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Few electrons in the Hubbard model

Thesis submitted for the degree of

"Magister Philosophiæ"

CANDIDATE

Michele Fabrizio

SUPERVISOR

Prof. Erio Tosatti

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Introduction

The physical properties of two dimensional interacting electronic systems, are of great current interest because of their possible relevance to explain the normal state and the mechanism of the HTc superconductors. These materials, in fact, are puzzling not only for the high critical temperature, but also because the normal state exhibits a number of anomalous properties. This is the reason why some authors[1,2,3] suggested that they can not be described in terms of the Landau Fermi liquid picture. According to the usual Landau theory, the low lying single particle excitations are well described in terms of weak interacting quasiparticles, whose decay rate, proportional to the imaginary part of the self energy, should vanish near the Fermi level like $(\epsilon - \epsilon_f)^2$ in $3d$, and $-(\epsilon - \epsilon_f)^2 \log|\epsilon - \epsilon_f|$[4] in $2d$. These well defined Fermi-like excitations imply also that the momentum distribution has a jump at the non-interacting Fermi surface, whose value is however reduced with respect to the non-interacting one[5]. There may be, instead, some experimental suggestion that the imaginary part of the self energy in these materials vanish only as $|\epsilon - \epsilon_f|$, which would implies a zero jump in the momentum distribution, and therefore a non-Landau liquid behaviour. While one would expect that a weakly interacting electron gas indeed should be a Landau Fermi liquid, it is an open question if a strongly correlated $2d$ gas is well described in terms of the Landau picture. One simple model which describes a strongly correlated fermionic system, is the two dimensional *Hubbard model*, which is a single tight-binding band plus a strong on-site repulsion. In spite of its simplicity, this model is both very rich, and very hard to study in $d > 1$ (in $1d$ there is an exact solution[6]). In fact, the strong on-site repulsion prevents double occupancies, and

acts like an effective Pauli principle even for electrons of different spin, therefore enhancing the correlations. This effect should be quantitatively stronger when the density of electrons, i.e. the probability of having a double occupied site, increases. While this is certainly true for sufficiently high dimensionality, exact results in $d = 1$ contradict this expectation, and show that strong coupling prevails at high density (close $\rho = 1$) and at low density (close to $\rho = 0$), and moreover non-Landau-Fermi liquid behaviour occurs for all densities. For this reason it is not clear if the expected non-Landau behaviour in $d = 2$ should be sought at high density, or rather if it should be independent of the number of electrons, as in $1d$. In addition, the possibility clearly remains, that the $d = 2$ Hubbard model has no anomaly, which in turn might or might not make it useless for the study of the HTc superconductors. In this context, Anderson[1] has recently argued that the $d = 2$ Hubbard model is indeed not a Landau liquid, at all densities of electrons, and also for all values of the interaction. In his opinion, the reason could be ascribed to the presence of anti-bound states, which appear at energies above the top of the band as soon as the interaction is turned on in $2d$ as well as in $1d$, while in $3d$ a critical value of the potential is needed (notice that in the continuum limit the band is not bounded, and these states would be pushed to infinity). In fact, as the scattering states have to be orthogonal to the anti-bound states, their nature should change completely with respect to the non-interacting ones, eventually becoming orthogonal. This should happen as soon as the on-site interaction is switched on. Therefore Anderson hypothesizes that this orthogonality catastrophe is a non-adiabatic effect, i.e. it can not be detected within perturbation theory. Based on an analogy to the problem of non-interacting electron scattering off an impurity[7], Anderson concludes that the system is not a Landau liquid, due to an *unrenormalizable Fermi-surface phase-shift.*

In this thesis we study the Hubbard model in $2d$ and $3d$ in the low density limit, where the analogy to a scattering problem is more apparent, in order to understand better if the anti-bound states are indeed so important as suggested in Ref. [1], or, otherwise, if the expected anomalies appear, if they appear, only at higher density. Moreover this limit seems to be less problematic than the high density limit (close to half filling), where in a square lattice the paramagnetic state is unstable toward antiferromagnetism due to the nesting property of the Fermi surface, so that it is less clear what should be the starting state of the perturbation theory.

In particular we have studied the few body problem in the Hubbard model on a non-frustrated lattice, in dimension $d \geq 2$. This problem can be approximately solved in the limit $1/N \to 0$, where N^d is the number of sites. The solution can be obtained for any number of electrons, provided that the density goes to zero for $N \to \infty$. We have checked the approximation by exact numerical solution of the three electrons problem, and the agreement is quite good. The results, *a priori*, have limited physical relevance, because they are obtained keeping fixed the number of particles, and sending to infinity the size of the system, i.e. they correspond to zero density in the thermodynamic limit. For example we find that the interaction correction to the total ground state energy, for the case of closed shells, is, to lowest order

$$\Delta E = \frac{N_e^2}{(aN)^d} \frac{f_0(k_f)}{4}$$

where a is the lattice constant, N_e the number of electrons, and f_0 is the low energy scattering amplitude at k_f, simply related to the Hubbard U. If we suppose this result to remain valid also at finite density in the thermodynamic limit, in spite of the approximation used, we obtain formally an extensive quantity, i.e.

$$\Delta E = N_e \frac{\rho f_0}{4}$$

This expression coincides with that obtained by usual perturbation theory in the dilute but finite density limit (Galitskii's continuum theory), both in $2d$[8], and in $3d$[9]. The agreement suggests a way to generalize a few body calculation to finite, but low, density. We find the same agreement for the momentum distribution $n(\mathrm{k})$. We further notice that, while our results are obtained as the leading terms of the perturbative expansion of a model which contains the anti-bound states, the quoted dilute limit results are derived in the continuum limit, i.e. in absence of explicit anti-bound states. The agreement suggests that the effects of these states are sub-leading, provided that the generalization, to finite density, of our few body result is indeed representative. The consequence of this viewpoint is that the dilute $2d$ Hubbard system is a Landau Fermi liquid, as it is in $3d$. The main difference between the two cases is that, in $2d$, the perturbation parameter is non-analytical in the density, being $1/\log \rho a^2$, where a is the lattice spacing, while in $3d$ the perturbation is in powers of $\rho^{1/3}$. Another difference, which depends just on the geometry of the Fermi surface in $2d$, is that the quasi-particle decay rate γ is proportional to $-(\epsilon-\epsilon_f)^2 \log|\epsilon-\epsilon_f|$. Support to the idea that anti-bound states do not cause the system to be a non Fermi liquid, comes from a recent article by Engelbrecht and Randeira[10]. They first of all show that, even in the continuum limit, bound excitations are present. In fact the band of the excitations of holes is upper bounded, and in analogy to the anti-bound states for electrons, there are anti-bound states of holes. They show up as isolated poles in the particle-particle channel, for negative energy of the pair. This implies that, in doing calculation, one encounters the effect of these states also in a model defined on a continuum. There are two main consequences of these bound excitations. The first one derives from the presence of isolated poles in the vertex functions. Secondly, the anti-bound states affect the states in the continuum, as was said before. Galitskii's

theory can not detect these effects, but a renormalization procedure like that one used in the Cooper Phenomenon[11] can do. Engelbrecht et *al.* claim that, by using the latter method, the effects of the anti-bound excitations are sub-leading with respect to the Galitskii's corrections, even if they are indeed non-analytical in the potential, as it usually happens for such bound excitations. Independently Anderson* suggested a possible use of the Cooper Phenomenon renormalization method, but he presumed that "this renormalization procedure cannot work if there are anti-bound states ...because the assumed *non-singular* part of the vertex are not harmless, but infinite". However, by inspection of the diagrammatic expansion, these problems seem not to be present, i.e. the perturbation theory in the density is well ordered as in $3d$, and contrary to the $1d$ case. The tentative conclusion that may be drawn from all the above is that: *at least at low density, the two dimensional Hubbard model is a Landau Fermi liquid.*

The thesis is organized as follows:

In Chapter 1. we study the few body problem. In section 1.1 we describe the two electron solution, which is very simple but instructive. We also show how the perturbation theory works. In section 1.2 we apply the perturbation theory to the three electron problem. We compare the results with those obtained by solving numerically the problem. Finally in section 1.3 we generalize the method to any number of electrons (but still in the zero density limit), and we calculate in this case the energy and the momentum distribution.

In Chapter 2. we discuss the $2d$ continuum Fermi gas at low but finite density. In section 2.1 we briefly introduce Galitskii's theory. In section 2.2 we review and discuss the pertinent results obtained by Ref. [10].

The three Appendices summarize some more technical issues. In Appendix I. we

* see the note (6) of Ref. [1]

show how the perturbation method works for four electrons. In Appendix II. we introduce the scattering theory in $2d$, which is necessary to apply Galitskii's theory. Finally in Appendix III. we show how the phase shifts are defined in a many body problem, and how it is possible, in the dilute limit, to relate them to some physical properties. Moreover we will briefly show how it is possible to perform perturbative calculations at low density.

Chapter 1

Few electrons in the Hubbard model

In this chapter we show a very simple method to solve approximately, in the limit of large size, the few electron Hubbard model in dimension greater than one.

In section 1.1 we describe the two body problem. This case is interesting because a perturbation expansion in $1/N$, where N^d is the number of sites of the system, appears quite natural and is also easier to apply than it is in the case of more than two electrons. Moreover all the features of the N-body problem appears already at this level.

In section 1.2 we solve the three body problem. It can be solved exactly with a little numerical effort, so that it is a good test for the validity of the perturbation method. We show how it is possible to solve a three body problem by reducing it to a one body problem. This indeed is a general property of a separable interaction, like the Hubbard repulsion, which allows the reduction of a N body problem to a $N-2$ problem. Then we apply to this case the perturbation method, and finally we compare the approximate results with the exact ones. We will see that the agreement is quite good.

In section 1.3 we generalize the method to any number of electrons. In particular we calculate the ground state energy and the momentum distribution for a system of N_e electrons whose unperturbed Fermi surface is not degenerate. The results are very interesting. In fact, even if they are obtained in the limit in which the size of the system goes to infinity faster than the number of electrons, i.e. strictly

at zero density in the thermodynamic limit, they are however formally finite in this limit. It is interesting to note that they also coincide, in $3d$ and $2d$ with well known perturbative results for dilute systems of electrons interacting through a strong short range repulsion[9,8]. These facts suggest that the results we have obtained may remain valid in the thermodynamic limit for low, but finite, density of electrons, at least up to second order in what turns out to be the effective perturbation parameter, i.e. $k_f^{d-2} f_0(k_f)$, where $f_0(k_f)$ is the scattering amplitude for two electrons calculated at k_f. Moreover the method suggests a simple way to generalize a few body result in the case of finite density in the thermodynamic limit.

All the results indicate that, at least for very low density and up to second order in perturbation theory, the $2d$ Hubbard model is compatible with a Landau Fermi liquid.

1.1 Two electrons in Hubbard model

Here we study the few electron problem on a lattice. This problem was intensively studied by D. C. Mattis and co-workers in a series of very early papers[12,13,14,15,16]. They were essentially interested in the bound states of few particles, and put less emphasis on scattering states. However some results we are going to derive, have been already obtained in these papers.

Let us consider two electrons in a d-dimensional hypercube of N^d sites. The Hubbard Hamiltonian in momentum space is

$$\hat{H} = \sum_{\mathbf{k}\sigma} \epsilon_{\mathbf{k}} c^{\dagger}_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} + \frac{U}{N^d} \sum_{\mathbf{p}\mathbf{p}'\mathbf{q}} c^{\dagger}_{\mathbf{p}+\mathbf{q}\uparrow} c^{\dagger}_{\mathbf{p}'-\mathbf{q}\downarrow} c_{\mathbf{p}'\downarrow} c_{\mathbf{p}\uparrow} \tag{1.1}$$

where

$$\epsilon_{\mathbf{k}} = -2t \sum_{i=1}^{d} \cos \mathrm{k}_i a$$

and a is the lattice spacing. Since this hamiltonian conserves the total momentum $\mathbf{P}$, we limit ourselves in a subspace in which $\mathbf{P}$ is fixed. A generic wave function is

$$|\Psi\rangle = \sum_{\mathbf{q}} L(\mathbf{q}) c^{\dagger}_{\mathbf{q}\uparrow} c^{\dagger}_{\mathbf{P}-\mathbf{q}\downarrow} |0\rangle \tag{1.2}$$

This function is the eigenvector of eigenvalue E if the coefficients $L(\mathbf{q})$'s satisfy the Schröedinger's equation

$$(E - \epsilon_{\mathbf{q}} - \epsilon_{\mathbf{P}-\mathbf{q}}) L(\mathbf{q}) = \frac{U}{N^d} \sum_{\mathbf{p}} L(\mathbf{p})$$

We call J the right hand side of this equation, so that

$$L(\mathbf{q}) = \frac{J}{(E - \epsilon_{\mathbf{q}} - \epsilon_{\mathbf{P}-\mathbf{q}})}$$

Substituting this expression in that for J it is easy to obtain a self consistent equation for the energy of the usual type, i.e.[12]

$$1 = \frac{U}{N^d} \sum_{\mathbf{q}} \frac{1}{(E - \epsilon_{\mathbf{q}} - \epsilon_{\mathbf{P}-\mathbf{q}})} \tag{1.3}$$

This equation can be solved analytically in $1d$ and numerically in $d \geq 2$. However one can also solve approximately this equation in the limit of very large systems, comparing afterwards the results with the exact ones. The approximation consists in the assumption, to be verified *a posteriori*, that *the correction* $\Delta E = E - E_0$ *of the energy with respect to the* $U = 0$ *case decreases faster when* $N \to \infty$ *then the bare energy level spacing.* This assumption implies that

$$\Delta E = \frac{\alpha(N)}{N^2} \qquad \text{with} \quad \lim_{N\to\infty} \alpha(N) = 0 \tag{1.4}$$

We will use this ansatz to solve approximately the self consistent equation (1.3). For simplicity we set $\mathbf{P} = 0$. Let's extract from the sum the term with $\mathbf{q} = 0$, which is the most divergent one, and in the remaining summation we take the continuum limit, making again use of (1.4). In this way Eq. (1.3) becomes

$$1 \simeq \frac{U}{\alpha(N) N^{d-2}} - U a^d \int_{\frac{2\pi}{Na}}^{k_0} \frac{d^d q}{(2\pi)^d} \frac{m}{\hbar^2 \mathbf{q}^2}$$

where $\frac{\hbar^2}{a^2 m} = 2t$ and k_0 is an upper cutoff of order a^{-1}. One then recognizes that the integral is related to the scattering amplitude $f_0(k)$ of two electrons at low energy (actually $k \sim N^{-1}$), that is

$$\int_{\frac{2\pi}{Na}}^{k_0} \frac{d^d q}{(2\pi)^d} \frac{m}{\hbar^2 \mathbf{q}^2} \simeq \frac{m}{f_0 \hbar^2}$$

where

$$f_0 = \begin{cases} 4\pi a & \text{in 3d} \\ -\dfrac{2\pi}{\log ka} \sim \dfrac{2\pi}{\log N} & \text{in 2d} \\ \dfrac{(2\pi)^2}{Na} & \text{in 1d} \end{cases} \tag{1.5}.$$

Then Eq. (1.3) reduces at last to

$$1 = \frac{U}{\alpha(N)N^{d-2}} - \frac{Ua^d m}{\hbar^2 f_0}$$

whose solution is simply

$$\alpha(N) = \frac{1}{N^{d-2}} \frac{U}{1 + \frac{Ua^d m}{\hbar^2 f_0}}$$

The general expression of $\alpha(N)$ shows that in $d \leq 2$ the limit $N \to \infty$ drives the system towards strong coupling (i.e. $U \to \infty$) and $\alpha(N)$ tends to a limiting value independent of U, as long as U is non-zero.

