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INFINITE-DIMENSIONAL INTEGRABLE SYSTEMS AND ALGEBRAIC GEOMETRY

A thesis submitted for the Degree of
Magister Philosophiæ
Academic Year 1987-'88

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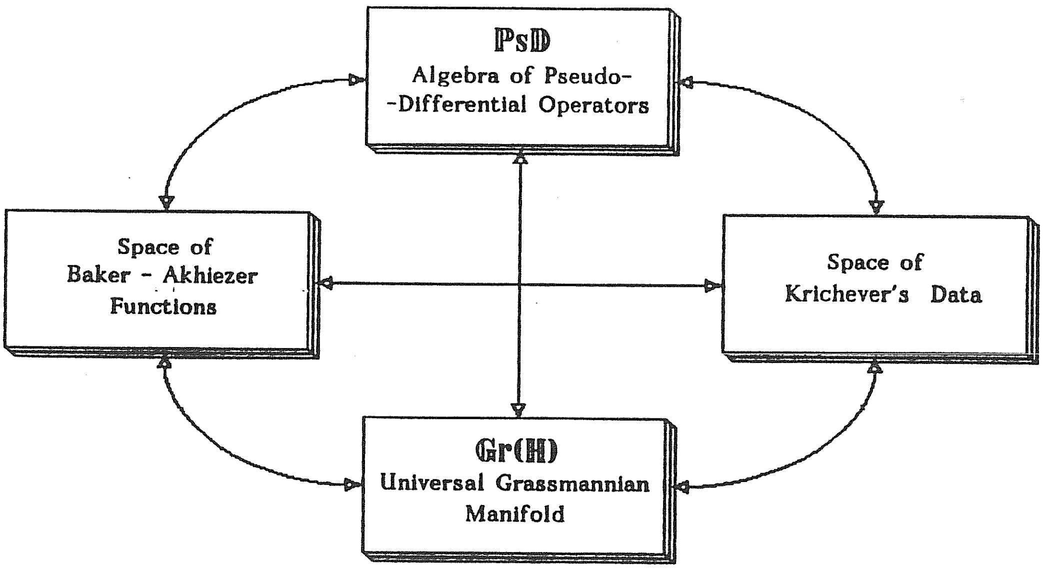

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0. INTRODUCTION

What are the Infinite-Dimensional Integrable Systems? To my knowledge, this fundamental question has not been given a final answer up to now. The notion of "complete integrability" for infinite-dimensional systems is commonly intended to be some sort of generalization of the classical notion of integrability "à la Liouville" in analytical mechanics; however, there is no criterion available in general to decide whether a nonlinear partial differential equation admits enough conservation laws to be considered as completely integrable. According to the common wisdom, the most direct test of complete integrability is the existence of the so-called "soliton solutions"; actually, this criterion has to be sharpened since there are PDEs admitting one- and two-soliton solutions, which nevertheless do not share all the features that one would expect for a completely integrable system, such as the existence of a canonical transformation to "action-angle" variables (in the infinite-dimensional case, this is commonly known as "Inverse Scattering Transform"), or the continuous dependence of solutions on initial data. Another well-known (and nevertheless mysterious) fact is the occurrence of the so-called "Painlevé property" in all classical examples of IDIS. The Painlevé property is the following one: the location of the essential singularities of the solutions is fixed, i.e. it does not depend on initial data; only polar singularities are affected by the choice of the initial conditions. This feature has been often proposed as a criterion for integrability, even if it is not clear at all why it should be so. However, the Painlevé property does not apply directly to the PDE one wants to test, but to the ODEs that can be extracted from it by imposing all possible symmetry requirements: therefore it is, in principle, very difficult to use the Painlevé property as a test. A thorough discussion on these subjects can be found in [1].

In spite of the previous observations, there are a number of PDEs occurring in the literature which are to be considered as IDIS beyond any possible doubt: in other words, any reasonable definition of IDIS should include them. Namely, these equations are those which can be extracted from some well- characterized "hierarchies" of PDEs, in particular from the so-called Kadomtsev-Petviashvilij hierarchy. These equations are presently acquiring a prominent position among the classical subjects of mathematical physics; this is due to many reasons, and I will try to list some of them below.

First, most of these equations arise directly as mathematical models in a number of rather different and apparently unrelated fields, such as fluid dynamics, condensed matter physics, plasma physics, quantum theory of particles and fields, gravitational theories. For instance, let me mention the ancestor of this family, the celebrated KdV equation, which was introduced to describe the solitary waves first observed in a channel by John Scott Russel in 1834; another example is provided by the theory of conductivity in solids, introduced by Fermi, Pasta and Ulam in 1940. It seems quite striking the fact that these models were built directly on the basis of the experimental observation, and only afterwards the distinctive features of the IDIS became apparent; however, exactly those features were able to explain the unexpected behavior of that models. Therefore, even a quite abstract setting of the theory of IDIS is likely, at least in principle, to provide some insight into a class of observable (and actually observed) phenomena.

On the other hand, some mathematical structures, introduced to deal with IDIS, have turned out to be suited also to different applications. As a matter of fact, the present burst of popularity of the algebro-geometric approach (which will be the subject of this thesis) is mainly due to the relevance of that method also for the string theory (see e.g. [2]); this is therefore another good motivation to investigate this subject.

A third reason may be found in the concurrence, in the various approaches to IDIS, of many different mathematical techniques. Dealing primarily with differential equations, the relevance of the analytical setting is immediately evident; but the latest developments have displayed also a charming interplay of algebra and geometry. It is not difficult to guess that investigating this favoured area of overlapping of many mathematical methods could provide useful indications also beyond the scope of the IDIS theory.

Let me further discuss this latter point. Roughly speaking, there are presently two main approaches to investigate the nature of the "complete integrability". The first one relies on the generalization to infinite-dimensional systems of the methods of hamiltonian mechanics, in particular the construction of suitable "Poisson structures". The remarkable amount of work done in this direction is summarized in various textbooks, such as [3], [1]. One of the most relevant issues of this work is probably the understanding of the intimate connection between Lie-algebra structures and hamiltonian structures. This interplay can be emphasized in two possible ways, either by writing the defining equations of the IDIS in matrix form (see e.g.[4]), or by regarding them as evolution equations for differential operators in one variable ([5], [6]). In Sect. 1.2 I will briefly mention the theory bearing the names of Adler, Kostant and Symes, which is a real cornerstone of that buiding. However, more general settings have been recently introduced. In particular, I believe that the theory of the so-called "Poisson-Nijenhuis structures" on a Lie algebra and their possible reductions (see [7]) could provide a fairly comprehensive framework, which also clarifies the role of the so-called "classical Yang-Baxter algebra" in this context.

A different approach relies on the algebro-geometric interpretation of IDIS. The first step in this direction was achieved by applying the methods of commutative algebra to the ring of differential operators, or more precisely to its commutative subrings. The modern algebraic geometry tells us how to investigate such algebraic structures in terms of the associated geometrical objects, such as algebraic curves and holomorphic line bundles. The evolution of the system is then interpreted as generating a geometrical deformation of the latter objects. The whole setting is mostly due to the Russian school, in particular to Krichever [8], while the idea of associating algebraic curves to rings of differential operators was first introduced around 1920 by Burchnall and Chaundy [9]. A typical issue of that construction is the recasting of the "evolution flows" as linear translational flows over a complex torus, the Jacobian variety, which parametrizes the structures of holomorphic line bundle (of degree zero) on a given algebraic curve. In particular cases, this can be achieved in purely geometrical terms, without relying on commutative algebra; for instance, for the KdV equation one can directly recover the "Jacobian flows" by means of the generalized Weierstrass $\wp$-function on hyperelliptic curves (see [10]).

In the algebro-geometrical picture a particular object has acquired a prominent role: the so-called Universal Grassmannian Manifold. In 1981 M. Sato proved that this infinite-dimensional manifold is particularly suited to parametrize solutions of the KP hierarchy. Sato's approach is not originally related with algebraic geometry; it relies mainly on purely algebraic considerations. The UGM setting, nevertheless, has deeply improved also the geometrical insight, and the investigation of its structure by various authors, like G. Segal, G. Wilson, M. Mulase, T. Shiota, E. Arbarello and C. De Concini (see [11] and refs. quoted therein), has eventually provided the solution of a classical problem of algebraic geometry, i.e. the "Schottky problem" about the characterization of the Jacobian varieties among the whole set of analytical complex tori. Of course, this sort of "by-products" are able to motivate by themselves the whole investigation of the subject, at least from the pure mathematicians' viewpoint. The UGM formalism was introduced to provide a geometrical interpretation of the "bilinear form" of the KP equations introduced by Hirota: the solution of the Schottky problem, for instance, has been obtained by regarding the Hirota equation as a condition for the existence of an algebraic curve whose Jacobian variety is a given complex torus.

Remarkably enough, Hirota's τ-function, which occurs in the bilinear form of the KP hierarchy, is also the starting point of the so-called "fermion-operator formalism" for IDIS, introduced by the Kyoto group ([12]) and dealing extensively with representations of infinite-dimensional Lie algebras. The difference between the "geometrical approach" à la Segal & Wilson and the "algebraic" setting by the Kyoto school seems however restricted merely to the way of introducing exactly the same objects; it has been claimed ([13]) that there is a complete correspondence between the two pictures. As it is well-known, both the UGM and the fermion- operator picture raised a great interest among theoretical physicists dealing with the string model and conformal field theories.

On the other hand, the Burchnall-Chaundy-Krichever framework, as well as the Kostant-Adler-Symes construction on the hamiltonian side, correspond to analogous treatments of the finite-dimensional case (see [14]). It is instructive to compare the difficulty in properly defining the class of "completely integrable systems" in the infinite-dimensional case, with the present occurrence in the literature of two possible notions of "complete integrability" in finite dimension. In fact, the development of algebro-geometrical methods for mechanical systems has led some authors to set a definition of "algebraic integrability" (see [15]) besides the classical "Liouville-integrability". It is still unclear whether the two notions of integrability do actually coincide, or the "algebraically integrable systems" form a proper subset of the "Liouville-integrable" ones. In a parallel way, for the moment nobody seems to know to which extent the methods of algebraic geometry can be applied to PDE system admitting a hamiltonian description. As a matter of fact, the relation between algebraic geometry and Poisson structures is still an open problem (some indications about the finite-dimensional case can be found in the recent paper [16]). For instance, the fact that the setting of the "spectral problem", which is quite relevant in the algebraic picture, arises in a natural way within the hamiltonian framework as the explicit expression of a well-defined "momentum mapping" seems not to have received the due attention in the literature.

Finding the link between algebraic geometry and hamiltonian picture is an exciting challenge, not only for aesthetical reasons, but also in view of the possible outcomes; for instance, the hamiltonian framework lies at the starting point of the geometrical approach to quantization, and the quantization of nonlinear systems is still a vaste, unexplored area. To mention a more restricted but quite fashionable problem, it has been recently suggested ([17]) that suitable modifications of the hamiltonian structure of KP equations could make possible to represent non-trivial central extensions of the Virasoro algebra in the space of differential operators. It is precisely in this sense that I said above that the study of IDIS could provide relevant indications to deal with even more general subjects.

This thesis should be intended as a preparatory step to deal with the aforementioned problem of recasting the algebro-geometric structures in terms of of the hamiltonian framework. As a matter of fact, I will not deal at all with the latter one here, but I will concentrate on the algebro-geometrical side and review the results appeared so far in the literature; the final aim of this thesis, however, is to present these results in a form particularly suited to compare them with the hamiltonian approach.

This is the reason for the particular organization of the material presented herein, which is represented in the diagram on the cover page. The first part of the thesis (Sects. 1 to 4) will be devoted to the description of the fundamental ingredients of the construction, namely: the algebra of differential operators and its extension to the algebra of (formal) pseudo-differential operators; the "Baker-Akhiezer functions"; the "Krichever data" consisting in an algebraic curve C, a distinguished point on it, and a line bundle L over C; and, finally, the Universal Grassmannian Manifold.

The presentation of the ring of differential operators will follow a classical approach (see [10], [18]). I will not deal with the analytic problems related to the action of these operators on different function spaces; since the properties which are relevant for the whole discussion are of algebraic character, one can rely on a purely formal construction. In particular, the pseudo-differential operators will not be introduced by means of the Fourier transform as it is currently done, so they are not to be interpreted, in principle, as integro-differential operators.

The algebro-geometric structures will be introduced in a way which is inspired by the subsequent construction; I will extensively describe the relation between algebraic varieties and commutative rings, since the link with IDIS, as I have anticipated below, relies mainly on this setting. On the other hand, a number of items of the theory of Riemann surfaces, which are needed only for technical reasons, will be simply recalled in a very sketchy way.

Sect. 3 deals with the abstract definition of the Baker-Akhiezer functions, anticipating some remarks which should contribute, in my hope, to clarify some points arising later.

In Sect. 4, before introducing the UGM, I will briefly recall the main properties of the well-known finite-dimensional Grassmannians; I will then present first the definition due to Sato, both for historical reasons and because the relation with differential operators is somewhat more direct in this setting; for the rest, I will follow almost faithfully the tractation by Segal & Wilson [13].

The second part of the thesis is devoted to the description of the links between the funtamental "blocks" listed above. I have tried to be as complete as possible in presenting each link in both directions; I regard this effort as the most significant feature of the present work, since each paper available on the subject follows in general only one arrow in describing these correspondences. I stress that having at hand such a completely "commutative" structure, even if apparently redundant, is a necessary starting point to make a comparison with the hamiltonian framework; in fact, we do not know in advance which object in the structure will be the most relevant to this purpose, and being able to recover the whole construction by starting from any point could be very important. From the reader's viewpoint, this organization of the material should make one able to obtain an almost self-consistent description of a single edge of the diagram without reading the other Sections.

1. DIFFERENTIAL OPERATOR ALGEBRA AND RELATED EQUATIONS

1.1 The Rings of Formal Differential and Pseudo-Differential Operators

A considerable insight into the deep structure of nonlinear IDIS such as KdV or KP equations has been obtained by regarding them as infinitesimal generators of flows in the space of differential operators, following the suggestions by Lax and Gelfand & Dikii ([19], [5]). The space of differential operators is naturally endowed with a ring structure, in terms of which these infinitesimal generators can be given a very simple description. A comprehensive reference for the material below is [10].

To define a differential operator (in one variable), one should first specify the space in which the coefficients of it do live. For the present purposes, it is enough to assume that this space is a commutative ring of functions **R**. According to the different situations, one could let **R** being the space of analytic functions in a neighborhood of the origin, the space of smooth functions with compact support, the space of smooth periodic functions, or a space of formal series, without concerning about convergence problems. For most purposes the latter choice is advisable, and I will consider alternatively the rings $\mathbb{C}[[x]]$ (Taylor series, not necessarily convergent) or $\mathbb{C}((x))$ (Laurent series); in general, however, I will not specify the ring of coefficients. In the sequel I will also consider rings of functions of more than one variable.

Let **R**[D] be the *space of formal differential operators with coefficients in* **R**. Each element $L \in \mathbf{R}[D]$ has the form $L = \sum_{k\geq 0} a_k(x)\, D^k$, $a_k \in \mathbf{R}$; the action of the elements of

R[D] as differential operators on **R** is defined through the identification $D \equiv d/dx$. In this way, the composition of operators $L \circ P$ is well-defined and makes the space **R**[D] a (non-commutative) ring.

R[D] is endowed with a natural grading, i.e. $\mathbf{R}[D]=\oplus_k \mathbf{R}[D]_k$, $\mathbf{R}[D]_k$ being the subspace of differential operators of degree k. The subspace $\mathbf{R}[D]_0$ can be obviously identified with **R**; however, they are not to be confused as rings: throughout this Section, I shall write f* to denote the multiplication operator associated to the function $f \in \mathbf{R}$.

The ring structure of **R**[D] is completely described by the Leibnitz rule:

$$D \circ f^* = f^* \circ D + (Df)^* \quad . \tag{1.1}$$

From this formula, in fact, one can derive the general multiplication rule; for notational convenience, let me define the operators ∂_x and ∂_D as acting on an element $L = \sum_{k\geq 0} a_k D^k$ as follows: $\partial_x L = \sum_{k\geq 0} (Da_k) D^k$, $\partial_D L = \sum_{k\geq 0} k\, a_k D^{k-1}$ (if L has degree zero, of course, I set $\partial_D L = 0$) the product of two operators $A, B \in \mathbf{R}[D]$ is then expressed by

$$A \circ B = \sum_{k\geq 0} (k!)^{-1} \partial_D{}^k A\, \partial_x{}^k B \quad , \tag{1.2}$$

whereby the product on the right-hand side is intended to be commutative (this formula can be proved by applying the Leibnitz rule to products of the form $D^m \circ (x^n)^*$). It is straightforward to check that the multiplication defined by (1.2) is associative.

In the next Sections I will deal with the following problem: let me consider a *one-parameter deformation* of a differential operator L, i.e. a family $L(t) = \sum_{k\geq 0} a_k(x,t) D^k$, which can be identified with an element of **R**'[D], **R**' being a ring of functions of the two variables (x, t). For reasons that will be explained below, to describe such a deformation it is convenient to extend the ring **R**[D] to the ring **PsD**(**R**) of *formal pseudo-differential operators with coefficients in* **R**. To this end one introduces the operator D^{-1}: by multiplying both sides of (1.1), from the left and from the right, by D^{-1}, one obtains

(1.3) $$f^* \circ D^{-1} = D^{-1} \circ f^* + D^{-1} \circ (Df)^* \circ D^{-1} \quad ;$$

iterating the application of this formula on $(Df)^* \circ D^{-1}$ in the last term, and so on, one eventually finds

(1.4) $$D^{-1} \circ f^* = \sum_{k\geq 0}(k!)^{-1}\, \partial_D{}^k(D^{-1})\, \partial_x{}^k f^* \quad .$$

The expression (1.4) shows that the multiplication rule (1.2) can be straightforwardly extended to the ring **R**((D)) of formal Laurent series with coefficients in **R**: this ring I will denote by **PsD**(**R**). Any element $A = \sum_{k\in\mathbb{Z}} a_k(x)D^k \in$ **PsD**(**R**) can be splitted into a *positive* (or *differential*) part $(A)_+ = \sum_{k\geq 0} a_k(x)D^k$ and a *negative* part $(A)_- = \sum_{k<0} a_k(x)D^k$. Let me remark that the meaning of the operator D^{-1} is merely algebraic in this setting: to make it correspond to the "integration operator" one should make a suitable choice of the ring **R**, for instance the ring of C^∞-functions with compact support.

