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Schroedinger Operators for Systems with Fractional Statistics

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The first aim of this brief report is to outline a relatively new topic in quantum mechanics, namely *Anyon systems*, on which my interest during this last year has been focused. Moreover, I would like to introduce the research field in which I'm going to work in the next future.

The study of Anyon systems is motivated on the observation that, for systems in two-dimensional space, particle statistics is described by representations of the *braid groups* ([S],[LD] and [D]). Systems with abelian braid statistics were first proposed and studied by Leinaas and Myrheim [LM]. This kind of "particles" has been called *anyons* by Wilczek [W1].

There is a quite large physics literature on this subject (see e.g. [W2],[L] and reference therein). Indeed, these models may have some interest in Solid State Physics. More precisely, they have been proposed for the description of the Fractional Quantum Hall Effect. Furthermore, some speculations on the role of anyons in high T_c superconductors appeared in the literature.

I was introduced to this subject by P.A. Marchetti. Froehlich and Marchetti have developed the quantization of such particles in the framework of three-dimensional Euclidean quantum field theories. They have analyzed rigorously a lattice model with Chern-Simons term and proved that anyon fields couple to the vacuum to a stable massive one-particle state ([FM1]).

As far as I know, (in the non-trivial case of $N \geq 3$ anyons) there are only few results concerning the mathematical foundations of these quantum mechanical models (i.e. self-adjointness of the Hamiltonian, spectral analysis and scattering theory)(see [BCMS]).

By the way, it is worth to mention here that a deep analysis of statistics based on fundamental postulates of local, relativistic quantum theory (algebraic approach) was carried out by Doplicher, Haag and Roberts ([DHR]). They classified all possible statistics (para-Bose and para-Fermi statistics of order d=1,2,...) compatible with locality and certain general assumptions on the nature of physical states, for theories in four and more space-time dimensions. The starting point of their analysis was reconsidered by Buchholz and Fredenhagen [BF]. They gave a more general foundation that includes gauge theories. Finally, Doplicher and Roberts [DR] succeeded in proving that the parastatistics of "charged" particles (in the sense of the theory of superselection sectors) in *local*, relativistic quantum theory could always be reinterpreted as ordinary Bose or Fermi statistics by introducing additional, internal degrees of freedom.

In this algebraic framework, a rigorous analysis of braid statistics in low dimensions (and under some assumption, a general connection between spin and statistics) has been performed ([FRS],[FGM]). On the basis of the analysis in [BF], one should not expect, in general, that one-particle states can be created by fields localizable in bounded regions. For *massive* relativistic field theories, these states can, however, be localizable in space-like cones of arbitrarily small opening angle. Non-local (in the sense described above) fields in 2+1 dimensions do not obey ordinary (Bose or Fermi) permutation statistics; rather, they obey braid statistics.

However, it seems that there is no complete understanding on how the geometric (quasiclassical) approach of Leinaas and Myrheim to braid-group statistics in quantum mechanics is related to the just mentioned algebraic features ([FM2]).

There are many (mathematical) open problems in this subject. But, first of all, I think one has to define, in a rigorous way, the quantum mechanical model of N anyons. Probably, the most natural way to face this problem is to study the Laplace operator acting on sections of an appropriate vector bundle (see below) over the configurations space of N identical partical, but this seems to be a difficult task (I would like to thank C. Reina for useful discussions on this point).

However, one can introduce a simplified framework. Indeed, it is possible to consider the Laplacian on an open subset of $\mathbb{R}^{Nd}$ with the "anyonic" boundary conditions as specified in the following.
Let me give a brief intoduction on this subject and the example of two anyons, keeping in mind this problem.

In quantum mechanical systems, in two-space dimensions, values are allowed for the spin (and type of statistics) which can not occur in higher dimensions where, as it is well know, only integer or half integer angular momentum (and correspondingly Bose or Fermi statistics) are allowed.