These results are consistent with the initial assumption (1.4) in $d \geq 2$, and are also in good agreement with the exact ones. In $1d$ (and more generally in $d < 2$) $\alpha(N)$ is instead finite for $N \to \infty$, therefore the condition (1.4) is not fulfilled. However this is consistent with what we already know about $1d$[6], that is the appearance of a finite phase shift in the limit $N \to \infty$ in the many body wave function as soon as U is switched on. The exact solution of Eq. (1.3) in $1d$ is[17] (for large size) $\Delta E \simeq 2t(2\pi/N)^2$, and indeed it does not satisfy (1.4), as the previous simple calculation predicts. This form of ΔE suggests that, in $1d$, the effect of the interaction can be adsorbed into a finite shift of the allowed k values. For $U = 0$ the ground state energy is just $E_0 = 2\epsilon_0$ while as soon as U is turned on the energy becomes $E = \epsilon_{\Delta k} + \epsilon_{-\Delta k}$, where $\Delta k = \frac{\pi}{Na}$ (π is the phase shift). Going back to $d \geq 2$, we can find the asymptotic form of the wave function. In this case the coefficients L's are given by

$$L(\mathbf{q}) = \begin{cases} \dfrac{JN^2}{\alpha(N)} & \text{if } \mathbf{q} = 0 \\ \dfrac{J}{2\epsilon_0 + \alpha(N)N^{-2} - 2\epsilon_{\mathbf{q}}} & \text{otherwise} \end{cases} \tag{1.6}$$

and the momentum distribution at $\mathbf{p}$

$$n(\mathbf{p}) = \frac{L^2(\mathbf{p})}{L^2(\mathbf{0}) + \sum_{\mathbf{q}\neq\mathbf{0}} L^2(\mathbf{q})}$$

If the hypothesis (1.4) is verified, the behaviour of the sum in the denominator, in the limit of very large sizes, is

$$\sum_{\mathbf{q}\neq\mathbf{0}} L^2(\mathbf{q}) \sim N^d \int_{\frac{2\pi}{N}}^{k_0} \frac{d^d q}{q^4} \sim N^4$$

Together with (1.6) this implies that inside the $U = 0$ Fermi surface (i.e. just for the wavevector $\mathbf{0}$)

$$n(\mathbf{0}) \simeq 1 - A\alpha^2(N)$$

that is

$$1 - n(\mathbf{0}) \sim \begin{cases} N^{-2} & \text{in } 3d \\ \log^{-2} N & \text{in } 2d \end{cases}$$

This result shows that the momentum distribution for $N \to \infty$ reaches the $U = 0$ case contrary to the $1d$ case where the momentum distribution remains spread all over the k space. In this case in fact, as Ref. [17] shows, the limiting behaviour of the momentum distribution for large lattice size is

$$n(k) = \frac{8}{\pi^2} \frac{(2\pi)^4}{(4a^2k^2 - (2\pi)^2)^2}$$

1.2 Three electrons

The ansatz (1.4) can be successfully applied to solve the same problem for more than two electrons. We briefly sketch how to find the eigenvalue equation for the three body problem, which again can be easily solved numerically.
As in the problem of two electrons we show that $2d$ looks more like $3d$ than $1d$, that is we will not find any finite phase shift in the limit $N \to \infty$.
Let us consider three electrons in the subspace of total momentum $\mathbf{P} = (\frac{2\pi}{Na}, 0, 0, ...)$. A generic wave function is

$$|\Psi\rangle = \sum_{\mathbf{qk}} L(\mathbf{q}, \mathbf{P}-\mathbf{q}-\mathbf{k}, \mathbf{k}) c^\dagger_{\mathbf{q}\uparrow} c^\dagger_{\mathbf{P}-\mathbf{k}-\mathbf{q}\downarrow} c^\dagger_{\mathbf{k}\uparrow} |0\rangle \tag{1.7}$$

The coefficients L's must be antisymmetric in the first and third variable. The eigenvalue equation for the L's is

$$\begin{aligned}(E - \epsilon_{\mathbf{q}} - \epsilon_{\mathbf{P}-\mathbf{q}-\mathbf{k}} - \epsilon_{\mathbf{k}}) L(\mathbf{q}, \mathbf{P}-\mathbf{q}-\mathbf{k}, \mathbf{k}) = \\ = \frac{U}{N^d} \sum_{\mathbf{p}} (L(\mathbf{p}, \mathbf{P}-\mathbf{p}-\mathbf{k}, \mathbf{k}) - L(\mathbf{p}, \mathbf{P}-\mathbf{p}-\mathbf{q}, \mathbf{q}))\end{aligned} \tag{1.8}$$

whose solution is

$$L(\mathbf{q}, \mathbf{P}-\mathbf{q}-\mathbf{k}, \mathbf{k}) = \frac{J(\mathbf{k}) - J(\mathbf{q})}{(E - \epsilon_{\mathbf{q}} - \epsilon_{\mathbf{P}-\mathbf{q}-\mathbf{k}} - \epsilon_{\mathbf{k}})}$$

where

$$J(\mathbf{k}) = \frac{U}{N^d} \sum_{\mathbf{q}} L(\mathbf{q}, \mathbf{P}-\mathbf{q}-\mathbf{k}, \mathbf{k})$$

Substituting the expression of L in that of J we obtain an homogeneous set of equations[12] (instead of a simple self consistent equation as for two electrons)

$$\begin{aligned}J(\mathbf{k}) = J(\mathbf{k}) \frac{U}{N^d} \sum_{\mathbf{q}} \frac{1}{(E - \epsilon_{\mathbf{q}} - \epsilon_{\mathbf{P}-\mathbf{q}-\mathbf{k}} - \epsilon_{\mathbf{k}})} + \\ - \frac{U}{N^d} \sum_{\mathbf{q}} \frac{J(\mathbf{q})}{(E - \epsilon_{\mathbf{q}} - \epsilon_{\mathbf{P}-\mathbf{q}-\mathbf{k}} - \epsilon_{\mathbf{k}})}\end{aligned} \tag{1.9}$$

which can be formally written in matrix form

$$\vec{J} = U\hat{T}(E) \cdot \vec{J}$$

where U is the Hubbard repulsion, and $\hat{T}(E)$ is a matrix which depends only on the energy E. This problem has solution only for a set of values of E which identify the eigenvalues. We have solved (1.9) numerically for the lowest eigenvalue; in table 1 we show the exact values of the energy and of the momentum distribution inside the non-interacting Fermi surface for this three body electron system with $U = 10 \cdot t$.

We have also applied the hypothesis (1.4) to solve approximately for $N \to \infty$ the same problem. We show the way to do it and at the end we compare the results with the exact ones. The agreement is quite good. In this problem the $U = 0$ ground state energy is $E_0 = 2\epsilon_0 + \epsilon_{\mathbf{P}} \simeq t(-6d + \frac{(2\pi)^2}{N^2})$. From (1.4) it follows that

$$\frac{1}{(E - \epsilon_{\mathbf{q}} - \epsilon_{\mathbf{P}-\mathbf{q}-\mathbf{k}} - \epsilon_{\mathbf{k}})} \sim \begin{cases} \frac{N^2}{\alpha(N)} & \text{if } (\mathbf{k},\mathbf{q}) = (\mathbf{0},\mathbf{0}) \text{ or } (\mathbf{0},\mathbf{P}) \text{ or } (\mathbf{P},\mathbf{0}) \\ N^2 & \text{otherwise} \end{cases}$$

Let us define the function $I(\mathbf{k})$

$$I(\mathbf{k}) = -N^{-d} \sum_{\mathbf{q} \not\subset F.S.} \frac{1}{(E - \epsilon_{\mathbf{q}} - \epsilon_{\mathbf{P}-\mathbf{q}-\mathbf{k}} - \epsilon_{\mathbf{k}})} \simeq \frac{a^d m}{f_0 \hbar^2}$$

the last equality holding only for small k. In the present case the Fermi sea ($F.S.$) has only the two wavevectors $\mathbf{0}$ and $\mathbf{P}$. Notice that the equation $U\hat{T} \cdot \vec{J} = \vec{J}$ can be decomposed into a part which diverges for $N \to \infty$ and another which does not. As a first approximation we can retain just the singular part (actually we have shown that the parameter that controls the expansion is the scattering amplitude, so that the next term is of order $f_0^2 \sim 1/\log^2 N$). This implies to solve the matrix

No. of sites	Energy	$n_\uparrow(\mathbf{0})$	$n_\uparrow(\mathbf{P})$	$n_\downarrow(\mathbf{0})$
8×8	-11.3074	0.9867	0.9858	0.9724
9×9	-11.4501	0.9879	0.9870	0.9749
10×10	-11.5532	0.9888	0.9880	0.9768
11×11	-11.6301	0.9895	0.9888	0.9783
12×12	-11.6889	0.9901	0.9894	0.9795
13×13	-11.7384	0.9906	0.9899	0.9805
14×14	-11.7714	0.9911	0.9903	0.9814
15×15	-11.8009	0.9914	0.9907	0.9822
16×16	-11.8250	0.9917	0.9911	0.9828
17×17	-11.8451	0.9920	0.9913	0.9834
18×18	-11.8619	0.9923	0.9916	0.9839
19×19	-11.8761	0.9925	0.9918	0.9843
20×20	-11.8883	0.9927	0.9921	0.9848
21×21	-11.8987	0.9929	0.9922	0.9851
22×22	-11.9078	0.9931	0.9924	0.9855
23×23	-11.9157	0.9932	0.9926	0.9858
24×24	-11.9226	0.9934	0.9927	0.9861

Table 1: Properties of the three electron ground state in the Hubbard model for U=10·t

equation $\hat{T}^{(s)} \cdot \vec{J} = 0$ (in $3d$ we take $U \to \infty$ for simplicity)

$$\hat{T}^{(s)} = \begin{pmatrix} \frac{1}{\alpha(N)N^{d-2}} - I(\mathbf{0}) & \frac{1}{\alpha(N)N^{d-2}} & 0 & 0 & 0 & . & . \\ \frac{1}{\alpha(N)N^{d-2}} & \frac{1}{\alpha(N)N^{d-2}} - I(\mathbf{P}) & 0 & 0 & 0 & . & . \\ 0 & 0 & -I(\mathrm{k}_3) & 0 & 0 & . & . \\ 0 & 0 & 0 & -I(\mathrm{k}_4) & 0 & . & . \\ . & . & . & . & . & . & . \end{pmatrix}$$

where some ordering of the k's is implicit in order to change a matrix of dimension $(N \times N \times N \times N)$ in one $(N^2 \times N^2)$. The solution of this equation gives

$$J(0) = -J(\mathbf{P}) \qquad J(\mathbf{k}) = 0 \ \text{ if } \mathbf{k} \neq \mathbf{0}, \mathbf{P} \tag{1.10a}$$

$$\alpha(N) \simeq \frac{2\hbar^2 f_0 N^2}{Vm} \tag{1.10b}$$

In $2d$ and $3d$ (1.10b) is compatible with the initial assumption (1.4). On the contrary, in $d < 2$ the result does not satisfy (1.4), exactly as we found for two electrons. For example in $2d$ the ground state energy should be, according to (1.10b) and using Eq. (1.5) for f_0

$$E \simeq -12t + \frac{t(2\pi)^2}{N^2} + \frac{16\pi t}{N^2 \log N^2} + \mathcal{O}(\frac{1}{N^2 \log^2 N}) \tag{1.11}$$

In fig. (1.1) we have plotted (1.11) versus N (solid line) together with the numerical solution of (1.9) at $U = 10t$ (open circles) and $U = 100t$ (open starred points) and $U = 1000t$ (diamond points). The dashed line is the total energy at $U = 0$. The agreement is quite good even if the size is not very large, confirming that (1.11) is asymptotically correct.

From (1.7) it is easy to derive the expression of the momentum distribution in terms of the function $J(\mathbf{k})$, i.e.

$$n_{\mathbf{k}\uparrow} = 4 \sum_{\mathbf{q}} \left(\frac{J(\mathbf{q}) - J(\mathbf{k})}{E - \epsilon_{\mathbf{k}} - \epsilon_{\mathbf{q}} - \epsilon_{\mathbf{P}-\mathbf{q}-\mathbf{k}}} \right)^2 \tag{1.12a}$$

$$n_{\mathbf{k}\downarrow} = 2 \sum_{\mathbf{q}} \left(\frac{J(\mathbf{q}) - J(\mathbf{P}-\mathbf{q}-\mathbf{k})}{E - \epsilon_{\mathbf{k}} - \epsilon_{\mathbf{q}} - \epsilon_{\mathbf{P}-\mathbf{q}-\mathbf{k}}} \right)^2 \tag{1.12b}$$

We can use our approximate solution to calculate the momentum distribution simply substituting (1.10a-b)and (1.11) in (1.12a-b). Defining

$$A = \frac{1}{(E - \epsilon_0 - \epsilon_{\mathbf{P}} - \epsilon_0)} \simeq \frac{N^2}{\alpha(N)}$$

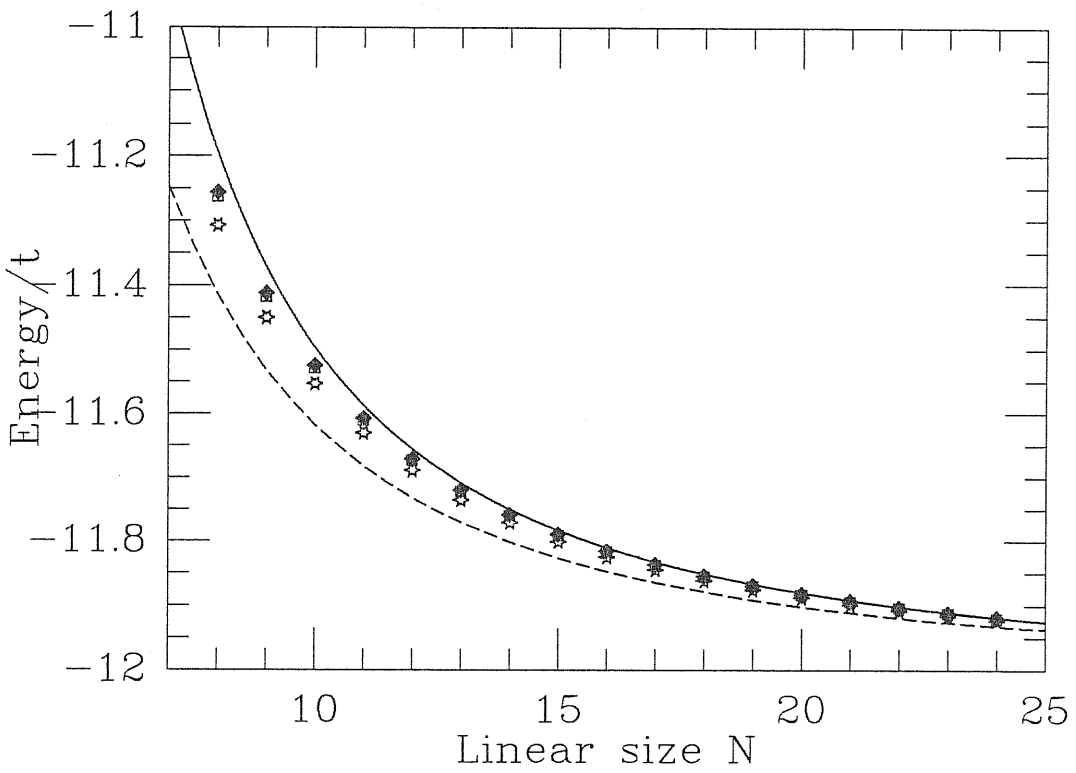

Fig. 1.1: Energy of three electrons on a $2d$ $N\times N$ lattice versus N. The points are the exact results for U=10·t (starred points), U=100·t (open squares), U=1000·t (diamond filled points). The solid curve is the U independent approximation (Eq. 1.11), while the dashed curve is the U=0 energy.

