ISAS - INTERNATIONAL SCHOOL FOR ADVANCED STUDIES

Some aspects of integrable systems in (1+1) dimensions

Thesis submitted for the degree of

"Magister Philosophiæ"

CANDIDATE

Andrea Danani

SUPERVISOR

Prof. Olivier Babelon

Academic year 1989/90

TRIESTE

SISSA - SCUOLA INTERNAZIONALE SUPERIORE I STUDI AVANZATI

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L'AMOUR ET LE CRANE

L'Amour est assis sur le crâne
De l'Humanité,
Et sur ce trône le profane,
Au rire effronté,

Souffle gaiement des bulles rondes
Qui montent dans l'air,
Comme pour rejoindre les mondes
Au fond de l'éther.

Le globe lumineux et frêle
Prend un grand essor,
Crève et crache son âme grêle
Comme un songe d'or.

J'entends le crâne à chaque bulle
Prier et gémir:
-"Ce jeu féroce et ridicule,
Quand doit-il finir?

Car ce que ta bouche cruelle
Eparpille dans l'air
Monstre assassin, c'est ma cervelle
Mon sang et ma chair!"

Charles Baudelaire

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PART TWO: QUANTUM SYSTEMS

Introduction

The initial idea which professors Bonora and Babelon suggested me was a review about the physical applications of integrable systems. After a general study of the argument, I realized that such a work was too disparate for a magister work, and I finally decided to restrict the field to some models among the most significative ones.

Even though it can seem contradictory, the final structure of the work has grown from the last chapter (from end to beginning and back again!). Indeed, investigating toroughly the solution of the Kondo model with the "Quantum Inverse Scattering"(QIS), I went back in a very natural way to the classical systems, since the QIS unify the traditional Bethe Ansatz and the classical inverse scattering, and these notes summarize all the material I looked through about the argument in the last months.

At this point, it is maybe opportune to stress that this work does not intend to be a rigorous treatise about integrable systems (in fact, only one proof is present). The main scope is much more modest: I just wish to illustrate some general features (physical and mathematical) of these systems using concrete examples.

This work deals only with (1+1)-dimensional models and is fundamentally divided into two parts. The first one is devoted to classical systems, here intended as integrable nonlinear differential equations. We first give some examples about the physical origin of the three most important equations of the subject. Then, after a short introduction to the concept of soliton and the inverse scattering method, using principally the nonlinear Schrödinger equation as basic example, we illustrate the so called *zero curvature method*, emphasizing the hamiltonian structure of these particular equations. In this part, the important concepts of *fundamental Poisson bracket* and *r-matrix* are also described.

The second part is dedicated to quantum systems. The chapter three illustrates, with

the help of the quantum nonlinear Schrödinger equation, the main characteristics of the Bethe Ansatz method and the QIS. The important concept of factorization equation (or Yang-Baxter equation) is introduced. Here, one sees that the approach to classical systems presented in chapter two can be readily generalized to quantum systems and the classical r-matrix is substituted by the quantum R-matrix. Chapter four is entirely dedicated to a description of the various lattice models and the relation between vertex models and quantum theories in one dimension is explained. An example of the construction of an integrable lattice model starting from the Yang-Baxter equation is also given.

The last two chapters are completely devoted to a particular class of models which have been successfully applied to the theory of magnetic alloys. The low-energy phenomena of these models cannot be treated by conventional perturbation theory because of their asymptotic freedom, but their complete integrability can avoid this handicap. First we give a detailed derivation of the hamiltonians for particular physical situations, preserving in a certain sense the initial spirit of the work. Then, in chapter six, we solve in detail the so-called s-d exchange model which describes a magnetic atom in a non-magnetic host metal. This allows us to see the techniques previously illustrated at work, to solve the model. To conclude we deduce the explicit formulas for the impurity magnetization.

Chapter 1

Physical motivations.

1.1 Nonlinear dispersive waves.

One of the most remarkable developments in the study of nonlinear dispersive waves is the discovery of a variety of explicit exact solutions for some of the simple canonical equations of the subject.

The main equations concerned are the Korteweg-de Vries equation

$$u_t + \alpha u u_x + u_{xxx} = 0, \tag{1.1}$$

the nonlinear Schrödinger equation

$$iu_t + \beta u_{xx} + \gamma |u|^2 u = 0 \tag{1.2}$$

and the sine-Gordon equation

$$u_{tt} - u_{xx} + \sin \beta u = 0. \tag{1.3}$$

These equations are canonical in that they combine some of the simplest types of dispersion with the simplest types of nonlinearity and are therefore quite simple in structure. However, the original equations of motions of most physical systems which lie behind these equations are not so simple and generally contain several dependent variables. If we look at the dispersion relation of the above-mentioned equations, we see that they can be viewed as a Taylor series approximation to a more general dispersion relation and for this reason, the equations are not mere models but frequently can be derived as a valid approximation for long waves, that is waves whose wavelenght are long compared to a typical lenght scale. Mathematically, in order to build this lenght scale into the original equations of motion, we need to rescale both space and time in order to introduce space and time variables which are appropriate for the description of long wave phenomena. This rescaling enables us to

isolate from the system the relevant equations of motion which describe how the system reacts on the new space and time scales. Before discussing this procedure in section 1.3., let us briefly introduce the concept of soliton which characterize these equations.

1.2 Solitons.

The nonlinear differential equations introduced in the previous section have a number of special properties which the vast majority of field equations do not have. One of the most important of these properties is the existence of solutions having finite energy and stability properties, called generally *solitons*. The solitons are pulse formed as a result of balancing occuring between the steepening effect due to the nonlinearity and the smoothing effect of the dispersion. It should be noted that solitary waves occur also on propagating systems that are characterized by nonlinearity and dissipation but here we consider only cases where dissipative effects are small enough to be neglected.

Traveling waves for the KdV equation.

Let us take for example the KdV equation. We examine travelling wave solutions assuming that

$$\phi(x,t) = \phi(x - vt), \quad v > 0. \tag{1.4}$$

We obtain after substitution in (1.1):

$$(\alpha\phi - v)\phi_\xi + \phi_{\xi\xi\xi} = 0 \tag{1.5}$$

where $\phi = \phi(\xi)$, $\xi = x - vt$. Integrating two times (with a multiplication by ϕ_ξ after the first time), the general traveling wave solution can be written in the form of an elliptic integral

$$\int_{\phi_0}^{\phi} \frac{d\phi}{\sqrt{P(\phi)}} = x - vt \tag{1.6}$$

where ϕ_0 is the value of ϕ at $(x - vt) = 0$, and

$$P(\phi) \equiv K_1 + K_2\phi + v\phi^2 - \frac{\alpha}{3}\phi^3. \tag{1.7}$$

Since a solitary wave is localized, its first and second derivative must vanish as $\xi \to \pm\infty$. This implies that $K_1 = K_2 = 0$ and we obtain the solitary wave

$$\phi(\xi) = \frac{3v}{\alpha}\text{sech}^2(\frac{\sqrt{v}}{2}\xi). \tag{1.8}$$

The following properties of the travelling wave can be readily seen:

1) The amplitude of the wave is directly proportional to its velocity. That is, the taller the wave the faster it moves. This actually is observed in the case of solitary water waves.

2) The wave travels only to the right. (In fact if we change $v \to -v$, the wave which would be moving to the right becomes oscillatory).

3) Finally, the most important of all is that the wave has no dispersion. That is, it mantains its shape as it moves. This is, in fact, the definition of a soliton solution.

Traveling wave solutions for the NS equation.

Let us find the traveling wave solutions of the nonlinear Schrödinger equation in the form

$$i\phi_t + \phi_{xx} + k|\phi|^2\phi, \quad k > 0. \tag{1.9}$$

We write ϕ in the form

$$\phi = \Phi(x,t)e^{i\theta(x,t)} \tag{1.10}$$

where Φ and θ are real, and seek a traveling wave solution for which the *carrier* travels at velocity v_c:

$$\theta = \theta(x - v_c t) \tag{1.11}$$

and the *envelope* travels with velocity v_e:

$$\Phi = \Phi(x - v_e t). \tag{1.12}$$

Equating real and imaginary parts, we obtain the ordinary differential equations

$$\Phi_{xx} - \Phi\theta_x^2 + v_c\Phi\theta_x + k\Phi^3 = 0 \tag{1.13a}$$

$$\Phi\theta_{xx} + 2\Phi_x\theta_x - v_e\Phi_x = 0. \tag{1.13b}$$

Integration of (1.13b) yields

$$\Phi^2(2\theta_x - v_e) = \text{const.} \tag{1.14}$$

Choosing for convenience the constant to be zero, (1.13a) integrates to the elliptic form

$$\int_{\Phi(0,0)}^{\Phi(x,t)} \frac{d\Phi}{\sqrt{P(\Phi)}} = x - v_e t \tag{1.15}$$

where

$$P(\Phi) = -\frac{k}{2}\Phi^4 + \frac{1}{4}(v_e^2 - 2v_e v_c)\Phi^2 + C. \tag{1.16}$$

We consider for simplicity only the case $C = 0$. In the range $v_e(v_e - 2v_c) > 0$, we have

$$\phi(x,t) = \Phi(x - v_e t)\exp\left[i\frac{v_e}{2}(x - v_c t)\right] \tag{1.17}$$

where

$$\Phi = \Phi_0 \,\mathrm{sech}\left[\sqrt{\frac{k}{2}}\Phi_0(x - v_e t)\right] \quad \Phi_0 = \sqrt{\frac{v_e^2 - 2v_e v_c}{2k}}. \tag{1.18}$$

These solutions decrease rapidly as $|x| \to \infty$, so that by analogy with the argument for the KdV equation, they may be considered to represent solitons. Unlike the KdV equation, however, the nonlinear Schrödinger equation does not admit a permanent progressive wave.

The steady solutions of the sine-Gordon equation will be derived in the section 2.2. For example, the so-called kink solutions have the form

$$\phi(x,t) = 4\tan^{-1}\left(C\exp\pm\frac{x - vt}{(1 - v^2)^{1/2}}\right), \quad (|v| < 1). \tag{1.19}$$

1.3 The reductive perturbation method.

The procedures which reduce a set of standard equations to simpler forms are usually perturbative in nature and are consequently called reductive perturbative theories. In this section, we shall illustrate with very simple examples how this procedure works for the KdV equation and the nonlinear Schrödinger equation. At the end, we will derive the sine-Gordon equation through a one-dimensional lattice model.

1.3.1 The KdV equation .

The KdV equation

$$\phi_t + \alpha\phi\phi_x + \phi_{xxx} = 0 \tag{1.20}$$

was first derived by Korteweg and de Vries to describe the propagation of shallow water waves. In fact, it is a model equation which describes nonlinear waves where the nonlinearity and the dispersion are balanced one against the other. Among the various other applications, we mention 1) the anharmonic lattice [Z69,73]; 2) pressure waves in liquid-gas bubble mixtures [vW68]; 3) thermally excited phonon packets in low- temperature nonlinear crystals [TV70].

The ion acoustic waves.

Here we will use as example a description of ion acoustic waves, that is, fluctuations in the ion density of a two-component plasma [Ta72,WT66]. The high-temperature plasma is a fully ionized gas comprising electrons and ions, which are governed by the two hydrodynamic equations for electrons and ions

$$\begin{cases} \dfrac{\partial n_j}{\partial t} + \nabla \cdot (n_j \mathbf{v}_j) = 0 \\ n_j m_j \dfrac{\partial \mathbf{v}_j}{\partial t} + (\mathbf{v}_j \cdot \nabla)\mathbf{v}_j = n_j(\mathbf{E} + \dfrac{\mathbf{v}_j}{c} \times \mathbf{B}) - \nabla p_j \qquad (j = i, e) \end{cases} \tag{1.21}$$

coupled with the Maxwell equations

$$\nabla \cdot \mathbf{E} = 4\pi(q_i n_i + q_e n_e), \tag{1.22a}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{1.22b}$$

$$\frac{\partial \mathbf{B}}{\partial t} + c\nabla \times \mathbf{E} = 0, \tag{1.22c}$$

$$-\frac{\partial \mathbf{E}}{\partial t} + c\nabla \times \mathbf{B} = 4\pi(\mathbf{j}_i + \mathbf{j}_e). \tag{1.22d}$$

Finally we have the equation of state

$$p_j = k n_j T_j \qquad (j = i, e). \tag{1.23}$$

Here, n and $\mathbf{v}$ are the number density and average velocity respectively.

Cold ions and hot electrons.

Let us consider a longitudinal wave ($\nabla \times \mathbf{E} = 0, \mathbf{B} = 0$) propagating in one dimension with $m_e \to 0$ and $T_{e,i}$ constant with $T_e \gg T_i$. Writing $\mathbf{v} = (v, 0, 0)$, we obtain the following equations:

$$en_e \frac{\partial \phi}{\partial x} = kT_e \frac{\partial n_e}{\partial x} \tag{1.24a}$$

$$\frac{\partial n_i}{\partial t} + \frac{\partial (n_i v_i)}{\partial x} = 0 \tag{1.24b}$$

$$\frac{\partial v_i}{\partial t} + v_i \frac{\partial v_i}{\partial x} = -\frac{e}{m_i} E - \frac{1}{m_i n_i} \frac{\partial (n_i T_i)}{\partial x} \tag{1.24c}$$

$$\frac{\partial^2 \phi}{\partial x^2} = 4\pi e (n_e - n_i) \tag{1.24d}$$

where $E = -\partial \phi / \partial x$.

Eq (1.24a) yields the Boltzmann distribution for the electrons:

$$n_e = n_0 \exp\left(\frac{e\phi}{kT_e}\right). \tag{1.25}$$

At this point, it is more convenient to scale out constants introducing new variables as follows:

$$\begin{aligned} n_i &\to n_i/n_0 \qquad & x &\to x/\lambda_d \\ v_i &\to v_i/c_s \qquad & t &\to \omega_p t \\ \phi &\to e\phi/kT_e \end{aligned} \tag{1.27}$$

where

$$\lambda_d^{-1} = \sqrt{\frac{4\pi n_0 e^2}{kT_e}}, \quad \omega_p = \sqrt{\frac{4\pi n_0 e^2}{m_i}}, \quad c_s = \lambda_d \omega_p. \tag{1.28}$$

Note that now, the space, time and velocity variables are dimensionless. w_p is called the plasma frequency and c_s the ion sound speed. Setting $T_i = 0$, the equations (1.24) reduce then to

$$\begin{cases} \dfrac{\partial n_i}{\partial t} + \dfrac{\partial (n_i v_i)}{\partial x} = 0 \\ \dfrac{\partial v_i}{\partial t} + v_i(\dfrac{\partial v_i}{\partial x}) = -\dfrac{\partial \phi}{\partial x} \\ \dfrac{\partial^2 \phi}{\partial x^2} = e^{\phi} - \phi_i \end{cases} \tag{1.29}$$

Next, we expand n, ϕ and v in terms of a perturbation parameter ε :

$$n = 1 + \varepsilon n^{(1)} + \varepsilon^2 n^{(2)} + \cdots \tag{1.30a}$$

$$\phi = \varepsilon \phi^{(1)} + \varepsilon^2 \phi^{(2)} + \cdots \tag{1.30b}$$

$$v = \varepsilon v^{(1)} + \varepsilon^2 v^{(2)} + \cdots \tag{1.30c}$$

where $n^{(i)}, \phi^{(i)}, v^{(i)} \to 0$ as $|x| \to \infty$ and we linearize the system (1.29) keeping only $O(\varepsilon)$:

$$\phi^{(1)}_{xxtt} + \phi^{(1)}_{xx} - \phi^{(1)}_{tt} = 0. \tag{1.31}$$

This yields the dispersion relation

$$\omega^2(k) = \frac{k^2}{(1+k^2)}. \tag{1.32}$$

Since we are looking for long waves, these have small wave number k. Consequently we write k as $k = \varepsilon^p \kappa$ where κ is O(1) and p is some unknown number which is to be determined later. Looking for solutions in the form $\exp(i\theta)$, we have $\theta(x,t) = \kappa \varepsilon^p x - \omega(\varepsilon^p \kappa)t$. In this case $w \simeq k - 1/2k^3$ and we are led to a natural scaling for x and t:

$$\xi = \varepsilon^p (x - at), \quad \tau = \varepsilon^{3p} t. \tag{1.33}$$

These new variables are long in the sense that it needs a large change in x and t in order to change ξ and τ appreciably.

Substituting everything in the system (1.29) and taking into account that

$$\frac{\partial}{\partial t} = \varepsilon^{3p} \frac{\partial}{\partial \tau} - a\varepsilon^p \frac{\partial}{\partial \xi}, \quad \frac{\partial}{\partial x} = \varepsilon^p \frac{\partial}{\partial \xi} \tag{1.34}$$

we obtain the following set of equations corresponding to the various powers of ε (without, as yet, setting any to zero).

For equation (1.29a), we obtain

$$\begin{aligned}
\varepsilon^{p+1}: &\quad -an_\xi^{(1)} + v_\xi^{(1)} \\
\varepsilon^{p+2}: &\quad -an_\xi^{(2)} + v_\xi^{(2)} + (n^{(1)}v^{(1)})_\xi \\
\vdots &\quad \vdots \\
\varepsilon^{3p+1}: &\quad n_\tau^{(1)}
\end{aligned}$$

For equation (1.29b)

$$\begin{aligned}
\varepsilon^{p+1}: &\quad -av_\xi^{(1)} + \phi_\xi^{(1)} \\
\varepsilon^{p+2}: &\quad -av_\xi^{(2)} + \phi_\xi^{(2)} + v^{(1)}v_\xi^{(1)} \\
\vdots &\quad \vdots \\
\varepsilon^{3p+1}: &\quad v_\tau^{(1)}.
\end{aligned}$$

For equation (1.29c)

$$\begin{aligned}
\varepsilon: &\quad -(\phi^{(1)} - n^{(1)}) \\
\varepsilon^2: &\quad -(\phi^{(2)} - n^{(2)} + \frac{1}{2}(\phi^{(2)})^2) \\
\varepsilon^{2p+1}: &\quad \phi_{\xi\xi}^{(1)}. \\
\varepsilon^{2p+2}: &\quad \phi_{\xi\xi}^{(2)}.
\end{aligned}$$

The lowest order terms $(\varepsilon^{p+1}, \varepsilon)$ just give

$$n^{(1)} = v^{(1)} = \phi^{(1)} \tag{1.35}$$

for $a = 1$ ($a = -1$ corresponds to time reversal).

In order to determine the value of p, a plausibility argument is needed. For example, if p is chosen too large $(3p + 1 > p + 2)$ then derivatives in τ will not occur at all at order $p+2$ of ε and it would be necessary to go to higher orders of perturbation theory to obtain an evolution equation for $n^{(1)}$.

However, setting $3p + 1 = p + 2$, then $p = 1/2$, and the $n^{(1)}$ and $v^{(1)}$ terms are of the same order as the ε^{p+2} terms where quadratic nonlinearities in $n^{(1)}$ occur. Setting the

terms at each order to zero we obtain the equations:

$$\begin{aligned} v_\xi^{(2)} - n_\xi^{(2)} + 2n^{(1)}n_\xi^{(1)} + n_\tau^{(1)} &= 0 \\ \phi_\xi^{(2)} - v_\xi^{(2)} + n^{(1)}n_\xi^{(1)} + n_\tau^{(1)} &= 0 \\ \phi_{\xi\xi}^{(1)} - \frac{1}{2}(\phi^{(1)})^2 &= \phi^{(2)} - n^{(2)}. \end{aligned} \tag{1.36}$$

The three equations together give

$$\phi_\tau^{(1)} + \phi^{(1)}\phi_\xi^{(1)} + \frac{1}{2}\phi_{\xi\xi\xi}^{(1)} = 0 \tag{1.37}$$

which is exactly the KdV equation.

The physical interpretation of this result is the following: if, by means of an electrostatic probe, a disturbance is introduced into an initially uniform plasma (which constitutes an initial value problem for the KdV equation) the number of solitons which will emerge is exactly the number of bound states of the initial disturbance. This has been tested, for instance, for a square wave input, which breaks down into as many solitons as it has discrete eigenvalues[HRM72].

1.3.2 A general procedure.

The above-mentioned reductive perturbation method has been established in a very general form by Taniuti and Wei [TaW68] who have shown that a large class on nearly hyperbolic mathematical systems reduce to the KdV equation and other canonical ones. The basic system of equations considered by them is of the form:

$$\frac{\partial U}{\partial t} + A\frac{\partial U}{\partial x} + \{\sum_{\beta=1}^{s}\prod_{\alpha=1}^{p}(H_\alpha^\beta\frac{\partial}{\partial t} + K_\alpha^\beta\frac{\partial}{\partial x})\}U = 0 \qquad p \geq 2, \tag{1.38}$$

where H_α^β and K_α^β are $n \times n$ matrices, all of which are functions of U. An expansion about a constant solution $U^{(0)}$ in terms of a small parameter ε is assumed.

By means of a so-called G-M transformation

$$U = U^{(0)} + \varepsilon U^{(1)} + \cdots, \tag{1.39a}$$

$$\xi = \varepsilon^a(x - \lambda_0 t), \quad \tau = \varepsilon^{a+1} t \quad (a = \frac{1}{p-1}), \tag{1.39b}$$

eq (1.38) can be reduced to

$$\frac{\partial \phi^{(1)}}{\partial \tau} + \alpha \phi^{(1)} \frac{\partial \phi^{(1)}}{\partial \xi} + \mu \frac{\partial^p \phi^{(1)}}{\partial \xi^p} = 0. \tag{1.40}$$

Here we have used the boundary condition as $x \to \infty$ that $U^{(1)} \to 0$ and, consequently, that $\phi^{(1)} \to 0$, while $U^{(1)}$ is given by

$$U^{(1)} = \phi^{(1)} R_0 \tag{1.41}$$

with R_0 the right eigenvector of $A_0 \equiv A(U_0)$ corresponding to λ_0, so that

$$(A_0 - \lambda_0 I) R_0 = 0. \tag{1.42}$$

The coefficients α and μ are given by

$$\alpha = \frac{L_0 (R_0 \cdot \nabla_u A_0) R_0}{L_0 \cdot R_0} = \nabla_u \lambda_0 \cdot R_0, \tag{1.43a}$$

$$\mu = \frac{L_0 K_0 R_0}{L_0 \cdot R_0}, \tag{1.43b}$$

where L_0 denotes the left eigenvector of A_0 corresponding to λ_0, and

$$K_0 = \sum_{\beta=1}^{s} \prod_{\alpha=1}^{p} (-\lambda_0 (H_\alpha^\beta)_0 + (K_\alpha^\beta)_0). \tag{1.44}$$

Therefore we see that for $p = 3$ we obtain the *KdV equation.* For $p = 2$, we obtain the *Burger's equation.*

For example, if we eliminate the n_i and ϕ from eqns (1.24), then we obtain a system of equations corresponding to system (1.38) with $p = 3, s = 1$ and U, A, H and K given by

$$\begin{aligned} U &= \begin{pmatrix} n_e \\ v_i \end{pmatrix}, \qquad A = \begin{pmatrix} v_i & n_e \\ 1/n_e & v_i \end{pmatrix}, \\ H_1 &= 0, \qquad K_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad H_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \\ K_2 &= \begin{pmatrix} v_i & 0 \\ 0 & 0 \end{pmatrix}, \qquad H_3 = 0, \qquad K_3 = \begin{pmatrix} 1/n_e & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned} \tag{1.45}$$

For $U^{(0)} = \binom{1}{0}$, the eigenvalues of A_0 become $\lambda_0 = \pm 1$. For example, considering the case of $\lambda_0 = 1$, we have

$$R_0 = \binom{1}{1}, \qquad L_0 = (1,1), \qquad \nabla_u \lambda_0 = (0,1), \tag{1.46}$$

so that we find $\alpha = 1, \mu = 1/2$, as expected.

1.3.3 The Nonlinear Schrödinger equation.

The nonlinear Schrödinger equation (NLS)

$$i\frac{\partial \phi}{\partial t} + \beta \frac{\partial^2 \phi}{\partial x^2} + \gamma \phi |\phi|^2 \tag{1.47}$$

is a generic equation for describing the modulation of a wave in a nonlinear medium. Among the various applications in physics of continuous systems, we mention 1) fluid physics [BN67] 2) one-dimensional self-modulation of a monochromatic wave [WT68]; 3) Langmuir waves in plasmas [Za72]; 4) relation to the Ginzburg-Landau equation of superconductivity [G66]; and 5) the more recent light pulses in optical fibers [HK85].

It has also found applications in discrete systems, such as atomic lattices [Ts77], magnetic chains [Las77] and electrical networks [MSW82]. In these systems the solutions of the NLS equation describe the slow space time evolution of the envelope of a career wave with fast oscillations.

Here we will relate the NLS equation to an effect occuring in nonlinear optics: the self-focusing of waves [T65].

The self-focusing of waves.

In nonlinear optics, there are some phenomena in which the refractive index change in proportion to the intensity of the light. For instance, we refer to the optical Kerr effect, in which anisotropy of the refractive index results from the total rotation of each molecule with anisotropy under the action of the light and the electrostriction effect, in which the pressure varies due to the electric field; the associated density variation then causes a

change of the refractive index, proportional to the square of the modulus of the electric field. Let us consider, for simplicity

$$\mathbf{D} = \alpha(\omega)\mathbf{E} + \beta|\mathbf{E}|^2\mathbf{E}. \tag{1.48}$$

The Maxwell's equations read:

$$\nabla \wedge (\nabla \wedge \mathbf{E}) + \frac{1}{c^2}\frac{\partial^2 \mathbf{D}}{\partial t^2} = 0 \tag{1.49a}$$

$$\nabla \cdot \mathbf{D} = 0. \tag{1.49b}$$

The nonlinearity is supposed to be weak. Let us consider a linearly polarized, quasi-monochromatic wave of the form

$$\mathbf{E}(\mathbf{x}, t) = \mathbf{E}_0(\mathbf{x}, t)e^{i(k_0 x - \omega t)}, \tag{1.50}$$

where $\mathbf{E}_0(\mathbf{x}, t)$ is a function of space and time which varies very slowly (the variations are small in the intervals $\triangle t \sim 1/\omega$ and $\triangle x \sim 1/k_0$). The wave vectors in the Fourier development are distributed in a small interval around the vector $\mathbf{k}_0$. Furthermore, we shall consider steady propagation in which the spatial variation of the wave amplitude is slower in the x-direction of propagation than normal to it. This suggests to write $\mathbf{E}_0$ in the form

$$\mathbf{E}_0(\mathbf{x}, t) = \varepsilon\varphi(\xi, \eta, \zeta)\mathbf{i} \tag{1.51}$$

$$\xi = \varepsilon^2 x \qquad \eta = \varepsilon y, \qquad \zeta = \varepsilon z$$

where $\mathbf{i}$ is the unit vector along the x-direction (i.e. the direction of polarization of the electric field).

Inserting eq(1.51) into eq (1.49), we obtain the linear dispersion relation

$$k_0^2 = \frac{\omega^2}{c^2}\alpha(\omega) \tag{1.52}$$

from the first order terms in ε. The second order terms do not exist, and from the third order terms in ε, we get the nonlinear Schrödinger equation

$$i\varphi_\xi + \frac{1}{2k_0}\nabla_\perp^2\varphi + \frac{k_0}{2}(\frac{\beta}{\alpha})\varphi|\varphi|^2 = 0, \tag{1.53}$$

where $\nabla_\perp^2 \equiv \partial^2/\partial\eta^2 + \partial^2/\partial\zeta^2$.

In the event that the amplitude does not change in the y and z-direction, the solutions of eq(1.53) becomes

$$\varphi = \varphi_0 \exp\left[i(\frac{k_0}{2})(\frac{\beta}{\alpha})^2|\varphi_0|^2\xi\right]. \tag{1.50}$$

Thus the change of phase, which is proportional to the intensity, results from the nonlinear effect. Since the effective wave number becomes $k = k_0[1 + (1/2)(\varepsilon\beta/\alpha)^2|\varphi_0|^2]$, the phase velocity ω/k becomes smaller in the regions of larger amplitude, and consequently the wave converges to the place of largest amplitude. When the amplitude attains a maximum at the centre of the light beam, the wave focuses on its centre. This is called the *self-focusing effect.* However, diffusion also occurs in waves due to diffraction (which is represented by the second term in (1.53)), so that there exists a threshold value of the amplitude for self-focusing to occur. It is then possible that the focusing properties are compensated precisely by the diffraction, allowing for a solution in the form of a stationary beam which doesn't spread.

1.3.4 The sine-Gordon equation.

The sine-Gordon equation

$$\frac{\partial^2\phi}{\partial x^2} - \frac{\partial^2\phi}{\partial t^2} = \sin\phi \tag{1.55}$$

has been used to describe 1) self-induced transparency [L67]; 2) Bloch wall motion of magnetic crystals [BB59]; 4) a unitary theory for elementary particles [Sk58]; and 5) propagation of magnetic flux on a Josephson line [Sc70]. The example which we will use here is the propagation of a crystal dislocation[KS50].

Propagation of a crystal dislocation.

Some of the phenomena displayed by a layer of atoms in a solid are exhibited by the classical model of a row of particles attached to each other by springs. The effect of the adjacent layers of atoms (the substrate) is represented by a periodic potential. The simplest equilibrium situation is obviously the one in which there is a particle in each

trough of the potential. However the balance between the potential energies may lead to other equilibrium configurations. For example, over some distance the chain may be expanded (or contracted) so that the number of particles is one less (or more) than the number of troughs. Such a configuration is referred to as a negative (positive) dislocation.

