + f_j(x,\partial)\partial^j := \Delta f| + f_j(x,\partial)\partial^j,$$

$$\bar{\Delta} g = \bar{g}'(x) + \bar{g}_j(x,\bar{\partial})\bar{\partial}^j := \bar{\Delta} g| + \bar{g}_j(x,\bar{\partial})\bar{\partial}^j, \qquad f,g \in O_q^N(\mathbf{R}) \tag{7.17}$$

and the relation $\bar{\Delta} = q^{2N}\Delta^*$ it is straightforward to show that

$$(\Delta f|)^* g := f^{'*} g = f^*(q^{-2N}\bar{\Delta} g|) - \partial^{j*} f_j^* g|$$

$$(\bar{\Delta} f|)^* g := \bar{f}^{'*} g = f^*(q^{2N}\Delta g|) - \bar{\partial}^{j*} \bar{f}_j^* g|. \tag{7.18}$$

Hence

$$(u, P^2 v) = -q^N < \bar{\psi}_u^* \Delta \psi_v| > -q^{-N} \ll \psi_u^* \bar{\Delta} \bar{\psi}_v| \gg \tag{7.19}$$

and

$$(P^2 u, v) = -q^{-N} < (\bar{\Delta}\bar{\psi}_u|)^* \psi_v > -q^N \ll (\Delta \psi_u|)^* \bar{\psi}_v \gg =$$

$$= -q^N < \bar{\psi}_u^* \Delta \psi_v| > -q^{-N} \ll \psi_u^* \bar{\Delta} \bar{\psi}_v| \gg +$$

$$+q^{-N} < \bar{\partial}^{j*}\bar{\psi}^*_{uj}\psi_v| > +q^N \ll \partial^{j*}\psi^*_{uj}\bar{\psi}_v| \gg; \tag{7.20}$$

The last two terms vanish because of Stoke's theorem (5.7) (in fact $\bar{\partial}^{j*}, \partial^{j*}$ are respectively derivatives of ∂ and $\bar{\partial}$ type), therefore

$$(u, P^2 v) = (P^2 u, v) \tag{7.21}$$

as claimed. As an immediate consequence of the hermiticity of the hamiltonian

$$(u, H_\omega v) = (H_\omega u, v), \tag{7.22}$$

if $|u>, |v>$ are two eigenvectors of H_ω with different eigenvalues, then

$$(u, v) = 0. \tag{7.23}$$

Looking back at the previous proof we see that in fact a stronger property holds:

$$n \neq m \quad \Rightarrow \qquad < \bar{\psi}_n^* \psi_m >= 0, \qquad \ll \psi_n^* \bar{\psi}_m \gg = 0 \psi_p \in \Psi_p, \quad \bar{\psi}_p \in \bar{\Psi}_p \tag{7.24}$$

For the evaluation of the scalar products (,) it is only necessary to find out integrals of the type $< (xCx)^k f(xCx) >$ with

$$f = exp_{q^{-2}}[-\frac{\omega q^{N+k} xCx}{\bar{\mu}}] exp_{q^2}[-\frac{\omega q^{-N-k} xCx}{\mu}], \tag{7.25}$$

since their tensor structure is already determined by the general knowledge of the tensors $S^{i_1 \dots i_{2n}}$, $\bar{S}^{i_1 \dots i_{2n}}$; this will be done in Appendix A.

For later use we derive the formulas

$$< (\bar{a}^{i\dagger}_{n+1} \bar{a}^{j\dagger}_n \bar{\psi})^* \psi >= q^{-1-N} < \bar{\psi}^* a^{j'}_{n-1} a^{i'}_n \psi > C_{j'j} C_{i'i} \tag{7.26}$$

$$\ll (a^{i\dagger}_{n+1} a^{j\dagger}_n \psi)^* \bar{\psi} \gg = q^{1+N} \ll \psi^* \bar{a}^{j'}_{n-1} \bar{a}^{i'}_n \bar{\psi} \gg C_{j'j} C_{i'i}. \tag{7.27}$$

They can be proved using the definitions (7.2), (7.3) of $a, \bar{a}, a^\dagger, \bar{a}^\dagger$ and the scaling property (6.31) of the integrations $< \ >, \ll \ \gg$.

In the classical case the N coordinate operators Y^i are a complete set of commuting observables; their action on a wave-function $\psi(y)$ is purely multiplicative

$$Y^i \psi = y^i \psi, \qquad\qquad q = 1. \tag{7.28}$$

In formula (7.28) new real coordinates $y^i = y^{i*}$ have been introduced (recall the fact that the coordinates x^i's are not real even for q=1) through a nondegenerate linear transformation A; this can be done also for $q \neq 1$, and in general such a transformation will take the form:

$$y^i := A^i_j x^j, \qquad y^{i*} = y^i \ \Leftrightarrow\ A^i_j = C_{jk} A^{i*}_k = A^{i*}_{j'} q^{\rho_j}, \tag{7.29}$$

where $\rho_j$is given in (1.10) and $j' := N+1-j$. Nevertheless, for $q \neq 1$ no coordinate operator acting purely multiplicatively in both the Π and the $\bar{\Pi}$ representation exists, as a glance to examples (7.11) shows. In fact, in that case $y^i \psi_{uj} = A^i_l \psi_{u^{lj}}$ does not contain terms of degree zero in x (or, equivalently, in y), whereas the corresponding vector $A^i_l \bar{\psi}_{u^{lj}}$ in the $\bar{\Pi}$ representation does, namely a purely multiplicative action in the Π representaion is not represented as such in the $\bar{\Pi}$ one, and viceversa. Then it is hard to imagine how one could define a set of N hermitean operators corresponding to the coordinates. The same sort of difficulty arises for the momenta and, as we shall see, for the components of the angular momentum, whereas we will be able to define an hermitean square angular momentum (Sect. 6). In other terms, so far we have not succeded in finding observables being covariant but not invariant (w.r.t. $SO_q(N,\mathbf{R})$. We don't know whether this difficulty can be overcome. If not, such a circumstance may be considered frustrating for finite dimensional systems like the one considered in this paper; in a gauge field theory it would not be a problem, since all observables are gauge-invariant.

8. The observable L^2

We look for some other hermitean operators such that they commute with the hamiltonian H_ω and with each other. To this end in this section we search the analog of the angular momenta. As a primary requirement they should commute with any scalar function of the coordinates and of the momenta. In the classical case they are antisymmetrized products of coordinates and derivatives of the type $\frac{1}{i}(y^i\partial^{y^j} - y^j\partial^{y^i})$ or their combinations. Therefore we first look at the commutation relations of the operators $\mathcal{L}^{ij} := \mathcal{P}_{A}{}^{ij}_{hk} x^h \partial^k = -q^{-2}\mathcal{P}_{A}{}^{ij}_{hk}\partial^h x^k$ and $\bar{\mathcal{L}}^{ij} = \mathcal{P}_{A}{}^{ij}_{hk} x^h \bar{\partial}^k = -q^2\mathcal{P}_{A}{}^{ij}_{hk}\bar{\partial}^h x^k$ with xCx, Δ and $xCx, \bar{\Delta}$ respectively. Using formulas (2.43), (2.44),(1.38),(2.31), (2.32) we find

$$\mathcal{L}^{ij} xCx = q^2 xCx \mathcal{L}^{ij}, \qquad \mathcal{L}^{ij}\Delta = q^{-2}\Delta\mathcal{L}^{ij} \tag{8.1}$$

and

$$\bar{\mathcal{L}}^{ij}xCx = q^{-2}xCx\bar{\mathcal{L}}^{ij}, \qquad \bar{\mathcal{L}}^{ij}\bar{\Delta} = q^2\bar{\Delta}\bar{\mathcal{L}}^{ij} \tag{8.2}$$

respectively. It immediately follows that

$$[G_{q^2}\mathcal{L}^{ij}, xCx] = 0 = [G_{q^2}\mathcal{L}^{ij}, \Delta] \tag{8.3}$$

$$[G_{q^{-2}}\bar{\mathcal{L}}^{ij}, xCx] = 0 = [G_{q^{-2}}\bar{\mathcal{L}}^{ij}, \bar{\Delta}], \tag{8.4}$$

where G_q was defined in (3.24). Next, it is easy to show that $G_{q^2}\mathcal{L}^{ij}$ (resp. $G_{q^{-2}}\bar{\mathcal{L}}^{ij}$) commutes with any scalar polynomial I (resp. $\bar{I}$) obtained combining x^i's and ∂^i's (resp. x^i's and $\bar{\partial}^i$'s):

$$[I(x,\partial), G_{q^2}\mathcal{L}^{ij}] = 0, \qquad [\bar{I}(x,\bar{\partial}), G_{q^{-2}}\bar{\mathcal{L}}^{ij}] = 0 \tag{8.5}$$

Actually any such polynomial can be written as a polynomial in xCx, Δ (resp. xCx, $\bar{\Delta}$) alone (see Appendix B).

By squaring the $\mathcal{L}^{ij}$, $\bar{\mathcal{L}}^{ij}$ we obtain scalar operators $\mathcal{L}^2$, $\bar{\mathcal{L}}^2$:

$$\mathcal{L}^2 := \mathcal{L}^{ij}\mathcal{L}_{ji} = x^h\partial^k\mathcal{P}_A{}^{ij}{}_{hk}x_j\partial_i \tag{8.6}$$

$$\bar{\mathcal{L}}^2 := \bar{\mathcal{L}}^{ij}\bar{\mathcal{L}}_{ji} = x^h\bar{\partial}^k\mathcal{P}_A{}^{ij}{}_{hk}x_j\bar{\partial}_i \tag{8.7}$$

To obtain the last expressions in (8.6), (8.7), we have used the property (1.19), and $\mathcal{P}_A^2 = 0$. Of course $G_{q^4}\mathcal{L}^2$ (resp. $G_{q^{-4}}\bar{\mathcal{L}}^2$) commutes with any scalar function $I(x,\partial)$ (resp. $I(x,\bar{\partial})$) and in particular with h_ω (resp. with $\bar{h}_\omega$).