In the sequel I will deal mostly with *monic* operators, i.e. operators whose highest-order coefficient is a constant. Without loss of generality, one can suppose this constant to be equal to one. In the paper [20] this setting is generalized to *elliptic* operators, i.e. operators whose highest-order coefficient is non-vanishing at $x = 0$. The relevance of the ring **PsD**(**R**) is mainly due to the following remarkable properties:

(1.5) **Proposition** : Let $Q \in \mathbf{PsD(R)}$ be a monic pseudo-differential operator of arbitrary degree n, $Q = D^n + \sum_{k<n} q_k(x)\, D^k$; then Q has a unique inverse Q^{-1}, which is monic and of degree (-n).

This proposition can be proved by induction. If one can find an "approximate inverse" $X_{(m)} = D^{-n} + \sum_{1\leq k\leq m} a_k\, D^{-n-k}$ such that $X_{(m)} \circ Q = 1 + b\, D^{-m-1} +$ (l.o.terms), then, setting $X_{(m+1)} = (X - bD^{-k-m-1})$, one finds $X_{(m+1)} \circ Q = 1 + c\, D^{-m-2} +$ (l.o.terms). Of course, $X_{(0)} = D^{-n}$. ∎

(1.6) **Corollary** : The monic degree-zero elements $K = 1 + \sum_{k<0} a_k(x) D^k$ form a multiplicative group (the *Volterra group*).

(1.7) **Proposition** (Schur): Every element $Q \in \mathbf{PsD(R)}$, monic and of degree n, has a unique n-th root $Q^{1/n}$. The centralizer $\mathbf{Z}(Q) = \{A \in \mathbf{PsD(R)} \,/\, [A,Q] = 0\}$ of Q is the ring of the Laurent series $\sum_{k\in\mathbb{Z}} c_k\, Q^{k/n}$, $c_k \in \mathbb{C}$. $\mathbf{Z}(Q)$ is commutative; furthermore, any commutative subring of $\mathbf{PsD(R)}$ is of this type.

The proof of this proposition can be found, for instance, in [10].

The reason for having introduced the rings $\mathbf{R}[D]$ and $\mathbf{PsD(R)}$ can now be illustrated by the following example. Let me consider again a deformation of an operator L, as above, and assume that this deformation is generated by the tangent vector $\partial_t L$ defined by the so-called Lax equation:

(1.8) $$\partial_t L = [P, L] \qquad (P, L \in \mathbf{R}'[D]);$$

for suitable choices of the "Lax pair" (P, L) , the equations for the coefficients of P and L which follow from (1.8) turn out to reproduce some well-known IDIS, for instance the KdV and the KP equations.

In fact, let $L = D^2 + u^*(x,t)$ and $P = D^3 + (3/2)(u(x,t)D + (Du(x,t))^*)$; calculating the expression of the commutator by the formula (1.2), one finds

(1.9) $$[P, L] = (1/4)\,(D^3u + 6uDu)^* \quad .$$

In a similar way one can recover the KP equation, the Boussinesq equation and other IDIS; this motivates the introduction of the ring **R**[D]. Now, let me remark that the choice of the second term for the Lax pair, once fixed the first one, is not arbitrary, since (1.8) can be consistent only if deg ([P, L]) ≤ deg(L) , since the degree of $\partial_t L$ cannot exceed deg(L) (more precisely (deg(L) - 1), if L is monic); while in general, for arbitrary P, one has deg([P, L]) ≤ (deg(P) + deg(L) - 1). As I will show in the next Section, the characterization of the admissible Lax pairs can be easily discussed by means of the properties of the ring **PsD(R)** described above.

1.2 Lax Equation, Spectral Problem and related topics.

The Lax equation (1.8) has a deep meaning, and the fact that it reproduces the PDEs of some well-known IDIS is far from being accidental. Let me list some relevant situations in which the Lax equation plays a central role.

In Part B of this thesis I will present the Burchnall-Chaundy-Krichever construction that I have outlined in the introduction. The Lax equation acquires an intrinsic geometrical meaning in that context, since it represents, in a suitable sense, the generators of linear flows on the Jacobian torus of spectral curves. This will be explained later.

As a second instance, let me briefly recall the main ideas of the Adler-Kostant-Symes approach to the complete integrability of Hamiltonian systems. Let $\mathcal{g}$ be a (possibly infinite-dimensional) Lie algebra. $\mathcal{g}$, or rather its dual $\mathcal{g}^*$ (which is often identified with $\mathcal{g}$ itself, by assuming the existence of an ad-invariant metric), is naturally endowed with a Poisson structure, called *Lie-Poisson bracket.* In fact, $\mathcal{g}^*$ can be identified with the cotangent space to the identity of the corresponding Lie group $\mathcal{G}$, and one can prove that the canonical symplectic form on $T^*\mathcal{G}$ admits a well-defined restriction to $\mathcal{g}^*$, which is however not of maximal rank (actually, not even of constant rank; therefore, it defines a Poisson structure but not a symplectic structure). The explicit expression of the Lie-Poisson bracket of two functions f, g $\in C^\infty(\mathcal{g}^*)$, calculated in $\mu \in \mathcal{g}^*$, is the following:

$$\{f, g\}(\mu) = \langle \mu, [\nabla f, \nabla g] \rangle \quad , \tag{1.10}$$

where ∇ denotes the gradient defined by the relation $df(\mu) = \langle\mu, \nabla f\rangle$. The characteristic leaves of this Poisson structure, i.e. the submanifolds of $\mathfrak{g}^*$ such that the restriction of the Lie-Poisson tensor has maximal rank, turn out to be the orbits of the coadjoint action ad*. Thus, on these orbits one can define a symplectic structure, which is commonly known as the *Kostant-Kirillov form*. Now, if one fixes a Hamiltonian H on a characteristic leaf and writes the corresponding Hamilton equations in Poisson form, one tipically finds Lax equations, due to the occurrence of the commutator in the expression (1.10).

It could seem that the hamiltonian systems on coadjoint orbits in a Lie algebra constitute a very particular case; this is not so, for at least two reasons. First, a theorem by Kirillov shows that such systems are the general model for hamiltonian systems with symmetries: namely, if the group $\mathcal{G}$ acts transitively on a symplectic manifold M, preserving the symplectic form, then M is isomorphic to a coadjoint orbit (in $\mathfrak{g}^*$) either of $\mathcal{G}$ or of a central extension of it. The second reason lies in the AKS construction. From the arguments outlined above it follows (I skip the details; see e.g. [14]) that any two ad*-invariant fuctions on $\mathfrak{g}^*$ have vanishing Lie-Poisson bracket. However, the restriction of these functions to any coajoint orbit Ω does not provide any non-trivial integral of motion for the hamiltonian systems above described, since these functions are just identically constant on Ω. The AKS construction relies on the introduction of a second coadjoint action. Namely, one starts from a splitting of a semisimple Lie algebra $\mathfrak{g}$ into two orthogonal subspaces N and M, closed with respect to the commutator operation, and considers the coadjoint action ad_N^* of N on its dual (which can be identified with M); this action is related with the coadjoint action of the whole algebra through a projection operator, so it is easy to see that the orbits of the two actions do not coincide. On the orbits of the "new" coadjoint action ad_N^* one can define in the standard way another Kostant-Kirillov form: it turns out that the ad*-invariant functions commute also with respect to the new symplectic structure, but they are not identically constant on the new orbits. It turns out that, for a hamiltonian system defined on these latter ones, one can obtain in this way enough non-trivial constants of the motion in involution to fulfill Liouville's condition for complete integrability.

Also this construction could seem to deal only with some particular cases of completely integrable hamiltonian systems, and in fact it is not expected to cover all the possible cases; however, one can treat in this way most of the known examples, included the infinite-dimensional case of KdV equation, whereby the Lie algebra involved is just **PsD(R)** and the splitting is the natural one, into the space $\mathbf{PsD(R)}_+ \equiv \mathbf{R}(D)$ and its complementary space $\mathbf{PsD(R)}_-$ (see e.g. [21]). Now, the AKS construction is directly related with the Lax equation because all the integrable systems which can be obtained through the AKS framework admit a Lax representation. Let me remark that Lax equations occur in more general approaches to the complete integrability problem, so they are to be regarded as one of the main contact points of the various theories.

The third and final argument that I will rise to support the investigation of Lax equations is that they express integrability conditions for systems of linear PDEs, which are commonly known as *spectral problems*. Before doing that, however, I wish to retake the question posed at the end of Sect. 1.1, i.e., how can one determine the admissible Lax pairs? The answer is provided by the following proposition:

(1.11) **Proposition** : Let $L \in \mathbf{R}[D]$, monic and of degree r; for each p<0, the space of the elements $P \in \mathbf{R}[D]$, of degree p, such that $\deg([P, L]) \leq (r-1)$ is generated by the elements $(L^{k/r})_+$, (k=1, ..., p), the differential part of the powers of the r-th root of L.

The fundamental ingredient of the proof of (1.11) is provided by the following lemma:

(1.12) **Lemma** : The space described in Proposition (1.11) is (p+1)-dimensional.

The explicit form of the commutator of $P = \sum v_k D^k$ and $L = \sum u_k D^k$ is the following:

$$[P, L] = \sum_{i,j,k\geq 0} [\binom{i}{k} v_i \partial_x{}^k u_j - \binom{j}{k} u_j \partial_x{}^k v_i] D^{i+j-k} . \qquad (1.13)$$

For each k, the coefficient of D^{r+k} depends on the coefficients v_i, and their derivatives, only for $p\geq i\geq k$. Therefore, once we have solved the condition for the vanishing of the coefficient of D^{r+k+1}, the analogous condition relative to D^{r+k} becomes a linear first-order ODE for v_k, which has a unique solution up to a constant. Since the coefficients of D^{r+k} must vanish for $p\geq k\geq -1$, and the equation for k=p is trivially satisfied (L is assumed to be monic), the lemma is proved by induction. To prove the theorem, one has to show that the commutators of the elements $(L^{k/r})_+$, (k=1, ..., p) with L itself have the prescribed degree. This follows from Proposition (1.7), which implies $[(L^{k/r})_+, L] = - [(L^{k/r})_-, L]$; since $(L^{k/r})_-$ has negative degree by definition, $\deg([(L^{k/r})_-, L]) \leq r-1$. ∎

Now I can discuss the setting of the spectral problem. Let me cosider two operators L, P $\in \mathbf{R}'[D]$; it is well known that the existence of a (formal) solution ψ of the joint eigenvalue problem for L and P

$$\begin{aligned} L\psi &= \lambda\psi \\ P\psi &= \rho\psi \end{aligned} \qquad (1.14)$$

depends on the the condition

$$[L, P] = 0 \quad . \tag{1.15}$$

For differential operators, (1.15) is known as *Novikov equation,* or *stationary KdV equation.* To obtain the standard KdV equation (1.8), one can set the following problem:

$$\begin{aligned} L\psi &= \lambda\psi \\ P\psi &= \partial_t \psi \quad , \end{aligned} \tag{1.16}$$

ψ being now intended as a function of two variables (x,t). The compatibilty condition becomes in fact

$$[L, \partial_t - P] = 0 \quad \Leftrightarrow \quad \partial_t L = [P, L] \quad . \tag{1.17}$$

Further generalizing to the case of $L, P \in \mathbf{R}''[D]$, where $\mathbf{R}'' \equiv \mathbf{R}'[[y]]$, one can consider the problem

$$\begin{aligned} L\psi &= \partial_y \psi \\ P\psi &= \partial_t \psi \quad ; \end{aligned} \tag{1.18}$$

this time, the compatibility condition leads to the *Zacharov-Shabat equation* ([22]), which some authors also call "Lax equation" or "zero-curvature condition":

(1.19) $$[\partial_y - L, \partial_t - P] = 0 \quad \Leftrightarrow \quad \partial_t L - \partial_y P = [P, L] \quad .$$

If one sets $L = D^2 + u(x,t)$ (from now on, I will write u instead of u* since there is no more possible ambiguity), and chooses P to be a third-order operator, one finds for the coefficients the Kadomtsev-Petviashvilij equation (see e.g. [23]).The proof that (1.19) is indeed a compatibility condition for the linear system (1.18) will be postponed to Part B.

I now wish to generalize even more this setting by considering a coefficient ring **R** formed by functions depending on an arbitrary number of variables, which I will collectively denote by $\boldsymbol{t} \equiv (x, t_2, t_3, \ldots)$. Let Q be a monic pseudo-differential operator of degree one, and P_n (n = 2, 3, ...) be an infinite family of differential operators such that $\deg(P_n) = n$. Let the following relations hold for any $n \leq 2$:

(1.20) $$Q\,\psi = \lambda\,\psi$$
$$P_n \psi = \partial_n \psi$$

(∂_n standing for $\partial/\partial t_n$); the related compatibility conditions are an infinite number of Lax equations

(1.21) $$\partial_n Q = [P_n, Q] \quad ,$$

together with infinitely many ZS equations

(1.22) $$\partial_m P_n - \partial_n P_m = [P_n, P_m] \quad .$$

In spite of having so many condition to be fulfilled, the problem admits solutions: in fact, let me assume (for the moment, as an *ansatz* suggested by the considerations above) $P_n = (Q^n)_+$; equations (1.22) are then automatically satisfied:

$$(1.23) \qquad (\partial_m Q^n - \partial_n Q^m)_+ - [P_m, P_n] = ([P_m, Q^n] - [P_n, Q^m] - [P_m, P_n])_+ =$$

$$= ([P_m - Q^m, P_n - Q^n])_+ = 0 \quad .$$

This problem, which will be re-examined from a different viewpoint in Sect. 5, leads therefore to the *KP hierarchy* (see [12]):

$$(1.24) \qquad \partial_n Q = [(Q^n)_+, Q] \quad .$$

2. ALGEBRAIC VARIETIES AND COMMUTATIVE RINGS.

2.1 Algebraic Curves.

One of the most beautiful results of the algebraic theory of the differential operators is the connection with the classical theory of algebraic curves. The root of this correspondence, as I have anticipated in the Introduction, is the relation between spectral problems and commutative subrings of differential operators. The modern approach to algebraic geometry suggests how to associate geometrical objects to commutative rings and modules, and this will be the subject of this Section. The main references on this material are the textbooks [24], [25] and [26]. In these books the definitions are usually given in terms of an arbitrary field $\mathbb{K}$ (even not algebraically closed). For my purposes, it is enough to deal with the particular case $\mathbb{K}\equiv\mathbb{C}$; however, most of the definitions hold in the general case.

(2.1) **Definition** : An *affine algebraic variety* $V\subset\mathbb{C}^n$ is the common zero locus of a set of polynomials $f_1, ..., f_k \in \mathbb{C}[x_1, ..., x_n]$.

The equations $f_i(x_1, ..., x_n) = 0$ are called the *defining equations* of V. The *algebraic curves* are the varieties of (complex) dimension one. The *plane algebraic curves* are the curves in $\mathbb{C}^2$ defined by one polynomial $f \in \mathbb{C}[x, y]$.

Of course, the zero locus of f is also the common zero locus of all polynomials $g\in\mathcal{I}(\{f_i\})$, $\mathcal{I}(\{f_i\})\subset\mathbb{C}[x_1, ..., x_n]$ being the principal ideal generated by the polynomials

f_i. Therefore, each algebraic variety V is in correspondence with a proper ideal of a polynomial ring $\mathbb{C}[x_1, ..., x_n]$; the ideal $\mathfrak{J}(\{f_i\})$ is called the *ideal of the variety* V and will be denoted by $\mathfrak{J}(V)$. A variety V is said to be *irreducible* if $\mathfrak{J}(V)$ is a prime ideal, or, equivalently, if $\mathfrak{J}(V)$ is generated by irreducible polynomials. Geometrically, being irreducible means that V cannot be split into the union of two proper subvarieties. In the sequel I will assume in general that V is irreducible.

An algebraic variety V carries two topological structures which deserve a particular attention: one is the "standard" topology induced on V by the usual topology of $\mathbb{C}^n$; the other one is the *Zariski topology*. The latter topology is obtained by defining the closed sets in V to be the algebraic subvarieties $X \subset V$. The two topologies are far from being equivalent. For instance, it is easy to see that the Zariski topology is not Hausdorff. As a matter of fact, these two structures reflect different attidudes towards the geometrical object V. For instance, one can alternatively define a (nonsingular) curve C to be an analytic complex manifold of dimension one; this definition is not strictly equivalent with the previous one, since there are affine curves which are not smooth manifolds, and there are analytic manifolds which are not the zero locus of a set of polynomials in $\mathbb{C}^n$: but most of the examples lie in the intersection of the two classes of objects. In the analytical construction C comes automatically equipped with the first topology mentioned above. From the algebraic viewpoint, however, the most "natural" topology is the Zariski one, as I will show below.

In the sequel, let me restrict to algebraic curves, although the definition below hold for affine varieties of arbitrary dimension.

(2.2) **Definition** : The *coordinate ring* $A(C)$ of a plane curve C is the quotient ring $\mathbb{C}[x,y]/\mathfrak{J}(C)$.