In order to describe the dynamics of the chain within a completely classical framework, we shall consider a row of particles, each of mass m, attached to one another by linear springs with equal spring constant k. The particles slide over a sinusoidally corrugated surface so that the periodic potential is provided by gravity. A discussion of the relation of the model to the macroscopic properties of a solid may be found in [In58] and [SS66]. The various quantities involved in setting up the governing equations are:

a = period of the substrate,

b = space between particles with unstrained spring,

X_n = location of n-th particle,

$\bar{x}_n$ = location of n-th trough = $a(n + 3/4)$,

$x_n = X_n - \bar{x}_n$ = displacement of n-th particle from n- trough.

Introducing the following abbreviations

$$\begin{aligned} \xi_n &= x_n/a, \qquad \kappa = (b-a)/a, \\ W &= 2mgh, \qquad L_0^2 = ka^2/2W, \end{aligned} \tag{1.56}$$

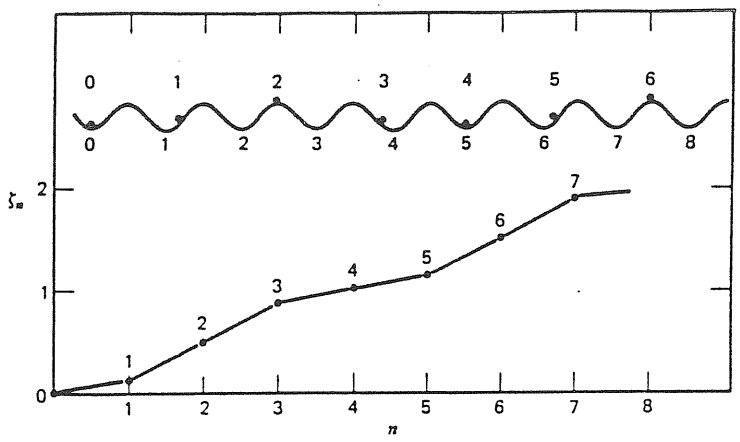
we find that the potential energy $V = V_g + V_{string}$ may be written

$$V = WL_0^2 \sum_n (\xi_{n+1} - \xi_n - \kappa)^2 + \frac{1}{2} W(1 - \cos 2\pi\xi_n). \tag{1.57}$$

Various possible equilibrium configurations of the masses are obtained by solving the system of equations that arise when we set $\partial V/\partial \xi_j = 0$. Admissible equilibrium configurations are therefore the solutions of the equations

$$\xi_{j+1} - 2\xi_j + \xi_{j-1} = \frac{\pi}{2L_0^2} \sin 2\pi\xi_j. \tag{1.58}$$

Choosing two adjacent values of ξ_j as well as L_0, the other displacements are readily obtained from (1.58). As an example, we set $\xi_0 = 0$, $\xi_1 = \frac{1}{8}$ and $L_0^2 = \pi\sqrt{2}$ so that $\xi_2 = \frac{1}{2}$.



Dislocation on chain of masses located over sinusoidal troughs.

Figure 1.1.

Then (1.58) yields $\xi_3 = \frac{7}{8}$, $\xi_4 = 1$, $\xi_5 = \frac{9}{8}$, and so on. A graph of these results as well as a diagram indicating the location of the masses with respect to the potential troughs is shown in fig 1.1.

Assuming that ξ varies slowly from one site to the next, we may replace the discrete labeling ξ_n by a continously varying parameter $\xi(n)$. It results that

$$\frac{d\xi}{dn} = \pm\frac{\sin \pi\xi}{L_0} \tag{1.59}$$

is the amount by which the distance between successive particles, measured in units of a, exceeds unity. Let us consider only the expansion of the chain (positive sign). Integrating and arbitrarly setting $\xi = 1/2$ at $n = 0$, we obtain

$$\xi = \frac{2}{\pi}\tan^{-1}(e^{n\pi/L_0}). \tag{1.60}$$

Therefore, in a region of magnitude L_0, the displacement increases from 0 to 1.

Let us consider now the motion of the particles on the chain. The kinetic energy of the chain is obviously

$$T = \frac{1}{2}ma^2\sum_n(\xi_n)^2 \tag{1.61}$$

and the Lagrangian equation of motion for the j-th particle have the form

$$ma^2\xi_j - 2L_0^2W(\xi_{j+1} - 2\xi_j + \xi_{j-1}) = 0. \tag{1.62}$$

In the continuum limit introduced above, this equation becomes

$$\frac{a^2}{c^2}\frac{\partial^2 \xi}{\partial t^2} - \frac{\partial^2 \xi}{\partial x^2} + \frac{\pi}{2L_0^2}\sin 2\pi\xi = 0 \tag{1.63}$$

wher $c^2 = ka^2/m$. If we now introduce the new dependent variable $\sigma = 2\pi\xi$ as well as the independent variables $u = \pi x/L_0$ and $v = \pi ct/L_0 a$, we obtain

$$\frac{\partial^2 \sigma}{\partial u^2} - \frac{\partial^2 \sigma}{\partial v^2} = \sin\sigma \tag{1.64}$$

which is exactly the sine-Gordon equation. We now see that the single-soliton solution

$$\sigma = 4\tan^{-1} e^{m(u-\beta v)}, \quad m = (1-\beta^2)^{-1/2} \tag{1.65}$$

describes the uniform translation of the dislocation given in eq (1.56). The energy required to create the static one soliton solution is, from eq (1.57), $E = 4L_0 W/\pi$. It is interesting to note that it is found from experiments that metals with a low value of V are more plastic. This may be explained by their increased ease of production of soliton-like dislocations.

Chapter 2

Hamiltonian methods in the theory of solitons

There is deep reason behind the solvability of certain nonlinear equations. They are actually completely integrable systems: if one looks at them in the right way, they have an Hamiltonian structure and a transformation exists to cyclic action-angle variables, which are essentially the scattering data of the associated scattering problem. In this chapter, after a short introduction to the inverse scattering method, following [FT86,D89], we outline the main features of the Hamiltonian approach to the method, using as basic example the Nonlinear Schrödinger equation (NS).

This method is based on the so-called *zero curvature representation*, which results to be the more elegant way to calculate Poisson brackets and has the great advantage that it can be readily generalized to the corresponding quantum systems.

2.1 The inverse scattering method.

There exist well known methods like Fourier and Laplace transformations for solving a given linear system with fixed initial conditions. These methods are, however, inapplicable to a nonlinear system.

Gardner, Greene, Kruskal and Miura [GGKM67] were the first to solve the initial value problem for the KdV equation and in the subsequent years, their method has become the standard one for solving nonlinear systems. This goes by the name of inverse scattering theory (IST).

In this section, we shall briefly summarize the Lax formulation [La68] which generalizes the GGKM method and recall the main steps of the IST procedure.

2.1.1 The theory of the Lax pair.

The main results of Lax can be formulated in terms of the following theorem.

Theorem: *Let a non linear evolution equation be given by*

$$u_t = K(u) \tag{2.1}$$

in which K is a non linear operator acting on a scalar function u. Suppose that a pair of self adjoint linear operators L and A (the Lax pair) which depend on the solution u(x,t) of eq(2.1) exist and satisfy the operator equation

$$iL_t = [A, L] = AL - LA. \tag{2.2}$$

Then the eigenvalues λ of the operator L given by

$$L\psi = \lambda\psi \tag{2.3}$$

are time-independent. Also, the time-evolution of the eigenfunction ψ is determined by

$$i\psi_t = A\psi. \tag{2.4}$$

For a proof, see for example [TN83].

It is sometimes possible to associate a scattering problem with the linear operator L. Let us look at the KdV example. If L is taken to be the Schrödinger operator $\partial^2/\partial x^2 + V(x,t)$, then $\partial L/\partial t = V_t$ is a multiplication with V_t. If $A = \partial/\partial x$ is taken, we obtain the result that the spectrum is time independent if $V_t = V - x$, which just means a shift in x. But if we take

$$A = -4i(\frac{\partial^3}{\partial x^3} + b\frac{\partial}{\partial x} + \frac{\partial}{\partial x}b), \quad b = -\frac{3}{4}V, \tag{2.5}$$

a simple calculation shows that terms depending on $\partial/\partial x$ and $\partial^2/\partial x^2$ cancel. We finally obtain

$$V_t = [A, L] = -V_{xxx} + 6VV_x, \tag{2.6}$$

which is just the KdV equation (1.1) with $\alpha = -6$. It is worth while deducing a certain correspondence between the KdV equation and the Schrödinger equation. When u is

infinitesimal in the KdV equation (2.6), the dispersion relation becomes $\omega + k^3 = 0$. Consequently, the phase velocity $\lambda_p = \omega/k$ is equal to $-k^2$, and hence the solution is a plane wave proceeding in the negative x direction. When $|u|$ is large, but $u \to 0$ as $|x| \to \infty$, the linear approximation is also valid as $|x| \to \infty$, because in that case $|u|$ is sufficiently small. For the case of a soliton, u decays exponentially as $|x| \to \infty$, so that k must be purely imaginary. That is, $k = i\kappa_p$, $\kappa_p \geq 0 (\leq 0)$ for $x \to +\infty(-\infty)$, and λ_p becomes κ_p^2, so that the soliton proceeds in the positive x direction.

This is of course readily seen from the soliton solution (1.8)

$$u = -2\kappa^2 \operatorname{sech}^2[\kappa(x - 4\kappa^2 t)],$$

which becomes proportional to $e^{\pm 2\kappa x}$ as $|x| \to \infty$. Hence putting $2\kappa = \kappa_p$ yields the soliton velocity $4\kappa^2 (= \kappa_p^2)$. On the other hand, if $|u|$ is sufficiently small, the Schrödinger equation reduces to $-\psi_{xx} \simeq \lambda\psi$ to give $\lambda \simeq k^2$ and $\psi \sim e^{\pm ikx}$. In general, λ is positive and arbitrary for the scattering state (the continous spectrum), while for the bound state, λ becomes negative and discrete, and ψ decays exponentially as $|x| \to \infty$, so that k becomes purely imaginary. In particular, for the one-soliton solution (eq.(1.8)), the Schrödinger equation admits one and only one bound state, with the eigenvalue $\lambda = -\kappa^2$, which implies that the soliton velocity is determined by the eigenvalue.

When $u(x, 0)$ is given, we can find $u(x, t)$ by the following procedure.

1) *The direct problem:* From the given $u(x, 0)$, calculate the scattering parameters for L: the eigenvalues of the bound states $(N; \kappa_n, C_n, n = 1, \cdots, N)$ and the reflection and transmission coefficients $(R(k), T(k), 0 \leq k^2 < \infty)$..

2) *Time evolution of the scattering data:* By means of (2.4) and the asymptotic form of ψ as $|x| \to \infty$, obtain the time evolution of the reflection and transmission coefficients for the scattered state and normalization factors for the bound states which will be called collectively the scattering data.

3) *The inverse problem:* From a knowledge of the scattering data of L as a function of time, construct $u(x, t)$. This step is accomplished through use of the so-called Gel'fand-

Levitan- Marchenko (GLM) equation

$$K(x,y;t) + G(x+y;t) + \int_x^\infty K(x,z;t)G(z+y;t)dz = 0, \quad x < y, \tag{2.7}$$

where

$$G(x;t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R(k,t)e^{ikx}dk + \sum_{n=1}^{N} c_n^2(t)e^{-\kappa_n x}. \tag{2.8}$$

The GLM equation is an integral equation for $K(x,y)$. The required solution $u(x,t)$ to the KdV equation is given by

$$u(x,t) = -2\frac{d}{dx}K(x,x;t). \tag{2.9}$$

This procedure is shown graphically in figure 2.1.

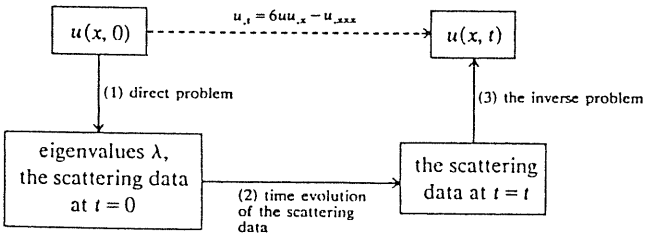


Figure 2.1

Although not many solutions with $R(k) \neq 0$ of (2.7) are known explicitely, all reflectionless potentials can be constructed. Assuming that $\kappa_l \neq \kappa_m$ for $l \neq m$ and making the following ansatz for K:

$$K(x,y) = -\sum_{l=1}^{N} c_l \psi_l(x) e^{-\kappa_l y}, \tag{2.10}$$

where ψ_l turns out to be the normalized bound state wavefunction, the GLM equation becomes

$$(1 + C(x))_{ml}\psi_l(x) = c_m e^{-\kappa_m x}, \; C_{lm}(x) = \frac{c_l c_m}{\kappa_l + \kappa_m} e^{-(\kappa_l+\kappa_m)x}. \tag{2.11}$$

From (2.11), we obtain all reflectionless potentials with eigenvalues ε_l and wavefunction normalization constants c_l:

$$u(x) = -2\frac{d^2}{dx^2}\ln \det(1 + C(x)), \quad \kappa^2 = -\varepsilon_l. \tag{2.12}$$

All soliton solutions of the KdV equation will be reflectionless potentials.

There is an entire hierarchy of operators A, all of odd order, for which the commutator $[A, L]$ is a multiplication operator. Thus there is an infinite family of nonlinear evolution equations which leave the Schrödinger operator invariant. They are called the higher KdV equations. This infinite family is closely tied to the infinite number of conservation laws for the KdV equation. This fact will be discussed in the next chapter.

2.1.2 Multi-soliton solutions.

In this section, we describe how to construct multi-soliton solutions through the application of so-called *Bäcklund transformations.*

Bäcklund transformations originated in the study of surfaces of constant negative curvature. Roughly speaking they can be described as follows. Given a higher order differential equation in the variable $u(x,t)$, namely

$$P(u(x,t)) = 0 \tag{2.13}$$

a Bäcklund transformation is a transformation to a new variable $v(x,t)$ defined by a pair of first order equations

$$\begin{cases} \dfrac{\partial u}{\partial x} = f(u(x,t), v(x,t)) \\ \dfrac{\partial v}{\partial t} = g(u(x,t), v(x,t)) \end{cases} \tag{2.14}$$

where f and g depend on u, v and their derivatives in such a way that the higher order equation eq(2.13) arises as the integrability condition of the two first order equations. Bäcklund transformations may relate the solution of the original equation to that of another which is easier to solve or one solution to another of the same equation which we may already know. Let us illustrate the latter with the sine-Gordon example. We introduce the light cone variables defined by

$$x^{\pm} = x \pm t$$

so that the SG equation (1.55) becomes

$$\partial_+ \partial_- u = \sin u. \tag{2.15}$$

The Bäcklund transformations, in the present case, to the variable $u_1(x^+, x^-)$, are defined by the pair of equations

$$\partial_+ u_1 = \partial_+ u + 2a \sin(\frac{u_1 + u}{2}) \tag{2.16a}$$

$$\partial_- u_1 = -\partial_- u + \frac{2}{a} \sin(\frac{u_1 - u}{2}) \tag{2.16b}$$

where a is a constant. Differentiating eqns(2.16) with respect to $x^{\pm}$, we obtain respectively

$$\partial_- \partial_+ u_1 - \partial_- \partial_+ u = (\sin u_1 - \sin u) \tag{2.17a}$$

$$\partial_+ \partial_- u_1 + \partial_+ \partial_- u = (\sin u_1 + \sin u). \tag{2.17b}$$

The integrability condition for eqns(2.16) then give

$$\partial_+ \partial_- u = \sin u \tag{2.18a}$$

$$\partial_+ \partial_- u_1 = \sin u_1. \tag{2.18b}$$

Thus we see that not only u but u_1 also satisfies the sine- Gordon equation. For example, taking the trivial solution $u = 0$, we obtain from (2.16) that

$$\frac{d(u_1/2)}{\sin(u_1/2)} = a\, dx_+ = \frac{1}{a} dx_- \tag{2.19}$$

so that after integration

$$u_1 = 4 \tan^{-1}(e^{\xi + C}), \tag{2.20a}$$

$$\xi \equiv a x_+ + \frac{1}{a} x_- = \frac{1}{2}(a + \frac{1}{a})(x - \frac{1 - a^2}{1 + a^2} t), \tag{2.20b}$$

where C is an integration constant which for simplicity will be set equal to zero. Defining

$$v = \frac{1 - a^2}{1 + a^2} \tag{2.21}$$

so that $|v| < 1$, u_1 takes the form

$$u_1 = 4 \tan^{-1} \left(\exp \frac{x - vt}{(1 - v^2)^{1/2}} \right) \tag{2.22}$$

and we see that we have obtained a topological kink solution moving to the right from the vacuum solution. This process can be further carried out to generate more complicated solutions. Normally, such a procedure would appear to be formidable if not for the fact that the Bäcklund transformations satisfy the theorem of permutability . This theorem states that two successive Bäcklund transformations with distinct parameters a_1 and a_2 are commutative and consequently, one can construct higher order solution algebraically.

The Bäcklund transformations are, therefore, of great help in constructing solutions. The difficuly lies, of course, in first finding a Bäcklund transformation. There exist a method, due to Clairin [Cl09], to construct Bäcklund transformations systematically. However, it is not always simple and straightforward.

Another useful method of solution which yields multisoliton solutions by directly solving nonlinear evolution equations has been established by Hirota and is called *Hirota's method.* This method basically involves transforming a nonlinear evolution equation into a bilinear differential equation. It can be applied not only to equations solvable by the inverse scattering method, but also to some equations which are not completely integrable. The method is also being extended so that it is applicable to initial value problems, periodic solutions and multi-dimensional problems. For an illustration, we refer to [H76,HS76].

2.2 Complete integrability.

Let us consider an Hamiltonian system with n degrees of freedom, Poisson Bracket $\{\ ,\ \}$ and Hamiltonian H on the phase space Γ_{2n} with coordinates $\xi = (p_i, q_i)$, $i = 1, \cdots, n$. The system is said to be integrable if it possesses n independent functions $I_i(p, q)$ in involution, i.e:

$$\{H, I_i\} = 0, \quad \{I_i, I_j\} = 0. \tag{2.23}$$

The I_i are called the (commuting) integrals of motion or conserved quantities. The Hamiltonian depends on p, q only through them,

$$H = H(I). \tag{2.24}$$

Morever, one can in principle find a change of variables

$$(p, q) \rightarrow (I, \varphi) \tag{2.25}$$

such that the following relations are true

$$\{\varphi_i, I_k\} = \omega_{ik}(I), \tag{2.26}$$

and, in particular

$$\{H, \varphi_i\} = \omega_i(I). \tag{2.27}$$

In the new variables the equations of motion simplify to

$$\begin{aligned} \dot{I} &= \{H, I\} = 0, \\ \dot{\varphi} &= \{H, \varphi\} = \omega(I), \end{aligned} \tag{2.28}$$

so that

$$I = \text{ constant}; \quad \varphi(t) = \varphi(0) + \omega \cdot t \tag{2.29}$$

is a solution.

The (φ, I) are known as the *angle-action* variables. Finding them explicitly could be difficult but a general theorem of mechanics says that they exist whenever the full system of conserved quantities is known.

In what follows, we shall consider field theoretical models and so the number of degrees of freedom will be infinite.

2.3 The zero curvature representation.

Let us rewrite the three equations introduced in chapter one in the following form:
-The Korteweg de Vries (KdV) equation:

$$\varphi_t - 6\varphi\varphi_x + \varphi_{xxx} = 0. \tag{2.30}$$

-The sine-Gordon (SG) equation:

$$\varphi_{tt} - \varphi_{xx} + \frac{m^2}{\beta}\sin\beta\varphi = 0. \tag{2.31}$$

-The nonlinear Schrödinger (NS) equation:

$$i\psi_t = -\psi_{xx} + 2\kappa|\psi|^2\psi. \tag{2.32}$$

The basic point is that the above three equations turn out to be the compatibilty condition for the overdetermined system of equations

$$\frac{\partial\phi}{\partial x} = A(x,t;\lambda)\phi \tag{2.33a}$$

$$\frac{\partial\phi}{\partial t} = B(x,t;\lambda)\phi. \tag{2.33b}$$

Here $\phi = \binom{\phi_1}{\phi_2}$ is a vector valued function of x and t, and A, B are 2×2 matrices. The compatibility is ensured by the zero curvature condition

$$F_{xt} \equiv \frac{\partial A}{\partial t} - \frac{\partial B}{\partial x} + [A,B] = 0. \tag{2.34}$$

Equations (2.33) and (2.34) have a natural geometric interpretation. In fact, the matrix functions $A(x,t;\lambda)$ and $B(x,t;\lambda)$ may be considered as local connection coefficients in the trivial vector bundle $\mathbb{R}^2 \times \mathbb{C}^2$ where the space time $\mathbb{R}^2$ is the base and the vector function ϕ taks values in the fiber $\mathbb{C}^2$. Here, λ is a subsidiary complex parameter. In this case, the $(A-B)$-connection has zero curvature and for this reason the representation of a nonlinear equation in the form (2.34) is called a zero curvature condition.
For the **KdV equation** we have:

$$A = \frac{\lambda}{2i}\sigma_3 + \sigma_+ + \varphi\sigma_- \tag{2.35}$$

$$B = \frac{1}{2i}(\lambda^3 + 2\lambda - 2i\varphi_x)\sigma_3 + (\lambda^2 + 2\varphi)\sigma_+ + (\lambda^2\varphi - i\lambda\varphi_x + 2\varphi^2 - \varphi_{xx})\sigma_- \tag{2.36}$$

and F_{xt}^{12} yields the KdV equation.

For the **NS equation:**

$$A = A_0 + \lambda A_1 \tag{2.37}$$

where

$$A_0 = \sqrt{\kappa}(\bar{\psi}\sigma_+ + \psi\sigma_-), \quad A_1 = \frac{1}{2i}\sigma_3 \tag{2.38}$$

and

$$B = B_0 + \lambda B_1 + \lambda^2 B_2 \tag{2.39}$$

where

$$\begin{aligned} B_0 &= i\kappa \mid \psi \mid^2 \sigma_3 - i\sqrt{\kappa}(\frac{\partial\bar{\psi}}{\partial x}\sigma_+ - \frac{\partial\psi}{\partial x}\sigma_-) \\ B_1 &= -A_0, \quad B_2 = -A_1. \end{aligned} \tag{2.40}$$

Here, the coefficients of $\lambda, \lambda^2, \lambda^3$ vanish identically by construction and the constant term is equivalent to the NS equation (2.32).

For the **sine-Gordon equation:**

$$A = \frac{1}{4i}\left((\beta\pi\sigma_3 + m(\lambda + \frac{1}{\lambda})\sin(\frac{\beta\varphi}{2})\sigma_1 + m(\lambda - \frac{1}{\lambda})\cos(\frac{\beta\varphi}{2})\sigma_2\right) \tag{2.41}$$

$$B = \frac{1}{4i}\left((\beta\frac{\partial\varphi}{\partial x}\sigma_3 + m(\lambda - \frac{1}{\lambda})\sin(\frac{\beta\varphi}{2})\sigma_1 + m(\lambda + \frac{1}{\lambda})\cos(\frac{\beta\varphi}{2})\sigma_2\right). \tag{2.42}$$

As usually

$$\begin{aligned} \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ \sigma_+ = \frac{\sigma_1 + i\sigma_2}{2} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \\ \sigma_- = \frac{\sigma_1 - i\sigma_2}{2} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \end{aligned} \tag{2.43}$$

2.4 Conserved quantities.

The consequences of the zero curvature condition are far reaching. For example, let us consider the parallel transport from (x_1, t_1) to (x_2, t_2) along the curve γ , given by the following path-ordered exponential:

$$\Omega_\gamma(1 \to 2) = \exp\left(\int_\gamma (Adx + Bdt)\right). \tag{2.44}$$

Then the vanishing of the curvature implies, for the Stokes theorem, that Ω_γ depends only on the initial and final points and not on γ, i.e. $\Omega_\gamma = 1$ if γ is closed.

Obviously, a superposition formula holds

$$\Omega_{\gamma_1+\gamma_2}(1 \to 3) = \Omega_{\gamma_2}(2 \to 3)\Omega_{\gamma_1}(1 \to 2). \tag{2.45}$$

Now we introduce an object of fundamental importance in the study of integrable systems, the so called *transition matrix* given by

$$T(x, y; \lambda) = \exp \int_y^x A(z, \lambda)dz. \tag{2.46}$$

To understand the role of $T(x, y; \lambda)$ let us analyze the auxiliary linear problem given by the equation (2.33a) at a fixed time t_0:

$$\frac{\partial \phi(x)}{\partial x} = A(x, \lambda)\phi(x) \tag{2.47}$$

where $A(x, \lambda) \equiv A(x, t_0; \lambda)$. Then the solution is given by

$$\phi(x) = T(x, y; \lambda)\phi(y); \tag{2.48}$$

that is $T(x, y; \lambda)$ translates the solution of the auxiliary problem along the x-axis for a fixed time. Let us mention some important properties of the matrix T:

$$T(x, y; \lambda)T(y, z; \lambda) = T(x, z; \lambda) \tag{2.49a}$$

$$T^{-1}(x, y; \lambda) = T(y, x; \lambda) \tag{2.49b}$$

$$T(x, x; \lambda) = I \tag{2.49c}$$

$$\det T(x, y; \lambda) = 1. \tag{2.49d}$$

Furthermore, T also satisfies the auxiliary problem, namely

$$\frac{\partial T(x,y;\lambda)}{\partial x} = A(x,\lambda)T(x,y;\lambda) \tag{2.50a}$$

$$\frac{\partial T(x,y;\lambda)}{\partial y} = -T(x,y;\lambda)A(y,\lambda). \tag{2.50b}$$

Now, let us prove that the zero curvature condition implies the existence of non trivial conserved quantities. Consider the situation illustrated in figure 2.2:

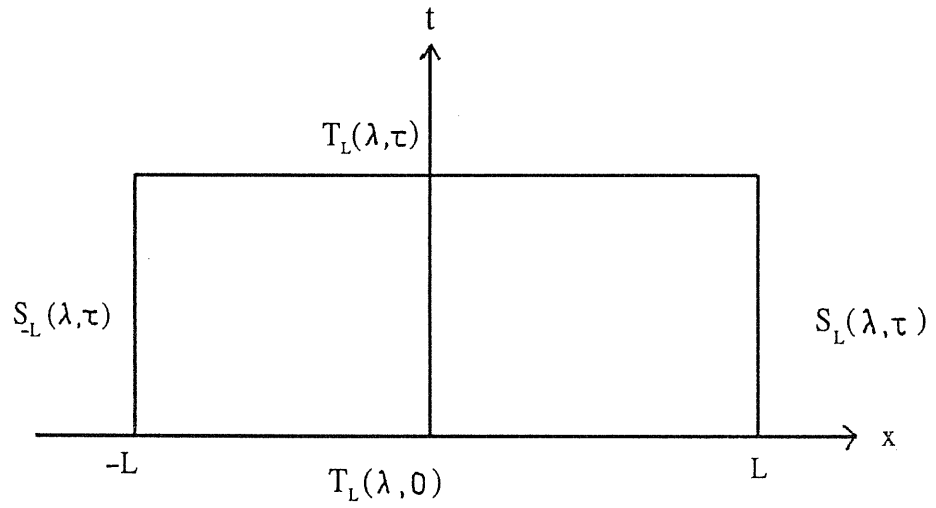


Figure 2.2

where

$$T_L(\lambda,t) = \exp\left[\int_{-L}^{L} A(x,t;\lambda)dx\right] \tag{2.51}$$

$$S_x(\lambda,\tau) = \exp\left[\int_0^{\tau} B(x,t;\lambda)dt\right]. \tag{2.52}$$

The zero curvature implies

$$S_L(\lambda,\tau)T_L(\lambda,0) = T_L(\lambda,\tau)S_{-L}(\lambda,\tau). \tag{2.53}$$

If we impose periodic boundary conditions on the interval $[-L,L]$, then $S_{-L}(\lambda,\tau) = S_L(\lambda,\tau)$ and therefore

$$T_L(\lambda,\tau) = S_L(\lambda,\tau)T_L(\lambda,0)S_L^{-1}(\lambda,\tau). \tag{2.54}$$

Taking the trace of this relation, we find that

$$\mathcal{T}(\lambda) = tr\ T_L(\lambda,\tau) = tr\ T_L(\lambda,0) \tag{2.55}$$

is time-independent; λ being arbitrary, we obtain an infinite set of conserved quantities.

In order to calculate these quantities, we use the fact that performing a gauge transformation $\phi = g\phi'$ in the linear system (2.33), the zero curvature condition is preserved. The transformation properties of A and the monodromy matrix T_L are

$$^gA = g^{-1}Ag - g^{-1}\partial_x g \tag{2.56}$$

$$^gT_L(\lambda,t) = g^{-1}(L,t)T_L(\lambda,t)g(-L,t). \tag{2.57}$$

Therefore, if $g(x,t)$ is periodic, i.e $g(L,t) = g(-L,t)$, one has

$$tr\ ^gT_L(\lambda,t) = tr\ T_L(\lambda,t) \tag{2.58}$$

and $\mathcal{T}(\lambda)$ will be easier to calculate if ^gA is diagonal. In the three examples considered, the matrices A and B belong to the sl_2 algebra, whose fundamental representation is

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ E_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ E_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{2.59}$$

In particular

$$A = A_h H + A_- E_- + A_+ E_+ \tag{2.60}$$

where $A_h(x,t), A_\pm(x,t)$ are periodic functions of x if we impose periodic boundary conditions.

