ISAS - INTERNATIONAL SCHOOL
FOR ADVANCED STUDIES

Thesis submitted for the degree of "Magister Philosophiae"

RECONSTRUCTION OF A HOMOGENEOUS RIEMANNIAN MANIFOLD FROM THE CURVATURE AND ITS COVARIANT DERIVATIVES

Candidate:
Lorenzo NICOLODI

Supervisor:
Prof. Franco TRICERRI

Academic Year 1988/89

TRIESTE

SISSA - SCUOLA INTERNAZIONALE SUPERIORE DI STUDI AVANZATI
TRIESTE
Strada Costiera 11

Scuola Internazionale Superiore di Studi Avanzati
International School for Advanced Studies

Thesis submitted for the degree of "Magister Philosophiae"

RECONSTRUCTION OF A HOMOGENEOUS RIEMANNIAN MANIFOLD FROM THE CURVATURE AND ITS COVARIANT DERIVATIVES

Candidate : Lorenzo NICOLODI   Supervisor : Professor Franco TRICERRI

Trieste, Academic Year 1988-89

CONTENTS

INTRODUCTION

A well-known theorem of Elie Cartan (see f. ex. [2], p.97) states that the curvature tensor R and its successive covariant derivatives D^sR with respect to the Levi Civita connection are a complete set of local invariants of the analytic Riemannian metrics. In a more precise manner, let (M,g) be an analytic Riemannian manifold (i.e. M and g are both analytic) and $(U,x_1,...,x_n)$ a normal coordinate system centered at $p \in M$, then the coefficients of the Taylor series expansion at p of $g_{ij} = g(\partial/\partial x_i, \partial/\partial x_j)$ are polynomials in the components of D^sR_p, $s = 0,1,2,...$, $D^0R = R$, that do not depend on the given manifold. We remember that a normal coordinate system of the type above always exists and is given, in terms of the exponential map exp_p of (M,g), by the formulae

$$x_i(exp_p(\textstyle\sum_{j=1}^{n} a_j u_j)) = a_i, \quad i = 1,...,n,$$

where $(u_1,...,u_n)$ is an orthonormal basis of the tangent space T_pM.

As a corollary we have that if (M,g) and (M',g') are two analytic Riemannian manifolds and if there exists a linear isometry $F : T_pM \to T_{p'}M'$ such that $F^*D'^sR'_{p'} = D^sR_p$ for each $s = 0,1,2,...$, then $f = exp_{p'} \circ F \circ exp_p^{-1}$ is a local isometry of a neighbourhood of p onto a neighbourhood of p' such that f(p) = p' and $f_{*|p} = F$. This isometry is unique, since an isometry is univocally determined by the image of a point and the corresponding differential map at that point.

Therefore, the preceding theorem says that the knowledge of R and D^sR, for all $s = 1,2,...$ at a point p determines uniquely (up to local isometries) the germ of the metric at p and solves the problem that historically is known as the *equivalence problem* (when are two Riemannian manifolds locally isometric ?).

Thus, in principle, we can recover all the relevant properties of the metric from R and its covariant derivatives.

In this thesis we deal with the problem of recognizing the *homogeneity* of a Riemannian manifold (M,g) from the curvature and its covariant derivatives.

If (M,g) is locally homogeneous, then, for each p and q of M, there exists an isometry $F : T_pM \to T_qM$, such that

$$F^*D^sR_q = D^sR_p, \tag{0.1}$$

for all s = 0,1,2,... , that is (M,g) is *strongly curvature homogeneous* in the terminology of [19]. So these tensors are the "same" in all the points of M.
Conversely, let (M,g) be a strongly curvature homogeneous Riemannian manifold and suppose, furthermore, that (M,g) is analytic. Then, by the previous theorem, (M,g) is locally homogeneous.

Actually, a stronger result holds. In fact, it can be proved that there exists an integer $k_M \leq n(n-1)/2 - 1$, $n = \dim M$, such that, if for each point p and q of M there exists an isometry $F : T_pM \rightarrow T_qM$ verifying (0.1) for all $0 \leq s \leq k_M + 1$, then the manifold (M,g) is locally homogeneous. Hence, in particular, a strongly curvature homogeneous Riemannian manifold is locally homogeneous.

This result was originally proved by I.M. Singer ([20]) when M is complete and simply connected. The local version as stated above was later obtained by Nomizu ([19]).

The proof of this theorem does not require the analyticity of the manifold and of the metric. Only a general reduction lemma and some general results from the theory of the infinitesimal connections on principal fibre bundles are necessary.

This theorem raises certain natural questions and problems ([20], [6]). For instance one can ask if there are spaces with the same curvature and the same covariant derivatives up to some order (less than $k_M + 1$), but which are not locally homogeneous. Examples of Riemannian manifolds with the same curvature (*curvature homogeneous Riemannian* manifolds), but that are not locally homogeneous are studied in [21],[23]. Another problem is treated in [15].

In chapter 2, section A, we reprove the theorem of Singer by a more direct approach providing a simplified treatment of some crucial steps of the proof.
By using the same approach we also prove, in addition, that a homogeneous Riemannian manifold is completely determined, among the class of all Riemannian manifolds, by the curvature and its covariant derivatives at some point up to order $k_M + 2$ (theorem 2.5).

On account of this, it is natural to ask how it is possible to recover the homogeneous space M and the metric g from these curvature data, i.e. how one can algebraically recapture the Lie algebra of the group of isometries and the isotropy group from these curvature data. This would provide a generalization to the homogeneous Riemannian spaces of the analogous theorem of E. Cartan concerning the Riemannian symmetric spaces (see f. ex. [3], p. 296), for which the curvature alone determines all.

The answer to this question is given, without proof, at the end of the quoted paper of Singer, where the necessary and sufficient conditions on a set of tensors are stated that they be the curvature tensor and its covariant derivatives at a point of a homogeneous Riemannian manifold. At our knowledge, a proof of this result has not been yet exhibited.
As a matter of fact this statement, indicated here as the second Singer's theorem, is incorrect ; in chapter 3, section B, we will present a counter-example to it given by O. Kowalski (see [12]). We will provide with theorem 3.1 the correct version of the second Singer's theorem and a demonstration of it. Further we show how to reconstruct a homogeneous Riemannian manifold only from these curvature data (see [17]).

The first part of chapter 3 is devoted to the theory of *infinitesimal models* of a homogeneous Riemannian manifold as developed in [23]. An infinitesimal model is a triplet (V,T,K), where V is an Euclidean vector space and T, K are tensors which verify the same algebraic conditions verified, respectively, by the torsion and curvature tensors of the canonical connection of a homogeneous Riemannian manifold with respect to a reductive decomposition.

We show, in particular, that the study of a homogeneous Riemannian manifold is equivalent to the study of a *regular* infinitesimal model by means of the *Nomizu construction* (see chapter 3, section A).

We shall supply the demonstration of the main theorem 3.1 proving that it is always possible to associate an infinitesimal model, in a unique way, to the set of tensors candidate to be the curvature tensor and its covariant derivatives at a point of a homogeneous Riemannian manifold. If this model is regular, the Nomizu construction will always produce a homogeneous Riemannian manifold (M,g) and the set of tensors at issue are just the curvature and the covariant derivatives of the curvature evaluated at a point of M.

In chapter 1 we collect the basic material which will be used throughout the thesis work recalling the definitions of the main notions in Riemannian and homogeneous geometry.

In chapter 2 we discuss the theorem of Singer. First of all we prove that in the hypothesis of the theorem the structure group O(n) of the principal fibre bundle of orthonormal frames OM over (M,g) can be reduced to the group $H = \{a \in O(n) : a(D^sR_{|p}) = D^sR_{|p},\ 0 \leq s \leq k_M+1\}$ via the reduction lemma 2.3. The reduction lemma 2.3, which gives a necessary and sufficient condition, under whom the structure group of a principal fibre bundle is reducible to a closed subgroup, has been proved by Singer in a different form, in the case of compact structure group, but it holds without that assumption. Next we show, using the previous lemma, that there exists a linear metric connection ∇ on TM such that $\nabla_X D^sR = 0$ for $0 \leq s \leq k_M+1$ (lemma 2.4). Finally, we prove that ∇ can be substituted with a linear metric connection

∇' such that the Riemann curvature tensor R and the difference tensor $S' = D - \nabla'$ between the Levi Civita connection D and ∇' are parallel with respect to ∇', i.e. $\nabla' R = \nabla S' = 0$. Consequently, the existence of such a connection on (M,g) implies that (M,g) is locally homogeneous (see theorem 1.11). In addition to the theorem of Singer we prove theorem 2.5.

The connection constructed on a locally homogeneous Riemannian manifold (M,g) by the procedure just described above is canonical in the sense that it is *unique* and depends only on the Riemannian structure of (M,g) (see [14]). Thus, in principle, on a locally homogeneous Riemannian manifold it is natural to work with this connection instead of the Levi Civita connection.

Section B of chapter 2 is devoted with the study of this canonical connection.

In chapter 3 we deal with the main theorem as pointed out above.

Acknowledgements.

I wish to express my sincere gratitude to professor Franco Tricerri for his generous and expert guide.

Chapter 1. BASIC MATERIAL

A. A brief review on the theory of connection.

We recall briefly some well-known concepts and formulae from the theory of connections on the tangent bundle TM and on the principal fibre bundle of orthonormal frames OM over a Riemannian manifold (M,g).

Let M be a differentiable manifold and $\Gamma(TM)$ the algebra of C^∞ vector fields on M (i.e. the sections of the tangent bundle TM). A *linear connection* (or *covariant derivative*) on M or more precisely on TM is a map $\nabla : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$, $(X,Y) \to \nabla_X Y$ such that

(i) $\nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z$,

(ii) $\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z$,

(iii) $\nabla_X fY = X(f)Y + f\nabla_X Y$,

for each $X,Y,Z \in \Gamma(TM)$, $f,g \in C^\infty(M)$.

For $p \in M$ and $X,Y \in \Gamma(TM)$, $(\nabla_X Y)_p$ depends only on $X_{|p}$ and on the germs of Y at p. It may occur that for $Y, Z \in \Gamma(TM)$ with $Y_{|p} = Z_{|p}$, $(\nabla_X Y)_p \neq (\nabla_X Z)_p$. In spite of this difficulty it is easy to prove (writing down in coordinates) that $(\nabla_X Y)_p$ depends only on the value of Y along a curve $\alpha : (-\varepsilon,\varepsilon) \to M$ such that $\alpha(0) = p$, $\dot\alpha(0) = X_{|p}$.

Let $X \in \Gamma(TM)$, X is said to be *parallel along* a smooth curve $\alpha : (a,b) \to M$ if $\nabla_{\dot\alpha} X = 0$ on α.y

Applying the theory of ordinary differential equations to the equation $\nabla_{\dot\alpha} X = 0$ with the initial condition $X_{|\alpha(a)} = v$, we have

Proposition 1.1. (*parallel transport*) *Let* $\alpha : [a,b] \to M$ *be a curve in* M, *and* $v \in T_{\alpha(a)}M$. *Then there exists a unique vector field* X *parallel along* α *such that* $X_{|\alpha(a)} = v$.

Notice that the theorem above gives a mapping from the tangent space at α(a) to the tangent space at α(t) for all t in the interval [a,b]. The linearity of the equation that gives the solution guarantees that this mapping is linear. By the uniqueness of solution this map is one-to-one and counting dimensions it is a linear isomorphism. This linear isomorphism is the *parallel transport* of v to $\alpha(t)$ along the curve α.

A curve α is a *geodesic* if $\dot{\alpha}$ is parallel. It is well-known that, for each point $p \in M$ and each tangent vector $v \in T_pM$, there exists a unique maximal geodesic $\alpha_v : (-\varepsilon,\varepsilon) \to M$ with $\alpha_v(0) = p$, $\dot{\alpha}_v(0) = v$. The linear connection ∇ is said to be *complete* if all maximal geodesics are defined on the whole real line **R**.

The operator $\nabla_X : \Gamma(TM) \to \Gamma(TM)$, $Y \to \nabla_X Y$, can be uniquely extended to the algebra of C^∞ tensor fields over M as a derivation, preserving type of tensors and commuting with contractions.

For example, considering a tensor field K of type (0,r) (resp. type (1,r)) on M as a multilinear mapping of $\Gamma(TM) \times \cdots \times \Gamma(TM)$ (r times) into $C^\infty(M)$ (resp. $\Gamma(TM)$), we have

$$(\nabla_X K)(X_1,\ldots,X_r) = \nabla_X K(X_1,\ldots,X_r) - \sum_{i=1}^{r} K(X_1,\ldots,\nabla_X X_i,\ldots,X_r),$$

where in the case (0,r) $\nabla_X f = X(f)$.

Given a tensor field K of type (s,r), s = 0,1, the *covariant derivative* ∇K of K is the tensor field of type (s,r+1) defined as follows :

$$(\nabla K)(X_1,\ldots,X_{r+1}) = (\nabla_{X_1} K)(X_2,\ldots,X_{r+1}).$$

The m-*th covariant derivative* $\nabla^m K$ of K is the tensor field of type (s,r+m) defined inductively by :

$$(\nabla^m K)(X_1,\ldots,X_m,X_{m+1},\ldots,X_{m+r}) =$$

$$\nabla_{X_1}((\nabla^{m-1}K)(X_2,\ldots,X_{m+r})) - \sum_{i=2}^{m+r}(\nabla^{m-1}K)(X_2,\ldots,\nabla_{X_1}X_i,\ldots,X_{m+r}).$$

We also write

$$(\nabla^m{}_{X_1\ldots X_m}K)(X_{m+1},\ldots,X_{m+r}) = (\nabla^m K)(X_1,\ldots,X_m,X_{m+1},\ldots,X_{m+r}).$$

Remark 1.2. Let we write $\Omega^r(M;TM)$ for the space $\Gamma(TM \otimes \Lambda^r(T^*M))$ of smooth sections of the bundle $TM \otimes \Lambda^r(T^*M)$, where T^*M is the cotangent bundle of M ; this is the space of differential forms on M with values in the bundle TM. At a point p of M an element μ of $\Omega^r(M;TM)$ is an alternating multilinear map $\mu_p : T_pM \times \cdots \times T_pM \to T_pM$, where there are r factors.

In this language a linear connection on TM is completely determined by a differential operator

$$\nabla : \Omega^0(M;TM) = \Gamma(TM) \to \Omega^1(M;TM)$$

such that for $f \in C^\infty(M)$ and $X \in \Gamma(TM)$

$$\nabla fX = df \otimes X + f\nabla X.$$

It is enough to set $\nabla_X Y = (\nabla Y)X$.

The *torsion* and *curvature* tensor fields of a linear connection ∇ are defined, respectively, by

$$(T_\nabla)_X Y = \nabla_X Y - \nabla_Y X - [X,Y]$$

$$(R_\nabla)_{XY} Z = \nabla_{[X,Y]} Z - [\nabla_X, \nabla_Y](Z).$$

For $X,Y,Z \in \Gamma(TM)$ the following identities hold :

$$\mathfrak{S}_{XYZ} \{(R_\nabla)_{XY}Z + (T_\nabla)_{(T_\nabla)_X Y}Z + (\nabla_X(T_\nabla))_Y Z\} = 0 \qquad \textit{(first Bianchi identity)}$$

$$\mathfrak{S}_{XYZ} \{\nabla_X(R_\nabla)_{YZ} + (R_\nabla)_{(T_\nabla)_X Y}Z\} = 0 \qquad \textit{(second Bianchi identity)},$$

where $\mathfrak{S}$ denotes the cyclic sum.