The algebraic notion of coordinate ring is related with the structure of the space of the fuctions defined on C. This space is described in terms of sheaf structures; recalling the definitions and theorems related with sheaf theory would exceed the scope of this thesis, so I refer the reader to standard textbooks such as [24], [25].

(2.3) **Definition** : A function $r : C \longrightarrow \mathbb{C}$ is said to be *regular at a point* $p \in C$ if there exist a (Zariski-)open neighborhood $U \subset C$ of p and two polynomials f, $g \in \mathbb{C}[x,y]$ such that, on U, g is nowhere vanishing and r = f/g.

A function is called *regular on C* if it is regular at every point of C. The regular functions form the *structure sheaf,* denoted by $\mathcal{O}_C$. The *local ring* $\mathcal{O}_{C,p}$ at a point $p \in C$ is the ring of germs of regular functions at p (i.e. the ring of equivalence classes of functions under the following relation: f ~ g if f = g on some neighborhood of p); equivalently, $\mathcal{O}_{C,p}$ is the stalk of $\mathcal{O}_C$ at p.

Now comes the relation between the structure sheaf and the coordinate ring. It is evident that the minimal algebraic subvarieties of C are its points. In general, any point $p \in \mathbb{C}^2$ is the zero locus of a maximal ideal of $\mathbb{C}[x,y]$; if $p \in C$, this maximal ideal contains the ideal $\mathcal{I}(C)$. Passing to the quotient by $\mathcal{I}(C)$, to each point p on the curve corresponds a maximal ideal in A(C); this maximal ideal turns out to be the ideal $m_p \equiv \{f \in A(C) / f(p)=0\}$. Let me recall the definition of the *localization* R_m of a ring R at a maximal ideal m: R_m is the ring formed by the equivalence classes of fractions f/g, where f, $g \in R$ and $g \notin m$, f/g being equivalent to f'/g' if $r \in R$ exists such that r (f'g - fg') = 0 . From the definition (2.3) it follows that $\mathcal{O}_{C,p} \cong A(C)_{m_p}$.

(2.4) **Proposition** : Let $\mathcal{O}(C)$ denote the ring $\Gamma(C, \mathcal{O}_C)$ of the *global* sections of the structure sheaf; then $\mathcal{O}(C) \cong A(C)$.

To prove (2.4), consider for each polynomial $f \in \mathbb{C}[x, y]$ the restriction $f|_C \in \mathcal{O}(C)$; this defines a map $\mathbb{C}[x, y] \to \mathcal{O}(C)$, the kernel of which is just $\mathcal{I}(C)$, and therefore an injective map $A(C) \to \mathcal{O}(C)$. Now, one can show that, in a suitable sense, $\mathcal{O}(C) \subseteq \bigcap_{p \in C} \mathcal{O}_{C,p} \cong \bigcap_{m} A(C)_{m}$, where m runs over all maximal ideals of $A(C)$; consequently, $A(C) \subseteq \mathcal{O}(C) \subseteq \bigcap_{m} A(C)_{m}$. A standard algebraic theorem says that, for any integral domain D, $D \cong \bigcap_{m} D_{m}$ Thus Proposition (2.4) is proved. ∎

In the next Section I will discuss the reconstruction of an affine curve from the knowledge of its coordinate ring. In the sequel, however, I will need to deal also with projective varieties. In fact, although any affine variety is compact in the Zariski topology, in general the affine varieties are not compact with respect to the complex topology. To see this, consider for instance the affine line y = 0 ; the Zariski-open sets are just the complements of discrete sets of points on the line, and it is easy to check that any covering of the line by such open sets admits a finite subcovering. Of course, the complex line is not compact in the usual topology. In the algebraic approach, a geometrical object which is compact in the usual sense is obtained by passing from $\mathbb{C}^n$ to the projective compactification $\mathbb{P}\mathbb{C}^n$.

(2.5) **Definition** : A *projective variety* $V \subset \mathbb{P}\mathbb{C}^n$ is the common zero locus of a set of *homogeneous* polynomials $f_1, ..., f_k \in \mathbb{C}[x_0, ..., x_n]$.

The natural embedding of V into $\mathbb{P}\mathbb{C}^n$ is obtained by identifying $\{x_0, ..., x_n\}$ with a system of homogeneous coordinates in $\mathbb{P}\mathbb{C}^n$.

One can think to a *projective curve* as an affine curve compactified by adding a "point at infinity". This compactification is achieved, in practice, by passig from the defining equation $f(x_0, x_2) = 0$ to the following homogeneous equation in $\mathbb{P}\mathbb{C}^n$:

$$(2.6) \qquad 0 = \tilde{f}(x_0, x_1, x_2) \equiv x_0^{\deg f} f(x_1/x_0, x_2/x_0) \quad .$$

In the sequel, I will mainly deal with affine curves arising from projective curves by subtracting a distinguished point. One can find in the reference textbooks the detailed construction of the rings and sheaves related to projective varieties; I will not deal with this subject since it is not strictly needed for the discussion of Part B.

2.2 The Spectrum of a Ring.

Since the points of a variety and the maximal ideals of its coordinate ring are in one-to-one correspondence, one is led to ask whether it is possible to endow the set of maximal ideals of an abstract commutative ring A with a structure of algebraic variety. There is, however, a technical problem in dealing with maximal ideals: one can see that to each morphism of algebraic varieties (i.e. to each continuous map between two varieties X and Y such that the pull-back of a regular function on Y is regular on X) should correspond a ring homomorphism of the respective coordinate rings; but the image of a maximal ideal under a generic homomorphism of rings is not necessarily a maximal ideal. The prime ideals, on the contrary, are preserved by ring homomorphisms, so one can overcome the difficulty by taking the set of prime ideals of A. Let me recall that a prime ideal $p \subset A$ can be defined in an equivalent way by any one of the following two properties:

(2.7) **Definition** : $p \subset A$ is a *prime ideal* iff either

(i) for any pair of elements a, b $\in$ A, ab $\in p \Rightarrow$ a $\in p$ or b $\in p$;

(ii) the quotient ring A/p is an integral domain.

For a maximal ideal m, A/m turns out to be a field; therefore, any maximal ideal is prime.

(2.8) **Definition** : The *spectrum* of a commutative ring A is defined (as a set) by setting $\mathrm{Spec}(A) \equiv \{ p \subset A \,/\, p \text{ is a prime ideal}\}$.

The next task is to topologize Spec(A). To do this, one considers for each ideal $a \subset A$ the set $U(a) \equiv \{ p \in \mathrm{Spec}(A) \,/\, a \subseteq p \}$. The sets $U(a)$, as a runs over all the ideals of A, are defined to be the *closed sets* in the *Zariski topology* on Spec(A). In this topology, the points p of Spec(A) can have different properties: if p is a maximal ideal, $U(p) \equiv \{p\}$, so that the point p itself is a closed set; such a point will be called a *closed point.* If p is not maximal, it can happen that the closure of p is the whole space Spec(A). This happens, for instance, for the zero ideal $0 \equiv \{0\}$, which is prime if A is an integral domain. Such points are called *generic points.*

The occurrence of generic points comes from having considered the prime ideals instead of the maximal ones; from the previous considerations, one can expect that only the closed points of Spec(A) correspond to the points of a variety.

So far, Spec(A) is only a topological space. To make it correspond to an algebraic variety I still have to define the sheaf of regular functions on it.

(2.9) **Definition** : A *regular function* on an open set $U \subset \mathrm{Spec}(A)$ is a map $f : U \to \coprod_{p \in U} A_p$ from U to the disjoint union of the localizations A_p at the points of U, such that: (i) $f(p) \in A_p$; (ii) locally, i.e. on some neighborhood V of any point $p \in U$, f coincides with a fraction a/b , a,b ∈ A and b does not belong to any ideal q contained in V.

By carefully reading this definition, one sees that it is the natural counterpart of Definition (2.3). One obtains in this way a sheaf $\mathcal{O}_A$ on Spec(A); its stalk at any point p is isomorphic to A_p. It is easy to check that the ring of global sections $\mathcal{O}(\mathrm{Spec}(A))$ is isomorphic to A itself. This procedure will be generalized in Sect. (2.4) to build the sheaf of sections of a line bundle on Spec(A) starting from a rank one A-module M.

To complete the picture of the correspondence between geometric and algebraic data, let me recall which is the algebraic counterpart of the notion of *dimension* and *singular locus* of a variety. In a ring A, the *height* of a prime ideal p is the maximal integer n such that there exist n prime ideals $p_1, \ldots, p_n$, with the property $p_1 \subset p_2 \subset \ldots \subset p_n \subset p$. The *Krull dimension* of a ring is the maximal height among all prime ideals. One can prove that the dimension of an affine variety equals the Krull dimension of its coordinate ring. Furthermore, take a point p on an affine variety V and consider the maximal ideal m_p defined above. The quotient m_p/m_p^2 is a vector space over $\mathbb{C}$, which is the algebraic counterpart to the tangent space to V in p. It can be proved that, according to the common geometrical intuition, the dimension over $\mathbb{C}$ of m_p/m_p^2 equals the dimension of V, i.e. the Krull dimension of $A(V)$, if and only if p is a nonsingular point.

So far I have been dealing with the reconstruction of an affine variety from a commutative ring; now I will briefly mention that a projective variety can be built by starting from a graded ring, by taking the space of homogeneous prime ideals. Namely, let $G = \oplus_{i \in \mathbb{N}} G_i$ a commutative graded ring; one defines $G_+ = \oplus_{i>0} G_i$ (G_+ is a homogeneous prime ideal, i.e. it is generated by homogeneous elements $a_i \in G_i$, $i>0$).

(2.10) **Definition** : The *projective spectrum* of a graded ring G is defined (as a set) by

$$\mathrm{Proj}(G) \equiv \{ p \subset G \,/\, p \text{ is a homogeneous prime ideal not containing } G_+ \} .$$

It is evident that Proj(G) ⊂ Spec(G), so one can endow Proj(G) with the relative Zariski topology of Spec(G). A construction which is similar to the previous one allows to define the structure sheaf on Proj(G). Further details can be found in [24] or [26] but will not be needed in the sequel.

For evident reasons, I cannot present here a complete exposition of the theory of algebraic varieties. In particular, I did not prove that Spec(A), or rather the set of its closed points, is an affine algebraic variety in the sense of Definition (2.1). As a matter of fact, I have implicitly claimed that a variety can be defined by the datum of the sheaf of regular functions. I will not discuss this point: for my purposes, it is enough to know that there is an homeomorphism between the underlying topological space of a variety V and the set of closed points of Spec($A(V)$), and this homeomorphism pulls back regular functions to regular functions (according to the respective definitions); this can be seen from the statements above. Therefore, whenever an abstract ring A can be realized as a quotient $\mathbb{C}[x_1, ..., x_n]/\mathfrak{I}$ for some ideal $\mathfrak{I}$, the above cosiderations allow to identify (the set of closed points of) Spec(A) with the variety defined as being the zero locus of $\mathfrak{I}$.

2.3 Divisors, Line Bundles, Jacobian Variety.

In Part B, I will make use of a number of standard notions concerning the geometry of algebraic curves. However, it would seem unreasonable to present here all the details and proofs, which can be easily found in any basic textbook on the subject ([27], [28]). Therefore, I simply list below, for the reader's convenience, the main definitions and properties which will be mentioned in the sequel.

I will assume in this Section a more "geometrical" approach to algebraic curves; in other words, let me regard a (nonsingular) algebraic curve C as a complex manifold with a holomorphic atlas. A *(holomorphic) line bundle* over C is a complex vector bundle of rank one defined by holomorphic transition functions.

(2.11) **Definition** : A *divisor* $\mathcal{D}$ on C is a finite formal linear combination, with integer coefficients, of points of C : $\mathcal{D} = \sum_{k\in\mathbb{Z}} a_k p_k$, $p_k \in C$. The degree of $\mathcal{D}$ is the sum of its coefficients: $\deg(\mathcal{D}) = \sum_k a_k$. An *effective* divisor is a divisor with only positive coefficients.

To each meromorphic function f on C is associated a divisor, denoted by $(f)= \sum_k a_k p_k$, where each p_k is either a zero or a pole of f, and a_k is either the order of the zero, or minus the order of the pole. Such a divisor is said to be *principal*. A principal divisor has always degree zero. The set of divisors of degree zero has a natural structure of additive group, and is denoted by $\mathrm{Div}_0(C)$. One can define an equivalence relation, called *linear equivalence*, by setting $\mathcal{D}_1 \sim \mathcal{D}_2$ iff $\mathcal{D}_1 - \mathcal{D}_2$ is principal. The *Picard group* $\mathrm{Pic}^0(C)$ is the quotient $\mathrm{Div}_0(C)/\sim$.

Let $H_1(C)$ be the first (singular-) homology group of C (as a topological manifold). The *genus* g of C is half the rank of $H_1(C)$ (which is always even). In the sequel I will deal with the case $g > 0$. Let $\{a_i, b_i\}$ (i =1, ..., g) a *symplectic basis* for $H_1(C)$, that is a system of generators such that the intersection matrix $\|a_i . b_j\|$ is the symplectic matrix $J \equiv \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$. Such a basis always exists on C.

One can choose a set $\{\omega_i\}$ (i =1, ...,g) of independent holomorphic one-forms (also called *abelian differentials of the first kind*) such that $\langle\omega_n, a_m\rangle = \delta_{mn}$ ($\langle,\rangle$ denoting integration of a one-form over a one-cycle).

(2.12) **Definition** : The *period matrix* Ω of C is the ($g \times g$)-matrix $\|\langle\omega_n, b_m\rangle\|$.

The rows of Ω define independent vectors in $\mathbb{C}^g$; let Λ denote the lattice generated by these vectors.

(2.13) **Definition** : The *Jacobian torus* Jac(C) is the abelian group $\mathbb{C}^g/\Lambda$.

Jac(C) has a structure of algebraic variety. Fixing a basepoint p_0 on C, one can define a mapping from C to Jac(C) by setting:

(2.14) **Definition** : The *Abel map* $\mathfrak{A} : C \rightarrow \mathrm{Jac}(C)$ is defined by

$$\mathfrak{A}(p) = \left(\int_{p_0}^{p} \omega_i\right) e_i \qquad \mathrm{mod}\ \Lambda$$

($\{e_i\}$ being the canonical basis in $\mathbb{C}^g$). In fact, different choices of the path from p_0 to p lead to the same equivalence class in $\mathbb{C}^g/\Lambda$. The Abel map extends by linearity to divisors on C :

(2.15) $$\mathcal{A}(\mathcal{D}) = \sum_k a_k \left(\int_{p_0}^{p_k} \omega_i\right) e_i \qquad \text{mod } \Lambda$$

In particular, a classical theorem states that $\mathrm{Pic}^0(C)$ and $\mathrm{Jac}(C)$ are isomorphic as abelian varieties.

An important object related to Jac(C) is the Riemann θ-function:

(2.16) **Definition** : $\theta(z) = \sum_{v \in \mathbb{Z}^g} [\, \exp(\, i\pi(v, \Omega v)\,) \cdot \exp(\, 2\pi i(v, z)\,)\,]$,

where (,) stands for the standard euclidean scalar product in $\mathbb{C}^g$. $\theta(z)$ is not periodic mod Λ (a periodic holomorphic function would be identically constant), so it does not define a function on Jac(C); however, it can be thought as a section of a line bundle on Jac(C), obtained by taking a suitable quotient (mod Λ) of the trivial bundle $\mathbb{C}^g \times \mathbb{C}$.

Going back to divisors on C, a relevant point is the correspondence between divisors and line bundles. On one hand, to each line bundle L on C one can associate the divisor of any one of its meromorphic sections (the divisor of a section of L is defined in the same way as for functions on C): this map is well defined modulo linear equivalence, since two sections of L always differ by a meromorphic function. Conversely, let $\mathcal{D}$ be a divisor; one can cover C by open sets U_α, and find a local meromorphic function f_α on each U_α such that (f_α) coincides with the restriction to U_α of $\mathcal{D}$. The functions $g_{\alpha\beta} \equiv f_\alpha/f_\beta$ are thus holomorphic on $U_\alpha \cap U_\beta$ and define the transition functions of a line bundle $L(\mathcal{D})$. Finally, one can check that, for any meromorphic section s of L,

$L \cong L((\mathcal{D}))$. By definition, the degree of a line bundle is the degree of its associate divisor.

The cohomology of sheaves of sections of line bundles on a curve C is one of the classical subjects of algebraic geometry. A fundamental result is the *Riemann-Roch formula*:

(2.17) **Proposition** (Riemann-Roch) : Given a curve C of genus g and an effective divisor $\mathcal{D}$, the zeroth and the first cohomology spaces of the sheaf $\mathcal{L}(\mathcal{D})$ of sections of $L(\mathcal{D})$ are related by the following equality:

$$\dim \mathrm{H}^0(C, \mathcal{L}(\mathcal{D})) - \dim \mathrm{H}^1(C, \mathcal{L}(\mathcal{D})) = \deg(\mathcal{D}) + 1 - g \quad .$$

In particular, for meromorphic functions on C the following result holds: if $\ell(\mathcal{D})$ ($\mathcal{D}$ a *generic* effective divisor), is the dimension of the space of meromorphic functions f such that (f) $+\mathcal{D}$ is effective or zero, then $\ell(\mathcal{D}) = 1$ if $\deg(\mathcal{D}) \leq g$, while $\ell(\mathcal{D}) = \deg(\mathcal{D}) + 1 - g$ if $\deg(\mathcal{D}) \geq g$ (note that also the constant functions are taken into account). The divisors of degree less than g for which $\ell(\mathcal{D})$ is greater than one are said to be *special*. The points on C which are special divisors are called *Weierstrass points*. If C is *hyperelliptic*, i.e. its defining equation has the form $y^2 = \mathrm{F}(x)$, F being some polynomial of degree $2g + 1$, the Weierstrass points coincide with the branching points of the function y.