We want to find out eigenvalues and eigenfunctions of $G_{q^4}\mathcal{L}^2$ (resp. $G_{q^{-4}}\bar{\mathcal{L}}^2$) in V (resp. in $\bar{V}$). From the above property it is clear that if $P(x)$ is an eigenvector of $G_{q^4}\mathcal{L}^2$ (resp. $G_{q^{-4}}\bar{\mathcal{L}}^2$) , then for any function $f = f(xCx)$ $g := P(x)f(xCx)$ is an eigenvector of $G_{q^4}\mathcal{L}^2$ (resp. $G_{q^{-4}}\bar{\mathcal{L}}^2$) with the same eigenvalue. A little thinking will convince the reader that, just as in the classical case, after factorizing a possible function $p(xCx)$ the eigenvectors $P(x)$ (resp. $\bar{P}(x)$) of $G_{q^4}\mathcal{L}^2$ (resp. of $G_{q^{-4}}\bar{\mathcal{L}}^2$)

$$G_{q^4}\mathcal{L}^2P(x)| = cP(x), \qquad G_{q^{-4}}\bar{\mathcal{L}}^2\bar{P}(x)| = \bar{c}\bar{P}(x) \tag{8.8}$$

can be written as homogeneous polynomials:

$$P(x) = p(xCx)A_{i_1i_2\ldots i_n}x^{i_1}x^{i_2}\ldots x^{i_n} \tag{8.9}$$

$$\bar{P}(x) = p'(xCx)\bar{A}_{i_1i_2\ldots i_n}x^{i_1}x^{i_2}\ldots x^{i_n}. \tag{8.10}$$

Actually $G_{q^4}\mathcal{L}^2$ (resp. $G_{q^{-4}}\bar{\mathcal{L}}^2$) is homogeneous in both x, ∂ (resp. x, $\bar{\partial}$) with the same degree (two), hence it transforms any homogeneous polynomial of degree n into another one of the same degree. Now we specify the form of the $A_{i_1i_2...i_n}$ (resp. $\bar{A}_{i_1i_2...i_n}$) coefficients. We are going to prove that just as in the classical case and up to factors $f(xCx)$ the set of homogeneous polynomials

$$M_{n,0}^S = \{P_S^{l_1l_2...l_n} := \mathcal{P}_{n,S}{}^{l_1l_2...l_n}_{i_1i_2...i_n}x^{i_1}x^{i_2}...x^{i_n}, \quad l_i = 1,...,N\} \tag{8.11}$$

is a complete set of eigenvectors of degree n of both $G_{q^4}\mathcal{L}^2$ and $G_{q^{-4}}\bar{\mathcal{L}}^2$. Here $\mathcal{P}_{n,S}$ is the q-deformed symmetric projector acting on $\bigotimes^n \mathbf{C}$ (in particular $\mathcal{P}_{2,S} \equiv \mathcal{P}_S$), whose existence will be briefly discussed in Appendix C. The main property of these projectors is that

$$\begin{aligned}
&\mathcal{P}_{n,S}\mathcal{P}_{A\ i,(i+1)} = 0 = \mathcal{P}_{n,S}\mathcal{P}_{1\ i,(i+1)}, && 1 \le i \le n-1\\
&\mathcal{P}_{n,S}\hat{R}^{\pm1}_{i,(i+1)}\mathcal{P}_{A\ (i+1),(i+2)} = 0 = \mathcal{P}_{n,S}\hat{R}^{\pm1}_{i,(i+1)}\mathcal{P}_{1\ (i+1),(i+2)} && 1 \le i \le n-2\\
&........
\end{aligned} \tag{8.12}$$

where, for any matrix F defined on $C \otimes C$ $F_{i,i+1}$ $(1 \le i \le n-1)$ is the matrix acting on $\otimes^n C$ defined by $F_{i,i+1} := \mathbf{1} \otimes ...\mathbf{1} \otimes F \otimes \mathbf{1} \otimes ... \otimes \mathbf{1}$ (F at the i_{th} and $(i+1)^{th}$ place). Since $\mathcal{P}_{n,S}^T = \mathcal{P}_{n,S}$, the above properties hold also if we multiply $\mathcal{P}_{n,S}$ by

$$\mathcal{P}_{A\ i,(i+1)},\ \mathcal{P}_{1\ i,(i+1)},\ \mathcal{P}_{A\ (i+1),(i+2)}\hat{R}^{\pm1}_{i,(i+1)},\ \mathcal{P}_{A\ (i+1),(i+2)}\hat{R}^{\pm1}_{i,(i+1)}, ... \tag{8.13}$$

from the left. Relations (8.12) imply

$$\begin{aligned}
&\mathcal{P}_{n,S}\mathcal{P}_{s\ i,(i+1)} = \mathcal{P}_{n,s} && 1 \le i \le n-1\\
&\mathcal{P}_{n,S}\hat{R}^{\pm1}_{i,(i+1)}\mathcal{P}_{S\ (i+1),(i+2)} = \mathcal{P}_{n,S}q^{\pm1} && 1 \le i \le n-2\\
&........
\end{aligned} \tag{8.14}$$

To reach the goal we first transform $\mathcal{L}^2$, $\bar{\mathcal{L}}^2$ into more suitable forms, which explicitly show their scalar character. By quite a lenghty calculation one can show that

$$\begin{aligned}
\mathcal{L}^2 &= \alpha_N(q)x^i\partial_i + \beta_N(q)x^ix^j\partial_j\partial_i + \gamma_N(q)xCx\Delta\\
\bar{\mathcal{L}}^2 &= \alpha_N(q^{-1})x^i\bar{\partial}_i + \beta_N(q^{-1})x^ix^j\bar{\partial}_j\bar{\partial}_i + \gamma_N(q^{-1})xCx\bar{\Delta}
\end{aligned} \tag{8.15}$$

where

$$\alpha_N(q) := \frac{(q^{2-\frac{N}{2}} + q^{\frac{N}{2}-2})}{(q^{1-\frac{N}{2}} + q^{\frac{N}{2}-1})}\frac{(q^{1-N} - q^{N-1})}{(q^{-2} - q^2)},$$

$$\beta_N(q) := \frac{q^3 + q^{N-1}}{\mu(q + q^{-1})}, \qquad \gamma_N(q) := -\frac{(q^{5-N} + q)(1 + q^{-N})}{\mu^2(q + q^{-1})}. \tag{8.16}$$

Now notice that property (8.12) together with the relations

$$\mathcal{P}_S{}^{ij}_{hk} x^h \partial^k = \mathcal{P}_S{}^{ij}_{hk} \partial^h x^k, \qquad \mathcal{P}_S{}^{ij}_{hk} x^h \bar{\partial}^k = \mathcal{P}_S{}^{ij}_{hk} \bar{\partial}^h x^k \tag{8.17}$$

implies

$$\mathcal{P}_{n,S}{}^{a_1a_2\ldots a_n}_{b_1b_2\ldots b_n} x^{b_1}\ldots x^{b_{i-1}} \partial^{b_i} x^{b_{i+1}}\ldots x^{b_n}| = 0$$

$$\mathcal{P}_{n,S}{}^{a_1a_2\ldots a_n}_{b_1b_2\ldots b_n} x^{b_1}\ldots x^{b_{i-1}} \bar{\partial}^{b_i} x^{b_{i+1}}\ldots x^{b_n}| = 0, \quad 1 \le i \le n. \tag{8.18}$$

Similarly, upon use of formulas (2.43), (2.44),

$$\mathcal{P}_{n,S}{}^{a_1a_2\ldots a_n}_{b_1b_2\ldots b_n} x^{b_1}\ldots x^{b_{i-1}} \Delta x^{b_i}\ldots x^{b_n}| = 0$$

$$\mathcal{P}_{n,S}{}^{a_1a_2\ldots a_n}_{b_1b_2\ldots b_n} x^{b_1}\ldots x^{b_{i-1}} \bar{\Delta} x^{b_i}\ldots x^{b_n}| = 0, \quad 1 \le i \le n. \tag{8.19}$$

This means that when applying $\mathcal{L}^2$ (resp. $\bar{\mathcal{L}}^2$) to $\mathcal{P}_{n,S}{}^{l_1l_2\ldots l_n}_{i_1i_2\ldots i_n} x^{i_1}x^{i_2}\ldots x^{i_n}$ we can forget all the terms (which we will denote by dots) containing powers of Δ or where the index b_i of a derivative ∂^{b_i} (resp. $\bar{\partial}^{b_i}$) is contracted with an index of $\mathcal{P}_{n,S}$. Now let us consider only the case of $\mathcal{L}^2$ explicitly. The term with coefficient γ_N in the RHS of (8.15) can be ignored, whereas

$$(x^a\partial_a)x^{b_i} = x^{b_i} + q^2 x^{b_i}(x^a\partial_a) + \ldots$$

$$(x^a x^b \partial_b\partial_a)x^{b_i} = (1+q^2)x^{b_i}(x^a\partial_a) + q^4 x^{b_i}(x^a x^b\partial_b\partial_a) + \ldots, \tag{8.20}$$

Hence

$$G_{q^4}\mathcal{L}^2\mathcal{P}_{n.S}{}^{l_1l_2\ldots l_n}_{i_1i_2\ldots i_n} x^{i_1}x^{i_2}\ldots x^{i_n}| =$$

$$= q^{-2n}[\alpha_N x^a\partial_a + \beta_N x^a x^b\partial_b\partial_a]\mathcal{P}_{n.S}{}^{l_1l_2\ldots l_n}_{i_1i_2\ldots i_n} x^{i_1}x^{i_2}\ldots x^{i_n}| =$$

$$= q^{-2n}\{\alpha_N[x^{i_1}x^{i_2}\ldots x^{i_n} + q^2 x^{i_1}(x^a\partial_a)x^{i_2}\ldots x^{i_n}] + \beta_N[(1+q^2)x^{i_1}(x^a\partial_a)x^{i_2}\ldots x^{i_n}$$

$$+q^4 x^{i_1}(x^a x^b\partial_b\partial_a)x^{i_2}\ldots x^{i_n}]\}\mathcal{P}_{n.S}{}^{l_1l_2\ldots l_n}_{i_1i_2\ldots i_n} = \ldots. =$$

$$= q^{-2n}\{\alpha_N(1+q^2+\ldots+q^{2(n-1)}) + \beta_N(1+q^2)[(1+q^2+\ldots+q^{2(n-2)})+$$

$$+q^4(1+q^2+\ldots+q^{2(n-3)}) + \ldots + q^{4(n-2)}\cdot 1]\mathcal{P}_{n,S}{}^{l_1l_2\ldots l_n}_{i_1i_2\ldots i_n} x^{i_1}x^{i_2}\ldots x^{i_n} \tag{8.21}$$

$P_S^{l_1\dots l_n}(x)$ are therefore eigenvectors of $G_{q^4}\mathcal{L}^2$ with eigenvalues (depending only on n)

$$c_n := q^{-2n}\{\alpha_N n_{q^2} + \beta_N(1+q^2)[\sum_{h=0}^{n-2}(n-h-1)_{q^2}q^{4h}]\} =$$

$$= q^{-2n}\{n_{q^2}[\alpha_N + \beta_N \frac{q^{2(n-1)}-1}{q^2-1}]\}, \tag{8.22}$$

where we have used the formula $(2.47)_b$ and the relation

$$\sum_{k=1}^{n} q^{2k}k_{q^2} = q^2\frac{n_{q^2}(n+1)_{q^2}}{2_{q^2}}, \tag{8.23}$$

