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FOR ADVANCED STUDIES

The Problem of Nonattainability of Nodes for Nelson Diffusions

Thesis submitted for the degree of
"Magister Philosophiæ"

CANDIDATE
Andrea Posilicano

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INTRODUCTION

Stochastic Quantization is an algorithm that permits to associate a stochastic process (we will call such process a Nelson diffusion) to a solution of the Schrödinger equation in such a way that the density of the process ρ corresponds to the usual probability density of Quantum Mechanics. In this correspondence the drift field of the diffusion process is singular on the nodes of the given solution of the Schrödinger equation, so that a problem of existence for all times for the corresponding diffusion process arises. In particular the process may reach the nodes in a finite time with a certain probability strictly greater than zero. Since the paradigm of Stochastic Quantization tells us that the process represents the trajectory of the particle, and since Quantum Mechanics tells us that $|\psi_t(x)|^2$ represents the probability density to find the particle in x at time t, it seem very unphysical that the process may reach the nodes. The idea to solve this problem is that the drift vector field b may points away from the nodes. In such case it will give rise to a strong repulsion, and will prevent the particle from approaching the nodes. This suggests that the distance function d of the nodal set will be increasing under the flow of b, and so one is lead to use d as a Lyapunov function for the process. In the stationary case, i.e. when the process corresponds to an eigenvalue of the Schrödinger equation, we show, under suitable regularity hypotheses, that the drift b has, on a sufficiently small nbh. of the nodal set, a very simple form that indicates that the distance function will be increasing under its flow. So we can apply the Lyapunov–type theorem obtaining nonattainability of nodes. In the non stationary case the hypotheses of the Lyapunov theorem are not satisfied. We prove this fact by exhibiting a counterexample. This counterexample shows that there is no hope to apply Lyapunov theorem, at least as it stands, to non stationary Nelson diffusions. The key fact for resolving this problem is that for Nelson diffusions the density of the process is a priori known. This permits us to state a Lyapunov–type theorem for Nelson diffusions in which the pointwise condition of the usual theorem is replaced by a more manageable condition in the mean. Using again the distance function as a Lyapunov function, but now employing this modified version of the Lyapunov theorem, we can prove the nonattainability of the nodal set when

such set is the union of a numerable family of regular submanifolds under the hypotheses $\rho/d^2 \in L^1_{loc}$ and $\rho\|b\|/d \in L^1_{loc}$.

In an attempt to make this thesis self–contained, we report, in the first four chapters, all the standard results on the theory of stochastic process that we use in the following chapters. In chapter five we give some regularity and nonattainability theorems for stochastic differential equations with unbounded drifts, making use of the Lyapunov function approach. In particular we prove a nonattainability theorem for regular submanifolds using the distance function, and some estimates on its Laplacean. Chapter six contains a review of Nelson's theory of Stochastic Quantization that permits us to state and discuss, in chapter seven, the problem of nonattainability of nodes. Chapter seven closes with a less probabilistic approach, due to E.Carlen, to prove existence of Nelson diffusion. His approach, that uses methods of the theory of partial differential equations, in particular maximum principle, permits to prove an existence theorem for Nelson diffusions with drifts that satisfy a "finite energy condition", without assuming regularity hypotheses on ρ and b.

CONTENTS

1

STOCHASTIC PROCESSES

1.1 Probability spaces and random variables

A *probability space* is a triple $(\Omega, \mathcal{S}, Pr)$, where Ω is a set, $\mathcal{S}$ is a σ-algebra of subsets, and Pr is a positive measure of total mass one on $(\Omega, \mathcal{S})$. In the triple $(\Omega, \mathcal{S}, Pr)$, Ω is called the *sample·space*, or *outcome space*, the points of Ω are *outcomes*, sets in $\mathcal{S}$ are *events*, and Pr is the *probability*. Intuitively a random variable is a quantity, obtained from a random experiment whose value depends upon the outcomes of the experiment. If a probability space $(\Omega, \mathcal{S}, Pr)$ represents the experiment, with the points in Ω corresponding to outcomes, then we can see how a random variable is represented mathematically as a function on Ω . Clearly the probability structure must come into play, and this is accomplished by requiring the function to be $\mathcal{S}$-measurable. In conclusion a *random variable* will be a $(E, \mathcal{B})$-valued measurable function on $(\Omega, \mathcal{S}, Pr)$, where $(E, \mathcal{B})$ is a measurable space. If X is a random variable on Ω, the *distribution* of X is the probability measure on $(E, \mathcal{B})$ given by