A Riemannian manifold (M,g) is a differentiable manifold M with a positive definite Riemannian metric g on it. The Riemannian metric give rise to an inner product g_p on the tangent space T_pM at each point p of M. The *Levi Civita connection* D of a Riemannian manifold (M,g) is the unique linear *metric* connection (i.e. $Dg = 0$) without torsion, that is defined by the formula

$$2g(D_X Y,Z) = Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) - g(X,[Y,Z]) + g(Y,[Z,X]) + g(Z,[X,Y]),$$

for $X, Y, Z \in \Gamma(TM)$.

For every linear connection ∇, the condition $\nabla g = 0$ express the compatibility of ∇ with the netric g and is equivalent to saying that the parallel transport along any piece-wise differentiable curve between two points p and q of M is a linear isometry of the tangent space T_pM onto T_qM. In particular, the curvature tensor R corresponding to the Levi Civita connection, also called *Riemann curvature tensor*, satisfies the following identities :

(i) $R_{XY}Z = -R_{YX}Z$;

(ii) $g(R_{XY}Z,W) + g(R_{XY}W,Z) = 0$;

(iii) $g(R_{XY}Z,W) = g(R_{ZW}X,Y)$;

(iv) $\mathfrak{S}_{XYZ} R_{XY}Z = 0$;

(v) $\mathfrak{S}_{XYZ} \nabla_X(R)_{YZ} = 0$.

Now, we review some aspects of connections on a principal fibre bundle. For our scope it is sufficient to confine ourselves to the *principal fibre bundle of orthonormal frames* over a Riemannian manifold (M,g), denoted by OM. A point u of OM is a pair $(p; u_1,...,u_n)$ where $p \in M$ and $(u_1,...,u_n)$ is an orthonormal frame of T_pM. $u \in$ OM can be interpreted as a linear isometry

$$u : (\mathbf{R}^n, < , >) \rightarrow (T_pM , g_p),$$

given by $u(e_j) = u_j$, where $< , >$ is the standard inner product of $\mathbf{R}^n$ and $(e_1,...,e_n)$ is the standard basis of $\mathbf{R}^n$. The projection $\pi : OM \rightarrow M$ is determined by $\pi(u) = p$.
The orthogonal group $O(n) = \{ a \in GL(n;\mathbf{R}) : {}^t aa = e\}$ is the structure group of OM. It acts *freely* (without fixed points) on OM by

$$(p; u_1,...,u_n)a = (p; \sum_{j=1}^{n} a_{1j}u_j,...,\sum_{j=1}^{n} a_{nj}u_j),$$

where $a = (a_{ij}) \in O(n)$.
Further, let U be an open neighbourhood of $p = \pi(u)$ and let $(E_1,...,E_n)$ be a local orthonormal frame field on U (a local cross section of OM). Then , for all $v = (q;v_1,...,v_n)$ of $\pi^{-1}(U)$, we have

$$v_i = \sum_{j=1}^{n} a_{ij}E_{j|q},$$

where $a = (a_{ij})$ is an element of O(n).
Hence the map $v \rightarrow (\pi(v),a)$ is a diffeomorphism of $\pi^{-1}(U)$ onto $U \times O(n)$. So we may identify $\pi^{-1}(U)$ with $U \times O(n)$ and the tangent space of OM at $v \in \pi^{-1}(U)$ can be identified with the direct sum

$$T_{\pi(v)}M \oplus T_aO(n).$$

A tangent vector X' of T_vOM can be expressed as $X' = X + aA$, where $X = \pi_*(X')$ and $A \in so(n) = T_eO(n) = \{A \in gl(n;R) : A = -{}^tA\}$, the Lie algebra of O(n) identified with the tangent space of O(n) at the identity.

A *connection* or *infinitesimal connection* on OM is an assignment to each $u \in OM$, of a subspace H_u of T_uOM such that

(i) $T_uOM = V_u \oplus H_u$;

(ii) for each $a \in OM$ $H_{ua} = (R_a)_*H_u$, where R_a is the transformation of OM induced by $a \in O(n)$, $R_au = ua$;

(iii) H_u depends differentially on u, i.e. if X is a C^∞ vector field on OM, $X_u = vX_u + hX_u$, with $vX \in V_u$ and $hX \in H_u$ for each $u \in OM$, then hX is a C^∞ vector field.

For each $u \in OM$, H_u is called *horizontal space* and V_u *vertical space.*

Let

$$u(t) = (p(t) ; u_1(t),...,u_n(t))$$

be a curve of OM and ∇ a linear metric connection on TM. One says that u(t) is a *horizontal curve* w.r.t. ∇ if all vector fields $u_i(t)$, $1 \le i \le n$, are parallel along the curve p(t) of M. We can define a connection on OM by taking as horizontal vectors at any $u \in OM$ the tangent vectors of the horizontal curves through that point.

An infinitesimal connection on OM defines a 1-form ω on OM with values in the Lie algebra so(n) of O(n) as follows. Let $A \in so(n)$ and consider the fundamental vector field A^* on OM induced by the one-parameter subgroup $\{\exp(tA)\}_{t \in R}$ acting on OM on the right.
The map $\tau_u : A \to A^*(u)$ is a linear isomorphism of so(n) onto V_u, for each $u \in OM$. Then, for each $X \in T_uOM$ we define $\omega(X)$ to be the unique $A \in so(n)$ such that $A^*(u)$ is equal to the vertical component of X, i.e. $\omega_u(X) = \tau_u^{-1}(vX)$; clearly $\omega(X) = 0$ if and only if X is horizontal. ω is called *connection form.* ω satisfies the following conditions:

(1.1) $\omega(A^*) = A$;

(1.2) $$(R_a)^* (\omega)(X) = a^{-1}\omega(X)a .$$

Conversely every 1-form with values in **so**(n) verifying (1.1) and (1.2) defines a connection on OM by the position $H_u = \text{Ker}\ \omega_u$.

Next we see how the notion of connection on OM, that is described geometrically, carry over to the tangent bundle TM in terms of a covariant derivative.

Let O(n) act on $\mathbf{R}^n$ by matrix multiplication on column vectors. The tangent bundle TM is isomorphic to the *associated vector bundle* $E = OM \times_{O(n)} \mathbf{R}^n$ with standard fibre $\mathbf{R}^n$ and structural group O(n), consisting of equivalence classes [u,v] with $[u,v] = [ua,a^{-1}v]$ for any $a \in O(n)$. Let $\sigma(x) = \psi^{-1}(x,e)$, e identity of O(n), the local section of OM induced by the bundle chart $\psi : \pi^{-1}(U) \to U \times O(n)$, U open in M. $\sigma(x) = (x; E_{i|x},...,E_{n|x})$, where $(E_{i|x},...,E_{n|x})$ is an orthonormal basis of T_xM for each $x \in U$.

Let ω be a connection form on OM and ω_U the **so**(n)-valued 1-form on U given by $\sigma^*(\omega)$. Take the local section σ_v of E = TM (i.e. a local vector field on M) determined by σ and a fixed $v \in \mathbf{R}^n$ as follows:

$$\sigma_v(x) = \sigma(x)(v) = [\sigma(x),v].$$

The map $v \to \sigma_v$ gives a linear immersion of $\mathbf{R}^n$ in $\Omega^0(U;TM)$, i.e. the local sections on U of TM.

Then there exists a unique covariant derivative ∇, *induced* by the connection on OM, characterized by (see also remark 1.2 and (1.2))

$$(\nabla\sigma_v)(x) = \sigma(x)(\omega_U\, v).$$

In particular, since $E_j = \sigma_{e_j}$, $1 \le j \le n$, where $(e_1,...,e_n)$ is the standard basis of $\mathbf{R}^n$,

$$(\nabla_X E_i)(x) = \sigma(x)(\omega_U(X)\, e_i) = \sum_{j=1}^{n} \omega_U(X)_{ij}(x)E_j(x),$$

$X \in T_xM$, $\omega_U(X) = (\omega_U(X)_{ij}) \in \mathbf{so}(n)$.

Proposition 1.3. *A linear connection* ∇ *on* TM *is induced by a connection on* OM *if and only if is metric, i.e.* $\nabla g = 0$.(see f.ex. [10], vol.1, p.158)

B. Homogeneous Riemannian manifolds.

We say that a connected Lie group G is a *Lie transformation group* on a manifold M or that G acts differentially on M if there exists a C^∞ map

$$G \times M \rightarrow M, \ (a,p) \rightarrow ap = L_a(p),$$

such that for all $a,b \in G$:

(i) L_a is a diffeomorphism of M ;

(ii) $L_a \circ L_b = L_{ab}$.

G acts *transitively* provided that for each $p,q \in M$ there is an $a \in G$ such that $ap = q$. G acts *effectively* when L_a is the identity transformation of M if and only if a is the identity e of G.
We say that a *Lie transformation group* G *acts by isometries* on a Riemannian manifold (M,g) or that the metric g is G-*invariant* if, for all $a \in G$, L_a is an isometry of (M,g).

If p is any point of M, $K = \{a \in G : L_a(p) = p\}$ is called the *isotropy subgroup* at p. K is a closed abstract subgroup of G, then it is well-known that there exists a unique analytic structure on K such that K is a Lie subgroup of G with the induced topology, i.e. is a topological Lie subgroup of G.
We can consider the set of all left cosets aK of K in G with the quotient topology. The *origin* o of G/K is the subgroup K considered as an element of G/K. The *projection* $\pi : G \rightarrow G/K$ sends each $a \in G$ to the coset aK containing it. For each $a \in G$ the *translation* $\tau_a : G/K \rightarrow G/K$ sends each bK to abK for $a,b \in G$. Further

$$\pi \circ L_a = \tau_a \circ \pi \quad \text{and} \quad \tau_{ab} = \tau_a \circ \tau_b.$$

We recall now the following.

Proposition 1.4. ([24]) *Let* K *be a closed subgroup of a Lie group* G. *Then* G/K *has a unique manifold structure such that :*
(a) π *is* C^∞ ;
(b) *there exist local smooth sections of* G/K *in* G, *i.e. given* $a \in G$ *there exists a neighbourhood* U *of* $\pi(a) = aK$ *in* G/K *and a smooth map* $\sigma : U \rightarrow G$ *such that* $\pi \circ \sigma = \text{id}$ *and* $\sigma(\pi(a)) = a$.

G/K is called *homogeneous space.*

In view of proposition 1.4 we have that $\pi : G \to G/K$ is a *principal fibre bundle* over the base manifold G/K with structural group K. K acts on G on the right as follows. Every $a \in K$ maps $g \in G$ into ga. The local triviality follows from the existence of local smooth sections. This fact is quite important since the topological and geometrical properties of G/K, G, K and their relations can be investigated by the application of fibre bundle methods (e.g. exact homotopy sequence).

From (b) follows that for a manifold N, a map $\varphi : G/K \to N$ is smooth if and only if $\varphi \circ \pi : G \to N$ is smooth. In fact if $\varphi \circ \pi$ is C^{∞}, then φ, which can locally be represented as the composition of a smooth section of G/K in G with $\varphi \circ \pi$, is also C^{∞}. The converse is clear.
For example, the identity $\pi \circ L_a = \tau_a \circ \pi$ shows that τ_a is smooth. Hence τ_a is a diffeomorphism since it has inverse map $\tau_{a^{-1}}$. This implies that there is a natural action of G on G/K given by the map $G \times G/K \to G/K$ sending (a,bK) to abK. So G is a Lie transformation group of G/K.

We see now that every transitive action can be represented in this way. If $G \times M \to M$ is an action and K the isotropy subgroup of a point p, there is a natural map from G/K into M that sends each coset aK = ao to the point ap. This map is well defined since

$$aK = bK \Rightarrow b^{-1}aK = K \Rightarrow b^{-1}a \in K \Rightarrow b^{-1}ap = p \Rightarrow ap = bp.$$

Further it holds :

Proposition 1.5. ([24]). *Let* G *be a Lie group that acts transitively on* M. *Then the map* $G/K \to M$, $aK \to ap$ *is a diffeomorphism.*

Let (M,g) a connected Riemannian manifold. Then (M,g) is said to be a *homogeneous Riemannian manifold* if the group I(M) of all isometries acts transitively on M, i.e. for every two points $p, q \in M$ there exists an isometry f such that f(p) = q.
More generally, one says that (M,g) is *locally homogeneous* if, for each $p, q \in M$, there exists a neighbourhood U of p, a neighbourhood V of q and a local isometry $f : U \to V$ such that f(p) = q.

A classical theorem of Myers and Steenrod (see [10], vol.1, p.239) states that the group I(M) of all isometries of a Riemannian manifold is a Lie transformation group on M and that the isotropy subgroup of any point of M is compact. Then we can also say that (M,g) is G-

homogeneous or homogeneous with respect to G if there exists a Lie transformation group G acting transitively by isometries on M.
Replacing G by the quotient group G/N where N is the kernel of the map $G \to I(M)$, $a \to L_a$, we can always suppose that G acts effectively on M. This permits to identify G with a Lie subgroup of I(M) (see f. ex. [4]). Clearly I(M) acts effectively om M.

Remark 1.6. a) Note that if G acts transitively on M it is not restrictive to consider G connected. In fact, since the projection $\pi : G \to M$ is an open map, the connected component G_o of G also acts transitively on M.

b) A homogeneous Riemannian manifold can ever be represented as a homogeneous space G/K with a G-invariant metric on it, where G is a connected Lie group and K the isotropy subgroup of some point of M. In fact, let φ be the diffeomorphism between G/K and M given by proposition 1.5 and let $g' = \varphi^*g$ be the induced metric on G/K. g' is G-invariant, since for each $a \in G$, X, Y vector fields on G/K,

$$(L_a)^*g'(X,Y) = (L_a)^*(\varphi^*g)(X,Y) = (L_a)^*(g(\varphi_*X,\varphi_*Y)) = g(\varphi_*X,\varphi_*Y) = g'(X,Y).$$

In particular the geometric properties of (M,g) can be described in terms of Lie groups theory.

c) A homogeneous Riemannian manifold is always complete (see f.ex. [10], vol.1, p.176)

d) It is worth mentioning that a simply connected, complete, locally homogeneous Riemannian manifold is homogeneous (see [22], p.16).

Next let **g** denote the Lie algebra of G and **k** the Lie algebra of K. A homogeneous space G/K is called *reductive* if there is a subspace **m** of **g** such that $\mathbf{g} = \mathbf{m} \oplus \mathbf{k}$ (direct sum of vector spaces) and $Ad(k)(\mathbf{m}) \subseteq \mathbf{m}$ for all $k \in K$, where Ad(K) denotes the adjoint representation of K in **g**.

Proposition 1.7.([11]) *A homogeneous space* G/K *with a* G-*invariant metric is reductive.*

The subspace **m** can be identified with T_oM at the origin $o \in M$ by means of the map $X \to X^*_{|o} = \pi_{*|e}X_{|e}$, where X^* denotes the Killing vector field on (M,g) generated by the one-parameter subgroup $\{\exp(tX)\}$ acting on M. The Lie subalgebra **k** of G is identified with the subalgebra of those Killing vector fields vanishing at o. In fact let $X \in \mathbf{g}$, then $X^*_{|o} = 0 \Leftrightarrow$ $(\exp tX)o = o$ for each $t \in \mathbf{R} \Leftrightarrow \exp tX \in K$ for each $t \in \mathbf{R} \Leftrightarrow X \in \mathbf{k}$.

Requiring π_* to be a linear isometry establishes a one-to-one correspondence between Ad(K)-invariant inner products on **m** and G-invariant metrics on M (see f.ex. [4]).