(the latter can be easily proved iteratively). Finally, replacing the explicit expressions (7. for α_N, β_N we find

$$c_n = \frac{q^{-2}}{q+q^{-1}}\frac{q^{\frac{N}{2}-2}+q^{2-\frac{N}{2}}}{q^{\frac{N}{2}-1}+q^{1-\frac{N}{2}}}[n]_q[N+n-2]_q. \tag{8.24}$$

In the same way we can show that the eigenvalues $\bar{c}_n$ of $G_{q^{-4}}\bar{\mathcal{L}}^2$ are given by

$$\bar{c}_n = \frac{q^{2}}{q+q^{-1}}\frac{q^{\frac{N}{2}-2}+q^{2-\frac{N}{2}}}{q^{\frac{N}{2}-1}+q^{1-\frac{N}{2}}}[n]_q[N+n-2]_q. \tag{8.25}$$

We see that the operators $2q^2G_{q^4}\mathcal{L}^2$ and $2q^{-2}G_{q^{-4}}\bar{\mathcal{L}}^2$ have the same eigenvectors $P_S(x)$ and the same eigenvalues

$$l_n^2 = \frac{2}{q+q^{-1}}\frac{q^{\frac{N}{2}-2}+q^{2-\frac{N}{2}}}{q^{\frac{N}{2}-2}+q^{1-\frac{N}{2}}}[n]_q[N+n-2]_q; \tag{8.26}$$

we have included in their definition a factor 2 so that for $N=3$ and q=1 the eigenvalues reduce to the classical ones $n(n+1)$ of the classical square angular momentum in three dimensions. Notice that, as the energies E_n, the eigenvalues (8.26) are invariant under the transformation $q \to q^{-1}$.

Consider M_k (the linear span of $\{x^{i_1}x^{i_2}\dots x^{i_k}\}$, see sect. 2) and its two projections

$$M_{k,0}^S = \mathcal{P}_{k,S}M_k \qquad M_k^1 = (\mathcal{P}_1\otimes\mathbf{1}_{k-2})M_k \tag{8.27}$$

Because of formula (8.12) their direct sum is M_k itself:

$$M_k = M_{k,0}^S \oplus M_k^1 = [\mathcal{P}_{k,S}\oplus(\mathcal{P}_1\otimes\mathbf{1}_{k-2})]M_k. \tag{8.28}$$

We can evaluate the dimensions of these subspaces in a straightforward way, since we know the dimension of M_l as a function of l (see formula (4.4)) and M_k^1 is generated by $\{x^{i_1}...x^{i_{k-2}}x^{j_{k-1}}x^{j_k}\mathcal{P}_1{}^{i_{k-1}i_k}_{j_{k-1}j_k}\}$, i.e. by $\{x^{i_1}...x^{i_{k-2}}(xCx)\}$:

$$dim(M_k^1) = dim(M_{k-2}) = \binom{N+k-3}{N-1}$$

$$dim(\mathcal{P}_k^S) := dim(M_k^S) = dim(M_k) - dim(M_k^1) = dim(M_k) - dim(M_{k-2}) \quad (8.29)$$

By repeated application of formula (8.28) we find

$$M_n = \bigoplus_{0\le m\le \frac{n}{2}} M^S_{n,n-2m} \qquad M^S_{n,n-2m} := (\mathcal{P}_1 \otimes ... \otimes \mathcal{P}_1 \otimes \mathcal{P}_{n-2m,S})M_n. \quad (8.30)$$

In other words

$$\mathbf{1}_{M_n} := \bigoplus_{0\le m\le \frac{n}{2}} (\mathcal{P}_1 \otimes ... \otimes \mathcal{P}_1 \otimes \mathcal{P}_{n-2m,S}) \quad (8.31)$$

is the identity operator on M_n. Relation (8.30) states that any eigenvector of $G_{q^4}\mathcal{L}^2$ (and $G_{q^{-4}}\bar{\mathcal{L}}^2$ belonging to M_n must belong to $M^S_{n,n-2m}$ for some $m \le \frac{n}{2}$, i.e it must be combination of the vectors $P_S^{l_1...l_{n-2m}}$ modulo a scalar factor $((xCx)^m)$, as claimed.

Since $[h_\omega, G_{q^4}\mathcal{L}^2] = 0$ (resp. $[\bar{h}_\omega, G_{q^{-4}}\bar{\mathcal{L}}^2] = 0$), it is possible to find eigenvectors of h_ω, $G_{q^4}\mathcal{L}^2$ (resp. $\bar{h}_\omega$, $G_{q^{-4}}\bar{\mathcal{L}}^2$) at the same time. Using again property (8.12) it is quite easy to realize that

$$(\mathcal{P}_{n,S}\psi_n)^{l_1l_2...l_n} \propto \mathcal{P}_{n,S}{}^{l_1l_2...l_n}_{i_1i_2...i_n} x^{i_1}x^{i_2}...x^{i_n} exp_{q^2}[-\frac{\omega q^{-n-N}xCx}{\mu}], \quad (8.32)$$

$$(\mathcal{P}_{n,S}\bar{\psi}_n)^{l_1l_2...l_n} \propto \mathcal{P}_{n,S}{}^{l_1l_2...l_n}_{i_1i_2...i_n} x^{i_1}x^{i_2}...x^{i_n} exp_{q^{-2}}[-\frac{\omega q^{+n+N}xCx}{\bar{\mu}}]; \quad (8.33)$$

therefore these functions are eigenfunctions respectively of $2q^2G_{q^4}\mathcal{L}^2$ and $2q^{-2}G_{q^{-4}}\bar{\mathcal{L}}^2$ with eigenvalue l_n^2. In general

$$[(\mathcal{P}_1 \otimes ... \otimes \mathcal{P}_1 \otimes \mathcal{P}_{n-2m,S})\psi_n]^{l_1...l_n} \propto$$

$$\propto \mathcal{P}_{(n-2m),S}{}^{l_{2m+1}...l_n}_{i_{2m+1}...i_n} x^{i_{2m+1}}...x^{i_n} p_{n,m}(xCx) exp_{q^2}[-\frac{\omega q^{-n-N}xCx}{\mu}] \quad (8.34)$$

and

$$[(\mathcal{P}_1 \otimes ... \otimes \mathcal{P}_1 \otimes \mathcal{P}_{n-2m,S})\bar{\psi}_n]^{l_1...l_n} \propto$$

$$\propto \mathcal{P}_{(n-2m),S}\,{}^{l_{2m+1}\dots l_n}_{i_{2m+1}\dots i_n} x^{i_{2m+1}}\dots x^{i_n}\bar{p}_{n,m}(xCx)exp_{q^{-2}}[-\frac{\omega q^{n+N}xCx}{\bar{\mu}}] \tag{8.35}$$

(with suitable polynomials $p_{n,m}, \bar{p}_{n,m}$, $0 \leq m \leq \frac{n}{2}$) are respectively eigenvectors of $2q^2 G_{q^4}\mathcal{L}^2$ and $2q^{-2}G_{q^{-4}}\bar{\mathcal{L}}^2$ with eigenvalue l^2_{n-2m}. Using the property (4.13) $(\mathcal{P}_A\psi_n = 0)$ we see that $\mathbf{1}_{M_n}$ is the identity operator in $\Psi_n, \bar{\Psi}_n$. Therefore

$$\Psi_n = \bigoplus_{0\leq m\leq\frac{n}{2}} \Psi_{n,n-2m} \qquad (resp. \quad \bar{\Psi}_n = \bigoplus_{0\leq m\leq\frac{n}{2}} \bar{\Psi}_{n,n-2m}) \tag{8.36}$$

where

$$\Psi_{n,n-2m} := (\mathcal{P}_1 \otimes \dots \otimes \mathcal{P}_1 \otimes \mathcal{P}_{n-2m,S})\Psi_n$$

$$(resp. \quad \bar{\Psi}_{n,n-2m} := (\mathcal{P}_1 \otimes \dots \otimes \mathcal{P}_1 \otimes \mathcal{P}_{n-2m,S})\Psi_n) \tag{8.37}$$

is the eigenspace of $h_\omega, 2q^2G_{q^4}\mathcal{L}^2$ (resp. of $\bar{h}_\omega, 2q^{-2}G_{q^{-4}}\bar{\mathcal{L}}^2$) with eigenvalues E_n, l^2_{n-2m}.

The above discussion shows that we are in the right condition to define a square angular momentum operator L^2 in $\mathcal{H}$. We set

$$L^2 \begin{array}{l} \overset{\Pi}{\nearrow} \; 2q^2G_{q^4}\mathcal{L}^2 \\ \underset{\bar{\Pi}}{\searrow} \; 2q^{-2}G_{q^{-4}}\bar{\mathcal{L}}^2 \end{array} \quad . \tag{8.38}$$

We introduce the subspaces $\mathcal{H}_{n,n-2m} \subset \mathcal{H}$ by

$$\mathcal{H}_{n,n-2m} \begin{array}{l} \overset{\Pi}{\nearrow} \; \Psi_{n,n-2m} \\ \bar{\Psi}_{n,n-2m} \; 2q^{-2}G_{q^{-4}}\bar{\mathcal{L}}^2 \end{array} \quad . \tag{8.39}$$

We summarize the preceding results in the

Proposition

The vectors

$$(\mathcal{P}_1 \otimes \dots \otimes \mathcal{P}_1 \otimes \mathcal{P}_{n-2m,S})^{l_1 l_2 \dots l_n}_{i_1 i_2 \dots i_n} |i_1 i_2 \dots i_n> \in \mathcal{H}_{n,n-2m} \tag{8.40}$$

$(n \geq 0, \quad 0 \leq m \leq \frac{n}{2})$ are eigenvectors of H_ω, L^2 with eigenvalues E_n, l^2_{n-2m} (see (7.7),(8.26)) respectively. Moreover

$$\mathcal{H} = \bigoplus_{n=0}^{\infty} \bigoplus_{0\leq m\leq\frac{n}{2}} \mathcal{H}_{n,n-2m} \tag{8.41}$$