$$\mu_X(A) := Pr\{\omega \in \Omega : X(\omega) \in A\} .$$

For every integrable $f : (E, \mathcal{B}) \to \mathbb{R}$ one has

$$\int_\Omega f(X(\omega))\, dPr(\omega) = \int_E f(x)\, d\mu_X(x) .$$

Suppose the $\mathbb{R}^n$-valued random variable X is integrable. The *expectation*, or *mean* of X , constitutes the coarsest probabilistic information concerning the random variable X , and is defined by

$$E(X) := \int_\Omega X(\omega)\, dPr(\omega) .$$

If X is in $L^2(\Omega,S,Pr)$, then one define the variance of X by

$$Var(X) := E((X-E(X)) \otimes (X-E(X))) = E(X \otimes X) - E(X) \otimes E(X) .$$

If X is $\mathbb{R}$–valued the *Chebyshev's inequality* holds:

$$Pr\{(X-E(X))^2 \geq \lambda\} \leq \frac{Var(X)^{1/2}}{\lambda} \qquad \lambda > 0 .$$

1.2 Conditional probability, independence, conditional expectation

Suppose an experiment represented by a probability space (Ω,S,Pr) is performed, and results in an outcome $\omega \in \Omega$. Let A and B denote events in S, and suppose we known that the outcome ω results in the event A, i.e. $\omega \in A$. What is the probability that B occurred ? It is clear that this probability is no longer $Pr(B)$, because if $A \cap B = \emptyset$, then $\omega \in A$ implies that $\omega \notin B$, and we are certain that B did not occur. We will define a function $Pr(\cdot \mid A)$ on S so that only those events which have non empty intersection with A have positive probability, and $Pr(\cdot \mid A)$ gives the probability of an event, knowing that an outcome resulted in A. Then $(A, A \cap S, Pr(\cdot \mid A))$, where $A \cap S$ denotes the subsets of A that are in S, will be a probability space that contains all the uncertain after we know the event A has occurred. In case $Pr(A) > 0$, it is easy to obtain a definition for $Pr(\cdot \mid A)$. In this case we define the *conditional probability*

$$Pr(B \mid A) := \frac{Pr(A \cap B)}{Pr(A)} \qquad B \in S .$$

An event B is said to be independent of an event A if the probability of B is unaffected by the knowledge that an outcome resulted in the event A. Thus B is *independent* of A if and only if

$$Pr(B \mid A) = Pr(B) .$$

The definition of conditional probability implies that B is independent of A

if and only if

$$\Pr(A \cap B) = \Pr(A)\Pr(B) \quad .$$

Analogously two sub-σ-algebras are said to be independent if

$$\Pr(A \cap A') = \Pr(A)\Pr(A') \qquad \forall A \in \mathfrak{D}\ , \forall A' \in \mathfrak{D}' \quad .$$

The random variables X, Y are independent if the corresponding σ-algebras they generate $\sigma(X), \sigma(Y)$ are independent. Returning to conditional probability, to deal with the more general situations in which $\Pr(A) = 0$, or A is replaced by a sub-σ-algebra , the more general concept of conditional expectation is introduced. Let $X \in L^1(\Omega, \mathcal{S}, \Pr)$, and let $\mathfrak{D}$ be a sub-σ-algebra. Note that X is not necessarily a random variable on $(\Omega, \mathfrak{D}, \Pr)$. We denote by $E(X | \mathfrak{D})$ the random variable on $(\Omega, \mathfrak{D}, \Pr)$ satisfying

$$\int_D E(X | \mathfrak{D})\, d\Pr = \int_D X\, d\Pr \qquad D \in \mathfrak{D} \quad .$$

The existence of $E(X | \mathfrak{D})$ is guaranteed by the Radon–Nikodim theorem, and we will call it the *conditional expectation* of X given $\mathfrak{D}$. It constitutes that portion of all information carried by X which is related to the sub-σ-algebra $\mathfrak{D}$. We now list some properties of the conditional expectation.