Let M = G/K. The action of K on the tangent space T_oM at the origin of M is called *linear isotropy representation.* For each $a \in K$ it is given by

$$a \rightarrow (L_a)_{*|o}.$$

The linear isotropy representation of a reductive homogeneous space corresponds under π_* to the adjoint representation Ad(K) on **m**, because for $k \in K$ $\tau_k \circ \pi = \pi \circ C_k$, $C_k : G \rightarrow G$, $b \rightarrow kbk^{-1}$ and $Ad(K)(\mathbf{m}) \subseteq \mathbf{m}$.

Since an isometry is determined by giving only the image of a point and the corresponding differential map at that point (see f.ex. [22]), the linear isotropy representation of a homogeneous Riemannian manifold (M,g) is *faithful*, i.e. is a monomorphism. Then the isotropy group at any point p of (M,g) can be identified with a subgroup of $O(T_pM)$, that is called *linear isotropy group* at p.

Remark 1.8. Given a simply connected homogeneous Riemannian manifold (M,g), the Lie algebra of the isotropy subgroup of I(M) at a point $p \in M$ can be identified with the Lie algebra **k** of all skew-symmetric endomorphisms of the tangent space T_pM, which as derivations of the tensor algebra of T_pM leave the curvature tensor R_p and all its covariant derivatives invariant, i.e.

$$\mathbf{k} = \{ A \in \mathbf{so}(T_pM) \ : \ A \cdot D^sR_p = 0 \ , s = 0, 1, 2, \dots \}.$$

In fact, let $K = \{a \in O(n) : a\,(D^sR_{|p}) = D^sR_{|p},\ s = 0, 1, 2, \dots \}$.The linear isotropy group at p is clearly contained in K. Conversely, suppose that $F \in K$. Then, since a homogeneous Riemannian manifold is complete and analytic, there exists an isometry (unique) f of M such that $f_{*|p} = F$ and $f(p) = p$ (see [10], vol.1, p. 261). Thus the isotropy group at p can be identified via the linear isotropy representation with K and the Lie algebra of K coincide with **k**.

A. The canonical connection.

A linear connection ∇ on G/K is G-*invariant* if for each $a \in G$, $X,Y \in \Gamma(T(G/K))$, L_a is an *affine transformation* of ∇,i.e.

$$(L_a)_*(\nabla_X Y) = \nabla_{(L_a)_* X}(L_a)_* Y.$$

For example the Levi Civita connection of a G-invariant metric is G-invariant.

With the notation of the preceding section we give the fundamental theorem for the G-invariant linear connection on a reductive homogeneous space G/K.

Theorem 1.9.([18]) *Let* G/K *be a reductive homogeneous space with a fixed reductive decomposition* $\mathbf{g} = \mathbf{m} \oplus \mathbf{k}$. *There exists a one-to-one correspondence between the set of* G-*invariant linear connections on* G/K *and the set of all bilinear functions* $\alpha : \mathbf{m} \times \mathbf{m} \to \mathbf{m}$ *which are invariant by* Ad(K), *that is,* $\mathrm{Ad}(h)\alpha(X,Y) = \alpha(\mathrm{Ad}(h)X,\mathrm{Ad}(h)Y)$ *for* $X,Y \in \mathbf{m}$ *and* $h \in K$. *The correspondence is given by*

$$\nabla_{X^*}Y^*{}_{|o} = \alpha(X,Y)^*{}_{|o}$$

Consider the bilinear function $\alpha : \mathbf{m} \times \mathbf{m} \to \mathbf{m}$, $(X,Y) \to -[X,Y]_{\mathbf{m}}$, where $[X,Y]_{\mathbf{m}}$ denote the projection of $[X,Y]$ on $\mathbf{m}$. Then for each $h \in K$

$$\alpha(\mathrm{Ad}(h)X,\mathrm{Ad}(h)Y) = -[\mathrm{Ad}(h)X,\mathrm{Ad}(h)Y]_{\mathbf{m}} = -\mathrm{Ad}(h)([X,Y]_{\mathbf{m}}) = \mathrm{Ad}(h)\alpha(X,Y),$$

since G/K is reductive. From the theorem 1.9 follows that there exists a unique G-invariant linear connection ∇ on G/K such that

$$\nabla_{X^*}Y^*{}_{|o} = -([X,Y]_{\mathbf{m}})^*{}_{|o},\ X,Y \in \mathbf{m}.$$

If $X \in \mathbf{k}$, $X^*{}_{|o} = 0$, then $\nabla_{X^*}Y^*{}_{|o} = -[X,Y]_{|o}$.

∇ is called the *canonical connection* of G/K relatively to the decomposition $\mathbf{g} = \mathbf{m} \oplus \mathbf{k}$.

In general, for a homogeneous Riemannian manifold (M,g), *the canonical connection is not uniquely determined*, but it depends on the group G of isometries acting transitively on the manifold and also on the reductive decomposition of the Lie algebra of G.

Let (M,g) be a homogeneous Riemannian manifold and $\mathbf{g} = \mathbf{m}\oplus\mathbf{k}$ a fixed reductive decomposition. We recall now some properties of the canonical connection ∇ of (M,g) with respect to this given decomposition.

1) ∇ is *complete* because the geodesic through $q = ao$ is given by $\gamma(t) = L_a(\exp(tX)o)$ for some $X \in \mathbf{m}$..

2) If a tensor field L on M is invariant by K, then $\nabla L = 0$. In particular ∇ is a metric connection, $\nabla T_\nabla = 0$ and $\nabla R_\nabla = 0$, where T_∇ and R_∇ are, respectively, the torsion and curvature tensor fields of ∇. Identifying the subspace $\mathbf{m}$ of $\mathbf{g}$ with the tangent space T_oM at the origin of M we have the standard formulae

(1.3) $$(T_\nabla)_X Y_{|o} = -[X,Y]_{\mathbf{m}},$$

(1.4) $$(R_\nabla)_{XY}Z_{|o} = -[[X,Y]_{\mathbf{k}},Z], \quad X,Y,Z \in \mathbf{m}.$$

Moreover, we have the following algebraic conditions :

(1.5) $T_\nabla : \mathbf{m} \to \mathrm{End}(\mathbf{m})$, $X \to (T_\nabla)_X$ and $(T_\nabla)_X Y = -(T_\nabla)_Y X$;

(1.6) $R_\nabla : \mathbf{m}\times\mathbf{m} \to \mathrm{End}(\mathbf{m})$, $(X,Y) \to (R_\nabla)_{XY}$ and $(R_\nabla)_{XY} = -(R_\nabla)_{YX}$;

(1.7) $(R_\nabla)_{XY}\cdot g = 0$;

(1.8) $(R_\nabla)_{XY}\cdot T_\nabla = 0$;

(1.9) $(R_\nabla)_{XY}\cdot R_\nabla = 0$;

(1.10) $\mathfrak{S}_{XYZ}((R_\nabla)_{XY}Z + (T_\nabla)_{(T_\nabla)_X Y}Z) = 0.$

(1.11) $\mathfrak{S}_{XYZ}(R_\nabla)_{(T_\nabla)_X Y}Z = 0$

In these formulae the curvature operator $(R_\nabla)_{XY} : Z \to (R_\nabla)_{XY}Z$ acts as a derivation on the tensor algebra of $\mathbf{m}$ and $\mathfrak{S}$ denotes the cyclic sum. (1.5), (1.6) and (1.7) are clear from the definitions ; (1.8) and (1.9) follows from the Ricci identity (see appendix)

$$(R_\nabla)_{XY} = -\nabla^2_{XY} + \nabla^2_{YX} - \nabla_{(T_\nabla)_X Y},$$

using the fact that $\nabla T_\nabla = \nabla R_\nabla = 0$. The conditions (1.10) and (1.11) are special cases of the classical Bianchi identities (see p.7), and it will possible to obtain both from (1.3) and (1.4) using the Jacoby identity for the Lie algebra g.

Observe that the algebraic condition (1.5) - (1.11) also holds for every linear metric connection ∇ on (M,g) such that $\nabla T_\nabla = \nabla R_\nabla = 0$.

Thus we have proved that on a homogeneous Riemannian manifold there exists a metric connection ∇ such that its torsion and curvature tensor fields are parallel with respect to ∇, namely the canonical connection. A connection which satisfies these properties is an *Ambrose-Singer connection* in the terminology of ([23}). The motivation is given by the following.

Theorem 1.10. (*local version of the Ambrose-Singer theorem,* [1],[22]) *Let* (M,g) *be a connected Riemannian manifold. If there exists a linear metric connection* ∇, *whose curvature and torsion tensor fields are parallel with respect to* ∇ *itself, then* (M,g) *is locally homogeneous.*

Proof. Let $\alpha(t)$ be a differentiable curve joining $p = \alpha(0)$ to $q = \alpha(1)$ and $F : T_pM \to T_qM$ the parallel transport along α w.r.t. ∇. F is an isometry which preserves the curvature and torsion tensors. Therefore, there exist two neighbourhoods U and V of p and q, respectively, and a ∇-affine transformation $f : U \to V$ such that $df_{|p} = F$ (see [10], vol.1, p. 261). Because F is an isometry and ∇ is metric, the transformation f is actually an isometry (see for ex. [22], p.3).

Q.e.d.

If we set $S = D - \nabla$ whit D the Levi Civita connection, it can be easily checked that the conditions $\nabla T_\nabla = \nabla R_\nabla = 0$ can be replaced by $\nabla S = \nabla R = 0$, where R is the Riemann curvature tensor of (M,g) (see [22]). In chapter 2 we will use theorem 1.10 in the following form.

Theorem 1.11. *Let* (M,g) *be a smooth Riemannian manifold. If there exists a linear metric connection* ∇ *on* (M,g) *such that the Riemann curvature tensor* R *of* (M,g) *and the difference tensor* $S = D - \nabla$ *between the Levi Civita connection* D *and* ∇ *are parallel with respect to* ∇, *then* (M,g) *is locally homogeneous.*

Chapter 2. THE THEOREM OF I.M. SINGER

A. The theorem of I.M. Singer

Let (M,g) be a differentiable Riemannian manifold of dimension n and p a point of M. We denote with $\mathbf{so}(T_pM)$ the Lie algebra of the endomorphisms of T_pM which are skew-symmetric with respect to the inner product g_p.

Let $\mathbf{g}(p; s)$ be the Lie subalgebra of $\mathbf{so}(T_pM)$ defined by

(2.1) $$\mathbf{g}(p; s) = \{ A \in \mathbf{so}(T_pM) \ : \ A\cdot R_p = A\cdot DR_p = \cdots = A\cdot D^sR_p = 0 \},$$

where A acts as a derivation on the tensor algebra on T_pM.

Of course, $\mathbf{g}(p; s) \supseteq \mathbf{g}(p; s+1)$, and there exists a *first integer* $k(p) \geq 0$ for which

(2.2) $$\mathbf{g}(p; k(p)) = \mathbf{g}(p; k(p)+1).$$

In general, k(p) depends on p and $k(p)+1 \leq n(n-1)/2$.

Remark 2.1. a) If (M,g) is a Riemannian symmetric space we have $DR = 0$. Then

$$\mathbf{g}(p; 0) = \mathbf{g}(p; 1) = \{ A \in \mathbf{so}(T_pM) \ : \ A\cdot R_p = 0 \}$$

coincide with the *isotropy algebra* of M if M is simply connected (see also remark 1.8).

b) $\mathbf{g}(p; 0) = \mathbf{so}(T_pM)$ if and only if (M,g) is *isotropic* in p, i.e. the sectional curvature c of the manifold does not depend on the choice of the tangent plane to M in p or equivalently if

$$(R_p)_{X_1X_2X_3X_4} = c\{g(X_1,X_3)g(X_2,X_4) - g(X_1,X_4)g(X_2,X_3)\},$$

where $(R_p)_{X_1X_2X_3X_4} = g((R_p)_{X_1X_2}X_3,X_4)$, $X_1, X_2, X_3, X_4 \in T_pM$.
In fact $A\cdot R_p = 0$ for each $A \in \mathbf{so}(T_pM)$ if and only if $aR_p = R_p$ for each $a \in O(T_pM)$. But from invariant theory we know that every invariant tensor F of type (0,4) is expressed as follows (see f.ex. [9], [25]) :

$$F(X_1,X_2,X_3,X_4) = c_1 g(X_1,X_2)g(X_3,X_4) + c_2 g(X_1,X_3)g(X_2,X_4) + c_3 g(X_1,X_4)g(X_2,X_3).$$

Then, since $(R_p)_{X_1X_2X_3X_4}$ is skew-symmetric with respect to X_3, X_4, we have

$$(R_p)_{X_1X_2X_3X_4} = c\{g(X_1,X_3)g(X_2,X_4) - g(X_1,X_4)g(X_2,X_3)\}.$$

Following Singer, we say that (M,g) is *infinitesimally homogeneous* if for each p and q of M there exists an isometry

$$F : T_pM \to T_qM$$

such that (0.1) holds for $0 \le s \le k(p)+1$.

The isometry F induces an isomorphism between $g(p; s)$ and $g(q; s)$ if $0 \le s \le k(p)+1$. This isomorphism is given by

$$g(p; s) \to g(q; s), \;\; A \to FAF^{-1}.$$

In fact for $X_1, ..., X_{s+4} \in T_qM$,

$$(FAF^{-1})\cdot(D^sR_{|q})(X_1,...,X_{s+4}) = -\sum_{i=1}^{s+4}(D^sR_{|q})(X_1,...,FAF^{-1}X_i,...,X_{s+4}) =$$

$$= -\sum_{i=1}^{s+4}F^*(D^sR_{|q})(F^{-1}X_1,...,AF^{-1}X^i,...,F^{-1}X_{s+4}) =$$

$$-\sum_{i=1}^{s+4}(D^sR_{|p})(F^{-1}X_1,...,AF^{-1}X_i,...,F^{-1}X_{s+4}) = A\cdot(D^sR_{|p})(F^{-1}X_1,...,F^{-1}X_{s+4}) = 0.$$

Therefore, for an infinitesimally homogeneous Riemannian manifold the integer k(p) is actually a *constant*. We put

$$k_M = k(p).$$

k_M is called *Singer's number*. Note that $k_M \le 3n/2$ (see [5], p.165).

Then we have :

Theorem 2.2. *An infinitesimally homogeneous Riemannian manifold is locally homogeneous.*

Theorem 2.2 was originally proved by Singer ([20]) when M is complete and simply connected. In this form was later obtained by Nomizu ([19]) using the notion of Killing generator at a point. We will not be concerned with this notion. Recall also remark 1.6, d).

The main tools in proving theorem 2.2 are the following lemmas.

Lemma 2.3. *Let* (P,π,M,G) *be a principal fibre bundle over a connected manifold* M *with structural group* G. *Then,* G *is reducible to a closed subgroup* H *if and only if there exists a differentiable map* $\Phi : P \to G/H$ *which is equivariant, i.e.*

$$\Phi(ua) = a^{-1}\Phi(u)$$

for all $u \in P$ *and* $a \in G$. *If this hypothesis is verified, then* $Q_u = \Phi^{-1}\Phi(u)$ *is a principal subbundle of* P *for each* $u \in P$. *Its structural group is the isotropy subgroup of* $\Phi(u)$ *(conjugate to* H *in* G).

Proof of lemma 2.3. The structural group G of P is reducible to H if and only if there exists a global section σ of the associated fibre bundle $E = P \times_G G/H$ with standard fibre G/H (see [KN], Prop. 5.6, p.57, vol.1). The bundle E can be identified in a natural way with P/H. In fact, E is the quotient of $P \times G/H$ with respect to the action of G given by

$$(u,\bar{a})b = (ub,b^{-1}\bar{a})$$

where $\bar{a}$ denotes the left coset represented by a.