and

$$B(\mathbf{k}) = \sum_{\mathbf{q}\not\subset F.S.} (E - \epsilon_{\mathbf{k}} - \epsilon_{\mathbf{q}} - \epsilon_{\mathbf{P}-\mathbf{q}-\mathbf{k}})^{-2} \sim N^4$$

then it is simple to show that

$$n_{0\uparrow} \simeq n_{\mathbf{P}\uparrow} \simeq 16A^2 + 4B$$

$$\sum_{\mathbf{q}\not\subset F.S.} n_{\mathbf{q}\uparrow} \simeq 8B$$

If we properly normalize the wave function, as we did for two electrons, then

$$n_{0\uparrow} \simeq n_{\mathbf{P}\uparrow} \simeq \frac{16A^2 + 4B}{16A^2 + 8B} \simeq 1 - \frac{I}{16A^2}$$

This result implies that the jump Z in momentum distribution at the fermi wavevector goes to 1 for $N \to \infty$ as it does for two electrons. In two dimensions this means

$$1 - Z \sim \frac{1}{\log^2 N} \qquad \text{for } N \to \infty$$

In fig. 1.2 we plot $n_\downarrow(\mathbf{0})$ for the exact numerical solution at $U = 10 \cdot t$ (black points) and for the U-independent approximation obtained by substituting Eq. (1.10a-b) in Eq. (1.12b) (white points).

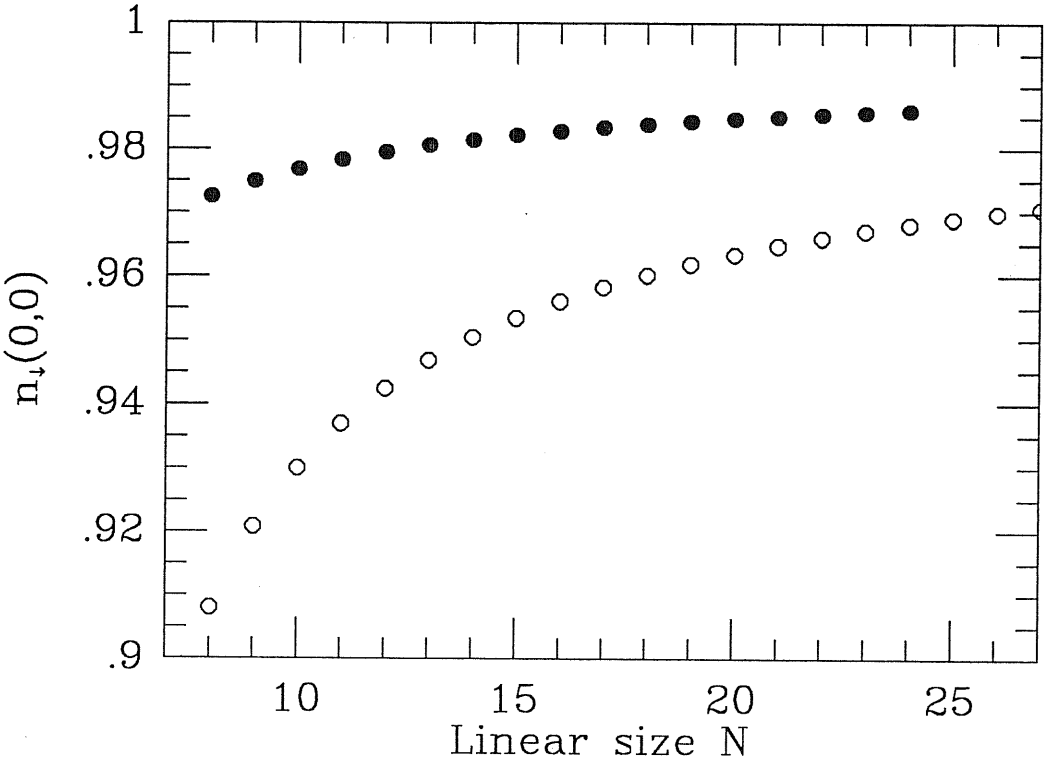

Fig. 1.2: $n_\downarrow(\mathbf{0})$ as a function of the linear size for the exact numerical solution at U=10·t (black points) and for the approximate one (white points)

The agreement between the exact and the approximate solution, is not so good as the one we find for the total energy. The reason is that the convergence towards

the asymptotic solution in the case of the momentum distribution is very slow, the convergency parameter being $1/\log N$, in contrast to the total energy, where the parameter is $1/(N^2 \log N)$.

1.3 Generalization to N_e electrons

We have seen that also the three electron problem can be approximately solved using as an ansatz the condition (1.4). The result is compatible with the initial hypothesis if the size of the system is very large, and it agrees quite well with the exact numerical solution. In this section we generalize the method to any number of electrons N_e and solve approximately the problem for a general closed shell, i.e. when the $U = 0$ Fermi surface is not degenerate. As an application we show the calculation for 4, 6, 8 electrons in appendix I. The solution of the non-degenerate case is interesting because, even if derived when in principle (1.4) is fulfilled (i.e. strictly at zero density in thermodynamic limit) it still gives results that remain finite in the thermodynamic limit. This suggests that these results have a more general validity. This is further confirmed by the fact that in $3d$ and $2d$ the energy and the jump in the momentum distribution as derived by (1.4), coincide with the calculations done by perturbation theory in the low density limit for systems of electrons interacting through a strong short range repulsion[9,8]. In the next chapter we will show how these calculations are done and how one re-obtains the same results of this section.

1.3.1 Ground state energy

Let's suppose to have N_e electrons in a subspace of $\mathbf{P}$ total momentum (for simplicity N_e is taken even). A generic wave function can be written as

$$|\Psi_{N_e}\rangle = \sum_{\mathbf{k}_1\mathbf{k}_2..} L(\mathbf{k}_1, \mathbf{k}_2, ..., \mathbf{k}_{N_e}) c^\dagger_{\mathbf{k}_1\uparrow} c^\dagger_{\mathbf{k}_2\downarrow} .... c^\dagger_{\mathbf{k}_{N_e}\downarrow} |0\rangle \qquad (1.13)$$

The coefficients L's are odd functions with respect to the interchange of two odd(even) momenta. Notice that they depend on $N_e - 1$ momenta because of

the conservation of the total momentum. The eigenvalue equation is

$$(E-\epsilon_{\mathbf{k}_1}-...-\epsilon_{\mathbf{k}_{N_e}})L(\mathbf{k}_1,...,\mathbf{k}_{N_e})=\frac{U}{N^d}\sum_{\mathbf{q}}L(\mathbf{k}_1+\mathbf{q},\mathbf{k}_2-\mathbf{q},\mathbf{k}_3,...,\mathbf{k}_{N_e})+$$
$$+L(\mathbf{k}_1+\mathbf{q},\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4-\mathbf{q},...,\mathbf{k}_{N_e})+... \tag{1.14}$$

where the dots correspond to all the possible pairings of momenta corresponding to one up and one down electron. Analogously to the three electron problem we define a function of N_e-2 momenta

$$J(\mathbf{k}_3,...,\mathbf{k}_{N_e})=\frac{U}{N^d}\sum_{\mathbf{q}}L(\mathbf{q},\mathbf{P}-\mathbf{q}-\sum_{i=3}^{N_e}\mathbf{k}_i,\mathbf{k}_3,...,\mathbf{k}_{N_e})$$

In terms of these functions, the coefficients L's can be written as

$$L(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,...,\mathbf{k}_{N_e})=\frac{J(\mathbf{k}_3,\mathbf{k}_4,...,\mathbf{k}_{N_e})-J(\mathbf{k}_3,\mathbf{k}_2...,\mathbf{k}_{N_e})+\cdots}{(E-\epsilon_{\mathbf{k}_1}-...-\epsilon_{\mathbf{k}_{N_e}})} \tag{1.15}$$

Again, if we then write a self consistent set of equations that the J's have to satisfy, this turns out of the general form

$$\vec{J}=U\hat{T}(E)\cdot\vec{J} \tag{1.16}$$

The main advantage in working with the J's rather than the L's is that by using the J's it is more clear the way to perform the perturbation expansion in powers of the system size N. To solve the set of equations (1.16) we generalize (1.4) to a finite number of electrons. This implies again that

$$E=E_0+\frac{\alpha(N)}{N^2}\qquad\text{where}\quad\lim_{N\to\infty}\alpha(N)=0 \tag{1.4}$$

and moreover that the energy difference between the Fermi level and the first excited state goes to zero, in the same limit, as $1/N^2$. Both the conditions actually mean zero density for $N\to\infty$. In fact at finite density in the thermodynamic

limit, the interaction correction to the energy no longer goes to zero for $N \to \infty$ (it remains finite and proportional to the volume), and the one particle excitation energy at Fermi level goes to zero only like $\sin(k_f)/(Na) \sim 1/N$, being k_f finite. As in the case of three electrons, the matrix $\hat{T}(E)$ has a singular part $\hat{T}(E)^{(s)}$ which include terms like

$$(E - \epsilon_{\mathbf{k}_1} - \ldots - \epsilon_{\mathbf{k}_{N_e}})^{-1} \sim \frac{N^2}{\alpha(N)}$$

and

$$\frac{1}{N^d} \sum_{\mathbf{q} \not\subset F.S.} \sum_{\mathbf{q}'} \delta(\mathbf{q}' + \mathbf{q} + \sum_{i=3}^{N_e} \mathbf{k}_i - \mathbf{P}) \frac{1}{(E - \epsilon_{\mathbf{q}} - \epsilon_{\mathbf{q}'} - \epsilon_{\mathbf{k}_3} - \cdots - \epsilon_{\mathbf{k}_{N_e}})} \sim f_0^{-1}(k_f)$$

where all the k's belong to the $F.S.$. At first approximation we can solve just the equation $\hat{T}(E)^{(s)} \cdot \vec{J} = 0$ (in $3d$, as before, we take $U \to \infty$). This again implies that the only J's different from zero are those whose arguments k's are included in the Fermi sphere of the unperturbed N_e electrons (their number N_j is $(\frac{N_e}{2})^2$ for not degenerate closed shell, counting the possible pairings of different spin states in the Fermi sphere. In the case of open shells this number is bigger, because the pairings involve all the degenerate Fermi surfaces). In general the equations these J's have to satisfy look like an usual eigenvalue problem

$$\hat{M} \cdot \vec{J} = \lambda \vec{J} \tag{1.17}$$

where λ is expressed in terms of the unknown $\alpha(N)$ by

$$\lambda = \left(\frac{m\alpha(N) a^d N^{d-2}}{\hbar^2 f_0(k_f)} \right)$$

and the energy expressed as a function of λ is

$$E = E_0 + \frac{\lambda \hbar^2 f_0}{mV} \tag{1.18}$$

In (1.17) the matrix $\hat{M}$ is independent of E and has dimension N_j. The method to construct $\hat{M}$ is very simple and can be deduced by simple inspection of (1.14) and by using the symmetry properties of the L's.

First let us suppose that the $U = 0$ Fermi surface is not degenerate. Then $\hat{M}$ is the matrix of dimension $(\frac{N_e}{2})^2 \times (\frac{N_e}{2})^2$

$$\hat{M} = \begin{pmatrix} 1 & -1 & 1 & -1 & 1 & -1 & . & . & . \\ -1 & 1 & -1 & 1 & -1 & 1 & . & . & . \\ 1 & -1 & 1 & -1 & 1 & -1 & . & . & . \\ . & . & . & . & . & . & . & . & . \end{pmatrix} \tag{1.19}$$

The case in which the Fermi surface is degenerate (open shell) is a little bit more complicated. Let us suppose that it is n-fold degenerate. First we construct a block matrix made up of n boxes like (1.19) each of which is defined in a particular Fermi surface.. However some rows(columns) of different boxes correspond indeed to the same k vectors. For this reason we move these rows(columns) in order to let them coincide with just one. The final matrix will look much more complicated than (1.19), and in general it will be more difficult to diagonalize. In this case we expect that the interaction partially removes the degeneracy, as we will show for 4, 6, 8 electrons in Appendix I, even if we expect that this open shell effect should be negligible in the thermodynamic limit. In table 2. we show the value of λ and the possible (degenerate) values of the total spin of the ground state of 4, 6 and 8 electron.

For the non degenerate closed shell case however the solution is very simple. In fact the matrix (1.19) has only one physical $\lambda \neq 0$ solution, which is $\lambda = (\frac{N_e}{2})^2$. For this solution the energy for the ground state is

$$E = E_0 + \frac{N_e \rho f_0(k_f) \hbar^2}{4m} \tag{1.20}$$

No. of electrons	λ	Total spin
4	3	0,1,1
6	7	0,2
8	15	0,1,1

Table 2: Values of λ and of the corresponding total spin for the ground state of 4, 6 and 8 electrons

and

$$\vec{J} = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ . \\ . \\ . \end{pmatrix}$$

We point out again that this result is obtained supposing that the condition (1.4) is true, i.e. at zero density in thermodynamic limit. Nevertheless Eq. (1.20) remains an extensive quantity, if we consider a finite density ρ in the thermodynamic limit. This fact suggests that (1.20) has a more general validity. There are two main reasons which support this hypothesis:

1) suppose we expand the $1d$ ground state energy of a finite number of electrons, which fill a closed shell, in powers of $1/N$, when $N \to \infty$. Then we express each power in terms of the density, i.e. $N^{-1} = N_e^{-1}\rho$. This expansion, at least up to order ρ^3, coincides with that obtained expanding in powers of ρ the energy of a system with finite density in the thermodynamic limit (cfr. Ref. [17]).This suggests that the few electron problem, even in a very pathological case like $1d$,

can give meaningful results in the thermodynamic limit, at least for the energy. The same does not happen for the momentum distribution, where one needs to do a finite size scaling, keeping the density fixed and varying the size, to recognize that the jump in the momentum distribution at the Fermi surface goes to zero.

2) for $3d$ and $2d$ (1.20) coincides with the well known result obtained by perturbation theory in dilute limit for electronic systems interacting through a strong short range repulsion[9,8], discussed in next chapter.