Theorem 1.1: *Let* $X \in L^1(\Omega, \mathcal{S}, \Pr)$. *Then*

i) $X \geq 0$ *implies* $E(X | \mathfrak{D}) \geq 0$ a.s.

ii) $E(E(X | \mathfrak{D})) = E(X)$ a.s.

iii) $\mathfrak{D}' \subset \mathfrak{D}$ *implies* $E(E(X | \mathfrak{D}) | \mathfrak{D}') = E(X | \mathfrak{D}')$ a.s.

iv) Y $\mathfrak{D}$ *–measurable implies* $E(XY | \mathfrak{D}) = Y E(X | \mathfrak{D})$ a.s.

v) X *independent of* $\mathfrak{D}$ *implies* $E(X | \mathfrak{D}) = E(X)$ a.s.

The *conditional probability* $\Pr(A | \mathfrak{D})$ of an event A given the sub-σ-algebra $\mathfrak{D}$ is defined by

$$\Pr(A | \mathfrak{D}) := E(\chi_A | \mathfrak{D}) \quad ,$$

and it is an easy exercise to see that if

$$\mathfrak{D} = \{\phi, A, A^c, \Omega\} \ , \quad 0 < \Pr(A) < 1 \quad ,$$

then

$$Pr(B|\mathcal{D})(\omega) = \begin{cases} Pr(A\cap B)/Pr(A) & \omega \in A \\ Pr(A^c \cap B^c)/Pr(A^c) & \omega \in A^c \end{cases} .$$

Now, let Y be a $\mathbb{R}^n$–valued random variable with distribution μ_Y . Similarly to the previous definition we may define $E(X|Y=y)$ by

$$\int_B E(X|Y=y)\, d\mu_Y(y) = \int_{Y^{-1}(B)} X \, dPr \qquad B \in \mathcal{B}$$

and $Pr(A|Y=y) = E(\chi_A | Y=y)$.

1.3 Stochastic processes

A stochastic process is a mathematical abstraction of random phenomena which develops according to a parameter, usually time, and whose development is governed by probabilistic laws. More precisely a *stochastic process* is a collection of objects

$$(\Omega, \mathcal{S}, (\mathcal{F}_t)_{t\in T}, (X_t)_{t\in T}, Pr)$$

where

i) $T \subseteq \mathbb{R}_+$
ii) $(\Omega, \mathcal{S}, Pr)$ *is a probability space*
iii) $(\mathcal{F}_t)_{t\in T}$ *is an increasing family of sub–*σ*–algebras of* $\mathcal{S}$
iv) $(X_t)_{t\in T}$ *is a family of* $(E,\mathcal{B})$ *–valued random variables, (* E *is a topological space and* $\mathcal{B}$ *is its Borel* σ*–algebra), such that* X_t *is* $\mathcal{F}_t$ *–measurable for all* $t\in T$ *.*

If the $\mathcal{F}_t$'s are not explicitly defined then $\mathcal{F}_t = \sigma(X_s, s\le t)$ (*the natural filtration*). The space E is called the *state space*. By the application

$$\Omega \ni \omega \longmapsto \{ t \mapsto X_t(\omega) \}$$

one may always suppose $\Omega \subset E^T$, and so Ω is often called the *trajectory space.*

Two stochastic processes $(\Omega_j, \mathcal{S}_j, (\mathcal{F}^j_t), (X^j_t), Pr^j)$, j=1,2 , are said *equivalent in law* if $\forall t_1,\dots,t_n \in T$, the random variables $(X^1_{t_1},\dots,X^1_{t_n})$ and $(X^2_{t_1},\dots,X^2_{t_n})$

have the same distribution. We say that they are *stochastically equivalent* if $(\Omega_1, \mathcal{S}_1, Pr^1) = (\Omega_2, \mathcal{S}_2, Pr^2)$, and $Pr(X_t^1 = X_t^2) = 1 \ \forall t \in T$. A stochastic process X_t is said to be *continuous* if

$$t \longmapsto X_t(\omega)$$

is continuous $\forall \omega \in \Omega$.