Proposition. *There exists periodic gauge transformations such that*

$$^gA_- = {}^gA_+ = 0$$

$$^gA_h = \frac{i}{L}P_L(\lambda)H$$

with $P_L(\lambda)$ independent of x.

PROOF: the desired gauge transformation is constructed through three successive steps, i.e:

$$g = g_1 g_2 g_3 \tag{2.61}$$

with

$$g_1 = \exp(f_+ E_+),\ g_2 = \exp(f_- E_-),\ g_3 = \exp(hH). \tag{2.62}$$

The coefficient of E_+ vanishes if f_+ is a solution of the Ricatti equation

$$f'_+ - 2A_h f_+ + A_- f_+^2 - A_+ = 0. \tag{2.63}$$

Setting

$$f_+ = \frac{1}{A_-}(v + \Phi), \quad \Phi = A_h + \frac{1}{2}\frac{A'_-}{A_-} \tag{2.64}$$

this equation becomes

$$v' + v^2 = V, \quad V = \Phi^2 - \Phi' + A_- A_+. \tag{2.65}$$

Finally, the substitution $v = y'/y$ linearizes the equation which becomes

$$y'' - Vy = 0. \tag{2.66}$$

The potential V being periodic, one can take for y any one of the two Bloch waves $y_\pm$:

$$y_\pm(x + L) = \exp(\pm i P_L(\lambda)) y_\pm(x - L) \tag{2.67}$$

where we assume that the Wronskian is normalized to one.

Then we obtain for the other two coefficients that

$$f_- = A_- y_+ y_- \tag{2.68}$$

$$h = -\frac{1}{2}\ln(A_- y_+^2 e^{-2iP_L(\lambda)\frac{x}{2L}}) \tag{2.69}$$

so that we have

$$tr\ T_L(\lambda) = 2\cos\ P_L(\lambda) \tag{2.70}$$

with

$$P_L(\lambda) = i\int_{-L}^{L} v\ dx = i\ln\frac{y(L)}{y(-L)}. \tag{2.71}$$

The functional $P_L(\lambda)$ can serve as well as $tr\ T_L(\lambda)$, as a generating functional for conserved quantities.

At this point, to show the complete integrability of the system, we have to define a Poisson bracket and see if these quantities are independent, commute and contain the hamiltonian.

2.5 Hamiltonian approach to ISM through an example: the NS equation.

In this section, we emphasize the hamiltonian structure of the NS equation and we calculate the conserved quantities and the action-angle variables of the system. This will be done using the r-matrix approach of the Leningrad school which allows one to transform the Poisson-brackets of the transition-matrix elements into a commutator.

2.5.1 The fundamental Poisson-bracket relations.

Let us show how the NS model can be considered as an Hamiltonian system. The phase space Γ_0 is here an infinite-dimensional real linear space with complex coordinates defined by pairs of functions $\psi(x), \bar{\psi}(x)$ in $S(\mathbb{R}^1)$. The algebra of observables on Γ_0 is given by smooth real analytic functionals. A *Poisson structure* on this algebra will be then defined by the following Poisson bracket

$$\{F, G\} = i \int_{-\infty}^{\infty} \left(\frac{\delta F}{\delta \psi(x)} \frac{\delta G}{\delta \bar{\psi}(x)} - \frac{\delta F}{\delta \bar{\psi}(x)} \frac{\delta G}{\delta \psi(x)} \right) dx. \tag{2.72}$$

The coordinates $\psi(x)$ and $\bar{\psi}(x)$ themselves may be considered as functional on Γ_0. However, their variational derivatives are generalized functions

$$\frac{\delta \psi(x)}{\delta \psi(y)} = \delta(x-y), \qquad \frac{\delta \bar{\psi}(x)}{\delta \bar{\psi}(y)} = \delta(x-y). \tag{2.73}$$

Using these relations , we obtain the canonical Poisson brackets

$$\begin{aligned} \{\psi(x), \psi(y)\} &= \{\bar{\psi}(x), \bar{\psi}(y)\} = 0 \\ \{\psi(x), \bar{\psi}(y)\} &= i\delta(x-y) \end{aligned} \tag{2.74}$$

and choosing the Hamiltonian to be

$$H = \int_{-\infty}^{\infty} (\mid \frac{\partial \psi}{\partial x} \mid^2 + \kappa \mid \psi \mid^4) dx \tag{2.75}$$

the NS equation of motion (3.1) is represented through the *Hamiltonian equation of motion*

$$\frac{\partial \psi}{\partial t} = \{H, \psi\} \tag{2.76}.$$

The next step consists in calculating the Poisson brackets between the matrix elements of the transition matrix considered as functionals of the dynamical variables $\psi(x), \bar{\psi}(x)$, for different values of λ. For this purpose, let us introduce the following notation

$$\{A \overset{\otimes}{,} B\}_{ik;jl} = \{A_{ij}, B_{kl}\}. \tag{2.77}$$

Recalling that

$$A(x,\lambda) = \sqrt{\kappa}(\bar{\psi}\sigma_+ + \psi\sigma_-) - \frac{i}{2}\lambda\sigma_3$$

we obtain

$$\{A(x,\lambda) \overset{\otimes}{,} A(y,\mu)\} = i\kappa(\sigma_- \otimes \sigma_+ - \sigma_+ \otimes \sigma_-)\delta(x-y). \tag{2.78}$$

Let us next define the so-called permutation matrix as

$$P = \frac{1}{2}(I + \sum_{i=1}^{3} \sigma_i \otimes \sigma_i) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.79}$$

Given any two arbitrary 2×2 matrices C and D, we have

$$P(C \otimes D) = (D \otimes C)P. \tag{2.80}$$

Using the properties of the matrix P , we obtain

$$[P, A(x,\lambda) \otimes I + I \otimes A(x,\mu)] = i(\lambda - \mu)(\sigma_+ \otimes \sigma_- - \sigma_- \otimes \sigma_+) \tag{2.81}$$

and comparing it with equation (2.78), we can write

$$\{A(x,\lambda) \overset{\otimes}{,} A(y,\mu)\} = \delta(x-y)[r(\lambda-\mu), A(x,\lambda) \otimes I + I \otimes A(y,\mu)] \tag{2.82}$$

where

$$r(\lambda - \mu) = \frac{\kappa}{(\lambda - \mu)} P \tag{2.83}$$

The relation (2.82) plays a fundamental role in proving the integrability of the system and is consequently called the *fundamental Poisson bracket relation* (FBR).

The FBR must obviously satisfy the Jacobi identity. Using the antisymmetry of the FBR which takes the form

$$\{A(x,\lambda)\overset{\otimes}{,}A(y,\mu)\} = -P\{A(y,\mu)\otimes A(x,\lambda)\}P \tag{2.84}$$

we obtain the consistency condition to be

$$[r_{12}(\lambda-\mu),r_{13}(\lambda)] + [r_{12}(\lambda-\mu),r_{23}(\mu)] + [r_{13}(\lambda),r_{23}(\mu)] = 0 \tag{2.85}$$

where r_{12} means that the operator r is embedded into the tensor product space. This relation, which is a sufficient but not necessary condition for having the Jacobi identity, is also known as the classical Yang-Baxter equation (CYBE).

From the local equality (2.82), the global relation for the transition matrix can be obtained:

$$\{T(x,y;\lambda)\overset{\otimes}{,}T(x,y;\mu)\} = [r(\lambda-\mu),T(x,y;\lambda)\otimes T(x,y;\mu)]. \tag{2.86}$$

To do this, one has to divide the interval $[y,x]$ into infinitesimal sub-intervals of lenght Δ and take the limit $\Delta \to 0$ after having used the general formula

$$\{A\overset{\otimes}{,}BC\} = I\otimes B\{A\overset{\otimes}{,}C\} + \{A\overset{\otimes}{,}B\}\otimes C. \tag{2.87}$$

Taking $x = L$ and $y = -L$, we get the same relation for the monodromy matrix $T_L(\lambda)$:

$$\{T_L(\lambda)\overset{\otimes}{,}T_L(\mu)\} = [r(\lambda-\mu),T_L(\lambda)\otimes T_L(\mu)]. \tag{2.88}$$

2.5.2 The L$\longrightarrow\infty$ limit.

Taking into account the particular form of A given by (2.37-38) and that $\psi(x),\bar{\psi}(x) \overset{x\to\pm\infty}{\longrightarrow} 0$, we see that

$$\lim_{x,y\to\infty} T(x,y;\lambda) = e^{-i/2\lambda\sigma_3(x-y)} \equiv T_0(x-y;\lambda). \tag{2.89}$$

The transition matrix $T(\lambda)$ for the infinite interval $(-\infty, \infty)$ is then given by

$$T(\lambda) = \lim_{L\to\infty} T_0(-L,\lambda)T_L(\lambda)T_0(-L,\lambda). \tag{2.90}$$

Due to the fact that $A(x,\lambda)$ satisfy the involution relation

$$\bar{A}(x,\lambda) = \sigma A(x,\bar{\lambda})\sigma \tag{2.91}$$

where $\sigma = \sigma_1$ if $\kappa > 0, \sigma = \sigma_2$ if $\kappa < 0$ and $a_{ik}^* \in \bar{A}$ if $a_{ik} \in A$, the transition matrix looks therefore like

$$T(\lambda) = \begin{pmatrix} a(\lambda) & \epsilon b^*(\lambda) \\ b(\lambda) & a^*(\lambda) \end{pmatrix} \tag{2.92}$$

with $\epsilon = \text{sign } \kappa$. We call $a(\lambda)$ and $b(\lambda)$ the transition coefficients. From (2.49d), they satisfy

$$\mid a(\lambda) \mid^2 - \epsilon \mid b(\lambda) \mid^2 = 1. \tag{2.93}$$

Using the identity (2.80), written as

$$(CD^{-1} \otimes DC^{-1})P = P(DC^{-1} \otimes D^{-1}C) \tag{2.94}$$

where $C = \exp(i\lambda\sigma_3 x)$, $D = \exp(i\mu\sigma_3 x)$, together with the formula

$$\lim_{\lambda\to\pm\infty} \frac{e^{i\lambda x}}{\lambda} = \pm i\pi\delta(\lambda)$$

and the definition (2.90) of $T(\lambda)$, it follows that

$$\{T(\lambda) \otimes T(\mu)\} = r_+(\lambda-\mu)(T(\lambda)\otimes T(\mu)) - (T(\lambda)\otimes T(\mu))r_-(\lambda-\mu) \tag{2.95}$$

where

$$r_\pm(\lambda-\mu) = \frac{\kappa}{2(\lambda-\mu)}(I\otimes I + \sigma_3\otimes\sigma_3) \pm i\pi\kappa\delta(\lambda-\mu)(\sigma_+\otimes\sigma_- - \sigma_-\otimes\sigma_+). \tag{2.96}$$

Equation (2.96) contains all the Poisson brackets between the coefficients of $T(\lambda)$. Explicitely we obtain

$$\{a(\lambda), a(\mu)\} = \{a(\lambda), \bar{a}(\mu)\} = \{b(\lambda), b(\mu)\} = 0 \tag{2.97a}$$

$$\{a(\lambda), b(\mu)\} = \frac{\kappa}{\lambda-\mu+i0} a(\lambda)b(\mu) \tag{2.97b}$$

$$\{a(\lambda), \bar{b}(\mu)\} = -\frac{\kappa}{\lambda-\mu+i0} a(\lambda)\bar{b}(\mu) \tag{2.97c}$$

$$\{b(\lambda), \bar{b}(\mu)\} = 2\pi i \mid \kappa \mid\mid a(\lambda) \mid^2 \delta(\lambda-\mu). \tag{2.97d}$$

The action-angle variables.

It can be shown, using eq(2.97) that if we define

$$\begin{aligned} Q(\lambda) &= -\arg b(\lambda) \\ P(\lambda) &= \frac{1}{2\pi\kappa}\log|a(\lambda)| \end{aligned} \tag{2.98}$$

then

$$\begin{aligned} \{Q(\lambda), Q(\mu)\} &= \{P(\lambda), P(\mu)\} = 0 \\ \{Q(\lambda), P(\mu)\} &= \delta(\lambda-\mu). \end{aligned} \tag{2.99}$$

We can analytically continue the Poisson bracket relations in eqns (2.97) to the upper half of the complex λ-plane and deduce for the discrete spectrum that the variables

$$q_j = \log|b_j| \quad p_j = -\frac{2}{\kappa}\mathrm{Re}\,\lambda_j, \tag{2.100}$$

$$\varphi_j = -\arg\, b_j \quad \varrho_j = -\frac{2}{\kappa}\mathrm{Im}\,\lambda_j \tag{2.101}$$

satisfy

$$\{p_k, q_j\} = \delta_{kj}, \qquad \{\varrho_k, \varphi_j\} = \delta_{kj}. \tag{2.102}$$

Together the variables $\{Q(\lambda), P(\lambda), q_j, p_j, \varrho_j, \varphi_j\}$ defined above constitute the action angle variables of the system.

In particular, let us note that in terms of the action variables, the Hamiltonian H becomes

$$H = \int_{-\infty}^{\infty}\lambda^2 Q(\lambda)d\lambda + \frac{\kappa^2}{4}\sum_{j=1}^{n}(\varrho_j p_j^2 - \frac{1}{3}\varrho_j^3) \tag{2.103}$$

which allows us to evaluate the time evolution of various quantities:

$$\begin{aligned} \dot{P}(\lambda) &= \dot{p}_j = \dot{\varrho}_j = 0, \\ \dot{Q}(\lambda) &= \{H, Q(\lambda)\} = \lambda^2, \\ \dot{q}_j &= \{H, q_j\} = \frac{\kappa^2}{2}\varrho_j p_j, \\ \dot{\varphi}_j &= \{H, \varphi_j\} = \frac{\kappa^2}{4}(p_j^2 - \varrho_j^2). \end{aligned} \tag{2.104}$$

These relations can be shown to be equivalent to the following formulae:

$$b(\lambda,t) = e^{-i\lambda^2 t} b(\lambda,0), \quad b_j(t) = e^{-i\lambda_j^2 t} b_j(0),$$
$$\lambda_j(t) = \lambda_j(0); \quad j = 1,\cdots,n. \tag{2.105}$$

Therefore, the passage to the new variables completely trivializes the dynamics of the NS model.

2.5.3 Construction of the conserved quantities.

We use the notation introduced in section 2.4. In this specific case we have

$$A_h = \frac{i}{2}\lambda, \quad A_- = \sqrt{\kappa}\psi, \quad A_+ = \sqrt{\kappa}\bar{\psi} \tag{2.106}$$
$$V = (\frac{i\lambda}{2})^2 - \frac{\psi'}{\psi}\frac{i\lambda}{2} - \frac{1}{2}\frac{\psi''}{\psi} + \frac{3}{4}(\frac{\psi'}{\psi})^2 + \kappa|\psi|^2. \tag{2.107}$$

When $\lambda \to \infty$, v admits the asymptotic expansion

$$v = \frac{i\lambda}{2} + \sum_{n=0}^{\infty} \frac{v_n}{(i\lambda)^n}. \tag{2.108}$$

We find

$$v_0 = -\frac{1}{2}\frac{\psi'}{\psi}, \quad v_1 = \kappa\bar{\psi}\psi \tag{2.109}$$

and for $n \geq 1$:

$$v_{n+1} = -\psi(\frac{v_n}{\psi})' - \sum_{p=1}^{n-1} v_p v_{n-p} \tag{2.110}$$
$$v_2 = -\psi\partial_x\bar{\psi} \tag{2.111}$$
$$v_3 = \psi\partial_x^2\bar{\psi} - |\psi|^2 \tag{2.112}$$

Therefore

$$P_L(\lambda) = \lambda L - \kappa(\frac{N}{\lambda} + \frac{P}{\lambda^2} - \frac{H}{\lambda^3} + \cdots) \tag{2.113}$$

where

$$N = \int_{-L}^{L} dx\ |\psi|^2 \tag{2.114}$$
$$P = \frac{i}{2}\int_{-L}^{L} dx(\psi\partial_x\bar{\psi} - \bar{\psi}\partial_x\psi) \tag{2.115}$$
$$H = \int_{-L}^{L} dx(\partial_x\bar{\psi}\partial_x\psi + \kappa|\bar{\psi}\psi|^2). \tag{2.116}$$

Notice that the coefficient of λ^0 in the expansion of $P_L(\lambda)$ is given by

$$i\int_{-L}^{L} dx\ v_0 = -\frac{i}{2}\ln\frac{\psi(L)}{\psi(-L)} \tag{2.117}$$

which vanishes when one uses periodic boundary conditions, but is in general a topological charge.

Using the fact that $tr\ A\otimes B = tr\ A\cdot tr\ B$, an immediate consequence of the FBR for the generating functional $\mathcal{T}$ is the following

$$\{\mathcal{T}(\lambda),\mathcal{T}(\mu)\} = 0 \tag{2.118}$$

from which we deduce the integrability of the system.

We see that in this approach, a new tool comes into play, the **r-matrix**, and the criterion of complete integrability is given by the eq(2.82). Although its validity for any model is rather a matter of fortune, the r-matrix approach can be extended to a broad variety of completely integrable models. Among them, one should mention the sine-Gordon model and the Heisenberg ferromagnet. For the SG equation in the form

$$\Box\varphi + \frac{m^2}{\beta}\sin\beta\varphi = 0 \tag{2.119}$$

we have

$$A(\lambda) = \frac{1}{4i}(\beta\dot{\varphi}\sigma_3 + k_0\sin\frac{\beta\varphi}{2}\sigma_1 + k_1\cos\frac{\beta\varphi}{2}\sigma_2) \tag{2.120}$$

and parametrizing the vector $k = (k_0, k_1)$ by

$$k_0 = m\text{ch}\ \lambda, \quad k_1 = m\text{sh}\ \lambda \tag{2.121}$$

we obtain the FPR with a trigonometric r-matrix given by

$$r(\lambda) = \frac{\sigma_1\otimes\sigma_1 + \sigma_2\otimes\sigma_2}{\text{sh}\ \lambda} + \frac{\text{ch}\ \lambda}{\text{sh}\ \lambda}\sigma_3\otimes\sigma_3. \tag{2.122}$$

For the second model, the equation of motion is

$$\frac{\partial}{\partial t}\mathbf{S} = \mathbf{S}\wedge\frac{\partial^2}{\partial x^2}\mathbf{S},\ \mathbf{S} = S^a(x),\ a = 1,2,3,\ \mathbf{S}^2 = 1 \tag{2.123}$$

where the field variables are subject to

$$\{S^a(x), S^b(y)\} = \varepsilon^{abc} S^c(x)\delta(x-y).$$

This implies, for the auxiliary linear problem, a matrix A given by

$$A(\lambda) = \frac{i}{2\lambda} S^a \sigma^a \tag{2.124}$$

and the r-matrix coincides with that of the NS model:

$$r(\lambda) = \frac{P}{\lambda}. \tag{2.125}$$

These two models have one common feature: the FPR are ultralocal. It has been shown in [Tsy81] that for nonultralocal models such as the KdV equation, the r-matrix is also apllicable. In this case, the eq(2.82) must be somehow generalized but the eq(2.86) remains valid.

The concept of r-matrix has also been used to classificate the integrable models, in connection to the Lie bracket formalism for current algebras (or loop algebras). We do not enter in this subject and refer to [FT87] and literature therein.

2.5.4 Time evolution of the transition matrix.

First, let us comment on the relationship with the scattering theory and the analytic properties of the transition coefficients. We first rewrite the auxiliary problem (2.33a) as the eigenvalue problem

$$L\phi = \frac{1}{2}\lambda\phi \tag{2.126}$$

for the first order matrix differential operator

$$L = i\sigma_3 \frac{d}{dx} + i\sqrt{\kappa}(\psi(x)\sigma_- - \bar{\psi}(x)\sigma_+). \tag{2.127}$$

We note that this is nothing but the scattering problem for the stationary massless Dirac equation. Let us define the following matrix functions:

$$T_\pm(x,\lambda) = \lim_{y\to\pm\infty} T(x,y;\lambda)T_0(y;x) \tag{2.128}$$

which satisfy eq(2.50a) with the asymptotic behaviour

$$T_{\pm}(x,\lambda) \overset{x\to\pm\infty}{\longrightarrow} T_0(x,\lambda) = e^{-i\lambda/2\sigma_3 x}. \tag{2.129}$$

If $\kappa > 0$, L is self-adjoint operator and the corresponding spectral problem is the main object of scattering theory. In particular, the matrices $T_{\pm}(x,\lambda)$ are called the matrix Jost solutions of the auxiliary problem with the boundary condition (2.129) and taking the limit $L \to \infty$ we can write

$$T(\lambda) = T_+^{-1}(x,\lambda)T_-(x,\lambda). \tag{2.130}$$

It can be shown, using certain properties of the integral representations for $T_{\pm}$, that the first column of T_- and the second of T_+ may be analytically extended into the upper half-plane, while the first of T_+ and the second of T_- may be into the lower half-plane. Denoting the columns of $T_{\pm}$ by $T_{\pm}^{(1,2)}$, so that

$$T_{\pm}(x,\lambda) = (T_{\pm}^{(1)}(x,\lambda), T_{\pm}^{(2)}(x,\lambda)), \tag{2.131}$$

we get, from (2.92), the following expressions for $a(\lambda)$ and $b(\lambda)$:

$$a(\lambda) = \det(T_-^{(1)}(x,\lambda), T_+^{(2)}(x,\lambda)) \tag{2.132a}$$

$$b(\lambda) = \det(T_+^{(1)}(x,\lambda), T_-^{(2)}(x,\lambda)). \tag{2.132b}$$

Using the analytic properties of the columns $T_-^{(1)}$ and $T_+^{(2)}$ and their asymptotic behaviours for $|\lambda| \to \infty$, it results that $a(\lambda)$ has an analytic continuation into the upper half-plane $\mathrm{Im}\lambda \geq 0$ with the asymptotic behaviour $a(\lambda) = 1 + o(1)$ as $|\lambda| \to \infty$. The coefficient $\bar{a}(\lambda)$ has an analytic continuation into the lower half-plane, which is denoted by $a^*(\lambda)$ and we have

$$a^*(\lambda) = \bar{a}(\bar{\lambda}), \qquad \mathrm{Im}\lambda \leq 0. \tag{2.133}$$

Furthermore, it results that $b(\lambda)$ has no analytic continuation off the real line.

Let us now investigate the zeros of $a(\lambda)$ in the upper λ half-plane. If $\kappa > 0, a(\lambda)$ has no zeros because of the selfadjointness of the operator L. If $\kappa < 0$, L is not selfadjoint and $a(\lambda)$ may have zeros. Let $\lambda_1, \cdots, \lambda_n$ be the zeros of $a(\lambda)$, $\mathrm{Im}\lambda_j > 0$, $j = 1, ..., n$. Here we

assume for simplicity that no zeros occur on the real axis and that all zeros are simple. Then it follows that

$$T_-^{(1)}(x,\lambda_j) = b_j T_+^{(2)}(x,\lambda_j), \qquad j=1,\cdots,n \tag{2.134}$$

with $b_j \neq 0$. It is clear from (2.133) that $\bar{\lambda}_1,\cdots,\bar{\lambda}_n$ are the zeros of $a^*(\lambda)$ in the lower half-plane. From the involution property, we have

$$T_-^{(2)}(x,\bar{\lambda}_j) = -\bar{b}_j T_+^{(1)}(x,\bar{\lambda}_j), \qquad j=1,\cdots,n \tag{2.135}$$

and using eq(2.130), we obtain

$$b_j = b(\lambda_j), \qquad \bar{b}_j = \bar{b}(\lambda_j), \qquad j=1,\cdots,n. \tag{2.136}$$

The set $\{\lambda_j,\bar{\lambda}_j\}$ is the discrete part of the spectrum of (2.126) for $\kappa<0$. Furthermore, for any κ, L has continous spectrum of multiplicity two on the whole real line, according to the existence, for real λ, of two linearly independent solutions of (2.126) given by the columns of $T_\pm(x,\lambda)$.

The analyticity of $a(\lambda)$ and the normalization relation can be used to express $a(\lambda)$ through its zeros (if there are any) and $b(\lambda)$. Namely, for $\mathrm{Im}\lambda>0$, we have

$$\begin{aligned}
&\kappa>0: \qquad a(\lambda) = \exp\left\{\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\log(1+|b(\mu)|^2)}{\mu-\lambda}d\mu\right\}\\
&\kappa<0: \qquad a(\lambda) = \exp\left\{\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\log(1-|b(\mu)|^2)}{\mu-\lambda}d\mu\right\}\prod_{j=1}^{n}\left(\frac{\lambda-\lambda_j}{\lambda-\bar{\lambda}_j}\right).
\end{aligned} \tag{2.137}$$

Let us conclude the chapter deriving the time evolution of $T(x,y;\lambda)$. By differentiating the auxiliary linear problem with respect to t and using the zero curvature condition together with the initial condition (2.49c), we obtain

$$\frac{\partial}{\partial t}T(x,y) = B(x)T(x,y) - T(x,y)B(y). \tag{2.138}$$

For $\psi(x),\bar{\psi}(x)$ rapidly decreasing, we can take the limit of (2.138) as $y\to-\infty, x\to+\infty$, for real λ, to obtain the evolution equations for the transition coefficients.

For this purpose, let us recall that

$$B(x,\lambda) \overset{|x|\to\infty}{\longrightarrow} B(\lambda) = \frac{i\lambda^2}{2}\sigma_3 \tag{2.139}$$

and consequently

$$[B(\lambda), T_0(x,\lambda)] = 0. \tag{2.140}$$

Using the definitions of the Jost functions

$$T_\pm(x,\lambda) = \lim_{y\to\pm\infty} T(x,y;\lambda)T_0(y;\lambda) \tag{2.141}$$

we find by multiplying by $T_0(y;\lambda)$ from the right and taking the limit as $y \to \pm\infty$ that

$$\frac{\partial}{\partial t}T_\pm(x,\lambda) = B(x,\lambda)T_\pm(x,\lambda) - \frac{i\lambda^2}{2}T_\pm(x,\lambda)\sigma_3. \tag{2.142}$$

The time evolution of the transition matrix is then easily obtained from $T = T_+^{-1}T_-$ to be

$$\frac{\partial}{\partial t}T(\lambda) = \frac{i\lambda^2}{2}[\sigma_3, T(\lambda)]. \tag{2.143}$$

This equation is remarkable in that the dependence on $\psi(x), \bar{\psi}(x)$ is completely eliminated. For the *continous spectrum*, we obtain

$$\frac{\partial}{\partial t}a(\lambda,t) = 0, \qquad \frac{\partial}{\partial t}b(\lambda,t) = -i\lambda^2 b(\lambda,t). \tag{2.144}$$

For real λ, we deduce that $a(\lambda)$ is time-independent and by virtue of analicity, the same holds for $\mathrm{Im}\lambda > 0$. Thus, as expected, in the rapidly decreasing case, the generating function for the conservation laws is just $a(\lambda)$.

For the *discrete spectrum*, an analogous analysis brings to

$$\frac{d}{dt}b_j(t) = -i\lambda_j^2 b_j(t), \qquad j = 1,\cdots,n. \tag{2.145}$$

and therefore, the time dependence of transition coefficients is perfectly equivalent to the formulae (2.105).

Chapter 3

Quantum integrability

In a quantum theory, the dynamical variables must be quantized as operators and various commutation relations must take the place of Poisson brackets. Furthermore, since the quantized variables do not commute, we must adress the question of operator ordering and wherever necessary, the quantum expressions must be regularized.

By integrability of a quantum system, we understand that we can determine the spectrum of the Hamiltonian as well as its scattering matrix. In this chapter, we will discuss the tradional approach to quantum integrable systems, the so-called *Bethe ansatz method*, and the more recent *Quantum inverse scattering method* (QSTM) which combines the main ideas and methods of both the classical and quantum theory of integrable systems.

3.1 The Bethe Ansatz.

3.1.1 Generalities.

By "Bethe Ansatz", we mean a wave function with a particular structure which provides the exact solution of many interesting physical systems.