1.3.2 Momentum distribution

The expression of the momentum distribution can be easily derived from (1.13) and (1.15). Apart from a normalization factor, it is

$$n_{\mathbf{k}\uparrow} = \sum_{(\mathbf{k}_3,\cdots,\mathbf{k}_{N_e})} L^2(\mathbf{k}, \mathbf{P} - \mathbf{k} - \sum_{i=3}^{N_e} \mathbf{k}_i, \mathbf{k}_3, \cdots, \mathbf{k}_{N_e}) \tag{1.21}$$

for spin up electrons, and analogously for spin down ones. The approximate solution we have found means that the only L's different from zero are

$$L^2(\{F.S.\}) \simeq V^2/f_0^2(k_f) = \Delta$$

$$L^2(\{F.S. - \mathbf{k}_i - \mathbf{k}'_i + \mathbf{k}_o + \mathbf{k}'_o\}) \simeq \frac{1}{(E - \epsilon_{k_i} - \epsilon_{k'_i} + \epsilon_{k_o} + \epsilon_{k'_o})^2} = \Lambda(\mathbf{k}_i, \mathbf{k}'_i, \mathbf{k}_o, \mathbf{k}'_o)$$

where $\{F.S.\}$ means the set of k vectors defining the Fermi sea, the $\mathbf{k}_i$'s are wavevectors inside the $F.S.$ while $\mathbf{k}_o$'s are outside, and the prime refers to down spin electrons. Notice that $\mathbf{k}_i + \mathbf{k}'_i = \mathbf{k}_o + \mathbf{k}'_o$. Let us consider first the case

$\mathrm{k} \subset F.S.$. Then

$$n_{\mathrm{k}\uparrow} = L^2(\{F.S.\})+$$
$$\sum_{\mathrm{k}_o,\mathrm{k}'_o \not\subset F.S.}^{\mathrm{k}_i,\mathrm{k}'_i \subset F.S.,\mathrm{k}_i \neq \mathrm{k}} L^2(\{F.S. - \mathrm{k}_i - \mathrm{k}'_i + \mathrm{k}_o + \mathrm{k}'_o\})\delta(\mathrm{k}_i + \mathrm{k}'_i - \mathrm{k}_o - \mathrm{k}'_o) =$$
$$= \Delta + \sum_{\mathrm{k}_o,\mathrm{k}'_o \not\subset F.S.}^{\mathrm{k}_i,\mathrm{k}'_i \subset F.S.} \Lambda(\mathrm{k}_i,\mathrm{k}'_i,\mathrm{k}_o,\mathrm{k}'_o)\delta(\mathrm{k}_i + \mathrm{k}'_i - \mathrm{k}_o - \mathrm{k}'_o)+$$
$$- \sum_{\mathrm{k}_o,\mathrm{k}'_o \not\subset F.S.}^{\mathrm{k}'_i \subset F.S.} \Lambda(\mathrm{k},\mathrm{k}'_i,\mathrm{k}_o,\mathrm{k}'_o)\delta(\mathrm{k} + \mathrm{k}'_i - \mathrm{k}_o - \mathrm{k}'_o) = \Delta + \Omega - \Xi(\mathrm{k})^-$$

When $\mathrm{k} \not\subset F.S.$ then

$$n_{\mathrm{k}\uparrow} = \sum_{\mathrm{k}'_o \not\subset F.S.}^{\mathrm{k}_i,\mathrm{k}'_i \subset F.S.} L^2(\{F.S. + \mathrm{k} + \mathrm{k}'_o - \mathrm{k}_i - \mathrm{k}'_i\})\delta(\mathrm{k}_i + \mathrm{k}'_i - \mathrm{k} - \mathrm{k}'_o) = \Xi(\mathrm{k})^+$$

where the $+$ or $-$ refers to $\mathrm{k} \not\subset F.S.$ or viceversa. It is easy to realize that

$$\sum_{\mathrm{k} \subset F.S.} n_{\mathrm{k}\uparrow} = \frac{N_e}{2}(\Delta + \Omega) - \Omega$$

and

$$\sum_{\mathrm{k} \not\subset F.S.} n_{\mathrm{k}\uparrow} = \Omega$$

If we normalize properly the momentum distribution, i.e. we divide by

$$\frac{2\sum_{\mathrm{k}} n_{\mathrm{k}\uparrow}}{N_e} = \Delta + \Omega$$

then we find

$$n_{\mathrm{k}} = \frac{\Delta + \Omega - \Xi(\mathrm{k})^-}{\Delta + \Omega} \simeq 1 - \frac{\Xi(\mathrm{k})^-}{\Delta} \qquad (1.22a)$$

for $\mathrm{k} \subset F.S.$, and outside the Fermi surface

$$n_{\mathrm{k}} = \frac{\Xi(\mathrm{k})^+}{\Delta + \Omega} \simeq \frac{\Xi(\mathrm{k})^+}{\Delta} \qquad (1.22b)$$

To obtain (1.22a-b) we have assumed $\Delta \gg \Omega$. This is actually true in our approximation (i.e. $N \to \infty$ while N_e is finite), because $\Omega \simeq N_e \cdot \Xi^-(\mathrm{k})$, where, as we will show, $\Xi^-(\mathrm{k}) \sim N^d$, so that $\Omega \sim N^d \ll \Delta \sim N^{2d} \log N$. These expressions coincide exactly with those found by perturbation theory in the low density limit (cfr. the general expression given by Ref. [18]). This is very intriguing, because we have needed just the first order of our perturbation expansion to find the correction to the momentum distribution up to second order. Notice that the usual perturbation theory in the low density limit needs a tricky diagram resummation and the second order calculation to obtain the same result. We can evaluate the approximate behaviour of n_{k} close to the Fermi surface. In this case

$$\Xi(\mathrm{k})^- \simeq \Xi(\mathrm{k})^+ \simeq \frac{N_e}{2}(Na)^d \int_{\mathrm{q} \not\subset F.S.} \frac{d^d q}{(2\epsilon_{\mathrm{q}} - 2\epsilon_f)^2} \sim V N_e k_f^{d-4}$$

(the factor $N_e/2$ counts the down spin electrons, i.e k'_i, inside the $F.S.$). This means that

$$\frac{\Xi(\mathrm{k})^-}{\Delta} \sim k_f^{2(d-2)} f_0^2(k_f)$$

and that the jump Z in the momentum distribution at k_f is close to one. In particular

$$1 - Z = 1 - (n_{k_f - 0\uparrow} - n_{k_f + 0\uparrow}) \sim k_f^{2(d-2)} f_0^2(k_f) \sim \begin{cases} (k_f a)^2 & \text{in } 3d \\ \log^{-2} \rho a^2 & \text{in } 2d \end{cases} \qquad (1.23)$$

We have shown that a few body calculation can give meaningful results in the thermodynamic limit, and this is achieved by doing an appropriate perturbation theory. In the next chapter we describe the diagrammatic tecnique which allows to obtain these results directly at finite, but low, density.

Chapter 2

Non ideal Fermi gas in two dimensions

In this chapter we show how it is possible to recover the results we have obtained in chapter 1 by usual perturbation theory. In particular we use the method introduced by Galitskii[9] in 1958 for the problem of a dilute system of electrons interacting through a strong short range repulsion. This method was originally introduced in $3d$, and applied to the $2d$ problem by Bloom in 1975[8]. It is reasonable to believe that this formulation, i.e. a continuous model with short range repulsion, has the same properties of the Hubbard model, at least in the low density limit. In particular, by calculating the self energy up to second order in $f_0(k_f)k_f^{d-2}$, the system is a Landau Fermi liquid with a quasi-particle decay rate $\gamma(p)_{q.p.} \sim -f_0^2 \cdot (p-p_f)^2 \log|p-p_f|$ and a residue Z that goes to one as the density goes to zero as $1-Z \sim f_0^2 \sim 1/\log^2 \rho a^2$, where a is related to the range of the potential (see appendix II). These results coincide with those obtained in the previous chapter, where few electrons at strictly zero density were discussed.

In the section 2.1 we describe briefly the method by which it is possible to apply the perturbation theory in a dilute system of electrons with short range repulsion, even if the interaction diverges. Then we show the results obtained by this method.

In the section 2.2 we review some recent results obtained by Engelbrecht and Randeira[10] about the possible effects of anti-bound states on the physical properties of such models. This problem arose after the suggestion of Anderson[1] that

the presence of anti-bound states causes these systems to deviate radically from a Landau Fermi liquid behaviour. This idea represents a many body analog to the problem of noninteracting fermions scattering off a single impurity. Engelbrecht and Randeria have claimed that this is not the case, i.e. that the correction to the self energy coming from these states is negligible, in the dilute limit, with respect to the results obtained by neglecting the anti-bound states.

2.1 Galitskii's theory

Let us consider a low density system of electrons interacting through a strong, possibly infinite, short range repulsion $U(r)$, of finite range R (from now on we take $\hbar = m = 1$). The hamiltonian is

$$\hat{H} = \int d\mathbf{x}\Psi^\dagger(\mathbf{x})\left(-\frac{\nabla^2}{2}\right)\Psi(\mathbf{x}) + \int d\mathbf{x}d\mathbf{y}\Psi^\dagger(\mathbf{x})\Psi^\dagger(\mathbf{y})U(|\mathbf{x}-\mathbf{y}|)\Psi(\mathbf{y})\Psi(\mathbf{x}) \quad (2.1)$$

In (2.1) we neglect discreteness of the lattice. A remarkable consequence of having a discrete lattice is the presence of anti-bound states that appear above the top of the band as soon as U is switched on. Yet, we believe that, in the dilute limit, this model behaves like the Hubbard model, which is defined on a lattice with on site repulsion. We justify this assumption by using results of our $2d$ few electron calculations. In fact we have seen that the interaction changes the momentum distribution, but this distortion goes to zero either far away from the Fermi level or when the size of the system goes to infinity. This suggests that the excitations of the non-interacting ground state which enter in the perturbation theory are mainly localized around the Fermi level, and their contribution to the actual ground state decreases to zero when the density or the range of the potential goes to zero. This does not happen in one dimension where, in fact, the solution of the few electron problem shows that the distortion of the $U = 0$ momentum distribution does not go to zero for the size going to infinity. This implies that, at finite density, the excitations that must be considered in a perturbative calculation are too many and too spread in the k space to be handled.

Let us go back to the model (2.1). As the potential is very large, we can not do any perturbation theory in powers of U, and simply truncate the expansion. Nevertheless, some kind of perturbative approach is still possible. We follow the method introduced by Galitskii[9] in 1958 for the three-dimension version of the

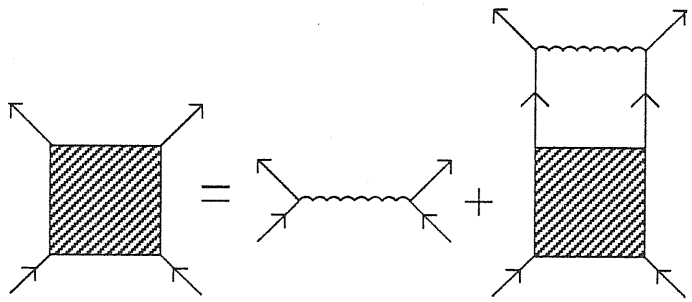

Fig. 2.1: Ladder approximation for the vertex correction

problem, generalized in $2d$ by Bloom[8], even if we do not explain it in detail.* The basic idea is that, although the repulsion may be very strong, the scattering amplitude for such a potential can still be small, playing the role of an effective perturbative parameter (see also appendix II).

This property suggests that the first thing one has to do, is to take into account the full effect of the interaction on the two particle wave function, neglecting any other many body effect. Once this has been done, the resulting effective theory is not singular anymore, even in the limit of diverging potential. Furthermore it is possible to treat it perturbatively, and the small parameter which controls the expansion turns out to be $k_f^{d-2} f_0(k_f)$, where $f_0(k_f)$ is the scattering amplitude at k_f, in the limit $k_f \cdot a \ll 1$**. In $2d$ this parameter is just $f_0(k_f) \simeq -2\pi/\log k_f a$, where a is a characteristic length of order $R\exp(-b/UR^2)$ (R is the range of the poten-

* A good reference for this approach is, besides the original paper, the chap. 4 section 11 of "Quantum theory of many particle systems" by A. L. Fetter and J. D. Walecka, McGraw Hill.

** Actually the fact that, by taking into account just the two particle correlation, one gets rid of the large Hartree-Fock correction, proportional to the potential, it is very interesting. It is also the basis of a systematic approach to the problem by including variationally in the many body wave function first the two particle correlations, then the three particle correlations and so

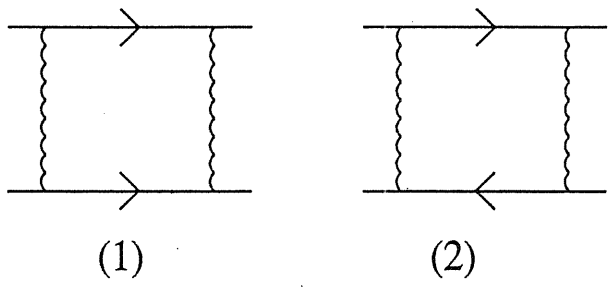

Fig. 2.2: Second order vertex corrections

tial, b a constant of order $\hbar^2/2m$ and U the Hubbard repulsion) (cfr. Appendix II or Ref. [20]). Note that in $1d$ the perturbation parameter $k_f^{d-2} f_0(k_f) \to const$ in the dilute limit (cfr. eq. 1.5), and therefore no simplification occur at low density. Actually the $2d$ case is not exactly the same as the $3d$ case. Engelbrecht and Randeira[10] have shown that the $2d$-vertex has an unusual singularity in the particle-particle channel for all momenta $q < 2p_f$, which they interpreted as collective modes of anti-bound holes. The energies of these modes have a term proportional to $\epsilon_f^2 a^2$, i.e. not analytical in U. However the contribution to the self energy of these modes is of order $\epsilon_f \rho a^2$, i.e. sub-leading with respect to that obtained by using the Galitskii's approximation, that is why we neglect these contributions. We will go back to this aspect in the section 2.2.

In practice Galitskii's method consists in taking for the vertex corrections only the contributes coming from the particle-particle (hole-hole) channels (fig. (2.1)).

In fig. (2.2) we draw two generic second order diagrams contributing to the vertex function. The approximation we adopt implies that only the first diagram is considered. It is not difficult to realize that the second diagram is negligible with respect to the first one in the low density limit, even in $2d$. This is true at least up to second order in the parameter $k_f^{d-2} f_0(k_f)$. Up to this order the full set of

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on (cfr. Ref. [19]).