Then, we identify E with P/H by identifying the element represented by the pair $(u,\bar{a})$ with the element $\overline{ua} \in P/H$ represented by ua. Therefore, let us write $\overline{ua} = u\bar{a}$.

Let Φ be a C^∞ equivariant map of P to G/H. Then $u\Phi(u) \in P/H$ does not depend from u, but only from $\pi(u)$, since

$$(ua)\Phi(ua) = uaa^{-1}\Phi(u) = u\Phi(u)$$

for each $a \in G$. Therefore, it is possible to define a section σ of E in the following way :

$$\sigma(\pi(u)) = u\Phi(u).$$

The section σ is C^∞. In fact, if $\eta : U \to P$ is a local smooth section of P, we have

$$\sigma(x) = \eta(x)\Phi(\eta(x)),$$

for each $x \in U$.

Conversely, if $\sigma : M \to E$ is a smooth section of E, for each $u \in P$, there exists a $g(u) \in G$ such that

$$\sigma(\pi(u)) = u\,\overline{g(u)}.$$

Of course, g(u) is not uniquely determined. Nevertheless, if

$$\sigma(\pi(u)) = u\overline{g_1(u)},$$

then

$$ug_1(u) = ug(u)h$$

for some $h \in H$. Since G acts freely on P, we have

$$g_1(u) = g(u)h.$$

Therefore, the map $\Phi : P \to G/H,\ u \to \overline{g(u)}$ is well defined. This map is C^∞. In fact, let U be an open set of M trivializing P. Let ψ be a diffeomorphism between $\pi^{-1}(U)$ and $U \times G$ such that

$$\psi(u) = (\pi(u), \varphi(u)),$$

where $\varphi(ua) = \varphi(u)a$. Then, ψ induces a diffeomorphism $\overline{\psi}$ between $\pi_E^{-1}(U)$ and $U \times G/H$ given by

$$\overline{\psi}(u\overline{a}) = (\pi(u), \varphi(u)\overline{a}).$$

Note that $\overline{\psi}^{-1}$ is defined by

$$\overline{\psi}^{-1}(x,\overline{a}) = u_o\varphi(u_o)^{-1}\overline{a},$$

where $u_o = \psi^{-1}(x,e)$ and e is the identity of G.

Therefore, we get

$$\bar{\psi}\circ\sigma(\pi(u)) = (\pi(u),\bar{\sigma}(u))$$

where $\bar{\sigma} : U \to G/H$ is C^∞. On the other hand, we have

$$\bar{\psi}\circ\sigma\ (\pi(u)) = \bar{\psi}(\sigma(\pi(u))) = \bar{\psi}(u\overline{g(u)}) = (\pi(u),\varphi(u)\Phi(u)).$$

From this we get

$$\Phi(u) = \varphi(u)^{-1}\bar{\sigma}(u),$$

and Φ is smooth.

Now, we prove that

$$Q = \Phi^{-1}(o),$$

where o is the origin of G/H (i.e. the coset H), *is a principal subbundle of* P *with structural group* H. It is sufficient to prove the following three points.

1. $\pi(Q) = M$; if $u \in \pi^{-1}(x)$, $x \in M$, there exists $a \in G$ such that $\Phi(u) = ao$ and $\Phi(ua) = a^{-1}\Phi(u) = o$. Hence $ua \in Q$.

2. $(\pi_{|Q})^{-1}(x) = u\cdot H$; if $u,v \in (\pi_{|Q})^{-1}(x)$, $v = ua$ for some $a \in G$ and $\Phi(ua) = a^{-1}\Phi(u)$, i.e. $o = a^{-1}o$ or equivalently $a \in H$.

3. There exist local sections of P with values in Q ; let V be an open set of M and $\eta : V \to P$ a local section of P on V. $\Phi\circ\eta : V \to G/H$ is C^∞. We can choose V small enough such that $(\Phi\circ\eta)(V)$ is contained in an open set U of G/H with a local section $\sigma : U \to G$. Then the map $\lambda = \sigma\circ\Phi\circ\eta : V \to G$ is C^∞. Finally,the map

$$\tau : V \to P,\ x \to \eta(x)\lambda(x)$$

is a local section of P on V. Since

$$\Phi(\eta(x)\lambda(x)) = \lambda(x)^{-1}\Phi(\eta(x))$$

and

$$\Phi(\eta(x)) = p(\sigma(\Phi(\eta(x)))) = p(\lambda(x)) = \lambda(x)o,$$

where p is the projection of G onto G/H, it is proved that τ takes values in Q.
Therefore, if $\Phi(u_o) = o$, $\Phi^{-1}\Phi(u_o) = Q$ is a principal subbundle of P with structural group H which is just the isotropy group of $\Phi(u_o)$. In general, if $\Phi(u_o) = ao$, then $Q_{u_o} = \Phi^{-1}\Phi(u_o) = Qa^{-1}$. This shows that Q_{u_o} is a principal subbundle of P too. Its structural group is aHa^{-1} which is the isotropy group of $\Phi(u_o)$.

Q.e.d.

Lemma 2.3 has been proved by I.M.Singer in a different form by supposing that G is compact, but it also holds without that assumption (see also [8]).

Let $OM = (OM,\pi,M,O(n))$ be the bundle of orthonormal frames over (M,g).
An element $u = (p; u_1,\dots,u_n)$ of OM induces an *isometry* , still denoted by u, of $V = \mathbf{R}^n$ on T_pM as follows :

(2.3) $$u(\xi) = u(\xi^1,\dots,\xi^n) = \sum_{i=1}^n \xi^i u_i \ .$$

For each $s \geq 0$, the tensor field D^sR defines a map K^s from OM to $\overset{s+4}{\otimes} V^*$ given by

(2.4) $$K^s(u)(\xi_1,\dots, \xi_{s+4}) = D^sR_{|\pi(u)} (u\xi_1,\dots,u\xi_{s+4}),$$

where $\xi_1,\dots,\xi_{s+4} \in V = \mathbf{R}^n$.

The orthogonal group O(n) acts on the left on the tensor product $\overset{m}{\otimes} V^*$ as follows

(2.5) $$(aT)(\xi_1,\dots,\xi_m) = T(a^{-1}\xi_1,\dots a^{-1}\xi_m).$$

It is easy to verify that K^s is an *equivariant* map with respect to the actions of O(n) on OM and on $\overset{s+4}{\otimes} V^*$, i.e.

(2.6) $$K^s(ua) = a^{-1}K^s(u),$$

for each $u \in OM$ and $a \in O(n)$.

Following I.M. Singer, we consider now the map

$$(2.7) \qquad \Phi_m : OM \to W_m = \oplus_{s=0}^{m+1} (\overset{s+4}{\otimes} V^*), \;\; u \to \Phi_m(u) = \sum_{s=0}^{m+1} K^s(u).$$

Clearly, also Φ_m is *equivariant*, i.e.

$$(2.8) \qquad \Phi_m(ua) = a^{-1}\Phi_m(u),$$

for all $u \in OM$ and $a \in O(n)$.

We are now able to prove the following lemma :

Lemma 2.4. ([20])) *If* (M,g) *is connected and infinitesimally homogeneous then there exists a linear metric connection* ∇ *such that, for each* $0 \le s \le k_M + 1$,

$$\nabla_X D^s R = 0.$$

Proof of lemma 2.4. Put $\Phi = \Phi_{k_M}$. It is easy to verify that (M,g) is infinitesimally homogeneous if and only if $\Phi(OM)$ is a single O(n)-orbit of W_{k_M}. Let u be a point of OM. Then, the orbit $\Phi(OM)$ is the homogeneous space O(n)/H, where H is the isotropy group of $\Phi(u)$. Therefore lemma 2.3 implies that $\Phi^{-1}\Phi(u)$ is a principal fibre subbundle of OM (u is a fixed point of OM). Its structural group H is

$$H = \{ \; a \in O(n) \;:\; a\,(D^s R_{|p}) \;= D^s R_{|p}, \;\; 0 \le s \le k_M + 1 \;\}.$$

Here $p = \pi(u)$, and T_pM is identified with $V = \mathbf{R}^n$ via the isometry u.

This means that (M,g) has an H-*structure*, and $P_u = \Phi^{-1}\Phi(u)$ is a bundle of *adapted frames* of this structure. Let now ω a *connection form* defining a connection on $P_u = \Phi^{-1}\Phi(u)$. ω takes values in the Lie algebra $\mathbf{h} = g(p; k_M+1)$ of H. As showed in chapter 1, section A, there exists a linear metric connection ∇ on M, corresponding to ω, such that,

$$\nabla_X E_i = \sum_{i=1}^{n} (\sigma^* \omega(X))_{ij} \, E_j \, ;$$

here $\sigma : U \subset M \to P_u$ is a local section of P_u given by $\sigma(x) = (x;E_1(x),\ldots,E_n(x))$, where $(E_1,\ldots,E_n)$ is a local orthonormal frame on U and X a vector field on M. Since $\Phi_{|P_u}$ is constant, the components of the tensor fields D^sR, $0 \le s \le k_M+1$, with respect to a local adapted frame are constant, i.e.

$$(D^sR_{|x})(E_{i_1|x},\ldots,E_{i_{s+4}|x}) = (D^sR_{|p})(u_{i_1},\ldots,u_{i_{s+4}}) = K^{(s)}{}_{i_1\ldots i_{s+4}}$$

Then

$$(\nabla_X D^sR)(E_{i_1},\ldots,E_{i_{s+4}}) = X((D^sR)(E_{i_1},\ldots,E_{i_{s+4}})) - \sum_{\alpha=1}^{s+4} D^sR(E_{i_1},\ldots,\nabla_X E_{i_\alpha},\ldots,E_{i_{s+4}}) =$$

$$-\sum_{\alpha=1}^{s+4}\sum_{m=1}^{n} (D^sR_{|p})(u_{i_1},\ldots,u_m,\ldots u_{i_{s+4}})(\sigma^*\omega(X))_{mi_\alpha} = (A\cdot D^sR_{|p})(u_{i_1},\ldots,u_{i_{s+4}}) = 0,$$

because $A = ((\sigma^*\omega(X))_{mi_\alpha}) \in \mathbf{h} = \mathbf{g}(p;\, k_M+1)$.

Therefore, all these tensor fields are parallel with respect to each linear connection ∇ which is adapted to the H-structure of M (i.e. induced by an infinitesimal connection on the bundle P_u, so with holonomy group contained in H).

Q.e.d.

Proof of theorem 2.2. We shall exhibit a linear metric connection ∇' which will satisfy the conditions of theorem 1.11. Thus, the theorem will follow directly from this theorem.

In order to construct ∇', let us consider first a linear metric connection ∇ such that $\nabla_X D^sR = 0$ for $0 \le s \le k_M +1$. It exists in virtue of lemma 2.4. Let S be the difference tensor between the Levi Civita connection D and ∇, i.e.

$$\nabla = D - S.$$

Since ∇ is a metric connection, for each $X \in T_qM$ the operator $Y \to (S_q)_X Y$, $Y \in T_qM$, is skew-symmetric.

We consider now the *trace form* of the Lie algebra $\mathbf{so}(T_qM)$ of the skew-symmetric endomorphisms of T_qM. It is defined by

$$(A,B) = -\operatorname{trace} AB = \sum_{i=1}^{n} g_q(A(e_i),B(e_i)),$$

where $(e_1,\ldots,e_n)$ is an orthonormal basis of T_qM. Of course, the definition does not depend on the choice of this basis and the form is positive definite. Note that the *Killing form* of $\mathbf{so}(T_qM)$ is proportional to the trace form (see [7]).

We put

$$\mathbf{h}(q) = \mathbf{g}(q;\ k_M+1).$$

$\mathbf{h}(q)$ is a Lie subalgebra of $\mathbf{so}(T_qM)$ and we denote by $\mathbf{h}^{\perp}(q)$ its orthogonal complement with respect to the trace form defined above. Then, we can decompose S_q in a unique way as follows

$$(S_q)_X = (S'_q)_X + (S''_q)_X,$$

where $(S'_q)_X$ (resp. $(S''_q)_X$) is the projection of the operator $(S_q)_X$ on $\mathbf{h}(q)$ (resp. $\mathbf{h}^{\perp}(q)$).

The tensor fields $q \to S'_q$, $q \to S''_q$ are C^∞. In order to prove this, let us consider a normal neighborhood U of q (w.r.t. ∇). Let $(e_1,\ldots,e_n)$ be an orthonormal basis of T_qM and $(E_1,\ldots,E_n)$ the local orthonormal frame in U obtained by parallel transport (w.r.t. ∇) along the ∇-geodesics through q.

Let $(H_1(q),\ldots,H_r(q))$ be a basis of $\mathbf{h}(q)$ which is orthonormal with respect to the trace form of $\mathbf{so}(T_qM)$. If $H_\alpha(q)$ ($\alpha = 1,\ldots, r$) has the components $H_{\alpha ij}$ ($i,j = 1,2,\ldots,n$) with respect to $(e_1,\ldots,e_n)$, we define the operators $H_1(x),\ldots, H_r(x)$ on $T_x(M)$ as having the same components with respect to $(E_{1|x},\ldots,E_{n|x})$. Because ∇ is a metric connection, we have $H_\alpha(x) \in \mathbf{so}(T_xM)$, $\alpha = 1,\ldots, r$. Moreover, these operators are mutually orthogonal with respect to the trace form. Since $H_\alpha(x)$ is obtained by parallel transport of $H_\alpha(q)$ along the ∇-geodesics through q, and the tensor fields D^sR are parallel with respect to the same connection ∇, we get that $H_1(x),\ldots, H_r(x)$ belong to $\mathbf{h}(x)$ and they form just an orthonormal basis of $\mathbf{h}(x)$.
Therefore we have

$$(S'_x)_X = \sum_{\alpha=1}^{r} ((S_x)_X, H_\alpha(x))H_\alpha(x)$$

for each $X \in T_xM$ and $x \in U$. This shows that S' and S'' = S – S' are C^∞.

Let us consider now the connection $\nabla' = D - S'' = \nabla + S'$.
It is a metric connection because $(S''_q)_X$ is skew-symmetric for each $q \in M$, $X \in T_qM$. Moreover

$$\nabla' D^sR \ = 0,$$

for $s \leq k_M+1$. In fact we have, for each vector field X :

$$\nabla'_X D^sR \ = \nabla_X D^sR + S'_X \cdot D^sR = 0,$$

since $(S'_q)_X \in \mathbf{h}(q)$.

Finally we have

$$\nabla' S'' = 0.$$

In fact, let U' be a normal neighborhood of q (w.r.t. ∇'). Let $(e_1,\ldots,e_n)$ be an orthonormal basis of T_qM. The choice of this basis allows us to identify T_qM with $\mathbf{R}^n$, $\mathbf{so}(T_qM)$ with $\mathbf{so}(n)$ and $\mathbf{h}(q)$ with the Lie subalgebra of $\mathbf{so}(n)$ whose elements are the matrices $A = (A_{ij})$ verifying (for $s \leq k_M+1$) the identities

$$\sum_{\alpha=1}^{s+4} \sum_{m=1}^{n} A_{im}\, K^{(s)}{}_{i_1\ldots i_{\alpha-1} m\, i_{\alpha+1}\ldots i_{s+4}} = 0,$$

where $K^{(s)}{}_{i_1\ldots i_{s+4}}$ are the components of $D^sR_{|q}$ with respect to $(e_1,\ldots,e_n)$.