In rotation-invariant quantum systems, in $d \geq 2$ space dimensions the spin S labels the irreducible unitary representations of the covering group $S\tilde{O}(d)$. In $d > 2$ these groups are non-abelian and the non-trivial commutation relations lead to the quantization of the angular momentum. In this case, the spin S can take only integer (bosons) or half integer (fermions) values. On the other hand, in $d = 2$, the fact that the rotation group is abelian leads to a larger range of possible values for the spin of particles. We have that $S\tilde{O}(2) \simeq \mathbb{R}$ and so its irreducibile representations are labelled by real numbers $S \in \mathbb{R}$, fixed in the following way. Let's consider the rotation by an angle of 2π: it is implemented on the Hilbert space of states by a unitary operator $U(2\pi)$ which is required to commute with all physical observables. Since the representation is irreducible it follows that $U(2\pi)$ must be a multiple of the identity : $U(2\pi) = e^{i2\pi S} I$. Hence, we have a theory of superselection sectors labelled by S and in each sector the angular momentum is quantized as $\hbar(S+integer)$. From a Schroedinger quantization point of view, one can look at the (one-particle) angular momentum operator in two dimensions. The operator $i\frac{\partial}{\partial\theta}$ on $L^2((0,2\pi))$ is symmetric and well-defined on $\mathcal{D} = \{f \in AC[0,2\pi] : f(0) = f(2\pi) = 0\}$.It is well known that this operator is not essentially self-adjoint. Indeed, it has a one-parameter family of self-adjoint extensions characterized by the following domains labelled by $S \in [0,1)$ (which corresponds to the spin):

$$\mathcal{D}_S = \{f \in AC[0,2\pi] : f(2\pi) = e^{i2\pi S} f(0)\}.$$

These simple remarks give rise to a natural question: what about statistics in two dimensions ? One can introduce the concept of quantum statistics in the following way.

Let M be the classical configurations space (can be not simlpy connected) of a physical system. In quantum mechanics the states of the system can be described by the sections of a complex vector bundle $E(M, \mathbb{C}^N)$ associated to the universal covering bundle of M. These sections correspond, in a natural way, to single-valued functions on $\tilde{M}$, the universal covering space of M, with values in $\mathbb{C}^N$ and having certain covariance properties under the action of covering transformations. This means that there exists a unitary representation U of the fundamental group $\pi_1(M)$ on $\mathbb{C}^N$ such that

$$\psi([\omega] \cdot q) = U([\omega])\psi(q), \qquad (*)$$

$\forall q \in \tilde{M}$ and $\forall [\omega] \in \pi_1(M)$.
The Hilbert space stucture on the space of states is :

$$\mathcal{H}_U = \{\psi \in L^2(\tilde{M}, d\mu|_{D_0}) \otimes \mathbb{C}^N \ : \ (*) \text{ holds}\},$$

where D_0 is a fundamental domain for the action of $\pi_1(M)$ on $\tilde{M}$ and $d\mu$ is a measure on $\tilde{M}$, invariant under the action of $\pi_1(M)$. In this setting one has that different inequivalent choices of the representation U lead to different quantizations of the underlying classical system. Note that the choice of the representation is equivalent to the choice of a flat connection on the bundle E, whose holonomy is given by the representation itself.
Now, the classical configurations space of N identical particles in d-space dimensions is

$$M_N = \frac{(\mathbb{R}^d)^{\times N} \backslash D_N}{S_N}$$

where $D_N = \{(x_1, ..., x_N) \in (\mathbb{R}^d)^{\times N} : x_i = x_j, i \neq j\}$ and S_N is the permutation group of N elements. From this point of view, a choice of particle statistics corresponds to a choice (up to equivalence) of an irreducible unitary representation of $\pi_1(M_N)$ in the quantization procedure described above. The fundamental group of M_N is :

$$\pi_1(M_N) = B_N \qquad for \ \ d = 2$$
$$\pi_1(M_N) = S_N \qquad for \ \ d \geq 3$$

where B_N is the *braid group of N identical strands* (Artin's Braid group).
B_N is the group generated by N-1 generators $\{\tau_i\}$ with the following relations :

(i) $\tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1}$
(ii) $\tau_i \tau_k = \tau_k \tau_i \quad |i-k| \geq 2$.

