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NON-CONVEX SYMMETRIC PROBLEMS IN CALCULUS OF VARIATIONS

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Academic Year 1989/90

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A mis abuelos:
Olegario y Felipa, Víctor y Vicenta.

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1. INTRODUCTION

Calculus of Variations has a long history, starting when Johann Bernoulli proposed the famous *brachystochrone problem* to the contemporaries. It was solved by the Bernoullis and others mathematicians at the end of the 17th century.

In this thesis, we deal with a class of integrals of the Calculus of Variations without convexity assumptions on their integrands. More precisely, we consider problems of the form :

$$(P) \quad \begin{aligned} & \text{Minimize } \int_B g(|x|, u(x)) dx + \int_B h(|x|, \Delta u(x) - \lambda u(x)) dx \\ & u \in W^{2,p}(B) \cap W_0^{1,p}(B) \\ & \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial B, \end{aligned}$$

where B is the unit ball and λ is a non negative number and we seek radially symmetric solutions.

The Direct Method permit us to obtain the existence of a minimum if the two following conditions hold:

- (i) The lower semicontinuity of the functional.
- (ii) Any minimizing sequence admits a convergent subsequence in the same topology in which the functional is semicontinuous.

In that direction, one consider the Sobolev space endowed with its weak topology. It is well known that convexity of the map $s \mapsto h(x, s)$ for almost all x , is a necessary and sufficient condition for the lower semicontinuity of the functional. Since we consider non convex integrands the above argument cannot be used. However, we will take in account the fact that:

$$(1.1) \quad \text{Min}(P^{**}) = \inf(P),$$

where (P^{**}) is the relaxed problem obtained convexifying $h(x, \cdot)$. In order to gain insight into the mathematical difficulties raised by this type of problems, it is worthwhile to con-

sider first the case $g \equiv 0$. For this case, let us remark that the problem of seeking the minimum on the larger space $W^{2,p} \cap W_0^{1,p}$ offers no difficulty. In fact, it is enough, in this case, to take any (radially symmetric) selection σ , in L^p , from the map $x \mapsto \operatorname{argmin}\{h(|x|, \cdot)\}$ and consider the (radially symmetric) solution u_1 to the Dirichlet problem

$$(DP) \quad \begin{aligned} \Delta u - \lambda u &= \sigma(|x|) \\ u &= 0 \quad \text{on } \partial B. \end{aligned}$$

Then u_1 is a solution to the given minimization problem. Hence, in general, this problem admits several solutions, obtained simply as solutions to Dirichlet problems. However this procedure cannot be used for the same minimization problem under the additional condition $\frac{\partial u}{\partial n} = 0$ on ∂B , since the corresponding Dirichlet problem would be overdetermined, and, even in this case, a more complex approach is needed.

There are many papers devoted to the existence of solutions to problem (P) that avoid the convexity assumption on h : however all of them prove existence of solutions to problem (P) by imposing conditions implying that every solution to problem (P^{**}) is in fact a solution to problem (P) . The method of proof goes by showing that along any solution to (P^{**}) , the functions h and h^{**} have to coincide almost everywhere, otherwise the Euler-Lagrange equation would be violated. This method cannot possibly be applied to cases where there are solutions to (P^{**}) that are not solutions to (P) : in particular it cannot be applied to the simple case when $g \equiv 0$ and $\lambda = 0$, because, in general, (P^{**}) has solutions that are not solutions to (P) . As an example, take $n = 2$, $h(s) = i(s)$, i the indicator function of the set $\{-1, +1\}$. A computation shows that the function u_2 defined by

$$u_2(x) = \begin{cases} \frac{1}{2}(-\frac{|x|^2}{2} + \log \sqrt{2}), & \text{if } |x| \leq \frac{1}{\sqrt{2}} \\ \frac{1}{2}(\frac{|x|^2}{2} - \log |x|) - \frac{1}{4}, & \text{if } \frac{1}{\sqrt{2}} < |x| \leq 1, \end{cases}$$

satisfies the boundary conditions: $u_2 = 0$ on ∂B and $\frac{\partial u_2}{\partial n} = 0$, and has a Laplacian taking values either $+1$ or -1 , i.e. u_2 is a solution to the original problem. However, the convexified problem, where h^{**} is the indicator function of the interval $[-1, +1]$ has, among others, the solution u_3 identically zero, i.e., in this simple case, there are solutions to the

relaxed problem that are not solutions to the original problem. Moreover, since the method mentioned above uses the Euler-Lagrange equation pointwise, some regularity conditions on the functions g and h have to be imposed. In that spirit Aubert-Tahraoui[A-T2], in case h independent of x and $g(x, \cdot)$ convex, proposed a method based on Duality Theory as presented in [E-T], by generalizing their earlier idea in dimension one (see[A-T1]), where the required hypothesis was $g_u(x, u) \neq 0$. Raymond ([R3],Annexe 1) gives a direct proof derived from the Euler-Lagrange equation by imposing a more general condition, besides regularity, such as

$$(1.2) \quad \sum_{i=1}^n h_{sx_i x_i}^{**}(x, s) + g_u(x, u) \neq 0,$$

again using the Euler-Lagrange equation for the case h depending on x . All of them sought the minimum on the space $W_0^{2,p}$.