Let $(E_1,\ldots,E_n)$ be the local orthonormal frame in U' obtained by parallel transport (w.r.t. ∇') along the ∇'-geodesics through q from the orthonormal basis $(e_1,\ldots,e_n)$ of T_qM. Since the tensor fields D^sR , $s \leq k_M+1$, are still parallel with respect to ∇', their components are constant and we have

$$(D^sR_{|x})(E_{i_1|x},\ldots,E_{i_{s+4}|x}) = K^{(s)}{}_{i_1\ldots i_{s+4}}.$$

Let $X = \sum_{i=1}^{n} a^iE_i$, a^i = constant, then we have

$$0 = (\nabla'_X D^sR)(E_{i_1},\ldots,E_{i_{s+4}}) = (D^{s+1}R)(X,E_{i_1},\ldots,E_{i_{s+4}})$$

$$+ \sum_{\alpha=1}^{s+4} \sum_{m=1}^{n} g(S''_X E_{i_\alpha},E_m) \cdot (D^sR)(E_{i_1},\ldots,E_m,\ldots, E_{i_{s+4}}) =$$

$$\sum_{m=1}^{n} a^m K^{(s+1)}{}_{m i_1\ldots i_{s+4}} + \sum_{\alpha=1}^{s+4} \sum_{m=1}^{n} g(S''_X E_{i_\alpha},E_m)\, K^{(s)}{}_{i_1\ldots m\ldots i_{s+4}},$$

where all the components $K^{(s+1)}{}_{m i_1\ldots i_{s+4}}$ and $K^{(s)}{}_{i_1\ldots m\ldots i_{s+4}}$ are constant for $s \leq k_M$. By evaluating the previous formula at the points of a radial geodesic of ∇' and by differentiating with respect to t, we get for all integers $s \leq k_M$:

$$0 = \sum_{\alpha=1}^{s+4} \sum_{m=1}^{n} (d/dt) g_{\gamma(t)}(S''_{X(t)} E_{i_\alpha}(t),E_m(t))\, K^{(s)}{}_{i_1\ldots i_{\alpha-1} m\, i_{\alpha+1}\ldots i_{s+4}} = 0,$$

where $X(t) = X_{|\gamma(t)}$, $E_i(t) = E_{i|\gamma(t)}$. We put

$$A_{ij}(t) = g_{\gamma(t)}(S''_{X(t)} E_i(t),E_j(t)).$$

Then, from the previous formulae it follows that the matrix

$$(d/dt)\, A(t) = (\, (d/dt)A_{ij}(t)\,)$$

is an element of $\mathbf{g}(p;\, k_M)$. Since $\mathbf{g}(q;\, k_M) = \mathbf{g}(q;\, k_M{+}1) = \mathbf{h}(q)$, we get that

$$(d/dt)A(t) \in \mathbf{h}(q)$$

(This is the only point where we use this property of k_M). On the other hand $(S''_{\dot\gamma(t)})_{X(t)}$ is an element of $\mathbf{h}^{\perp}(\gamma(t))$, therefore

$$A(t) \in \mathbf{h}^{\perp}(q).$$

By differentiating with respect to t, we get

$$(d/dt)A(t) \in \mathbf{h}^{\perp}(q).$$

From these results it follows that $(d/dt)A(t) = 0$. Thus

$$0 = (d/dt)g_{\gamma(t)}(S''_{X(t)}E_i(t),E_j(t)) = g((\nabla'_{\dot\gamma(t)}S'')_{X(t)}\, E_i(t),E_j(t)).$$

Hence, $\nabla' S'' = 0$.

Q.e.d.

From the proof of the theorem 2.2 we get also the

Theorem 2.5. *Let* (M,g) *and* (M',g') *be two Riemannian manifolds. Suppose that* (M',g') *is locally homogeneous. If for each point* p *of* M *there exists an isometry* $F : T_pM \to T_oM'$ ($o \in M'$ *is supposed fixed) such that*

$$F^{*}D'^{s}R'_{|o} = D^{s}R_{|p}$$

for $0 \le s \le k(p){+}1$, *then* (M,g) *is locally homogeneous and locally isometric to* (M',g').

Proof. First of all (M,g) is infinitesimally homogeneous. Therefore it is locally homogeneous. Then, there exist two metric connections ∇ and ∇' on M and M', respectively, such that

(i) their curvature and torsion tensors are parallel.

(ii) $\nabla_X D^s R = 0$ and $\nabla'_{X'} D'^s R' = 0$ for $0 \le s \le k_M+1 = k_{M'}+1$.

(iii) If $\nabla = D - S$ and $\nabla' = D' - S'$, then

$$(S_q)_X \in g(q;\, k_M+1)^\perp = \mathfrak{h}(q)^\perp,\quad (S'_{q'})_{X'} \in g'(q';\, k_M+1)^\perp = \mathfrak{h}'(q')^\perp,$$

for each $q \in M$, $q' \in M'$, $X \in T_qM$, $X' \in T_{q'}M'$.

From (ii) we get

$$i_X D^{s+1}R = S_X \cdot D^s R \qquad i_X D'^{s+1}R' = (S')_{X'} \cdot D'^s R',$$

where i_X denotes the *interior product* by the vector field X. These identities hold for $s \le k_M$.

Let p be a point of M and $F : T_pM \to T_oM'$ an isometry such that $F^* D'^s R'_{|o} = D^s R_{|p}$ for all $s \le k_M+1$. From the previous identities we get easily that :

$$F^*(S'_o)_X \cdot (D^s R_{|p}) = (S_p)_X \cdot (D^s R_{|p}),$$

for all $s \le k_M$, where $F^*(S'_o)_X = F^{-1} \circ (S'_o)_{FX} \circ F$.

This implies that

$$F^*(S'_o)_X - (S_p)_X \in \mathfrak{h}(p),$$

since $g(p;\, k_M) = g(p;\, k_M+1) = \mathfrak{h}(p)$. On the other hand $(S_p)_X$ and $F^*(S'_o)_X$ are elements of $\mathfrak{h}^\perp(p)$, thus we have $F^*(S'_o)_X - (S_p)_X = 0$, i.e.

$$S_p = F^*(S'_o).$$

Since the torsion of ∇ is parallel, a simple computation gives us its curvature tensor R_∇ (see f.ex. [22]). Explicitly, we have

(2.9) $$(R_\nabla)_{XY} = R_{XY} + S_{(T_\nabla)_X Y} + [S_X, S_Y],$$

where

(2.10) $$(T_\nabla)_X Y = S_Y X - S_X Y$$

is the torsion of ∇. Similar formulae hold for ∇' and therefore we get

$$F^*(R_{\nabla'|o}) = R_{\nabla|p}$$

and

$$F^*(T_{\nabla'|o}) = T_{\nabla|p}.$$

In other terms, F is an isometry between T_pM and T_oM' which preserves the curvature and the torsion tensors of ∇ and ∇'. Since these tensors are parallel, a standard result (see [10] vol. 1, p. 261) implies that there exist two neighbourhoods U and U' of p and o, respectively, and an affine transformation $f : U \to U'$ such that $df_{|p} = F$. Because F is an isometry and the involved connections are metric, the transformation f is actually an isometry (see f. ex. [22], p. 3). This proves the theorem.

Q.e.d.

Generalizing what is known for Riemannian manifolds with constant sectional curvature and for Riemannian symmetric spaces, we have

Corollary 2.6. *Let* (M,g) *and* (M'g') *be two locally homogeneous Riemannian manifolds. Suppose that there exist two points* $p \in M$, $p' \in M'$ *and a linear isometry* $F : T_pM \to T_{p'}M'$ *such that*

$$F^* D'^s R'_{|p'} = D^s R_{|p}$$

for $0 \le s \le k(p)+1$, *then* (M,g) *and* (M',g') *are locally isometric.*

B. A canonical connection for locally homogeneous Riemannian manifold

The connection constructed on a locally homogeneous Riemannian manifold (M,g) by the procedure described in the proof of the theorem 2.2 above is canonical in the sense that it is *unique* and depends only on the Riemannian structure of (M,g) (see [14]). Thus, in principle, on a locally homogeneous Riemannian manifold it is natural to work with this connection instead of the Levi Civita connection. Here we describe this connection.

Let (M,g) be a locally homogeneous Riemannian manifold of dimension n and k_M its Singer's number. The Lie algebra $\mathbf{g}(q; s)$ defined by (2.1) are all isomorphic for different points of M. Nevertheless, it is better for our purposes to avoid any identification and to keep distinct these algebras. Let $\mathbf{so}(M)$ be the vector fibre bundle over M whose fibres are the vector spaces $\mathbf{so}(T_qM)$.

The Riemann metric g of M induces a fibre metric on $\mathbf{so}(M)$ (i.e. an inner product on the fibres $\mathbf{so}(T_qM)$ which depends smoothly on q) given by

$$\bar{g}_q(A,B) = -\operatorname{trace} AB = \sum_{i=1}^{n} g_q(A(e_i),B(e_i)),$$

for each $A,B \in \mathbf{so}(T_qM)$, where $(e_1,\ldots,e_n)$ is an orthonormal basis of T_qM. Of course, the definition does not depend on the choice of this basis and the form is positive definite. It is important to remark that the restriction of $\bar{g}$ to the fibres $\mathbf{so}(T_qM)$ of $\mathbf{so}(M)$ is the trace form of $\mathbf{so}(T_qM)$, that is proportional with the Killing form of $\mathbf{so}(T_qM)$ (see also p. 25).

Put

$$\mathbf{h}(q) = \mathbf{g}(q; k_M)$$

and consider the set

$$E = \bigcup_{q\in M} \mathbf{h}(q).$$

Then we have

Lemma 2.7. *The set* E *has a natural structure of vector fibre subbundle of* $\mathbf{so}(M)$.

Proof. We have to show that each point q of M has a neighbourhood U where n(n–1)/2 linearly independent smooth sections of $\mathbf{so}(M)$ are defined in such a way that the first $r = \dim\mathbf{h}(q)$ sections give a basis of $\mathbf{h}(x)$, for each $x \in U$. This is proved during the proof of theorem 2.2.

Q.e.d.

Since $\mathbf{so}(M)$ carries the metric $\bar{g}$, it splits as direct sum of E and its orthogonal complement $E^{\perp}$, $\mathbf{so}(M) = E \oplus E^{\perp}$. Moreover, we have the

Lemma 2.8. *Let* ∇ *be any linear metric connection on* M *such that* $\nabla_X D^s R = 0$ *for* $0 \le s \le k_M+1$, X *any vector fields on* M, *then*

$$\nabla_X\Gamma(E) \subseteq \Gamma(E),\ \ \nabla_X\Gamma(E^\perp) \subseteq \Gamma(E^\perp),$$

where $\Gamma(E)$ *(resp.* $\Gamma(E^\perp)$ *) is the space of the smooth sections of* E *(resp.* $E^\perp$).

Proof. Let U be a normal neighbourhood of q (w.r.t. ∇), $(e_1,\dots,e_n)$ an orthonormal basis of T_qM and $(E_1,\dots,E_n)$ the local orthonormal frame in U obtained by parallel transport (w.r.t. ∇) along the ∇-geodesics through q. Let Φ be the **so**(M)-valued local 1-form defined by $\Phi(X)(E_i) = \nabla_X E_i$. Actually, Φ is E-valued. In fact, from $\nabla_X D^s R = 0$, we get

$$\Phi(X)\cdot D^s R = 0,$$

for $0 \le s \le k_M+1$. Therefore, $\Phi_{|x}$ belongs to the fibre $\mathbf{h}(x)$ of E, for all $x \in U$.
If σ is a section of E (resp. $E^\perp$) obtained by parallel transport along the ∇-geodesics (thus the components of $\sigma(E_i)$ w.r.t. $(E_1,\dots,E_n)$ are constant), we have

$$(\nabla_X\sigma)(E_i) = \Phi(X)(\sigma(E_i)) - \sigma(\Phi(X)(E_i)).$$

So,

$$\nabla_X\sigma = \Phi(X)\circ\sigma - \sigma\circ\Phi(X) = \mathrm{ad}(\Phi(X))(\sigma).$$

Since $\mathbf{h}(x)$ is a Lie subalgebra of $\mathbf{so}(T_xM)$, we get in particular that both $\mathbf{h}(x)$ and $\mathbf{h}(x)^\perp$ are $\mathrm{ad}(\mathbf{h}(x))$-invariant for all $x \in U$. This proves that $\nabla_X\sigma$ is still a section of E (resp. $E^\perp$). This proves the lemma.

Q.e.d.

We are now ready to prove the

Theorem 2.9.([14]) *Let* (M,g) *a locally homogeneous Riemannian manifold, then there exists a unique metric connection* ∇ *on* (M,g) *such that* :

1) *the torsion and the curvature of* ∇ *are parallel w.r.t.* ∇;

2) *if* $\nabla = D - S$, D *is the Levi Civita connection, then* S_X *is a section of* $E^\perp$ *for each vector fields* X.

Proof. First the *unicity*. Let ∇ and ∇' two connections which satisfy the hypothesis of the theorem. Then $S_X - S'_X$ is a section of $E^\perp$. On the other hand, we have

$$\nabla(D^sR) = \nabla'(D^sR) = 0,$$

for all s, since $\nabla S = \nabla R_\nabla = 0$ and $\nabla' S' = \nabla' R_{\nabla'} = 0$. Therefore,

$$0 = \nabla_X D^sR - \nabla'_X D^sR = (S_X - S'_X)\cdot D^sR,$$

for all s. So, $S_X - S'_X$ is also a section of E. Hence it is zero and $\nabla = \nabla'$.

The *existence* is just proved in theorem 2.2, where we construct a liner connection with the required properties. Nevertheless, following [14] we give here a more intrinsic proof of this fact.

Let us consider a linear metric connection $\overline{\nabla}$ such that $\overline{\nabla} D^sR = 0$, for $s \le k_M + 1$ (see lemma 2.4). Let $\overline{S}$ be the difference tensor between D and $\overline{\nabla}$. $\overline{S}$ decomposes uniquely as follows

$$\overline{S} = S + \widetilde{S},$$

where $S_X \in \Gamma(E^\perp)$ and $\widetilde{S}_X \in \Gamma(E)$ for all vector fields X on M. Let ∇ be the connection defined by

$$\nabla_X = D_X - S_X = \overline{\nabla}_X + \widetilde{S}_X.$$

Then $\nabla_X D^sR = 0$ for $s \le k_M + 1$. In fact

$$\nabla_X D^sR = \overline{\nabla}_X D^sR + \widetilde{S}_X \cdot D^sR = 0,$$

for $s \le k_M + 1$, because $\widetilde{S}_X$ is a section of E. Therefore

$$i_X D^{s+1}R = S_X \cdot D^sR$$

for all $s \le k_M$. Thus, $D^{s+1}R$ is obtainable from S and D^sR by tensor products, contractions and suitable permutations of the arguments. Since ∇ acts on the tensor algebra as a derivation, commuting with the contractions and with the permutations of the arguments, we get

$$\nabla_X D^{s+1}R = (\nabla_X S)\cdot D^sR$$

for $0 \le s \le k_M$. Therefore, $(\nabla_X S)_Y \cdot D^sR = 0$ if $0 \le s \le k_M$, and $(\nabla_X S)_Y$ is a section of E for each vector fields X,Y on M.