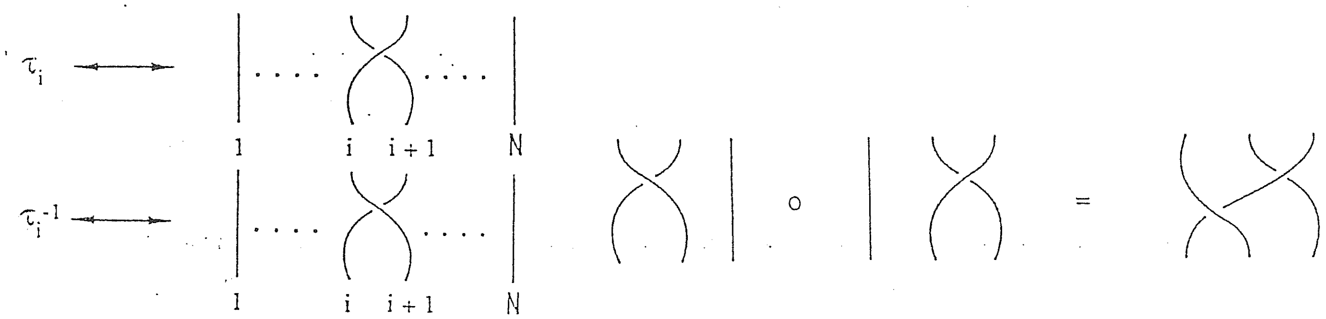

fig.1

Now, following the quantization procedure descibed above, we have to look at the one-dimensional, unitary, irreducible representations of $\pi_1(M_N)$. In the case of the permutation group S_N ($d \geq 3$), one has only two representations of this kind:

$$1.U_N(\pi) = I \qquad (\forall \pi \in S_N) \Longrightarrow \text{ bosons}$$
$$2.U_N(\pi) = (-1)^{|\pi|} \quad (\forall \pi \in S_N) \Longrightarrow \text{ fermions}$$

(where $|\pi|$ is the signature of π).
For $d = 2$ spatial dimensions we have to consider the braid group B_N . It has a family of one-dimensional, unitary, irreducibile representations labelled by a real parameter. The latter interpolates between the two representations that factorize through the subgroup of permutations which corresponds to boson and fermion systems:

$$U_N(\tau_i) = e^{i2\pi\alpha} I \quad \alpha \in [0,1).$$

In this case we have the so called *Anyon systems.*

For boson and fermion systems, as it is well known, we can incorporate the effect of statistics as a symmetry property of a many-particle wave function under exchange of particle variables. One can construct the N particle Hilbert space of states as a tensor product of one-particle Hilbert spaces: $\mathcal{H}_N = \otimes^N \mathcal{H}_1$. It carries a representation of S_N which decomposes on the irreducibile ones: the restriction of $\mathcal{H}_N$ to the subspace of symmetric (bosons) or antisymmetric (fermions) tensor products.

On the other hand, $\mathcal{H}_N$ is not a representation space for B_N . As a matter of fact, the Hilbert space of N anyons has the structure of a tensor product of one-particle states only when, in the Schroedinger picture, it is restricted to simply connected regions of the configurations space M. Indeed, the effect of statistics on the many-particle wave functions depends not only upon the permutation of particle variables but also on extra data given by the braiding. However, we can again incorporate the effect of statistics as a boundary conditions on the wave functions. But we have to consider the wave functions as single-valued functions defined on a fundamental domain D_0 (w.r.t. the action of the braid group) of the universal covering space $\tilde{M}$ $(*)$.

The above considerations show that in two dimensions, due to non-trivial topology property of the classical configurations space of N particle , it is possible to have anyon-statistics. Now, one has to consider the dynamics for such systems. Let's use Schroedinger quantization procedure and face the problem of defining in some sense a free-dynamics. It seems reasonable to consider, at least formally, the usual free-particle Hamiltonian, but acting on wave functions defined on a fundamental domain D_0 :

$$D_0 = \{(\vec{x}_1, ..., \vec{x}_N) \in (\mathbb{R}^2)^{\times N} \ : \ x_1 > x_2 > ... > x_N\}.$$

Then, the following boundary conditions on $\tilde{\psi}_N(q) \in L^2(\tilde{M}, d\mu|_{D_0})$ take into account the effect of statistics:

$$\tilde{\psi}_N(b \cdot q) = U_N(b)\tilde{\psi}_n(q)$$

$\forall b \in B_N$ and $\forall q \in D_0$.