Its origin goes back to Bethe who constructed in 1931 [Be] the exact wave function of the isotropic spin-1/2 Heisenberg chain with nearest-neighbour interaction:

$$H = J \sum_{n=1}^{N} \sigma_n \cdot \sigma_{n+1}. \tag{3.1}$$

Bethe proved that the eigenstates of the Hamiltonian (3.1)

$$|\Psi >= \sum_{x_i} \Psi(x_1, \cdots, x_M) \sigma^{\dagger}_{x_1} \cdots \sigma^{\dagger}_{x_M} |0>; \quad (x_i = 1, ..., N) \tag{3.2}$$

are given by a function of a special form:

$$\Psi(x_1, \cdots, x_M) = \sum_{P} \exp \left\{ i \sum_{j=1}^{M} k_{P_j} x_j + i \sum_{i>j=1}^{M} \text{sign}(x_i - x_j) \Phi(k_{P_j}, k_{P_j}) \right\}, \tag{3.3}$$

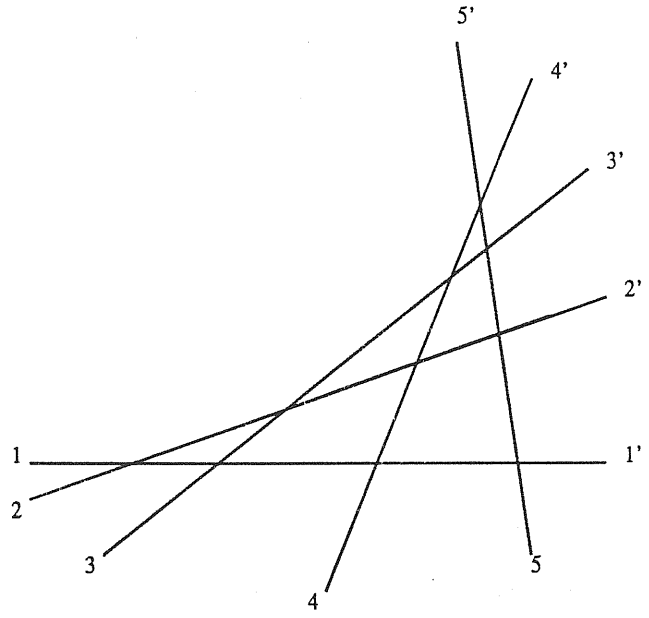


Figure 3.1

where $P = \{P_1, \cdots, P_M\}$ is a permutation of the integers $\{1, \cdots, M\}$, $|0>$ is the state with all spins down, x_j are coordinates on the lattice, k_j are quasimomenta of magnons and

$$\Phi(2k, 2p) = 2\cot^{-1}\left[\frac{\sin(k-p)}{\cos(k+p) - \cos(k-p)}\right], \tag{3.4}$$

is a two particle scattering phase. The spin and energy of this state are equal to

$$S^z = N/2 - M, \quad E = J\sum_{j=1}^{M} \cos k_j. \tag{3.5}$$

It results very convenient to use a parametrization for $k(\lambda)$ and $p(\lambda)$ in which the scattering phase depends on only the difference between the corresponding arguments λ and λ'. For the Heisenberg model, this parametrization has the following form [Hu38]:

$$k(\lambda) = 2\tan^{-1}\lambda, \tag{3.6}$$

$$\Phi(k(\lambda), p(\lambda')) = 2\tan^{-1}(\frac{\lambda - \lambda'}{2}). \tag{3.7}$$

The quantity λ is generally called 'rapidity.

The wavefunction describes the so-called factorized scattering: when a particle passes from a region $X_Q = \{x_{q_1} < \cdots < x_{q_M}\}$ to another region $X_{Q'}$, which differs from the first one by a permutation of some pair (x_p, x_q), only two-particle processes occur. Furthermore, any N-particle scattering process is expressed standardly in terms of the product of $N(N-1)/2$ two-particle processes (see Fig. 3.1 for N=5).

The absence of multiparticle processes is in fact the Bethe hypothesis. For its self-consistency, it is necessary that the several ways in which one can decompose the permutation connecting X_Q with another region $X_{Q'}$ lead to the same result. For systems without internal symmetry, the scattering reduces only to a phase shift. Therefore Bethe hypothesis is valid provided that the N-particle scattering phase has the following form:

$$\Phi(k_1, \cdots, k_N) = \sum_{i>j=1}^{N} \Phi(k_i, k_j). \tag{3.8}$$

Until the early 1960's, Bethe's work had no considerable impact on either physicists or mathematicians. Then in 1963 Liniger revived the Bethe method by solving the problem of the interacting Bose gas. Generalizing this solution the general many-body wavefunction for a problem with a δ -function interaction was derived definitively in 1967 by McGuire, Lieb,Yang and Gaudin [Ga67,Ya67].

In the course of this solution, the general special function relations between two-particle scattering amplitudes were derived. These relations are necessary and sufficient conditions for the validity of the Bethe Ansatz and they constitute the foundations of the theory of integrable two-dimensional statistical models (Baxter [Ba72]) and of the theory of factorized scattering (Zamolodchikov [Zm79]).

These relations are generally called the Yang-Baxter equations. In the general case of the one-dimensional quantum theory of particles with n different colours, they have the following form:

$$S_{a_2 a_2'}^{a_1 a_1'}(k_1, k_2) S_{a_3 a_3'}^{a_1' a_1''}(k_1, k_3) S_{a_3' a_3''}^{a_2' a_2''}(k_2, k_3) =$$
$$S_{a_3 a_3'}^{a_2 a_2'}(k_2, k_3) S_{a_3' a_3''}^{a_1' a_1''}(k_1, k_3) S_{a_2' a_2''}^{a_1' a_1''}(k_1, k_3); \tag{3.9}$$

here $S_{a_2 a_2'}^{a_1 a_1'}(k_1, k_2)$ is the two-particle scattering matrix. The subscripts (a_1, a_2) and (a_1', a_2') correspond to the particle colours in the $|in>$ and $|out>$ states with momenta k_1, k_2.

These factorization equations imply that all the possible decompositions of the N-particles S matrix as a product of two particle S matrices give the same result.

Obviously, for systems without an internal symmetry, the S-matrix is diagonal and condition (3.9) is trivially fulfilled. For systems possessing an internal symmetry, the Bethe

Ansatz has the following form. Let $Q = \{q_1, .., q_N\}$ and $P = \{p_1, .., p_N\}$ be permutations of the integers $\{1, ..N\}$. Then in the region $X_Q = \{x_{q_1} < x_{q_2} < \cdots < x_{q_N}\}$, the wavefunction is

$$\psi(X_Q) = \sum_P A_{a_1..a_N}(P \mid Q) \exp\{i \sum_{j=1}^{N} k_{p_j} x_j\}. \tag{3.10}$$

The summation is carried over all permutations. $A(P|Q)$ in the given region does not depend on coordinates. $A(P|Q)s$ from different regions are connected with each other by S matrix elements. For example, if the region X_Q and $X_{Q<ij>}$ differ by the permutation of i and j particles, then

$$A_{..kl..}(P \mid Q) = S_{lj}^{ki} A_{..ij..}(P \mid Q_{<ij>}). \tag{3.11}$$

Once the two-particle S-matrix is known, all $A(P|Q)'s$ can be expressed in terms of $A(P \mid I)$, $I = \{1, .., N\}$.

The parameters $\{k_j\}$ of the wavefunction (3.10) are so far arbitrary quantities. In order to find the spectrum of the system, some boundary conditions have to be imposed. The more convenient choice are the periodic boundary conditions (PBC) where one puts the system into a box with length L and demand that

$$\psi(x_1, \cdots, x_j, \cdots, x_N) = \psi(x_1, \cdots, x_j + L, \cdots, x_N) \tag{3.12}$$

for any $j = 1, \cdots, N$. The resulting equation for the coefficients $A(P \mid Q)$ leads to the following eigenvalue problem

$$\exp(ik_j L)\xi = T_j \xi, \qquad \xi = A(I \mid I) \tag{3.13}$$

where

$$T_j = S_{jj+1}...S_{jN}S_{j1}..S_{jj-1} \tag{3.14}$$

and $S_{jl} = S(k_j, k_l)$ denotes the spin entries of the scattering matrix.

For models without an internal symmetry, the S matrix is not an operator but a c-number and eq(3.13) leads straight to the algebraic equation for k_i:

$$\exp(ik(\lambda_\alpha)L) = \prod_{\substack{j=1 \\ j \neq i}}^{M} \exp(i\Phi(\lambda_\alpha - \lambda_\beta)). \tag{3.15}$$

Equation (3.15) describes the whole spectrum of the system. The most useful form of PBCs is found by taking the logarithm of (3.15), for which one must choose a branch of the function $\Phi(\lambda)$. We obtain

$$k(\lambda_\alpha)L = 2\pi N_\alpha + \sum_{\substack{\beta=1 \\ \beta\neq\alpha}}^{M} \Phi(\lambda_\alpha - \lambda_\beta), \tag{3.16}$$

where N_α = integer are the quantum numbers of the system. The transcendental equations (3.16) were first found by Bethe in his work about the 1/2-spin chain and we shall call them *Bethe-Ansatz equations* in the following. For systems possessing an internal symmetry, PBC imply the simultaneous diagonalization of N operators, and using the triangle equations, it can be shown that all operators T_j commute with each other so that the problem may indeed be solved. We will return in more details to this argument in chapter 6 by examining the solution of the Kondo model.

3.1.2 The Bethe Ansatz for the quantum NS.

As a simple example, let us look at the traditional Bethe ansatz for the quantum nonlinear Schrodinger system (QNS).

The dynamical variables $\psi(x)$ and $\bar{\psi}(x)$ are replaced by the annihilation and creation operators $\psi(x)$ and $\psi^\dagger(x)$ with

$$\begin{aligned} [\psi(x), \psi(y)] &= [\psi^\dagger(x), \psi^\dagger(y)] = 0 \\ [\psi(x), \psi^\dagger(y)] &= \delta(x-y) \end{aligned} \tag{3.17}$$

and the functionals of the dynamical variables are defined to be normal ordered, so that all the ψ operators stand to the right of $\psi^\dagger$. The normal ordered quantum hamiltonian takes the form

$$H = \int_{-\infty}^{\infty} dx[\partial_x\psi^\dagger\partial\psi + \kappa\psi^\dagger\psi^\dagger\psi\psi] \tag{3.18}$$

and the Heisenberg equation of motion becomes

$$i\partial_t\psi = -\partial_x^2\psi + 2\kappa\psi^\dagger\psi\psi \tag{3.19}$$

which we recognize to be the quantum nonlinear Schrodinger equation. Let us define the Fock vacuum $\mid 0 >$ by

$$\psi(x) \mid 0 >= 0 \qquad \forall x. \tag{3.20}$$

It will be called the pseudovacuum and it is to be distinguished from the physical vacuum which is the ground state of the Hamiltonian, representing the Dirac sea.

Since the operator $\widehat{N}$ of number of particles

$$\widehat{N} = \int dx \psi^{\dagger}(x)\psi(x) \tag{3.21}$$

is an integral of motion, i.e [H,N]=0, we may consider each N-body sector of the Hilbert space separately and look for common eigenfunctions $\Psi(k_1, \cdots, k_N)$ of operator H and $\widehat{N}$:

$$\begin{aligned} &\Psi(k_1, \cdots, k_N) \\ &= \frac{1}{\sqrt{N!}} \int_{-\infty}^{\infty} d^N x \phi_N(x_1, \cdots, x_N \mid k_1, \cdots, k_N) \psi^{\dagger}(x_1) \cdots \psi^{\dagger}(x_N) \mid 0 > \end{aligned} \tag{3.22}$$

Here ϕ is a symmetrical function of all x_j. The eigenvalue equation

$$H \mid \Psi >= E_N \mid \Psi >; \qquad \widehat{N} \mid \Psi >= N \mid \Psi > \tag{3.23}$$

results in the fact that ϕ_N is an eigenfunction of the many- body Schrödinger operator with delta-function interaction:

$$H = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + 2\kappa \sum_{i>j=1}^{N} \delta(x_i - x_j); \tag{3.24}$$

$$H\phi_N = E_N \phi_N. \tag{3.25}$$

Let us consider the domain in the coordinate space defined by

$$x_1 < x_2 < \cdots < x_N. \tag{3.26}$$

In this domain, ϕ_N is an eigenfunction of the free hamiltonian with the same eigenvalue and the delta-function potential leads to a discontinuity in the first derivative of the wave function at the interaction points:

$$\left(\frac{\partial}{\partial x_{i+1}} - \frac{\partial}{\partial x_i} \right) \phi_N = \kappa \phi_N \qquad (x_{i+1} = x_i + 0; \quad i = 1, \cdots, N). \tag{3.27}$$

We can easily see that the function defined in domain (3.26) as

$$\phi_N = \prod_{i>j} \left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j} + \kappa \right) \det(\exp\{ik_i x_j\}) \tag{3.28}$$

satisfies all the conditions with $E_N = \sum_{i=1}^{N} k_i^2$. Using this we can check that the solution of eq(..) can be written in the following form:

$$\phi_N(x_1, \cdots, x_N \mid k_1, \cdots, k_N) = \sum_P \exp(i \sum_{n=1}^{N} k_{P_n} x_N) \prod_{i>j} \frac{k_{P_i} - k_{P_j} - i\kappa\epsilon(x_i - x_j)}{k_{P_i} - k_{P_j}} \tag{3.29}$$

where as usually $\{P_1, \cdots, P_N\}$ denote the permutations of the numbers $\{1, \cdots, N\}$ and the summation is over all such permutations. Therefore the N-particle states ϕ_N are the exact eigenstates of the full hamiltonian and the scattering phase is given by

$$e^{i\Phi(k_i - k_j)} = \left(\frac{k_i - k_j + i\kappa}{k_i - k_j - i\kappa} \right) \tag{3.30}$$

or equivalently

$$\Phi(k) = i \ln(\frac{k - i\kappa}{k + i\kappa}) = -2 \tan^{-1} \frac{k}{\kappa}. \tag{3.31}$$

Let us rewrite the BA equation (3.16) ignoring the λ-parametrization. We have:

$$k_i L = \sum_{j \neq i} \Phi(k_i - k_j) + 2\pi N_i. \tag{3.32}$$

A change in the branch of (3.32) is just equivalent to a redifinition of the N_i's. Let us note that $\Phi(k)$ in (3.31) is not precisely the phase shift due to the interaction which has discontinuity of -2π at $k = 0$ for any finite κ. In fact, this definition introduces a fermionic description of the spectrum in terms of the N_i's because a state with two identical N's vanishes identically.

Ground state distribution and excitations.

The ground-state k distribution is obtained from the BA equations (3.32) by choosing the N_i's to be as closely spaced as possible, i.e. $N_{i+1} = N_i + 1$. Subtracting the PBC's for adjacent k_i's gives

$$k_{i+1} - k_i = \frac{1}{L} \sum_j \Phi(k_{i+1} - k_j) - \Phi(k_i - k_j) + \frac{2\pi}{L}. \tag{3.33}$$

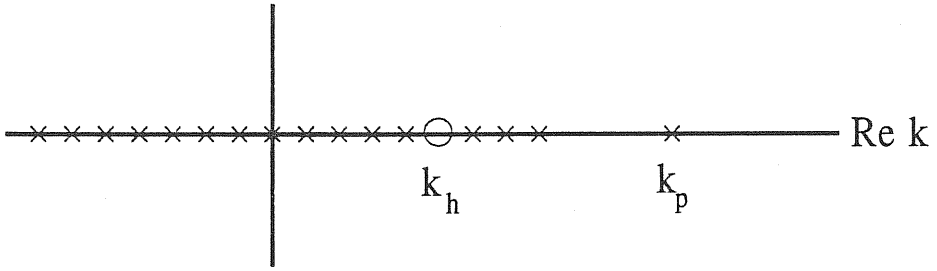


Figure 3.2

As $L \to \infty$, the k_i's become infinitesimally spaced, and the quantity

$$\rho(k_i) = \frac{1}{L(k_{i+1} - k_i)} \tag{3.34}$$

approaches a continuous function. The sum in (3.33) can be replaced by an integral:

$$\frac{1}{L}\sum_j \cdots \to \int_{-k_F}^{k_F} dk\rho(k)\cdots, \tag{3.35}$$

where k_F is determined from the particle density by

$$\int_{-k_F}^{k_F} dk\rho(k) = \frac{N}{L}. \tag{3.36}$$

Equation (3.33) becomes an integral equation for the ground-state density $\rho(k)$,

$$2\pi\rho(k) = 1 - \int_{-k_F}^{k_F} R(k-k')\rho(k')dk', \tag{3.37}$$

where the kernel R is given by

$$R(k) = \Phi'(k) = -\frac{2\kappa}{k^2+\kappa^2}. \tag{3.38}$$

The ground state energy is obtained from the solution to (3.37),

$$\frac{E_0}{L} = \int_{-k_F}^{k_F} k^2\rho(k)dk. \tag{3.39}$$

The excitations above the ground state consist of "particles" which are filled modes above the Fermi surface and "holes" which are empty modes below the Fermi surface. Let us consider for simplicity a single particle-hole excitation, as shown in Figure 3.2.

In response to such an excitation, the Fermi sea will shift slightly, compatibly with the PBC's. Denoting the modes in the excited state by $\tilde{k}_i$, we have

$$\tilde{k}_i L = \sum_{j\neq i} \Phi(\tilde{k}_i - \tilde{k}_j) + \Phi(\tilde{k}_i - k_p) - \Phi(\tilde{k}_i - k_h) + 2\pi N_i. \tag{3.40}$$

Subtracting (3.33) gives

$$(\tilde{k}_i - k_i)L = \sum_j [\Phi(\tilde{k}_i - \tilde{k}_j) - \Phi(k_i - k_j)] + \Phi(\tilde{k}_i - k_p) - \Phi(\tilde{k}_i - k_h). \tag{3.41}$$

Let us define a function $\omega(k)$ by

$$(\tilde{k}_i - k_i)L \to \omega(k_i). \tag{3.42}$$

Denoting $F(k) \equiv \omega(k)\rho(k)$, in the thermodynamic limit we obtain

$$2\pi F(k) + \int_{-k_F}^{k_F} R(k-k')F(k')dk' = \Phi(k-k_p) - \Phi(k-k_h) \tag{3.43}$$

So it is possible to describe the vacuum polarization caused by a particle and a hole. Observables for the excitations over the ground states (energy, momentum, scattering matrix,..) are obtained by adding the contributions due the vacuum polarization to the corresponding "bare" quantities. For example, let us calculate the excitation energy ΔE

$$\begin{aligned} \Delta E \equiv E - E_0 &= k_p^2 - k_h^2 + \sum_{sea}(\tilde{k}_i^2 - k_i^2) \\ &= k_p^2 - k_h^2 + \int_{-k_F}^{k_F} 2kF(k)dk. \end{aligned} \tag{3.44}$$

Defining the function $\varepsilon(k)$ as the solution of the linear integral equation

$$\varepsilon(k) + \frac{1}{2\pi}\int_{-k_F}^{k_F} R(k-k')\varepsilon(k')dk' = k^2 - \mu_0, \tag{3.45}$$

where μ_0 is the chemical potential fixed by the requirement $\varepsilon(\pm k_F) = 0$, it is possible to prove that

$$\Delta E(k_p, k_h) = \varepsilon(k_p) - \varepsilon(k_h). \tag{3.46}$$

Therefore, $\varepsilon(k)$ is just the energy of the one-particle excitation over the ground state and eq(3.45) can be regarded as the fundamental spectral equation of the theory. The

generalization of (3.45) for finite temperatures was obtained by Yang and Yang [YY69] using a variational method. More details and rigorous results about this subject are well summarized in [BIK85].

3.2 Other models solved by BA wavefunction.

3.2.1 Models with diagonal S matrices.

Here we enumerate the main models described by the Bethe-Ansatz wave function (3.3) with a special two-particle scattering phase.

(i) The *spin 1/2 anisotropic Heisenberg ring (XXZ-chain)* [Or58]

$$\mathcal{H} = I \sum_{n=1}^{N} [(\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y) + \cos\mu \ \sigma_n^z \sigma_{n+1}^z]. \tag{3.47}$$

$$\begin{aligned} k(\lambda) &= 2\tan^{-1}(\cot\frac{\mu}{2}\tanh\mu\lambda), \\ \Phi(\lambda) &= 2\tan^{-1}(\cot\mu\tanh\mu\lambda), \\ E &= I\sum_{\alpha=1}^{M}\cos k(\lambda_\alpha). \end{aligned} \tag{3.48}$$

(ii) The *interacting Bose-gas* [LL63]

$$\mathcal{H} = \int \left[-\varphi^* \frac{\partial^2}{\partial x^2}\varphi + g(\varphi\varphi^*)^2\right] \ dx \tag{3.49}$$

φ is a Bose-field and

$$\begin{aligned} \Phi(k) &= 2\tan^{-1}(k/g), \\ E &= \sum_{\alpha=1}^{N} k_j^2. \end{aligned} \tag{3.50}$$

(iii) The *massive Thirring model* [BS65,BT79]

$$\mathcal{L} = \int (\bar{\psi}\partial_\mu\gamma^\mu\psi - m\bar{\psi}\psi + g \ j_\mu j^\mu)dx; \tag{3.51}$$

here $j_\mu = \bar{\psi}\gamma_\mu\psi$, $\psi = (\psi_1, \psi_2)$ is a spinor, $\bar{\psi} = \psi^\dagger\gamma_0$ and γ_μ are two-dimensional Dirac matrices

$$\{\gamma_\mu, \gamma_\nu\} = \delta_{\mu\nu}.$$

This model is equivalent to the *Sine-Gordon model* which was directly solved in [STF79]:

$$\mathcal{L} = \int \left[\frac{1}{2}(\partial_\mu \varphi)^2 - m_0(1 - \cos\beta\varphi)\right] dx, \tag{3.52}$$

$$\begin{aligned} k(\lambda) &= m \sinh\lambda, \\ \Phi(\lambda) &= 2\tan^{-1}(\tan g \tanh\lambda/2), \\ E &= m\sum_{\alpha=1}^{M} \cosh\lambda_\alpha, \end{aligned} \tag{3.53}$$

where $m = m_0/2\Lambda$ (Λ is an ultraviolet cutoff), and

$$\beta^2 = 4\pi + 8g \quad \text{at } |g| \leq 1.$$

(iv) The *resonance-level model* [FWi80]

$$\mathcal{H} = \int dx(-ic^\dagger(x)\frac{\partial}{\partial x}c(x) + V\delta(x)(c^\dagger(x)d + d^\dagger c(x)) + U\delta(x)c^\dagger(x)c(x)d^\dagger d), \tag{3.54}$$

$$\begin{aligned} k(\lambda) &= V\exp(\lambda), \\ \Phi(\lambda) &= 2\tan^{-1}(\tan U \tanh\lambda/2), \\ E &= \sum_{\alpha=1}^{N} \tan^{-1}(V^2/k(\lambda_\alpha)). \end{aligned} \tag{3.55}$$

(v) The generalization of the *Heisenberg model for an arbitrary spin S* [ZF80,KS81]

$$\mathcal{H} = \sum_{n=1}^{N} \mathcal{P}(\mathbf{S}_n \cdot \mathbf{S}_{n+1}); \tag{3.56}$$

here $\mathcal{P}(x)$ is a polynomial of the order $2S$ of a special form,

$$\begin{aligned} k(\lambda) &= 2\tan^{-1}\lambda/S, \\ \Phi(\lambda) &= 2\tan^{-1}\lambda, \\ E &= \sum_{\alpha=1}^{M} \cos k(\lambda_\alpha). \end{aligned} \tag{3.57}$$

3.2.2 Models with internal symmetries.

The factorization conditions (3.9) are very strict constraints on the scattering kinematics. Some non-trivial solutions correspond to different models of the 1+1-dimensional quantum-field theory. Among them, we mention:

(i) *Many-body problem with the δ- function interaction* [Y67,Su68]:

$$\mathcal{H} = \int \left[-\sum_{a=1}^{n} \psi_a^* \frac{\partial^2}{\partial x^2} \psi_a + g \sum_{a>b} \psi_a^* \psi_b^* \psi_b \psi_a \right] dx, \tag{3.58}$$

where n is the number of colours.The two-particle S matrix of this model is

$$S_{a_1 a_1'}^{a_2 a_2'}(k,p) = \frac{k - p + i g P_{a_1 a_1'}^{a_2 a_2'}}{k - p + i g}, \tag{3.59}$$

where $P_{a_1 a_1'}^{a_2 a_2'} = \delta_{a_1 a_2'} \delta_{a_2 a_1'}$ is a permutation operator.
The energy of the system is $E = \sum_{j=1}^{N} k_j^2$.

(ii) *Hubbard model* [LW68]:

$$\mathcal{H} = t \sum_{n,\sigma} (c_{n\sigma}^\dagger c_{n+1\sigma} + c_{n+1\sigma}^\dagger c_{n\sigma}) + U \sum_n c_{n\uparrow}^\dagger c_{n\uparrow} c_{n\downarrow}^\dagger c_{n\downarrow}; \tag{3.60}$$

$$S_{a_1 a_1'}^{a_2 a_2'}(k,p) = \frac{t(\sin k - \sin p) + i U P_{a_1 a_1'}^{a_2 a_2'}}{t(\sin k - \sin p + i U)}, \tag{3.61}$$

where $c_{n\sigma}^\dagger$ is the creation operator of an electron with spin σ at the nth site of a one-dimensional lattice. The energy of the system is

$$E = t \sum_{j=1}^{N} \cos k_j.$$

(iii) *Chiral invariant Gross-Neveu model* [Be79,AL79]:

$$\mathcal{L} = \int dx \left[\sum_{a=1}^{n} i \bar{\psi}_a \partial_\mu \gamma^\mu \psi_a - g \left((\sum_{a=1}^{n} \bar{\psi}_a \psi_a)^2 - (\sum_{a=1}^{n} \bar{\psi}_a \gamma_5 \psi_a)^2 \right) \right], \tag{3.62}$$

$$S_{a_1 a_1'}^{a_2 a_2'}(\tau, \tau') = \frac{\tau - \tau' + i g P_{a_1 a_1'}^{a_2 a_2'}}{\tau - \tau' + i g}. \tag{3.63}$$

$\tau = \pm 1$ is a particle chirality and the energy is given by

$$E = \sum_{\tau=\pm 1} \tau \sum_{i=1}^{N_\tau} k_{\tau_i}, \tag{3.64}$$

where N_τ and k_{τ_i} are the number and momenta of particles with the given chirality τ.

The physically relevant *s-d exchange model* and *Anderson model* will be treated in more details in chapter 6.

3.3 Quantum Inverse Scattering Method for the Nonlinear Schrödinger equation.

The inverse scattering tecnique analyzed in chapter 2 which had been developed in classical field theory could be formulated as an exact operator method for solving quantum field theory ([SF78],[Skl79], [ThW79],[Ho79]). Morever, this method called quantum inverse scattering method is closely related to the Bethe Ansatz technique previously discussed and provides an elegant algebraic formulation of those results.

As said in the introduction and illustrated in the previous section, the quantum version of NS was originally solved by means of the traditional coordinate Bethe Ansatz. In this section we briefly illustrate how the QISM works in this special case.

The essence of the QISM consists in formulating the problem of quantum generalization of the classical IST as the problem of constructing the quantum operators $\hat{a}(\lambda), \hat{a}^\dagger(\lambda)$, $\hat{b}(\lambda), \hat{b}^\dagger(\lambda)$ from the transition coefficients after replacing $\psi, \bar{\psi}$ by $\psi, \psi^\dagger$ and subsequent normal ordering.

Most of the relations obtained in the classical case readily generalize to the quantum system but the operator character will introduce new structures into the theory.

Since P is a $c-$ number matrix, eq (2.80) still holds and we obtain

$$R(\lambda-\mu)(\widehat{A}(x,\lambda)\otimes I + I\otimes\widehat{A}(x,\mu) + \hbar\kappa\sigma_-\otimes\sigma_+) = (\widehat{A}(x,\lambda)\otimes I + I\otimes\widehat{A}(x,\mu) + \hbar\kappa\sigma_+\otimes\sigma_-)R(\lambda-\mu) \tag{3.65}$$

where $R(\lambda-\mu)=1-i\hbar r(\lambda-\mu)$. According to the correspondence principle

$$\frac{1}{i\hbar}[\ ,\ ]\stackrel{\hbar\to 0}{\longrightarrow}\{\ ,\ \}$$

eq (3.65) reduces to eq (2.82) in the classical limit. The terms $\kappa\sigma_-\otimes\sigma_+$ and $\kappa\sigma_+\otimes\sigma_-$ are the quantum corrections which arise due to the non commutativity of ψ and $\psi^\dagger$.

Let us now turn to the quantum T-matrix:

$$\widehat{T}(x,y;\lambda)=:T(x,y;\lambda):=:\exp\int_y^x\widehat{A}(z,\lambda)dz: \tag{3.66}$$

and define the following two products of the $\widehat{T}$ operators:

$$\widehat{T}_1(x,y;\lambda)=\widehat{T}(x,y;\lambda)\otimes 1 \tag{3.67}$$

$$\widehat{T}_2(x,y;\lambda)=1\otimes\widehat{T}(x,y;\lambda) \tag{3.68}$$

which represent particular embeddings of the $\widehat{T}$ operators into the tensor product space.

Classically, we have

$$T_1(x,y;\lambda)T_2(x,y;\mu)=T_2(x,y;\mu)T_1(x,y;\lambda). \tag{3.69}$$

At the quantum level, however, this is no longer true. Therefore, if we succeed in finding a relation between the two products in eq (3.69), we would determine the quantum noncommutativity.

The result is the following:

$$R(\lambda-\mu)\widehat{T}_1(x,y;\lambda)\widehat{T}_2(x,y;\mu)=\widehat{T}_2(x,y;\mu)\widehat{T}_1(x,y;\lambda)R(\lambda-\mu). \tag{3.70}$$

This relation is of fundamental importance in the study of quantum integrability and is the quantum analogue of eq (2.86).