2.4 $\mathcal{O}(C)$-Modules and Line Bundles

The reconstruction of a line bundle, starting from a commutative ring A and a rank-one A-module M, involves two steps. First, one defines the sheaf associated to M on Spec(A), which I will denote by $\mathcal{M}$; then, it is possible to associate to $\mathcal{M}$ an algebraic variety which is fibered over Spec(A). This variety turns out to be a line bundle $L(\mathcal{M})$; its sheaf of sections is isomorphic to $\mathcal{M}$, while the original A-module M is isomorphic to the A-module of the global sections of $L(\mathcal{M})$.

The first step is analogous to the construction of the structure sheaf on Spec(A). For the moment, let me assume more generally that M has rank n. One considers the prime ideals $p \in$Spec(A) and define the *localization* M_p of the module M at a prime ideal p to be the ring formed by the equivalence classes of fractions m/a, where m$\in$M, a$\in$A and a$\notin$ p, m/a being equivalent to m'/a' if b $\in$ A exists such that b (m'a-ma') = 0 in M. M_p is an A_p-module with the scalar multiplication (a/b)·(m/c) = (ma/bc).

(2.18) **Definition** : The sheaf $\mathcal{M}$ is obtained by associating to each open set $U \subset$ Spec(A) the set of functions $s : U \to \coprod_{p \in U} M_p$ such that: (i) $(p) \in M_p$; (ii) s is locally a fraction m/a (in a sense analogous to definition (2.9)).

(2.19) **Proposition :** (i) $\mathcal{M}$ is a sheaf of modules over the structure sheaf of Spec(A);

(ii) For each p, the stalk $\mathcal{M}_p$ is isomorphic to M_p.

(iii) The module of global sections Γ(Spec(A), $\mathcal{M}$) is isomorphic to M.

I will not give the proof of (2.19) (see [24]). Now comes the second step:

(2.20) **Definition** : For k>0, let $S^k M$ denote the symmetrized k-fold tensor product of M with itself. The symmetric algebra S(M) is the direct sum $\oplus_{k\geq 0} S^k M$, $S^0 M$ being identified with A.

In the general case in which M is free and of rank n , $S(M) \cong A[x_1, ..., x_n]$.

M being a free A-module, $\mathcal{M}$ is locally free by construction, so that for each open set $U \subset Spec(A)$ on which $\mathcal{M}$ is free, I can choose a basis for $\mathcal{M}(U)$. Let now $\mathcal{S}_M$ be the sheaf on Spec(A) associated to the module S(M); from the considerations above it follows:

(2.21) **Proposition** : $Spec(\mathcal{S}_M(U)) \cong Spec(\mathcal{O}_A(U) [x_1, ..., x_n])$.

Starting from a covering $\{U_\alpha\}$ of Spec(A), one can "glue" the spaces $Spec(\mathcal{S}_M(U_\alpha))$ by a standard technique (see [24]). The resulting object is denoted by **Spec**(S(M)), and is endowed with local trivializations provided by the identification (2.21). In this way one obtains a vector bundle of rank n: the fiber at a closed point m of Spec(A) turns out to be isomorphic to $Spec((\mathcal{O}_{A,m}/m)[x_1, ..., x_n]) \cong Spec(\mathbb{C}[x_1, ..., x_n]) \cong \mathbb{C}^n$. Therefore, if one stars from an A-module of rank one, one gets a line bundle.

The above construction should in principle be generalized to projective curves, to be really suited to the future discussion. However, a fundamental property of projective varieties is that they can be covered by affine open sets; one can therefore apply locally the construction above and then glue together the affine pieces (the non-triviality of the line bundle so obtained lies just on this glueing procedure).

3. THE BAKER-AKHIEZER FUNCTIONS

Let C be a smooth compact algebraic curve of genus g, with a distinguished point p and a local coordinate k in a neighborhood of that point. Since the point p is identified in general with the "point at infinity", it is common to use the local function $z = k^{-1}$ instead of the parameter k to express the local dependence of a function on C in a neighborhood of p.

(3.1) **Definition** : A *Baker-Akhiezer function* ψ is a function on C having an essential singularity at p such that, in a neighborhood of p, $\psi \sim \exp[Q(z)]$ for some polynomial $Q \in \mathbb{C}[z]$, and being meromorphic on the affine part $C \setminus p$, with exactly g poles.

The Baker-Akiezer functions provide solutions of the spectral problems listed in Sect. (1.2), as it will be shown in Sect. 5 (see [13], [29]). For the moment, let me only anticipate two points. First, let me assume that a differential operator L (with constant coefficients) exist such that the BA function ψ is the solution of the eigenvalue problem

$$L \psi = \lambda \psi. \tag{3.2}$$

If one wants to study a deformation of L, by allowing its coefficients to depend on a number of parameters $(x, t_2, t_3, \ldots, t_n)$, one is led to deal with a family of BA functions ψ, depending on the same parameters through the condition (3.2). I will regard such a family as a single object $\psi(x, t_2, t_3, ..., t_n)$. An useful parametrization is provided by the coefficients of the polynomial Q(z) expressing the singular behavior at p, so that:

$$(3.3) \qquad \psi(x, t_2, t_3, ...; z) = \exp(xz + t_2 z^2 + t_3 z^3 + ...) \cdot \sum_{k \in \mathbb{N}} a_k (x, t_2, ...) \, z^{-k} \quad .$$

In general, ψ is normalized by requiring $a_0 \equiv 1$. The degree of Q(z), and consequently the number of parameters, is directly related with the number of equations defining the spectral problem.

The second remark is the following. When dealing with spectral problems, one is often led to *formal* expressions of the type (3.3). These *formal Baker-Akhiezer functions* (whereby z plays the role of *spectral parameter*) do solve the spectral problem, but there are in principle no indications about the actual convergence of the series on the right-hand side of (3.3). As a matter of fact, if one started with *given* differential operators, the convergence of ψ can be ensured by the functional-analytic properties of the operators themselves; but the problem I will deal with consists in *finding* the differential operators which solve the compatibility equations for the spectral problem. Therefore, I wish to obtain both the differential operators and the BA function at the same time; the whole algebro-geometrical approach is actually intended to provide methods to construct these objects from geometrical data, and ensure in that way that the solutions are not just formal ones. Thus, the future strategy will consist mainly in trying to identify formal solutions with well-defined BA functions over suitable *spectral curves*.

4. THE UNIVERSAL GRASSMANNIAN MANIFOLD

4.1 The Finite-Dimensional Grassmannian

The fourth block of the ideal diagram underlying this thesis concerns the Universal Grassmannian Manifold. Since this is, in some sense, a generalization of the usual notion of finite-dimensional Grassmannian, I begin by recalling the definition and some relevant features of the latter one. For the sake of simplicity, I will assume that the vector spaces involved in the definitions are vector spaces over $\mathbb{C}$ rather than over a generic field $\mathbb{K}$.

(4.1) **Definition** : The *Grassmannian* Gr(n, k) is the set of linear k-dimensional subspaces of $\mathbb{C}^n$. It can be described as the quotient

$$\mathrm{Gr}(n, k) \equiv \{\text{k-frames in } \mathbb{C}^n\}/GL(k, \mathbb{C})$$

(a *k-frame* is a system of k independent vectors). It is easy to see that $GL(n, \mathbb{C})$ acts transitively on Gr(n, k). I will show that Gr(n, k) is an algebraic variety by embedding it in $\mathbb{P}\mathbb{C}^m$, $m=\binom{n}{k}-1$. To each k-frame $(v_1, \ldots, v_k)$ one associates the exterior product $v_1\wedge v_2\wedge \ldots \wedge v_k \in \wedge\mathbb{C}^n \cong \mathbb{C}^{m+1}$; needless to say, this exterior product is independent of the action of $GL(k, \mathbb{C})$, up to a multiplicative factor. Therefore, we can assume each point of Gr(n, k) to correspond to a point in $\mathbb{P}\mathbb{C}^m$. By choosing a basis $\{e_i\}$ in $\mathbb{C}^n$ and setting $v_i = \sum_j v_i^j e_j$, one can write

(4.2) $$v_1\wedge \ldots \wedge v_k = \sum_{1\leq i_1<\ldots<i_k<n} w_{i_1\ldots i_k}\, e_{i_1}\wedge \ldots \wedge e_{i_k} \;;$$

denoting the coordinates $w_{i_1 \cdots i_k}$ by w_I, $I \equiv (i_1, ..., i_k)$, one has $w_I = \det \| v_i^j \|$, $i \in I$, $j = 1, ..., k$.

The set of coordinates w_I allows to write explicitly the defining equations of the (image of) the Grassmannian Gr(n, k) in $\mathbb{P}\mathbb{C}^m$. In fact, the point identified by the coordinates w_I lies in the image of Gr(n, k) iff the multivector $v = \sum_I w_I e_I \in \wedge \mathbb{C}^n$ is decomposable, i.e. it is of the form $v = v_1 \wedge \ \dots \wedge v_k$. It is possible to prove (see [27]) that the condition of being decomposable yields the following coordinate expression:

$$(4.3) \qquad \sum_{1 \le p \le k+1} w_{i_1 \dots i_{k-1} i_p} \, w_{j_1 \dots (j_p) \dots j_{k+1}} = 0$$

(where (.) means that the corresponding index has to be suppressed). These are the well-known *Plücker relations.*

The same embedding of Gr(n, k) into $\mathbb{P}\mathbb{C}^m$ can be described in a more abstract way: in fact, there is a general method of embedding into a projective space $\mathbb{P}\mathbb{C}^r$ any algebraic variety endowed with an invertible sheaf $\mathcal{L}$, provided $\mathcal{L}$ is generated by (r+1) global sections. In the case of Gr(n, k), one first define the *universal* (or *tautological*) *bundle* S(n, k) to be the subbundle of the trivial bundle Gr(n, k) × $\mathbb{C}^n$, obtained by setting the fiber over a point p ∈ Gr(n, k) to be just the k-plane in $\mathbb{C}^n$ corresponding to p. To this vector bundle one further associates the *determinant bundle* Det S(n, k) , simply by taking the k-th exterior power of each fiber of S(n, k): the transition functions of the determinant bundle so obtained coincide with the determinant of the transition functions of S(n, k). One can prove that the sheaf of sections of the *dual* of the determinant bundle, Det*S(n, k), is generated by the $\binom{n}{k}$ sections associating to each point of Gr(n, k) its Plücker coordinates w_I. Thus, this sheaf provides the embedding described above.

4.2 Sato's Definition of UGM

M. and Y. Sato's approach to the Universal Grassmannian Manifold ([30]) relies mainly on the generalization of the Plücker embedding. Let me consider the system of inclusions

(4.4) $$\ldots \subset Gr(n, k) \subset Gr(n', k') \subset \ldots \quad ,$$

for k'>k and n'-k'>n-k. These inclusions are obtained by identifying $\mathbb{C}^n$ with the subspace of $\mathbb{C}^{n'}$ spanned by the first n vectors of the canonical basis $\{e_i\}$, and then mapping a k-space into a k'-space by taking the direct sum with the linear span of the vectors $\{e_j\}$ (j=n+1, ..., n+k'-k). This is represented, in terms of the Plücker coordinates, by setting

(4.5) $$(w_{i_1 \ldots i_k}) \longmapsto (w_{i_1 \ldots i_k j_1 \ldots j_{k'-k}}) \ , \quad j_p \equiv n+p \ .$$

And assuming the remaining coordinates w_I to be equal to zero. One can check that the Plücker relations are preserved. The inverse limit of the sequence of inclusions (4.4) can thus be represented as follows:

(4.6) **Definition** : Let $Gr \equiv \{ \{w_I\} / \{w_I\}$ fullfill the Plücker relations$\}$, where I runs over all ordered subsets of $\mathbb{N}$ (such subsets will be called *partitions*). The Universal Grassmannian Manifold is the quotient

$$\mathrm{Gr} \equiv (Gr - \{0\}) / \mathbb{C}^{\times} \ .$$

Subtracting the point with identically zero coordinates and taking the quotient by $\mathbb{C}^\times$ correspond to regarding the w_I as homogeneous coordinates.

Let V° be the subspace of $\mathbb{C}^{n'}$ spanned by the k' vectors $\{e_i\}$, $i = n'-k',...,n'$. From the definition of the inclusions (4.4), one sees directly that the image of a k-plane of $\mathbb{C}^n$ is contained in the set of k'-spaces whose projection on V° has a kernel of dimension k at most; the cokernel has obviously the same dimension. Therefore, since the dimension of the kernel and the cokernel above are independent of n' and k', passing to the limit for $n',k'\to\infty$, one can set the following alternative definition:

(4.7) **Definition** : Let **V** be the infinite-dimensional space generated by the vectors $\{e_i\}$, $i\in\mathbb{Z}$, **V**° be the subspace spanned by $\{e_j\}$, $j<0$, and $\pi°$ be the orthogonal projection on **V**°. Gr is the set of the subspaces W of **V** such that

$$\dim(\ker \pi°|_W) = \dim(\operatorname{coker} \pi°|_W) < \infty .$$

Let me remark that the space **V**° has been defined here by taking the basic vectors with *negative* index. In the next Section I will adopt Segal & Wilson's viewpoint, and the space H_+, playing the same role as **V**° here, will be the one generated by vectors with *positive* index. The occurrence of the two conventions in the literature is somewhat confusing, although it is always easy to realize which one has been adopted in a particular paper.

4.3 Segal-Wilson's Approach to UGM

In their well-known paper [13], G. Segal and G. Wilson adopt a definition of the UGM which is close to (4.7). However, their approach relies mostly on functional-analytic and algebro-geometrical arguments, while Sato, and more generally the Kyoto group, tend to stress the purely algebraic side of the construction.

According to Segal & Wilson, the Grassmannian Gr(H) can be defined for an arbitrary separable complex Hilbert space H, with a suitable decomposition $H=H_+\oplus H_-$; however, in the applications one usually deals with $H \equiv L^2(S^1)$, so I will restrict the definitions to this case. Let S^1 be identified with the unit circle $|z|=1$ in $\mathbb{C}$, and consider the orthonormal basis of $L^2(S^1)$ provided by the functions $\{z^k\}_{k\in\mathbb{Z}}$. $H \equiv L^2(S^1)$ can be splitted in two closed, infinite-dimensional orthogonal subspaces $H_+\equiv\mathbb{C}\{z^k\}_{k\geq 0}$ and $H_-\equiv\mathbb{C}\{z^k\}_{k<0}$. For an arbitrary subspace $W \subset H$, let π_+ and π_- denote the orthogonal projection operators from W to H_+ and to H_- respectively.

(4.8) **Definition** : The Grassmannian Gr(H) is the set of all closed subspaces $W \subset H$ such that π_+ is a Fredholm operator and π_- is a compact operator.

Let me recall that a Fredholm operator has by definition finite-dimensional kernel and cokernel; the *index* of the operator is the difference between these dimensions. According to this definition, the index of π_+ is not required to be zero, as it was in (4.7). Gr(H) is split into connected components labeled by the index of π_+, (this index is also called *virtual dimension* of the space W). For many purposes, it is enough to deal with the connected component of the spaces with zero virtual dimension.

The requirement on π_- does not occur in Sato's purely algebraic construction, since this requirement is mainly intended to define the functional-analytic features of the manifold Gr(H). Namely, requiring π_- to be compact allows to give Gr(H) the structure of a Banach manifold, while other choices would lead only to a Fréchet manifold. In some situations, it is advisable to require π_- to be an Hilbert-Schmidt operator, so that Gr(H) becomes an Hilbert manifold. To see how the restrictions on π_- affect the structure of Gr(H), it is enough to observe that each point W of the "compact" (resp. "Hilbert-Schmidt") Grassmannian has a neighborhood $U(W) \equiv \{W' \subset Gr(H) \,/\, W'$ is the graph of a compact (resp. Hilbert-Schmidt) operator $W \to W^{\perp}\}$. Such neighborhoods are diffeomorphic to the Banach (resp. Hilbert) space of compact (resp. Hilbert-Schmidt) operators $H_+ \to H_-$, which therefore provides a local model for the analytic structure of Gr(H).

Due to the restrictions on the projections $\pi_\pm$, the group GL(H) does not act on Gr(H); instead, its subgroup $GL_{res}(H)$ acts transitively. $GL_{res}(H)$ is defined as follows: assume that $g \in GL(H)$ has a block decomposition, with respect to the splitting $H=H_+ \oplus H_-$, of the form $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$; $g \in GL_{res}(H)$ iff b and c are compact operators (a and d are then automatically Fredholm). In view of the relation with the KP flows, the following subgroups of $GL_{res}(H)$ are very relevant:

(4.9) **Definition** : Let Γ be the group of continuous maps g: $S^1 \to \mathbb{C}^\times$, and Γ_+ be the subgroup formed by the maps g which extend to holomorphic functions on a neighborhood of the unit disk $|z| \leq 0$, with the condition $g(0)=1$. Γ and Γ_+ can be viewed as subgroups of $GL_{res}(H)$, by letting their elements act as multiplication operators on H.

Each element $g \in \Gamma_+$ admits a representation of the form

(4.10) $$g = \exp [\Sigma_{i>0} t_i z^i] \quad ;$$

the coefficients $(t_1, t_2, ...)$ provide an useful parametrization of Γ_+ ; in the sequel I will often write x instead of t_1 .