Notice that one can approach this problem by analysing an equivalent one: let $\psi(x_1, ..., x_N) = \Theta(x_1, ..., x_N) f(x_1, ..., x_N) \in C_0^\infty(D_0)$, where

$$\Theta(x_1, ..., x_N) = \prod_{i<j}^{N} e^{i\alpha\theta_{ij}},$$

with $\theta_{ij} = \arctan(\frac{y_i - y_j}{x_i - x_j})$ and $f(x_1, ..., x_N)$ a symmetric function.
Then, one obtain on $C_0^\infty(D_0)$

$$-\sum_{k=1}^{N} \Delta_k \psi = -\sum_{k=1}^{N} \Theta(\nabla_k + i\alpha\vec{A}_k)^2 f,$$

with $\vec{A}_k(x_1, ..., x_N) = \sum_{i \neq k} \left(\frac{y_i - y_k}{|\vec{x}_i - \vec{x}_k|^2}, -\frac{x_i - x_k}{|\vec{x}_i - \vec{x}_k|^2} \right)$.

Hence, the effect of statistics can be realized, formally, by a singular, pure gauge vector potential (Aharonov-Bohm effect). Note that $\vec{A}_k$ is not locally square-integrable so there are not at our disposal general theorems stating essential self-adjointness of this covariant Laplacian. Moreover, as we will see at least in the two body case, this operator requires further "radial" boundary condition specifying the self-adjoint extension.

In the particular case of two anyons, one can handle with this problem in a quite simple way. Indeed, in the center-of-mass frame we are dealing, as usual, with an effective one-particle system. Moreover, for the radial symmetry, one can decompose the Hilbert space of relative motion on its angular momentum eigenspaces.

On each eigenspace one has to analyse the problem of self-adjoint extensions of the Schroedinger operator $-\frac{d^2}{dr^2} + \frac{\lambda(\lambda-1)}{r^2}$ on $C_0^\infty(\mathbb{R}_+)$. This problem was exaustively discussed in the literature (i.e. [R],[BG] and [AGHH]). In particular, the theory of self-adjoint extensions of this kind of operators, for $\frac{1}{2} \leq \lambda < \frac{3}{2}$ (values of no essential self-adjointness), gives rise to a mathematically rigorous definition of systems with "zero-range interactions".

The effect of statistics, in the two body case, consists in a shift of the angular momentum eigenvalues proportional to α and correspondingly in a centrifugal barrier increasing with the statistical parameter.

In the $N \geq 3$ case, the pairs angular momentum are no longer a conserved quantity. This fact makes the problem more difficult but also more interesting.

In order to analyse the general N body case, we have tried to obtain the well known results for the two-body case with a different method which do not implies the use of relative motion and angular momentum decomposition.

As we have noticed above, the problem of self-adjointness of the Laplacian with these "anyonic" boundary conditions seems to be intimately related to the problem of "zero-range interactions".

It is worth to emphasize that an important difference exists between the self-adjoint extensions of the Laplacian in two dimension with bosonic boundary conditions and that of the Laplacian with these "anyonic" boundary conditions. In the former, no extension is needed: an extension represents an additional interaction ("zero-range interaction"). On the other hand, for generic anyons an extension is required also to define "free" dynamics and it represents further informations that must be specified when describing the physics of the system.

An idea to face this problem is to look at the methods used in the study of point interactions. In particular, Dell'Antonio, Figari and Teta [DFT] have developed a method using renormalization techniques of singular quadratic forms to analyze Hamiltonians for N particles interacting through zero-range forces in two dimensions.

It was pointed out by themselves that in the problem at hand the singular locus is of codimension one, while in the former problem it was of codimension two. So that, they suggest to consider also the extensions defining δ' -interactions [AGHH]. These extensions could give the right discontinuity ("anyonic" boundary conditions) on this singular locus. So that a possible direction could be to generalize the techniques in [DFT] to δ' -interactions.