The commutation relations for the infinite interval can be obtained performing the limits $x\to+\infty$, $y\to-\infty$. Defining

$$R_\pm(\lambda-\mu)=\left(1+\frac{i\hbar\kappa}{2(\lambda-\mu\pm i0)}\sigma_+\otimes\sigma_-\right)R(\lambda-\mu)\left(1+\frac{i\hbar\kappa}{2(\lambda-\mu\pm i0)}\sigma_-\otimes\sigma_+\right) \tag{3.71}$$

we can write

$$R_+(\lambda-\mu)\widehat{T}_1(\lambda)\widehat{T}_2(\mu) = \widehat{T}_2(\mu)\widehat{T}_1(\lambda)R_-(\lambda-\mu). \tag{3.72}$$

If we further recall the form of the transition matrix, namely

$$\widehat{T}(\lambda) = \begin{pmatrix} \hat{a}(\lambda) & \hat{b}(\lambda) \\ -\hat{b}^\dagger(\lambda) & \hat{a}^\dagger(\lambda) \end{pmatrix} \tag{3.73}$$

then the quantum commutation relations can be read out from eq (3.72). They have the form

$$[\widehat{a}(\lambda),\widehat{a}(\mu)] = [\widehat{a}(\lambda),\widehat{a}^\dagger(\mu)] = [\widehat{b}(\lambda),\widehat{b}^\dagger(\mu)] = 0 \tag{3.74a}$$

$$[\widehat{a}(\lambda),\widehat{b}(\mu)] = \frac{i\hbar\kappa}{(\lambda-\mu+i0)}\widehat{b}(\mu)\widehat{a}(\lambda) \tag{3.74b}$$

$$[\widehat{a}(\lambda),\widehat{b}^\dagger(\mu)] = -\frac{i\hbar\kappa}{(\lambda-\mu+i0)}\widehat{a}(\lambda)\widehat{b}^\dagger(\lambda) \tag{3.74c}$$

$$\begin{aligned}[\widehat{b}(\lambda),\widehat{b}^\dagger(\lambda)] &= 2\pi\kappa\delta(\lambda-\mu)\widehat{a}(\lambda)\widehat{a}^\dagger(\lambda) \\ &\quad - \hbar^2\kappa^2\left(\frac{1}{(\lambda-\mu)(\lambda-\mu+i0)} - \frac{i\pi\delta(\lambda-\mu)}{\lambda-\mu+i0}\right)\widehat{b}(\lambda)\widehat{b}^\dagger(\mu)\end{aligned} \tag{3.74d}$$

Note that in the limit of $\hbar \to 0$, the relations in eqns(3.74) reduce to the classical relations (2.97)

Furthermore, it follows from (3.74a) that $\log \hat{a}(\lambda)$ is a commutative family of operators and it can be shown that it includes the number of particles N, the momentum P and the Hamiltonian H.

Let us next define the operators:

$$\widehat{\phi}(\lambda) = (2\pi\hbar \mid \kappa \mid \widehat{a}^\dagger(\lambda)\widehat{a}(\lambda))^{-1/2}\widehat{b}^\dagger(\lambda) \tag{3.75}$$

$$\widehat{\phi}^\dagger(\lambda) = \widehat{b}(\lambda) = (2\pi\hbar \mid \kappa \mid \widehat{a}^\dagger(\lambda)\widehat{a}(\lambda))^{-1/2}. \tag{3.76}$$

Then we can show that

$$[\widehat{\phi}(\lambda),\widehat{\phi}(\mu)] = [\widehat{\phi}^\dagger(\lambda),\widehat{\phi}^\dagger(\mu)] = 0 \tag{3.77}$$

$$[\widehat{\phi}(\lambda),\widehat{\phi}^\dagger(\mu)] = \delta(\lambda-\mu) \tag{3.78}$$

as well as

$$[\log\widehat{a}(\lambda),\widehat{\phi}^\dagger(\mu)] = \log\left(1+\frac{i\hbar\kappa}{2(\lambda-\mu+i0)}\right)\widehat{\phi}^\dagger(\mu). \tag{3.79}$$

Thus we can think of $\widehat{\phi}^\dagger$ and $\widehat{\phi}$ as raising and lowering operators and construct the Fock space as

$$| k_1, k_2, \cdots, k_N >= \widehat{\phi}^\dagger(k_1)\widehat{\phi}^\dagger(k_2)\cdots\widehat{\phi}^\dagger(k_N)|0> \tag{3.80}$$

where we assume the vacuum to satisfy

$$\widehat{a}(\lambda)|0> \tag{3.81}$$

so that all the conserved quantum numbers of the vacuum would be vanishing. In terms of these new operators, the Hamiltonian of the system can be shown to take the form

$$H = \int_{-\infty}^{\infty} dk \quad k^2\widehat{\phi}^\dagger(k)\widehat{\phi}(k) \tag{3.82}$$

so that

$$H|k_1, k_2, \cdots, k_N >= \left(\sum_{i=1}^{N} k_i^2\right) | k_1, k_2, \cdots, k_N > . \tag{3.83}$$

We recognize these states to be nothing other than the Bethe ansatz states up to a constant. Therefore we have seen why the QSTM can be really interpreted as the algebraization of the Bethe method. We shall not enter into more details since the solution of the s-d exchange model in chapter 6 will provide a complete illustration of the method. Let us only mention at this point that the Bethe equations (3.15) can be obtained recquiring that the state (3.80) be an eigenvector of Tr $\hat{T}(\lambda)$ and equating the remaining 'unwanted' terms to zero.

Chapter 4

Integrable Statistical models

One of the most remarkable aspects of the Bethe's Ansatz is that it also provides an exact treatment of certain lattice models for which operators are defined on discrete lattice sites. The various developments, starting from the famous Onsager's solution of the two-dimensional Ising model, were unified and extended in the remarkable works of Baxter.

Baxter constructed the solution of the 'eight-vertex model' including all the previously solved models as special cases, and put forward the method which became the ground of the modern development of the Bethe Ansatz.

It has been found that certain operators which emerge in the formulation of the QSTM are directly related to the so-called transfer matrix (to be defined below) of the lattice models. Thus the study of soluble lattice models provides important new insights into the nature of exact integrability in quantum systems.

For 2-dimensional statistical mechanics, we have two types of models: the vertex models and the IRF (interaction round a face) models, which we shall introduce in the following two sections.

4.1 Lattice models.

4.1.1 Vertex models.

Consider a square lattice. The fluctuating variables $a = 1, .., n$ ('spins' or 'colours') are attached to each bond connecting the nearest-neighbour lattice site. The vertex Boltzmann weight $S_{a_1 a_1'}^{a_2 a_2'}$ (fig.4.1) correspond to each colour configuration around any lattice site. Denoting the energy of the vertex by $\varepsilon(a_1 a_1' a_2 a_2')$, we have the relation:

$$S_{a_1 a_1'}^{a_2 a_2'} = \exp[-\beta\varepsilon(a_1 a_1' a_2 a_2')], \quad \beta = 1/k_B T. \tag{4.1}$$

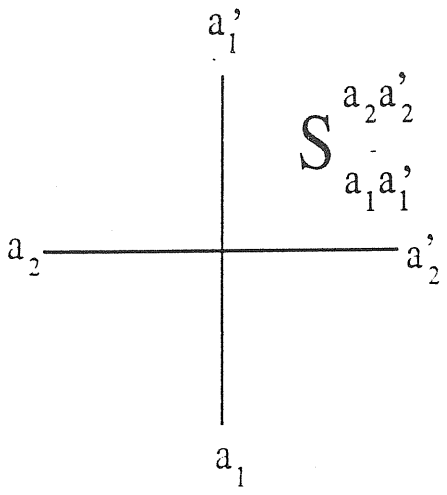


Figure 4.1

The (row-to-row) transfer matrix of a vertex model is the operator connecting the configurations on the vertical bonds placed between the rows $(n-1,n)$ and $(n,n-1)$ characterized respectively by the configurations $\{i_k\}$ and $\{j_k\}$. Then

$$T \equiv T_{i_1 \ldots . i_N}^{j_1 \ldots . j_N} = \sum_{\{p_k\}} \prod_{k=1}^{N} S_{i_k j_k}^{p_k p_{k+1}} \tag{4.2}$$

and the partition function for a lattice with M rows is given by

$$Z = Tr\{T^M\}. \tag{4.3}$$

This last equation shows how important is the knowledge of the eigenvalues of T. Namely just the largest eigenvalue Λ_{max} gives the free energy in the thermodynamic limit:

$$\begin{aligned} f &= -\lim_{N,M\to\infty} \frac{1}{NM} \log Z \\ &= \lim_{N\to\infty} \frac{1}{N} \log \Lambda_{max} \end{aligned} \tag{4.4}$$

Frow now on, let us assume for simplicity that the number of colours are only two, like in the usual vertex models. Then the transfer matrix becomes of order 2×2.

Suppose we have two sets of different Boltzmann weights and corresponding transfer matrices T and T'. Under what conditions do we have $[T,T'] = 0$? Can we answer this

question in terms of Boltzmann weights only? The main idea is to embed the transfer matrix into a family of commuting transfer-matrices and to try to diagonalize the whole family simultaneously.

We introduce the so-called monodromy matrix

$$L(\alpha) \equiv L^{\{j_1,\ldots,j_N\}j}_{\{i_1,\ldots,i_N\}i}(\alpha;\alpha_1,..,\alpha_N) = \\ S^{ip_1}_{i_1j_1}(\alpha_1-\alpha)S^{p_1p_2}_{i_2j_2}(\alpha_2-\alpha)\cdots S^{p_{n-1}j}_{i_Nj_N}(\alpha_N-\alpha) \tag{4.5}$$

which is a 2×2 matrix with operators in $V_N = \bigotimes_{k=1}^{N} V_v^{(k)}$ as matrix elements ($V_v^{(k)}$ is the vertical space associated to the k th column and in our specific case $V_v^{(k)} = \mathbb{C}^2$). The summation over all p_j indices is implicit.

It follows immediately that

$$T(\alpha) = Tr_h L(\alpha) = \sum_i L^{\{j\}i}_{\{i\}i}(\alpha). \tag{4.6}$$

The variable α is called spectral parameter and can be considered as a coupling constant also depending on the temperature.

Let us assume now that there exists a non singular matrix $R = R^{cd}_{ab}(\alpha,\alpha')$ such that

$$R(\alpha,\alpha')[S(\alpha)\otimes S(\alpha')] = [S(\alpha')\otimes S(\alpha)]R(\alpha,\alpha'). \tag{4.7}$$

Here, $\otimes$ means tensor product of matrices acting on V_h. R acts on $V_h\otimes V_h$.

More explicitely, this equation reads:

$$R^{bf}_{ae}(\alpha,\alpha')S^{fd}_{lm}(\alpha)S^{ec}_{mn}(\alpha') = S^{lm}_{ae}(\alpha')S^{mn}_{bf}(\alpha)R^{fd}_{ec}(\alpha,\alpha') \tag{4.8}$$

In both equations, there is an implicit matrix product on the vertical spaces V_v. This is the Yang-Baxter equation for the vertex models. If we regard R^{bf}_{ae} as a Boltzmann weight with state variables (a,e,b,f), we can represent the equation by the picture depicted in figure 4.2.

It follows from the definition of the monodromy matrix L and the equation (4.8) that

$$R(\alpha,\alpha')[L(\alpha)\otimes L(\alpha')] = [L(\alpha')\otimes L(\alpha)]R(\alpha,\alpha'). \tag{4.9}$$

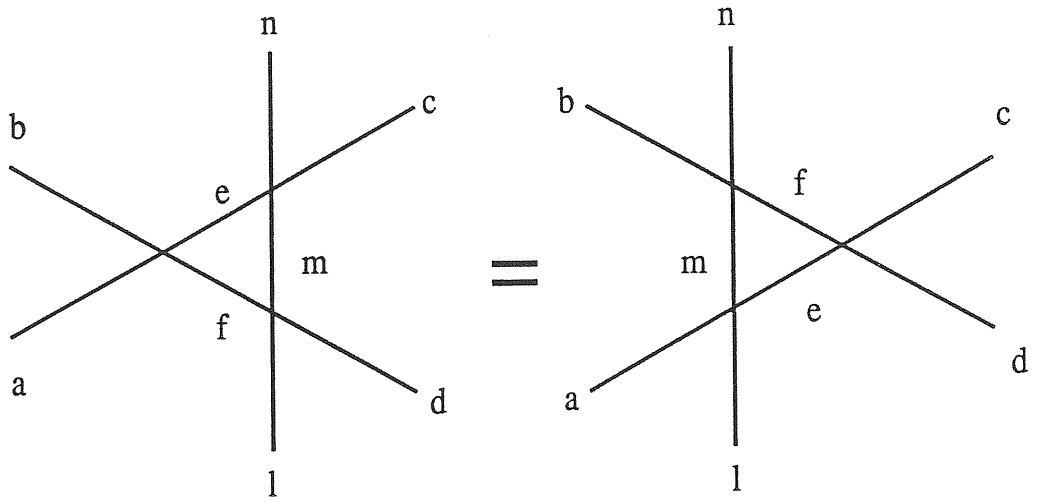


Figure 4.2

This equation holds in $V_h \otimes V_h \otimes V_v$. Leftmultiplying eq(4.9) by $R^{-1}(\alpha, \alpha')$ and taking the trace on $V_h \otimes V_h$ yields

$$T(\alpha)T(\alpha') = T(\alpha')T(\alpha) \quad \text{or} \quad [T(\alpha), T(\alpha')] = 0. \tag{4.10}$$

We have therefore a one-parameter family of commuting transfer matrices. This equivalence between the factorization relation (4.8) and the selfcommutativity (4.10), shown by Baxter [Ba72], will be very useful later.

Let us conclude this section with the identification of the vertex models to quantum theories in one-dimension. One can consider the space V_v as a quantum space of states for each site of a given horizontal line. The matrix $T(\theta)$ will be now a quantum operator in the total space V of quantum states. It can be shown that the operators

$$Q_k = -\frac{\delta^k}{\delta \theta^k} \ln T(\theta)_{\theta=0} \tag{4.11}$$

couple $(k+1)$ nearest neighbours on the horizontal line. Usually Q_1 can be identified with a quantum hamiltonian. The commutativity property of the transfer matrices implies that $[Q_k, Q_l] = 0 \; \forall k, l$. So we have an infinite number of commuting and conserved magnitudes and one can then conclude to have an integrable theory.

4.1.2 IRF models.

In this way of defining lattice models, the state variables σ_i are located on the lattice points (sites) of a square lattice. The Boltzmann weight is assigned to each unit face (or plaquette) depending on the state variable configuration round a face, the energy of which we denote by $\varepsilon(a,b,c,d)$. The corresponding Boltzmann weight is defined to be

$$\omega(a,b,c,d) = \exp(-\beta\varepsilon(a,b,c,d)), \quad \beta = 1/k_B T \tag{4.12}$$

so that the partition function of a square lattice of N sites becomes

$$Z_N = \sum_{\sigma_i} \cdots \sum_{\sigma_N} \prod_{faces} \omega(\sigma_i\sigma_j\sigma_k\sigma_l). \tag{4.13}$$

4.2 Description of the principal models.

Almost all important two-dimensional models can be expressed in the form of IRF models. Below, we give a brief description of the most important ones.

Nearest neighbour Ising model.

The Hamiltonian is

$$H = -J_1 \sum_{h.edges} \sigma_i\sigma_j - J_2 \sum_{v.edges} \sigma_k\sigma_l \tag{4.14}$$

where $\sigma_1, \cdots, \sigma_N = \pm 1$.

The minus sign before the interaction coefficients J_1, J_2 means we have ferromagnetic interaction, i.e.the system prefers equal spin. This model, originally solved by Onsager [On44] can be incorporated into an IRF model through the identification

$$\varepsilon(a,b,c,d) = -\frac{1}{2}J_1(ab+cd) - \frac{1}{2}J_2(bc+ad). \tag{4.15}$$

Six- and Eight-Vertex models.

Let us first define the models as vertex models. If we consider two-state bond variables characterized by an arrow, there are 16 distinct types of combinations of arrows at each lattice point. Let us consider only those configurations of arrows where the number of in-arrows at each lattice point is even. The eight allowed configurations are then

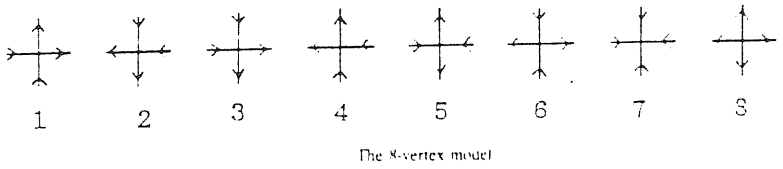


Figure 4.3

where we assume that the energy is invariant under the simultaneous inversion of the directions of all the arrows:

$$\varepsilon_1 = \varepsilon_2, \quad \varepsilon_3 = \varepsilon_4, \quad \varepsilon_5 = \varepsilon_6, \quad \varepsilon_7 = \varepsilon_8. \tag{4.16}$$

This defines the 8-vertex model. The 6-vertex model is obtained eliminating the last two configurations corresponding to ε_7 and ε_8. Indeed, the 6-vertex model has been introduced by Pauling and Slater to calculate the residual entropy of ice and to describe phase transitions of ferroelectric and antiferroelectric systems. The oxygen atoms in ice are tetrahedrally arranged, so that each atom is hydrogen-bonded to four others. But only two H-atoms are attached to each O-atom and this gives an orientation to the bonds: the two bonds where the H-atoms belonging to the water molecule are located, are represented by incoming arrows, according to the so-called *ice-rule*. More recent is the application of the 6-vertex model to surface phase transitions (roughening and surface melting) through the van Beijeren's construction [Bj77], which maps surface models to certain special cases of 6V-models. The IRF identification of the 8V-model is given by

$$\varepsilon(a,b,c,d) = -Jac - J'bd - J_4abcd. \tag{4.17}$$

The 6V-model can be thought as a special case of this model in which $-J, -J', -J_4$ tend to infinite, the appropriate differences remaining finite.

Hard Hexagons model.

Here, the state variable can take on the value either 0 or 1, and the Boltzmann weight is nonzero only if there are no neighbouring pairs with state variables equal to one. This means

$$\sigma_i = 0,1 \qquad 0 \leq \sigma_i + \sigma_j \leq 1 \quad \textit{for adjacent sites } i \textit{ and } j. \tag{4.18}$$

It is called hard hexagons because gluing the six triangles surrounding a particle they form a hexagon and no two such hexagons can overlap. The energy can be shown to have the form

$$\varepsilon(a,b,c,d) = -\beta \log\{z^{(a+b+c+d)/4}(1-ab)(1-bc)(1-cd)(1-da)(1-bd)\} \tag{4.19}$$

where z is the activity. The model has been successfully applied to the adsorption problem of a He^4 monolayer on a carbon surface.

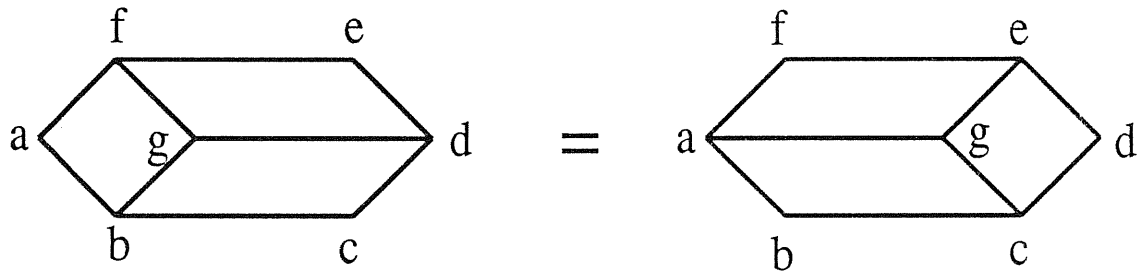


Figure 4.4

These four exactly solved IRF models are the most significant ones. There are obviously other exactly solvable models. Among them we mention the next-nearest-neighbour Ising model, the q-state model, the 3-spin model and the 6V with an external electric field.

Similarly to (4.8), we can obtain the Yang-Baxter relation for the IRF model. Writing $\omega(a,b,c,d)$ as S^{dc}_{ab}, we have

$$S^{fg}_{ab}(\alpha)S^{ed}_{fg}(\alpha')S^{dc}_{gb}(\alpha'') = S^{eg}_{fa}(\alpha'')S^{gc}_{ab}(\alpha')S^{ed}_{gc}(\alpha) \tag{4.20}$$

where summation over g is assumed. Relation (4.20) is also called star-triangle equation (a term conied by Onsager in his work on the Ising model) and has a graphical interpretation depicted in the figure 4.4).

4.3 Mathematical methods.

When the Boltzmann weights satisfy the Yang-Baxter equation ((4.8):vertex,(4.20):IRF), the row-to-row transfer matrices commute and then the model is exactly solvable. This fact offers a very powerful method to construct exactly solvable models through the two following steps:

(1) Introduction of a model with appropriate physical requirements (symmetries, number of state variables, etc.)

(2) Solution of the Yang-Baxter equation for the model.

Point (1) is crucial in order to obtain meaningfull solutions of (2). Let us take IRF models for example. In order to solve the functional equation (4.20), we assume that

$$\alpha = u,\ \alpha' = u + v,\ \alpha'' = v, \tag{4.21}$$

so that

$$\sum_g S_{ab}^{fg}(u) S_{fg}^{ed}(u+v) S_{gb}^{dc}(v) = \sum_g S_{gc}^{ed}(u) S_{ab}^{gc}(u+v) S_{fa}^{eg}(v) \tag{4.22}$$

and reflection symmetry among the Boltzmann weights

$$S_{ab}^{dc}(u) = S_{cb}^{da}(u) = S_{ad}^{bc}(u). \tag{4.23}$$

Then the YBE (4.23) may be regarded as addition theorems for S_{ab}^{dc}. The solutions are expressed in terms of functions whose genus is less or equal than one. They are classified into three cases: (1) elliptic (2) trigonometric and (3) rational. Recently, it was shown that without assuming the additive parametrization, we obtain solutions with genus larger than one [AY87].

4.3.1 Hard square model with diagonal interactions.

The elliptic function parametrization.

In order to illustrate the strategies mentioned before, we use the hard square model with diagonal interactions. This is a special case of IRF model where

$$\begin{aligned}\omega(a,b,c,d) &= mz^{(a+b+c+d)/4}e^{Lac+Mbd}t^{-(a-b+c-d)} \\ &\quad \text{if } ab=bc=cd=da=0 \\ &= 0 \quad \text{otherwise.}\end{aligned} \tag{4.24}$$

m is a trivial normalization factor; t cancels out of the partition function; L and M are diagonal interactions coefficents. The hard hexagon model is obtained by taking $m=1,\ L=0$ and $M=-\infty$.

Substituing these quantities into the YB equations and reducing appropriately the seven equations obtained, we are left with the following three equations:

$$\Delta_i = \Delta_i', \quad i=1,2,3 \tag{4.25}$$

where

$$\begin{aligned}\Delta_1 &= z^{-1/2}(1-ze^{L+M}), \\ \Delta_2 &= z^{1/2}(e^L+e^M-e^{L+M}), \\ \Delta_3 &= z^{-1/2}(e^{-L}+e^{-M}-e^{-(L+M)}-ze^{L+M}),\end{aligned} \tag{4.26}$$

and similarly for Δ_i', z', L', M'.

In order to obtain nontrivial solutions of (4.25), we have to suppose that $\Delta_1, \Delta_2, \Delta_3$ satisfy the constraints

$$\Delta_2 = \Delta_1^{-1},\ \Delta_3 = \Delta_1 + \Delta_1^{-1}. \tag{4.27}$$

This implies that

$$z = \frac{(1-e^{-L})(1-e^{-M})}{e^{(L+M)}-e^L-e^M}. \tag{4.28}$$

Let us set $\Delta = \Delta_1$. Then if two models differ in their values of z, L, M, but have the same value of Δ and both satisfy (4.28), then their transfer-matrices commute. Note that (4.28) is satisfied for all z in the limit $L \to 0$ and $M \to -\infty$, which is the hard hexagon model. Eliminating z between (4.27) and (4.28) gives

$$\Delta^{-2} e^{L+M} = (e^L - 1)(e^M - 1)(e^{L+M} - e^L - e^M). \tag{4.29}$$

Given Δ, this is a symmetric biquadratic relation between e^L and e^M, which can be parametrized in terms of elliptic functions (see [Ba82b], section 15.10). We denote the elliptic theta function Θ_1 by

$$\Theta_1(u,p) \equiv \Theta_1(u) = 2p^{1/4} \sin u \prod_{n=1}^{\infty} (1 - 2p^{2n} \cos 2u + p^{4n})(1 - p^{2n}) \tag{4.30}$$

where q is the nome and u the argument. Usually q is regarded as a real constant, $0 < q < 1$, while u is a complex number. Then we obtain for the Boltzmann weights:

$$\begin{aligned}
\omega_1 &= \omega(0,0,0,0) = \frac{\Theta_1(3\lambda - u)}{\Theta_1(3\lambda)}, \\
\omega_2 &= \omega(0,1,0,0) = \omega(0,0,0,1) = \frac{\Theta_1(\lambda - u)}{\Theta_1(\lambda)} \\
\omega_3 &= \omega(1,0,0,0) = \omega(0,0,1,0) = \frac{\Theta_1(u)}{[\Theta_1(\lambda)\Theta_1(2\lambda)]^{1/2}} \\
\omega_4 &= \omega(0,1,0,1) = \frac{\Theta_1(4\lambda - u)}{\Theta_1(4\lambda)} \\
\omega_5 &= \omega(1,0,1,0) = \frac{\Theta_1(2\lambda - u)}{\Theta_1(2\lambda)}
\end{aligned} \tag{4.31}$$

where $\lambda = \pi/5$.

If L', M', z' are given for u' and the same with L'', M'', z'' for u'', the YB equation (4.25) are satisfied provided only that

$$u + u' + u'' = \frac{\pi}{5}. \tag{4.32}$$

In fact, we can regard the parametrization (4.31) as a mapping from the variables L, M to the variables p^2, u. Taking p^2 and u to satisfy

$$-1 < p^2 < 1, \quad -\pi/5 < u < 2\pi/5 \tag{4.33}$$

this mapping is then one-to-one.

The model has four regimes depending on the values of u and p. Defining $\Delta_c = [\frac{1}{2}(1+\sqrt{5})]^{-5/2}$, we have

$$I : \Delta > \Delta_c, \qquad p^2 < 0, -\pi/5 < u < 0, \tag{4.34}$$

$$II : 0 < \Delta < \Delta_c, \quad p^2 > 0, -\pi/5 < u < 0, \tag{4.35}$$

$$III : -\Delta_c < \Delta < 0, \; p^2 > 0, \quad 0 < u < \pi/5, \tag{4.36}$$

$$IV : \Delta < -\Delta_c, \qquad p^2 < 0, \quad 0 < u < \pi/5. \tag{4.37}$$

The states in I and III are disordered. The ones in II and IV are ordered, in that the translation invariance of the lattice is spontaneously broken. Therefore it results that the system is critical on the boundary given by $p^2 = 0$ and $\Delta = \pm\Delta_c$. For practical purpose, it results convenient to define new parameters x and w in the following way:

$$I, IV : \quad p^2 = -\exp(-\varepsilon), \; x = -\exp(-\pi^2/5\varepsilon), \; w = \exp(2\pi u/\varepsilon) \tag{4.38}$$

$$II, III : \quad p^2 = \exp(-\varepsilon), \; x = -\exp(-4\pi^2/5\varepsilon), \; w = \exp(-4\pi u/\varepsilon). \tag{4.39}$$

Therefore, we have seen that if two models have the same value of x, but different values of u, then their row-to-row transfer matrices commute and the Boltzmann weights are entire functions of u.

4.3.2 The corner transfer-matrices.

Analytical methods to calculate physical quantities have been developed. The free energy can be obtained by the *inversion method* [Sh81,Ba82a]. Another method which uses integration over Grassmann variables [Sa80] can be applied for some models and simplifies in a very elegant way the computations, showing explicitely the fermion algebra hidden in the problem.

The one point function (magnetization, density,etc.) is generally obtained by the *corner transfer-matrix method* [Ba82b]. Let us introduce the corner transfer-matrices. The lattice is divided into four quadrants. State variables on the boundary are fixed to

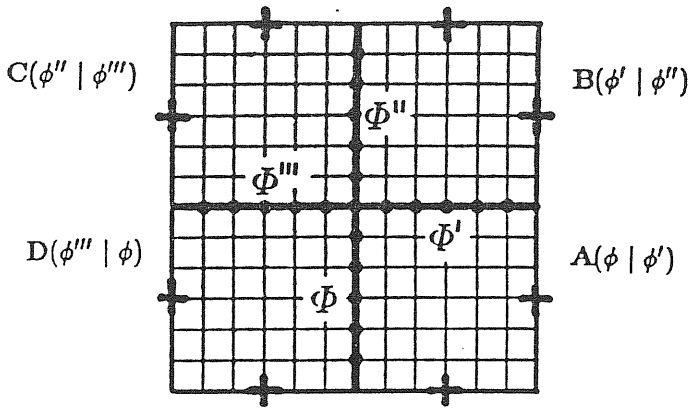


Figure 4.5

their ground state values. For configurations in the lower-right quadrant, we define the corner transfer-matrix **A** whose matrix elements are

$$A(\underbrace{\sigma_1,\cdots,\sigma_m}_{\phi}\,|\,\underbrace{\sigma_1',\cdots,\sigma_m'}_{\phi'}) = \delta(\sigma_1,\sigma_1')\sum\prod_{faces}\omega(\sigma_i,\sigma_j,\sigma_k,\sigma_l). \tag{4.40}$$

The sets of variables ϕ and ϕ' are not summed over. We write $\mathbf{A} = A(\phi,\phi')$ and similarly, we introduce $\mathbf{B},\mathbf{C}$ and $\mathbf{D}$ for the other three quadrants so that the partition function (4.13) of the full lattice becomes (see figure 4.5):

$$Z_N = \sum_{\phi\cdots\phi'''} A(\phi|\phi')\cdots D(\phi'''|\phi) = Tr\ \mathbf{ABCD}. \tag{4.41}$$

The summation in (4.41) is over all spin sets $\phi\cdots\phi'''$ subject to the restriction that $\sigma_1 = \sigma_1' = \sigma_1'' = \sigma_1'''$. This can be taken into account writing **A**,**B**,**C**,**D** in the following block diagonal form:

$$A = \begin{array}{c} \\ + \\ - \end{array}\begin{array}{c} \begin{array}{cc} + & - \end{array} \\ \begin{pmatrix} // & 0 \\ 0 & // \end{pmatrix} \end{array}$$

and the same for **B**,**C** and **D**. Using these definitions, we can write

$$\begin{aligned} M_0 = <\sigma_1> &= \frac{1}{Z_N}\sum_{\substack{all\\spins}}\sigma_1\prod_{\substack{all\\faces}}\omega(\sigma_i,\sigma_j,\sigma_k,\sigma_l) \\ &= \frac{Tr\ \mathbf{SABCD}}{Tr\ \mathbf{ABCD}}, \quad \mathbf{S} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \end{aligned} \tag{4.42}$$

By using the analytic properties of corner transfer-matrices and the explicit form of the Boltzmann weights, it is then possible to calculate the one-point function $< \sigma_1 >$. Let us only recall the main steps leadind to the result.