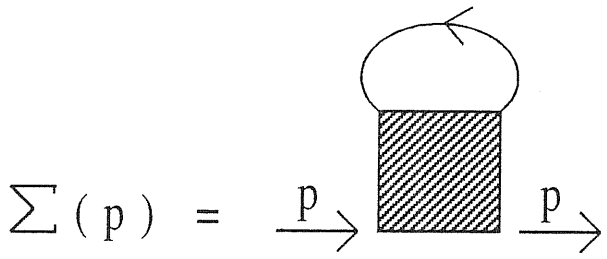

Fig. 2.3: Self energy in the ladder approximation

diagrams contributing to the self energy is only that drawn in fig. (2.3). In fact let us consider a generic diagram for the self energy not included in the approximation used, for example that of fig. (2.4). This class of diagrams implies an infinite series corresponding to the ladder approximation for the vertex. The consequence is that fig. (2.4) is substituted by the diagram of fig. (2.5), which is of third order in f_0[21], i.e. the order at which the ladder approximation for the vertex is no more valid. Once we realize that the ladder approximation is a good one even in two dimensions, we have just to repeat the general arguments used by Galitskii in $3d$. The only thing we need is the asymptotic behaviour of the scattering amplitude at low energy and the optical theorem in two dimensions, i.e. (see appendix II)

$$f(\mathrm{k},\mathrm{k}') \simeq -\frac{2\pi}{\log ka} + O(log^{-2}) \quad \text{for } \mathrm{k},\mathrm{k}' \to 0$$

and for $|\mathrm{k}| = |\mathrm{k}'|$

$$\mathrm{Im}f(\mathrm{k},\mathrm{k}') = -\frac{1}{4\pi}\int d\hat{\mathrm{n}} f(\mathrm{k},k\hat{\mathrm{n}}) f^*(\mathrm{k}',k\hat{\mathrm{n}})$$

These two relations are enough to justify even in $2d$ the analytical expressions (25) and (26) in Ref. [9] for the self energy correction up to second order in f_0 (see Eq. (A.10) of Appendix III). From the self energy calculation one can evaluate in $2d$ the quasi-particle energy ($\epsilon_{q.p.}(\mathrm{k})$), the leading part of its decay rate (proportional to the ImΣ) and the residue of the Green function Z. The results are

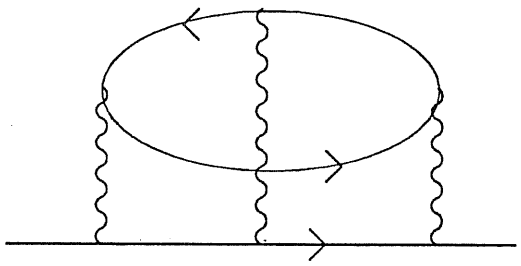

Fig. 2.4: Third order diagram contributing to self energy

$$\epsilon_{q.p.} = \frac{k_f^2}{2}\left(1 - \frac{2}{\log \rho a^2} + \cdots\right) \tag{2.2}$$

$$\mathrm{Im}\Sigma(k, \epsilon_{q.p.}) = \beta \frac{(k-k_f)^2 \log|k-k_f| \mathrm{sign}(k-k_f)}{\pi \log^2 \rho a^2} \tag{2.3}$$

$$Z \simeq 1 - \frac{\delta}{log^2 \rho a^2} \tag{2.4}$$

We have found that $\beta = 4$ and $\delta = 4\log 2$. These two results do not coincide exactly with those obtained by Ref. [8] ($\beta = 5$ and $\delta \simeq 1.04$), but the discrepancy has no important consequence. We have also calculated the total energy by the usual tecnique[21], and the result coincides with (1.20) and with Ref. [8], i.e.

$$\frac{E}{V} = \frac{E_0}{V} - \frac{\pi \rho^2}{\log \rho a^2} + \cdots$$

These results suggest that, at least up to second order in the scattering amplitude, the $2d$ non-ideal Fermi gas is a Landau Fermi liquid, even if the physical quantities are not analytical in the density. The logarithmic corrections to the quasi-particle lifetime are characteristic of the $2d$[4]. It is also very interesting that the results coincide with those obtained in the previous chapter. There we did a few particle calculation in the limit $k_f \sim 1/N$ (N is the number of sites per side), and after we took a very peculiar thermodynamic limit (i.e. the number of electron divided by the volume has been always interpreted as a finite density, even if the

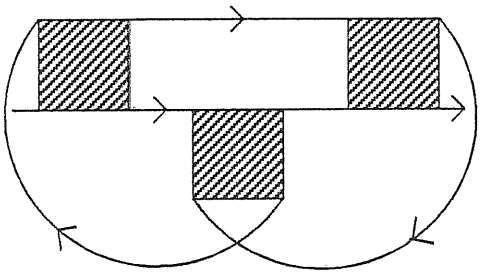

Fig. 2.5: The summed diagram whose starting contribution is fig. (2.4)

result was valid strictly at zero density). The agreement between the two calculations suggests a way to generalize few particle calculations to finite density in the thermodynamic limit.

2.2 The effects of the anti-bound states

In this section we discuss a recent paper by Engelbrecht and Randeria[10], who studies the effects of possible anti-bound states in the particle-particle excitation spectrum. This problem has been raised by Anderson who suggested[1] that these anti-bound states give raise to a non-Fermi liquid behaviour of the Hubbard model in $2d$. Notice that in $2d$ as well as in $1d$, anti-bound states appear for any value of the Hubbard U, while in $d > 2$ a value of U larger than a critical one is needed. If we have understood well Anderson's idea, these anti-bound states reduce strongly the Hilbert space available to the scattering states. This should happen because each anti-bound state must be orthogonal to the others and to all the states of the continuum. For this reason each scattering state has no more a well defined momentum k. The consequence may be that, if we add one more particle in a state of fixed k ($k > k_f$) to a system of N-electrons, the ground state wave function is orthogonal to the one obtained by applying a creation operator of momentum k on the N-particle ground state. This is another way of saying that the residue is zero, i.e. the system is not a Landau Fermi liquid. This should be analogous to the problem of noninteracting fermion scattering off a single impurity, i.e. to the infrared catastrophe[7].

The first problem one encounters in applying Galitskii's theory to the Hubbard model, is the presence of anti-bound states for the two body problem in the vacuum, which does not allow a simple generalization of the method. In fact the method requires the completeness of the set of scattering states (cfr. Ref. [21] pag. 140 eq. 11.44), which is not satisfied in presence of bound (anti-bound) states. Actually this is not a problem. Completeness is needed because in the continuum limit it is not possible to sum the whole series of diagrams for the

ladder. However in the case of a discrete lattice the summation is possible. This means that *a priori* one could use the exact ladder summation scheme for the Hubbard model including also the anti-bound states (see appendix III eq. A.1). However, even in the continuum limit (which is preferable for doing calculations) some anti-bound states[10] are present. In fact, as soon as the density of electrons becomes finite, anti-bound states for the excitations of holes appear because the hole energy band is bounded. This means that one can find isolated poles of the vertex function for energies lower than the bottom of the two hole excitation band. Clearly in the Hubbard model, which is symmetric under particle-hole transformation, one expects analogous poles above the top of the two particle excitations. The problem is: are these isolated poles, which also exist in the continuum limit, really important ? Do they lead to a non-Fermi liquid behaviour ? The first thing to notice is that, as usual, these states have an excitation energy which is not analytical in the potential (it is proportional to $a^2 \sim R^2 \mathrm{e}^{-2b/R^2U}$), so that the usual perturbation theory is not able to detect their effects, as Ref. [1] suggested. However, as Ref. [10] shows, by applying the so-called Cooper phenomenon renormalization method[11] in order to take into account isolated poles in the vertex, the contribution coming directly from these states (which Galitskii's theory misses) to the ground state energy, is obviously positive, as these states have a large excitation energy, and is proportional to $(k_f \cdot a)^2$. The origin of this contribution it is easy to understand. In fact we know that a generic anti-bound state has an energy of the form

$$\Delta = E_c \mathrm{e}^{-1/(\mathrm{N}(\epsilon_f)\lambda)}$$

where E_c is a characteristic energy (usually the smallest energy scale), λ the effective coupling constant and $\mathrm{N}(\epsilon_f)$ the density of states at the Fermi energy. In our case ($U \to \infty$) there are two energy scales, i.e. $\hbar^2/(ma^2)$ and ϵ_f, of which the

latter is the smallest. Moreover in $2d$ we have shown that

$$\mathrm{N}(\epsilon_f)\lambda = -\frac{1}{2\pi}\frac{4\pi}{\log \rho a^2} = -\frac{1}{\log k_f a}$$

This implies that the energy of an anti-bound state is $\propto \epsilon_f k_f a$. The contribution to the total energy of these states is usually $N^d\Delta^2 \sim N^d\rho^3 a^2$, which is also the result obtained in Ref. [10]. This contribution is clearly sub-leading in the low density limit with respect to that obtained by Galitskii's perturbation theory, which is proportional to $N^d\rho^2/\log \rho a^2$. The same happens to the self energy corrections. In this latter case the contribution is also of higher order in $\epsilon - \epsilon_f$. For example, according to Ref. [10], the correction to the $\mathrm{Im}\Sigma$ coming directly from these states is proportional to $(\epsilon - \epsilon_f)^{5/2}$, while the one obtained by using Galitskii's theory to $(\epsilon - \epsilon_f)^2 \log|\epsilon - \epsilon_f|$, i.e. the latter is the leading one. These results confirms what we have found in the few particle problem, namely we recover the same contributions obtained by Galitskii's perturbation theory as the leading terms in a low density expansion of the $2d$ Hubbard model.

Conclusions

The nature of a two dimensional interacting electron gas, has became a problem of current interest after the interesting suggestion that the normal state of the HTc superconductors can not be described as a Landau Fermi liquid, and that such a type of behaviour might be intrinsic to the $2d$ Hubbard model, much as it does in $1d$.

In this thesis we have studied the few body problem on a $2d$ and $3d$ lattice. The model we have used, is the Hubbard model, i.e. a tight-binding model plus an on site repulsion, which, according to Anderson, is believed to be relevant for the basic understanding of these new superconductors and enough to detect the non-Landau behaviour. In particular we have performed an approximate few body calculation in the limit $1/N \to 0$, where N^d is the number of sites of the lattice. The approximation consists in solving the Schröedinger equation using an ansatz, to be verified *a posteriori*. The ansatz is that the correction to the $U = 0$ ground state energy goes to zero faster than the energy level spacing. The method can be applied to any number of electrons N_e, provided the density $\rho = N_e/N^d$ goes to zero for $N \to \infty$. In fact this is the necessary condition for the ansatz to be valid. This implies that, *a priori*, the results obtained have no physical relevance in the thermodynamic limit, corresponding to a $\rho = 0$ case. Nevertheless if we take a very peculiar thermodynamic limit on the results, which consists in interpreting as a finite density the number of electrons divided by the volume (let us note that this is not the usual thermodynamic limit, because the results have been obtained keeping the number of electron fixed and sending the volume to infinity), we obtain intensive quantities, which may be of physical relevance. The interesting thing is

that the results, obtained in $d \geq 2$, coincide with those obtained in the dilute limit by Galitskii's continuum perturbation theory, both in $3d$[9] and in $2d$[8], and both for the total energy and the momentum distribution.

The method suggests a way to generalize a few body calculation to a finite density one.

With respect to the nature of the ground state, our calculation confirms the result of the Galitskii's perturbative approach, according to which the $2d$ interacting fermion system is a Landau Fermi liquid. Moreover our results, if our approximation is correct, are the leading terms of an expansion in the inverse of the volume of a model which contains anti-bound states. This implies that the effects of such anti-bound states, whose role might have been crucial in destroying the Landau picture (see Ref. [1]), are indeed sub-leading, and do not appear to actually break the Landau Fermi liquid theory.

The present work strictly applies to the dilute corner of the phase diagram, i.e. zero, or weakly zero electron density. The possibility of non-Fermi liquid behaviour remains very well open in the rest of the phase diagram, particularly in the neighborhood of half-filling, where other approaches must be applied.

Appendix I

Four electrons in Hubbard model

The four body problem is again a very simple one and can be used to clarify the method introduced before. In this appendix we describe how it can be approximately solved in $2d$ (it can be simply generalized in $3d$), by using the method described in chapter 1.. Then we show the results of an analogous calculation for 6 and 8 electrons.

A generic wave function of four electrons is

$$|\Psi\rangle = \sum L(\mathrm{k}_1, \mathrm{k}_2, \mathrm{k}_3, \mathrm{k}_4) c^\dagger_{\mathrm{k}_1\uparrow} c^\dagger_{\mathrm{k}_2\downarrow} c^\dagger_{\mathrm{k}_3\uparrow} c^\dagger_{\mathrm{k}_4\downarrow} |0\rangle$$

where $L(\mathrm{k}_1, \mathrm{k}_2, \mathrm{k}_3, \mathrm{k}_4) = -L(\mathrm{k}_3, \mathrm{k}_2, \mathrm{k}_1, \mathrm{k}_4) = -L(\mathrm{k}_1, \mathrm{k}_4, \mathrm{k}_3, \mathrm{k}_2)$. The eigenvalue equation for the coefficients is

$$\begin{aligned}
&(E - \epsilon_{\mathrm{k}_1} - \epsilon_{\mathrm{k}_2} - \epsilon_{\mathrm{k}_3} - \epsilon_{\mathrm{k}_4}) L(\mathrm{k}_1, \mathrm{k}_2, \mathrm{k}_3, \mathrm{k}_4) = \\
&= \frac{U}{N^d} \sum_{\mathrm{q}} (L(\mathrm{k}_1 + \mathrm{q}, \mathrm{k}_2 - \mathrm{q}, \mathrm{k}_3, \mathrm{k}_4) + L(\mathrm{k}_1 + \mathrm{q}, \mathrm{k}_2, \mathrm{k}_3, \mathrm{k}_4 - \mathrm{q}) + \\
&+ L(\mathrm{k}_1, \mathrm{k}_2 - \mathrm{q}, \mathrm{k}_3 + q, \mathrm{k}_4) + L(\mathrm{k}_1, \mathrm{k}_2, \mathrm{k}_3 + \mathrm{q}, \mathrm{k}_4 - \mathrm{q}))
\end{aligned}$$

If we define the function

$$J(\mathrm{k}_1, \mathrm{k}_2) = \frac{U}{N^d} \sum_{\mathrm{q}} L(\mathrm{q}, \mathrm{P} - \mathrm{q}, \mathrm{k}_1, \mathrm{k}_2)$$

then the coefficients L's are

$$L(\mathrm{k}_1, \mathrm{k}_2, \mathrm{k}_3, \mathrm{k}_4) = \frac{J(\mathrm{k}_3, \mathrm{k}_4) - J(\mathrm{k}_3, \mathrm{k}_2) - J(\mathrm{k}_1, \mathrm{k}_4) + J(\mathrm{k}_1, \mathrm{k}_2)}{E - \epsilon_{\mathrm{k}_1} - \epsilon_{\mathrm{k}_2} - \epsilon_{\mathrm{k}_3} - \epsilon_{\mathrm{k}_4}}$$

The self consistent set of equations the J's have to satisfy is

$$J(\mathbf{k}_1,\mathbf{k}_2) = J(\mathbf{k}_1,\mathbf{k}_2)\frac{U}{N^d}\sum_{\mathbf{q}}\frac{1}{E-\epsilon_{\mathbf{k}_1}-\epsilon_{\mathbf{k}_2}-\epsilon_{\mathbf{q}}-\epsilon_{\mathbf{P}-\mathbf{k}_1-\mathbf{k}_2-\mathbf{q}}}+$$
$$-\frac{U}{N^d}\sum_{\mathbf{q}}\frac{J(\mathbf{k}_1,\mathbf{q})+J(\mathbf{q},\mathbf{k}_2)}{E-\epsilon_{\mathbf{k}_1}-\epsilon_{\mathbf{k}_2}-\epsilon_{\mathbf{q}}-\epsilon_{\mathbf{P}-\mathbf{k}_1-\mathbf{k}_2-\mathbf{q}}}+ \qquad (A.1)$$
$$+\frac{U}{N^d}\sum_{\mathbf{q}}\frac{J(\mathbf{q},\mathbf{P}-\mathbf{q}-\mathbf{k}_1-\mathbf{k}_2)}{E-\epsilon_{\mathbf{k}_1}-\epsilon_{\mathbf{k}_2}-\epsilon_{\mathbf{q}}-\epsilon_{\mathbf{P}-\mathbf{k}_1-\mathbf{k}_2-\mathbf{q}}}$$