It is shown in the paper [C-C] for the first time, via a Liapunov's Theorem, that in some classical problems the convexity of $h(x, \cdot)$ can be replaced by the concavity of $g(x, \cdot)$, and no regularity was required. Recently, Raymond in his paper [R2] carried out this idea to deal with optimal control problems governed by an elliptic equation but he only considered one boundary condition. Our results neither contain nor are contained in any of the papers mentioned above.

In the chapter 2 we present some notations and preliminary results. The chapter 3 is divided into two sections, the first one, deals the simplest case i.e. when the integrand does depend on the state function u explicetely and the second section is devoted to solve the linear case. Lastly, in the chapter 4, by assuming that the relaxed problem has a radially symmetric solution we show that the original problem admits a solution with the same symmetry under the hypothesis that $g(|x|, \cdot)$ is concave.

2. NOTATIONS AND PRELIMINARIES

Throughout this thesis, n is an integer, p is a real number such that $2 \leq n < p$. B is the unit ball of \mathbf{R}^n with boundary ∂B . Fixed $\lambda \geq 0$ we equip the space $W^{2,p}(B) \cap W_0^{1,p}(B)$ with the norm $\|\Delta u - \lambda u\|_{L^p}$.

We now list some preliminary results.

Proposition 2.1 .- ([Ad] th. 6.2; th. 7.53)

- (a) Given $n \in \mathbf{N}, p \in \mathbf{R}$ such that $2 \leq n < p$, the immersion $W^{2,p}(B) \subset C^1(\overline{B})$ is compact.
- (b) The operator $\Gamma : W^{2,p}(B) \rightarrow W^{2-\frac{1}{p},p}(\partial B) \times W^{1-\frac{1}{p},p}(\partial B)$ defined as

$$\Gamma(u) = \left\{ u|_{\partial B}, \frac{\partial u}{\partial n} \right\}$$

is linear, continuous and surjective. Moreover, $W_0^{2,p}(B) = \text{Ker} \Gamma$, i.e.

$$W_0^{2,p}(B) = \left\{ u \in W^{2,p}(B) : u = \frac{\partial u}{\partial n} = 0 \text{ on } \partial B \right\}.$$

$\text{SO}(n)$ denotes the Rotation group in \mathbf{R}^n which has as elements the orthogonal matrices $A \in M(n)$ such that $\det(A) = 1$, it is a compact and connected topological group (see [DNF]). Therefore, given $u \in W^{2,p}(B)$ the integral

$$\int_{\text{SO}(n)} u(Ax) d\mu(A)$$

is well defined, where μ is a left (or right) Haar measure on $\text{SO}(n)$ with $\mu(\text{SO}(n)) = 1$ (see [C]). Remark by the definition of $\text{SO}(n)$, we have that its elements preserve the inner product, i.e.

$$\langle Ax, Ay \rangle = \langle x, y \rangle \quad \forall A \in \text{SO}(n).$$

Hence $|Ax| = |x|$. Furthermore, fixed $x \in \mathbf{R}^n$, it is not difficult to show that:

$$\left\{ Ax \in \mathbf{R}^n : A \in \text{SO}(n) \right\} = \left\{ y \in \mathbf{R}^n : |y| = |x| \right\}.$$

We have the following

Proposition 2.2 .-

Let $u \in W^{2,p}(B) \cap W_0^{1,p}(B)$ such that $\frac{\partial u}{\partial n} = 0$ on ∂B . Define $\bar{u} : B \rightarrow \mathbb{R}$ by

$$\bar{u}(x) = \int_{SO(n)} u(Ax) d\mu(A).$$

Then

- (a) $\bar{u} \in W^{2,p}(B) \cap W_0^{1,p}(B)$;
- (b) $\frac{\partial \bar{u}}{\partial n} = 0$ on ∂B ;
- (c) $\bar{u}(x) = \bar{u}(|x|)$;
- (d) $\Delta \bar{u}(x) = \int_{SO(n)} \Delta u(Ax) d\mu(A)$.

Proof.—

Since $|Ax| = |x|$ we have that \bar{u} vanishes on ∂B since u does so. We use Tonelli-Fubini's theorem (see [C]) to prove that $\bar{u} \in W^{2,p}(B)$. Let $A \in M(n)$, $A = (a_{ij})$, $|x| = 1$

$$\begin{aligned} \frac{\partial \bar{u}}{\partial n}(x) &= \langle \nabla \bar{u}(x), x \rangle \\ &= \sum_{j=1}^n \frac{\partial \bar{u}}{\partial x_j}(x) x_j \\ &= \sum_{j=1}^n \int_{SO(n)} \sum_{i=1}^n \frac{\partial u}{\partial \xi_i}(Ax) a_{ij} x_j d\mu(A) \\ &= \int_{SO(n)} \sum_{i=1}^n \frac{\partial u}{\partial \xi_i}(\xi) \xi_i d\mu(A) \\ &= \int_{SO(n)} \langle \nabla u(\xi), \xi \rangle d\mu(A) \end{aligned}$$

where $\xi = Ax$ and $\xi_i = \sum_{j=1}^n a_{ij} x_j$. Since $|\xi| = |x|$ claim (b) is proved. (c) follows from the remark above and (d) is a consequence of the definition of $SO(n)$. ■

Let I be any interval in \mathbb{R} and denote by \mathcal{L} the σ -algebra of (Lebesgue) measurable subsets of I and, by $\mathcal{B}(\mathbb{R})$ the σ -algebra of \mathbb{R} . Finally by $\mathcal{L} \otimes \mathcal{B}(\mathbb{R})$ we denote the product σ -algebra on $I \times \mathbb{R}$ generated by all the sets of the form $A \times B$ with $A \in \mathcal{L}$ and $B \in \mathcal{B}(\mathbb{R})$.