It is convenient to consider the following typical example of spaces W belonging to Gr(H): let S be an ordered sequence $(s_1, s_2, ...)$ of integer numbers, such that $s_k = k$ for all but a finite number of values of k. The space H_S spanned by the functions $\{z^k\}_{k\in S}$ belongs to Gr(H) and has virtual dimension zero. In fact, the kernel and the cokernel of π_+ are spanned by the negative powers $\{z^k\}_{k\in S,k<0}$ and by the "lacking" positive powers $\{z^k\}_{k\notin S,k>0}$ respectively, so they have the same finite dimension. To obtain a space of virtual dimension d, one should take a sequence S for which $s_k = k\text{-}d$ holds for almost all k). The spaces H_S are good models for the points of the Grassmannian, since for each $W \in Gr(H)$ there exists a sequence S such that W is the graph of a compact operator $H_S \to H_S^{\perp}$. This is expressed by saying that W is *transverse* to $H_S^{\perp}$: it is evident that in this case $W \cong H_S$ as Hilbert spaces. One can prove in this way that in any space W the *elements of finite order* $\Sigma_{k<n} a_k z^k$ are dense, since they are dense in H_S.

In view of the connection with KP flows, the following definition is very important:

(4.11) **Definition** : The subspace $Gr^{(n)} \subset Gr(H)$ is the closed subspace formed by the elements such that $z^n W \subset W$ (z^n acting as a multiplication operator).

Not every point W of the Grassmannian belongs to such a subspace. For the spaces H_S it is easy to check which are the values of n for which $H_S \in Gr^{(n)}$; for instance, let $S = (-2, -1, 1, 2, 4, 5, 6, \ldots)$; then H_S belongs to $Gr^{(3)}$ and to any $Gr^{(n)}$ for $n \geq 5$. It is quite evident that $W \in Gr^{(n)}$ implies $W \in Gr^{(kn)}$, for any $k \in \mathbb{N}$.

To each point W one can associate a commutative ring according to the following definition:

(4.12) **Definition** : The ring A_W is the ring of analytic functions $f: S^1 \to \mathbb{C}$ such that $f \cdot W^{alg} \subset W^{alg}$, W^{alg} being the set of elements of finite order in W.

Of course, $W \in Gr^{(n)}$ and $z^n \in A_W$ are equivalent statements. For the spaces H_S it is very easy to find generators for A_W ; I will discuss an example in Part B, where I will show how to associate algebraic curves to points of Gr(H).

Let me now describe how one can recover a space W belonging to some $Gr^{(n)}$ from the knowledge of a sufficient number of independent functions belonging to it. To this aim it is important to observe that a Grassmannian $Gr(H^n)$ can be defined, analogously to the case of Gr(H), starting from the Hilbert space H^n of the (L^2-integrable) vector-valued functions $\mathbf{f} : S^1 \to \mathbb{C}^n$. Letting $\{e_i\}_{1 \leq i \leq n}$ be the canonical basis in $\mathbb{C}^n$, a basis for H^n is provided by the functions $\{z^k e_i\}_{k \in \mathbb{Z}}$. One can set a correspondence between H^n and H by associating to each element $z^k e_i$ the element z^{nk+i-1} of the basis of H. More generally, to each vector-valued function $\mathbf{f} = \sum_i f_i(z)\, \mathbf{e}_i$ one can associate the scalar function

$$f = \sum_i f_i(z^n)\, z^{i-1} \qquad (4.13)$$

Conversely, let $\zeta_1, ..., \zeta_n$ be the n-th roots of z; to an element f ∊ H corresponds the function $\mathbf{f} = \sum_i f_i(z)\, \mathbf{e}_i$, where

(4.14) $$f_i(z) = n^{-1} \sum_k (\zeta_k)^{-i} f(\zeta_k) \ .$$

Under this correspondence, $Gr^{(n)}$ is the image of the subspace of $Gr(H^n)$ formed by elements $W^{(n)}$ such that $z\, W^{(n)} \subset W^{(n)}$. It is now easy to see that such elements are completely determined by the generators of the n-dimensional quotient space $W^{(n)}/(z\, W^{(n)})$.

This allows to associate in an unique way to n independent functions $\psi_1, ..., \psi_n \in H$ a space $W \in Gr^{(n)}$, which contains them. In fact, one considers the vector-valued function $\boldsymbol{\psi}_k = \sum_i \psi_k{}^i(z)\, \mathbf{e}_i$ associated to each ψ_k and forms the matrix

(4.15) $$\gamma = \| \psi_k{}^i \| = \begin{pmatrix} 1 & \dots & 1 \\ (\zeta_1)^{-1} & \dots & (\zeta_1)^{-n+1} \\ \dots & \dots & \dots \\ (\zeta_n)^{-1} & \dots & (\zeta_n)^{-n+1} \end{pmatrix} \begin{pmatrix} \psi_1(\zeta_1) & \dots & \psi_n(\zeta_1) \\ \psi_1(\zeta_2) & \dots & \psi_n(\zeta_2) \\ \dots & \dots & \dots \\ \psi_1(\zeta_n) & \dots & \psi_n(\zeta_n) \end{pmatrix}$$

At this point, a space $W^{(n)} \in Gr(H^n)$ is provided by $W^{(n)} = \gamma H^{(n)}{}_+$, $H^{(n)}{}_+$ being the space spanned by $\{z^k \mathbf{e}_i\}_{k \geq 0}$. Under the map $H^{(n)}{}_+ \mapsto \gamma H^{(n)}{}_+$, the constant functions $\mathbf{e}_i$ become just the functions $\boldsymbol{\psi}_i$; therefore, the space W ∊ Gr(H), obtained from $W^{(n)}$ according to the prescription (4.13), will contain the functions ψ_k. Of course, the procedure fails if the functions ψ_k are such that γ has determinant equal to zero; the necessary condition is thus the regularity of the matrix $\|\psi_i(\zeta_k)\|$, where the ζ_k are related by the condition $(\zeta_k)^n = z$.

4.4 The Dual Determinant Bundle and the τ-Function

The geometric meaning of Hirota's τ-function is connected with the generalization to Gr(H) of the Plücker embedding. For each point $W \in Gr(H)$, consider pairs $(\mathbf{w}, \lambda)$ formed by a basis $\mathbf{w} \equiv \{w_k\}$ for W and a complex number λ.

(4.16) **Definition** : The dual determinant bundle Det* on Gr(H) is defined by assigning to each point $W \in Gr(H)$ the one-dimensional linear space formed by the equivalence classes of pairs $(\mathbf{w}, \lambda)$ under the following equivalence relation: $(\mathbf{w},\lambda) \sim (\mathbf{w}', \lambda')$ iff $\mathbf{w}'=T\mathbf{w}$ (T being a linear operator which differs from the identity by a trace-class operator) and $\lambda' = (\det T)\lambda$.

I omit the explicit description of the transition functions of Det*, which would involve a number of technical points which are not directly relevant for the present purposes. The condition on the operator T must be imposed in order T to have a well-defined determinant; furthermore, since not any two bases $\mathbf{w}$, $\mathbf{w}'$ are related by such an operator, one should also restrict the possible choice of the basis. In order to do this, one can use the property, stated in the previous Section, that each $W \in Gr(H)$ is the graph of an operator $H_S \to H_S{}^{\perp}$, for some sequence S as in the last Section (there is a unique such S which is minimal with respect to a suitable ordering); one can therefore define a reference basis $\mathbf{w}^{\circ}$ to be the preimage, under the projection $W \to H_S$, of the standard basis $\{z^k\}_{k \in S}$ of H_S. The allowed representatives $(\mathbf{w}, \lambda)$ are those for which the basis $\mathbf{w}$ can be obtained from $\mathbf{w}^{\circ}$ by applying an operator T having a determinant ($\mathbf{w}$ will then be called an *admissible basis*).

The role of Plücker coordinates is played by the following sections of Det* :

(4.17) **Definition** : For any $W \in Gr(H)$, let $\pi_S[W]$ be the projection operator $\pi_S[W]$: $W \rightarrow H_S$. The *Plücker coordinates* in Gr(H) are the sections w_S of Det* defined by setting $w_S: W \mapsto$ (w, det $\pi_S[w]$) ,**w** being an admissible basis for W and $\pi_S[w]$ standing for the matrix form of the operator $\pi_S[W]$ with respect to the basis **w** (and to the standard basis in H_S).

This observation follows from the definitions:

(4.18) **Proposition** : The sections w_S can be identified with the coordinates w_I introduced in Sect. 4.2 by associating to each sequence S the partition $I \equiv (i_1, i_2, ...)$, $i_k = k - s_k$. Each Plücker coordinate $w_S(W)$ is different from zero iff W is transverse to $H_S{}^{\perp}$.

I shall now consider the section $\sigma \equiv w_{\mathbb{N}}$, i.e. the determinant of the projection π_+. In particular, let me examine the action on σ of the group Γ_+ defined by (4.9). It is useful to this purpose to parametrize elements $g \in \Gamma_+$ by the coefficients h_i of the series expansion

$$g = 1 + \Sigma_{i>0}\, h_i\, z^i \quad . \tag{4.19}$$

The action of g on the admissible basis **w** is simply given by $\{w_k\} \mapsto \{g \cdot w_k\}$. For example, for $W \equiv H_+$, one has $\{z^k\} \mapsto \{z^k + \Sigma_{i>0}\, h_i\, z^{i+k}\}$; the projection $g \cdot H_+ \rightarrow H_+$ is then expressed by the upper triangular matrix $G \equiv \| h_{j-i} \|$ (having set $h_0 \equiv 1$). This matrix has determinant equal to one, therefore for H_+ one has $\sigma(g\cdot H_+)= g\cdot\sigma(H_+)$, since the left-hand and right-hand sides of this equality are represented can be represented by $(\{g\cdot w_k\}, \det G)$ and $(\{g\cdot w_k\}, 1)$ respectively. However, on a generic point W , $\sigma(W)$ is not equivariant under the action of Γ_+. Let me consider the case of a space $H_S \neq H_+$. The standard basis is indeed an admissible basis; now, $\sigma(H_S)$ should vanish, according to Proposition 4.18, since H_S is not transverse to H_-, while the projection $g\cdot H_S \to H_+$ is expressed by the matrix $G_S \equiv \|h_{j-s_i}\|$, and it is easy to see that the determinant of this matrix is equal to the determinant of its $(r \times r)$ upper left block, r being the maximum integer for which $s_k \neq k$, and this determinant does not vanish in general. Therefore, for an arbitrary point W, $\sigma(g\cdot W) \neq g\cdot\sigma(W)$.

(4.20) **Definition** : For each $W \in Gr(H)$, the function $\tau_W : \Gamma_+ \to \mathbb{C}$ is defined (up to a constant factor) by setting:

$$\tau_W(g) \equiv \frac{\sigma(g^{-1}\cdot W)}{g^{-1}\cdot\rho} \quad ,$$

ρ being an arbitrary nonzero element of the fiber of Det* over W. If W is transverse to H_-, $\sigma(W)$ does not vanish and one can choose $\rho \equiv \sigma(W)$. In that case the τ-function can be regarded as properly expressing the "defect of equivariance" of the action of Γ_+ on the section $\sigma(W)$:

(4.21) $$\tau_W(g)\, g^{-1}\cdot\sigma(W) = \sigma(g^{-1}\cdot W) \quad .$$

If W is transverse to $H_S{}^\perp$, one can choose $\rho \equiv w_S(W)$. With this choice, one easily finds:

(4.22) **Proposition** : For any sequence S, τ_{H_S} coincides with the determinant of the finite matrix $\|h_{j-s_i}\|$ described above, having set

$$g^{-1} = 1 + \sum_{i>0} h_i z^i \ .$$

Such a determinant is known in the literature as the *Schur function* $F_I(h)$, I being the partition associated to S as in (4.18). The Schur functions were originally introduced within the group representation theory, as character polynomials of the irreducible representations of GL(n, $\mathbb{C}$). The authors from the Kyoto school have often emphasized this fact in their papers, while Segal & Wilson consider this connection as an irrelevant coincidence. This clearly reflects the different attitude of the various authors towards the subject of the UGM.

For a generic point $W \in Gr(H)$, the τ-function can be expressed in terms of its Plücker coordinates; namely, one finds

(4.23) $$\tau_W(g) = \sum_S w_S(W)\, F_S(h) \quad ,$$

where $F_S(h)$ denotes (by abuse of notation) the Schur function relative to the partition associated to S, h represents the coefficients of the expansion of g^{-1} as in (4.22), and the sum runs over all the sequences S as above. Actually, only the sequences for which W is transverse to $H_S{}^{\perp}$ do actually contribute to the sum.

4.5 The free-fermion representation

The so called "free-fermion operator formalism" for the KP hierarchy is one of the main results of the Kyoto group (see [12], [31]). I will recall here this construction, which is extremely interesting by itself, even if I will not deal with it in Part B.

(4.24) **Definition** : The *free fermion operator algebra* $\mathcal{A}$ is the non-commutative algebra generated by elements φ_n and φ^*_n, $n \in \mathbb{Z}$, satisfying the follwing Canonical Anticommutation Relations:

$$[\varphi_m , \varphi_n]_+ = [\varphi^*_m, \varphi^*_n]_+ = 0 \qquad ; \qquad [\varphi_m , \varphi^*_n]_+ = \delta_{mn} \quad .$$

The *Fock space* $\mathfrak{F}$ is the left $\mathcal{A}$-module generated by a cyclic (*vacuum*) vector $|\Omega_0>$ satisfying the identities

(4.25) $$\varphi^*_n |\Omega_0> = 0 \quad \text{if } n<0 \quad ; \quad \varphi_n |\Omega_0> = 0 \quad \text{if } n\geq 0 \quad .$$

One can build in a symmetric way a right $\mathcal{A}$-module $\mathfrak{F}^*$, with vacuum vector $<\Omega_0|$; these spaces are endowed with a bilinear pairing $\mathfrak{F}\times\mathfrak{F}^* \to \mathbb{C}$ defined by

(4.26) $$(<\Omega_0| a \ , \ b |\Omega_0>) \longmapsto <\Omega_0| ab |\Omega_0> \qquad a,b \in \mathcal{A}$$

($<\Omega_0| ab |\Omega_0>$ turns out to be a complex number due to the CAR and to the identities

(4.25)). The pairing is normalized by setting $<\Omega_0|\Omega_0> = 1$.

Let $t \equiv (t_1, t_2, ...)$ be an infinite set of parameters; one defines the *Hamiltonian* $H(t)$ by setting:

$$H(t_1, t_2, ...) \equiv \Sigma_{i\in\mathbb{Z},\, k\in\mathbb{N}}\ t_i\, \varphi_i\, \varphi^*_{k+i} \qquad (4.27)$$

Let $\mathcal{G} \subset \mathcal{A}$ be the group formed by the elements $\gamma\in\mathcal{A}$ such that

$$\gamma\, \varphi_n = \Sigma_{m\in\mathbb{Z}}\ \varphi_m\, \gamma\, a_{mn} \quad , \qquad \varphi_n^*\, \gamma = \Sigma_{m\in\mathbb{Z}}\ \gamma\, \varphi^*_n\, a_{mn} \quad ; \qquad (4.28)$$

to each element $\gamma\in\mathcal{G}$ one associates a τ-function:

$$\tau_\gamma(t) \equiv <\Omega_0|\, e^{H(t)}\gamma\, |\Omega_0> \quad . \qquad (4.29)$$

How is this framework related with the construction of the previous Sections? The interpretation given by Segal & Wilson is the following one. Let Gr(H) be the "Hilbert-Schmidt" Grassmannian, and let $\mathcal{H}$ be the L^2-closure, with respect to a suitable inner product (see[2]), of the space spanned by the Plücker coordinates w_S in Γ(Det*) . To each point W∈Gr(H) one lets correspond the section Ω_W of Det* defined by

$$\Omega_W\, (W') \equiv \det \|<w_i\, ,\, w'_j>\| \quad , \qquad (4.30)$$

where $\{w_i\}$ and $\{w'_i\}$ are admissible basis for W and W' respectively, and < , > denotes the scalar product in $\mathcal{H}$. The map $W \longmapsto |\Omega_W>$ defines a Plücker embedding of Gr(H) into the projective space $\mathbb{P}(\mathcal{H})$, $|\Omega_W>$ being the ray correspondig to the section Ω_W. Under this map, the image of the section σ ($\equiv \Omega_{H+}$) will be identified with the vacuum vector $|\Omega_0>$.

One introduces at this point the elements $q_\zeta \in \Gamma_+$ and $p_\zeta \in \Gamma_-$, defined for each $\zeta \in \mathbb{C}$, $|\zeta|>1$, as follows:

(4.31) $$q_\zeta = 1 - \frac{z}{\zeta} \quad ; \quad p_\zeta = (1 - \frac{1}{z\,\bar{\zeta}})^{-1} \quad .$$

(where $\bar{}$ stands for complex conjugation). The element $p_\zeta q_\zeta \in \Gamma$ acts on W and consequently on the section Ω_W: setting $\zeta = \rho e^{i\theta}$, one defines the operator-valued distribution $\varphi(\theta)$ by letting $\rho \rightarrow 1$. One sets as usual

(4.32) $$\varphi(f) \equiv (2\pi)^{-1} \int_0^{2\pi} \varphi(\theta)\, f(\theta)\, d\theta$$

for any $f \in H \equiv L^2(S^1)$. In particular, let $\varphi_n \equiv \varphi(z^n)$ and $\varphi^*_n \equiv \varphi(\bar{z}^n)$; one can check from the definition (4.32) that

(4.33) $$[\varphi(\theta), \varphi(\theta')]_+ = 0 \quad ; \quad [\varphi(\theta), \varphi^*(\theta')]_+ = \delta(\theta - \theta') \quad ,$$

and that consequently φ_n and φ^*_n fulfill the CAR stated in (4.24).