The following theorem is due to Kolmogorov

Theorem 1.2: *Let X_t be a $\mathbb{R}^n$–valued stochastic process such that*

$$E \, \|X_t - X_s\|^\beta \le c \, |t-s|^{\alpha+n} \qquad c, \alpha, \beta > 0 \, .$$

Then X_t is stochastically equivalent to a continuous stochastic process. More precisely such process is of Hölder class C^γ, with $\gamma < \alpha/\beta$.

Given a $(E, \mathcal{B})$ –valued stochastic process $(\Omega, \mathcal{S}, (\mathcal{F}_t), (X_t), Pr)$, a n–uple $(t_1, \dots, t_n)$ of elements of T with $t_1 < \dots < t_n$, one may define the probability measure on E^n

$$\mu_\pi (A_1 \times \dots \times A_n) := Pr(X_{t_1} \in A_1, \dots, X_{t_n} \in A_n) \, .$$

The measures μ_π are said to be the finite *dimensional distributions* of X_t. The knowledge of the finite dimensional distributions characterizes Pr in the following way:

Theorem 1.3: *Let $(\Omega, \mathcal{S}, (\mathcal{F}_t), (X_t), Pr^j)$, j=1,2 , have the same finite dimensional distributions. Then Pr^1 and Pr^2 coincide on $\sigma(X_t, t \in T)$.*

The following celebrated *Kolmogorov's construction theorem* permits us to construct stochastic from distribution functions:

Theorem 1.4: *Let $\{F_n(x_1, \dots, x_n)\}$ be a sequence of consistent distribution functions, i.e.*

i) for any intervals $I_k = [a_k, b_k) \quad 1 \le k \le n$

$$\Delta_{I_1} \cdots \Delta_{I_n} F_n(x) \ge 0$$

where

$$\Delta_{I_k} f(x) := f(x_1, \dots, x_{k-1}, b_k, x_{k+1}, \dots x_n) - f(x_1, \dots, x_{k-1}, a_k, x_{k+1}, \dots x_n)$$

ii) if $x_j^{(k)} \uparrow x_j$ *as* $k \uparrow \infty$ $(1 \le j \le n)$, *then*

$$F_n(x_1^{(k)}, \dots, x_n^{(k)}) \uparrow F_n(x_1, \dots, x_n) \quad \text{as} \quad k \uparrow \infty$$

iii) if $x_j \downarrow -\infty$ *for some j, then*

$$F_n(x_1, \dots, x_n) \downarrow 0$$

and if $x_j \uparrow \infty$ *for all j* $1 \le j \le n$, *then*

$$F_n(x_1 \dots, x_n) \uparrow 1$$

iv)

$$\lim_{x_n \uparrow \infty} F_n(x_1, \dots, x_n) = F_{n-1}(x_1, \dots, x_{n-1}) \qquad n > 1 \quad .$$

Then there exists a stochastic process such that the given distribution functions are its finite dimensional distributions.

1.4 Martingales

A stochastic process $(\Omega, S, (F_t)_{t \in T}, (X_t)_{t \in T}, Pr)$ is called a *martingale* if, for each $t \in T$, $X_t \in L^1(\Omega, S, Pr)$, and for each $s > 0$,

$$E(X_{t+s} \mid F_t) = X_t \quad . \tag{1.1}$$

Such processes are sometimes referred to as 'fair games' processes; if X_t represents a gambler's fortune at time t , the game is fair if his expected fortune at a future time $t+s$, given the game history up to some previous time t , is precisely the fortune at time t . *Sub–(super–) martingales* are stochastic processes which satisfy (1.1) with $\ge$ ($\le$) instead of =.