Example : *N=2 Anyons System (free dynamics)*

The classical configurations space in the center-of-mass frame is:

$$M_2 = \frac{\mathbb{R}^2\backslash\{0\}}{\mathbb{Z}_2} = \mathbb{R}_+ \times S^1$$

(w.r.t. the following equivalence relation : $x \sim -x$).
Its fundamental group is $\pi_1(M_2) = \mathbb{Z}$ and its universal covering space is : $\tilde{M}_2 = \mathbb{R}_+ \times \mathbb{R}$.
So we have a fundamental domain $D_0 = \mathbb{R}_+ \times [-\frac{\pi}{2}, \frac{\pi}{2}) \ni x = (r, \theta)$.
Formally, the free Hamiltonian for α -anyons is the operator:

$$h^{(\alpha)} = -\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2}\frac{d^2}{d\theta^2}, \quad \alpha \in [0,1)$$

on the Hilbert space $L^2(\mathbb{R}_+) \otimes L^2((-\frac{\pi}{2}, \frac{\pi}{2}))$. The index (α) means that the operator $\frac{d^2}{d\theta^2}$ is considered on the following domain:

$$\mathcal{D}_\alpha = \{f \in AC(-\frac{\pi}{2}, \frac{\pi}{2}) : f(\frac{\pi}{2}) = e^{i2\pi\alpha} f(-\frac{\pi}{2})\}.$$

Using the decomposition in (relative) angular momentum eigenspaces we have $\mathcal{H}^{(\alpha)} = \oplus_{m\in 2\mathbb{Z}} \mathcal{H}_{m+\alpha}$, where $\mathcal{H}_{m+\alpha} = L^2(\mathbb{R}_+, rdr) \otimes [Y_{\alpha+m}]$ and $[Y_{\alpha+m}] = \{e^{i(\alpha+m)\theta}, m \in 2\mathbb{Z}\}|_{L^2((-\frac{\pi}{2}, \frac{\pi}{2}))}$.
On each $\mathcal{H}_{m+\alpha} \sim L^2(\mathbb{R}_+, rdr)$ we can use the unitary transformation:

$$\tilde{U} : L^2(\mathbb{R}_+, rdr) \longrightarrow L^2(\mathbb{R}_+, dr), \quad (\tilde{U}f)(r) = r^{1/2} f(r);$$

then we have to analyze the following formal operator :

$$h_{m+\alpha} = -\frac{d^2}{dr^2} - \frac{1}{4r^2} + \frac{(m+\alpha)^2}{r^2}.$$

on $L^2(\mathbb{R}_+)$.
For this type of operator we have these results [RS II]:
If $|m+\alpha| \geq 1$ (i.e. $\forall m \neq 0$) then $h_{m+\alpha}$ is essentially self adjoint on $C_0^\infty(\mathbb{R}_+)$; while if $|m+\alpha| < 1$ (i.e. m=0) then it is not essentially self-adjoint. In particular, it has deficiency indices (1,1), so that it admits a one-parameter family of self-adjoint extensions. In the s-wave eigenspace (m=0), we have to characterize the one-parameter family of self-adjointness domains. In the case at hand, this can be explicitly seen: the functions φ_α in the kernel of $(H_\alpha|_{C_0^\infty((0,\infty))})^* \pm i$ are solutions of the following ordinary differential equation

$$[-\frac{d^2}{dr^2} - \frac{1}{4r^2} + \frac{\alpha^2}{r^2}]\varphi_{\alpha\pm} = \mp i\varphi_{\alpha\pm} .$$

The requirement that $\varphi_\alpha(r)$ be in $L^2(\mathbb{R}_+, dr)$ implies (up to a constant) for $|\alpha| < 1$

$$\varphi_{\alpha\pm}(r) = r^{1/2} H_\alpha^{(i)}(e^{\mp i\frac{\pi}{4}} r),$$

where $H_\alpha^{(i)}$ (i=1,2) are the Hankel functions. With these explicit solutions one can determine the domains of self-adjointness in a standard way ([RSII]).

Nevertheless, one can give a more concrete characterization of these domains, using a general method developed for a large class of Schroedinger operators on the real half line in the paper of Bulla and Gesztesy[BG].

In our case, the domains of self-adjointness can be characterized in the following way:

$$\mathcal{D}(h_{\alpha,\nu}) = \{f \in L^2(\mathbb{R}_+) : f, f' \in AC_{loc}(\mathbb{R}_+) \ ; \ \nu f_{0,\alpha} = f_{1,\alpha} \ ; \ h_{\alpha,\nu} f \in L^2(\mathbb{R}_+)\},$$

with $-\infty < \nu \leq \infty$; where the boundary values $f_{0,\alpha}$ and $f_{1,\alpha}$ are defined as:

$$f_{0,\alpha} = \lim_{r \downarrow 0} \frac{2\alpha f(r) r^\alpha}{r^{1/2}},$$
$$f_{1,\alpha} = \lim_{r \downarrow 0} \left[\frac{f(r)}{r^{\alpha+1/2}} - \frac{f_{0,\alpha}}{2\alpha r^{2\alpha}} \right]$$

for $\alpha \neq 0$. The case $\alpha = 0$ corresponds to two bosons in two dimensions (see [AGHH] for a detailed analysis).