A further step consists in taking the exterior algebra $\wedge(H_+ \oplus H_-)$, which is generated by the products

(4.34) $$z^{i_1} \wedge \ldots \wedge z^{i_m} \wedge \bar{z}^{j_1} \wedge \ldots \wedge \bar{z}^{j_n} .$$

Let me now associate to each space H_S in Gr(H) an element $z^S \in \wedge(H_+ \oplus H_-)$ of the form (4.34), where the indices i_k coincide (with opposite sign) with the negative integers in the sequence S, while the indices j_k coincide with the lacking positive integers in S. To a more general space $W \in Gr(H)$ one associates the element

$$(4.35) \qquad z_W \equiv \sum_S w_S(W)\, z^S \quad .$$

Gr(H) can now be realized as a left $\mathcal{A}$-module via the map

$$(4.36) \quad z^S \equiv (z^{i_1} \wedge \ldots \wedge z^{i_m} \wedge \bar{z}^{j_1} \wedge \ldots \wedge \bar{z}^{j_n}) \longmapsto \varphi^S \equiv \varphi_{i1} \cdots \varphi_{im}\, \varphi^*_{j1} \cdots \varphi^*_{jm};$$

in this way, $|\Omega_{HS}\rangle = \varphi^S|\Omega_0\rangle$, and more generally $|\Omega_W\rangle = \sum_S w_S(W)\varphi^S|\Omega_0\rangle$. In particular, $\tau_W(g)$ corresponds exactly to the τ-function τ_γ defined by (4.29), having set $|\Omega_W\rangle = \gamma\,|\Omega_0\rangle$ and $g = \exp[H(t)]$.

The same construction could also be carried for $Gr(H^n)$, leading to a n-component fermion representation; this can be important for the physical interpretation of such constructions according to quantum field theory, but from the viewpoint of the integrable systems this generalization seems not to add any relevant piece of information with respect to the ordinary scalar construction, because of the correspondence between $Gr(H^n)$ and Gr(H) stated in Sect. 4.3 , which can easily be implemented in this formalism.

5. BAKER-AKHIEZER FUNCTIONS AND DIFFERENTIAL OPERATORS

5.1 Novikov Equation

I have introduced in Sect. 1.2 the Novikov equation

(5.1) $$[L, P] = 0$$

for a pair of monic operators L, P $\in$ **R**[D], as being the compatibility condition for the spectral problem

(5.2) $$L\psi = \lambda\psi \qquad P\psi = \mu\psi \quad .$$

I will now consider L and P as acting on the space of formal Baker-Akhiezer functions, letting the reader refer to the remark at the end of Sect. 3 about the general meaning of this setting.

(5.3) **Proposition** : For any monic operator L $\in$ **R**[D] of degree n, there exists a pseudo-differential operator $K = 1 + \sum_{k>0} a_k(x)D^{-k}$, belonging to the Volterra group, such that $L \circ K = K \circ D^n$.

This proposition can be checked by direct computation: the equation in (1.17) allows to find iteratively all the coefficients a_k of K. K is defined up to multiplication by an arbitrary element of the Volterra group with constant coefficients , which obviously commutes with D^n. ∎

For the rest of this Section and the next one, I will drop out the symbol $\circ$ for multiplication (composition) of operators, and I will assume all the operators to be monic.

(5.4) **Proposition** : The formal Baker-Akhiezer function

$$\psi = K\, e^{xz} = e^{xz}\, [1 + \textstyle\sum_{k>0} a_k(x)\, z^{-k}]$$

is a solution of the spectral problem (5.2), if K is defined according to (5.3) and L and P fulfill the condition (5.1).

In fact, $L\psi = (LK)\, e^{xz} = (KD^n)\, e^{xz} = z^n\, K\, e^{xz} = z^n\, \psi$; for the second equation in (5.2), let $R \in$ **PsD(R)** be defined by $R = K^{-1}\, P\, K$. Since $0 = [P, L] = [K\, R\, K^{-1}, L] =$ $= K[R, D^n]\, K^{-1}$, R should commute with D^n and is therefore an operator with constant coefficients, $R = \sum_{k\in\mathbb{Z}} c_k D^k$, $c_k \in \mathbb{C}$. Let me denote R by $\mu(D)$: it follows that $P =$ $= K\mu(D)\, K^{-1}$ and $P\psi = K\mu(D)\, e^{xz} = \mu(z)\psi$. ∎

As a consequence of these observations, one is led to associate to the BA function ψ the whole centralizer $\mathcal{Z}(L)$ in **R**[D] (which is not to be confused with the centralizer Z(L) in **PsD(R)**). An important remark is the following one:

(5.5) **Proposition** : If two commuting differential operators L and P are of relatively prime degree, the space formed by the eigenfunctions ψ of L and P corresponding to a given pair of eigenvalues (λ, μ) is one-dimensional for almost all pairs $(\lambda, \mu) \in \mathbb{C}^2$.

This proposition will be proved in Sect. 7.

5.2 The KP Hierarchy

I will now consider the infinite set of linear differential equations

(5.6) $$P_n \psi = \partial_n \psi$$

for all $n \geq 2$, where both P_n and ψ depend on infinitely many variables $(x, t_2, t_3, \ldots)$ and ∂_n stands for $\partial/\partial t_n$ as in Sect. 1.2. I have showed above that the associated compatibility conditions

(5.7) $$\partial_m P_n - \partial_n P_m = [P_n, P_m]$$

admit the solution $P_n = (Q^n)_+$, Q being a pseudo-differential operator of degree one. Let me prove that if we restrict the system (5.6) to the space of formal BA functions

(5.8) $$\psi = \exp(xz + t_2z^2 + t_3z^3 + ...)[1 + \textstyle\sum_{k>0} a_k(t)\, z^{-k}]$$

all the solutions of (5.7) are of the type stated above.

Assume K to be the pseudo-differential operator $K= 1+ \sum_{k>0} a_k(t)\, D^{-k}$, such that $\psi = K \exp(xz + t_2z^2 + t_3z^3 + ...)$; then

(5.9) $$\partial_n\psi = [\partial_n K + K z^n] \exp(xz + t_2z^2 + t_3z^3 + ...) = [(\partial_n K)K^{-1}+KD^nK^{-1}]\, \psi$$

Therefore, $P_n = [(\partial_n K)K^{-1}+KD^nK^{-1}]$ and since P_n is a differential operator, and the degree of $(\partial_n K)$ cannot exceed (-1), one finds

(5.10) $$P_n = (KD^nK^{-1})_+ \quad , \quad (\partial_n K)K^{-1} = - (KD^nK^{-1})_- .$$

Setting $Q \equiv (KD^nK^{-1})$, one obtains the desired result.

The BA function technique allows also to prove that the ZS equation

(5.11) $$\partial_3 L - \partial_2 P = [P, L]$$

is the compatibility condition for the spectral problem

(5.12) $$L\, \psi = \partial_2\, \psi$$
$$P\, \psi = \partial_3\, \psi \quad ;$$

in fact, assume (to fix the ideas) that L and P are of degree 2 and 3 respectively; suppose that for a given such pair (L, P), not necessarily fulfilling (5.11), there exists a BA functions which solves the spectral problem: this BA function should have the form

$$\psi = \exp(xz + t_2z^2 + t_3z^3)[1 + \textstyle\sum_{k>0} a_k(x, t_2, t_3)\, z^{-k}] \,. \tag{5.14}$$

Now, ψ is defined up to multiplication by a Laurent series in z with constant coefficients; therefore, the kernel of the operator $A = [\partial_3\text{-P}, \partial_2\text{-L}]$ contains an infinite-dimensional subspace of the space of BA functions. However, on this space, A can be written as a differential operator with respect to the variable x only. In fact, by introducing the operator K related with ψ in the usual way, one can substitute to each ∂_i the expression $[(\partial_i K)K^{-1}+KD^iK^{-1}]$; but a differential operator in one variable cannot have an infinite-dimensional kernel, unless it is the trivial operator. Thus, the existence of a solution of the system (5.12) implies that (L, P) must fulfill (5.11) as it was claimed.

6. BAKER-AKHIEZER FUNCTIONS AND KRICHEVER DATA

In Section 3 I have introduced "axiomatically" the BA functions as objects defined on a given smooth algebraic curve. In this Section I mean to point out that there is a "faithful" correspondence between BA functions and algebraic curves: this correspondence is ensured by the property of existence and uniqueness of the BA function associated to a given Krichever datum.

(6.1) **Definition** : A *Krichever datum* is a set (C, p, k, $\mathcal{D}$), where C is a smooth projective curve of genus g, p a distinguished point on it, k a local coordinate in a neighborhood of p, and $\mathcal{D}$ is a generic effective divisor of degree g on C.

Thanks to the correspondence between divisors and line bundles described in Sect. 2.3, the Krichever data can be re-expressed as quintuples (C, p, k, L, φ), L being a line bundle over C and φ being a local trivialization of L over the domain of the coordinate k. This setting can be further generalized to singular curves, but this generalization will not be relevant for this Section.

As in Section 3, let $z = k^{-1}$. Given a polynomial $R(z) \in \mathbb{C}[z]$, I can define ω_R to be the unique meromorphic differential on C having a pole in p as the only singularity, such that the principal part of ω_R in p has the form $dR(z)$ and such that all the a-periods $<\omega_R, a_i>$ (see def. in Sect. 2.3) vanish. Let $\mathbf{B} \equiv (B_1, ..., B_g)$ be the vector of b-periods $B_i \equiv <\omega_R, b_i>$.

(6.2) **Proposition** : Given a Krichever datum $(C, \mathrm{p}, k, \mathcal{D})$, a polynomial $\mathrm{R}(z) \in \mathbb{C}[z]$, and an arbitrary point $\mathrm{q} \neq \mathrm{p}$ on C, the Baker-Akhiezer function defined by (3.1) is given uniquely by the following expression:

$$\psi(\mathrm{s}) \equiv \exp\left[\int_{\mathrm{q}}^{\mathrm{s}} \omega_{\mathrm{R}}\right] \frac{\theta(\mathfrak{A}(\mathrm{s}) - \mathbf{K} + \mathbf{B})}{\theta(\mathfrak{A}(\mathrm{s}) - \mathbf{K})} ,$$

θ being the Riemann theta-function and $\mathfrak{A}$ being the Abel map, defined by (2.16) and (2.14) respectively, and $\mathbf{K}$ being the constant vector such that $\theta(\mathfrak{A}(\mathrm{s}) - \mathbf{K})$ vanishes exactly when $\mathrm{s} \in \mathcal{D}$.

It is easy to check that all the requirements of def. 3.1 are fulfilled. The relevant point to prove is the uniqueness statement. Suppose ψ' to be another function on the same curve, with the same divisor of poles and the same singular behavior at p. The ratio $\varphi \equiv \psi'/\psi$ is therefore a meromorphic function on C. One can see from the explicit formula above that the divisor of zeros of ψ, $(\psi)_0$, has degree g and is non-special. Now, $(\psi)_0$ coincides with the divisor of poles of φ; however, a non-special divisor cannot be the divisor of poles of a meromorphic function on C (recall Sect. 2.3). Therefore φ has to be constant.

If one chooses another parametrization of C, the coefficients of the polynomial R should change accordingly. For instance, let $\mathrm{R} = (xz + t_2 z^2 + t_3 z^3)$; if I change the parametrization in such a way to have $z \longmapsto \mathrm{a}z + \mathrm{b} + \mathrm{c}z^{-1} + \mathrm{O}(z^{-2})$, $\mathrm{a} \neq 0$, b and c being constant, the corresponding change of the coefficients is given by

$$\begin{aligned} x &\longmapsto \mathrm{a}x + 2\mathrm{ab}\, t_2 + (3\mathrm{ab}^2 + 3\mathrm{a}^3\mathrm{c})\, t_3 \\ t_2 &\longmapsto \mathrm{a}^2\, t_2 + 3\, \mathrm{a}^2\mathrm{b}\, t_3 \\ t_3 &\longmapsto \mathrm{a}^3\, t_3 \end{aligned} \qquad (6.3)$$

However, as I have anticipated in Sect. 3, I will consider the family of BA functions $\psi(x, t_2, t_3, ...; z)$ as a single object, and I shall still call it a BA function. Such an object is defined uniquely by the geometrical datum (C, p, $\mathcal{D}$), without regard to the parametrization, nor to the polynomial R(z); as I will show below, it corresponds to the orbit in the space of BA functions (in the strict sense) of the flows generated by Lax equations. To anticipate a concrete example, Let R(z) be the third-order polynomial above and note that the associated BA function $\psi(x, t_2, t_3; z)$ solves the spectral problem

$$L\psi = \partial_2 \psi$$

(6.4)

$$P\psi = \partial_3 \psi \quad ;$$

here L and P are determined from ψ, according to the methods explained in Sect.5, through the Volterra operator K defined by means of the coefficients of the series expansion of ψ (so that $\psi = K \exp[R(x, t_2, t_3; z)]$), by setting $L=KD^2K^{-1}$ and $P=(L^{3/2})_+$. With these assumptions, one could check that ψ leads to a solution of the Kadomtsev-Petviashvilij equation. If one furthermore assumes that the function z^2 extends to a meromorphic function on C having the only singularity at p (which can happen only if C is hyperelliptic and p is a Weierstrass point), one can define the function

(6.5) $$\psi' = \exp[-t_2 z^2]\, \psi \quad .$$

ψ' has the same polar divisor as ψ, while its singular behavior at p is expressed by the polynomial exponent $(xz + t_3 z^3)$. Due to the uniqueness property, ψ' must be the BA function on C associated with the latter polynomial. The system (6.4) thus becomes

(writing now ψ for ψ')

$$L\psi = z^2 \psi$$

(6.6)

$$P\psi = \partial_3 \psi \quad .$$

As I have already stressed, (6.4) lead to the Lax equation $\partial_t L= [P, L]$, having set $t \equiv t_3$. By a change of variable $x \longmapsto x'$, any differential operator L of order two can be put into the form $L = D^2 + u(x, t)$, in order to recover exactly the KdV equation.

The fact that u does not depend on t_2 in the present case can be proved in the following way: let $Q = KDK^{-1}$; by construction, $Q\psi = z\psi$: but the first equation in (6.4) reads $(Q^2)_+\psi = z^2 \psi$, so that $(Q^2)_- = 0$. Now, it has been shown in Sect. 5.2 that $(Q^2)_- =$ $= (\partial_2 K)K^{-1}$; therefore K, and thus L, do not depend on t_2. As a consequence of that construction, to find a solution to the KdV equation, one can start from a given hyperelliptic curve C, set $R(z)=(xz + tz^3)$, and write down explicitly ψ according to (6.2). Let now $\psi = \exp[R(z)][1+\sum_{k>0} a_k (x, t)z^{-k}]$ be the series expansion for ψ; from the relation $L = KD^2K^{-1}$ one easily finds $u(x, t)$:

(6.7)
$$u = - 2\, Da_1 \quad .$$

7. DIFFERENTIAL OPERATORS AND ALGEBRAIC CURVES

7.1 The Burchnall-Chaundy-Krichever Theory

Burchnall, Chaundy and Baker first introduced around 1920-1930 a relationship between pairs of commuting differential operators and algebraic curves. Their remarkable papers [9] were however forgotten, and more recently Krichever recovered independently the same result. On the other hand, Burchnall and Chaundy did not relate their construction to Lax equations (which were introduced later), and consequently to the IDIS. The connection with the latter ones is actually due to Krichever. The starting point of the BCK theory is the following observation:

(7.1) **Proposition** : Let L and P be two differential operators of degree m and n respectively, m, n being relatively prime, and let $[L, P]=0$. Then there exists an irreducible polynomial $Q(x,y) \in \mathbb{C}[x, y]$ such that $Q(L, P)=0$.

In fact, let me consider the equation $(P - \mu)\psi=0$, defining the m-dimensional eigenspace V_μ of P in the space $C^\infty((-1, 1))$. Since P and L commute, $\psi \in V_\mu$ implies that $L\psi \in V_\mu$. Let $\{\psi_i\}_{i=1, ..., m}$ be the basis in V_μ such that $(\partial^j/\partial x^j)\psi_i(0)=\delta_{ij}$. With respect to that basis, the restriction of L to V_μ is represented by a m×m matrix $\mathbf{L}_\mu$ with polynomial entries in μ. Let $Q(\lambda, \mu)$ be the characteristic polynomial $Q(\lambda, \mu) = \det(\mathbf{L}_\mu - \lambda\mathbf{1})$. The differential operator $Q(L, P)$ vanishes on each $\psi \in V_\mu$, for any μ: therefore it has an infinite-dimensional kernel and must vanish identically. To see that Q is irreducible, suppose $Q(x, y)=Q_1(x, y)Q_2(x, y)$; then, either $Q_1(L, P)$ or $Q_2(L, P)$ should be the zero operator; in fact, if $Q_2(L, P)$ is not zero, $Im(Q_2(L, P))$ is infinite-dimensional and then $Q_1(L, P)$ must vanish for the same argument as above. In either cases, one of the two polynomials, say Q_1, has exactly the same zeros as Q; thus the zero locus of Q_2 is properly included in the zero locus of Q_1, and Q_2 must be a factor of Q_1. Iterating the argument until one is left with irreducible factors only, one concludes that Q must be the r-th power of an irreducible polynomial, for some integer r. Now, Q contains the monomials λ^m and μ^n, so r should be a common divisor of m and n. If m, n are relatively prime, then r = 1 and the proposition is proved. ∎

Consequently, the equation $Q(\lambda, \mu) = 0$ has exactly n distinct solutions for each value of λ, except for a finite number of branching points. Since V_μ is n-dimensional, each eigenspace $V_{\mu,\lambda}$ of L_μ is one-dimensional. These one-dimensional joint eigenspaces of L and P form a line bundle on the curve C defined by $Q(\lambda, \mu)=0$. In this way, one associates to the pair (L, P) a Krichever datum, once the curve C has been compactified by adding the point at infinity. To be precise, one should check that the line bundle so obtained has degree g; although this could be done within the framework described so far, I will postpone the discussion about this point to Sect. 7.3, where I will rely on a more abstract setting. Before introducing the latter one, however, I wish to sketch a different example of algebraic curves arising as spectral curves, namely the Floquet theory of the Hill's operator.