We recall that a function $f : I \times \mathbf{R} \rightarrow \overline{\mathbf{R}}$ is called $\mathcal{L} \otimes \mathcal{B}(\mathbf{R})$ -measurable or simply measurable if the inverse image under f of every closed subset of $\overline{\mathbf{R}}$ is measurable, in other words $f^{-1}(C) \in \mathcal{L} \otimes \mathcal{B}(\mathbf{R})$ for every closed subset C of $\overline{\mathbf{R}}$.

Let $\xi \mapsto h^{**}(r, \xi)$ be the bipolar of the function $\xi \mapsto h(r, \xi)$. We have the following

Proposition 2.3 .- ([E-T] Prop. I.4.1; Lemma IX.3.3; Prop. IX.3.1)

(a) $h^{**}(r, \xi)$ is the largest convex (in ξ) function not larger than $h(r, \xi)$.

(b) Suppose that $h : I \times \mathbf{R} \rightarrow \overline{\mathbf{R}}$ is such that;

(h_1) h is $\mathcal{L} \otimes \mathcal{B}(\mathbf{R})$ -measurable;

(h_2) $\xi \mapsto h(r, \xi)$ is lower semicontinuous for almost all r in I ;

(h_3) there exists a positive constant α_1 such that

$$h(r, \xi) \geq \alpha_1 |\xi|^p - \beta_1(r) \text{ where the function } r \mapsto r^{n-1} \beta_1(r) \text{ is in } L^1(I).$$

Then

$$h^{**}(r, \xi) = \min \left\{ \sum_{i=1}^2 \lambda_i h(r, \xi_i) : \xi = \sum_{i=1}^2 \lambda_i \xi_i; \lambda_i \geq 0; \sum_{i=1}^2 \lambda_i = 1 \right\}.$$

(c) Let $z(\cdot)$ be measurable . Then there exist measurable $p_i : I \rightarrow [0, 1]$ and measurable $v_i : I \rightarrow \mathbf{R}$, $i=1,2$, such that:

$$\sum_{i=1}^2 p_i(r) = 1; \quad z(r) = \sum_{i=1}^2 p_i(r) v_i(r); \quad h^{**}(r, z(r)) = \sum_{i=1}^2 p_i(r) h(r, v_i(r)).$$

Finally, we state a version of the famous Liapunov's theorem.

Proposition 2.4 .- ([Ce] 16.1.V)

Let A be a measurable subset of \mathbf{R}^n with finite (Lebesgue) measure, let f_1, \dots, f_k be a integrable functions from A to \mathbf{R}^m and let p_1, \dots, p_k be a measurable functions from A to $[0, 1]$ such that:

$$\sum_{i=1}^k p_i(x) = 1 \text{ a.e. on } A.$$

Then, there exists a measurable partition of A , $(A_i)_i$, $i = 1, \dots, k$ such that

$$\sum_{i=1}^k \int_A p_i(x) f_i(x) dx = \sum_{i=1}^k \int_{A_i} f_i(x) dx.$$

3. EXISTENCE THEOREMS FOR A CLASS OF NON-CONVEX SYMMETRIC PROBLEMS.

We first establish a result when the integrand does not depend on the state function u and in the second section we extend this result to the linear case. Our main tools are: the notion of Rotation group in \mathbf{R}^n , in order to obtain a radially symmetric solution to the relaxed problem from one which is not so, and Liapunov's theorem on the range of vector measures as presented in [Ce]16.1.V (see also Prop. 2.4 of this thesis), which allows to construct a solution to the original problem from a solution to the relaxed problem.

3.1. EXISTENCE FOR THE SIMPLEST CASE.

We shall assume the following hypothesis.

HYPOTHESIS (H).- Set $I = [0, 1]$. The map $h : I \times \mathbf{R} \rightarrow \overline{\mathbf{R}}$ is such that

(h_1) h is $\mathcal{L} \otimes \mathcal{B}(\mathbf{R})$ -measurable;

(h_2) $\xi \mapsto h(r, \xi)$ is lower semicontinuous for a.e. $r \in I$.

Moreover:

(h_3) there exist a positive constant α_1 , such that

$h(r, \xi) \geq \alpha_1 |\xi|^p - \beta_1(r)$ where the function $r \mapsto r^{n-1} \beta_1(r)$ is in $L^1(I)$.

THEOREM 3.1.- Let h satisfy hypothesis (H) and λ be non-negative. Assume that the functional $\int_B h(|x|, \Delta u(x) - \lambda u(x)) dx$ has a finite value for some u in $W^{2,p}(B) \cap W_0^{1,p}(B)$ such that $\frac{\partial u}{\partial n} = 0$ on ∂B . Then the problem

$$(P_0) \quad \begin{aligned} & \text{Minimize } \int_B h(|x|, \Delta u(x) - \lambda u(x)) dx \\ & u \in W^{2,p}(B) \cap W_0^{1,p}(B), \\ & \frac{\partial u}{\partial n} = 0 \text{ on } \partial B, \end{aligned}$$

admits at least one radially symmetric solution.