Let us define the functions

$$A(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4) = \frac{U}{N^d}\frac{1}{E-\epsilon_{\mathbf{k}_1}-\epsilon_{\mathbf{k}_2}-\epsilon_{\mathbf{k}_3}-\epsilon_{\mathbf{k}_4}}$$

$$I(\mathbf{k}_1,\mathbf{k}_2) = -\frac{U}{N^d}\sum_{\mathbf{q}}\frac{1}{E-\epsilon_{\mathbf{k}_1}-\epsilon_{\mathbf{k}_2}-\epsilon_{\mathbf{q}}-\epsilon_{\mathbf{P}-\mathbf{k}_1-\mathbf{k}_2-\mathbf{q}}} \simeq \frac{Ua^d m}{f_0\hbar^2}$$

and the wave vectors (for simplicity we work in $2d$)

$$\mathbf{0}=(0,0)\quad \mathbf{1}=(0,\frac{2\pi}{Na})\quad \mathbf{2}=(0,-\frac{2\pi}{Na})\quad \mathbf{3}=(\frac{2\pi}{Na},0)\quad \mathbf{4}=(-\frac{2\pi}{Na},0)$$

The unperturbed Fermi sea for total momentum zero is fourfold degenerate. If we assume the condition (1.4) to be valid , then the value of A when the arguments belong to one of the possible Fermi surfaces is more singular than in the other cases, i.e.

$$A(\mathbf{0},\mathbf{0},\mathbf{1},\mathbf{2}) = A(\mathbf{0},\mathbf{0},\mathbf{2},\mathbf{1}) = A(\mathbf{0},\mathbf{0},\mathbf{3},\mathbf{4}) = A(\mathbf{0},\mathbf{0},\mathbf{4},\mathbf{3}) = \cdots = \frac{U}{\alpha(N)}$$

To leading order, just the J's whose arguments belong to the degenerate Fermi surfaces are not zero . In this approximation we can write explicitly the system

eq. (A.1)

$$
\begin{aligned}
&J(0,0) = \\
&= J(0,0)(-I + A(0,0,1,2) + A(0,0,2,1) + A(0,0,3,4) + A(0,0,4,3))+ \\
&- A(0,0,1,2)(J(0,2) + J(1,2)) - A(0,0,2,1)(J(0,1) + J(2,0))+ \\
&+ A(0,0,1,2)J(1,2) + A(0,0,2,1)J(2,1) + A(0,0,3,4)J(3,4)+ \\
&+ A(0,0,4,3)J(4,3)
\end{aligned}
$$

$$
\begin{aligned}
&J(0,1) = \\
&= J(0,1)(-I + A(0,1,0,2) + A(0,1,2,0)) - A(0,1,0,2)(J(0,2) + J(0,1)) \\
&- A(0,1,2,0)(J(0,0) + J(2,1)) + A(0,1,2,0)J(2,0) + A(0,1,0,2)J(0,2)
\end{aligned}
$$

$$
\begin{aligned}
&J(1,2) = \\
&= J(1,2)(-I + A(1,2,0,0)) - A(1,2,0,0)(J(1,0) + J(0,2))+ \\
&+ A(1,2,0,0)J(0,0)
\end{aligned}
$$

All the other equations can be easily obtained by these ones. As both A and I are logarithmically divergent we can neglect the terms on the left hand sides. Then we can divide the equations by A, so that the resulting system looks like an eigenvalue equation, where the eigenvalue is

$$\lambda = \frac{I}{A} \simeq \frac{a^2 m \alpha(N)}{\hbar^2 f_0(k_f)}$$

and the matrix $\hat{M}$ is

$$\begin{pmatrix}
4 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 \\
-1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & -1 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1
\end{pmatrix}$$

The first row(column) corresponds to $J(\mathbf{0},\mathbf{0})$, while the others in order to $J(\mathbf{0},\mathbf{1})$, $J(\mathbf{2},\mathbf{1})$, $J(\mathbf{2},\mathbf{0})$, $J(\mathbf{0},\mathbf{2})$, $J(\mathbf{1},\mathbf{2})$, $J(\mathbf{1},\mathbf{0})$, $J(\mathbf{0},\mathbf{3})$, $J(\mathbf{4},\mathbf{3})$, $J(\mathbf{4},\mathbf{0})$, $J(\mathbf{0},\mathbf{4})$, $J(\mathbf{3},\mathbf{4})$ and $J(\mathbf{3},\mathbf{0})$. One can immediately realize that the matrix has the form we predict in sec. 1.3, where in the present case the state $(\mathbf{0},\mathbf{0})$ belongs to all the degenerate Fermi surfaces. The matrix can be easily diagonalised to find two eigenvalues, i.e $\lambda = 3$ (three fold degenerate), and $\lambda = 7$. For $\lambda = 3$ the eigenvectors are

$$\vec{J_1} = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \vec{J_2} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \quad \vec{J_3} = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

$\vec{J_1}$ and $\vec{J_2}$ correspond to triplet states, while $\vec{J_3}$ is a singlet. The eigenvector corresponding to $\lambda = 7$ is

$$\vec{J_4} = \begin{pmatrix} 4 \\ -1 \\ 1 \\ -1 \\ -1 \\ 1 \\ -1 \\ -1 \\ 1 \\ -1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

and it is a singlet. The result shows that the interaction removes partially the degeneracy and this makes λ for the ground state lower than the predicted value for a non degenerate Fermi surface, i.e. $\lambda = (N_e/2)^2 = 4$. By using the method to build up the matrix which has been described in section 1.3, it is easy to solve even the 6 and 8 electron problem. We do not describe these calculations in detail, we just show the results.

For 6 electrons the $U = 0$ ground state is sixfold degenerate, having 2 state of total spin 1, 3 of spin 0 and 1 of spin 2. The approximation we have used to find the perturbed ground state shows that

1) the ground state is twofold degenerate and has $\lambda = 7$ and spin 0 or 2;

2) the first excited state is threefold degenerate and has $\lambda = 9$. There is one state of spin 0 and two of spin 1;

3) the second excited state has $\lambda = 13$ and spin 0 .

For 8 electrons the $U = 0$ ground state is fourfold degenerate, in the same way as the 4 electrons case, and the degeneracy is removed. In this case we have

1) the ground state has $\lambda = 15$ and it is threefold degenerate. One state has

spin 0, two have spin 1;

2) the other state, which is a singlet, has $\lambda = 19$.

It is interesting to note that, according to this approximation, the ground state has spin 0 or maximum (in $1d$ there is only a possible non degenerate Fermi surface, i.e. a fourfold degenerate state, whose spin is 1, i.e. the maximum allowed).

Appendix II

Low energy scattering in two dimensions

In this appendix we calculate the low energy scattering amplitude in two dimensions*. This calculation is needed to apply Galitskii's theory.
Let us consider two interacting particles in two dimensions. The Schröedinger's equation in the relative coordinates is

$$(k^2 + \nabla^2)\Psi_{\mathbf{k}}(\mathbf{r}) = V(\mathbf{r})\Psi_{\mathbf{k}}(\mathbf{r}) \qquad (A.1)$$

where the energy $E = k^2$. Let us suppose that the range of the potential R is such that $kR \ll 1$. The most general solution of (A.1) is

$$\Psi_{\mathbf{k}}(\mathbf{r}) = \sum_l \mathrm{e}^{\mathrm{i}l\theta} R_l(kr)$$

where $r = \sqrt{x^2 + y^2}$. When $R \ll r \ll 1/k$ the equation R_l has to satisfy becomes

$$R_l^{''} + R_l^{'}/r - l^2 R_l/r^2 = 0$$

whose solution is

$$\begin{aligned} R_l &= a_l r^l + b_l r^{-l} \qquad \text{for } l \neq 0 \\ R_0 &= b\log(r/a) \end{aligned} \qquad (A.2)$$

* The calculations are just the generalization in two dimensions of the chapter 17 of Landau and Liftshitz's "Quantum mechanics, not relativistic theory". The results we derive have been already obtained by several authors (see Ref. [22], Ref. [23] or Ref. [8] for example)

For $r \gg 1/k$ the solution of (A.1) is

$$R_l(kr) = c_l J_l(kr) + d_l N_l(kr) \tag{A.3}$$

where J_l and N_l are respectively the Bessel's and the Neumann's functions of order n**. The functions (A.2) and (A.3) match in the intermediate region if

$$c_l \left(\frac{kr}{2}\right)^l \frac{1}{l!} - d_l \left(\frac{kr}{2}\right)^{-l} \frac{(l-1)!}{\pi} = a_l r^l + b_l r^{-l}$$

which implies for $l \neq 0$

$$\begin{aligned} c_l &= a_l l! \left(\frac{k}{2}\right)^{-l} \\ d_l &= -b_l \left(\frac{k}{2}\right)^l \frac{\pi}{(l-1)!} \end{aligned} \tag{A.4a}$$

and for $l = 0$

$$\begin{aligned} d_0 &= \frac{\pi b}{2} \\ c_0 &= -b \log ka \end{aligned} \tag{A.4b}$$

When $kr \gg 1$ the asymptotic expression of the wave function, by using (A.3) is

$$\Psi_{\mathbf{k}}(\mathbf{r}) \simeq \sqrt{\frac{1}{2\pi kr}} \sum_l \mathrm{e}^{\mathrm{i}l\theta} N_l \left(\mathrm{e}^{\mathrm{i}(kr-\pi/4-\pi l/2+\delta_l)} + \mathrm{e}^{i(kr-\pi/4-\pi l/2+\delta_l)}\right) \tag{A.5a}$$

where

$$\tan \delta_l = -d_l/c_l \tag{A.5b}$$

We know that at large distances the wave function has the usual form of a scattered one, i.e.

$$\Psi_{\mathbf{k}}(\mathbf{r}) \simeq \mathrm{e}^{\mathrm{i}\mathbf{kr}} + \sqrt{\frac{2}{\pi kr}} f(\theta) \mathrm{e}^{\mathrm{i}(\mathbf{kr}-\pi/4)} \tag{A.6}$$

** All the properties of the Bessel's, Neumann's and Henkel's functions we use in this appendix are taken by " Table of integral, series and product" by I. S. Gradshteyn/I. M. Ryzhik, Academic Press.

For $kr \gg 1$ however

$$e^{ikr\cos\theta} = \frac{1}{2}(H_0^{(1)}(kr) + H_0^{(2)}(kr)) + \sum_{l\neq 0}(\mathrm{i})^l \cos l\theta (H_l^{(1)}(kr) + H_l^{(2)}(kr)) \simeq$$

$$\simeq \sqrt{\frac{2}{\pi kr}} e^{\mathrm{i}(kr-\pi/4)} \left(\frac{1}{2} + \sum_{l\neq 0}(\mathrm{i})^l \cos l\theta e^{-\mathrm{i}\pi l/2}\right) +$$

$$+ \sqrt{\frac{2}{\pi kr}} e^{-\mathrm{i}(kr-\pi/4)} \left(\frac{1}{2} + \sum_{l\neq 0}(\mathrm{i})^l \cos l\theta e^{\mathrm{i}\pi l/2}\right)$$

where the $H_0^{(1)}$ and $H_0^{(2)}$ are the Henkel's functions of first and second kind. Comparing (A.6) with (A.5) we get at last

$$N_l = (\mathrm{i})^l e^{\mathrm{i}\delta_l}$$
$$f(\theta) = \frac{1}{2}\sum_l e^{\mathrm{i}l\delta_l}(e^{2\mathrm{i}\delta_l} - 1) \qquad (A.7)$$

When $k \ll 1$ then $\tan\delta_l \simeq \delta_l$ that is

$$\delta_0 \simeq \frac{\pi}{2\log ka}$$
$$\delta_l = \frac{b_l}{a_l}\pi\left(\frac{k}{2}\right)^{2l}\frac{1}{l!(l-1)!}$$

In this limit

$$f(\theta) \simeq \mathrm{i}\delta_0 \simeq \frac{\mathrm{i}\pi}{2\log ka} \qquad (A.8)$$

Now we find the relation between $f(\theta)$ and the scattering amplitude $f(\mathrm{q},\mathrm{k})$. This function is defined through the relations

$$\Psi_{\mathbf{k}}(\mathbf{r}) = e^{\mathrm{i}\mathbf{kr}} + \int \frac{d^2q}{(2\pi)^2} f(\mathrm{q},\mathrm{k}) \frac{e^{\mathrm{i}\mathbf{qr}}}{k^2 - q^2 + \mathrm{i}\eta} = e^{\mathrm{i}\mathbf{kr}} + \Phi_k(\mathbf{r})$$

$$f(\mathrm{q},\mathrm{k}) = \int d\mathbf{r} e^{-\mathrm{i}\mathbf{qr}} V(\mathbf{r})\Psi_{\mathbf{k}}(\mathbf{r})$$

The function Φ_k is

$$\int \frac{qdq}{(2\pi)^2}\frac{1}{k^2 - q^2 + \mathrm{i}\eta}\int_0^{2\pi} d\theta e^{\mathrm{i}qr\cos\theta} f(q\hat{\mathrm{n}},\mathrm{k}) \qquad (A.9)$$

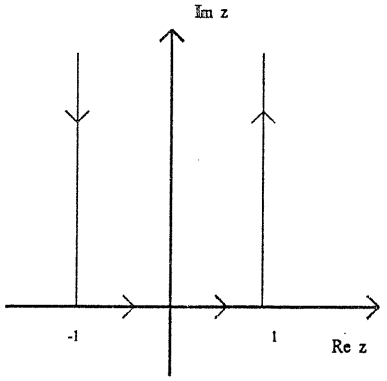

Fig. A.1: Contour in the complex plane.

the second integral can be rewritten as

$$\int_{-1}^{1} \frac{dz}{\sqrt{1-z^2}} \mathrm{e}^{\mathrm{i}qrz} (f(q\hat{\mathbf{n}}, \mathbf{k}) + f(q\hat{\mathbf{n}}', \mathbf{k}))$$

where $\hat{\mathbf{n}} = (z, \sqrt{1-z^2})$ and $\hat{\mathbf{n}}' = (z, -\sqrt{1-z^2})$. We calculate this integral going to the complex plane.