7.2 The spectral curve of the Hill's operator

In the mathematical literature, the operator $D^2 + q(x)$ is known as the *Hill's operator*. The so-called *Floquet theory* of the Hill's operator with a periodic potential $q(x)$ leads directly to an algebraic curve, without relying on the algebraic properties of commutative subrings of differential operators. This subject is widely discussed in [32], and is the starting point for the thorough investigation due to McKean and his collaborators ([33]); these authors have showed, within the framework of spectral theory, that the space of all potentials $q(x)$ leading to the same spectrum of the Hill's operator is to be identified with (the real part of) the Jacobian torus of an hyperelliptic curve of possibly infinite genus. I will summarize below the main elementary ideas of the Floquet theory.

Let $q(x)$ be a piecewise continuous periodic function of minimal period π. The differential equation

$$(7.2) \qquad (D^2 + q)\, y = 0$$

admits two independent solutions y_1, y_2 such that $y_1(0) = y_2'(0) = 1$, $y_1'(0) = y_2(0) = 1$. With these "normalized solutions" one defines the *characteristic equation*

$$(7.3) \qquad s^2 - [\, y_1(\pi) + y_2'(\pi)]\, s + 1 = 0 \qquad .$$

Let s_1, s_2 be the roots of (7.3), and let $\alpha \in \mathbb{C}$ be such that $s_1 = e^{i\alpha\pi}$ and $s_2 = e^{-i\alpha\pi}$.

(7.4) **Proposition** (Floquet) : If $s_1 \neq s_2$, then the Hill's equation (7.2) has two linearly independent solutions $f_1 = e^{i\alpha x} p_1(x)$ and $f_2 = e^{-i\alpha x} p_2(x)$, p_1 and p_2 being periodic functions of period π .

The proof can be found in [32]. Let me now define $Q(x, \lambda) = q(x) - \lambda$, assuming q and λ to be real. The eigenvalue problem for $(D^2 + q)$ is translated into the Hill's equation

$$[\, D^2 + Q(x, \lambda)\,]\, y(x, \lambda) = 0 \,. \tag{7.5}$$

For each value of λ there are two normalized solutions $y_1(x, \lambda)$, $y_2(x, \lambda)$, and two characteristic roots $s_1(\lambda)$ and $s_2(\lambda)$; $s_1 \neq s_2$ for almost all λ. The values λ_i for which $s_1 = s_2$ carry a relevant information, since the solutions of Hill's equation (7.5) are bounded if the characteristic exponent α is real, while they are unbounded for $Im(\alpha) \neq 0$. Thus the branch points λ_i are the endpoints of the interval of instability of the spectrum of $(D^2 + q)$ (such intervals are called *gaps*).

Let now $\Delta(\lambda)$ be the *discriminant*

$$\Delta(\lambda) = y_1(\pi, \lambda) + y_2'(\pi, \lambda) \,. \tag{7.6}$$

If $\Delta(\lambda) = \pm 2$ one has $s_1 = s_2 = \pm 1$; it is possible to show that in this case there exists a nontrivial solution of minimal period π (if $\Delta(\lambda) = 2$) or 2π (if $\Delta(\lambda) = -2$). Therefore, one can split the set of branch points $\{\lambda_i\}$ into two groups, $\{\lambda_i^\circ\}$ and $\{\lambda_i'\}$, were $\Delta(\lambda_i^\circ) = 2$ and $\Delta(\lambda_i') = -2$. The order of these values on the real axis is the following:

$$\lambda_0^\circ < \lambda_1' \leq \lambda_2' < \lambda_1^\circ \leq \lambda_2^\circ < \lambda_3' \leq \lambda_4' < \dots \quad ; \tag{7.7}$$

both sequences $\{\lambda_i^\circ\}$ and $\{\lambda_i'\}$ tend to infinity, and the point at infinity itself is a branch point. The intervals of instability are those of the type $(\lambda_i^\circ, \lambda_{i+1}^\circ)$ or $(\lambda_i', \lambda_{i+1}')$; if the endpoints of one of these intervals coincide, the gap disappears; the intervals of stability, on the contrary, cannot disappear.

There are potentials q(x) such that the number of gaps is finite. For these *finite-gap* potentials, almost all the roots of the equation $\Delta^2(\lambda) - 4 = 0$ are double roots; the gaps correspond to the simple roots μ_i in the periodic spectrum of $(D^2 + q)$. One is therefore naturally led to consider the hyperelliptic curve defined by the equation

$$\rho^2 = \prod_{0}^{N} (\mu - \mu_i) \quad . \tag{7.8}$$

The genus of this curve is determined by the number of gaps; in the general case, the genus is infinite (assuming that such objects have a meaning). To get a compact curve one adds the point at infinity which turns out to be a Weierstrass point.

To see the relation of this construction with the BCK one, one could observe that the functions f_1 and f_2 introduced in (7.4) are simultaneous eigenfunctions of the Hill's operator $L = (D^2 + q)$ and of the *translation operator* T_π, defined by setting $T_\pi f(x) = f(x+\pi)$. T_π is not a differential operator: however, if one restricts the coefficient ring of $\mathbb{R}[D]$ to contain only periodic functions, then the operators L and P involved in the BCK setting both commute with T_π. One can therefore consider the matrix expression of any two of these operators acting on the eigenspaces of the third one, so that both the construction of this Section and the BCK one are recast in a purely algebraic form; in this way (see e.g. [34]) one can check the equivalence of the two approaches for the particular case of the Hill's operator. From the viewpoint of the spectral theory, we learn in this way that hyperelliptic curves of finite genus lead to finite-gap potentials.

7.3 Abstract description of the BCK theory

In this Section I will summarize the abstract setting of the BCK construction, following essentially the presentation by D. Mumford in the paper [35]. The subject is also discussed in [20], [36]. The main result can be stated as follows:

(7.9) **Proposition** (Krichever) : To each commutative subring $A \subset \mathbf{R}[D]$ ($\mathbf{R}$ being the ring of formal series $\mathbb{C}[[x]]$), containig at least two monic operators of relatively prime degree, corresponds a (generalized) Krichever datum (C, p, k, $\mathfrak{L}$), where $\mathfrak{L}$ is a torsion-free rank-one sheaf such that $\dim H^0(\mathfrak{L}) = \dim H^1(\mathfrak{L}) = 0$. Conversely, to each such Krichever datum there corresponds a commutative ring of differential operators as above; two rings A_1, A_2 being identified, however, if $A_1 = u \circ A_2 \circ u^{-1}$ for some function $u(x)$.

If the curve C is smooth, we can consider the line bundle L associated to the sheaf $\mathfrak{L}$, instead of $\mathfrak{L}$ itself; from the Riemann-Roch formula we see that the degree of L should be equal to g - 1, g being as usual the genus of C. This seems in contrast with the previous setting, where one was dealing with line bundles of degree g. However, the difference is only apparent; it corrresponds essentially to a different choice of the normalization of the BA functions. As a matter of fact, since the main point is to study equations which are related with *deformations* of the line bundle L, one can always tensorize the latter one with a *fixed* bundle L_0 to change the degree, without affecting the result; actually, one tipically wants to end up with a line bundle of degree zero, i.e. with a point of the Jacobian variety. Let me now outline the proof of (7.9).

The starting point is a deformation $\mathcal{L}^*$ of the sheaf $\mathcal{L}$, defined in this way: let U be an open neighborhood of p in C; the variety $C\times\mathbb{C}$ can be covered by the two open sets $U_1 \equiv U \times \mathbb{C}$ and $U_2 \equiv (C \setminus p) \times \mathbb{C}$. $\mathcal{L}^*$ is obtained by glueing the two restrictions of $\mathcal{L} \otimes \mathcal{O}_{\mathbb{C}}$ to U_1 and U_2 through the transition function e^{xz}, z being the inverse of the local coordinate k, as in Sect. 3. The sections of this sheaf could be thought of as (local) BA functions with the normalization $a_0(x) \equiv 0$ instead of $a_0(x) \equiv 1$ as usual (there are no global sections of this kind).

The next step consists in defining the operator ∇: $\mathcal{L}^* \to \mathcal{L}^*(p)$, $\mathcal{L}^*(p)$ being the sheaf of meromorphic sections of $\mathcal{L}^*$ with at most one pole of order one in p. ∇ is defined over U_2 as being just $D \equiv (d/dx)$, and it extends to U_1 by means of the formula

$$e^{-xz}\, D(e^{xz}\, f(z,x)) = z\, f(z,x) + D\, f(z,x) \quad . \tag{7.10}$$

Therefore, $\nabla f = z\, f +$ (sect. of $\mathcal{L}^*$). Iterating ∇, one obtains maps $\nabla^n : \mathcal{L}^* \to \mathcal{L}^*(np)$.

From the hypothesis on the cohomology of $\mathcal{L}$, it follows that also $\dim H^0(\mathcal{L}^*) =$ $= \dim H^1(\mathcal{L}^*) = 0$; from the cohomology sequence associated with the exact sequence $0 \to \mathcal{L}^*(p) \to \mathcal{L}^* \to \mathcal{L}^*(p)/\mathcal{L}^* \to 0$ one deduces that $\dim H^0(\mathcal{L}^*(p)) = 1$. Choose a generator $s_0 \in \Gamma(C\times\mathbb{C}, \mathcal{L}^*(p))$; by applying ∇ one can obtain a sequence of sections $s_n \equiv \nabla^n s_0 \in \Gamma(C\times\mathbb{C}, \mathcal{L}^*((n+1)p))$.

Let me now consider the ring $A \equiv \Gamma(C \setminus p, \mathcal{O}_C)$: it is possible to see, from the fact that $s_n = s_0 z^n$ + (l.o. terms), that the set of sections $\{s_i\}_{i=0,\dots,n}$ form a basis of $\Gamma(C\times\mathbb{C}, \mathcal{L}^*((n+1)p))$ as an A-module. Take $a \in A$; any such function can be extended to a meromorphic function on C by letting the pole in p have an order large enough: $a = \alpha z^n$ + (l.o. terms). Then the section $a\, s_0 \in \Gamma(C\times\mathbb{C}, \mathcal{L}^*((n+1)p))$ can be expanded as follows:

$$(7.11) \qquad a\, s_0 = \alpha s_n + \sum_{0\le k\le n-1} a_k(x)\, s_k = [\alpha \nabla^n + \sum_{0\le k\le n-1} a_k(x)\, \nabla^k]\, s_0 .$$

This formula defines an embedding of A in **R**[D], mapping $a \longmapsto [\alpha D^n + \sum a_k(x) D^k]$. If one chooses a different generator s_0, the image of the ring is unchanged up to the identification stated in (7.9). If n is large enough, there always exists a global meromorphic section on C having only a pole of order n in p; therefore A contains elements of almost any degree. The image of A must then include differential operators of relatively prime degree (all the elements of A are monic, as one can see from (7.11)). This proves the proposition in one direction.

Conversely, consider the commutative subring of differential operators A as an abstract ring; according to the discussion in Sect. 2.2, one recovers the affine part $C \setminus p$ simply by taking the closed points of Spec(A). To get also the point p, one has to build the projective curve associated with the graded ring $G = \oplus_{i\in \mathbb{N}} G_i$, whereby $G_i \equiv \{L \in A /\ \deg(L) \le i\}$. Furthermore, one can check that **R**[D] itself is a free A-module of rank one, provided A has one element for each degree n above some finite degree n_0; this is easily checked to be true if A contains two operators of relatively prime degree (in fact, the equation am + bn = c, m and n fixed integers, has a solution $(m, n) \in \mathbb{N}^2$ for any $c \ge m+n$ if and only if [n] is an invertible element in the ring Z_m, i.e. n is prime w.r.t. to m). The sheaf $\mathcal{L}$ (or rather its restriction to $C \setminus p$) can thus be identified with the sheaf over Spec(A) associated to **R**[D], according to the technique presented in Sect. 2.4. To see this, just define a correspondence $\mathbf{R}[D] \to \Gamma(C \setminus p, \mathcal{L})$ by setting $D \longmapsto s_0$. ∎

What happens if we change the sheaf $\mathcal{L}$? One expects that the embedding of A, viewed as an abstract ring, into **R**[D], change accordingly. If we assume C to be smooth and $\mathcal{L}$ to be the sheaf of sections of a line bundle L, we can describe a deformation of that sheaf by letting the Jacobian variety act as follows: to each point on Jac(C) it corresponds an equivalence class of divisors $[\mathcal{D}]$ of degree zero, and the action on L is given by $L\otimes L(\mathcal{D})$, $L(\mathcal{D})$ being the line bundle associated with $\mathcal{D}$.

An infinitesimal deformation is therefore described by a tangent vector to Jac(C). This tangent vector can be represented by a line bundle L' over the variety $C\times\mathbb{C}[t]/(t^2)$, with transition function $1+t\,\mathrm{c}$, c being an element of $\mathrm{H}^1(\mathcal{O}_C)$. Omitting the details, I simply recall that s_0 lifts to a section $\mathrm{s}_0{}^*$ of the "deformed sheaf" $\mathcal{L}^{**}$, obtained by tensoring (over $\mathcal{O}_C$) $\mathcal{L}^*$ with the sheaf of sections of L'. The restriction of $\mathrm{s}_0{}^*$ to the affine part $C\setminus\mathrm{p}$ has the form $\mathrm{s}_0{}^*=(\mathrm{s}_0+t\,\mathrm{s}_0')$, s_0' being defined as follows: observe that the local section cs_0 of $\mathcal{L}^*$ can be written in the form $\mathrm{cs}_0=\sum_{k\in\mathbb{Z}}\mathrm{c}_k(x)\,\mathrm{s}_k$: then s_0' coincides with (minus) the "positive part" $(\mathrm{cs}_0)_+$, i.e. $\mathrm{s}_0'=-\sum_{k\geq 0}\mathrm{c}_k(x)\,\mathrm{s}_k$.

For any $\mathrm{a}\in\Gamma(C\setminus\mathrm{p},\mathcal{O}_C)$ as above, one can expand $\mathrm{as}_0{}^*$ in the form

$$\mathrm{as}_0{}^*=\sum_{0\leq k\leq n}(\mathrm{a}_k(x)+t\,\mathrm{a}'_k(x))\,\nabla^k\,\mathrm{s}_0{}^*\quad. \tag{7.12}$$

The variation in the coefficients of the image of a in R[D] is then obtained by solving

$$\mathrm{as}_0'=\sum_{0\leq k\leq n}(\mathrm{a}'_k(x)\,\nabla^k\,\mathrm{s}_0+\mathrm{a}_k(x)\,\nabla^k\,\mathrm{s}_0')\quad. \tag{7.13}$$

The left-hand side of (7.13) can be expanded in the form

$$\mathrm{as}_0'=-(\sum_{i\geq 0}\mathrm{c}_i(x)\,\nabla^i\,)(\sum_{k\geq 0}\mathrm{a}_k(x)\,\nabla^k\,)\,\mathrm{s}_0\quad, \tag{7.14}$$

while

(7.15) $\sum_{k\geq 0} a_k(x)\, \nabla^k s_0' = -(\sum_{k\geq 0} a_k(x)\, \nabla^k)(\sum_{i\geq 0} c_i(x)\, \nabla^i)\, s_0$.

Having associated to a the differential operator $L(a) = \sum_{k\geq 0} a_k(x)\, D^k$, it is natural to associate to c the *pseudo-differential* operator $W(c)=\sum_{k\in \mathbb{Z}} c_k(x)\, D^k$, so that equation (7.13) becomes

(7.16) $\partial_t L(a) = [(W(c))_+, L(a)]$.

This setting illustrates therefore the geometrical meaning of the Lax equation, as it was anticipated in the Introduction; the Lax equation represents the image of the infinitesimal action of Jac(C) on line bundles over C, under the Krichever's correspondence (7.9).

8. UNIVERSAL GRASSMANNIAN MANIFOLD AND DIFFERENTIAL OPERATORS

In the paper [30], M. and Y. Sato have presented a method to make a given pseudo-differential operator K (belonging to the Volterra group) correspond to a point $W_K \in Gr$. This is the starting point for the "grassmannian approach" to infinite-dimensional integrable systems. I will recall in this Section the direct construction due to Sato, while in Sect. 9 I will present the slightly different strategy of Segal & Wilson, consisting in associating to a point $W \in Gr(H)$ a Baker-Akhiezer function. In any case, it should be clear at this point (even if a precise statement in this sense has not been formulated) that dealing with the space of BA functions is equivalent to dealing with the Volterra group.

In Sect. 4.2 it was stated that Gr is the set of linear subspaces (with suitable restrictions) of an infinite-dimensional vector space V. The crucial point is now the following:

(8.1) **Proposition :** V admits a structure of left **PsD(R)**-module; with this structure, V is isomorphic to the quotient module **PsD(R)/M** , where **M** is the maximal ideal **PsD(R)** $\circ\, x$.

First, one lets **PsD(R)** act on V by setting

$$x \circ \sum v_k e_k = \sum k\, v_{k-1} e_k \quad ; \quad D \circ \sum v_k e_k = \sum v_{k+1} e_k \tag{8.2}$$

It is straightforward to see that the commutation rule $[D, x] = 1$ is preserved by this representation as well as the other algebraic relations holding in **PsD(R)**. The next step consists in setting the correspondence

$$v = \sum v_k e_k \mapsto V = \sum v_k D^{-k-1} \tag{8.3}$$

which maps a vector $v \in \mathbf{V}$ into a pseudo-differential operator with constant coefficients $V \in \mathbf{PsD}(\mathbb{C})$. Let me prove at this point the following

(8.4) **Lemma** : Any operator $Q \in \mathbf{PsD(R)}$ is equivalent mod **M** to an operator with constant coefficients $V \in \mathbf{PsD}(\mathbb{C})$. Furthermore, to each equivalence class $[Q] \in \mathbf{PsD(R)/M}$ belongs a unique operator with constant coefficients.