On the other hand, we have

$$(\nabla_X S)_Y = [\nabla_X, S_Y] - S_{\nabla_X Y}.$$

$S_{\nabla_X Y}$ is a section of $E^{\perp}$. The bracket $[\nabla_X, S_Y]$ is just the covariant derivative of the section $\sigma = S_Y$ of $E^{\perp}$. So, from the lemma 2.8 we obtain that $[\nabla_X, S_Y]$ is still a section of $E^{\perp}$.
Therefore, $(\nabla_X S)_Y$ is also a section of $E^{\perp}$. For this reason we have $\nabla S = 0$. From this follows that the torsion and the curvature of ∇ are parallel w.r.t. ∇ (see chapter 1, section C).

Q.e.d.

Remark 2.10. The tensor field S at a point q is uniquely determined by the following two conditions :

(1) $i_X D^{s+1} R_{|q} = (S_q)_X \cdot D^s R_q$;

(2) $(S_q)_X \in \mathbf{h}(q)^{\perp}$,

for all $s \le k_M$.

Further ∇ is *invariant for local isometries* of (M,g). Indeed, let f a local isometry of M. We put

$$\tilde{S}_X = f_*^{-1} \circ S_{f_* X} \circ f_*$$

for each vector field X. Since f is a local isometry, $\tilde{S}_X$ is still a section of $E^{\perp}$.
Moreover,

$$\tilde{S}_X \cdot D^s R = S_X \cdot D^s R,$$

for all $s \ge 0$, because the tensor fields $D^s R$ are invariant by local isometry and $i_X D^{s+1} R = S_X \cdot D^s R$ for all s, since the torsion and the curvature of ∇ are parallel. It follows that $\tilde{S}_X - S_X$ is a section of E, so $\tilde{S}_X = S_X$ for all X. From this we get that f is an *affine transformation* of ∇.

Then, the connection ∇ is a *purely Riemannian invariant.* It depends only on the Riemannian structure of M and if M is globally homogeneous does not depend, unlike the canonical connection of a homogeneous Riemannian manifold (see chapter 1, section C), on a reductive decomposition of the Lie algebra of a transitive Lie group of isometries of M.

Chapter 3.
RECONSTRUCTION OF A HOMOGENEOUS RIEMANNIAN MANIFOLD FROM THE CURVATURE AND ITS COVARIANT DERIVATIVES.

A. Infinitesimal models, homogeneous spaces and the main theorem.

The theorem 2.5 suggests that it is possible to reconstruct a homogeneous Riemannian manifold from the knowledge of R and its covariant derivatives up to some order. To prove this, we must first carefully examine the algebraic properties of R and D^sR.

Let (M,g) be a homogeneous Riemannian manifold, and p a fixed point of M. We put $V = T_pM$, $< , > = g_p$ and $R^0 = R_{|p}$, $R^1 = DR_{|p}$, $R^s = D^sR_{|p}$, for $s \geq 2$. Then we have

(3.1) $$R^0_{XYZW} = -R^0_{YXZW} = R^0_{ZWXY},$$

(3.2) $$\mathfrak{S}_{XYZ}\, R^0_{XYZW} = 0 \qquad (\textit{first Bianchi identity}),$$

(3.3) $$R^1_{XYZVW} = -R^1_{XZYVW} = R^1_{XVWYZ},$$

(3.4) $$\mathfrak{S}_{YZV} R^1_{XYZVW} = 0,$$

(3.5) $$\mathfrak{S}_{XYZ} R^1_{XYZVW} = 0 \qquad (\textit{second Bianchi identity}).$$

Moreover, if we denote with R^0_{XY} the skew-symmetric endomorphism of V given by

$$< R^0_{XY}Z, W > = R^0_{XYZW},$$

we also have

(3.6) $$R^{s+2}_{XY\dots} - R^{s+2}_{YX\dots} = -R^0_{XY}\cdot R^s \qquad (\textit{Ricci identities}),$$

for each $s \geq 0$. Here R^0_{XY} acts as a derivation on the tensor algebra on V.

Of course, the Riemann curvature tensor and its covariant derivatives of every Riemannian manifold always satisfy these conditions. But, if (M,g) is also homogeneous, then there exists a linear metric connection ∇ such that

$$\nabla_X D^s R = 0$$

for $0 \leq s \leq k_M+1$ (see lemma 2.4). Recall that k_M is the integer defined in chapter 2 for the infinitesimally homogeneous spaces. If we put

$$\nabla = D - S,$$

we get, as already remarked,

$$i_X D^{s+1} R = D_X D^s R = S_X \cdot D^s R. \tag{3.7}$$

Recall that i_X denotes the interior product with X and S_X acts as a derivation on the tensors. Since ∇ is metric, S_X is a skew-symmetric endomorphism of V. We denote with **so**(V) the Lie algebra of these endomorphisms. Following Singer, we define the maps μ_s and ν by

$$\mu_s : \mathbf{so}(V) \rightarrow W_s = \oplus_{\alpha=0}^{s} \left(\overset{\alpha+4}{\otimes} V^* \right), \quad A \rightarrow \mu_s(A) = (A \cdot R^0, A \cdot R^1, \ldots, A \cdot R^s), \tag{3.8}$$

and

$$\nu : V \rightarrow W_{k_M+1}, \quad X \rightarrow \nu(X) = (i_X R^1, \ldots, i_X R^{k_M+2}). \tag{3.9}$$

By evaluating (3.7) at p, we get

$$\nu(V) \subseteq \mu_{k_M+1}(\mathbf{so}(V)). \tag{3.10}$$

Finally, because the Lie algebra **g**(p; s) defined by (2.1) is the kernel of μ_s and **g**(p; k_M) = **g**(p;k_M+1), we have

$$\mathrm{Ker}(\mu_{k_M}) = \mathrm{Ker}(\mu_{k_M+1}). \tag{3.11}$$

Conversely, we have the following correct version of [20], p.697 (see [17]) :

Theorem 3.1. *Let* V *be a vector space with a scalar product* < , > *and* k_M *a non-negative integer. Let* $R^0, R^1, ..., R^{k_M+2}$ *be tensors of degrees* 4, 5, ..., k_M+6, *respectively, given on* V *and such that*

(a) *the conditions* (3.1), ..., (3.5), (3.6) *hold for all* $s \le k_M$, *and* (3.10), (3.11) *are also satisfied;*

(b) *the associated infinitesimal model is regular (for the definitions see below).*

Then there is a homogeneous Riemannian manifold (M,g) *with the "Singer's number" equal to* k_M, *such that* < , >, $R^0, R^1, ..., R^{k_M+2}$ *will coincide, respectively, with the tensors* g_p, R_p, ..., $(D^{k_M+2}R)_p$ *at a point* $p \in M$, *assuming that* V *is properly identified with the corresponding tangent space* T_pM.

The condition (b) is missing in [20]. It is always satisfied if $R^0, R^1, ..., R^{k_M+2}$ are the curvature tensor and the covariant derivatives of the curvature of a homogeneous Riemannian manifold at some point. In the general (abstract) case (b) is not a consequence of (a) and without this hypothesis the theorem is false. In fact O.Kowalski was able to produce a 5-dimensional counter-example by exhibiting three tensors R^0, R^1 and R^2 over $V = \mathbf{R}^5$, satisfying only (a), which cannot be the curvature and the covariant derivatives of the curvature tensor of any Riemannian homogeneous manifold (see the next section).

In order to prove and clarify theorem 3.1, let us recall first the notion of *infinitesimal model* (see [23]) and define the *regular* one.

The definition of the infinitesimal model *associated* to $R^0, R^1, ..., R^{k_M+2}$ will be given after Theorem 3.3.

Let V be a real n-dimensional vector space endowed with a positive definite scalar product < , >. A pair of tensors (T,K) over V is an *infinitesimal model* if it satisfies the following axioms :

(3.12) $\quad T : V \to \mathrm{End}(V), \; X \to T_X \quad \text{and} \quad T_X Y = -T_Y X,$

(3.13) $\quad K : V \times V \to \mathrm{End}(V), \; (X,Y) \to K_{XY} \quad \text{and} \quad K_{XY} = -K_{YX},$

(3.14) $\quad \langle K_{XY}Z, W \rangle = -\langle K_{XY}W, Z \rangle,$

(3.15) $\quad K_{XY} \cdot T = 0,$

(3.16) $\quad K_{XY} \cdot K = 0,$

$$(3.17) \qquad \mathfrak{S}_{XYZ}\,(K_{XY}Z + T_{T_X Y}Z) = 0,$$

$$(3.18) \qquad \mathfrak{S}_{XYZ}\, K_{T_X YZ} = 0.$$

Two infinitesimal models (T,K) on (V, < , >) and (T',K') on (V', < , >) are said to be *isomorphic* if there exists an isometry $a : (V, <,>) \to (V', <,>)$ such that aT = T', aK = K'.

It is clear that if (M,g) is a homogeneous Riemannian manifold and ∇ an *Ambrose-Singer connection* on it, then for an arbitrary point p of M, $T = (T_\nabla)_{|p}$ and $K = (R_\nabla)_{|p}$ determine an infinitesimal model on $(V = T_pM, <,> = g_p)$ (see p. 13). Varying p we obtain isomorphic infinitesimal models.

Conversely, we recall the so-called *Nomizu construction* (see [18]).
Let **k** be the Lie subalgebra of so(V) defined by

$$(3.19) \qquad \mathbf{k} = \{ A \in \mathbf{so}(V) \;:\; A\cdot T = A\cdot K = 0 \}.$$

Of course, $K_{XY} \in \mathbf{k}$ for each X and Y in V. Moreover, let **g** be the *direct sum* of V and **k** endowed with the following bracket:

$$(3.20) \qquad [X,Y] = -T_XY + K_{XY},$$

$$(3.21) \qquad [A,X] = A(X),$$

$$(3.22) \qquad [A,B] = A\circ B - B\circ A,$$

for all $X,Y \in V$ and $A,B \in \mathbf{k}$. Then, the axioms (3.12), ..., (3.18) implies that (**g**, [,]) is a *Lie algebra*. Let G be the connected and simply connected Lie group whose Lie algebra is g and H the connected subgroup of G corresponding to **k**. If H is *closed*, we say that (T,K) is a *regular* infinitesimal model. In this case, the homogeneous space M = G/H is simply connected and reductive with respect to the decomposition $\mathbf{g} = V\oplus\mathbf{k}$ because $[\mathbf{k},V] \subseteq V$ and H is connected. The scalar product < , > and the tensors T and K are invariant with respect to the isotropy representation of H (by the definition of **k**) and consequently Ad(H)-invariant (see chapter 1). Hence < , >, T and K extend to G-invariant tensor fields on M. As we saw in chapter 1 the canonical connection of G/H with respect to the reductive decomposition $g = V\oplus\mathbf{k}$ has parallel curvature and torsion tensor fields (it is an *Ambrose-Singer connection*). Moreover,

from the formulae (1.3), (1.4) and (3.20), (3.21), (3.22) follows that these tensors evaluated at the origin o of M coincide with T and K when T_oM is identified with V.
If T = 0, the Nomizu construction is the classical construction of E. Cartan of a Riemannian symmetric space by using the curvature tensor (see [3], [7]).

We insist on the fact that the infinitesimal model has to be regular, otherwise the Nomizu construction does not work. Note that if dimV = 3,4, the infinitesimal model is always regular. In fact an old result of Mostow [16] states that any connected subgroup of a simply connected Lie group is closed if its codimension is less or equal to 4.

When (M,g) is a homogeneous Riemannian manifold we can attach to it an infinitesimal model in such a way that the manifold can be reconstructed from this model. It suffices to make (M,g) a reductive homogeneous space G/K and consider the canonical connection of G/K as in chapter 1. Then T and K will represent the torsion and the curvature of this canonical connection at the origin of G/K.

These considerations show that the study of homogeneous Riemannian manifolds is equivalent to the study of a class of infinitesimal models.

Now, we suppose that R^0, R^1,..., R^{k_M+2} are tensors on V of degree 4,5,...,k_M+6, respectively. We suppose also that they satisfy the conditions (3.1), ..., (3.5), (3.6) if $s \leq k_M$, (3.10) and (3.11).

We shall prove theorem 3.1 by showing that it is always possible to associate to the tensors $R^0,R^1,...,R^{k_M+2}$ an infinitesimal model in a unique way. If this model is regular, the Nomizu construction will always produce a homogeneous Riemannian manifold.

First of all, let us remark that

$$\mathfrak{h} = \operatorname{Ker} \mu_{k_M+1} \tag{3.23}$$

is a *Lie subalgebra* of so(n). Let $\mathfrak{h}^\perp$ be the orthogonal complement of $\mathfrak{h}$ in so(n) with respect to the *trace form.* As we have already remarked in chapter 2, both $\mathfrak{h}$ and $\mathfrak{h}^\perp$ are ad($\mathfrak{h}$)-invariant.

From (3.10) we deduce that, for each X in V, there exists an element A(X) of so(n) (uniquely determined up to a component in $\mathfrak{h}$) such that

$$i_X R^{s+1} = A(X)\cdot R^s$$

if $0 \leq s \leq k_M+1$.

Since $\mathbf{so}(n) = \mathbf{h} \oplus \mathbf{h}^{\perp}$, we have

$$A(X) = A_1(X) + A_2(X),$$

where $A_1(X)$ and $A_2(X)$ are the projections of $A(X)$ on $\mathbf{h}$ and $\mathbf{h}^{\perp}$, respectively. Let S be the (uniquely defined) map

$$S : V \to \mathbf{h}^{\perp}, \quad X \to S_X = A_2(X). \tag{3.24}$$

Then we have :

Lemma 3.2. *S is a linear map and*

$$i_X R^{s+1} = S_X \cdot R^s$$

for $0 \leq s \leq k_M+1$.

Proof. Since $A_1(X) \cdot R^s = 0$ for $0 \leq s \leq k_M+1$, we have

$$i_X R^{s+1} = A_2(X) \cdot R^s = S_X \cdot R^s.$$

Moreover,

$$i_{X+Y} R^{s+1} = S_{X+Y} \cdot R^s$$

and

$$i_{X+Y} R^{s+1} = i_X R^{s+1} + i_Y R^{s+1} = S_X \cdot R^s + S_Y \cdot R^s,$$

for $0 \leq s \leq k_M+1$. From these identities we get that $S_{X+Y} - S_X - S_Y$ belongs to $\mathbf{h}$. Therefore, it is zero, i.e.

$$S_{X+Y} = S_X + S_Y.$$

In the same way we prove that

$$S_{aX} = aS_X,$$

for each real number a. So, S_X is linear.

Q.e.d.

We put now

(3.25) $$T_XY = -S_XY + S_YX,$$

(3.26) $$K_{XY} = R^0{}_{XY} + [\,S_X, S_Y] + S_{T_XY},$$

then we have :

Theorem 3.3. *The pair* (T,K) *is an infinitesimal model on* V.

(T,K) is called the *infinitesimal model associated* to $R^0, R^1, \ldots, R^{k_M+2}$.

Remark that the tensors $R^0, R^1, \ldots, R^{k_M+2}$ can be recovered from (T,K) by the formula (3.26), lemma 3.2 and

(3.27) $$2< S_XY, Z > = -< T_XY, Z > + < T_YZ, X > - < T_ZX, Y >.$$

Therefore, if (M,g) is the homogeneous Riemannian manifold corresponding to (T,K), obtained by the Nomizu construction, the tensors $R^0, R^1, \ldots, R^{k_M+2}$ are just the curvature and the covariant derivatives of the curvature evaluated at the origin o of M.

We divide the proof of theorem 3.3 in several steps by proving first some useful lemmas.