$$\oint \frac{dz}{\sqrt{1-z^2}} \mathrm{e}^{\mathrm{i}qrz} (f(q\hat{\mathbf{n}}, \mathbf{k}) + f(q\hat{\mathbf{n}}', \mathbf{k}))$$

where the contour is drawn in fig. A.1

This implies that the integral is

$$\begin{aligned} &- 2\mathrm{i} \int_0^{\infty} \frac{dx}{\sqrt{x^2 - 2\mathrm{i}x}} f(q\hat{\mathbf{r}}, \mathbf{k}) \mathrm{e}^{\mathrm{i}qr - xqr} + \\ &- 2\mathrm{i} \int_{-\infty}^{0} \frac{dx}{\sqrt{x^2 + 2\mathrm{i}x}} f(-q\hat{\mathbf{r}}, \mathbf{k}) \mathrm{e}^{-\mathrm{i}qr - xqr} \end{aligned} \tag{A.10}$$

in the hypothesis $f(q\hat{\mathbf{n}}, \mathbf{k})$ does not vary much when the imaginary part of $\hat{\mathbf{n}}$ varies on a scale $1/qr$ (this is asymptotically true as $r \to \infty$, that is as $\mathrm{e}^{-xqr} \to 0$). Then one realizes that (A.10) is

$$\pi H_0^{(1)}(qr) f(q\hat{\mathbf{r}}, \mathbf{k}) + \pi H_0^{(2)}(qr) f(-q\hat{\mathbf{r}}, \mathbf{k})$$

This means that (A.9) in the limit $r \gg 1$ is

$$\frac{1}{4\pi}\int_0^\infty \frac{qdq}{k^2-q^2+\mathrm{i}\eta}\left(H_0^{(1)}(qr)f(q\hat{\mathbf{r}},\mathbf{k})+H_0^{(2)}(qr)f(-q\hat{\mathbf{r}},\mathbf{k})\right) \qquad (A.11)$$

The Henkel's functions satisfy the relation $H_0^{(2)}(\mathrm{e}^{-\mathrm{i}\pi}z) = -H_0^{(1)}(z)$, so that (A.11) becomes ***

$$\int_{-\infty}^\infty \frac{qdq}{4\pi}\frac{H_0^{(1)}(qr)f(q\hat{\mathbf{r}},\mathbf{k})}{k^2-q^2+\mathrm{i}\eta} = -\frac{\mathrm{i}}{4}H_0^{(1)}(kr)f(k\hat{\mathbf{r}},\mathbf{k})$$

This implies that for $rk \gg 1$ the wave function is

$$\Psi_{\mathbf{k}}(\mathbf{r}) \simeq \mathrm{e}^{\mathrm{i}\mathbf{k}\mathbf{r}} - \frac{\mathrm{i}}{4}\sqrt{\frac{2}{\pi kr}}f(k\hat{\mathbf{r}},\mathbf{k})\mathrm{e}^{\mathrm{i}(kr-\pi/4)}$$

and that the scattering amplitude is

$$f(k\hat{\mathbf{n}},k\hat{\mathbf{n}}') = -\frac{4}{\mathrm{i}}f(\theta) \simeq -\frac{2\pi}{\log ka} \qquad (A.12)$$

the last equality holding for small k.

Indeed the most general wave function is a superposition of $\Psi_{\mathbf{k}}$ of different k-directions, i.e.

$$\Psi(r\hat{\mathbf{n}}) \simeq \int_0^{2\pi} d\theta F(\hat{\mathbf{n}}')\left(\mathrm{e}^{\mathrm{i}kr\hat{\mathbf{n}}\hat{\mathbf{n}}'} - \frac{\mathrm{i}}{4}\sqrt{\frac{2}{\pi kr}}f(k\hat{\mathbf{n}},k\hat{\mathbf{n}}')\mathrm{e}^{\mathrm{i}(kr-\pi/4)}\right)$$

the first integral can be calculated going to the complex plane as we have shown before. The result is

$$\begin{aligned}
\Psi(r\hat{\mathbf{n}}) &\simeq \pi H_0^{(1)}(kr)F(\hat{\mathbf{n}}) + \pi H_0^{(2)}(kr)F(-\hat{\mathbf{n}}) + \\
&\qquad - \frac{\mathrm{i}}{4}\sqrt{\frac{2}{\pi kr}}\mathrm{e}^{\mathrm{i}(kr-\pi/4)}\int_0^{2\pi} d\theta F(\hat{\mathbf{n}}')f(k\hat{\mathbf{n}},k\hat{\mathbf{n}}') \simeq \\
&\simeq \sqrt{\frac{2}{\pi kr}}\left(\pi\mathrm{e}^{-\mathrm{i}(kr-\pi/4)}F(-\hat{\mathbf{n}}) + \pi\mathrm{e}^{\mathrm{i}(kr-\pi/4)}F(\hat{\mathbf{n}}) + \right. \\
&\qquad \left. - \frac{\mathrm{i}}{4}\mathrm{e}^{\mathrm{i}(kr-\pi/4)}\int_0^{2\pi} d\theta F(\hat{\mathbf{n}}')f(k\hat{\mathbf{n}},k\hat{\mathbf{n}}')\right) \simeq \\
&\simeq \sqrt{\frac{2}{\pi kr}}\left(\pi\mathrm{e}^{-\mathrm{i}(kr-\pi/4)}F(-\hat{\mathbf{n}}) + \pi\hat{S}\cdot F\mathrm{e}^{-\mathrm{i}(kr-\pi/4)}\right)
\end{aligned}$$

*** To calculate the integral we use the formulas 6.568 2. pag. 690, 6.532 4. and 6.534 pag. 678 of I. S. Gradshteyn/I. M. Ryzhik's book.

where

$$\hat{S} = \hat{I} - \frac{\mathrm{i}}{4\pi}\hat{f}(k\hat{\mathbf{n}}, k\hat{\mathbf{n}}')$$

is the scattering matrix. As it has to be unitary, i.e. $\hat{S} \cdot \hat{S}^{\dagger} = \hat{I}$, this means

$$-\frac{\mathrm{i}}{4\pi}\hat{f} + \frac{\mathrm{i}}{4\pi}\hat{f}^{\dagger} + \frac{1}{16\pi^2}\hat{f} \cdot \hat{f}^{\dagger} = 0$$

that is

$$\mathrm{Im}\hat{f} = -\frac{\mathrm{i}}{8\pi}\hat{f} \cdot \hat{f}^{\dagger} \tag{A.13}$$

which is the optical theorem in two dimension. In particular when $k \to 0$, by using (A.12) and (A.13) we get

$$\mathrm{Im}f \simeq -\frac{\pi}{2\log^2 ka} \tag{A.14}$$

We want now to discuss the behaviour of the constant a as a function of the range and of the amplitude of the potential.Let us suppose for simplicity that the potential is just $V(r) = V_0$ for $r < R$ and zero otherwise. The R_l function for $r > R$ is

$$R_l(kr) = A\left(\cot\delta_l J_l(kr) - N_l(kr)\right)$$

while for $r < R$

$$R_l(k'r) = BJ_l(ik'r)$$

where the J_l's and the N_l's are the Bessel and Neumann functions, $k = \sqrt{2E}$ and $k' = \sqrt{2(V_0 - E)}$. For $r = R$ we impose that the logarithmic derivative of the wave function is continuous, i.e.

$$k'R\frac{J_l'(\mathrm{i}k'R)}{J_l(\mathrm{i}k'R)} = \beta_l = kR\frac{\cot\delta_l J_l'(kR) - N_l'(kR)}{\cot\delta_l J_l(kR) - N_l(kR)}$$

Let us suppose that $x = Rk \ll 1$ and $V_0 R^2$ =constant. Then for $l = 0$

$$\begin{aligned}\cot \delta_0 &= \frac{xN_0'(x) - \beta_0 N_0(x)}{xJ_0'(x) - \beta_0 J_0(x)} = \\ &= \frac{-xN_1(x) - \beta_0 N_0(x)}{xJ_1(x) - \beta_0 J_0(x)} \simeq \\ &\simeq -\frac{2}{\pi}\left(\frac{1 - \beta_0(\log x/2 + \gamma)}{\beta_0}\right) + O(kR) = \frac{2}{\pi}\log ka + O(kR)\end{aligned}$$

where γ is the Euler's constant and

$$a = \frac{R}{2}\mathrm{e}^{\gamma - \frac{1}{\beta}}$$

When $V_0 R^2 \gg 1$ from the asymptotic expression of the imaginary argument Bessel's functions it is found that $\beta \simeq V_0 R^2$, so that

$$a = \frac{R}{2}\mathrm{e}^{\gamma}\mathrm{e}^{-\frac{1}{V_0 R^2}} \qquad (A.15)$$

The constant a is therefore non analytical in the potential.

Appendix III

Phase shifts and physical properties

In this appendix we show some useful relations between what it is defined a *phase shift* in a many body problem and some physical properties. First of all let us consider the expression of the vertex in the ladder approximation (fig. 2.1 in chapter 2). In the following we use the results obtained by Ref. [10]. If the system is defined on a lattice it is possible to sum exactly all the ladder diagrams, and the result is

$$\Gamma(\frac{P}{2}+q,\frac{P}{2}-q;\frac{P}{2}+q',\frac{P}{2}-q') = \Gamma(\mathbf{P},\Omega) = \frac{U}{1-U\chi(\mathbf{P},\Omega)} \qquad (A.1)$$

The function χ is

$$\begin{aligned}\chi(\mathbf{P},\Omega) &= -\frac{\mathrm{i}}{V}\sum_{\mathbf{q}}\int\frac{d\epsilon}{2\pi}G(\frac{P}{2}-\mathbf{q},\frac{\Omega}{2}-\epsilon)G(\frac{P}{2}+\mathbf{q},\frac{\Omega}{2}+\epsilon) = \\ &= \frac{1}{V}\sum_{\mathbf{q}}\frac{N(\mathbf{P},\mathbf{q})}{\Omega-\epsilon_{\frac{\mathbf{P}}{2}-\mathbf{q}}-\epsilon_{\frac{\mathbf{P}}{2}+\mathbf{q}}+\mathrm{i}\eta N(\mathbf{P},\mathbf{q})}\end{aligned} \qquad (A.2)$$

where $N(\mathbf{P},\mathbf{q}) = 1-n_{\frac{\mathbf{P}}{2}+\mathbf{q}}-n_{\frac{\mathbf{P}}{2}-\mathbf{q}}$ and the energy $\epsilon_{\mathbf{k}}$, are referred to the Fermi's energy. This is the expression at zero temperature. At finite T, in the Matsubara's tecnique, there is no imaginary term in the denominator, Ω is an imaginary Bose frequency, and the $n_{\mathbf{p}}$'s are Fermi functions. Let us sum and subtract from the denominator the term

$$\frac{U}{V}\sum_{\mathbf{q}}\frac{1}{\epsilon_{\frac{\mathbf{P}}{2}-\mathbf{p}'}+\epsilon_{\frac{\mathbf{P}}{2}+\mathbf{p}'}-\epsilon_{\frac{\mathbf{P}}{2}-\mathbf{q}}-\epsilon_{\frac{\mathbf{P}}{2}+\mathbf{q}}+\mathrm{i}\eta} = U\chi_0(\mathbf{p}')$$

It is easy to realize that $1/U - \chi_0(\mathbf{p}')$ is nothing but $f^{-1}(\mathbf{p}', \mathbf{p})$ (in the Hubbard's case there is no dependence from the incoming momentum). Then

$$\Gamma^{-1}(\mathbf{P}, \Omega) = f^{-1}(\mathbf{p}') + \chi_0(\mathbf{p}') - \chi(\mathbf{P}, \Omega) \qquad (A.3)$$

Now we are able to take the continuum limit. As in the limit of zero lattice spacing the inverse scattering amplitude diverges, we have to regularize the potential, i.e. the δ-function, introducing an ultraviolet cutoff ($\sim 1/a$), which we interpret as the inverse range of the potential. However the term $\chi_0 - \chi$ has no problem in the ultraviolet limit, so that the interaction inside remains a δ-function. The only quantity which needs a regularization is the scattering amplitude, and we take for it the expression given in appendix II. This is the essence of the Galitskii's method. From (A.3) we are able to study the analytical properties of the vertex. Following Ref. [10] it is easy to realize that the vertex has a branch cut which lies in the upper half-plane for $\Omega < 0$, and in the lower in the opposite. The cut starts at $\Omega = 2\epsilon_{\frac{\mathbf{P}}{2}}$ towards $\Omega = \infty$. When $2\epsilon_{\frac{\mathbf{P}}{2}} = \omega^*_{\mathbf{P}} < 0$ there is also an isolated pole for $\Omega < \omega^*_{\mathbf{P}}$ which corresponds to an anti-bound state of two holes. In fact, even if we have eliminated the electron anti-bound states by taking the continuum limit, as soon as the density is finite in the thermodynamic limit, there appear anti-bound states for the excitations of two holes, whose band is indeed bounded, in contrast to the electron one. These excitations have an energy[10] $\omega_b(\mathbf{P}) = -\omega^*_{\mathbf{P}}(1 - 2\omega^*_{\mathbf{P}} a^2)$, where a is the scattering length (cfr. eq. A.15 in appendix II), which is not analytical in U, as usually happens for this kind of excitations. Moreover these poles exist only in $d \leq 2$, because the effective interaction ($\propto \omega_b(\mathbf{P})$) is too weak to give the pole in $3d$. These poles do not lead to an instability of the system.

Now we show the relation between the vertex and the correction to the thermodynamic potential. What we are going to say follows Ref. [10] and a paper by

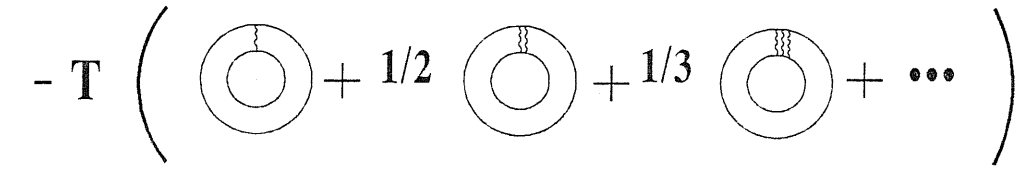

Fig. A.1: Diagrammatic expansion for $\langle \mathcal{S} \rangle_{\text{conn}} - 1$

Nozieres et *al.*[24]. Let us consider the correction $\Delta\Omega$ to the thermodynamic potential in the ladder approximation, by using the perturbation theory at finite temperature T[11]

$$\Delta\Omega = -T(\langle \mathcal{S} \rangle_{\text{conn}} - 1)$$

where $\langle \mathcal{S} \rangle_{\text{conn}} - 1$ is shown in fig. A.1.

It is very simple to realize that, by using eq. A.2

$$\langle \mathcal{S} \rangle_{\text{conn}} = \langle 1 + \chi + \frac{\chi^2}{2} + \frac{\chi^3}{3} + \cdots \rangle = \langle 1 - \log(1 - \chi) \rangle$$

where we are considering the function χ at finite temperature. The final result is

$$\Delta\Omega = T \sum_n \sum_{\mathbf{P}} \log(1 - \chi(\mathbf{P}, \mathrm{i}\Omega_n))$$

If we sum on the Matsubara frequencies by going in the complex Ω's plane and then we note that χ has a branch cut on the real axis, we find

$$\Delta\Omega = -\frac{1}{\pi} \sum_{\mathbf{P}} \int_{-\infty}^{\infty} d\Omega g(\Omega) \delta(\mathbf{P}, \Omega) \qquad (A.4)$$

where $g(\Omega)$ is the Bose function and

$$\delta(\mathbf{P}, \Omega) = -\mathrm{Arg}(1 - \chi(\mathbf{P}, \Omega + \mathrm{i}\eta) \qquad (A.5)$$

(it is important to note that $\delta(\mathbf{P}, 0) = 0$, so that there are no problems at $\Omega = 0$). The eq. (A.5) is the definition of the *phase shift* in a system of interacting electrons.