First, because of the symmetries of the YBE, we can relate **B**,**C** and **D** to **A** and write

$$M_0 = \frac{Tr\ \mathbf{S}\mathbf{A}^2(u)\mathbf{A}^2(\lambda - u)}{Tr\ \mathbf{A}^2(u)\mathbf{A}^2(\lambda - u)}. \tag{4.43}$$

Then, since $A(u)$ and $A(v)$ commute in the thermodynamic limit, they can be simultaneously diagonalized by a u-or v-independent transformation and also normalized so that their top left elements (the maximum eigenvalue) becomes one. Therefore

$$[\mathbf{A}(u), \mathbf{A}(v)] = 0 \tag{4.44}$$

$$\mathbf{A}(u)\mathbf{A}(v) = \ scalar \times \mathbf{X}(u+v) \tag{4.45}$$

and

$$\mathbf{A}_d(u) = \frac{P^{-1}\mathbf{A}(u)P}{\alpha_1(u)}, \quad \mathbf{X}_d(u) = \frac{P^{-1}\mathbf{A}(u)P}{x_1(u)}. \tag{4.46}$$

The $i = 1$ case implies that the scalar is 1; for $i \geq 2$, we have

$$\alpha_i(u)\alpha_i(v) = x_i(u+v). \tag{4.48}$$

These equations must be true for all complex numbers u and v in some domain and constitute a very strong condition on $\alpha_i(u)$. We prepare some mathematics to write down the result. Define a function $F(\sigma_0, q)$ by

$$F(\sigma_0, q) \equiv F(\sigma_0) = \sum{}' q^{\sigma_1 + 2\sigma_2 + 3\sigma_3 + \cdots} \tag{4.48}$$

where the sum $\sum'$ is over $\sigma_1, \sigma_2, \cdots$ under the restriction

$$R: \quad \sigma_i = 0, 1 \ \text{ and } \ 0 \leq \sigma_i + \sigma_{i+1} \leq 1, \ i \geq 0, \tag{4.49}$$

and two elliptic functions through

$$G(x) = F(0), \quad H(x) = F(1). \tag{4.50}$$

In regime I, we can write that

$$\alpha_i(u) = R^{1/2}\sigma_i e^{-3u(\sigma_1+2\sigma_2+\cdots+m\sigma_m)} \tag{4.51}$$

where

$$R^2 = -\frac{xG(x)}{H(x)}, \quad -1 < x < 0, \quad x = e^{-\lambda}. \tag{4.52}$$

Substitution in (4.43) yields

$$M_0 = <\sigma_1> = \frac{R^2F(1)}{F(0)+R^2F(1)}, \quad q = x^6 \tag{4.53}$$

and making further usage of identities found by Ramanujan, we finally arrive at

$$M_0 = \frac{xG(x)H(x^6)P(x^3)}{P(x)}, \quad P(x) = \prod_{n+1}^{\infty}(1-x^{2n-1}). \tag{4.54}$$

The one-point in other regimes and the free energy can also be obtained in a similar way without any approximation [Ba82b].

Chapter 5

A physical application: the theory of magnetic alloys

5.1 Introduction.

The subject of this chapter is the description of some models applied to the theory of dilute magnetic alloys.

The development of the theory started with the following questions: 'Under what circumstances does a localized moment exist in a metal?' and 'What are the consequences of the interaction between a localized moment and the conduction electrons?'.

Below, we shall discuss various model hamiltonians which can answer, at least partially, these questions: the simple s-d exchange model (or Kondo hamiltonian) and the more general Anderson model.

The great interest for these models is due to the following fact: they are quantum many body theories characterized by the growth of an effective coupling at low energies. As well known, the principal difficulty in solving these problems is that the relevant low energy phenomena cannot be treated in the framework of conventional perturbation theory.

In order to avoid these problems, various approaches have been used. Among them, phenomenological theories (Nozieres[N73]) and renormalization group tecniques (Wilson[W75]). Remarkably, it has been shown in the last years that many of these models are completely integrable and the relevant solutions have been obtained.

5.2 Transition metals and their compounds.

An interesting group of solids is the family of materials containing transition-group elements whose atoms have incomplete d or f subshells. The transition metals are divided into three large groups - d metals, f metals, and mixed d-f metals- which are further divided as follows:

a. *3d metals* Sc, Ti, V, Cr, Mn, Fe, Co, Ni (iron group).

b. *4d metals* Y, Zr, Nb, Mo, Tc, Ru, Rh, Pd (palladium group).

c. *5d metals* La, Hf, Ta, W, Re, Os, Ir, Pt (platinum group).

d. *4f metals* Ce, Pr, Nd, Pm, Sm, Eu, Gd, Tb, Dy, Ho, Er,Tm, Yb, Lu (rare earth metals)

e. *6d-5f metals* Ac, Th, Pa, U (actinides)

Metals and alloys in which atomic magnetic ordering (ferro-, antiferro- or ferrimagnetic) is observed in a certain temperature interval requires that at least one of the constituents be a transition element. In these materials, many-electrons effects are manifest very intensively and even when they possess no magnetic order, they still exhibit unusual thermal, magnetic, optical, electrical and even mechanical properties. The nature of these anomalies is due to the peculiar behavior of d and f states.

Let us consider the atom of a transition element. First we have to pay attention to the small radius of d electron and particularly f electron subshells in comparision with characteristic distance between nearest ions in the metallic state of a relevant element. Another interesting feature is that the filling of d and f subshells proceeds in jumps at the middle and end of each series. This indicates that the one-electron approach is inadequate to describe the atoms of the transition elements. In particular, we have to take account of the exchange correlation interaction, which leads to the formation of an atomic magnetic moment.

What happens to electronic states as atoms are united into a crystal? We are concerned primarly with the states of unfilled shells, for these states are responsible for almost all the properties of the crystal. When atomic states form a band we have a gain in kinetic energy and a loss in Coulomb repulsive energy. If the radius of a relevant electron subshell

exceeds the nearest atom or ion distance, Bloch states arise. This situation occurs for outer s - and p -shell electrons, giving rise to metals, covalent bonds or ionic crystals.

A different situation occurs in rare-earth metals for the states of f electrons, which are believed to be well localized ($\sim$ 0.3 Å). These electrons maintain their atomlike character and do not form a band, so that their magnetic moment in compounds or in the metallic state are normally close to those of the corresponding atoms.

Both cases may be realized for d states. The atomlike behavior of d -states persists in many semiconducting compounds (f.ex. NiO). In other cases metal-insulator transitions occur. But on the whole the problem as to the nature of d states is solved rather in favor of their band character.

5.3 The Kondo problem : s-d exchange model.

The interaction of magnetic moments with conduction electrons and the magnetic formation in metals has been a topic of large interest during the last decades.

It is well known that a small amount of magnetic impurities dissolved in a non-magnetic metal drastically affects its properties. The first striking experimental data on this subject was the occurence of a resistivity minimum at low temperatures found in certain, supposed to be pure, metals. Later it was realized that the effect was due to transition element impurities and is proportional to their concentration.

Then detailed studies of dilute magnetic alloys have shown that in all alloys below a certain temperature T_k , called the Kondo temperature, the impurity part of the magnetic susceptibility becomes temperature independent and remains finite at $T = 0$.

The properties of dilute alloys with localized moments are basically determined by the exchange interaction between the conduction electrons of the metal and the magnetic impurity. The conventional description of this interaction is based on the so-called s-d exchange (or Kondo) model:

$$H_{s-d} = \sum_{k,\sigma} e_k c^{\dagger}_{k\sigma} c_{k\sigma} + I \sum_{kk'\sigma\sigma'} c^{\dagger}_{k\sigma} \sigma^{\mu}_{\sigma\sigma'} c_{k'\sigma'} S^{\mu} \tag{5.1}$$

where $c^{\dagger}_{k\sigma}, c_{k\sigma}$ are the conduction electron creation and annihilation operators, e_k is the electron kinetic energy, S^{μ} is the impurity spin operator localized at $x = 0$.

Some considerations have to be done at this level. First of all, in deriving the Hamiltonian, the spectrum has been linearized around $\epsilon_F = 0$ since we are interested only in the low- temperature properties of the metal. As we will see later, this is fundamental for having the integrability of the model. Only the s-wave states around the impurity are kept and a cut-off D is imposed.

The Hamiltonian is thus built out of operators c_{klm} with $|k - \epsilon_F| \leq D$ and $l = m = 0$. Furthermore, in real alloys the exchange interaction is always antiferromagnetic:
$I > 0$, with $\rho(\epsilon_F)I \ll 1$.

In 1964, Jim Kondo observed that the spin-flip scattering amplitude computed to second order of perturbation theory rises as energy and temperature decreases. For example, the resistivity due to the magnetic impurity is proportional to the scattering amplitude squared:

$$R_i \sim cI^2(T) \simeq c[I + \rho I^2 \log \frac{D}{T} + O(\rho^2 I^3)]^2 \tag{5.2}$$

where c is the concentration of magnetic impurities.

When the temperature is of the order of the Kondo temperature

$$T_k \sim D \exp(-\frac{1}{\rho I}) \tag{5.3}$$

all the terms of the perturbation expansion are of the same order in magnitude and, despite the weak bare coupling, perturbation theory no longer holds.

Another example is the impurity susceptibility which attains its free value

$$\chi = \mu^2/T \qquad (\mu^2 = \mu_B^2 g^2 S(S+1)) \tag{5.4}$$

up to corrections that vanish logarithmically at high temperatures:

$$\chi^i = \frac{\mu^2}{T}(1 - \frac{1}{\ln T/T_k} - \frac{\ln \ln T/T_k}{2 \ln^2 T/T_k} + O(\frac{1}{\ln T/T_k})^3). \tag{5.5}$$

we see immediately that at $T \sim T_k$, perturbation theory breaks down and nothing can be learned about the ground state. This crossover from the strong coupling regime ($T \gg T_k$) with logarithmic behavior to the weak coupling regime ($T \ll T_k$) with Fermi liquid simple behavior, is the essence of the Kondo problem.

5.4 The Anderson model

The other interesting problem in the subject *Magnetic impurities in non magnetic metals* is the description of the electronic mechanism leading to the formation of the magnetic moment of an ion with an incomplete inner shell placed in a metal.

The experimental evidence is that various impurities in different hosts display widely different behaviours. For instance, Mn, Cr and Fe in Cu, Au and Ag matrices, like Ce in La, possess well-defined magnetic moments: the impurity susceptibility follows the Curie law at high temperatures. On the other hand, Ni and Ti in the same Cu, Au and Ag samples prove to be non-magnetic. Transition ions which proved to be magnetic in Cu, Au and Ag samples, lose this property in Al (see, for example [Ri74]). Similarly, Ce in La ceases to be magnetic after Th or Y is introduced (see [St77] for a review).

Whether the impurity is magnetic or non-magnetic, depends on the properties of both the impurity ion and the host metal. A free $3d$ or $4f$ ion is obviously magnetic. However, the metal overlap of the $3d$ or $4f$ electron wave function with that of the conduction electron band of the host metal leads to the delocalization of the impurity electron states and can destroy the magnetic moment [Fr58].

5.4.1 Derivation of the model.

The behaviour of the impurity is basically affected by two interactions: atomic Coulomb and exchange forces in a free atom and the admixture of the wavefunctions with the conduction electron band of the host. Let us consider the general form of the metal-impurity interaction:

$$H = H_0 + \sum_{j=1}^{N} V_{imp}(r_j) + \frac{1}{2}\sum_{i\neq j}\frac{e^2}{r_{ij}} + \sum_{j=1}^{N}\frac{1}{mcr_j}\frac{dV_{imp}(r_j)}{dr_j}(\mathbf{L}_j\cdot\sigma_j). \qquad (5.6)$$

Here H_0 is a hamiltonian of electrons placed in the potential of the crystal lattice; $m, \mathbf{L}$ and σ, N are the mass, orbital moment, spin and the number of electrons, respectively; $V_{imp}(r)$ is a potential of the impurity stripped of all the electrons of the outer shell. The third term in (5.6) is a Coulomb interaction, whilst the fourth term describes the spin-orbit coupling caused by the impurity potential.

A crucial observation, at this point, is that the $l = 2$ (or 3) levels of V_{imp} are very close to the Fermi surface, thus forming the narrow resonance. Its small width is due to the difference between the interatomic distance k_F^{-1} and the Bohr radius r_B of the $d(f)$-shell: $k_F r_B \ll 1$. This greatly simplifies the Hamiltonian (5.6) because the eigenfunctions can be expanded using as a basis the orbital and band wavefunctions. For the latter it is convenient to choose a spherical wave with the centre at the impurity site:

$$\Psi^\dagger = \sum_{l,m,\sigma} \Psi_{lm}(k,\mathbf{r}) a^\dagger_{klm\sigma} + \sum_{m,\sigma} \Psi_d(\mathbf{r}) d^\dagger_{m\sigma}, \tag{5.7}$$

where $k < r_B^{-1}$ and $a^\dagger_{klm\sigma}$ is a creation operator for the spherical wave with centre at the impurity point:

$$\Psi_{lm}(k,\mathbf{r}) = r_l(kr) Y_{lm}(\frac{\mathbf{r}}{r}). \tag{5.8}$$

The operator $d^\dagger_{m\sigma}$ corresponds to the localized components of the state with $l = 2, 3$:

$$\Psi_d(\mathbf{r}) = R_{l_0}(r/r_B) Y_{l_0 m}(\mathbf{r}/r), \tag{5.9}$$

where $l_0 = 2$ and 3 for the transition and rare-earth impurities, m is a z-component of the angular momentum: $m = -l_0, ..., l_0$.

It must be noted here that the set of functions so chosen is not orthogonal. However, it can be shown that the overlap integral relative to the non-orthogonality is sufficiently small. When rewritten in terms of $a^\dagger, a$ and $d^\dagger, d$ the hamiltonian contains a rather large numbers of terms.

(i) The terms with only a and $a^\dagger$ form the hamiltonian of the host metal:

$$H_0 = \sum_{k,l,m,\sigma} \epsilon(k) a^\dagger_{klm\sigma} a_{klm\sigma}. \tag{5.10}$$

(ii) The terms with only d operators represent the hamiltonian of a $3d(4f)$ ions in a crystal field:

$$H_{atom} = \sum_{m,m',\sigma} \epsilon_{mm'} d^\dagger_{m\sigma} d_{m'\sigma} + \sum_{m_i,\sigma_i} U^{m_3\sigma_3;m_4\sigma_4}_{m_1\sigma_1;m_2\sigma_2} d^\dagger_{m_1\sigma_1} d^\dagger_{m_2\sigma_2} d_{m_3\sigma_3} d_{m_4\sigma_4} - A\,\mathbf{L}_d \cdot \mathbf{S}_d \tag{5.11}$$

Here $\epsilon_{mm'}$ is the one-electron atomic energy (measured from the Fermi energy chosen at the origin). The second term is the non-relativistic part of interaction between *d(f)* electrons. The last term represent the spin-orbit coupling:

$$\mathbf{L}_d = \sum_{m,m',\sigma} d^\dagger_{m\sigma} \mathbf{I}_{mm'} d_{m'\sigma} \tag{5.12}$$

is the total angular momentum of *d(f)* shell and

$$\mathbf{S}_d = \sum_{m,\sigma,\sigma'} d^\dagger_{m\sigma} \sigma_{\sigma\sigma'} d_{m\sigma'} \tag{5.13}$$

is the total spin of the *d(f)* shell.

(iii) The terms with a a and a d operator represent the one-electron mixing interaction:

$$H_{mix} = \sum_{l,m,m',\sigma} v^{l_0 m}_{lm'} (a^\dagger_{klm'\sigma} d_{m\sigma} + h.c.). \tag{5.14}$$

Evaluating the hybridization matrix element, one can always neglect the crystal field and spin-orbit effects. Therefore, we have spherical symmetry and this interaction involve only conduction electrons in the partial wave $l = l_0$, requiring it to have the following form which conserves the angular momentum as well as spin:

$$H_{mix} = \sum_{k,m,\sigma} v_k (a^\dagger_{kl_0 m\sigma} d_{m\sigma} + h.c.)$$

$$v^{l_0 m}_{lm'} = v_k \delta_{ll_0} \delta_{mm'}. \tag{5.15}$$

(iv) Finally there are terms containing both d and a operators which have the following form: (1) $a^\dagger d^\dagger a d$ describes the contact exchange coupling. This interaction is relevant only for alloys which do not exhibit the Kondo effect.

(2) $a^\dagger a^\dagger dd, d^\dagger d^\dagger aa$ terms describe processes that in the cases of interset may only be virtual (the states with $(n_{d_0} \pm 2)$ corresponding to two additional orbital electrons or holes lie sufficiently high).

(3) $a^\dagger a^\dagger ad, d^\dagger d^\dagger da$ terms gives only trivial renormalization of virtual-mixing coupling [Hi78]

Thus, the behaviour of a magnetic impurity in a metal may be described by the so-called semiphenomenological Anderson hamiltonian [A61]:

$$H_A = \sum_{k,\sigma=\uparrow\downarrow} \sum_{m=-l_0}^{l_0} \epsilon(k) a^\dagger_{km\sigma} a_{km\sigma} + \frac{1}{N} \sum_{k,m,\sigma} v_k (a^\dagger_{km\sigma} d_{m\sigma} + d^\dagger_{m\sigma} a_{km\sigma}) + H_{atom}, \quad (5.16)$$

where H_{atom} is given by (5.11). Here we write only conduction electrons in partial waves with $l = l_0$:

$$a_{kl_0m\sigma} \equiv a_{km\sigma}. \quad (5.17)$$

The essence of this hamiltonian is that it describes the one- dimensional system. All quantities entering it depend only on the modulus of $\mathbf{k} : |\mathbf{k}| = k$. This one-dimensionality follows from the hypothesis about a spherical Fermi surface and impurity-ion potential. Obviously, one-dimensionality is held only until one can consider impurities as independent scatterers.

Let us now shortly discuss the energy scales involved in the hamiltonian (5.16). First we ignore the hybridization term.

Let $n_d^{(0)}$ be the occupation number in the ground state of H_{atom}, $|n_d^{(0)}>$ being the subspace of atomic states with that occupation number. The hybridization term in (5.16) couples $|n_d^{(0)}>$ to the subspaces $|n_d^{(0)} \pm 1>$ with the occupation number $n_d^{(0)} \pm 1$. Since the conduction band acts as a reservoir, one-electron energies corresponding to transitions from $|n_d^{(0)}>$ to $|n_d^{(0)} \pm 1>$ should be measured from the Fermi level. Typical ionization energy for $3d$ and Ce ions is $\sim 2 - 3eV$. Conversely, in some $4f$ compounds, one of these ionization energies may be sufficiently small, while the other is large.

Within a given valency subspace, the various terms are split according to Hund's rule, and if $L \neq 0$ one should then consider the crystal electric field and the spin-orbit splitting.

In transition impurities, it results that all the energies involved in the above-mentioned effects exceed the possible temperature interval. Therefore, one should consider in hamiltonian (5.16) only virtual transitions between the ground state of multiplet $|n_d^{(0)}>$ and the states of multiplets $|n_d^{(0)} \pm 1>$. The typical energy differences corresponding to these transitions are denoted by $E_\pm$.

The hybridization mixes the states with different occupation numbers and may break the localized moment. Resonances with adjacent occupation numbers acquire the width

$\Gamma \sim \rho(\epsilon_F)v^2$. The real processes with change of occupation number can be neglected provided that

$$\frac{\Gamma}{E_{\pm}} \ll 1. \tag{5.18}$$

If the above-mentioned condition holds, the system is in the local moment regime and we have the Kondo effect. Indeed, Scrieffer and Wolff [SW66] have shown that, in this limit, an exchange hamiltonian emerges as a result of projecting the Anderson model into a sector with a fixed number of particles $n_d^{(0)}$.

In the opposite case, two configurations $|n_d^{(0)}>$ and $|n_d^{(0)} \pm 1>$ have large probabilities of occupation, with the result that the system exhibits an intermediate impurity-level occupation, that is, a nonintegral valence. This state is non- magnetic and only the fluctuation of the occupation numbers for different spins can result in a magnetic susceptibility.

Therefore, changing the relative values of U, Γ and ϵ_d the non-magnetic regime can go over continuously to the magnetic one, but simultaneously, the impurity magnetic moment is compensated by the conduction electrons due to the Kondo effect. Hence, the Anderson model provides a unified description of the narrow many-body and the broad single-particle resonances.

5.4.2 The Anderson model for rare earth alloys.

Not all variants of the Anderson model turn out to be integrable. Some parameters in the model must be constrained in order to have an Hamiltonian diagonalizable by Bethe's ansatz.

In this section, we sall consider the so-called *degenerate Anderson model* which describes the Cerium or Ytterbium impurity in both the localized-moment and mixed valence regimes.

In this case, the strong $e^- - e^-$ repulsion in the f -shell enables to restrict ourselves to very large values of U. The f -level is then either empty or occupied by only one electron. The possible transition correspond here to $4f^0\ ^1S_0 \leftrightarrow 4f^1\ ^2F_{5/2}$ for Ce and $4f^0\ ^1S_0 \leftrightarrow 4f^{13}\ ^2F_{7/2}$ for Yb. We denote the siglet (non-magnetic) state by $|0>$. Due to the spin-orbit coupling, the magnetic state is characterized by the total angular momentum

J and its projection j. Furthermore, we assume the matrix element of hybridization

$$< J, j|H_{mix}|0 >= V \tag{5.19}$$

to be constant.

Let us define the operators $\mathbf{X}_{0j}, \mathbf{X}_{j0}$ changing the configuration of the impurity shell as follows:

$$|0 > = \mathbf{X}_{0j}|j > \tag{5.20a}$$

$$|j > = \mathbf{X}_{j0}|0 > . \tag{5.20b}$$

The other matrix elements of these operators are set to zero. These $\mathbf{X}$ operators satisfy the following parastatistics:

$$\mathbf{X}_{j0}\mathbf{X}_{0j'} + \mathbf{X}_{0j'}\mathbf{X}_{j0} = \mathbf{X}_{jj'} + \delta_{jj'}\mathbf{X}_{00} \tag{5.21}$$

or more generally

$$\mathbf{X}_{jk}\mathbf{X}_{pq} = \delta_{kp}\mathbf{X}_{jq} \tag{5.22}$$

and have matrix representation $\mathbf{X}_{jk}^{\alpha\beta} = \delta_{\alpha k}\delta_{\beta j}$. In terms of $\mathbf{X}$, the Anderson Hamiltonian takes the form:

$$H_A = \sum_{k,j} \epsilon(k) a_{kj}^\dagger a_{kj} + V \sum_{k,j} (a_{kj}^\dagger \mathbf{X}_{0j} + \mathbf{X}_{j0} a_{kj}) + \sum_{j=-J}^{J} \epsilon_f \mathbf{X}_{jj}. \tag{5.23}$$

where we omit the subscript J in a_{kJj}. Now, the infinite Coulomb repulsion in the f-shell is hidden in the algebra (5.21), since $\mathbf{X}_{j0}$ creates an f -electron subject to the condition that the f -level is empty. Depending on ϵ_f/Γ, the impurity has a magnetic moment ($\epsilon_f \ll \Gamma$), has a mixed valance ($\epsilon_f \sim \Gamma$), or is non-magnetic ($\epsilon_f \gg \Gamma$).

We shall discuss the Bethe Ansatz equation of this model in section (6.2).

Chapter 6

Solutions of the models.

6.1 Bethe Ansatz for the s-d exchange model.

In section (5.3), we formulated the canonical hamiltonian describing a magnetic atom in a non-magnetic host metal, the so-called *s-d exchange model*:

$$H_{s-d} = \sum_{k,\sigma} e_k c^{\dagger}_{k\sigma} c_{k\sigma} + I \sum_{kk'\sigma\sigma'} c^{\dagger}_{k\sigma} \sigma^{\mu}_{\sigma\sigma'} c_{k'\sigma'} S^{\mu}. \tag{6.1}$$

Shifting the hamiltonian to the coordinate representation,we obtain

$$H_{s-d} = \int dx \left(-i \sum_{\sigma} c^{\dagger}_{\sigma}(x) \frac{\partial}{\partial x} c_{\sigma}(x) + I\delta(x) \sum_{\sigma,\sigma'} c^{\dagger}_{\sigma}(x) \sigma^{\mu}_{\sigma\sigma'} c_{\sigma'}(x) S^{\mu} \right). \tag{6.2}$$

We consider the eigenstate in which there are electrons with spin components $\sigma_1, ..., \sigma_N$ and a localized moment with the components $s = -S, ..., S$:

$$\mid \Psi >= \int \Psi_{\sigma_1,..,\sigma_N,s}(x_1, ..., x_N) c^{\dagger}_{\sigma_1}(x_1) ... c^{\dagger}_{\sigma_N}(x_N)(S^{\dagger}) \prod_{j=1}^{s+S_N} dx_j \mid 0 >, \tag{6.3}$$

where $\mid 0 >$ is the state without particles and with a component of the impurity moment equal to $-S$.

The wave function $\Psi_{\sigma_1,..,\sigma_n,s}$ satisfies the Schrödinger equation

$$(-i \sum_{j=1}^{N} \frac{\partial}{\partial x_j} - E)\Psi_{\sigma_1,..\sigma_N,s} + \frac{1}{2} I \sum_{j=1}^{N} \delta(x_j) \sigma^{\mu}_{\sigma_j\sigma'_j} S^{\mu}_{ss'} \Psi_{\sigma_1,..\sigma_j,..\sigma_N,s'} = 0. \tag{6.4}$$

Bethe's hypothesis enable us to write down the solution of this equation for arbitrary N. Let us mention one more time the arguments anticipated in section (4.1) with a little modification:

Suppose that $Q = \{q_0, ..., q_N\}$ is a permutation of the numbers $\{0, 1, ..., N\}$ while $Q' = \{q'_1, ..., q_{N'}\}$ and $P = \{p_1, ..., p_N\}$ are permutations of the integers $\{1, ..., N\}$, with Q' coinciding with Q from which $q_i = 0$ is excluded, then in the region $X_Q = \{x_{q_0} < x_{q_1} < .... < x_{q_N}\}$, the wave function is *Bethe Ansatz* (see eq.(4.2)):

$$\Psi_{\sigma_1,..\sigma_N,s}(x_1, ..., x_N) = \sum_P A_{\sigma_1..\sigma_N,s}(Q, Q' \mid P) \exp\{i \sum_{j=1}^{N} k_{p_j} x_j\}. \tag{6.5}$$

where $\{k_j\}$ is a sequence of different values of k_j and $x_0 = 0$. The state (6.5) is the eigenstate of the hamiltonian (6.2) with the energy

$$E = \sum_{j=1}^{N} k_j. \tag{6.6}$$

The factors $A_{\sigma_1..\sigma_N,s}$ are obviously not independent. First we must require that Ψ be antisymmetric under permutations in the pair (x_i, σ_i). As a result

$$A_{\sigma_1..\sigma_N,s}(Q; Q' \mid P) = A_{\sigma_{q'_1}..\sigma_{q'_N},s}(Q; Q'P)(-1)^P, \tag{6.7}$$

where $Q'P$ is the product of the permutations and $(-1)^P$ is the parity of the permutation P. The factors A for different regions are linked through the Schrödinger equation (6.4), which determines the discontinuity of A on the boundary of the region X_Q. If X_Q and $X_{\widetilde{Q}}$ only differ by permutation of the particle x_j and the impurity $x_0 = 0$, then:

$$A_{...\sigma_j,...,s}(\widetilde{Q}) = R^{\sigma_j\sigma'_j}_{ss'} A_{...\sigma'_j...,s'}(Q). \tag{6.8}$$

where

$$R^{\sigma_j\sigma'_j}_{ss'} \equiv \mathbf{R}_{0j} = \exp\left(i\frac{I}{2}(\sigma_\mathbf{j} \cdot \mathbf{S})\right). \tag{6.9}$$

Here we define $(\sigma \cdot \mathbf{S})$ as $\sigma^\mu S^\mu$. If two regions differ by permutation of the particles x_i and x_j, then the factors $A(Q)$ and $A(\widetilde{Q})$ are connected via the permutation operator, i.e.:

$$\begin{aligned} A_{...\sigma_i...\sigma_j,...}(\widetilde{Q}) &= P^{\sigma_i\sigma'_i}_{\sigma_j\sigma'_j} A_{...\sigma'_i...\sigma'_j,...}(Q) \\ &= A_{...\sigma_j...\sigma_i...}(Q) \end{aligned} \tag{6.10}$$

where

$$P^{\sigma_i\sigma'_i}_{\sigma_j\sigma'_j} = \frac{1}{2}(\mathbf{1} \cdot \mathbf{1} + \sigma_{\sigma_i\sigma'_i} \cdot \sigma_{\sigma_j\sigma'_j}) = \delta_{\sigma_i\sigma'_j}\delta_{\sigma_j\sigma'_i} \tag{6.11}$$

is the permutation operator.