Lemma 3.4. *For* $0 \le s \le k_M$ *we have*

$$R^{s+2}{}_{XY\ldots} - R^{s+2}{}_{YX\ldots} = (\,[S_X,S_Y] + S_{T_XY}\,)\cdot R^s.$$

Proof. From lemma 3.2 we get

$$R^{s+2}{}_{XYZ_1\ldots Z_s} = (\,i_XR^{s+2}\,)_{YZ_1\ldots Z_s} = (\,S_X\cdot R^{s+1})_{YZ_1\ldots Z_s}$$

$$= -R^{s+1}{}_{S_XYZ_1\ldots Z_s} - \sum_{\alpha=1}^{s} R^{s+1}{}_{YZ_1\ldots S_XZ_\alpha\ldots Z_s}$$

$$= -(i_{S_XY}R^{s+1}\,)_{Z_1\ldots Z_s} - \sum_{\alpha=1}^{s} (i_YR^s)_{Z_1\ldots S_XZ_\alpha\ldots Z_s}$$

$$= -(S_{S_XY}\cdot R^s)_{Z_1\ldots Z_s} - \sum_{\alpha=1}^{s} (S_Y\cdot R^s\,)_{Z_1\ldots S_XZ_\alpha\ldots Z_s}$$

$$= \sum_{\alpha=1}^{s} R^s_{Z_1 \cdots S_{S_X Y} Z_\alpha \cdots Z_s} + \sum_{\alpha<\beta} R^s_{Z_1 \cdots S_Y Z_\alpha \cdots S_X Z_\beta \cdots Z_s}$$

$$+ \sum_{\alpha>\beta} R^s_{Z_1 \cdots S_X Z_\beta \cdots S_Y Z_\alpha \cdots Z_s} + \sum_{\alpha=1}^{n} R^s_{Z_1 \cdots S_Y S_X Z_\alpha \cdots Z_s}.$$

Then, the lemma follows easily from (3.6).

Q.e.d.

Lemma 3.5. *For* $0 \le s \le k_M$ *we have* $K_{XY} \cdot R^s = 0$.

Proof. From the lemma 3.4 and (3.6) we get

$$([S_X, S_Y] + S_{T_X Y}) \cdot R^s = - R^0_{XY} \cdot R^s.$$

Therefore, the lemma follows from the definition of K.

Q.e.d.

Lemma 3.6. *For each vector* X *and* Y *in* V, K_{XY} *belongs to the Lie algebra* **h**.

Proof. It is sufficient to recall that

$$\mathrm{Ker}\,(\mu_{k_M}) = \mathrm{Ker}\,(\mu_{k_M}+1) = \mathbf{h}.$$

Q.e.d.

Lemma 3.7. *For each* $A \in \mathbf{h}$ *we have* $A \cdot S = 0$. *In particular* $K_{XY} \cdot S = 0$.

Proof. By the definition of **h** and lemma 3.2 we have :

$$0 = (A \cdot R^{s+1})_{X_0 X_1 \cdots X_{s+4}}$$

$$= - R^{s+1}_{A X_0 X_1 \cdots X_{s+4}} - \sum_{\alpha=1}^{s+4} R^{s+1}_{X_0 X_1 \cdots A X_\alpha \cdots X_{s+4}}$$

$$= -\,(i_{AX_0} R^{s+1})_{X_1 \dots X_{s+4}} - \sum_{\alpha=1}^{s+4} (i_{X_0} R^{s+1})_{X_1 \dots AX_\alpha \dots X_{s+4}}$$

$$= -\,(S_{AX_0} \cdot R^{s})_{X_1 \dots X_{s+4}} - \sum_{\alpha=1}^{s+4} (S_{X_0} \cdot R^{s})_{X_1 \dots AX_\alpha \dots X_{s+4}}$$

$$= -\,(S_{AX_0} \cdot R^{s})_{X_1 \dots X_{s+4}} + (A \cdot (S_{X_0} \cdot R^{s}))_{X_1 \dots X_{s+4}}.$$

So, we get (see also the appendix)

$$([A\,,\, S_{X_0}] - S_{AX_0}) \cdot R^{s} = 0,$$

for $0 \le s \le k_M$. Since $\mathbf{h} = \mathrm{Ker}\,(\mu_{k_M})$, we have

$$(A \cdot S)_{X_0} = [\, A,\, S_{X_0}] - S_{AX_0} \in \mathbf{h}.$$

On the other hand, $S_{X_0} \in \mathbf{h}^{\perp}$ and $\mathbf{h}^{\perp}$ is $\mathrm{ad}(\mathbf{h})$-invariant.
So, $[\, A,\, S_{X_0}] - S_{AX_0}$ is an element of $\mathbf{h}^{\perp}$ too. This implies that

$$(A \cdot S)_{X_0} = [S_{X_0}, A] - S_{AX_0} = 0$$

Q.e.d.

Lemma 3.8. *The first reduced Bianchi identity holds, i.e.*

$$_{XYZ}\,(K_{XY}Z + T_{T_X Y}Z) = 0.$$

Proof. By direct computation we find

$$K_{XY}Z + T_{T_X Y}Z = R^0{}_{XY}Z + S_X S_Y Z - S_Y S_X Z + S_Z S_Y X - S_Z S_X Y.$$

The lemma follows easily from this formula.

Q.e.d.

Lemma 3.9. *The second reduced Bianchi identity holds, i.e.*

$$\mathfrak{S}_{XYZ}\, K_{T_XYZ} = 0.$$

Proof. From (3.26) we get

$$K_{T_XYZ} = R^0{}_{T_XYZ} + [\, S_{T_XY}, S_Z] + S_{T_{T_XY}Z}.$$

By taking the cyclic sum with respect to X,Y and Z we get

$$\mathfrak{S}_{XYZ}\, K_{T_XYZ} = -R^0{}_{S_XYZ} + R^0{}_{S_YXZ} - R^0{}_{S_YZX} + R^0{}_{S_ZYX} - R^0{}_{S_ZXY} + R^0{}_{S_XZY} +$$
$$+\mathfrak{S}_{XYZ}\,[\, S_{T_XY}, S_Z] + \mathfrak{S}_{XYZ} S_{T_{T_XY}Z}.$$

Let $R^1{}_{XYZ}$ be the endomorphism of V defined by

$$< R^1{}_{XYZ}V,W > = R^1{}_{XYZVW}.$$

Then, lemma 3.2 implies that

$$R^1{}_{XYZ} = (\, S_X{\cdot}R^0\,)_{YZ} = [\, S_X, R^0{}_{YZ}] - R^0{}_{S_XYZ} - R^0{}_{YS_XZ}.$$

Thus, we get

$$\mathfrak{S}_{XYZ}\; K_{T_XYZ} = R^1{}_{XYZ} + R^1{}_{YZX} + R^1{}_{ZXY} - [S_X, R^0{}_{YZ}] - [S_Y, R^0{}_{ZX}] - [S_Z, R^0{}_{XY}] +$$
$$+\mathfrak{S}_{XYZ}\,[S_{T_XY}, S_Z] + \mathfrak{S}_{XYZ}\, S_{T_{T_XY}Z}$$
$$= \mathfrak{S}_{XYZ}\,(\, [S_{T_XY}, S_Z] + S_{T_{T_XY}Z} - [S_X, R^0{}_{YZ}]\,),$$

since R^1 satisfies the second Bianchi identity (3.5).
From lemma 3.7 and formula (3.26) we obtain

$$[S_X, R^0{}_{YZ}] = [S_X, K_{YZ}] - [S_X, [S_Y, S_Z]] - [S_X, S_{T_YZ}] = -S_{K_{YZ}X} - [S_X, [S_Y, S_Z]] - [S_X, S_{T_YZ}].$$

So, the Jacobi identity gives

$$\mathfrak{S}_{XYZ}\, K_{T_XYZ} = \mathfrak{S}_{XYZ}\,(S_{T_{T_XY}Z} + S_{K_{XY}Z})$$

and the lemma follows from the previous one.

Q.e.d.

The proof of lemma 3.9 achieves the proof of theorem 3.3 and consequently of theorem 3.1.

Thus, in order to construct a homogeneous Riemannian manifold whose curvature tensor and its first k_M+2 covariant derivatives at some point are $R^0, R^1, \ldots, R^{k_M+2}$ satisfying (a) and (b) of theorem 3.1, we proceed as follows.

First, we determine the Lie algebra

$$\mathbf{h} = \mathrm{Ker}\,(\mu_{k_M+1})$$

and the map

$$S : V \to \mathbf{h}^{\perp}$$

given by (3.24). Then, we define an infinitesimal model (T,K) on V by the formulas (3.25) and (3.26). Finally, we apply the *Nomizu construction* to get the desired homogeneous Riemannian manifold G/H = M.

It is important to remark at this point that **h** is just the Lie algebra **k** of the *isotropy group* H. Indeed we have

Proposition 3.10. *The Lie algebra* **h** *coincides with the Lie subalgebra* **k** *of* so(n) *given by*

$$\mathbf{k} = \{\ A \in \mathbf{so}(n)\ :\ A\cdot T = A\cdot K = 0\ \}.$$

Proof. The lemma 3.7 implies that $A\cdot S = 0$ for each $A \in \mathbf{h}$. Of course, this gives $A\cdot T = 0$. Moreover, by the definition of **h** we also have $A\cdot R^0 = 0$.

From the formula (3.26) we get

$$(A\cdot K)_{XY} = [A\,,K_{XY}] - K_{AXY} - K_{XAY} =$$

$$= (A\cdot R^0)_{XY} + [A, [S_X,S_Y]] - [S_{AX},S_Y]$$

$$- [S_X,S_{AY}] + [A,S_{T_XY}] - S_{T_{AX}Y} - S_{T_XAY}.$$

By using the Jacobi identity we get easily

$$(3.28) \qquad (A\cdot K)_{XY} = (A\cdot R^0)_{XY} + [(A\cdot S)_X, S_Y] - [(A\cdot S)_Y, S_X] + S_{(A\cdot T)_X Y}.$$

So, if $A \in \mathbf{h}$, we have also $A\cdot K = 0$ and therefore $\mathbf{k} \supseteq \mathbf{h}$.

Conversely, if $A \in \mathbf{k}$, then $A\cdot S = 0$ by the formula (3.27). Moreover, (3.28) implies $A\cdot R^0 = 0$.

We shall prove by induction on s that $A\cdot R^s = 0$ for each $s \leq k_M+1$.
As A acts as derivation, a simple calculation (see the proof of lemma 3.7) gives us

$$(3.29) \qquad (A\cdot R^{s+1})_{X_0 X_1 \ldots X_{s+4}} = (\,[A,S_{X_0}]\cdot R^s\,)_{X_1 \ldots X_{s+4}} + $$
$$- (S_{AX_0}\cdot R^s)_{X_1 \ldots X_{s+4}} + (S_{X_0}\cdot(\,A\cdot R^s\,))_{X_1 \ldots X_{s+4}}.$$

So, if by the inductive hypothesis we have $A\cdot R^s = 0$, from (3.29) we get $A\cdot R^{s+1} = 0$ since

$$(A\cdot S)_{X_0} = [A,S_{X_0}] - S_{AX_0} = 0.$$

Q.e.d.

We close up with the following :

Remark 3.11. Let (M,g) be a homogeneous Riemannian manifold, and let $R^0, R^1, \ldots, R^{k_M+2}$ be its curvature tensor and its successive covariant derivatives at some point $p \in M$. Then the previous construction leads to an infinitesimal model (T,K). Changing the point p, we always obtain an isomorphic model. Hence, we can attach a unique infinitesimal model (T,K) (up to an isomorphism) to every homogeneous Riemannian manifold (M,g), and the model is always *regular*. This gives the reconstruction of a homogeneous Riemannian manifold from the curvature and its covariant derivatives (only up to a local isometry, in general).

B. Counter-example to the second Singer's theorem.

In this section we give the promised counter-example to the original statement of Singer ([20], p. 697), that we indicate as the second Singer's theorem. The second Singer's theorem consists of theorem 3.1 where the condition (b) is missing. The counter-example that we are going to exhibit is due to O. Kowalski ([12]) and is given in dimension five. It is worth mentioning that the second Singer's theorem is true for dimension less or equal than four; in fact as pointed out above all infinitesimal models of dimension less or equal than four are automatically regular.

At first we discuss some general facts that will be useful in the explanation of the counter-example.

Let us have a simply connected homogeneous Riemannian manifold (M,g) with a reductive representation $M = G/K$, $\mathbf{g} = \mathbf{m}\oplus\mathbf{k}$. Knowing only this last Lie algebra structure together with the inner product $<\, ,\, >$ induced on $\mathbf{m}$ by the metric g, one can reconstruct a reductive representation of the homogeneous manifold (M,g) as follows. Let $\tilde{G}$ a connected and simply connected Lie group with Lie algebra $\mathbf{g}$ and consider the homomorphism $\pi : \tilde{G} \to G$ of $\tilde{G}$ onto G corresponding to the identity transformation of $\mathbf{g}$. Here π is a universal covering map of G.
Let $\tilde{K}$ denote the connected component of the inverse image of K by the projection π. Because K was closed in G, $\tilde{K}$ is also closed in $\tilde{G}$. The corresponding homogeneous space $\tilde{G}/\tilde{K}$ is well-defined and is simply connected. Moreover, because the group $Ad_G(K)$ preserves the inner product $<\, ,\, >$ on $\mathbf{m}$, the algebra $ad_{\mathbf{g}}(\mathbf{k})$ consists of skew-symmetric endomorphisms of $\mathbf{m}$ and hence again $Ad_{\tilde{G}}(\tilde{K})$, whose Lie algebra is also $ad_{\mathbf{g}}(\mathbf{k})$, preserves the same inner product on $\mathbf{m}$. Thus $\tilde{G}/\tilde{K}$ is provided with a $\tilde{G}$-invariant metric (see chapter 1). If $\tilde{\pi}$ denotes the map of $\tilde{G}/\tilde{K}$ onto G/K induced by π, then the pair $(\tilde{G}/\tilde{K},\tilde{\pi})$ is a Riemannian covering manifold of G/K, i.e. $\tilde{\pi}$ is a covering map that is a local isometry. Actually, since G/K is simply connected, $\tilde{\pi}$ is an isometry. We can conclude that $\tilde{G}/\tilde{K}$ is a reductive representation of (M,g).

Let now G^* be a connected Lie group with Lie algebra $\mathbf{g}$ and with a *finite fundamental group*. Consider its universal covering group $(\tilde{G},\pi)$ (with finite fibre). Then a subgroup K^* is closed in G^* if and only if the corresponding subgroup $\tilde{K}$ is closed in $\tilde{G}$. In this case also G^*/K^* is provided with a G^*-invariant metric, but it may be only locally isometric to (M,g).

Let (T,K) be a regular infinitesimal model on (V,< , >) and consider the reductive homogeneous space G/H, with respect to the decomposition $\mathbf{g} = V \oplus \mathbf{k}$, obtained by the Nomizu construction as described in the previous section. Let $\mathbf{k}^\circ \subseteq \mathbf{k}$ be any subalgebra of $\mathbf{k} = \{A \in \mathbf{so}(V) : A \cdot T = A \cdot K = 0\}$ such that $K_{XY} \in \mathbf{k}^\circ$ for all $X,Y \in V$. Then $\mathbf{g}^\circ = V \oplus \mathbf{k}^\circ$ is a subalgebra of $\mathbf{g} = V \oplus \mathbf{k}$ and the corresponding connected subgroup $G^\circ \subseteq G$ acts transitively on M, because $\mathbf{g}^\circ \supseteq V$ and any point in M can be reached by a geodesic (w.r.t. the canonical connection) starting from the origin of the form $(\exp tX) \cdot o$, for some $X \in V$. Thus the connected subgroup $H^\circ \subseteq H$ corresponding to $\mathbf{k}^\circ$ is the isotropy group at the origin and hence is closed in G°. So we can conclude that $G/H = G^\circ/H^\circ$ (see also [13]).