We now show that L^2 is hermitean. Using formulas (2.41), (1.19) (where we take $f(\hat{R}) = \mathcal{P}_A$) it is easy to show that $\mathcal{L}^{2*}$ is related to $\bar{\mathcal{L}}^2$ by

$$\mathcal{L}^{2*} = q^{-2N-4}\bar{\mathcal{L}}^2. \tag{8.42}$$

From the definition (7.14) of the scalar product

$$(u, \frac{1}{2}L^2 v) = q^2 < \bar{\psi}_u^* G_{q^4} \mathcal{L}^2 \psi_v| > + q^{-2} \ll \psi_u^* G_{q^{-4}} \bar{\mathcal{L}}^2 \bar{\psi}_v| \gg \tag{8.43}$$

whereas

$$(\frac{1}{2}L^2 u, v) = q^{-2} < (G_{q^{-4}} \bar{\mathcal{L}}^2 \bar{\psi}_u|)^* \psi_v > + q^2 \ll (G_{q^4} \mathcal{L}^2 \psi_u|)^* \bar{\psi}_v \gg; \tag{8.44}$$

because of Stoke's theorems (5.7). We can rewrite the RHS of the latter formula as

$$RHS(8.44) = q^{2N+2} < (G_{q^{-4}} \bar{\psi}_u)^* \mathcal{L}^2 \psi_v| > q^{-2N-2} \ll (G_{q^4} \psi_u)^* \bar{\mathcal{L}}^2 \bar{\psi}_v| \gg =$$

$$= q^{2N+2} < G_{q^{-4}} (\bar{\psi}_u^* G_{q^4} \mathcal{L}^2 \psi_v|) > q^{-2N-2} \ll G_{q^4} (\psi_u^* G_{q^{-4}} \bar{\mathcal{L}}^2 \bar{\psi}_v|) \gg; \tag{8.45}$$

finally,using the fundamental property (6.31) of the integrals $< \;\; >$, $\ll \;\; \gg$ we get

$$(\frac{1}{2}L^2 u, v) = q^2 < \bar{\psi}_u^* G_{q^4} \mathcal{L}^2 \psi_v| > + q^{-2} \ll \psi_u^* G_{q^{-4}} \bar{\mathcal{L}}^2 \bar{\psi}_v| \gg = (u, L^2 v), \tag{8.46}$$

as claimed. The direct consequence of formulas (7.23),(8.46) is that $\mathcal{H}_{n,m}$ are orthogonal subspaces of $\mathcal{H}$, i.e.

$$u \in \mathcal{H}_{n,k}, \quad v \in \mathcal{H}_{n',k'} \quad and \quad (n,k) \neq (n',k') \Rightarrow (u,v) = 0. \tag{8.47}$$

Looking back at the previous proof we see that in fact a stronger property holds:

$$< \bar{\psi}_{n,k}^* \psi_{n',k'} > = 0 = \ll \psi_{n,k}^* \bar{\psi}_{n',k'} \gg \qquad if \quad (n,k) \neq (n',k'), \tag{8.48}$$

where $\psi_{p,h} \in \Psi_{p,h}$, $\bar{\psi}_{p,h} \in \bar{\Psi}_{p,h}$.

We have not succeded in defining (hermitean) components of the q-deformed angular momentum as operators on $\mathcal{H}$. The operators $G_{q^2}\mathcal{L}^{ij}$ and $G_{q^{-2}}\bar{\mathcal{L}}^{ij}$ have different eigenvalues, hence they cannot represent one and the same operator on $\mathcal{H}$.

9. Positivity of the scalar product

In this section we prove the positivity of the scalar product (,); in this way the proof that $\mathcal{H}$ is a pre-Hilbert space is finished. Then completion of $\mathcal{H}$ can be performed in the standard way.

The results of the preceding section imply that it is sufficient to prove positivity inside each subspace $\mathcal{H}_{n,n-2m}$. The most general $|u>\in \mathcal{H}_{n,k}$, $k = n-2m$, $0 \leq m \leq \frac{n}{2}$ is of the form

$$
\begin{array}{cc}
 & D_{l_1 l_2 \ldots l_k}\psi^{l_1 l_2 \ldots l_k}_{n,(k,S)} \\
 & \overset{\Pi}{\nearrow} \\
u & \\
 & \overset{\bar{\Pi}}{\searrow} \\
 & D_{l_1 l_2 \ldots l_k}\bar{\psi}^{l_1 l_2 \ldots l_k}_{n,(k,S)}
\end{array}
\qquad D_{l_1\ldots l_k} \in \mathbf{C} \tag{9.1}
$$

(see (8., (8.,), where

$$
\begin{aligned}
\psi^{l_1 l_2 \ldots l_k}_{n,(k,S)} &:= (a^\dagger_n C a^\dagger_{n-1})\ldots(a^\dagger_{k+2} C a^\dagger_{k+1})\psi^{l_1 l_2 \ldots l_k}_{k,S} \\
\bar{\psi}^{l_1 l_2 \ldots l_k}_{n,(k,S)} &:= (\bar{a}^\dagger_n C \bar{a}^\dagger_{n-1})\ldots(\bar{a}^\dagger_{k+2} C \bar{a}^\dagger_{k+1})\bar{\psi}^{l_1 l_2 \ldots l_k}_{k,S}
\end{aligned} \tag{9.2}
$$

and

$$
\begin{aligned}
\psi^{l_1 l_2 \ldots l_k}_{k,S} &:= \mathcal{P}_{k,S}{}^{l_1\ldots l_k}_{i_1\ldots i_k}\psi^{i_1\ldots i_k}_k = t_k(q)\mathcal{P}_{k,S}{}^{l_1\ldots l_k}_{i_1\ldots i_k} x^{i_1}\ldots x^{i_k} exp_{q^2}[-\frac{\omega q^{-k-N} xCx}{\mu}] \\
\bar{\psi}^{l_1 l_2 \ldots l_k}_{k,S} &:= \mathcal{P}_{k,S}{}^{l_1\ldots l_k}_{i_1\ldots i_k}\bar{\psi}^{i_1\ldots i_k}_k = t_k(q^{-1})\mathcal{P}_{k,S}{}^{l_1\ldots l_k}_{i_1\ldots i_k} x^{i_1}\ldots x^{i_k} exp_{q^{-2}}[-\frac{\omega q^{k+N} xCx}{\bar{\mu}}].
\end{aligned} \tag{9.3}
$$

Here $a_m C a_{m+1} := a^i_m C_{ij} a^j_{m+1}$ and $a_m i\dagger, a^i_m$ are the creation/destruction operators introduced in (7.2), (7.3). An easy calculation shows that $t_k(q)$ is given by

$$
t_k(q) = q^{\frac{1-k}{2}} \prod_{h=0}^{k-1}(1+q^{-N-2h}), \tag{9.4}
$$

therefore $t(q), t(q^{-1})$ are positive $\forall q \in \mathbf{R}^+$. In the rest of this section $a \propto b$ will mean $a = \sigma b$, $\sigma > 0$. The square norm of u is given by

$$
(u,u) = D^*_{p_1\ldots p_k} D_{l_1\ldots l_k}[< (\bar{\psi}^{p_1\ldots p_k}_{n,(k,S)})^* \psi^{l_1\ldots l_k}_{n,(k,S)} > + \ll (\psi^{p_1\ldots p_k}_{n,(k,S)})^* \bar{\psi}^{l_1\ldots l_k}_{n,(k,S)} \gg]. \tag{9.5}
$$

From the definition $(9.2)_a$ and formulas (7.26) it follows

$$< (\bar{\psi}^{p_1 \dots p_k}_{n,(k,S)})^* \psi^{l_1 \dots l_k}_{n,(k,S)} > \propto < (\bar{\psi}^{p_1 \dots p_k}_{k,S})^* f^{l_1 \dots l_k}_{k,S} > \tag{9.6}$$

where

$$f^{l_1 \dots l_k}_{k,S} := (a_k C a_{k+1}) \dots (a_{n-4} C a_{n-3})(a_{n-2} C a_{n-1}) \psi^{l_1 l_2 \dots l_k}_{n,(k,S)}. \tag{9.7}$$

Because of formula (7.26), only the component of $f^{l_1 \dots l_k}_{k,S}$ belonging to Ψ_k contributes to the integral (9.6). Looking at formula (A.10), we can decompose the operator $(a_{n-2} C a_{n-1})$ in the following way

$$(a_{n-2} C a_{n-1}) = \alpha_{n-1,2}(a^\dagger_{n+2} C a^\dagger_{n+1}) + \beta_{n-1,2}(a_n C a_{n+1}) + \gamma_{n-1,2}(a^\dagger_n C a_{n+1}) +$$
$$+ \delta_{n-1,2}(a_{n+2} C a^\dagger_{n+1}), \tag{9.8}$$

which is appropriate to clearly display the result of its action on Ψ_n: we see that it maps $\psi_{n,(k,S)}$ into a combination of functions $\psi'_{n+2}, \psi'_n, \psi'_{n-2}$ belonging respectively to $\Psi_{n+2}, \Psi_n, \Psi_{n-2}$. Next, the operator $(a_{n-4} C a_{n-3})$ acts on $\psi'_{n+2}, \psi'_n, \psi'_{n-2}$. For each of these three functions we choose the appropriate decomposition of $(a_{n-4} C a_{n-3})$. Doing the same job again and again, we end up with a combination of functions belonging to $\Psi_{2n-k}, \Psi_{2n-2-k}, \dots \Psi_k$. It is not difficult to realize that

$$\mathcal{P}_{\Psi_k}(f^{l_1 \dots l_k}_{k,S}) = \prod_{h=1}^{m} \beta_{n-2h+1,2}(a_{k+2} C a_{k+3}) \dots (a_n C a_{n+1}) \psi^{l_1 l_2 \dots l_k}_{n,(k,S)}, \tag{9.9}$$

where $\mathcal{P}_{\Psi_k}$ denotes the projector on Ψ_k. Since all coefficients $b_{l,m}$ are positive for $q \in \mathbf{R}^+$, by picking the explicit definition $(9.2)_a$ of $\psi_{n,(k,S)}$ we find

$$\mathcal{P}_{\Psi_k}(f^{l_1 \dots l_k}_{k,S}) \propto (a_{k+2} C a_{k+3}) \dots (a_n C a_{n+1})(a^\dagger_n C a^\dagger_{n-1}) \psi^{l_1 l_2 \dots l_k}_{n-2,(k,S)}. \tag{9.10}$$

In the appendix A it is proved that

$$(a_n C a_{n+1})(a^\dagger_n C a^\dagger_{n-1}) \psi^{l_1 l_2 \dots l_k}_{n-2,(k,S)} \propto \psi^{l_1 l_2 \dots l_k}_{n-2,(k,S)} \tag{9.11}$$

(see formula (A.12), (A.23)); hence

$$\mathcal{P}_{\Psi_k}(f^{l_1 \dots l_k}_{k,S}) \propto (a_{k+2} C a_{k+3}) \dots (a_{n-2} C a_{n-1}) \psi^{l_1 l_2 \dots l_k}_{n-2,(k,S)} \tag{9.12}$$

using $m = \frac{n-k}{2}$ times the same kind of argument we conclude that

$$\mathcal{P}_{\Psi_k}(f^{l_1 \dots l_k}_{k,S}) \propto \psi^{l_1 l_2 \dots l_k}_{k,S}. \tag{9.13}$$