Let us consider the scattering of two particles by the impurity. There are two ways in which we can go from the region $x_1 < x_2 < 0$ to the region $x_1 > x_2 > 0$. The first path, say

$$x_1 < x_2 < 0 \rightarrow x_1 < 0 < x_2 \rightarrow 0 < x_1 < x_2 \rightarrow 0 < x_2 < x_1$$

transforms A into $\mathbf{P}_{12}\mathbf{R}_{10}\mathbf{R}_{20}\ A$. The other path, namely

$$x_1 < x_2 < 0 \rightarrow x_2 < 0 < x_1 \rightarrow x_2 < 0 < x_1 \rightarrow 0 < x_2 < x_1$$

leads to $A \rightarrow \mathbf{R}_{20}\mathbf{R}_{10}\mathbf{P}_{12}A$. It is easy to check that now

$$\mathbf{P}_{12}\mathbf{R}_{10}\mathbf{R}_{20} = \mathbf{R}_{20}\mathbf{R}_{10}\mathbf{P}_{12}. \tag{6.12}$$

This so-called *factorization condition* , together with the unitarity conditions

$$\mathbf{P}_{ji}\mathbf{P}_{ij} = \mathbf{I}, \qquad \mathbf{R}_{j0}\mathbf{R}_{0j} = \mathbf{I}, \tag{6.13}$$

guarantees that all ways of factorizing Q into products of pair transpositions lead to the same result, ensuring the applicability of the Bethe hypothesis. Therefore, the general solution of the Schrodinger equation (6.4) is given by the formula (6.5) together with the conditions (6.7),(6.8) and (6.10) [An80;Wi80,81].

6.1.1 Periodic boundary conditions.

The parameters $\{k_j\}$ of the wavefunction (6.5) are so far arbitrary quantities. In order to find the spectrum of the hamiltonian, we put the system on a box with lenght L and impose periodic boundary conditions (PBC):

$$\Psi(x_1, ..., x_j, ...x_N) = \Psi(x_1, ..., x_j + L, ..., x_N). \tag{6.14}$$

for any $j = 1, .., N$.

The equation (6.14) means that the particle at x_j must be shifted through all other particles in the same order as they are spatially arranged. Each 'shifting through' corresponds to an exchange of the spatial order of two particles or one particle and the impurity

and involves the operators $\mathbf{P}_{ji}$ and $\mathbf{R}_{j0}$ respectively. Between collisions the wavefunction acquires the phase shift $k_j L$.

The resulting equation for the coefficients $A(P \mid Q)$ leads to the following eigenvalue problem

$$\exp(ik_j L)\xi = T_j \xi, \qquad \xi = A(I \mid I) \tag{6.15}$$

where

$$T_j = P_{jj+1}...P_{jN}R_{j0}P_{j1}...P_{jj-1} \tag{6.16}$$

Due to the factorization conditions (6.12) and (6.13) the operators $\mathbf{T}_j$ commute with each other and this guarantees the existence of a common set of the eigenvectors of the operators $\mathbf{T}_j$. In fact, due to their special form here, $\mathbf{T}_j$ do not only commute but are simply equal to each other:

$$T_{j\sigma_1...\sigma_N;s}^{\sigma_1'...\sigma_N';s'} = \delta_{\sigma_1'\sigma_2}\delta_{\sigma_1'\sigma_2}\cdots\delta_{\sigma_{N-1}\sigma_N}R_{\sigma_N\sigma_1}^{ss'} \qquad \forall j. \tag{6.17}$$

In order to diagonalize the matrix, we identificate our problem to a vertex model and use the strategy pointed out by Baxter [Ba72]. Namely we build a parametric set of commuting operators $\mathbf{T}(\alpha)$, the $\mathbf{T}_j$ being members of this set and we employ the quantum inverse-scattering method.

6.1.2 The set of commuting operators.

For an arbitrary impurity spin the factorization equation includes matrices of different rank. For this reason, we separate the matrices that refer only to the particles, $r_{\sigma_2\sigma_2'}^{\sigma_1\sigma_1'}$, from those that refer to a particle and an impurity, $R_{ss'}^{\sigma\sigma'}(\alpha)(\sigma = 1,2)$ and $(s = -S, ..., S)$. With these notations, we obtain

$$\mathbf{r}_{ij}(\alpha)\mathbf{R}_{i0}(\alpha+\alpha')\mathbf{R}_{j0}(\alpha') = \mathbf{R}_{j0}(\alpha')\mathbf{R}_{i0}(\alpha+\alpha')\mathbf{r}_{ij}(\alpha), \tag{6.18a}$$

$$\mathbf{r}_{ij}(\alpha)\mathbf{r}_{ik}(\alpha+\alpha')\mathbf{r}_{jk}(\alpha') = \mathbf{r}_{jk}(\alpha')\mathbf{r}_{ik}(\alpha+\alpha')\mathbf{r}_{ij}(\alpha), \tag{6.18b}$$

with the condition that at some α_0:

$$\mathbf{R}_{j0}(\alpha_0) = \exp\left(i\frac{I}{2}(\sigma\cdot\mathbf{S})\right). \tag{6.19}$$

This condition automatically ensures that within an arbitrary factor, $P_{ij} = \mathbf{r}_{ij}(0)$, that is P_{ij} belongs to the set $\mathbf{r}_{ij}(\alpha)$.

Let us write down the main steps leading to the solution of the eqns (6.18). First we generalize the unitarity conditions (6.13):

$$\mathbf{r}_{ij}(\alpha)\mathbf{r}_{ji}(-\alpha) = \mathbf{I}, \tag{6.20a}$$

$$\mathbf{R}_{0j}(\alpha)\mathbf{R}_{j0}(-\alpha) = \mathbf{I}. \tag{6.20b}$$

Then we write $\mathbf{R}(\alpha_0)$ in the form

$$\exp\left(i\frac{I}{2}(\sigma \cdot \mathbf{S})\right) = w_0' + 2w'(\sigma \cdot \mathbf{S}) \tag{6.21}$$

where

$$w_0' = \frac{(2S+3)e^{iIS/2} + (2S-1)e^{-iI(S+1)/2}}{2(2S+1)} \tag{6.22}$$

$$w' = \frac{e^{iIS/2} - e^{-iI(S+1)/2}}{2(2S+1)} \tag{6.23}$$

Using the $O(3)$-invariance of the condition (6.19), the matrices $\mathbf{r}$ and $\mathbf{R}$ must be sought for in the form

$$R_{ss'}^{\sigma\sigma'}(\alpha) = w_0'(\alpha)\delta_{\sigma\sigma'}\delta_{ss'} + 2w'(\alpha)(\sigma_{\sigma\sigma'} \cdot \mathbf{S}_{ss'}) \tag{6.24}$$

$$r_{\mu\mu'}^{\sigma\sigma'}(\alpha) = w_0(\alpha)\delta_{\sigma\sigma'}\delta_{\mu\mu'} + w(\alpha)(\sigma_{\sigma\sigma'} \cdot \sigma_{\mu\mu'}) = \delta_{\sigma_i\sigma_j'}\delta_{\sigma_j\sigma_i'}. \tag{6.25}$$

For convenience, we introduce the following notations:

$$a = \omega_0 + \omega, \qquad b = \omega_0 - \omega, \qquad c = 2\omega \tag{6.26}$$

and the same for primed indices.

Substituing (6.24-5) into eqns (6.18), we find that

$$h(\alpha) \equiv \frac{b(\alpha)}{c(\alpha)} = \frac{b'(\alpha)}{c'(\alpha)} \tag{6.27}$$

and

$$b'(\alpha)c'(\alpha+\alpha')c(\alpha') + c'(\alpha)c'(\alpha+\alpha')b(\alpha) = c'(\alpha)b'(\alpha+\alpha')c(\alpha'). \tag{6.28}$$

In terms of $h(\alpha)$, we obtain

$$h(\alpha) + h(\alpha') = h(\alpha + \alpha') \tag{6.29}$$

and without loss of generality, we can assume that

$$h(\alpha) = \frac{\alpha}{ig} \tag{6.30}$$

where g is a parameter to be determined from the initial conditions at the points $\alpha = 0$ and $\alpha = \alpha_0$. Choosing $\alpha_0 = 1$ and respecting the unitarity conditions (6.20) for any α, we obtain

$$b(\alpha) = \frac{\alpha}{\alpha + ig} a(\alpha) \tag{6.31a}$$

$$c(\alpha) = \frac{\alpha}{\alpha + ig} a(\alpha) \tag{6.31b}$$

and

$$g = \frac{2}{2S+1} \tan(IS/2 + 1/4). \tag{6.32}$$

6.1.3 Diagonalization of the monodromy matrix.

Let us construct the monodromy matrix

$$L^{\{j_1,\cdots,j_{N+1}\},k}_{\{i_1,\cdots,i_{N+1}\},l}(\alpha;\alpha_1,\cdots,\alpha_{N+1}) = r^{kk_1}_{i_1 j_1}(\alpha_1 - \alpha)$$
$$r^{k_1 k_2}_{i_2 j_2}(\alpha_2 - \alpha).....r^{k_{N-1} k_N}_{i_N j_N}(\alpha_N - \alpha) R^{k_N l}_{i_{N+1} j_{N+1}}(\alpha_{N+1} - \alpha), \tag{6.32}$$

which is graphically represented in Figure 6.1. For convenience, we rewrite it as

$$\mathbf{L}(\alpha;\alpha_1,....,\alpha_{N+1}) = \mathbf{r}_{*1}(\alpha_1 - \alpha)\cdots\mathbf{r}_{*N}(\alpha_N - \alpha)\mathbf{R}_{*0}(\alpha_{N+1} - \alpha), \tag{6.33}$$

where * corresponds to operator and matrix indices.

Then, it follows the important relation [Be79]:

$$\mathbf{T}(\alpha,\alpha_1,\cdots,\alpha_{N+1})\mid_{\alpha=\alpha_1=\cdots=\alpha_N=0,\alpha_{N+1}=\alpha_0} = \mathbf{T}_j. \tag{6.34}$$

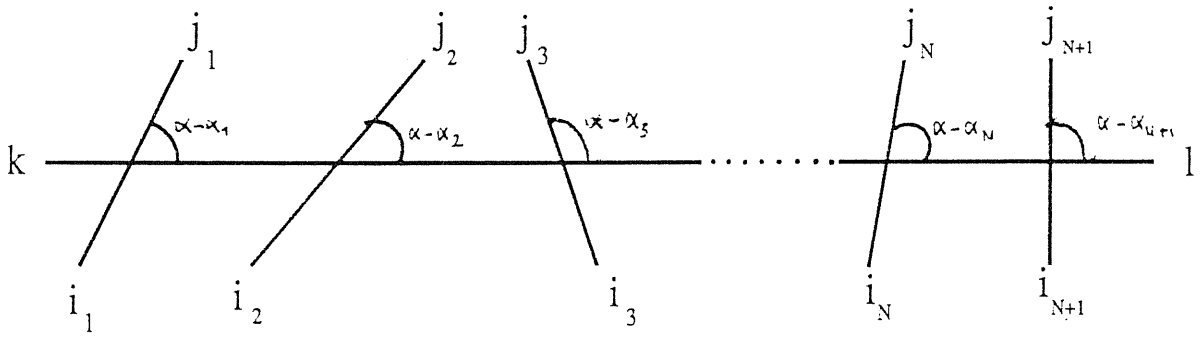


Figure 6.1

Let us then denote the matrix elements of the operator matrix L_l^k as

$$\mathbf{L}_1^1 = \mathbf{A}, \qquad \mathbf{L}_2^1 = \mathbf{B}, \qquad \mathbf{L}_1^2 = \mathbf{C}, \qquad \mathbf{L}_2^2 = \mathbf{D}. \tag{6.35}$$

The matrix $\mathbf{r}(\alpha - \alpha')$ in (6.33) according to formulae (6.24) and (6.26) is

$$r_{kk'}^{qq'}(\alpha) = b(\alpha)\delta_{qq'}\delta_{kk'} + c(\alpha)\delta_{qk'}\delta_{kq'}, \tag{6.36}$$

where $b(\alpha)$ and $c(\alpha)$ are given by (6.31). Rewriting (4.9) in components we have

$$b(\alpha - \alpha')\mathbf{L}_l^q(\alpha')\mathbf{L}_{l'}^k(\alpha) + c(\alpha - \alpha')\mathbf{L}_l^k(\alpha')\mathbf{L}_l'^q(\alpha)$$
$$= b(\alpha - \alpha')\mathbf{L}_{l'}^k(\alpha)\mathbf{L}_l^q(\alpha') + c(\alpha - \alpha')\mathbf{L}_l^k(\alpha)\mathbf{L}_{l'}^q(\alpha'). \tag{6.37}$$

We need only the following commutation relations from (6.37):

$$[\mathbf{A}(\alpha), \mathbf{A}(\beta)] = [\mathbf{B}(\alpha), \mathbf{B}(\beta)] = [\mathbf{C}(\alpha), \mathbf{C}(\beta)]$$
$$= [\mathbf{D}(\alpha), \mathbf{D}(\beta)] = [\mathbf{A}(\alpha), \mathbf{D}(\beta)] = 0, \tag{6.38a}$$

$$b(\alpha - \beta)\mathbf{A}(\alpha)\mathbf{B}(\beta) = a(\alpha - \beta)\mathbf{B}(\beta)\mathbf{A}(\alpha) - c(\alpha - \beta)\mathbf{B}(\alpha)\mathbf{A}(\beta), \tag{6.38b}$$

$$b(\beta - \alpha)\mathbf{D}(\alpha)\mathbf{B}(\beta) = a(\beta - \alpha)\mathbf{B}(\alpha)\mathbf{D}(\beta) - c(\beta - \alpha)\mathbf{B}(\alpha)\mathbf{D}(\beta), \tag{6.38c}$$

The commutativity of traces of the L matrices,

$$[\mathbf{A}(\alpha) + \mathbf{D}(\alpha), \mathbf{A}(\beta) + \mathbf{D}(\beta)] \tag{6.39}$$

is the consequence of eq (6.38a).

Let us consider the state Ω_0 with all particle's spins up, the impurity spin projection being $+S$. The matrices $\mathbf{r}_{*j}(\alpha_j - \alpha), \mathbf{R}_{*0}(\alpha_{n+1} - \alpha)$, when applied to the vacuum Ω_0, become triangular matrices:

$$\mathbf{r}_{*j}\Omega_0 = \begin{pmatrix} a(\alpha) & c(\alpha)\sigma_j^- \\ 0 & b(\alpha) \end{pmatrix} \Omega_0 \tag{6.40}$$

$$\mathbf{R}_{*0}(\alpha)\Omega_0 = \begin{pmatrix} (S+1/2)a'(\alpha) - (S-1/2)b'(\alpha) & 2c'(\alpha)S^- \\ 0 & -(S-1/2)a'(\alpha) + (S+1/2)b'(\alpha) \end{pmatrix} \Omega_0. \tag{6.41}$$

The action of the diagonal elements A and D of the matrices $\mathbf{r}_{*j}$ and $\mathbf{R}_{*0}$ to the vacuum will be

$$\mathbf{C}(\alpha)\Omega_0 = 0 \tag{6.42}$$

$$\mathbf{A}(\alpha)\Omega_0 = \Lambda_A(\alpha)\Omega_0, \qquad \mathbf{D}(\alpha)\Omega_0 = \Lambda_D(\alpha)\Omega_0, \tag{6.43}$$

where

$$\Lambda_A(\alpha) = \prod_{j=1}^{N} a(\alpha_j - \alpha)[(S+1/2)a'(\alpha_{N+1} - \alpha) - (S-1/2)b'(\alpha_{N+1} - \alpha)] \tag{6.44a}$$

$$\Lambda_D(\alpha) = \prod_{j=1}^{N} b(\alpha_j - \alpha)[-(S-1/2)a'(\alpha_{N+1} - \alpha)(S+1/2)b'(\alpha_{N+1} - \alpha)] \tag{6.44b}$$

Other eigenstates can be constructed by successive applications on Ω_0 of the "annihilation" operator $B(\alpha'_\beta)$ which decreases the total spin projection by one:

$$\Omega_M(\alpha'_1, ..., \alpha_{M'}) = \prod_{\beta=1}^{M} \mathbf{B}(\alpha'_\beta)\Omega_0. \tag{6.45}$$

It is possible to show that Ω_M is an eigenvector of the operator $\mathbf{A}(\alpha) + \mathbf{D}(\alpha)$, if $\{\alpha'_\beta\}$ satisfy a certain system of transcendental equations. Let us note that by virtue of (6.38a) and (6.38b), carrying $\mathbf{A}(\alpha), \mathbf{D}(\alpha)$ through all the $\mathbf{B}(\alpha'_\beta)$, we have

$$(\mathbf{A}(\alpha) + \mathbf{D}(\alpha)) \prod_{\beta=1}^{M} \mathbf{B}(\alpha'_\beta)\Omega_0 = \Lambda(\alpha, \{\alpha'_\beta\}) \prod_{\beta=1}^{M} \mathbf{B}(\alpha'_\beta)\Omega_0 + \sum_{\gamma=1}^{M} \Lambda_\gamma(\alpha, \{\alpha'_\beta\}) \prod_{\substack{\beta=1 \\ \beta\neq\gamma}}^{M} \mathbf{B}(\alpha'_\beta)\mathbf{B}(\alpha)\Omega_0, \tag{6.46}$$

where

$$\Lambda(\alpha,\{\alpha'_\beta\}) = \prod_{\beta=1}^{M} \frac{a(\alpha-\alpha'_\beta)}{b(\alpha-\alpha'_\beta)}\Lambda_A(\alpha) + \prod_{\beta=1}^{M} \frac{a(\alpha'_\beta-\alpha)}{b(\alpha'_\beta-\alpha)}\Lambda_D(\alpha); \tag{6.47a}$$

$$\begin{aligned}\Lambda_\gamma(\alpha,\{\alpha'_\beta\}) = &-\frac{c(\alpha-\alpha'_\gamma)}{b(\alpha-\alpha'_\gamma)} \prod_{\substack{\beta=1\\ \beta\neq\gamma}}^{M} \frac{a(\alpha'_\gamma-\alpha'_\beta)}{b(\alpha'_\gamma-\alpha'_\beta)}\Lambda_A(\alpha'_\gamma) \\ &- \frac{c(\alpha'_\gamma-\alpha)}{b(\alpha'_\gamma-\alpha)} \prod_{\substack{\beta=1\\ \beta\neq\gamma}}^{M} \frac{a(\alpha-\alpha'_\beta)}{b(\alpha-\alpha'_\beta)}\Lambda_D(\alpha'_\gamma).\end{aligned} \tag{6.47b}$$

The first summand on the right-hand side of (6.46) is obtained when we carry $A(\alpha), D(\alpha)$ through $B(\alpha'_\beta)$, using only the first term on the right hand-side of (6.38ab). The second summand is obtained from the remaining commutations. The fact that the 2^M terms arising from the subsequent commutations reduce to only M terms is due to the special form of the functions $a(\alpha), b(\alpha)$ and $c(\alpha)$. A proof by induction of the equations (6.46-47) will be given in Appendix (A).

As a result we have that $\Omega(\{\alpha'_\beta\})$ is an eigenvector of $\mathbf{A}(\alpha) + \mathbf{D}(\alpha)$ with a spin projection $S^z = N/2 - M + S$ and eigenvalue $\Lambda(\alpha,\{\alpha'_\beta\})$ if $\{\alpha'_\beta\}$ satisfy the following system of equations:

$$\begin{aligned}\prod_{j=1}^{N} \frac{a(\alpha_j-\alpha'_\gamma)}{b(\alpha_j-\alpha)} \frac{[(S+1/2)a'(\alpha_{N+1}-\alpha)-(S-1/2)b'(\alpha_{N+1}-\alpha'_\gamma)]}{b(\alpha_j-\alpha)[-(S-1/2)a'(\alpha_{N+1}-\alpha'_\gamma)(S+1/2)b'(\alpha_{N+1}-\alpha'_\gamma)]} \\ = -\prod_{\beta=1}^{M} \frac{a(\alpha'_\beta-\alpha'_\gamma)}{a(\alpha'_\gamma-\alpha'_\beta)} \frac{b(\alpha'_\gamma-\alpha'_\beta)}{b(\alpha'_\beta-\alpha'_\gamma)}.\end{aligned} \tag{6.48}$$

Setting

$$\alpha'_\beta = g(-\lambda+i/2), \qquad \alpha=0, \qquad \alpha_j = \delta_{N+1,j} \tag{6.49}$$

and using eqns (6.46) and (6.47a) with $b(0)=0$, we obtain

$$\exp(ik_jL) = \exp(iIS/2)\prod_{\alpha=1}^{M}\left(\frac{\lambda_\alpha+i/2}{\lambda_\alpha-i/2}\right). \tag{6.50A}$$

The condition removing the 'unwanted terms' yields the following equations for λ_α:

$$\left(\frac{\lambda_\alpha+i/2}{\lambda_\alpha-i/2}\right)^N\left(\frac{\lambda_\alpha+1/g+iS}{\lambda_\alpha+1/g-iS}\right) = -\prod_{\beta=1}^{M}\frac{\lambda_\alpha-\lambda_\beta+i}{\lambda_\alpha-\lambda_\beta-i}. \tag{6.50B}$$

The *Bethe Ansatz equations* (A) and (B) solves the problem of diagonalization of the s-d exchange hamiltonian and completely describe its spectrum.

The energy eigenvalue is determined by the momenta $\{k_j\}$ of the charge density waves and by the rapidities $\{\lambda_\alpha\}$ of the spin density waves:

$$E = \sum_{j=1}^{N} k_j \tag{6.51}$$

and the spin projection is given by

$$S^z = N/2 - M + S. \tag{6.52}$$

Equations (6.50-52) for $S = 1/2$ were obtained by Andrei [An80] and Wiegmann [Wi80a,81] and for arbitrary S, by Fateev and Wiegmann [FW81a] and Furuya and Lowenstein [FL82].

6.2 The Bethe ansatz for the degenerate Anderson model.

Rewriting the Hamiltonian (5.23) in coordinate space, we obtain

$$H_A = \sum_j \int dx \left\{ -ic_j^\dagger(x)\frac{\partial}{\partial x}c_j(x) + V\delta(x)[c_j^\dagger \mathbf{X}_{0j} + \mathbf{X}_{j0}c_j(x)] \right\} + \sum_j \epsilon_f \mathbf{X}_{jj}. \tag{6.53}$$

This Hamiltonian is an integrable variant of the degenerate Anderson model. As in the Kondo model, two approximations used are crucial for the integrability: the linearization of the spectrum and the contact interaction. Other approximations like the exclusion of the multiple occupancy of the f-level can be avoided but would considerably complicate the solution. As known, to prove the integrability of the hamiltonian and therefore the validity of the Bethe hypothesis, it suffices to find out whether the two particle S-matrix satisfies the factorization conditions.

The one particle problem.

Consider one electron with momentum k and spin m. This electron is either localized at the f-level or propagates through the crystal. The one-particle wavefunction has the form:

$$|\Psi_{km}(x)> = \left(\int dx g_k(x) c_m^\dagger(x) + e_k \mathbf{X}_{j0} \right) |0> . \tag{6.54}$$

Recquiring the Schrödinger equation $H|\Psi_{km}>=E|\Psi_{km}>$ to be satisfied yields

$$\begin{cases} -i\dfrac{\partial}{\partial x}g_k(x)+Ve_k\delta(x)=Eg_k(x) \\ Vg_k(0)+e_k\epsilon_f=Ee_k, \qquad g_k(0)=\dfrac{1}{2}[g_k(0^+)+g_k(0^-)] \end{cases} \tag{6.55}$$

The wave function $g_k(x)$ is a plane wave for $x>0$ and $x<0$ with a discontinuity at the origin:

$$g_k(x)=\exp(ikx+i\mathrm{sign}x\phi_k), \tag{6.56a}$$

$$\phi_k=\tan^{-1}\left(-\frac{1}{2}\frac{V^2}{k-\epsilon_f}\right) \tag{6.56b}$$

$$e_k=\frac{V}{k-\epsilon_f}g_k(0), \quad E=k. \tag{6.56c}$$

The two-particle problem.

The two-body wavefunction consists of three terms,

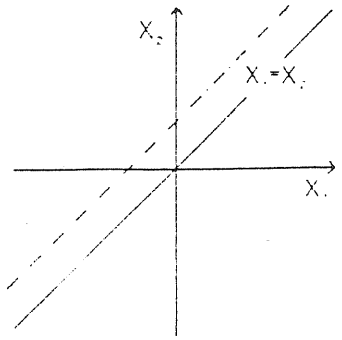
$$|\Psi>=\int dx_1dx_2g_{m_1m_2}(x_1,x_2)c^\dagger_{m_1}(x_1)c^\dagger_{m_2}(x_2)|0> \\ +\int dx_1e^{m_2}_{m_1}c^\dagger_{m_1}(x_1)\mathbf{X}_{m_20}|0>+\int dx_2e^{m_1}_{m_2}c^\dagger_{m_2}(x_2)\mathbf{X}_{m_10}|0>, \tag{6.57}$$

the first one corresponding to two traveling electrons and the other two to one electron being localized. The amplitudes $g_{m_1m_2}(x_1,x_2)$ and $e^{m_j}_{m_i}$ satisfy the equations:

$$\begin{cases} [-i(\partial_{x_1}+\partial_{x_2})-E]g_{m_1m_2}(x_1,x_2)+V(e^{m_2}_{m_1}(x_1)\delta(x_2)-e^{m_1}_{m_2}(x_2)\delta(x_1))=0 \\ (-i\partial_{x_1}+\epsilon_f-E)e^{m_2}_{m_1}(x_1)+Vg_{m_1m_2}(x_1,0)=0 \\ (-i\partial_{x_2}+\epsilon_f-E)e^{m_1}_{m_2}(x_2)-Vg_{m_1m_2}(0,x_2)=0 \\ E=k_1+k_2, \qquad g_{m_1m_2}(x_1,x_2)=-g_{m_2m_1}(x_2,x_1) \end{cases} \tag{6.58}$$

The solution for $g_{m_1m_2}$ is a superposition of plane waves in all six regions shown in fig 6.2. First, it is discontinous at the lines $x_1=0$ and $x_2=0$ where one of the electrons crosses the impurity; the jumps are given by the one-particle phase shift (6.56b). For the half-plane $x_1<x_2$, the function $g_{m_1m_2}(x_1,x_2)$ can then be written as

$$g_{m_1m_2}(x_1,x_2)=A_{m_1m_2}(1,1)g_{k_1}(x_1)g_{k_2}(x_2) \\ +A_{m_1m_2}(1,2)g_{k_1}(x_2)g_{k_2}(x_1). \tag{6.59}$$



The six sectors in the two-electron problem. The dash–dotted line represents a trajectory, $x_2 - x_1 = \text{const}$.

Figure 6.2

A similar expression holds for the other half-plane $x_2 < x_1$, where we call the coefficients $A_{m_1 m_2}(2,2)$ and $A_{m_1 m_2}(2,1)$, respectively. By antisymmetry of the wavefunction we have $A_{m_1 m_2}(1,1) = -A_{m_2 m_1}(2,1)$ and $A_{m_1 m_2}(1,2) = -A_{m_2 m_1}(2,2)$, such that the coefficients with interchanged spin indices are completely determined. The constants $A_{m_1 m_2}(i,j)$ are the amplitudes of the wavefunction when both electrons are to the left of the impurity.

The continuity of $g(x_1, x_2)$ and the discontinuity of its derivative across the boundary line $x_1 = x_2$ yields two relations among the four constants. The relations among the amplitudes for $x_1 < x_2$ and $x_2 < x_1$ determine the scattering matrix. The solutions of the system (6.58) are finally given by the following expressions:

$$g_{m_1 m_2}(x_1, x_2) = (g_{k_1}(x_1) g_{k_2}(x_2) - g_{k_1}(x_2) g_{k_2}(x_1)) \\ (A_{m_1 m_2}\theta(x_1 x_2) + S^{m_2 m_2'}_{m_1 m_1'}(k_1, k_2) A_{m_1' m_2'}\theta(x_2 - x_1)); \quad (6.60)$$

$$e^{m_2}_{m_1}(x) = (g_{k_1} e_{k_2} - g_{k_2} e_{k_1})(A_{m_1 m_2}\theta(x) + S^{m_2 m_2'}_{m_1 m_1'}(k_1, k_2) A_{m_1' m_2'}\theta(-x)), \quad (6.61)$$

where the functions $g_k(x)$ and e_k are given by (6.56) and the two-particle S matrix has the form

$$S^{m_2 m_2'}_{m_1 m_1'}(k, p) = \frac{(p - k + 2i\Gamma P^{m_2 m_2'}_{m_1 m_1'})}{(p - k + 2i\Gamma)}. \quad (6.62)$$

It can be verified that the matrix (6.62) satisfies the factorization condition.