Proof

We consider the relaxed problem

$$(P_0^{**}) \quad \text{Minimize } \int_B h^{**}(|x|, \Delta u(x) - \lambda u(x)) dx$$

$$u \in W^{2,p}(B) \cap W_0^{1,p}(B),$$

$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \partial B.$$

Clearly, h^{**} satisfies the growth condition (h_3) , therefore a well known result (see [E-T]) assures that problem (P_0^{**}) has a solution \hat{u} . We claim that we can assume the function \hat{u} to be radially symmetric. If it were not so, we consider the function $\bar{u} : B \rightarrow \mathbb{R}$ defined by

$$(3.1) \quad \bar{u}(x) = \int_{SO(n)} \hat{u}(Ax) d\mu(A)$$

instead of \hat{u} , which by Proposition 2.2 belongs to $W^{2,p}(B) \cap W_0^{1,p}(B)$, is such that $\frac{\partial \bar{u}}{\partial n} = 0$ on ∂B and $\bar{u}(x) = \bar{u}(|x|)$. Let us show that \bar{u} is another solution to problem (P_0^{**}) . Remark that on one hand Jensen inequality (see[C]) and (d) of Proposition 2.2 imply

$$(3.2) \quad h^{**}(|x|, \Delta \bar{u}(x) - \lambda \bar{u}(x)) \leq \int_{SO(n)} h^{**}(|x|, \Delta \hat{u}(Ax) - \lambda \hat{u}(Ax)) d\mu(A) < +\infty.$$

On the other hand, by using Tonelli-Fubini's theorem (see[C]) we have

$$\begin{aligned} & \int_B \int_{SO(n)} h^{**}(|x|, \Delta \hat{u}(Ax) - \lambda \hat{u}(Ax)) d\mu(A) dx = \\ & = \int_{SO(n)} \int_B h^{**}(|x|, \Delta \hat{u}(Ax) - \lambda \hat{u}(Ax)) dx d\mu(A), \end{aligned}$$

but

$$\int_B h^{**}(|x|, \Delta \hat{u}(Ax) - \lambda \hat{u}(Ax)) dx = \int_B h^{**}(|y|, \Delta \hat{u}(y) - \lambda \hat{u}(y)) dy \quad \forall A \in SO(n),$$

so that

$$\begin{aligned} & \int_B \int_{SO(n)} h^{**}(|x|, \Delta \hat{u}(Ax) - \lambda \hat{u}(Ax)) d\mu(A) dx = \\ &= \int_{SO(n)} \int_B h^{**}(|x|, \Delta \hat{u}(Ax) - \lambda \hat{u}(Ax)) dx d\mu(A) = \int_B h^{**}(|y|, \Delta \hat{u}(y) - \lambda \hat{u}(y)) dy. \end{aligned}$$

Hence, from (3.2) it follows that

$$\int_B h^{**}(|x|, \Delta \bar{u}(x) - \lambda \bar{u}(x)) dx \leq \int_B h^{**}(|y|, \Delta \hat{u}(y) - \lambda \hat{u}(y)) dy$$

i.e. \bar{u} is a radially symmetric solution to problem (P_0^{**}) .

Using spherical coordinates we obtain

$$(3.3) \quad \int_B h^{**}(|x|, \Delta \hat{u}(x) - \lambda \hat{u}(x)) dx = n\omega_n \int_0^1 r^{n-1} h^{**}(r, \hat{u}''(r) + \frac{n-1}{r} \hat{u}'(r) - \lambda \hat{u}(r)) dr$$

where ω_n denotes the volume of the unit ball B . By (c) of Proposition 2.3 there exist measurable functions p_i and v_i , $i = 1, 2$; such that

$$(3.4) \quad \sum_{i=1}^2 p_i(r) = 1; p_i(r) \geq 0, i = 1, 2;$$

$$(3.5.a) \quad \sum_{i=1}^2 p_i(r) v_i(r) = \hat{u}''(r) + \frac{n-1}{r} \hat{u}'(r) - \lambda \hat{u}(r);$$

$$(3.5.b) \quad \sum_{i=1}^2 p_i(r) h(r, v_i(r)) = h^{**}(r, \hat{u}''(r) + \frac{n-1}{r} \hat{u}'(r) - \lambda \hat{u}(r)).$$

On the other hand by Lusin's theorem there exist a sequence $(K_j)_j$ of disjoint compact subsets of I , and a null set N , such that $I = N \cup (\cup_j K_j)$ and the restriction of each of the maps $r \mapsto h(r, v_i(r))$ to each K_j is continuous. By Proposition 2.4 there exists a measurable partition of each K_j , $(E_{i,j})_i, i = 1, 2$, such that: for every j ,

$$(3.6) \quad \sum_{i=1}^2 \int_{K_j} p_i(r) r^{n-1} h(r, v_i(r)) dr = \sum_{i=1}^2 \int_{K_j} \chi_{E_{i,j}}(r) r^{n-1} h(r, v_i(r)) dr;$$

$$(3.7) \quad \sum_{i=1}^2 \int_{K_j} p_i(r) r^{n-1} v_i(r) dr = \sum_{i=1}^2 \int_{K_j} \chi_{E_{i,j}}(r) r^{n-1} v_i(r) dr;$$

$$(3.8) \quad \sum_{i=1}^2 \int_{K_j} p_i(r) r^{n-1} v_i(r) \varphi(r) dr = \sum_{i=1}^2 \int_{K_j} \chi_{E_{i,j}}(r) r^{n-1} v_i(r) \varphi(r) dr,$$

where $\varphi \in W^{2,p}(B)$ is the radially symmetric solution of the following Dirichlet problem:

$$(3.9) \quad \begin{aligned} \Delta \varphi - \lambda \varphi &= 1 \\ \varphi &= 0 \quad \text{on } \partial B. \end{aligned}$$