From eq.'s (9.6), (7.24), (9.13), (6.18) it follows that

$$< (\bar{\psi}^{p_1\dots p_k}_{n,(k,S)})^* \psi^{l_1\dots l_k}_{n,(k,S)} > \propto < (\bar{\psi}_{k,S}^{p_1\dots p_k})^* \psi_{k,S}^{l_1\dots l_k} > \propto$$

$$\propto \mathcal{P}_{k,S}\,{}^{l_1\dots l_k}_{i_1\dots i_k} \mathcal{P}_{k,S}\,{}^{p_1\dots p_k}_{j_1\dots j_k} C^{h_1 j_1}\dots C^{h_k j_k}\cdot$$

$$< x^{h_k}\dots x^{h_1} x^{i_1}\dots x^{i_k} exp_{q^2}[-\frac{\omega q^{-k-N} xCx}{\mu}] exp_{q^{-2}}[-\frac{\omega q^{k+N} xCx}{\bar{\mu}}] > =$$

$$= \mathcal{P}_{k,S}\,{}^{l_1\dots l_k}_{i_1\dots i_k} \mathcal{P}_{k,S}\,{}^{p_1\dots p_k}_{j_1\dots j_k} C^{h_1 j_1}\dots C^{h_k j_k}\cdot$$

$$\frac{S^{h_k\dots h_1 i_1\dots i_k}}{\mathcal{S}_{2k}} < exp_{q^{-2}}[-\frac{\omega q^{k+N} xCx}{\bar{\mu}}](xCx)^k exp_{q^2}[-\frac{\omega q^{-k-N} xCx}{\mu}] > \tag{9.14}$$

Similarly one can show that

$$\ll (\psi^{p_1\dots p_k}_{n,(k,S)})^* \bar{\psi}^{l_1\dots l_k}_{n,(k,S)} \gg \propto$$

$$\propto \mathcal{P}_{k,S}\,{}^{l_1\dots l_k}_{i_1\dots i_k} \mathcal{P}_{k,S}\,{}^{p_1\dots p_k}_{j_1\dots j_k} C^{h_1 j_1}\dots C^{h_k j_k}\cdot$$

$$\frac{\bar{S}^{h_k\dots h_1 i_1\dots i_k}}{\bar{\mathcal{S}}_{2k}} < exp_{q^{-2}}[-\frac{\omega q^{k+N} xCx}{\bar{\mu}}](xCx)^k exp_{q^2}[-\frac{\omega q^{-k-N} xCx}{\mu}] > . \tag{9.15}$$

Since

$$< exp_{q^{-2}}[-\frac{\omega q^{k+N} xCx}{\bar{\mu}}](xCx)^k exp_{q^2}[-\frac{\omega q^{-k-N} xCx}{\mu}] > \; > 0 \tag{9.16}$$

(see formula (A.24)), $\mathcal{S}_2 k > 0$, $\bar{\mathcal{S}}_{2k} > 0$ (see formula (6.25),(6.26)), we can reduce the positivity of (u, u) to the positivity of the two quantities

$$D^*_{p_1 p_2\dots p_k} \mathcal{P}_{k,S}\,{}^{p_1 p_2\dots p_k}_{j_1 j_2\dots j_k} D_{l_1 l_2\dots l_k} \mathcal{P}_{k,S}\,{}^{l_1 l_2\dots l_k}_{i_1 i_2\dots i_k} C^{h_1 j_1} C^{h_2 j_2}\dots C^{h_k j_k} S^{h_k\dots h_1 i_1\dots i_k} \tag{9.17}$$

and

$$D^*_{p_1 p_2\dots p_k} \mathcal{P}_{k,S}\,{}^{p_1 p_2\dots p_k}_{j_1 j_2\dots j_k} D_{l_1 l_2\dots l_k} \mathcal{P}_{k,S}\,{}^{l_1 l_2\dots l_k}_{i_1 i_2\dots i_k} C^{h_1 j_1} C^{h_2 j_2}\dots C^{h_k j_k} \bar{S}^{h_k\dots h_1 i_1\dots i_k}. \tag{9.18}$$

We prove the positivity of (9.17); the proof of the positivity of (9.18) is completely analogous. First, using property (C.6),(C.10) of the symmetric projectors we can rewrite (9.17) in the following way

$$[(\otimes^k C)\cdot D]^T_{j_1\dots j_k} \mathcal{P}_{k,S}\,{}^{j_k\dots j_1}_{h_k\dots h_1} \mathcal{P}_{k,S}\,{}^{l_1\dots l_k}_{i_1\dots i_k} S^{h_k\dots h_1 i_1\dots i_k} D_{l_1 l_2\dots l_k}. \tag{9.19}$$

In appendix D we prove the following

Lemma:

$$\mathcal{P}_{k,S}{}^{j_k\ldots j_1}_{h_k\ldots h_1}\,\mathcal{P}_{k,S}{}^{l_1 l_2\ldots l_k}_{i_1 i_2\ldots i_k}\,S^{h_k\ldots h_1 i_1\ldots i_k} =$$

$$= \sigma_k[\mathcal{P}_{k,S}\cdot(\otimes^k C)\mathcal{P}_{k,S}]^{l_1\ldots l_k}_{j_1\ldots j_k}, \qquad \sigma_k > 0; \tag{9.20}$$

a similar result could be proved in the barred case. Using property (C.6) (c.10) again, we can rewrite the LHS of (9.20) as a sum of positive terms

$$\sum_{m_1,m_2,\ldots,m_k} |[(\otimes^k C)\cdot\mathcal{P}_{k,S}\cdot D]^{m_1 m_2\ldots m_k}|^2; \tag{9.21}$$

this expression is always ≥ 0 and is zero if and only if $\mathcal{P}_{k,S}\cdot D = 0 \;\Leftrightarrow\; u = 0$ (in fact $(\otimes^k C)$ is a nondegenerate matrix). The proof of the positivity of the expression (9.17) is thus completed, and so the positivity of $(\ ,\ )$,

$$(u,u)\geq 0 \qquad (u,u)=0 \;\Leftrightarrow\; u=0, \qquad u\in\mathcal{H} \tag{9.22}$$

is thoroughly demonstrated.

Now we can introduce a norm $\|\ \|$ in $\mathcal{H}$ by setting

$$\|u\|^2 = (u,u). \tag{9.23}$$

The completion $[\mathcal{H}]$ of $\mathcal{H}$ w.r.t. this norm can be performed in the standard way. It induces completions $[V]$, $[\bar{V}]\subset O_q^N(\mathbf{R})$ of $V,\bar{V}$. It would be interesting to investigate if the latter can be characterized in an intrinsic way, e.g. by characterizing their (formal) power expansion in x^i's. This is left as a possible subject for some future work.

Notes

(1) To do this job one has to manage q-series. We hope to report useful results in this direction elesewhere [15].

Appendix to Chapter 4

Appendix A

In this appendix we first show how to evaluate integrals of the type

$$< (xCx)^m exp_{q^{-2}}[-\frac{\omega q^{N+k} xCx}{\bar{\mu}}] exp_{q^2}[-\frac{\omega q^{-N-k} xCx}{\mu}] > \tag{A.1}$$

taking $f_0 := exp_{q^{-2}}[-\frac{\omega q^{N} xCx}{\bar{\mu}}] exp_{q^2}[-\frac{\omega q^{-N} xCx}{\mu}]$ as reference function. The outcoming results, together with formulas (4.18),(4.19), will allow the determination of all integrals involved in the scalar products of vectors of $\mathcal{H}$. Second, we give some results concerning the action of creation/destruction operators on functions $\psi \in V$.

We start from

$$< (xCx)^m exp_{q^{-2}}[-\frac{\omega q^{k+N-2} xCx}{\bar{\mu}}] exp_{q^2}[-\frac{\omega q^{-k-N} xCx}{\mu}] >=$$

$$= q^{N+2m} < (xCx)^m exp_{q^{-2}}[-\frac{\omega q^{k+N} xCx}{\bar{\mu}}] exp_{q^2}[-\frac{\omega q^{2-k-N} xCx}{\mu}] > \tag{A.2}$$

which is a direct consequence of the scaling property (4.31) of the integrals. Using the q-derivatives properties (2. of the exponentials to expand the functions $exp_{q^{-2}}[-\frac{\omega q^{N-2} xCx}{\bar{\mu}}]$, $exp_{q^2}[-\frac{\omega q^{2-N} xCx}{\mu}]$ we find

$$< (xCx)^{m+1} exp_{q^{-2}}[-\frac{\omega q^{k+N} xCx}{\bar{\mu}}] exp_{q^2}[-\frac{\omega q^{-k-N} xCx}{\mu}] >= (q^{1-\frac{N}{2}} + q^{\frac{N}{2}-1}) \cdot$$

$$\cdot [\frac{N}{2} + m]_q \frac{[m]_q}{[2m]_q} < (xCx)^m exp_{q^{-2}}[-\frac{\omega q^{k+N} xCx}{\bar{\mu}}] exp_{q^2}[-\frac{\omega q^{2-k-N} xCx}{\mu}] >, \tag{A.3}$$

i.e.

$$< (xCx)^m exp_{q^{-2}}[-\frac{\omega q^{k+N} xCx}{\bar{\mu}}] exp_{q^2}[-\frac{\omega q^{-k-N} xCx}{\mu}] >=$$

$$= (\frac{q^{1-\frac{N}{2}} + q^{\frac{N}{2}-1}}{\omega})^m [\frac{N}{2}+m-1]_q [\frac{N}{2}+m-2]_q ... [\frac{N}{2}]_q \frac{[m]_q!}{[2m]_q!!} \cdot$$

$$< exp_{q^{-2}}[-\frac{\omega q^{k+N} xCx}{\bar{\mu}}] exp_{q^2}[-\frac{\omega q^{-k-N} xCx}{\mu}] > . \qquad (A.4)$$

Now consider the integral $< f_0 >$. If $k = 2l$, upon use of of the q-derivative property (2.) one finds

$$< f_0 > = < exp_{q^{-2}}[-\frac{\omega q^{2l+N} xCx}{\bar{\mu}}] exp_{q^2}[-\frac{\omega q^{-2l-N} xCx}{\mu}] \cdot$$

$$\cdot [\prod_{h=0}^{l-1} (1 - q^{2(h-l)-N} \omega \frac{q^2-1}{\mu} xCx)][\prod_{h=0}^{l-1} (1 - q^{2(l-h)+N} \omega \frac{q^{-2}-1}{\bar{\mu}} xCx)] > . \qquad (A.5)$$