The N-particle problem.

The above arguments are straightforwardly generalized to the N -particles problem. In each domain $X_Q = \{X_{q_1} < \cdots < X_{q_N}\}$, the wavefunction $g_{m_1 \cdots m_N}(x_1, \cdots, x_N)$ is of the form:

$$g_{m_1 \cdots m_N}(x_1, \cdots, x_N) = \sum_P A_{m_1, \cdots, m_N}(Q \mid P) \prod_{j=1}^{N} g_{k_P}(x_j) \tag{6.63}$$

where m_j are the spins of the particles, $\{k_j\}$ is a set of unequal numbers and $g_k(x)$ is given by (6.56). The energy of the state with the wavefunction (6.63) is:

$$E = \sum_{j=1}^{N} k_j. \tag{6.64}$$

There is a number of constraints imposed on $N! \times N!$ matrix $A(Q \mid P)$ by the Schrodinger equation, continuity conditions at the boundaries of the domain X_Q and antisymmetry. The number of these constraints exceeds the number of elements of the matrix $A(Q \mid P)$. The consistency of these constraints is guaranteed by the factorization condition which proves the Bethe hypothesis. Let $I = \{1, \cdots, N\}$, then $A(Q \mid P)$ is related to $A(I \mid P)$ by the matrix $\mathbf{S}(Q \mid P)$:

$$A_{m_1, \cdots, m_N}(Q \mid P) = S^{m'_1, \cdots, m'_N}_{m_1, \cdots, m_N}(Q \mid P) A_{m'_1, \cdots, m'_N}(I \mid P). \tag{6.65}$$

Then $\mathbf{S}(Q \mid P)$ is a multiple scattering matrix. Due to the validity of the factorization condition, it is the product of two-particle S matrices.

The coordinate Bethe Ansatz for this problem , in a similar manner as for the sd-exchange model, leads to the eigenvalue problem of the operator

$$\mathbf{T}_j = \mathbf{S}_{jj+1} \cdots \mathbf{S}_{jN} \mathbf{S}_{j1} \cdots \mathbf{S}_{jj-1} \qquad \mathbf{S}_{ij} \equiv \mathbf{S}_{ij}(k_i, k_j) \tag{6.66}$$

with the $\mathbf{S}$ given above.

The eigenvalues of the operators $\mathbf{T}_j$ are given by the so- called Bethe ansatz hierarchy. Let us denote by N_k the number of particles with spin component $m = -j + k$ and define

$$m_0 = N, \qquad m_n = 0, \qquad m_i = \sum_{k=i}^{n-1} N_k, \quad (i = 1, \cdots, n-1) \tag{6.67}$$

such that $N \geq m_1 \geq m_2 \geq \cdots \geq m_{2j} \geq 0$. Then the nested Bethe ansätze yield the following sets of coupled equations:

$$\exp(ik_jL)\left(\frac{k_j-\epsilon_f-i\Gamma}{k_j-\epsilon_f+i\Gamma}\right)=\prod_{\alpha=1}^{m_1}\frac{k_j/2\Gamma+\lambda_\alpha^{(1)}-i/2}{k_j/2\Gamma+\lambda_\alpha^{(1)}+i/2}, \tag{6.68a}$$

$$\prod_{\tau=\pm1}\prod_{\beta=1}^{m_{j+\tau}}\left(\frac{\lambda_\alpha^{(j)}-\lambda_\beta^{(j+\tau)}+i/2}{\lambda_\alpha^{(j)}-\lambda_\beta^{(j+\tau)}-i/2}\right)=-\prod_{\beta=1}^{m_j}\left(\frac{\lambda_\alpha^{(j)}-\lambda_\beta^{(j)}+i}{\lambda_\alpha^{(j)}-\lambda_\beta^{(j)}-i}\right), \tag{6.68b}$$

where $\lambda_\alpha^{(0)}=-k_\alpha/2\Gamma$. The Bethe-Ansatz hierarchy for the degenerate Anderson model was constructed by [Sch82].

6.3 Exact solution of the s-d exchange model.

In order to illustrate some standard procedures which are often used to solve Bethe-like equations, we finish this chapter with the main steps leading to the expression for the impurity magnetization.

In general, the solutions of the eqns (6.50A-B) lie in the complex plane of λ. However it can be shown that the ground state with a given spin projection S^z is formed by the real solutions.

Taking the logarithms of these equations, we get

$$k_jL=2\pi N_j-\sum_{\alpha=1}^{M}(\Phi(\lambda_\alpha)+\pi)+\frac{1}{2}IS \tag{6.69A}$$

$$N\Phi(\lambda_\alpha)+\Phi(\frac{\lambda+1/g}{2S})=2\pi J_\alpha+\sum_{\beta=1}^{M}\Phi(\frac{\lambda_\alpha-\lambda_\beta}{2}) \tag{6.69B}$$

where $\Phi(\lambda)=2\arctan(2\lambda)$,

J_α integer if M is odd, $J_\alpha+1/2$ integer if M even, $\mid 2J_\alpha \mid \leq N-M$. Each allowed choice of the integers N_j and J_α uniquely determines an eigenstate of the Hamiltonian. We shall refer to the $\{N_j, J_\alpha\}$ configurations as the quantum numbers of the state they determine.

First let us discuss some properties of the derived equations. In the absence of the impurity, they describe a non-interacting electron gas. The impurity term $(\frac{\lambda+1/g+iS}{\lambda+1/g-iS})$ is equal to unity for either $g=0$ or $S=0$ where the interaction vanishes. Equation (6.50A)

describes the 'scattering' of the charge subsystem particles with momenta k_j on the spin subsystem particles having so-called rapidities λ_α. Equation (6.50B) describes the spin-spin scattering. Both systems are fermionic since the wavefunctions vanish if any two values of either λ_α or k_j coincide [YY66]. The number of particles in the spin subsystem is equal to the number of flipped spins M.

We note from eq (6.69A) that all electrons are equally shifted from their free value. In other words the phase shift of the electrons due to their interaction does not depend on their momentum and the spin and charge excitations do not interact.

6.2.1 The ground state.

Let us find the configuration $\{N_j, J_\alpha\}$ corresponding to the ground state by minimizing the charge and spin parts of the energy. Two important propositions hold:

(i) The solutions of eq(6.69) are monotonous functions of the integers J_α.

(ii) The integers J_α are bounded by the interval $[(-N+M)/2, (N-M)/2)]$. The boundaries correspond to $\lambda = \mp\infty$.

According to (ii), the spin part of the energy is bounded. Therefore the integers J_α should begin with the highest: $J_1 = J_{max} = (N-M)/2$ and form a consecutive configuration:

$$J_{\alpha+1} = \frac{N-M}{2} - \alpha, \qquad \alpha = 0, 1, ..., M-1. \tag{6.70}$$

Furthermore, according to (i), the solution of eq (6.69B) for N, M large form a non-uniform dense distribution between $\lambda = +\infty$ and $\lambda = -B$, where $\lambda = -B$ corresponds to the lowest integer $J_M = (N-3M)/2$. When $S^z = 0$ $(M = N/2)$, $J_M = -N/4$ and we have $B = \infty$.

Obviously, the spectrum is not bounded from below, since the integers N_j can take arbitrarly large and negative values. To define the model, we introduce a cutoff by imposing that all charge energies should not exceed the Fermi energy $\epsilon_F = \pi N/L$. It means that the minimum integer should be $N_1 = -N/2$. Therefore, N_j are successive integers from $-N/2$ to $N/2$ and $E^{ch} = 0$.

The modification of the ground state configuration is obtained by putting *holes* on it, where by hole we mean an integer omitted from the consecutive sequence.

We shall name *holes rapidities* all those values of $\widetilde{\lambda}_\beta$ ($\beta = 1, ..., N-2M$) corresponding to a permissible quantum number coinciding with any number of $\{J_\alpha\}$. Therefore, for the ground state $\widetilde{J}_\beta = ((N-3M)/2) - \beta$. It is obvious that for the ground state with a given spin S^z, the rapidities take their values on $(-\infty, -B)$. Thus, $\widetilde{\lambda}_\beta = 0$ when $\lambda > -B$ and $\rho = 0$ when $\lambda < -B$. For $M = N/2$ holes are absent and $B = \infty$ ($\widetilde{\rho} = 0$).

As we are interested only in the thermodynamic limit ($N, M, L \to \infty$, $\pi N/L = \epsilon_F$), we define densities of particles and holes induced by each configuration$\{J_\alpha\}$ as follows:

$$N\rho(\lambda)d\lambda = number\ of\ \lambda's\ in\ d\lambda$$

$$N\widetilde{\rho}(\lambda)d\lambda = number\ of\ holes\ in\ d\lambda.$$

The discretized version is written as

$$\rho(\lambda_\alpha) = \frac{1}{N(\lambda_{\alpha+1} - \lambda_\alpha)}. \tag{6.71}$$

Putting ρ in eq (6.69), going to the continous limit and differentiating respect to λ, we obtain the following integral equation:

$$\rho(\lambda) + \widetilde{\rho}(\lambda) + \int_{-B}^{\infty} a_1(\lambda' - \lambda)\rho(\lambda')d\lambda' = a_{1/2}(\lambda) + \frac{1}{N}a_S(\lambda + 1/g) \tag{6.72}$$

where

$$a_n(\lambda) := \frac{1}{\pi}\frac{n}{\lambda^2 + n^2}. \tag{6.73}$$

The energy and the spin per particle for the ground state are

$$\frac{1}{N}E^{sp} = -\frac{N}{L}\int_{-B}^{\infty}(\Phi(\lambda) + \pi)\rho(\lambda)d\lambda \tag{6.74}$$

$$\frac{1}{N}S^z = \frac{1}{2} + \frac{S}{N} - \int_{-B}^{\infty}(\Phi(\lambda) + \pi)d\lambda. \tag{6.75}$$

The last three equations describe all the magnetic properties of the impurity. Using the linearity of the equations (6.72), we can write

$$\rho = \rho_h + \frac{1}{N}\rho_i \tag{6.76}$$

corresponding to the host metal and impurity.

These functions satisfy the following equations

$$\rho_h(\lambda) = a_{1/2}(\lambda) - \int_{-B}^{\infty} a_1(\lambda' - \lambda)\rho_h(\lambda')d\lambda' \tag{6.77a}$$

$$\rho_i(\lambda) = a_S(\lambda + 1/g) - \int_{-B}^{\infty} a_1(\lambda' - \lambda)\rho_i(\lambda')d\lambda' \tag{6.77b}$$

and one can decompose the spin in the same way:

$$\frac{1}{N}S_h^z = \frac{1}{2} - \int_{-B}^{\infty} \rho_h(\lambda)d\lambda \tag{6.78a}$$

$$S_i^z = S - \int_{-B}^{\infty} \rho_i(\lambda)d\lambda. \tag{6.78b}$$

In the leading order respect to $1/N$, the total spin in presence of a magnetic field is determined by the magnetism of the conduction band. Therefore

$$\frac{1}{N}S^z = \frac{H}{4\epsilon_F}. \tag{6.79}$$

Our task then is to solve for the λ density $\rho(\lambda)$. By minimizing the energy, we shall determine the parameter B in terms of the magnetic field H and thus finally find the magnetization curve $M = M(H)$.

At this point, it is instructive to solve the equations for the case $M = N/2$ where holes are absent and $B = \infty$ ($\widetilde{\rho} = 0$). Defining ρ_0 as the ground state distribution, we have

$$\rho_0(\lambda) + \int_{-\infty}^{\infty} a_1(\lambda' - \lambda)\rho_0(\lambda')d\lambda' = a_{1/2}(\lambda) + \frac{1}{N}a_S(\lambda + 1/g) \tag{6.80}$$

In Fourier space, we get

$$\widehat{\rho}^0(\omega) = \int_{-\infty}^{\infty} e^{i\omega\lambda}\rho_0(\lambda)d\lambda = \frac{1}{2\cosh\frac{1}{2}\omega} + \frac{1}{N}e^{iw/g}\frac{e^{-(2S-1)|\omega|/2}}{2\cosh\frac{1}{2}\omega}. \tag{6.81}$$

Therefore, at $H = 0$,

$$\frac{1}{N}S_h^z = \frac{1}{2} - \widehat{\rho}_h^0(0) = 0 \tag{6.62}$$

as it should be, and the equation(6.78b) gives

$$M_i = S - \widehat{\rho}_i^0(0) = S - \frac{1}{2}. \tag{6.83}$$

The impurity spin thus is only partially screened and the ground- state is $2S$ fold degenerate.

The case $H \neq 0$ and $B \neq \infty$.

For what follows, it is more convenient to count all physical quantities from their ground-state values. Let us make the following shift:

$$\sigma(\lambda) = \rho(\lambda - B) \tag{6.84a}$$

$$\widetilde{\sigma}(\lambda) = \widetilde{\rho}(\lambda - B) \tag{6.84b}$$

so that $\sigma(\lambda) \equiv 0$ for $\lambda > 0$ and $\widetilde{\sigma}(\lambda) \equiv 0$ for $\lambda < 0$ and define the following functions:

$$\rho^{+}(\omega) = \int_{0}^{\infty} e^{i\omega\lambda}\sigma(\lambda)d\lambda \tag{6.85a}$$

$$\rho^{-}(\omega) = \int_{-\infty}^{0} e^{i\omega\lambda}\widetilde{\sigma}(\lambda)d\lambda. \tag{6.85b}$$

$\rho^{\pm}(\omega)$ are analytical in the upper (lower) half plane of w.

Making the Fourier transformation of eqns (6.77-78), we obtain

$$(1 + e^{-|\omega|})\rho^{+}(\omega) + \rho^{-}(\omega) = e^{i\omega B}(e^{-|\omega|/2} + \frac{1}{N}e^{-S|\omega|}e^{-i\omega/g}) \tag{6.86}$$

$$\frac{1}{N}S^{z} = \frac{1}{2} + \frac{S}{N} - \rho^{+}(0) \tag{6.87}$$

and returning to the λ-space, we obtain

$$\sigma_h(\lambda) + \widetilde{\sigma}_h(\lambda) - \int_{-\infty}^{0} R(\lambda - \lambda')\widetilde{\sigma}_h(\lambda')d\lambda' = (2\cosh\pi(\lambda - B))^{-1}, \tag{6.88}$$

$$\sigma_i(\lambda) + \widetilde{\sigma}_i(\lambda) - \int_{-\infty}^{0} R(\lambda' - \lambda)\widetilde{\sigma}_i(\lambda')d\lambda' = \mathcal{S}_{2S}(\lambda - B + 1/g), \tag{6.89}$$

$$H/2\epsilon_F = \int_{-\infty}^{0} \widetilde{\sigma}_h(\lambda)d\lambda, \tag{6.90}$$

$$M_i = S - \frac{1}{2} + \frac{1}{2}\int_{-\infty}^{0} \widetilde{\sigma}_i(\lambda)d\lambda, \tag{6.91}$$

where

$$R(\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(-i\omega\lambda)(1 + \exp(|\omega|))^{-1}d\omega, \tag{6.92}$$

$$\mathcal{S}_{2S}(\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(-i\omega\lambda)\frac{\exp(-(2S-1)|\omega|/2)}{2\cosh\omega/2}d\omega. \tag{6.93}$$

This form of the equations is more convenient because all the characteristics of the system are expressed in terms of the excited states only (i.e. the tilded functions).

Let us deal with a magnetic field small in comparision with the Fermi energy ($H \ll \epsilon_F$ or $S^z/N \ll 1$). This means that $B \gg 1$ and it is sufficient to solve eq.(6.88) in the first order of $\exp(-\pi B)$. Since

$$\frac{1}{2\cosh \pi(\lambda - B)} \simeq e^{-\pi\lambda} e^{\pi B} \tag{6.94}$$

we write $\sigma_h(\lambda) = e^{-\pi B} r(\lambda)$ and thus

$$\frac{H(B)}{2\epsilon_F} = e^{-\pi B} \int_{-\infty}^{0} \widetilde{r}(\lambda) d\lambda \tag{6.95}$$

where $\widetilde{r}$ is the solution of the equation

$$r(\lambda) + \widetilde{r}(\lambda) - \int_{-\infty}^{0} R(\lambda - \lambda')\widetilde{r}(\lambda')d\lambda' = e^{\pi\lambda}. \tag{6.96}$$

These two equations allow us to find the relation between H and B and to calculate then the magnetization through eq (6.91).

6.2.2 The Wiener-Hopf method.

The equation (6.96) may be solved by the Wiener-Hopf method. Let us outline the general features of this procedure [MF53].

Consider the following linear integral equation:

$$\psi_+(x) + \psi_-(x) + \int_{-\infty}^{0} dx' \psi_-(x') K(x - x') = F(x) \tag{6.97}$$

where $\psi_+ \equiv 0$ for $x < 0$ and $\psi_- \equiv 0$ for $x > 0$. ψ_+ and ψ_- are then complementary functions. Performing the Fourier transformation of (6.97), we obtain

$$\widehat{\psi}_+(\omega) + \widehat{\psi}_-(\omega)(1 + \widehat{K}(\omega)) = \widehat{F}(\omega). \tag{6.98}$$

Next we rewrite the kernel as a product,

$$1 + \widehat{K}(\omega) = G^+(\omega) G^-(\omega) \tag{6.99}$$

wher $G^{\pm}(\omega)$ is analytic in the upper (lower) half-plane and $G^{\pm}(\infty)$ is a constant. Eq(6.98) can then be written as

$$\psi_+(\omega)/G^+(\omega) + \psi_-(\omega)G^-(\omega) = \widehat{F}(\omega)/G^+(\omega) \tag{6.100}$$

where now the first (second) term on the left-hand side is analytic in the upper (lower) half ω-plane.

A similar separation can be made on the right-hand side of (6.100) by means of a Cauchy transformation:

$$\frac{\widehat{F}(\omega)}{G^+(\omega)} = q^+(\omega) + q^-(\omega) \tag{6.101}$$

where

$$q^{\pm}(\omega) = \pm\frac{i}{2\pi}\int_{-\infty}^{\infty}\frac{\widehat{F}(\omega')}{G^+(\omega')}\frac{d\omega'}{\omega - \omega' \pm i0}. \tag{6.102}$$

This leads to

$$H(\omega) = \frac{\widehat{\psi}_+(\omega)}{G^+(\omega)} - q^+(\omega) = -\widehat{\psi}_-(\omega)G^-(\omega) + q^-(\omega). \tag{6.103}$$

The left-hand side is analytic at $\mathrm{Im}\omega \geq 0$, the right-hand side is analytic at $\mathrm{Im}\omega \leq 0$. The two sides are equal in an infinitesimal strip around the real axis so that one side can be undestood as the analytical continuation of the other. Morever, $H(x)$ is an entire function and tends to zero at large ω. The solution of (6.98) is then

$$\psi_+(\omega) = q^+(\omega)G^+(\omega) \tag{6.104a}$$

$$\psi_-(\omega) = q^-(\omega)/G^-(\omega) \tag{6.104b}$$

and $\psi_+(\omega)$ and $\psi_-(x)$ are obtained by inverting the Fourier transform.

The impurity magnetization.

Let us now apply the method to our specific example. Here, from (6.96), we have

$$\begin{aligned} &\psi_+(\lambda) = r(\lambda), \quad \psi_-(\lambda) = \widetilde{r}(\lambda), \\ &K(x) = -R(x), \quad F(\lambda) = e^{\pi\lambda} \end{aligned} \tag{6.105}$$

and the kernel is given by

$$1 - R(\omega) = \frac{e^{|\omega|/2}}{2\cosh \omega/2}. \tag{6.106}$$

Using the following two factorizations:

$$e^{-\pi|\omega|} = f_-(\omega) f_+(\omega), \quad f_\pm(\omega) = \left(\frac{\mp i\omega + 0}{e}\right)^{\pm i\omega}, \tag{6.107}$$

$$\frac{\pi}{\cosh \pi x} = \Gamma(1/2 - ix)\Gamma(1/2 + ix) \tag{6.108}$$

we obtain for the functions $G^\pm$ factorizing the kernel

$$G^+(\omega) = G^-(-\omega) = \frac{1}{\sqrt{2\pi}} f_+(\omega/2\pi)\Gamma(1/2 - i\omega/2\pi) \tag{6.109}$$

with the normalization $G^\pm(\infty) = 1$.

From the equation (6.102), the solution of (6.96) is

$$\rho_-(\omega) = \frac{q^-(\omega)}{G^-(\omega)} = (G^-(\omega)G^+(i\pi))^{-1} \frac{1}{i\omega + \pi}. \tag{6.110}$$

Using (6.95) and (6.110) we find the relation between B and H:

$$\frac{H}{2\epsilon_F} = \exp(-\pi B)\left(\frac{2}{\pi e}\right)^{1/2} \tag{6.111}$$

which we use to define a new scale T_H in the following way:

$$B - \frac{1}{g} = \frac{1}{\pi} \ln \frac{T_H}{H}. \tag{6.112}$$

Specifying the 'Kondo temperature' T_K as the quantity related to magnetic susceptibility at $H = T = 0$

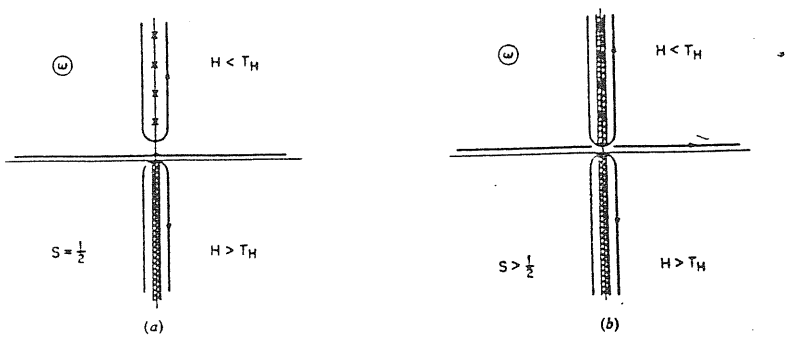
$$T_K = (2\pi\chi_i)^{-1} = \frac{2\epsilon_F}{\pi} \exp(-\pi/g) \tag{6.113}$$

and therefore the scale T_H is related to T_K as follows:

$$T_H = \left(\frac{2\pi}{e}\right)^{1/2} T_K. \tag{6.114}$$

Then using eq(6.93) and (6.112), we obtain the universal formula for the impurity magnetization at $T = 0$:

$$M_i(H) = S - \frac{1}{2} + \frac{i}{2\pi^{3/2}} \int_{-\infty}^{\infty} d\omega \times \exp(-2i\omega \ln H/T_H) \frac{\Gamma(1/2 + i\omega)}{\omega + i0} f_+^{2S}(\omega) f_-^{2S-1}(\omega). \tag{6.115}$$



The complex plane of the integrand (6.115) for (a) $S = 1/2$ and (b) $S > 1/2$.

Figure 6.3

The integral (6.115) has two different representations and asymptotic expansions for different ratios of H/T_H. If $H > T_H$, the contour of integration envelops the lower half-plane and one should calculate the integral over the contour encircling the cut $f_+(\omega)$ and pole $\omega = -i0$ (see Fig.6.3a). In this case, one does approach the single-spin magnetic moment, but very slowly. Defining a so-called 'invariant charge' $z(H/T_H)$ defined through

$$\frac{1}{z} - \frac{1}{2}\ln z = \ln \frac{H}{T_H}, \tag{6.116}$$

we obtain from (6.115) a power-series expansion in z

$$M_i(H > T_H) = S\left(1 + \sum_{n=1}^{\infty} a_n(S) z^n (H/T_H)\right). \tag{6.117}$$

The first term of this expansion $a_1 = -1/2$ coincides with the term obtained by means of the perturbation theory [AM70].

The most interesting region $H < T_H$ is inaccessible by perturbation theory. If $S = 1/2$ the only singularities of the integrand in the upper half-plane are the poles, leading to the power series in (H/T_H):

$$M_i^{(1/2)}(H < T_H) = \frac{2}{\sqrt{\pi}} \sum_{n=o}^{\infty} \left(\frac{n+1/2}{e}\right)^{(n+1/2)} \frac{(-1)^n}{n!(n+1/2)} \left(\frac{H}{T_H}\right)^{2n+1}. \tag{6.118}$$

We can deduce from (6.118) that the magnetization increases linearly for small H, in contrast to an isolated magnetic moment, for which M_i would be μ sign$H$ with a discontinuous jump at $H = 0$.

In the case $S \neq 1/2$, the integration over the cut $f_-(\omega)$ in the upper half-plane contributes to the low magnetic-field behaviour (Fig.6.3b). The poles of the Γ function now give only the exponentially small contribution to the impurity magnetization.

For more details about these calculations and the thermodynamics of the model we refer to [TW83] and [AFL83].

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Appendix

In this appendix, we show by induction the validity of the Bethe equations (6.46-47) in section 6.1.3. For this purpose, let us rewrite eq (6.38b) in the form

$$A(\alpha)B(\beta) = f(\alpha-\beta)B(\beta)A(\alpha) - g(\alpha-\beta)B(\alpha)A(\beta) \tag{A1}$$

where

$$\begin{aligned} f(\alpha) &= \frac{a(\alpha)}{b(\alpha)}, \quad g(\alpha) = \frac{c(\alpha)}{b(\alpha)} \\ f(\alpha) &= 1 + g(\alpha). \end{aligned} \tag{A2}$$

The proof will be done, carrying only the operator $A(\alpha)$ through all the $B(\alpha_\beta)$. Obviously, the proof for the operator $D(\alpha)$ goes in the same way. We keep the notation of chapter 6.

The identity (6.46) is trivial for $M = 1$. Let us look therefore at the step $M \to M+1$. We have:

$$A(\alpha)B(\alpha_{M+1})\Omega_M = f(\alpha-\alpha_{M+1})B(\alpha_{M+1})[A(\alpha)\Omega_M] - g(\alpha-\alpha_{M+1})B(\alpha)[A(\alpha_{M+1})\Omega_M]. \tag{A3}$$

We assume that

$$A(\alpha)\Omega_M = \Lambda(\alpha,\{\alpha_\beta\})\ \Omega_M + \sum_{\gamma=1}^{M} \Lambda_\gamma(\alpha,\{\alpha_\beta\})\ \widetilde{\Omega}_M(\alpha), \tag{A4}$$

where

$$\Lambda(\alpha,\{\alpha_\beta\}) = \prod_{\beta=1}^{M} f(\alpha-\alpha_\beta)\Lambda_A(\alpha), \tag{A5}$$

$$\Lambda_\gamma(\alpha,\{\alpha'_\beta\}) = -g(\alpha-\alpha_\gamma) \prod_{\substack{\beta=1 \\ \beta\neq\gamma}}^{M} f(\alpha_\gamma-\alpha_\beta)\Lambda_A(\alpha_\gamma), \tag{A6}$$

and we have defined

$$\widetilde{\Omega}_M(\alpha) \equiv \prod_{\substack{\beta=1 \\ \beta\neq\gamma}}^{M} B(\alpha)B(\alpha_\beta)\Omega_0. \tag{A7}$$

Substituting (A4-7) in (A3), we obtain

$$A(\alpha)\Omega_{M+1} = \prod_{\beta=1}^{M+1} f(\alpha-\alpha_\beta)\Lambda_A(\alpha)\ \Omega_{M+1}$$
$$-g(\alpha-\alpha_{M+1})\prod_{\beta=1}^{M} f(\alpha_{M+1}-\alpha_\beta)\Lambda_A(\alpha_{M+1})\ \Omega_M$$
$$-\sum_{\gamma=1}^{M}[g(\alpha-\alpha_\gamma)f(\alpha-\alpha_{M+1}) - g(\alpha_{M+1}-\alpha_\gamma)g(\alpha-\alpha_{M+1})]$$
$$\prod_{\substack{\beta=1\\ \beta\neq\gamma}}^{M} f(\alpha_\gamma-\alpha_\beta)\Lambda_A(\alpha_\gamma)\widetilde{\Omega}_{M+1}(\alpha). \qquad (A8)$$

We have to compare this expression with $A(\alpha)\Omega_{M+1}$ given by the equation (A4). We conclude that the last two terms on the right-hand side of (A8) must be equal to

$$-\sum_{\gamma=1}^{M+1} g(\alpha-\alpha_\gamma)\prod_{\substack{\beta=1\\ \beta\neq\gamma}}^{M+1} f(\alpha_\gamma-\alpha_\beta)\Lambda_A(\alpha_\gamma)\widetilde{\Omega}_{M+1}(\alpha). \qquad (A9)$$

Since the second term of (A8) is recognized to be the term in (A9) given by $\gamma = M+1$, the condition to be verified by the functions $f(\alpha)$ and $g(\alpha)$ results to be

$$f(\alpha-\alpha_{M+1}) - f(\alpha_\gamma-\alpha_{M+1}) = \frac{g(\alpha_{M+1}-\alpha_\gamma)g(\alpha-\alpha_{M+1})}{g(\alpha-\alpha_\gamma)}. \qquad (A10)$$

These are the functional equations for the functions $f(\alpha)$ and $g(\alpha)$ which define the non-commutativity of the operators $A(\alpha)$, $D(\alpha)$ and $B(\alpha)$ given generally by the quantum R-matrix.

In our case, the identity (A10) is then easily verified, using equation (A2) together with the additivity and the antisymmetry of the function $h(\alpha) = 1/g(\alpha)$.