In particular, the boundary condition $f_{0,\alpha} = 0$ (i.e. $\nu = \infty$) represents the Friedrichs extension of h_α. It is reasonable to take this extension as definition of "free dynamics" for this system. Moreover, these results show that anyons can have *point interactions* (except in the fermionic case, i.e. $\alpha = 1$).

To understand this kind of boundary conditions, let us look at the behaviour, as $r \downarrow 0$, of a general solution of the radial equations (on s-wave):

$$f(r) \xrightarrow{r \downarrow 0} c_1 r^{1/2-\alpha} + c_2 r^{1/2+\alpha}$$

The condition $\nu f_{0,\alpha} = f_{1,\alpha}$ gives the following relation between the constants c_1 and c_2: $\nu c_1 = \frac{1}{2\alpha} c_2$.

Notice that for $\alpha = \frac{1}{2}$ (semions) we have the radial equation for two bosons in three dimensions. The relation above shows, in this case, that the limit value of the first derivative at the origin is given by the limit value at the origin of the functions in the domain of the operator. These are, indeed, the mixed conditions defining "point interaction" boson systems in three dimensions.

In the next future, I have the opportunity to study, under the supervision of I.M.Sigal, some relatively new techniques developed in the analysis of scattering theory: *phase space analysis* and *propagation estimates.*

Consider an N-body system in the center-of-mass frame and let H be its self-adjoint Schroedinger operator. The *scattering theory* studies the large time behavior of the orbits $e^{-itH}\psi$, for the states ψ orthogonal to the bound states. One expects that as $t \to \pm\infty$ the system breaks down into independently moving, stable subsystems. Cast into rigorous terms this problem is called *asymptotic completeness* .

The scattering theory involves a comparison between two different dynamics for the same system: the given dynamics and a "free" dynamics. This comparison can be accomplished by means of *wave operators.* The main mathematical problem of the quantum-machanical scattering theory amounts to prove the existence of the wave operators and to establish their properties: isometry, mutual orthogonality (w.r.t. distinct clusters) and asymptotic completeness.

The existence proof goes back to a simple and very effective criterion of J.Cook. At the same time, it was shown by J.M. Jauch that existence implies readily isometry and mutual orthogonality. Asymptotic completeness, however, was found to be a very hard problem.

One can say that there are two basic parameters characterizing the qualitative behaviour of the asymptotic evolution: the number of particles, N , and the decay rate, μ , of the pair potential. As N goes from 2 to 3 and more, the geometry of many-body potentials $V(x) = \sum V_{ij}(x_i - x_j)$ changes considerably. Indeed, in the case $N \geq 3$, the potential V(x) does not vanish as $|x| \to \infty$ along certain hyperplanes X_a , where a labels different break-ups of the system into subsystems.

As far as the decay rate is concerned, when μ passes through 1, the threshold phenomena (motion with zero relative velocity) become important (see below) and the asymptotic behaviour of ψ_t as $t \to \pm\infty$ changes (one has to consider modified wave operators).

The two-particle systems were finally solved for all short-range potential ($o(|x|^{-1})$ at infinity) by S. Agmon [A] and V. Enss [E1], and for a large class of long-range potentials ($O(|x|^{-\alpha}$, $0 < \alpha \leq 1$ at infinity) by V.Enss[E2], P.Perry [P] and others (see [RS-III] and references therein).

In N-body systems the statement of asymptotic completeness expresses the fact that as $t \to \pm\infty$, $e^{itH}\psi$ approaches a superposition of waves propagating freely (classically) along the planes X_a while committed to a bounded (quantum) motion in the perpendicular directions.

For the three-body case the first result is due to L.D. Faddeev [F]. After his famous work, there were a number of papers improving considerably Faddeev's method. However, their results are very close to those of Faddeev. Namely, they proved asymptotic completeness for three-body systems with potentials $o(|x|^{-2})$ at infinity, under certain implicit assumptions on the potentials.