Expanding the products contained in the square brackets and using formula $(A.4)$ to evaluate all the integrals one finds

$$< f_0 > = z_k < exp_{q^{-2}}[-\frac{\omega q^{k+N} xCx}{\bar{\mu}}] exp_{q^2}[-\frac{\omega q^{-k-N} xCx}{\mu}] > \qquad (A.6)$$

with a suitable constant z_k. If k is even, this formula, together with $(A.4)$, allows to evaluate any integral $(A.1)$ in terms of $< f_0 >$ (which is taken as the normalization factor of te integral). If k is odd, by repeating the previous steps we obtain

$$< f'_0 > = z'_k < exp_{q^{-2}}[-\frac{\omega q^{k+N} xCx}{\bar{\mu}}] exp_{q^2}[-\frac{\omega q^{-k-N} xCx}{\mu}] >, \qquad (A.7)$$

where $f'_0 := exp_{q^{-2}}[-\frac{\omega q^{N+1} xCx}{\bar{\mu}}] exp_{q^2}[-\frac{\omega q^{-N-1} xCx}{\mu}]$. Following the line suggested at the end of sect. 4, it is possible to find the constant $\phi(q,\omega)$ such that

$$< exp_{q^{-2}}[-\frac{\omega q^{1+N} xCx}{\bar{\mu}}] exp_{q^2}[-\frac{\omega q^{-1-N} xCx}{\mu}] > =$$

and

$$\phi(q,\omega) < exp_{q^{-2}}[-\frac{\omega q^{N} xCx}{\bar{\mu}}] exp_{q^2}[-\frac{\omega q^{-N} xCx}{\mu}] > \qquad (A.8)$$

and to show that it is positive $\forall q \in \mathbf{R}^+$. We don't perform here this computation, but just notice that by continuity the positivity of ϕ must hold at least in a neighbourhood of $q = 1$, since $\phi(1,\omega) = 1$. In this way all the integrals $(A.1)$ are evaluated in terms of the normalization constant $< f_0 >$.

****

From the definition (5.2), (5.3) of the creation/destruction operators it immediately follows

$$a_n^i = \frac{q^{n+N}+q^{-n}}{q^{n+N+m}+q^{-n}}a_{n+m}^i + \frac{q^{n+N+m}-q^{n+N}}{q^{n+N+m}+q^{-n}}a_{n+m}^{i\dagger}$$

$$\bar{a}_n^i = \frac{q^{-n-N}+q^n}{q^{-n-N-m}+q^n}\bar{a}_{n+m}^i + \frac{q^{-n-N-m}-q^{-n-N}}{q^{n-N-m}+q^n}\bar{a}_{n+m}^{i\dagger}, \qquad m\in\mathbf{Z}, \qquad (A.9)$$

whence

$$a_{n-1}Ca_n := a_{n-1}^iC_{ij}a_n^j = \alpha_{n,m}(q)(a_{n+m+1}^\dagger Ca_{n+m}^\dagger) + \beta_{n,m}(q)(a_{n+m-1}Ca_{n+m})+$$

$$+\gamma_{n,m}(q)(a_{n+m-1}^\dagger Ca_{n+m}) + \delta_{n,m}(q)(a_{n+m+1}Ca_{n+m}^\dagger)$$

$$\bar{a}_{n-1}C\bar{a}_n = \alpha_{n,m}(q^{-1})(\bar{a}_{n+m+1}^\dagger C\bar{a}_{n+m}^\dagger) + \beta_{n,m}(q^{-1})(\bar{a}_{n+m-1}C\bar{a}_{n+m})+$$

$$+\gamma_{n,m}(q^{-1})(\bar{a}_{n+m-1}^\dagger C\bar{a}_{n+m}) + \delta_{n,m}(q^{-1})(\bar{a}_{n+m+1}C\bar{a}_{n+m}^\dagger), \qquad (A.10)$$

with

$$\beta_{n,m} := \frac{q^{-n-N}+q^n}{q^{-n-N-m}+q^n}\cdot\frac{q^{-n-N+1}+q^{n-1}}{q^{-n-N-m+1}+q^{n-1}} > 0 \qquad \forall q\in\mathbf{R}^+. \qquad (A.11)$$

****

We know that

$$(a_nCa_{n+1})(a_n^\dagger Ca_{n-1}^\dagger)\psi_{n-2,(k,S)}^{l_1\ldots l_k} = v_{n-2,k}\psi_{n-2,(k,S)}^{l_1\ldots l_k} \qquad (A.12)$$

(the function $\psi_{n-2,(k,S)}$ $(k=n-2m)$ was defined in (7.2)), since both sides are eigentuntions of $h_\omega, G_{q^4}\mathcal{L}^2$ with the same eigenvalues and have the same transformation properties under the coaction of the quantum group $SO_q(N,\mathbf{R})$. We now determine the constant $v_{n-2,k}$. Note that $\psi_{n-2,(k,S)}^{l_1\ldots l_k}$ can be written in the form

$$\psi_{n-2,(k,S)}^{l_1\ldots l_k} = [c(xCx)^{m-1}+\ldots]exp_{q^2}[-\frac{q^{-n-N+2}\omega xCx}{\mu}]\mathcal{P}_{k,S}{}^{l_1\ldots l_k}{}_{i_1\ldots i_k}x^{i_1}\ldots x^{i_k}, \quad (A.13)$$

where (as below) the dots in the square bracket denote lower degree powers of (xCx). The strategy will be to find out $v_{n-2,k}$ by only looking at the term of highest degree in xCx at each step of the derivation. From the definition (5.2),(5.3) of the creation/destruction operators and the definition (4.27) of the B operator we get

$$(a_n C a_{n+1})(a_n^\dagger C a_{n-1}^\dagger) = q^{-3}[xCx + \frac{q^{2(n+1+N)}}{\omega^2}\Delta + \frac{q^{n+2N}\mu^2}{\omega(q^2-1)}B + cost]\cdot$$

$$[xCx + \frac{q^{10-2n}}{\omega^2}\Delta - \frac{q^{4-n+N}\mu^2}{\omega(q^2-1)}B + cost]G_q^4 \tag{A.14}$$

The Δ's in the first and second square bracket have to act respectively on functions belonging to $G_{q^2}\Psi_n$ and to $G_{q^4}\Psi_{n-2}$, therefore they can be respectively replaced by $(q^{-N-4}\omega^2 xCx - q^{-N-2}E_n)$ and $(q^{-N-8}\omega^2 xCx - q^{-N-4}E_{n-2})$. Hence

$$(a_n C a_{n+1})(a_n^\dagger C a_{n-1}^\dagger) = E \cdot F, \tag{A.15}$$

where

$$E := q^{-3}[xCx(1 + q^{N+2(n-1)}) + \frac{q^{n+2N}\mu^2}{\omega(q^2-1)}B + cost]$$

$$F := [xCx(1 + q^{2(1-n)-N}) - \frac{q^{4-n+N}\mu^2}{\omega(q^2-1)}B + cost]G_q^4. \tag{A.16}$$

From formulas (6.18) and $(6.20)_a$ one easily derives the identity

$$B\mathcal{P}_{k,S}{}^{l_1\ldots l_k}_{i_1\ldots i_k} x^{i_1}\ldots x^{i_k}| = \frac{q^{2k} + q^{2-N}}{\mu}\mathcal{P}_{k,S}{}^{l_1\ldots l_k}_{i_1\ldots i_k} x^{i_1}\ldots x^{i_k} \tag{A.17}$$

Using the fundamental property (4.28) of B, formulas $(A.13)$ and $(A.17)$ we find

$$F\psi_{n-2,(k,S)}^{l_1\ldots l_k} = [c(xCx)^m q^{4-4m}(1 + q^{2(1-n)-N})exp_{q^2}[-\frac{q^{-n-N-2}\omega xCx}{\mu}]+$$

$$-c\frac{\mu q^{6-n-2m+N}(q^{2k} + q^{2-N})}{\omega(q^2-1)}(xCx)^{m-1}exp_{q^2}[-\frac{q^{-n-N}\omega xCx}{\mu}] + \ldots]\cdot$$

$$\mathcal{P}_{k,S}{}^{l_1\ldots l_k}_{i_1\ldots i_k} x^{i_1}\ldots x^{i_k}; \tag{A.18}$$

applying the q-derivative property (2.) to the exponential $exp_{q^2}[-\frac{q^{-n-N}\omega xCx}{\mu}]$ we get

$$F\psi_{n-2,(k,S)}^{l_1\ldots l_k} = f[c(xCx)^m + \ldots]exp_{q^2}[-\frac{q^{-n-N-2}\omega xCx}{\mu}]\mathcal{P}_{k,S}{}^{l_1\ldots l_k}_{i_1\ldots i_k} x^{i_1}\ldots x^{i_k} \tag{A.19}$$

with

$$f := q^{4-6m}(1 + q^{2m})(1 + q^{2(m+1-n)-N}). \tag{A.20}$$

After similar steps one can see that the result of the action of E on $\psi_{n-2,(k.S)}$ is

$$E \cdot F\psi^{l_1...l_k}_{n-2,(k,S)} = e \cdot f[c(xCx)^{m+1} + ...]exp_{q^2}[-\frac{q^{-n-N-2}\omega xCx}{\mu}]\mathcal{P}_{k,S}{}^{l_1...l_k}_{i_1...i_k} x^{i_1}...x^{i_k} \tag{A.21}$$

where

$$e := q^{-3}(1-q^{2m})(1-q^{2(n-m-1)+N}) \tag{A.22}$$

Using again the q-derivative property (2.), we increase by 4 the degree of the q-power in the exponent and we lower by 2 the degree of the polynomial in (xCx) contained in the square bracket, with the result that eq. $(A.12)$ holds with $v_{n-2,k}$ given by

$$v_{n-2,k} = e \cdot f(\frac{-\mu}{\omega(q^2-1)q^{-n-N-2}})(\frac{-\mu}{\omega(q^2-1)q^{-n-N}}) =$$

$$= \mu^2 q^{3-6m+2n+2N}(1+q^{2m})(1+q^{2(m+1-n)-N})\frac{1-q^{2m}}{1-q^2}\frac{1-q^{2(n-m-1)+N}}{1-q^2}. \tag{A.23}$$

We see that $v_{n-2,k} > 0 \quad \forall q \in \mathbf{R}^+$.