(see for instance [G-T]). We claim that the map

$$r \mapsto \sum_{j=1}^{\infty} \sum_{i=1}^2 \chi_{E_{i,j}}(r) r^{n-1} h(r, v_i(r))$$

belongs to $L^1(I)$. If it is so,

$$\begin{aligned} \left| \sum_{i,j} \chi_{E_{i,j}}(r) r^{\frac{n-1}{p}} v_i(r) \right|^p &= \sum_{i,j} \chi_{E_{i,j}}(r) r^{n-1} |v_i(r)|^p \\ &\leq \frac{1}{\alpha_1} \sum_{i,j} \chi_{E_{i,j}}(r) r^{n-1} (h(r, v_i(r)) + \beta_1(r)), \end{aligned}$$

i.e. the map

$$r \mapsto \sum_{i,j} \chi_{E_{i,j}}(r) r^{\frac{n-1}{p}} v_i(r)$$

belongs to $L^p(I)$ or, equivalently, the map

$$(3.10) \quad x \mapsto \sum_{i,j} \chi_{E_{i,j}}(|x|) v_i(|x|)$$

belongs to $L^p(B)$. To prove the previous claim, first, notice that, from (3.5.b), the map

$$r \mapsto \sum_i p_i(r) r^{n-1} h(r, v_i(r))$$

is integrable. On the other hand the sequence of maps

$$z_m(r) = \sum_{j \leq m} \left\{ \sum_i \chi_{E_{i,j}}(r) r^{n-1} \left(h(r, v_i(r)) + \beta_1(r) \right) \right\}$$

is monotone non decreasing and

$$\int_0^1 z_m(r) dr = \sum_{j \leq m} \int_{K_j} \sum_i \chi_{E_{i,j}}(r) r^{n-1} \left(h(r, v_i(r)) + \beta_1(r) \right) dr.$$

Set $S_m = \cup_{j \leq m} K_j$. By (3.6), the right hand side equals

$$\begin{aligned} & \sum_{j \leq m} \int_{K_j} \sum_i p_i(r) r^{n-1} \left(h(r, v_i(r)) + \beta_1(r) \right) dr = \\ &= \int_0^1 \chi_{S_m}(r) r^{n-1} \left(h^{**}(r, \hat{u}''(r) + \frac{n-1}{r} \hat{u}'(r) - \lambda \hat{u}(r)) + \beta_1(r) \right) dr \\ &\leq \int_0^1 r^{n-1} \left(h^{**}(r, \hat{u}''(r) + \frac{n-1}{r} \hat{u}'(r) - \lambda \hat{u}(r)) + \beta_1(r) \right) dr < \infty. \end{aligned}$$

Hence

$$\begin{aligned} & \int_0^1 \sum_{i,j} \chi_{E_{i,j}}(r) r^{n-1} \left(h(r, v_i(r)) + \beta_1(r) \right) dr = \int_0^1 (\lim_m z_m(r)) dr \\ &= \lim_m \int_0^1 z_m(r) dr = \int_0^1 r^{n-1} \left(h^{**}(r, \hat{u}''(r) + \frac{n-1}{r} \hat{u}'(r) - \lambda \hat{u}(r)) + \beta_1(r) \right) dr. \end{aligned}$$

The latter implies

$$(3.11) \quad \int_0^1 \sum_{i,j} \chi_{E_{i,j}}(r) r^{n-1} h(r, v_i(r)) dr = \int_0^1 r^{n-1} h^{**}(r, \hat{u}''(r) + \frac{n-1}{r} \hat{u}'(r) - \lambda \hat{u}(r)) dr,$$

so that the claim is proved. Since $E_{i,j}, i = 1, 2$, is a partition of K_j , we have

$$h(r, \sum_{i,j} \chi_{E_{i,j}}(r) v_i(r)) = \sum_{i,j} \chi_{E_{i,j}}(r) h(r, v_i(r)).$$

Therefore from (3.3) and (3.11) it follows that

$$(3.12) \quad \int_B h^{**}(|x|, \Delta \hat{u}(x) - \lambda \hat{u}(x)) dx = \int_B h(|x|, \sum_{i,j} \chi_{E_{i,j}}(|x|) v_i(|x|)) dx.$$

Now, let u be the (unique) radially symmetric solution to the Dirichlet problem

$$(3.13) \quad \Delta u - \lambda u = \sum_{i,j} \chi_{E_{i,j}}(|x|) v_i(|x|)$$

$$(3.14) \quad u = 0 \quad \text{on} \quad \partial B.$$

We actually know that $u \in W^{2,p}(B)$ ([G-T]), i.e. $u \in W^{2,p}(B) \cap W_0^{1,p}(B)$. Notice that from (3.10) the right hand side of (3.13) is in $L^p(B)$. We claim that the function u is a solution to problem (P_0) . To infer it, we shall prove that

$$(3.15) \quad \frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial B \quad \text{or, equivalently, that } u'(1) = 0 \quad \text{and}$$

$$(3.16) \quad \int_B h^{**}(|x|, \Delta \hat{u}(x) - \lambda \hat{u}(x)) dx = \int_B h(|x|, \Delta u(x) - \lambda u(x)) dx.$$