Martingales satisfy key estimates :

Theorem 1.5: *Let* X_t *(* Y_t *) be a sub–(super–)martingale. For arbitrary* *real numbers* $t>0, p\geq 1$ *, one has*

$$\Pr\{\sup_{[t_0,T]} X_t > t\} \leq E(\|X_T\|^p)/t^p$$

$$E(\sup_{[t_0,T]} \|X_t\|^t) \leq \left(\frac{t}{t-1}\right)^t E(\|X_T\|^t) \qquad t>1$$

$$\Pr\{\sup_{[t_0,T]} \|Y_t\|>t\} \leq E(\|Y_{t_0}\|^p)/t^p \quad .$$

1.5 Brownian motions

Let $X_1,\dots,X_n$ be n random variables on the same probability space. $X_1,\dots,X_n$ are said to have joint normal distribution (μ,Γ), where $\mu\in\mathbb{R}$, Γ is a symmetric matrix, if the characteristic function

$$f(u) := E\exp iu\cdot X \qquad u=(u_1,\dots,u_n) \quad X=(X_1\dots X_n)$$

has the form

$$f(u) = \exp(\mu\cdot u - \frac{1}{2}u\cdot\Gamma u) \quad .$$

Theorem 1.6: *Let* $X_1,\dots,X_n$ *have joint normal distribution* (μ,Γ). *Then*

i) $X_1,\dots,X_n$ *are independent iff* $\Gamma_{ij}=0$ *whenever* $i\neq j$

ii) if X_i, X_j *are independent for* $i\neq j$ *, then* $X_1,\dots,X_n$ *are mutually independent*

iii) if $Y=AX$ *, with* A *any linear operator, then* $Y_1,\dots,Y_n$ *have joint normal distribution*

iv) $\mu_j = E(X_j)$, $\Gamma_{ij} = E((X_i-\mu_i)(X_j-\mu_j))$

v) if $\det\Gamma\neq 0$ *,then* X *has a distribution with density*

$$p(y) = \frac{1}{(2\pi)^{n/2}(\det\Gamma)^{1/2}} \exp -\frac{1}{2}((y-\mu)\cdot\Gamma^{-1}(y-\mu)) \quad .$$

A $\mathbb{R}$-valued stochastic process $(\Omega, S, (\mathcal{F}_t)_{t\geq 0}, (X_t)_{t\geq 0}, P)$ is called a *Brownian motion* if

i) $X_0 = 0$ a.s.

ii) *for any* $0 \leq t_1 < \dots < t_n, X_{t_1} \dots X_{t_n}$ *have joint normal distribution* $(0, (t_j \wedge t_k)_{jk})$.

A $\mathbb{R}^n$-valued stochastic process X_t is said to be a $\mathbb{R}^n$*-valued Brownian motion* if each its component X_t^k is a Brownian motion and $\sigma(X_t^k)$ k=1,...,n are independent.

Since

$$E(\| X_t - X_s \|^4) \leq (n^2 + 2n)(t-s)^2$$

Brownian motion is a continuous process by Kolmogorov theorem. The following theorem of Doob, provides a criterion for determining when a continuous process is a Brownian motion.

Theorem 1.7: *Let* $(\Omega, S, (\mathcal{F}_t)_{t\geq 0}, (X_t)_{t\geq 0}, P)$ *be a continuous stochastic process such that for all* $t \geq s \geq 0$

i) $X_0 = 0$ a.s.

ii) $E(X_t - X_s | \mathcal{F}_s) = 0$ a.s.

iii) *Then* X_t *is a Brownian motion.*

For a Brownian motion the so called law of the iterated logarithm holds :

$$\limsup_{t \downarrow 0} \frac{X_t}{\sqrt{2t \log\log 1/t}} = 1 \quad \text{a.s.} \tag{1.2}$$

$$\liminf_{t \downarrow 0} \frac{X_t}{\sqrt{2t \log\log 1/t}} = -1 \quad \text{a.s.} \tag{1.3}$$

Almost all sample paths of Brownian motion are nowhere differentiable. To see this, note that, for fixed nonnegative s and t , $X_{t+s} - X_s$ is a Brownian motion; then observe that (1.2) and (1.3) imply that, for $0<\varepsilon<1$, and for almost every sample path, there are sequences of values of h (dependent on the path) tending to zero such that

$$\frac{X_{t+h} - X_t}{h} \geq (1-\varepsilon)\sqrt{\frac{2 \log\log 1/h}{h}} \tag{1.4}$$

and

$$\frac{X_{t+h} - X_t}{h} \leq (1-\varepsilon)\sqrt{\frac{2 \log\log 1/h}{h}} \quad . \tag{1.5}$$