****

We show that

$$< exp_{q^{-2}}[-\frac{\omega q^{k+N}xCx}{\bar{\mu}}](xCx)^k exp_{q^2}[-\frac{\omega q^{-k-N}xCx}{\mu}] > \; > 0 \qquad \forall q \in \mathbf{R}^+ \tag{A.24}$$

First we consider the case $k = 2h$. Using the scaling property (4.36) of the integral we find

$$LHS(A.24) = q^{-k(k+1)} < \bar{\psi}_0(xCxG_{q^2})^k\psi_0 > . \tag{A.25}$$

It is easy to prove that $(xCx)G_{q^2}$ can be decomposed in the following way

$$(xCx)G_{q^2} = \theta_{n+1}(a_nCa_{n+1}) + \lambda_{n+1}(a_n^\dagger Ca_{n+1}) +$$

$$+\rho_{n+1}(a_{n+2}Ca^\dagger_{n+1}) + \sigma_{n+1}(a^\dagger_{n+2}Ca^\dagger_{n+1}) \tag{A.26}$$

where

$$\theta_{n+1} := \frac{q^{\frac{1}{2}}}{(1+q^{N+2n})(1+q^{N+2n-2})}, \qquad \lambda_{n+1} := q^{N+2n-2}\theta_{n+1},$$

$$\rho_{n+1} := q^{N+2n}\frac{1+q^{2n+N-2}}{1+q^{2n+N+2}}\theta_{n+1}, \qquad \sigma_{n+1} := q^{2n+N+2}\rho_{n+1}. \tag{A.27}$$

Only the component $\mathcal{P}_{\Psi_0}((xCxG_{q^2})^k\psi_0)$ belonging to Ψ_0 of the function $(xCxG_{q^2})^k\psi_0$ gives a nonvanishing contribution to the integral (A.25), because of property (5.24). Using the decomposition (A.26) with $n = 0, 2, ..., 2(k-1)$ we see that

$$\mathcal{P}_{\Psi_0}((xCxG_{q^2})^k\psi_0) = \tau_k\psi_0, \tag{A.28}$$

where τ_k is given by a sum of products of constants (A.27), which are all positive for $q \in \mathbf{R}^+$, hence is positive as well. This proves (A.24) in the case $k = 2l$.

If $k = 2l+1$ an analogous reduction shows that

$$< exp_{q^{-2}}[-\frac{\omega q^{k+N}xCx}{\bar{\mu}}](xCx)^k exp_{q^2}[-\frac{\omega q^{-k-N}xCx}{\mu}] >=$$

$$= \tau_k' < exp_{q^{-2}}[-\frac{\omega q^{1+N}xCx}{\bar{\mu}}](xCx)exp_{q^2}[-\frac{\omega q^{-1-N}xCx}{\mu}] >, \tag{A.29}$$

where $\tau_k' > 0 \ \ \forall q \in \mathbf{R}^+$. Formulas $(A.3)$, $(A.8)$ imply

$$< exp_{q^{-2}}[-\frac{\omega q^{1+N}xCx}{\bar{\mu}}](xCx)exp_{q^2}[-\frac{\omega q^{-1-N}xCx}{\mu}] >=$$

$$= \frac{q^{\frac{2-N}{2}} + q^{\frac{N-2}{2}}}{\omega}[\frac{N}{2}]_q \frac{[k-m]_q}{[2(k-m)]_q}\phi(q,\omega)\cdot$$

$$< exp_{q^{-2}}[-\frac{\omega q^{N}xCx}{\bar{\mu}}]exp_{q^2}[-\frac{\omega q^{-N}xCx}{\mu}] > . \tag{A.30}$$

Since $\phi(q,\omega)$ is positive $\forall q \in \mathbf{R}^+$, $(A.24)$ is proved for any k.

Appendix B

In this appendix we show that any scalar polynomial $I(x,\partial)$ (resp. $I(x,\bar{\partial})$) in x^i, ∂^j (resp. $x^i, \bar{\partial}^j$) can be expressed as a ordered polynomial in the variables xCx, Δ (resp. $xCx, \bar{\Delta}$) alone. We limit ourselves to the nonbarred case; the proof for the barred case is a word by word repetition of the proof of the former, after obvious replacements.

To be a scalar I must be a polynomial in scalar variables of the type

$$\tilde{I}_{2n}(\varepsilon_i, \varepsilon_j') = (\eta_{\varepsilon_1})^{i_1}(\eta_{\varepsilon_2})^{i_2}...(\eta_{\varepsilon_n})^{i_n}(\eta_{\varepsilon_n'})_{i_n}...(\eta_{\varepsilon_2'})_{i_2}\eta_{\varepsilon_1',i_1}, \tag{B.1}$$

where $\varepsilon_i, \varepsilon'_j = +, -$, $\eta_+ := x$ and $\eta_- := \partial$. From here we see that I can only contain terms of even degree in η^i_ε; we denote by I_{2m} a scalar polynomial of degree $2m$ and containing only terms of even degree in η^i_ε. The only four independent I_2 are $1, xCx, \Delta, x^i\partial_i$, and they all can be expressed as polynomials in xCx, Δ because of formula (2.

Our claim amounts to showing that for any I_{2m} ($m \geq 0$) there exist an ordered polynomial $P_I(xCx, \Delta)$ in xCx, Δ such that

$$I_{2m} = P_I(xCx, \Delta) \tag{B.2}$$

The claim is obviously true for $m = 0$. The general proof is by induction: assume that it is true for $m = k$. Since any $I_{2(k+1)}$ can be written as a polynomial in $\tilde{I}_{2n}$ variables with $n \leq k+1$, it is sufficient to prove the claim for a $\tilde{I}_{2(k+1)}$ whatsoever. By the induction hypothesis and the very definition (A.1) of the $\tilde{I}$ variables $\tilde{I}_{2(k+1)}$ can be written in the form

$$\tilde{I}_{2(k+1)} = (\eta_\varepsilon)^i \tilde{P}(xCx, \Delta)(\eta_{\varepsilon'})_i \tag{B.3}$$

with some polynomial $\tilde{P}$. Decomposing the latter in a sum of monomials and using formulas

$$\partial^i(xCx) = \mu x^i + q^2(xCx)\partial^i \qquad\qquad x^i\Delta = q^{-2}\Delta x^i - \mu q^{-2}\partial^i \tag{B.4}$$

to move the η^i's step by step through all the factors xCx, Δ as far as the extreme right we will be able to write the RHS of (A.3) as a combination of terms of the type $\tilde{P}'(xCx, \Delta) \cdot (\eta_{\varepsilon''})^i(\eta_{\varepsilon'})_i$; but $(\eta_{\varepsilon''})^i(\eta_{\varepsilon'})_i$ is a polynomial of the type I_2 for which the claim (A.2) holds, hence it holds also for $\tilde{I}_{2(k+1)}$ and the statement (A.2) is completely proved.

Appendix C

In this appendix we give arguments witnessing for the existence of the projectors $\mathcal{P}^S_k$ defined in formula (6.12) and list a few properties of theirs.

The number of independent equations (6.12) is the same for any q, since we know that the dimension of the basic projextors $\mathcal{P}_A, \mathcal{P}_1, \mathcal{P}_S$ is q-independent. But we know that for $q = 1$ there exist $d_k := \binom{N+k-1}{N-1} - \binom{N+k-3}{N-1}$ independent solutions of equations (6.12), or, equivalently, there exists one projector $\mathcal{P}_{k,S}$ of dimension d_k. Then we can apply the implicit function theorem to say that such a projector must exist in a neighbourhood of $q = 1$. In this neighbourhood

$$\mathcal{P}_{k,S}\,{}^{a_1a_2...a_k}_{b_1b_2...b_k}\xi^{b_1}\xi^{b_2}...\xi^{b_k} = 0 \tag{C.1}$$

because of the defining properties (2.) of the wedge product of forms ξ. Applying the coaction to the LHS of $(C.1)$ we get

$$\mathcal{P}_{k,S}\,{}^{a_1a_2...a_k}_{b_1b_2...b_k}T^{b_1}_{b'_1}T^{b_2}_{b'_2}...T^{b_k}_{b'_k}\otimes\xi^{b'_1}\xi^{b'_2}...\xi^{b'_k} \tag{C.2}$$

This is a strong indication (maybe a proof!) that $\mathcal{P}_{k,S}$ must be a polynomial π_k in the $(k-1)$ variables $\hat{R}_{i,i+1}$, $i = 1, ..., k-1$; in fact in this case using formula (2.) we can rewrite $(C.2)$ as

$$T^{a_1}_{b_1}...T^{a_k}_{b_k}\otimes\mathcal{P}_{k,S}\,{}^{b_1...b_k}_{b'_1...b'_k}\xi^{b'_1}...\xi^{b'_k}, \tag{C.3}$$

which is zero because of $(C.1)$. In this case the fact that π_k satisfies eq.'s (6.12) must be a purely algebraic consequence of the relations involving $\hat{R}$-matrices and projectors $\mathcal{P}_A, \mathcal{P}_1, \mathcal{P}_S$, eq.'s (2. etc. But these relations are true $\forall q \in \mathbf{R}^+$, where $\hat{R}, \pi_k(\hat{R}_{i,i+1})$ are always well-defined, therefore setting $\mathcal{P}_{k,S} \equiv \pi_k$ provides a solution of (6.12) $\forall q \in \mathbf{R}^+$.

Here we give, as an example, the explicit form of $\mathcal{P}_{3,S}$ in terms of $\hat{R}_{i,i+1}, (\mathcal{P}_1)_{i,i+1}$:

$$\begin{aligned}\mathcal{P}_{3,S} = \frac{1}{3_{q^2}!}\{&\mathbf{1} + q(\hat{R}_{12}+\hat{R}_{23}) + q^2(\hat{R}_{12}\hat{R}_{23}+\hat{R}_{23}\hat{R}_{12}) + q^3\hat{R}_{12}\hat{R}_{23}\hat{R}_{12}+\\
&-\frac{(q^N-1)\mu}{2(q^{N+2}-1)}[(2q^2+1-q^4)((\mathcal{P}_1)_{12}+(\mathcal{P}_1)_{23})+2q^4Q_N((\mathcal{P}_1)_{12}(\mathcal{P}_1)_{23}+(\mathcal{P}_1)_{23}(\mathcal{P}_1)_{12})\\
&q^2(q+q^{-1})(\hat{R}_{12}(\mathcal{P}_1)_{23}+(\mathcal{P}_1)_{23}\hat{R}_{12}+\hat{R}_{23}(\mathcal{P}_1)_{12}+(\mathcal{P}_1)_{12}\hat{R}_{23})+\\
&+(\hat{R}_{12}(\mathcal{P}_1)_{23}\hat{R}_{12}+\hat{R}_{23}(\mathcal{P}_1)_{12}\hat{R}_{23})]\}\end{aligned} \tag{C.4}$$