Ad (3.15). First, remark that (3.13) in spherical coordinates implies

$$(3.17) \quad u'(r) = \frac{\lambda}{r^{n-1}} \int_0^r s^{n-1} u(s) ds + \frac{1}{r^{n-1}} \int_0^r s^{n-1} \sum_{i,j} \chi_{E_{i,j}}(s) v_i(s) ds.$$

Then

$$\begin{aligned} u'(1) &= \lambda \int_0^1 s^{n-1} u(s) ds + \int_0^1 s^{n-1} \sum_{i,j} \chi_{E_{i,j}}(s) v_i(s) ds \\ &= \lambda \int_0^1 s^{n-1} u(s) ds + \sum_j \int_{K_j} s^{n-1} \sum_i \chi_{E_{i,j}}(s) v_i(s) ds \\ &= \lambda \int_0^1 s^{n-1} u(s) ds + \sum_j \int_{K_j} s^{n-1} \sum_i p_i(s) v_i(s) ds \\ &= \lambda \int_0^1 s^{n-1} u(s) ds + \int_0^1 s^{n-1} \sum_i p_i(s) v_i(s) ds \\ &= \lambda \int_0^1 s^{n-1} u(s) ds + \int_0^1 (s^{n-1} \hat{u}'(s))' ds - \lambda \int_0^1 s^{n-1} \hat{u}(s) ds \\ &= \lambda \int_0^1 s^{n-1} (u(s) - \hat{u}(s)) ds, \end{aligned}$$

where we have used (3.7) and (3.5.a). On the other hand, by taking in account (3.9), we have

$$n\omega_n \int_0^1 s^{n-1}(u(s) - \hat{u}(s))ds = \int_B (u(x) - \hat{u}(x))dx = \int_B (\Delta\varphi(x) - \lambda\varphi(x))(u(x) - \hat{u}(x))dx.$$

From Green's Formula we have

$$\int_B \Delta\varphi(x)(u(x) - \hat{u}(x))dx = \int_B \varphi(x)(\Delta u(x) - \Delta\hat{u}(x))dx$$

so that

$$\begin{aligned} n\omega_n \int_0^1 s^{n-1}(u(s) - \hat{u}(s))ds &= \int_B \varphi(x)(\Delta u(x) - \lambda u(x) - \Delta\hat{u}(x) + \lambda\hat{u}(x))dx \\ &= n\omega_n \int_0^1 s^{n-1} \left\{ \sum_{i,j} \chi_{E_{i,j}}(s)v_i(s) - \sum_i p_i(s)v_i(s) \right\} \varphi(s)ds. \end{aligned}$$

The last integral equals

$$n\omega_n \sum_j \left\{ \int_{K_j} s^{n-1} \sum_i \chi_{E_{i,j}}(s)v_i(s)\varphi(s)ds - \int_{K_j} s^{n-1} \sum_i p_i(s)v_i(s)\varphi(s)ds \right\}$$

and, from (3.8) we conclude that

$$u'(1) = \lambda \int_0^1 s^{n-1}(u(s) - \hat{u}(s))ds = 0.$$

Ad (3.16). This is a straightforward consequence from (3.12) and (3.13). This proves that u is a radially symmetric solution to problem (P_0) . ■

Remark 3.2 .- The proof of theorem 3.1 corresponding to the integrability of the functions v_i induce us to formulate a slight extension of Proposition 2.4. Namely:

Let A be a measurable subset of \mathbb{R}^n with finite (Lebesgue) measure, let f_1, \dots, f_k be a measurable functions from A to \mathbb{R}^m and let p_1, \dots, p_k be a measurable functions from A to $[0, 1]$ satisfying:

$$\sum_{i=1}^k p_i(x) = 1 \text{ a.e. on } A \text{ and } \sum_{i=1}^k p_i f_i \in L^1(A).$$

Then, there exists a measurable partition of A , $(A_i)_{i=1, \dots, k}$ such that

$$(*) \quad \int_A \sum_{i=1}^k p_i(x) f_i(x) dx = \sum_{i=1}^k \int_A \chi_{A_i}(x) f_i(x) dx.$$

Moreover, for all $i = 1, \dots, k$, f_i is in $L^1(A_i)$.

In fact, what we actually have is that $\sum_{i=1}^k p_i f_i$ is in $L^1(A)$ and we only need equalities of the form (*). The proof is given in [R1].

Remark 3.3 .- In the scalar case; $n = 1$, $p > 1$, B is the interval $] -1, +1[$. In this situation the proof remains as before (setting $n = 1$), except only that we take

$$\bar{u}(x) = \frac{1}{2} \hat{u}(x) + \frac{1}{2} \hat{u}(-x)$$

instead of that defined in (3.1).

3.2 EXISTENCE FOR THE LINEAR CASE.

In this section I is again the interval $[0, 1]$, $a : I \rightarrow \mathbf{R}$ is such that $r \mapsto r^{n-1}a(r)$ is in $L^{p'}(I)$ with p' the conjugate exponent of p . We present here a result which is an extension of theorem 3.1.