By the preceding discussion we have the following.

Remark 3.12. Assuming that the original infinitesimal model (T,K) on (V,< , >) is *regular* and assuming that G° is any connected Lie group with Lie algebra $\mathbf{g}^\circ$ and with a *finite fundamental group*, then the subgroup $H^\circ \subseteq G^\circ$ corresponding to $\mathbf{k}^\circ$ must be closed.

We explain now the counter-example.

Consider the infinitesimal model given as follows.

Let (V, < , >) a 5-dimensional Euclidean vector space with a given orthonormal basis $\{X_1,\dots,X_5\}$; the algebra $\mathbf{so}(V)$ of skew-symmetric endomorphisms of V has the basis $\{A_{ij}\}$, $1 \le i < j \le 5$, where $A_{ij}(X_i) = X_j$, $A_{ij}(X_j) = -X_i$, $A_{ij}(X_k) = 0$ otherwise.
Fix constants $\lambda > 0$, $\rho > 0$, $r < 0$ and define T_XY and K_{XY} for $X,Y \in V$ by the formulae

$$T_{X_1}X_2 = 2\rho X_5,\ T_{X_3}X_4 = 2\lambda X_5,\ T_{X_1}X_3 = T_{X_1}X_4 = T_{X_2}X_3 = T_{X_2}X_4 = 0,$$

(3.30) $$T_{X_1}X_5 = -\rho X_2,\ T_{X_2}X_5 = \rho X_1,\ T_{X_3}X_5 = -\lambda X_4,\ T_{X_4}X_5 = \lambda X_3,$$

$$T_{X_i}X_j = -T_{X_j}X_i,\ 1 \le j \le i \le 5;$$

(3.31) $$K_{X_1X_2} = -K_{X_2X_1} = -2\lambda\rho(r^{-1}A_{12} + A_{34}),$$

$$K_{X_3X_4} = -K_{X_4X_3} = -2\lambda\rho(A_{12} + rA_{34}), \qquad K_{X_iX_j} = 0 \text{ otherwise.}$$

The conditions (3.12)-(3.18) are easily verified. Hence (T,K) defines an infinitesimal model over (V,< , >).

The tensors S and R^0 satisfying (3.25) and (3.26) are given explicitly by an easy calculation as follows:

(3.32) $S_{X_1} = -\rho A_{25}, S_{X_2} = \rho A_{15}, S_{X_3} = -\lambda A_{45}, S_{X_4} = \lambda A_{35}, S_{X_5} = 0.$

$R^0_{X_1X_2} = -(\rho^2 + 2\lambda\rho r^{-1})A_{12} - 2\lambda\rho A_{34}, R^0_{X_3X_4} = -2\lambda\rho A_{12} - (\lambda^2 + 2\lambda\rho r)A_{34},$

(3.33) $R^0_{X_1X_3} = -\lambda\rho A_{24}, R^0_{X_2X_4} = -\lambda\rho A_{13}, R^0_{X_1X_4} = \lambda\rho A_{23}, R^0_{X_2X_3} = \lambda\rho A_{14},$

$R^0_{X_1X_5} = \rho^2 A_{15}, R^0_{X_2X_5} = \rho^2 A_{25}, R^0_{X_3X_5} = \lambda^2 A_{35}, R^0_{X_4X_5} = \lambda^2 A_{45}.$

Now assume $\lambda \neq \rho$. One can check easily that the algebra $\mathbf{k} = \{A \in \mathbf{so}(V) : A\cdot T = A\cdot K = 0\}$ coincide with *span*(A_{12}, A_{34}).
Consider the algebras analogous to those defined by (2.1) :

$$\mathbf{g}(V; s) = \{ A \in \mathbf{so}(V) \ : \ A\cdot R^0 = A\cdot R^1 = \cdots = A\cdot R^s = 0 \},$$

where the tensors R^s for $s > 0$ are defined inductively by $i_X R^{s+1} = S_X \cdot R^s$, $X \in V$ (see lemma 3.2). Assuming $r \neq -\lambda/\rho$, $2r \pm 3 \neq -\lambda/\rho$, we get $\mathbf{g}(V; 0) = \mathbf{g}(V; 1) = \mathbf{k} =$ *span*(A_{12}, A_{34}). Further, the tensors R^0, R^1, R^2 satisfy all the conditions (a) from theorem 3.1 for $k_M = 0$ and the algebra $\mathbf{h} = \mathrm{Ker}\, \mu_1 = \{A \in \mathbf{so}(V) : A\cdot R^0 = A\cdot R^1 = 0\}$ coincide with $\mathbf{k} =$ *span*(A_{12}, A_{34}). Note that, due to (3.32), we have $S_X \in \mathbf{h}^\perp$ for all $X \in V$.

Hence the infinitesimal model (T,K) is the infinitesimal model *associated* in a unique way to R^0, R^1, R^2. If we suppose that R^0, R^1, R^2 can be identified with the tensors R_p, DR_p, D^2R_p at a point p of a Riemannian homogeneous manifold, then the infinitesimal model (T,K) must be regular.

Now let $\mathbf{k}^\circ \subseteq \mathbf{k}$ be the subalgebra generated by all endomorphisms K_{XY} for $X,Y \in V$ and let $\mathbf{g}^\circ = V \oplus \mathbf{k}^\circ = V \oplus [V,V]$. $\mathbf{g}^\circ$ is an ideal of $\mathbf{g}$. Using the Lie bracket $[X,Y] = -T_XY + K_{XY}$ for $X,Y \in V$, the Lie algebra structure of $\mathbf{g}^\circ$ can be described by a suitable normed basis $\{X_1^*, \ldots, X_4^*, W_1^*, W_2^*\}$ in the following way :

$[X_1^*, X_2^*] = -W_1^*$ $\quad$ $[X_1^*, W_1^*] = X_2^*$ $\quad$ $[X_2^*, W_1^*] = -X_1^*$

(3.34) $[X_3^*, X_4^*] = -W_2^*$ $\quad$ $[X_1^*, W_1^*] = X_4^*$ $\quad$ $[X_4^*, W_2^*] = -X_3^*$

$$[X_i^*, X_j^*] = [X_i^*, W_k^*] \quad \text{otherwise.}$$

On the other hand we have $\mathbf{k}° = span\ (W_1^* + rW_2^*)$

The basis (3.34) can be represented by a basis of vector fields (*infinitesimal isometries*) in a six-dimensional space $\mathbf{R}^6(x_j,y_j)$, j = 1,2,3 as follows :

$$W_1^* = x_1\partial/\partial x_2 - x_2\partial/\partial x_1, \quad X_1^* = x_2\partial/\partial x_3 - x_3\partial/\partial x_2, \quad X_2^* = x_3\partial/\partial x_1 - x_1\partial/\partial x_3$$

$$W_2^* = y_1\partial/\partial y_2 - y_2\partial/\partial y_1, \quad X_3^* = y_2\partial/\partial y_3 - y_3\partial/\partial y_2, \quad X_4^* = y_3\partial/\partial y_1 - y_1\partial/\partial y_3.$$

Thus we can realize $\mathbf{g}°$ by the Lie group SO(3) × SO(3),which has a *finite fundamental group*, and consequently $\mathbf{k}°$ by the subgroup H° of all product matrices of the form

$$\begin{bmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} \cos rt & -\sin rt & 0 \\ \sin rt & \cos rt & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now, if r is *irrational*, then H° is not closed in G°. Then, by remark 3.12, R^0, R^1, R^2 do not correspond to any 5-dimensional homogeneous Riemannian manifold.

APPENDIX.

The Ricci identity.

Let M be a differentiable manifold, ∇ a linear connection on TM and $\mathcal{T}(M)$ the algebra of C^∞ tensor fields over M. If we extend the covariant derivative to the algebra of C^∞ tensor fields $\mathcal{T}(M)$ as specified in chapter 1, section A, the second covariant derivative is expressed by

$$\nabla^2{}_{XY} = \nabla_X \nabla_Y - \nabla_{\nabla_X Y},$$

$X,Y \in \Gamma(M)$. From the definitions of the curvature R_∇ and the torsion T_∇ of ∇, follows immediately that

$$(R_\nabla)_{XY}Z = -\nabla^2{}_{XY}Z + \nabla^2{}_{YX}Z - \nabla_{(T_\nabla)_X Y}Z,$$

$X,Y,Z \in \Gamma(M)$. Now, if we consider the curvature operator $(R_\nabla)_{XY}$ acting as a derivation on $\mathcal{T}(M)$ we have for each function $f \in C^\infty(M)$ and $X,Y \in \Gamma(M)$

$$(R_\nabla)_{XY}(f) = (-\nabla^2{}_{XY} + \nabla^2{}_{YX} - \nabla_{(T_\nabla)_X Y})(f) = 0.$$

Thus, $(R_\nabla)_{XY}$ and $-\nabla^2{}_{XY} + \nabla^2{}_{YX} - \nabla_{(T_\nabla)_X Y}$ agree on differentiable vector fields and on differentiable functions. Since a derivation of $\mathcal{T}(M)$ is completely determined by its operation on the algebra of differentiable functions $C^\infty(M)$ and the module of differentiable vector fields $\Gamma(M)$ (see [10], prop.3.3, p.30) we have that

$$(R_\nabla)_{XY} = -\nabla^2{}_{XY} + \nabla^2{}_{YX} - \nabla_{(T_\nabla)_X Y} \qquad \textit{(Ricci identity)}$$

as derivations on $\mathcal{T}(M)$.

Some calculations.

Let V a real vector space. If A is an endomorphism of V, then A can be uniquely extended to the tensor algebra $\mathcal{T}(V)$ as a derivation, preserving type of tensors and commuting with contraction. Denoting this extension again by A we have, if $X \in V$, $\omega \in V^*$, V^* dual vector space of V, $A\cdot(\omega\otimes X) = A\cdot\omega\otimes X + \omega\otimes AX$. Applying the contraction $C : \mathcal{T}(V)^1_1 \to \mathbb{R}$,

$C(\omega\otimes X) = \omega(X) = \omega_X$ and noting that A annihilates scalars, we get $(A\cdot\omega)_X = -\,\omega_{AX}$. For a tensor K of type (0,r), respectively (1,r), it follows that, for $X_1,\dots,X_r \in V$,

$$(A\cdot K)_{X_1\cdots X_r} = -\sum_{i=1}^{r} K_{X_1\cdots AX_i\cdots X_r}\,, \text{ respectively}$$

$$(A\cdot K)_{X_1\cdots X_r} = A(K_{X_1\cdots X_r}) - \sum_{i=1}^{r} K_{X_1\cdots AX_i\cdots X_r}$$

Lemma. *Let* A, B *be two endomorphisms of* V *and* K *a tensor of type* (0,r) (or (1,r)) *over* V, *then*

$$A\cdot(B\cdot K) = B\cdot(A\cdot K) + ([A,B])\cdot K, \tag{1}$$

where the bracket [A,B] *of two endomorphisms denote the endomorphism* AB – BA.

Proof. Let K be a (0,r) tensor and $X_1,\dots,X_r \in V$,then

$$(A\cdot(B\cdot K))_{X_1\cdots X_r} = -\sum_{i=1}^{r} (B\cdot K)_{X_1\cdots AX_i\cdots X_r} = \tag{2}$$

$$\sum_{i=1}^{r}\sum_{j<i} K_{X_1\cdots BX_j\cdots AX_i\cdots X_r} + \sum_{i=j=1}^{r} K_{X_1\cdots BAX_j\cdots X_r} + \sum_{i=1}^{r}\sum_{j>i} K_{X_1\cdots AX_i\cdots BX_j\cdots X_r}.$$

Subtracting from (2) the corresponding expression for $B\cdot(A\cdot K)$ we obtain (1). The proof for a (1,r) tensor is analogous.

Q.e.d.

REFERENCES

[1] W. AMBROSE and I.M. SINGER, On homogeneous Riemannian manifolds, *Duke Math. J.*, **25** (1958), 647-669.

[2] M. BERGER, P. GAUDUCHON, E. MAZET, *Le spectre d'une variété riemannienne*, Lecture Notes in Math. 194, Springer, Berlin,Heidelberg,New York (1971).

[3] A.L. BESSE, *Einstein manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, Band 10, Springer, Berlin (1987).

[4] J. CHEEGER and D.G. EBIN, *Comparison theorems in Riemannian geometry*, North-Holland Publ. Co., Amsterdam (1975).

[5] M. GROMOV, *Partial differential relations*, Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, Band 9, Springer, Berlin (1986)

[6] M. GROMOV, *Structures métriques pour les variétés riemanniennes*, Cedic, Paris, (1981).

[7] S. HELGASON, *Differential geometry , Lie groups and symmetric spaces*, Academic Press, New York (1978).

[8] D. HUSEMOLLER, *Fibre bundles*, second edition, Graduate Texts in Mathematics 20, Springer, New York, Heidelberg, Berlin (1975).

[9] N. IWAHORI, Some remarks on tensor invariants of O(n), U(n), Sp(n), *J. Math. Soc. Japan*, **10** (1958), 145-160.

[10] S. KOBAYASHI and K. NOMIZU, *Foundations of differential geometry*, Interscience Publ., New York, vol. I (1963), vol. II (1969).

[11] B. KOSTANT, Holomomy of the Lie algebra of infinitesimal motions of a Riemannian manifold, *Trans. Amer. Math. Soc.*, **80** (1955), 528-542.

[12] O. KOWALSKI, Counter-example to the "Second Singer's Theorem", preliminary version.

[13] O. KOWALSKI, *Generalized symmetric spaces*, Lecture Notes in Math. 805, Springer, Berlin ,Heidelberg,New York (1980).

[14] O. KOWALSKI and F. TRICERRI, A canonical connection for locally homogeneous Riemannian manifolds, in preparation.

[15] F. LASTARIA, tesi di dottorato, in preparation.

[16] G.D. MOSTOW, The extensibility of local Lie groups of transformations and groups surfaces, *Ann. of Math.*, **52** (1950), 606-636.

[17] L. NICOLODI and F. TRICERRI, On two theorems of I.M.Singer about homogeneous spaces, Preprint S.I.S.S.A., Trieste (1989).

[18] K. NOMIZU, Invariant affine connections on homogeneous spaces, *Amer. J. Math.*, **76** (1954), 33-65.

[19] K. NOMIZU, Sur les algèbres de Lie de générateurs de Killing et l' homogénéité d' une variété riemannienne, *Osaka Math. J.*, **14** (1962), 45-51.

[20] I.M. SINGER, Infinitesimally homogeneous spaces, *Comm. Pure Appl. Math.*, **13** (1960), 685-697.

[21] H. TAKAGI, On curvature homogeneity of Riemannian manifolds, *Tôhoku Math J.*, **26** (1974), 581-585.

[22] F. TRICERRI and L. VANHECKE, *Homogeneous structures on Riemannian manifolds*, London Math. Soc. Lecture Notes Series 83, Cambridge Univ. Press, Cambridge (1983).

[23] F. TRICERRI and L. VANHECKE, Curvature homogeneous Riemannian manifolds, *Ann. Sci. École Norm. Sup.*, to appear.

[24] F.W. WARNER, Foundations of differential manifolds and Lie groups, Scott, Foresman, Glenview, Illinois (1971).

[25] H. WEYL, *Classical groups, their invariants and representations*, Princeton University Press, Princeton (1946).