From the above explicit expression we see that $\mathcal{P}_{3,S}$ is symmetric (namely $\mathcal{P}^T_{3,S} = \mathcal{P}_{3,S}$) since $\hat{R}, \mathcal{P}^1$ are, and it is invariant under the interchange of indices $(12) \leftrightarrow (23)$. In a forthcoming paper we will show that

$$\mathcal{P}_{k,S} = \pi_k(\hat{R}_{i,i+1}, \mathcal{P}^1_{i,i+1}), \qquad i = 1, 2, ..., k-1 \tag{C.5}$$

$$\mathcal{P}^T_{k,S} = \mathcal{P}_{k,S} \tag{C.6}$$

where π_k is a polynomial in $\hat{R}_{i,i+1}, \mathcal{P}^1_{i,i+1}$ such that

$$\pi_k(\hat{R}_{k-i,k-i+1}, \mathcal{P}^1_{k-i,k-i+1}) = \pi_k(\hat{R}_{i,i+1}, \mathcal{P}^1_{i,i+1}). \tag{C.7}$$

Let us denote by P_k the permutator on $\otimes^k \mathbf{C}^{\mathrm{N}}$ defined by

$$P_k(v_1 \otimes v_2 \otimes ... \otimes v_k) = v_k \otimes ... \otimes v_2 \otimes v_1, \qquad v_i \in \mathbf{C}^{\mathrm{N}}; \tag{C.8}$$

Using the relation (2. it is easy to check that

$$P_k \cdot (\otimes^k C)\hat{R}_{i,i+1} = \hat{R}_{k-i,k+1-i} P_k \cdot (\otimes^k C),$$

$$P_k \cdot (\otimes^k C)\mathcal{P}^1_{i,i+1} = \mathcal{P}^1_{k-i,k+1-i} P_k \cdot (\otimes^k C) \tag{C.9}$$

Relations (C.7), (C.9) imply

$$[P_k \cdot (\otimes^k C), \mathcal{P}_{k,S}] = 0 \tag{C.10}$$

Appendix D

We give a brief proof of Lemma (7.20). From relation (4.25) we infer that

$$\mathcal{P}_{k,S}{}^{j_k \dots j_1}_{h_k \dots h_1} \mathcal{P}_{k,S}{}^{l_1 l_2 \dots l_k}_{i_1 i_2 \dots i_k} S^{h_k \dots h_1 i_1 \dots i_k} =$$

$$= \sigma'_k \mathcal{P}_{k,S}{}^{j_k \dots j_1}_{h_k \dots h_1} \mathcal{P}_{k,S}{}^{l_1 \dots l_k}_{i_1 \dots i_k} \Delta^k x^{h_k} \dots x^{h_1} x^{i_1} \dots x^{i_k}|, \qquad \sigma'_k > 0. \tag{D.1}$$

Using relations (2.,(6.17),(6.18),(6.19) we can rewrite the RHS in the following way:

$$\sigma'_k \mathcal{P}_{k,S}{}^{j_k \dots j_1}_{h_k \dots h_1} \mathcal{P}_{k,S}{}^{l_1 \dots l_k}_{i_1 \dots i_k} \Delta^{k-1}(\mu \partial^{h_k} + q^2 x^{h_k} \Delta) x^{h_{k-1}} \dots x^{h_1} x^{i_1} \dots x^{i_k}| =$$

$$= \sigma'_k \mathcal{P}_{k,S}{}^{j_k \dots j_1}_{h_k \dots h_1} \mathcal{P}_{k,S}{}^{l_1 \dots l_k}_{i_1 \dots i_k} \Delta^{k-1}[\mu k_{q^2} x^{h_k} \dots x^{h_2} \partial^{h_1} x^{i_1} \dots x^{i_k}| +$$

$$\Delta^{k-1} q^{2k} x^{h_k} \dots x^{h_1} \Delta x^{i_1} \dots x^{i_k}||]. \tag{D.2}$$

The second term in the square brackets will yield a vanishing contribution. In fact, the operator Δ^{k-1} can transform at most $(k-1)$ of the k x^{h_i} into ∂^{h_i}, and the remaining x^{h_i}'s can be moved to the left of all derivatives using property (6.17); such an expression is zero, since it contains a number $l > k$ of derivatives acting on $x^{i_1}...x^{i_k}$ (i.e. on their left). Using k times the same kind of argument we end up with

$$\mathcal{P}_{k,S}{}^{j_k...j_1}_{h_k...h_1}\mathcal{P}_{k,S}{}^{l_1l_2...l_k}_{i_1i_2...i_k}S^{h_k...h_1i_1...i_k} =$$

$$\sigma_k''\mathcal{P}_{k,S}{}^{j_k...j_1}_{h_k...h_1}\mathcal{P}_{k,S}{}^{l_1...l_k}_{i_1...i_k}[\partial^{h_k}...\partial^{h_2}\partial^{h_1}x^{i_1}...x^{i_k}|, \qquad \sigma_k''>0. \tag{$D.3$}$$

Now let us perform the remaining derivations in the RHS of (D.3). Using the relation

$$\hat{R}^{-1}{}_{pq}^{h_1i_1}C^{qi_2} = C^{h_1q}\hat{R}^{i_1i_2}_{qp} \tag{$D.4$}$$

(see (2.) it becomes

$$\sigma_k''\mathcal{P}_{k,S}{}^{j_k...j_1}_{h_k...h_1}\mathcal{P}_{k,S}{}^{l_1...l_k}_{i_1...i_k}[\partial^{h_k}...\partial^{h_2}(C^{h_1i_1}x^{i_2}...x^{i_k}+q\hat{R}^{-1}_{pq}{}^{h_1i_1}x^pC^{qi_2}x^{i_3}...x^{i_k}+...)|, \tag{$D.5$}$$

and using relations (6.14) it can be written in the form

$$\sigma_k''\mathcal{P}_{k,S}{}^{j_k...j_1}_{h_k...h_1}\mathcal{P}_{k,S}{}^{l_1...l_k}_{i_1...i_k}[\partial^{h_k}...\partial^{h_2}k_{q^2}C^{h_1i_1}x^{i_2}...x^{i_k}| =$$

$$= .... =$$

$$= \sigma_k\mathcal{P}_{k,S}{}^{j_k...j_1}_{h_k...h_1}\mathcal{P}_{k,S}{}^{l_1...l_k}_{i_1...i_k}C^{h_1i_1}C^{h_2i_2}...C^{h_ki_k} =$$

$$= \sigma_k[\mathcal{P}_{k,S}(\otimes^kC)\mathcal{P}_{k,S}]^{j_k...j_1}_{l_1...l_k}, \qquad \sigma_k>0 \tag{$D.6$}$$

where for the last equality we have used property (C.6). Lemma (7.20) is thus proved.

Conclusions

We have shown that the quantum harmonic oscillator on $\mathbf{R}^N$ with symmetry $SO(N, \mathbf{R})$ admits a q-deformation into the harmonic oscillator on the quantum space $\mathbf{R}_q^N$ with symmetry $SO_q(N, \mathbf{R})$, for any $q \in \mathbf{R}^+$.

In fact this q-deformed harmonic oscillator has a lower bounded energy spectrum; generalizing the classical algebraic construction, the Hilbert space of physical states is built applying construction operators to the (unique) ground state. The scalar product is strictly positive for any $q \in \mathbf{R}^+$. Observables are defined as hermitean operators, as usual. In particular we have constructed the observables hamiltonian, square angular momentum, square lenght, square momentum; as in the classical case, the first two commute. We haven't found non-scalar observables (such as position, momentum and angular momentum components) yet.

Both spectra of the hamiltonian and of the square angular momentum are discrete, and the eigenvalues have the same degeneracy as in the non-deformed case. The q-deformed eigenvalues are invariant under the replacement $q \to q^{-1}$ and can be obtained from the classical ones essentially by the replacement $n \to [n]_q$, where $[n]_q$ is the q-deformed integer n given by $[n]_q := \frac{q^n - q^{-n}}{q - q^{-1}}$. Energy levels are no more equidistant; their difference increases with n.

Guiding ideas for the construction were $SO_q(N, \mathbf{R})$-covariance and correspondence principle in the classical limit $q \to 1$. Essential tools were the two differential calculi on $\mathbf{R}_q^N$, the corresponding two integrations on this quantum space and the corresponding two representations of the Hilbert space into the space of functions on $\mathbf{R}_q^N$. A sort of quantized scaling property of the integrals under dilatation of the integration variables has been singled out.

References

[1] J. Wess, Talk given on occasion of the Third Centenary Celebrations of the Mathematische Gesellschaft Hamburg, March 1990; S. L. Woronowicz, Publ. Rims. Kyoto Univ. **23** (1987) 117.

[2] V. G. Drinfeld, " Quantum Groups ", Proceedings of the International Congress of Mathematicians 1986, Vol. 1, 798; M. Jimbo, Lett. Math. Phys. **10** (1986), 63.

[3] L. D. Faddeev, N. Y. Reshetikhin and L. A. Takhtajan, " Quantization of Lie Groups and Lie Algebras ", Algebra and Analysis, **1** (1989) 178, translated from the Russian in Leningrad Math. J. **1** (1990), 193.

[4] G. Mack and V. Schomerus, " Quasi-Hopf Quantum Symmetry in Quantum Theory ", DESY 91-037 (May 1991).

[5] S. L. Woronowicz, Commun. Math. Phys. **122** (1989) 125-170.

[6] Yu. Manin, preprint Montreal University, CRM-1561 (1988); " Quantum Groups and Non-commutative Geometry ", Proc. Int. Congr. Math., Berkeley **1** (1986) 798; Commun. Math. Phys. **123** (1989) 163.

[7] J. Wess and B. Zumino, Nucl. Phys. Proc. Suppl. **18B** (1991) 302.

[8] W. Pusz and S. L. Woronowicz, Reports in Math. Phys. **27** (1990) 231.

[9] A. Connes, Noncommutative Differential Geometry, Publ. Math. IHES **62** (1986) 41; and references therein.

[10] W. Weich, " The Quantum Group $SU_q(2)$: Covariant Differential Calculus and a Quantum-Symmetric Quantum Mechanical Model ", Ph.D. Thesis (1990), Karlsruhe University.

[11] U. Carow-Watamura, M. Schlieker and S. Watamura, Z. Phys. C Part. Fields **49** (1991) 439.

[12] G. Fiore, " $SO_q(N, \mathbf{R})$-Symmetric Harmonic Oscillator on the N- dim Real Quantum Euclidean Space ", Sissa preprint 35/92/EP.

[13] See for example:
A. J. Macfarlane, J. Phys. A: Math. Gen. **22** (1989) 4581; L. C. Biedenharn, J. Phys. A: Math. Gen. **22** (1989) L873; Chang-Pu Sun and Hong-Chen Fu, J. Phys. A. Math. Gen. **22** (1989) L983.

[14] H. O'Campo, forthcoming paper.

[15] G. Fiore, in preparation.