THEOREM 3.4 .- Let h satisfy hypothesis (H) and λ be non-negative. Assume that the functional $\int_B h(|x|, \Delta u(x) - \lambda u(x)) dx$ has a finite value for some u in $W^{2,p}(B) \cap W_0^{1,p}(B)$ such that $\frac{\partial u}{\partial n} = 0$ on ∂B . Then the problem

$$(P_l) \quad \begin{aligned} & \text{Minimize } \int_B a(|x|)u(x)dx + \int_B h(|x|, v(x))dx \\ & u \in W^{2,p}(B) \cap W_0^{1,p}(B), v \in L^p(B) \\ & \Delta u - \lambda u = v, \\ & \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial B, \end{aligned}$$

admits at least one radially symmetric solution.

Proof.-

Let us consider the following minimization problem:

$$(P_l^{**}) \quad \begin{aligned} & \text{Minimize } \int_B a(|x|)u(x)dx + \int_B h^{**}(|x|, v(x))dx \\ & u \in W^{2,p}(B) \cap W_0^{1,p}(B), v \in L^p(B) \\ & \Delta u - \lambda u = v, \\ & \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial B, \end{aligned}$$

Let \hat{u} be a solution to this problem which we can assume radially symmetric, otherwise we can consider the function \bar{u} defined as in (3.1) instead of \hat{u} , noticing in addition that

$$\int_B a(|x|)\bar{u}(x)dx = \int_B a(|x|)\hat{u}(x)dx.$$

We proceed as in the proof of theorem 3.1. By (c) of Proposition 2.3, (3.4), (3.5.a) and (3.5.b) hold for some measurable functions p_i, v_i , $i = 1, 2$. Using Lusin's theorem, as above, we apply Proposition 2.4 to assure the existence of a measurable partition of each K_j , $(E_{i,j})_i$, $i = 1, 2$, such that: for every j , besides (3.6), (3.7), (3.8) also the following holds:

$$(3.18) \quad \sum_{i=1}^2 \int_{K_j} p_i(r) r^{n-1} v_i(r) \psi(r) dr = \sum_{i=1}^2 \int_{K_j} \chi_{E_{i,j}}(r) r^{n-1} v_i(r) \psi(r) dr,$$

where $\psi \in W^{2,p'}(B)$ is the (radially symmetric) solution of the following Dirichlet problem:

$$(3.19) \quad \begin{aligned} \Delta \psi - \lambda \psi &= a(|x|) \\ \psi &= 0 \quad \text{on } \partial B. \end{aligned}$$

We again claim that the function u , solution to (3.13), (3.14) is a solution to problem (P_l) . As before (3.15), (3.16) hold, so that it only remains to prove that

$$(3.20) \quad \int_B a(|x|) \hat{u}(x) dx = \int_B a(|x|) u(x) dx.$$

To that end, let us remark that

$$\int_B a(|x|) u(x) dx = \int_B (\Delta \psi(x) - \lambda \psi(x)) u(x) dx.$$

By means of Green's Formula the right hand side can be written as

$$\int_B (\Delta u(x) - \lambda u(x)) \psi(x) dx.$$

Taking in account (3.13) in spherical coordinates, the last integral equals

$$\begin{aligned}
n\omega_n \int_0^1 r^{n-1} \sum_{i,j} \chi_{E_{i,j}}(r) v_i(r) \psi(r) dr &= n\omega_n \sum_j \int_{K_j} r^{n-1} \sum_i \chi_{E_{i,j}}(r) v_i(r) \psi(r) dr \\
&= n\omega_n \sum_j \int_{K_j} r^{n-1} \sum_i p_i(r) v_i(r) \psi(r) dr \\
&= n\omega_n \int_0^1 r^{n-1} \sum_i p_i(r) v_i(r) \psi(r) dr \\
&= \int_B \left(\Delta \hat{u}(x) - \lambda \hat{u}(x) \right) \psi(x) dx \\
&= \int_B \left(\Delta \psi(x) - \lambda \psi(x) \right) \hat{u}(x) dx \\
&= \int_B a(|x|) \hat{u}(x) dx,
\end{aligned}$$

where we have used (3.18) and (3.5.a), so that the proof is complete. \blacksquare

4. A FURTHER EXISTENCE RESULT.

It is shown in the paper [C-C], for the first time, via Proposition 2.4, that in some classical problems the convexity of $h(x, \cdot)$ can be replaced by the concavity of $g(x, \cdot)$. Recently, Raymond [R2] carry over this idea to deal with optimal control problems governed by an elliptic equation but he only considered one boundary condition. Here, by assuming that the relaxed problem has a radially symmetric solution we show that the original problem admits a solution with the same symmetry under the hypothesis that $g(|x|, \cdot)$ is concave.

We assume the following hypothesis on the function g .

HYPOTHESIS (G) .- Set $I = [0, 1]$. The map $g : I \times \mathbf{R} \rightarrow \mathbf{R}$ is such that

- (g₁) g is $\mathcal{L} \otimes \mathcal{B}(\mathbf{R})$ -measurable;
- (g₂) $u \mapsto g(r, u)$ is lower semicontinuous for almost all r in I ;
- (g₃) $u \mapsto g(r, u)$ is concave for almost all r in I .