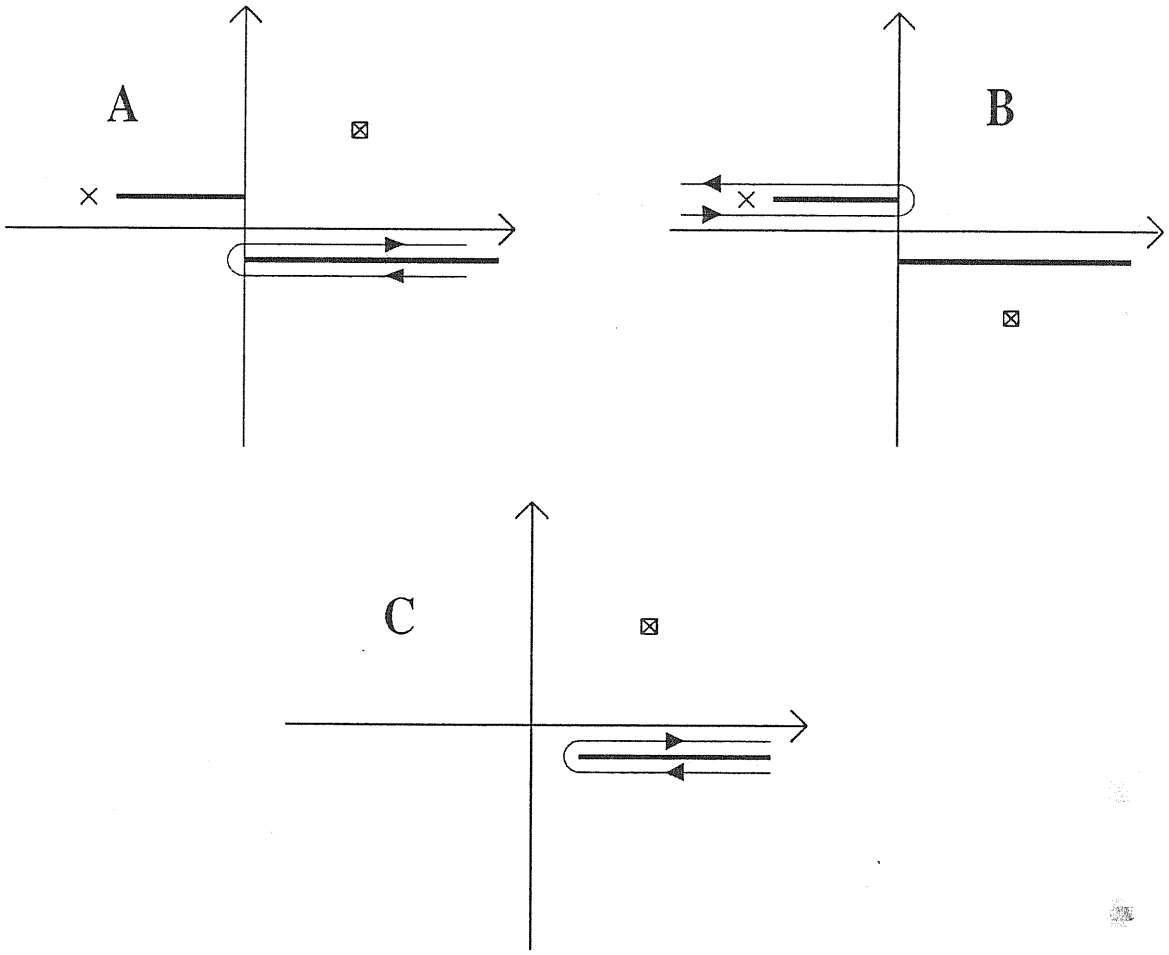

Fig. A.2: Integration contours for the self energy

Let us show now the relation between the phase shifts and the self energy. In the ladder approximation (cfr. fig. 2.5) the self energy is

$$\Sigma(\mathrm{k},\epsilon) = -\frac{\mathrm{i}}{V}\int_{-\infty}^{\infty}\frac{d\Omega}{2\pi}\sum_{\mathbf{P}} G(\mathbf{P}-\mathrm{k},\Omega-\epsilon)\Gamma(\mathbf{P},\Omega) \qquad (A.6)$$

The integral on the frequencies can be done by going in the complex plane. The contour we use depends on the analytical properties of the product $G(\mathbf{P}-\mathrm{k},\Omega-$

$\epsilon)\Gamma(\mathbf{P},\Omega)$. The different cases are drawn in fig. A.2. The contour A correspond to the case $2\epsilon_{\frac{\mathbf{P}}{2}} < 0$ and $|\mathbf{P}-\mathbf{k}| < k_f$, the B to the case $2\epsilon_{\frac{\mathbf{P}}{2}} < 0$ and $|\mathbf{P}-\mathbf{k}| > k_f$ and the C to $2\epsilon_{\frac{\mathbf{P}}{2}} > 0$ and $|\mathbf{P}-\mathbf{k}| < k_f$. The cross in the square is the pole of the Green's function, while the one without square is the isolated pole of the vertex (which exists only for $2\epsilon_{\frac{\mathbf{P}}{2}} < 0$). The bold line represents the branch cut of the vertex. The result is

$$\begin{aligned}\Sigma(\mathbf{k},\epsilon) = &\frac{1}{V\pi}\sum_{\epsilon_{\frac{\mathbf{P}}{2}}<0}^{\mathbf{P}}\theta(\epsilon_{\mathbf{P}-\mathbf{k}})\int_0^\infty d\Omega G(\mathbf{P}-\mathbf{k},\Omega-\epsilon)\mathrm{Im}\Gamma(\mathbf{P},\Omega)+\\ &-\frac{1}{V\pi}\sum_{\epsilon_{\frac{\mathbf{P}}{2}}<0}^{\mathbf{P}}\theta(-\epsilon_{\mathbf{P}-\mathbf{k}})\int_{2\epsilon_{\frac{\mathbf{P}}{2}}}^0 d\Omega G(\mathbf{P}-\mathbf{k},\Omega-\epsilon)\mathrm{Im}\Gamma(\mathbf{P},\Omega)+\\ &+\frac{1}{V}\sum_{\epsilon_{\frac{\mathbf{P}}{2}}<0}^{\mathbf{P}}\theta(\epsilon_{\mathbf{P}-\mathbf{k}})\mathrm{Res}\Gamma(\mathbf{P},\Omega)_{|\omega_b(\mathbf{P})}G(\mathbf{P}-\mathbf{k},\omega_b(\mathbf{P})-\epsilon)+\\ &+\frac{1}{V\pi}\sum_{\epsilon_{\frac{\mathbf{P}}{2}}>0}^{\mathbf{P}}\theta(\epsilon_{\mathbf{P}-\mathbf{k}})\int_0^\infty d\Omega G(\mathbf{P}-\mathbf{k},\Omega-\epsilon)\mathrm{Im}\Gamma(\mathbf{P},\Omega)\end{aligned} \tag{A.7}$$

Galitskii's approximation instead consists in expanding the ladder in f_0, that is (notice that $\mathrm{Im}f^{-1} = \mathrm{Im}\chi_0$)

$$\Gamma(\mathbf{P},\Omega) \simeq \mathrm{Re}f(\mathbf{p}') + \mathrm{Re}f(\mathbf{p}')^2\left(\chi(\mathbf{P},\Omega) - \mathrm{Re}\chi_0(\mathbf{p}')\right) + O(f_0^3) \tag{A.8}$$

where

$$\mathrm{Re}\chi_0(\mathbf{p}') = \wp\int\frac{d\mathbf{q}}{(2\pi)^2}\frac{1}{q^2-p'^2}$$

and in inserting this function in the expression of the self energy eq. (A.6). The two approaches give the same results, when expanded in powers of f_0 (notice that the ladder approximation is valid only up to second order in f_0). The main differences is that in Galitskii's approximation one neglects the effects of the anti-bound excitation. There are two main consequences of the presence of anti-bound

states. The first is the appearance of isolated poles in the vertex function (see the third term in eq. (A.7)). On the other hand the anti-bound states affect the scattering states. In fact if we consider the discrete version of the ladder, we realize immediately that the poles corresponding to the scattering states, do not coincide exactly with the corresponding non-interacting states. This causes in the continuum limit the appearance of the phase shifts. Notice that in Galitskii's approximation the poles of the vertex are just the non-interacting ones. However, as Ref. [10] shows, the contribution of the anti-bound states is sub-leading with respect to the corrections obtained by using Galitskii's theory. This implies that the Galitskii's approximation is good, at least in the low density limit, and that the system is indeed a Landau Fermi liquid. In both the approaches, however, the main ingredient is the function $\chi(\mathbf{P},\Omega)$, defined in eq. (A.2). In the remainder of this Appendix, we derive a compact form for this function, valid both in $3d$ and in $2d$. The first thing to calculate is the integration region defined by the function $N(\mathbf{P},\mathbf{q}) = 1 - n_{\frac{\mathbf{P}}{2}+\mathbf{q}} - n_{\frac{\mathbf{P}}{2}-\mathbf{q}}$. A simple calculation shows that (we normalize all the momenta to the Fermi one):

1) the integration regions defined by the condition $|\mathbf{P}/2+\mathbf{q}| > 1$ and $|\mathbf{P}/2-\mathbf{q}| > 1$ are:

1.a) for $P \geq 2$:

$$P/2-1 \leq q \leq P/2+1 \text{ and } \frac{1-q^2-P^2/4}{Pq} \leq \cos\theta_{\mathbf{Pq}} \leq \frac{q^2+P^2/4-1}{Pq}$$

$$q \leq P/2-1 \quad \text{and} \quad 0 \leq \theta_{\mathbf{Pq}} \leq 2\pi$$

$$q \geq P/2+1 \quad \text{and} \quad 0 \leq \theta_{\mathbf{Pq}} \leq 2\pi$$

1.b) for $P < 2$:

$$\sqrt{1 - P^2/4} \leq q \leq P/2 + 1 \quad \text{and}$$
$$\frac{1 - q^2 - P^2/4}{Pq} \leq \cos\theta_{\mathbf{Pq}} \leq \frac{q^2 + P^2/4 - 1}{Pq}$$

$$q \geq P/2 + 1 \quad \text{and} \quad 0 \leq \theta_{\mathbf{Pq}} \leq 2\pi$$

2) the integration regions defined by the condition $|\mathbf{P}/2+\mathbf{q}| < 1$ and $|\mathbf{P}/2-\mathbf{q}| < 1$ are:

2.a) for $P \geq 2$:

no phase space available

2.b) for $P < 2$:

$$1 - P/2 \leq q \leq \sqrt{1 - P^2/4} \quad \text{and}$$
$$\frac{q^2 + P^2/4 - 1}{Pq} \leq \cos\theta_{\mathbf{Pq}} \leq \frac{1 - q^2 - P^2/4}{Pq}$$

$$0 \leq q \leq 1 - P/2 \quad \text{and} \quad 0 \leq \theta_{\mathbf{Pq}} \leq 2\pi$$

These conditions are the same in $2d$ as in $3d$. Let us introduce the total energy of the pair in the center of mass frame

$$E = \Omega - P^2/4 + 1$$

and the function

$$f(q,P) = \begin{cases} 2\pi - 4\arccos\dfrac{q^2 + P^2/4 - 1}{Pq} & \text{in } 2d \\ 4\pi\dfrac{q^2 + P^2/4 - 1}{Pq} & \text{in } 3d \end{cases}$$

then it is very simple to obtain a simple expression for χ that is

$$\chi(\mathbf{P},\Omega) = \chi_1(\mathbf{P},\Omega)\theta(P-2) + \chi_2(\mathbf{P},\Omega)\theta(2-P) - \chi_3(\mathbf{P},\Omega)\theta(2-P) \qquad (A.9)$$

where

$$\chi_1(\mathbf{P},\Omega) = \int_0^{P/2-1} \frac{q^{d-1}dq}{(2\pi)^d}\frac{\Omega_d}{E-q^2+i\eta} +$$

$$+ \int_{P/2+1}^{\infty} \frac{q^{d-1}dq}{(2\pi)^d}\frac{\Omega_d}{E-q^2+i\eta} + \int_{P/2-1}^{P/2+1} \frac{q^{d-1}dq}{(2\pi)^d}\frac{f(q,P)}{E-q^2+i\eta}$$

$$\chi_2(\mathbf{P},\Omega) = \int_{P/2+1}^{\infty} \frac{q^{d-1}dq}{(2\pi)^d}\frac{\Omega_d}{E-q^2+i\eta} +$$

$$+ \int_{\sqrt{1-P^2/4}}^{P/2+1} \frac{q^{d-1}dq}{(2\pi)^d}\frac{f(q,P)}{E-q^2+i\eta}$$

$$\chi_3(\mathbf{P},\Omega) = \int_0^{1-P/2} \frac{q^{d-1}dq}{(2\pi)^d}\frac{\Omega_d}{E-q^2-i\eta} +$$

$$- \int_{1-P/2}^{\sqrt{1-P^2/4}} \frac{q^{d-1}dq}{(2\pi)^d}\frac{f(q,P)}{E-q^2-i\eta}$$

where Ω_d is the solid angle in d-dimensions. The knowledge of χ allows the calculation of the vertex, and consequently of the phase shifts. Let us notice that, according to eq. (A.5), the function $\chi(\mathbf{P},\Omega)$ we have to use in order to calculate the phase shifts, is not exactly the one defined in eq. (A.9). The difference is that, to obtain the phase shifts, also $\chi_3(\mathbf{P},\Omega)$ has to be calculated at $\Omega + i\eta$.

The last thing we want to show, is the expression of the self energy in Galitskii's approximation, by means of eq. (A.9), that is

$$\begin{aligned}\Sigma(\mathbf{p},\epsilon) &= -\mathrm{i}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int\frac{d\mathbf{k}}{(2\pi)^d}G(\mathbf{k},\omega)\Gamma(\mathbf{p}+\mathbf{k},\epsilon+\omega) = \\ &= f_0\int_{k<p_f}\frac{d\mathbf{k}}{(2\pi)^d}+ \\ &+ f_0^2\int\frac{d\mathbf{k}}{(2\pi)^d}\Big[\theta(p_f-k)\theta(|\mathbf{p}+\mathbf{k}|-2p_f)\chi_1(\mathbf{p}+\mathbf{k},\epsilon+\epsilon_{\mathbf{k}})+ \\ &+ \theta(p_f-k)\theta(2p_f-|\mathbf{p}+\mathbf{k}|)\chi_2(\mathbf{p}+\mathbf{k},\epsilon+\epsilon_{\mathbf{k}})+ \\ &+ \theta(k-p_f)\theta(|\mathbf{p}+\mathbf{k}|-2p_f)\chi_3(\mathbf{p}+\mathbf{k},\epsilon+\epsilon_{\mathbf{k}})+ \\ &- \theta(p_f-k)\mathrm{Re}\chi_0\Big]\end{aligned} \tag{A.10}$$

This expression is valid both in $3d$ and $2d$. To obtain eq. (A.10) we have integrated over ω by going to the complex plane and by catching the poles of the Green's function. The term linear in f_0 is just $f_0\rho/2$.

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