In fact, given a pseudo-differential operator Q, one can always find another operator P such that $[Q - (P \circ x), D] = 0$, which implies that $(Q - P \circ x)$ has constant coefficients (the equation can be solved iteratively); on the other hand, two operators with constant coefficients cannot differ by an operator of the form $(P \circ x)$, since such an operator cannot have constant coefficients, as it is easy to check. ∎

By means of this lemma, one can regard the map defined by (8.3) as a map $S : \mathbf{V} \to \mathbf{PsD(R)/M}$. It remains only to prove that S is an isomorphism of **PsD(R)**-modules. Assuming $\mathbf{R} \equiv \mathbb{C}[[x]]$, it is enough to check that $x \circ S(v) = S(x \circ v)$ and $D \circ S(v) = S(D \circ v)$: this follows from the straightforward application of (1.2):

$$x \circ S(v) = - S(v) \circ x + \sum (k-1)\, v_k\, D^{-k-2}$$

(8.5)

$$D \circ S(v) = \sum v_k\, D^{-k} .$$

∎

Under the isomorphism S, the subspace **V**° introduced in Sect. 4.2 corresponds to the subspace spanned by $\{D^k\}_{k\geq 0}$ (mod **M**) .

Now, the relation with the Grassmannian can be made clear. Let K be a Volterra operator with coefficients in $\mathbb{C}((x))$ (formal Laurent series); I will assume that, for both K and K^{-1}, all the coefficients have a finite number of negative-power terms which is bounded by some integers m and n, for K and K^{-1} respectively: this is equivalent to saying that $x^m \circ K$ and $K^{-1} \circ x^n$ belong to $\mathbf{PsD}(\mathbb{C}[[x]])$. To the operator K one associates the subspace $W_K = (K^{-1} \circ x^n) \circ \mathbf{V}°$. Of course, if the condition $K^{-1} \circ x^n \in \mathbf{PsD}(\mathbb{C}[[x]])$ holds for some n, it holds also for any n'>n; however, a different choice of the integer n does not affect the definition of W_K, since $x^n \circ \mathbf{V}° \equiv \mathbf{V}°$. From (8.2) one can see that the negative-index generators of **V** can be mapped into positive-index generators by applying x^{-1} or D only. Since K^{-1} does not contain any positive power of D, and it contains negative powers of x up to the n-th only, W_K contains a finite number of positive-index generators and thus defines a point in Gr.

In the paper [30], the image of the KP flows on the Grassmannian is also described; however, I will treat this subject, as well as the converse problem of recovering the operator K from a point in Gr, in the next Section, by relying on the Segal-Wilson's formalism.

9. GRASSMANNIAN MANIFOLD AND BAKER-AKHIEZER FUNCTIONS

9.1 BA function and τ-function

The relationship between points of Gr(H) and BA functions, according to Segal & Wilson [13], relies on the properties of the τ-function (4.20).

Let me recall that Γ_+ was defined by (4.9) as being the group of holomorphic maps $D_0 \to \mathbb{C}$, (D_0 being the unit disk $|z| \leq 1$) acting as multiplication operators on H (after restriction to S^1) and therefore on the points $W \in Gr(H)$. In particular, the operator q_ζ, belonging to Γ_+ for any given $\zeta \in \mathbb{C}$ such that $|\zeta| > 1$, has been introduced by (4.31).

(9.1) **Proposition** : Let W be transverse to H_-; then, the boundary value of $\tau_W(q_\zeta)$ for $|\zeta| \to 1$ (as a function of ζ) belongs to W.

To prove this statement, consider the compact operator A: $H_+ \to H_-$ whose graph is W (according to Sect. 4.3, A exists if W is transverse to H_-); let $\mathbf{w} \equiv \{w_k\}$ be an admissible basis for W, which shall be expanded in the form $w_k = \sum_{i \in \mathbb{Z}} w_{ik} z^i$. The matrix $W_+ \equiv \| w_{ik} \|_{i \geq 0}$ represents the projection operator $W \to H_+$, and one has $\sigma(W) = (\mathbf{w}, \det W_+)$. Write $g \in \Gamma_+$ in the block form $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ as in Sect. 4.3; then

(9.2) $$\tau_W(g) = \det(\mathbb{1} + a^{-1} b\, A) \quad .$$

A direct computation allows to see that for $g = q_\zeta$, the operator $a^{-1}b$ maps a function $f(z)$ into the constant function $f(\zeta)$. Since W is transverse to H_-, one can assume that each element w_k of the admissible basis w has order k. The element w_0 has therefore the form $w_0 = 1 + f_0(z)$, $f_0(z) \in H_-$. The matrix corresponding to the operator $a^{-1}bA$ has the only non-vanishing entry $(a^{-1}bA)_{00}$, which is equal to $f_0(\zeta)$. As a result, one finds

(9.3) $$\tau_W(q_\zeta) = 1 + f_0(\zeta) \quad .$$

At this point it is apparent that for $|\zeta| \to 1$, the boundary value of $\tau_W(q_\zeta)$ just coincides with w_0. ∎

I will now consider the subspace $\Gamma^W \subset \Gamma_+$, on which the τ-function τ_W does not vanish: $\Gamma^W \equiv \{g \in \Gamma_+ \,/\, \tau_W(g) \neq 0\}$; equivalently, Γ^W is characterized by saying that $g \in \Gamma^W$ iff $g^{-1} \cdot W$ is transverse to H_- (i.e. $g^{-1}W$ lies in the open set U_S; this makes evident the fact that Γ^W is dense in Γ_+). The formula (9.3) shows that for any $g \in \Gamma^W$ one can write

(9.4) $$\tau_{g^{-1}W}(q_\zeta) = 1 + \sum_{i>0} a_k(t)\, z^{-k} \quad ,$$

$t \equiv (x, t_2, t_3, \ldots)$ representing the set of parameters of g according to (4.10)

(9.5) **Definition** : The *Baker-Akhiezer function* ψ_W *associated to* $W \in Gr(H)$ is the function

$$\psi_W(z; t) \equiv g \cdot \tau_{g^{-1}W}(q_\zeta) = g \cdot \tau_W(x - z^{-1}, t_2 - (2z)^{-2}, t_3 - (3z)^{-3}, \ldots) \,/\, \tau_W(t) \ .$$

Formally, ψ_W admits an expansion of the form (3.3). From prop. (9.1) it follows that $\psi_W(z; t) \in W$ for any value of t : this is the crucial property of ψ_W. In principle, ψ_W has been defined only for those values $(x, t_2, t_3, ...)$ which correspond to elements of Γ^W; actually, ψ_W is an analytic function at these values, while it becomes singular outside Γ^W. However, since the singularities arise from the vanishing of $\tau_W(t)$ in the denominator of the right-hand side of the formula above, and one can check that the τ-function has only zeros of finite order, ψ_W extends meromorphically to the whole Γ_+.

(9.6) **Proposition** : For any $W \in Gr(H)$, $\psi_W(z; t)$ is the unique meromorphic function of the form $\exp[xz + t_2z^2 + ...] (1 + \sum_{i>0} a_k(t)\, z^{-k})$ which belongs to W (as a function of z) for any value of **t** .

In fact, let ψ' another such function; for $t = (0, 0, ...)$, ψ' must be proportional to w_0, since it is a function of order zero belonging to W; therefore it must be a multiple of ψ_W by a constant factor. By using the methods of Sect. 5.2, one can easily write a family of differential operators P_n such that ψ' is a solution of the spectral problem (5.6); these differential operators depend only on the expansion of ψ' at $t = (0, 0, ...)$ and must then coincide with those associated in the same way with ψ_W. Therefore ψ' and ψ_W satisfy the same linear differential equations, with the same initial value; thus they must coincide for all t. ∎

Let me consider more closely the family of differential operators P_n associated with ψ_W. A remarkable result of this setting is the following one:

(9.7) **Proposition** : If W belongs to $Gr^{(n)}$ for some n, the KP hierarchy associated with ψ_W (in the sense of Sect. 5.2) reduces to a subhierarchy not depending on the variable t_n.

From the very definition of $Gr^{(n)}$ it follows that $W \in Gr^{(n)}$ imply $z^n\psi_W \in W$. Then, also $(P_n - z^n)\psi_W \in W$; in fact $L\psi_W$ belongs to W for any differential operator P, since $\psi_W \in W$ for all x. Now, $\psi' \equiv (P_n - z^n)\psi_W$ must be a multiple of ψ_W, because it is easily seen to fulfill the hypotheses of (9.6); however, it is also evident that the coefficient of the expansions of ψ' and ψ_W do not coincide, except for the zeroth-order coefficient; then $\psi' \equiv 0$. I have already showed in Sect. 5.2 that if $P_n\psi_W = z^n\psi_W$, then the associate Volterra operator K, as well as the operator Q direclty occurring in the KP hierarchy (1.24), do not depend on t_n. ∎

Actually, I have already stressed that $Gr^{(kn)} \subset Gr^{(n)}$ for all positive integers k; consequently, infinitely many variables drop out from the KP hierarchy in the case of Prop. (9.7). In particular, if W belongs to two spaces $Gr^{(m)}$ and $Gr^{(n)}$, m and n being relatively prime, the KP hierarchy reduces to a hierarchy involving only finitely many variables. I will come back to this remark in the next Section; but it should be evident that the analogy of the condition above with the one occurring in the BCK construction is not accidental.

At this point, I wish to make two remarks. First, the procedure described above leads to an expression for the Volterra operator K associated to ψ_W, which is the same that Sato & Sato present in their paper [30]. Sato's approach seems more direct, but this depends only on the fact that [30] deals from the beginning with the image of the Grassmannian under the Plücker embedding. Secondly, let me recall that the introduction of the τ-function is due to Hirota, who was led to recast the KP hierarchy into a bilinear differential equation for τ. De Concini (see [11]) has shown that the Hirota bilinear equation can be derived from the observation that $\tau_W(-t) \equiv \tau_{W^\perp}(t)$; one can therefore define the two BA functions ψ_W and $\psi_{W^\perp}$ from the same τ-function, by simply reversing the sign of the parameters of the element $g \in \Gamma_+$; the two functions so obtained are orthogonal (since they belong to orthogonal subspaces). Writing the orthogonality relation for ψ_W and $\psi_{W^\perp}$ in terms of the τ-function, one obtains the Hirota equation.

9.2 From the BA function to Gr(H)

To find the point $W \in Gr(H)$ which is associated with a given BA function, one make use of the method presented in Sect. 4.3. The points which can be recovered in this way are those associated to BA functions such that, for some n,

$$\partial_n \psi = z^n \psi , \tag{9.8}$$

so that we expect to end up with a point $W \in Gr^{(n)}$. In general, the starting point is a differential operator L of degree n, rather that a BA function; one gets this latter one in the usual way, i.e. by defining a Volterra operator such that $L \circ K = K \circ D^n$, and setting the "initial value" $\psi(x, 0, 0, ...; z) = K\, e^{xz}$; (9.8) is then satisfied. As I have stressed above, the initial value of the BA function is sufficient to recover the whole spectral problem satisfied by ψ, and the spectral problem generates the evolution of ψ in a unique way; therefore, the problem is restricted to finding a space W which contains ψ and its derivatives calculated in $\mathbf{t} = (x, 0, 0, ...)$. Actually, the first (n-1) derivatives are enough to fix W; in fact, only n independent functions are needed to characterize a point in $Gr^{(n)}$, while, on the other hand, the other derivatives of ψ are dependent of the first (n-1) ones through the equation $L\psi = z^n \psi$. The only problem lies in the fact that, starting from a differential operator L, one gets in the way explained above a formal solution, whitout any information on the convergence of its expansion. To get a point W in the Grassmannian, one must assume that the series defining ψ, as well as its first (n-1) formal derivatives, actually converge. Segal & Wilson point out in [13] that this seems to be the only criterion available to single out the formal BA functions having an image in Gr(H). Having supposed that the derivatives $\psi^{(k)} \equiv D^k\psi(x, 0, ...; z)$, k=0, ..., n-1, actually define n functions of z, one obtains W through the formula (4.15). To ensure that $\gamma(z)$ is regular, one observes that $\psi^{(k)} \sim z^k$ for $z \to \infty$; therefore, up to a possible rescaling of z (ψ has nice transformation properties under rescaling, which I have not mentioned here), it is not restrictive to assume that $\gamma(z)$ is invertible for $|z|=1$. To complete the picture, I must show that ψ coincides with the BA function ψ_W. From the same argument as in the proof of (9.6), one can see that it is enough to prove that the initial values $\psi(x, 0, ...; z)$ and $\psi_W(x, 0, ...; z)$ should be the same; now, since both $\{\psi^{(k)}\}$ and $\{ D^k\psi_W(x, 0, ...; z)\}$ form a basis for W/z^nW, and each $\psi^{(k)}$ has exactly degree k, one can check by induction, starting from k=0, that the two sets of derivatives must coincide.

10. THE KRICHEVER MAP

One of the most relevant results about the UGM is the correspondence between points of Gr(H) and Krichever data.

Let (C, p, k, L, φ) be a Krichever datum as in (6.1), with $\deg(L)=g-1$, and assume that the range of the local coordinate k extends to the unit disk D_0 in C: the unit circle S^1 will thus be identified with its image on C through k. This setting is not restrictive, up to a possible rescaling of k which does not affect the construction below. Let U be a neighborhood of p included in the domain of k, and including the unit circle. By means of the trivialization φ, the local holomorphic sections of the line bundle L over $U \setminus p$ can be identified with local analytic functions : $\Gamma(U \setminus p, L) \cong \Gamma(U \setminus p, \mathcal{O}_C)$. Taking the restriction to S^1 of these functions, one can identify the Hilbert space H of Sect.4.3 with the L^2-closure of $\Gamma(U \setminus p, L)$, whereby the variable z is to be identified as usual with k^{-1}. The space H_-^{alg} (resp. H_-) can be identified with the subspace (resp. with the L^2-closure of the subspace) of functions on the unit circle which extend analytically to the unit disk, or, equivalently, which extend to holomorphic sections of L over U.

Observe that $C=(C \setminus p) \cup U$ and, up to retraction, $S^1=(C \setminus p) \cap U$, and consider the Mayer-Vietoris exact sequence

$$(10.1) \qquad 0 \to \Gamma(C, L) \to \Gamma(C \setminus p, L) \oplus H_-^{alg} \to H^{alg} \to H^1(C, L) \to 0 .$$

Taking the quotient by H_-^{alg}, one finds the following diagram:

$$
(10.2)\qquad
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & & H_-^{alg} & & H_-^{alg} & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & \Gamma(C,L) & \to & \Gamma(C\setminus p, L)\oplus H_-^{alg} & \to & H^{alg} & \to & H^1(C,L) \to 0\,. \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & \Gamma(C,L) & \to & \Gamma(C\setminus p, L) & \to & H_+^{alg} & \to & H^1(C,L) \to 0\,. \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0 & & 0
\end{array}
$$

which allows to prove that the lower row is also exact. From the Riemann-Roch formula one has:

$$(10.3)\qquad \dim \Gamma(C,L) - \dim H^1(C,L) = \deg(L) - g + 1 = 0 \ ,$$

and one concludes that $\pi_+ \colon \Gamma(C\setminus p, L) \to H_+$ is a Fredholm operator (I will omit the proof that π_- is compact; it can be found in [13]). In this way one identifies the subspace W^{alg} corresponding to the Krichever datum with the subspace of the functions on S^1 extending to holomorphic sections of L on $C\setminus p$; W is then recovered by taking the L^2-closure of W^{alg}.

Conversely, to get a curve and a line bundle from a point $W\in Gr(H)$, one has just to apply the standard procedure described in Sect. 2.2 and 2.4. This time, the relevant objects are the ring A_W introduced by (4.12) and the A_W-module W^{alg}. The construction does not always lead to non-trivial results; for instance, if W does not belong to any one of the subspaces $Gr^{(n)}$, A_W contains only the constant functions, and $Spec(A_W)=Spec(\mathbb{C})$ is only one point. On the other hand, if $W \in Gr^{(n)}$ for some n, and therefore $W \in Gr^{(kn)}$ for all positive integers k, all the monomials z^{kn} are contained in A_W. For example, consider the space H_S, S = (-3, -1, 1, 3, 4, 5, ...); we have $H_S \in Gr^{(n)}$ for n = 2, 4, 6, 7, ... In this case, A_W is the ring $\mathbb{C}[z^2, z^7]$, which can be identified with $\mathbb{C}[x, y]/(y^2-x^7)$. The associated curve C is the singular hyperelliptic curve $y^2=x^7$. One easily realizes why the points of $Gr^{(2)}$ lead to hyperelliptic curves.

However, if A_W does not contain functions of any order above some finite order n_0, W^{alg} cannot be an A_W-module of rank one. Krichever [8], Mumford [35], and Verdier [20] have discussed possible generalization of the BCK construction to higher-rank bundles, corresponding to rings of operators having higher-dimensional common eigenspaces, but this is beyond the scope of this thesis. A sufficient condition in order W^{alg} to be a rank-one A_W-module is that W belong to two subspaces $Gr^{(m)}$ and $Gr^{(n)}$, with m and n relatively prime. We already know from Prop. (9.7) that in this case only a finite number of variables occur in the KP hierarchy. This should not be surprising; in fact, once one has recovered a Krichever datum including a curve C of finite genus, the KP flows can be recast as linear flows on Jac(C), and the tangent space to this torus is finite-dimensional. The requirement on the integers m and n above is therefore the exact translation of the analogous requirement in the BCK setting. In fact, it is easy to generalize the argument of the proof of Prop. (9.7) to build an isomorphism between A_W and the commutative ring of differential operators having ψ_W as common eigenfunction. The latter statement allows to close the diagram of the cover page.

La nature est un temple où de vivants piliers
Laissent parfois sortir de confuses paroles;
L'homme y passe à travers des forêts de symboles
Qui l'observent avec des regards familers.

Ch. Baudelaire, *Correspondances*

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