Moreover:

- (g₄) there exist a positive constant α_2 such that

$$g(r, u) \geq -\alpha_2 |u|^p - \beta_2(r) \text{ where the function } r \mapsto r^{n-1} \beta_2(r) \text{ is in } L^1(I).$$

Let us consider the problem

$$(P_g) \quad \begin{aligned} & \text{Minimize } \int_B g(|x|, u(x)) dx + \int_B h(|x|, v(x)) dx \\ & u \in W^{2,p}(B) \cap W_0^{1,p}(B), v \in L^p(B) \\ & \Delta u - \lambda u = v, \\ & \frac{\partial u}{\partial n} = 0 \text{ on } \partial B. \end{aligned}$$

THEOREM 4.1 .- Let h satisfy hypothesis (H), g satisfy hypothesis (G) and λ be non-negative. Assume, that the corresponding relaxed problem to (P_g) has a radially symmetric solution, then problem (P_g) admits at least one solution with the same symmetry.

Proof .-

Let \hat{u} be a radially symmetric solution to problem (P_g^{**}) . Essentially the proof is the same as that of theorem 3.1. By (c) of Proposition 2.3, (3.4), (3.5.a) and (3.5.b) hold for some measurable functions $p_i, v_i, i = 1, 2$. On the other hand, the map $x \mapsto -\partial_u(-g(|x|, \hat{u}(x)))$ admits a selection $\sigma(\cdot)$ in $L^{p'}(B)$ (p' is the conjugate exponent of p , for the existence of such a selection see lemma 5.2 of [R2]) which we can suppose radially symmetric. Then after using Lusin's theorem, as above, we apply Proposition 2.4 to assure the existence of a measurable partition of each $K_j, (E_{i,j}), i = 1, 2$, such that: for every j , besides (3.6), (3.7), (3.8) also the following holds:

$$(4.1) \quad \int_{K_j} \sum_{i=1}^2 p_i(r) r^{n-1} v_i(r) \psi(r) dr = \int_{K_j} \sum_{i=1}^2 \chi_{E_{i,j}}(r) r^{n-1} v_i(r) \psi(r) dr,$$

where $\psi \in W^{2,p'}(B)$ is the (radially symmetric) solution of the following Dirichlet problem:

$$(4.2) \quad \begin{aligned} \Delta \psi - \lambda \psi &= \sigma(|x|) \\ \psi &= 0 \quad \text{on } \partial B. \end{aligned}$$

We again claim that the function u , solution to (3.13), (3.14) is a solution to problem (\bar{P}_g) . As before (3.15), (3.16) hold, so that it only remains to prove

$$(4.3) \quad \int_B g(|x|, \hat{u}(x)) dx = \int_B g(|x|, u(x)) dx.$$

To that end, notice that for any selection $\sigma(\cdot)$ of $x \mapsto -\partial_u(-g(|x|, \hat{u}(x)))$, we have by concavity,

$$(4.4) \quad g(|x|, u(x)) \leq g(|x|, \hat{u}(x)) + \sigma(x)(u(x) - \hat{u}(x)), \forall x \in B.$$

Assume we can show that

$$(+) \quad \int_B \sigma(x)(u(x) - \hat{u}(x)) dx = 0.$$

Then (4.4) implies

$$(4.5) \quad \int_B g(|x|, u(x)) dx \leq \int_B g(|x|, \hat{u}(x)) dx.$$

Therefore, since \hat{u} is a solution to problem (P_g^{**}) , by (a) of Proposition 2.3,

$$\begin{aligned} & \int_B g(|x|, u(x)) dx + \int_B h(|x|, \Delta u(x) - \lambda u(x)) dx \geq \\ & \geq \int_B g(|x|, \hat{u}(x)) dx + \int_B h^{**}(|x|, \Delta \hat{u}(x) - \lambda \hat{u}(x)) dx. \end{aligned}$$

Finally by (3.16) and taking in account (4.5), (4.3) follows, and the assertion is proved .

We now prove the claim (+).

$$\begin{aligned} \int_B \sigma(x)(u(x) - \hat{u}(x)) dx &= \int_B (\Delta \psi(x) - \lambda \psi(x))(u(x) - \hat{u}(x)) dx \\ &= \int_B (\Delta u(x) - \lambda u(x) - \Delta \hat{u}(x) + \lambda \hat{u}(x)) \psi(x) dx, \end{aligned}$$

where the last integral was obtained by applying Green's Formula. On the other hand from (3.13) and (5.a) it follows that, in spherical coordinates,

$$\begin{aligned} & \int_B (\Delta u(x) - \lambda u(x) - \Delta \hat{u}(x) + \lambda \hat{u}(x)) \psi(x) dx = \\ &= n\omega_n \int_0^1 r^{n-1} \left\{ \sum_{i,j} \chi_{E_{i,j}}(r) v_i(r) - \sum_i p_i(r) v_i(r) \right\} \psi(r) dr \end{aligned}$$

and the right hand side equals

$$n\omega_n \sum_j \left\{ \int_{K_j} r^{n-1} \sum_i \chi_{E_{i,j}}(r) v_i(r) \psi(r) dr - \int_{K_j} r^{n-1} \sum_i p_i(r) v_i(r) \psi(r) dr \right\}$$

then by using (4.1), claim (+) is proved, which completes the proof. ■

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