An important breakthrough was made by V.Enss who introduced time-dependent method and phase-space analysis to this problem [E1,2]. He was able to prove the asymptotic completeness and the existence of all wave operators of 3-body systems if the potentials decay like $|x|^{-\mu}$ with $\mu > \sqrt{3}-1$ (notice that this result includes long-range systems).

In the general N-body short-range case ($\mu > 1$) the proof of the asymptotic completeness was first given by I.M.Sigal and A.Soffer [SigSof1]. Their proof was based on the time dependent approach and on a detailed analysis of the propagation in phase space.

They have shown that, in a certain sense specified below, the evolution $e^{iHt}\psi$ with total energy E (i.e. with ψ in a small spectral interval around E), which is away from thresholds of H, concentrates as $|t| \to \infty$ (not uniformly in time) on a certain set of phase-space, called *propagation set* at energy E. This set is a collection of classical trajectories followed by free stable clusters resulting from various breaks-up of the system. The coordinates and momenta of the centers-of-mass of the clusters are parallel or antiparallel depending on whether the cluster are outgoing or incoming. The restriction on the kinetic energy of this free classical motion stems from the energy conservation law and the fact that the stable internal motion of the clusters is described by bound state of their internal Hamiltonian and, consequently, the energy of the internal motion is given by the corresponding eigenvalue.

Propagation estimates show that the probability for ψ_t to be in a phace-space region disjoint from the propagation set vanishes as $|t| \to \infty$ in the following sense: for any *phase-space operator* $J(x,p)$ supported outside the propagation set at energy E there is a small interval $\Delta \ni E$ s.t.

$$\int_{-\infty}^{+\infty} dt \parallel J(x,p) < x >^{-1/2} e^{-iHt}\psi \parallel^2 \leq C \parallel \psi \parallel^2$$

for any $\psi \in \mathrm{Ran}P_\Delta(H)$, where $< x >= (1+|x|^2)^{1/2}$ and $C < \infty$ and indipendent of ψ.

These estimates reflect the intutive picture that after collision a system disintegrates into a number of stable clusters whose centers-of-mass follow classical trajectories.

To connect the phace-space picture with the usual physical-space picture, they construct a particular *phase-space partition of unity* with elements $j_{a,E}(x,p)$ (symbols of phase-space operators) supported in phase-space regions where the system is broken into "well separated" clusters. Moreover, the x-boundaries of these regions lie outside the propagation set. Thus each element of this partition is associated with a certain break-up dynamically decoupled from other breaks-up (decoupling of channels). On each element of the partition the total evolution e^{-iHt} should behave as the cluster evolution e^{-iH_at}, where $H_a = H-$ (intercluster interaction) (i.e. H_a describes independently moving clusters associated with the break-up a). To compare these two evolutions one introduces the Deift-Simon wave operators for given E and sufficiently small $\Delta \ni E$:

$$W^{\pm}_{a,E} = s - lim_{t\to\pm\infty} e^{-iH_a t} j_{a,E}(x,p) e^{-iHt}.$$

If this limit exists on $\text{Ran}P_{\Delta}(H)$ then the system is asymptotically complete ([SigSof1]).

In subsequent papers ([Sig1] and [SigSof2-3]), the authors improved the propagation estimates described above. They have specified further the propagation set by replacing the arbitrary coefficient of proportionality between positions and momenta with the time. Using these techniques, they have shown asymptotic clustering for Coulomb-type potentials.

However, in the long-range case, even when the system breaks-up into indipendent subsystems (possibly unstable), namely when it is *asymptotically clustering*, the intercluster interaction cannot be entirely ignored, as in the short-range case. It leaves a trace in the form of an overall time-dependent potential in the internal coordinates of the clusters. As a result the energy for the broken-up system is not conserved and one cannot exclude the possibility of the build up of probability around the threshold energies of clusters (no asymptotic completeness).

Asymptotic completeness for many-body long-range systems is presently the main mathematical problem of many-body scattering theory, but it is clear that for $N \geq 4$ it requires conceptually new understanding .

However, in order to have deeper insight into the physical mechanism as well as into mathematical methods it is interesting to study the three-particle scattering problem for potentials with arbitrary slow decay. A combination of methods of the just mentioned papers can be a starting point in the treatment of this problem.

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