The right hand side of (1.4) and (1.5) approach $+\infty$ and $-\infty$ respectively, so $(X_{t+h} - X_t)/h$ has, with probability one, for many fixed t, the extended real line as its cluster set. Thus the path of X_t are nowhere differentiable, since the average rates of change experience arbitrarily large fluctuations as $h \downarrow 0$ at each time :

Theorem 1.8:
i) Almost all sample paths of a Brownian motion are nowhere differentiable
ii) Almost all sample paths of a Brownian motion have infinite variation on any finite interval.

1.6 The Wiener process

The most celebrated realization of a Brownian motion is the *Wiener process.* Here

$$\Omega = C_0(\mathbb{R}_+, \mathbb{R}^n) \equiv \{\gamma \in C(\mathbb{R}_+, \mathbb{R}^n) : \gamma(0) = 0\}$$

$X_t(\gamma) := \gamma(t)$, Σ is the Borel σ–algebra with respect to the topology on Ω given by the metric

$$d(\gamma, \gamma') := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\sup_{0 \leq t \leq n} \|\gamma(t) - \gamma'(t)\|}{1 + \sup_{0 \leq t \leq n} \|\gamma(t) - \gamma'(t)\|}$$

and $\Pr$ is the so called *Wiener measure.* The Wiener measure can be constructed in the following way due to Nelson:

let $\dot{\mathbb{R}}^n$ denote the one–point compactification of $\mathbb{R}^n$ and let Ω be the cartesian product space

$$\Omega := \prod_{0 \leq t \leq \infty} \dot{\mathbb{R}}^n \quad .$$

With the product topology, by Tychonoff theorem, Ω is a compact Hausdorff space. Let $C_{fin}(\Omega)$ be the set of function of the form

$$\varphi(\gamma) = F(\gamma(t_1), \ldots, \gamma(t_n)) \qquad t_1 < \cdots < t_n$$

F continuous. $C_{fin}(\Omega)$ contains 1 and separates points, so, by the Stone–Weirestrass theorem, it is dense in $C(\Omega)$. Define the linear functional L on $C_{fin}(\Omega)$ by

$$L\varphi := \int P_{t_1}(0)\, P_{t_2-t_1}(x_1) \cdots P_{t_n-t_{n-1}}(x_n)\, F(x_1, \ldots, x_n)\, dx_1 \ldots dx_n$$

with

$$P_t(x) := \frac{1}{(4\pi t)^{n/2}} \exp\left(-\|x\|^2/4t\right) \quad .$$

If

$$F(x_1 \ldots x_n) = \chi_{E_1}(x_1) \cdots\cdots \chi_{E_n}(x_n)$$

with $E_1, \ldots, E_n$ Borel sets in $\mathbb{R}^n$, then $L\varphi$ is interpreted as the probability that a particle performing a Brownian motion, starting at 0 in $t = 0$, is in E_j at time t_j. Let $\tilde{L}$ be the extension of L to $C(\Omega)$. By Riez's representation theorem, there exists a regular probability measure $\Pr$ on Ω such that

$$\tilde{L}\varphi = \int_{\Omega} \varphi(\gamma)\, d\Pr(\gamma) \quad .$$

The following theorem shows that $\Pr$ so constructed is a probability measure when restricted to $C_0(\mathbb{R}_+, \mathbb{R}^n)$:

Theorem 1.9: *For $0 < \alpha \leq 1$ the set Ω_α of Hölder continuous paths in $\mathbb{R}^n$ of order α is a Borel subset of Ω. Moreover*

$$\Pr(\Omega_\alpha) = 1 \qquad 0 < \alpha < 1/2$$

$$\Pr(\Omega_\alpha) = 0 \qquad 1/2 \leq \alpha \leq 1 \quad .$$

2

MARKOV PROCESSES

2.1 Definition of Markov process

Let $(\Omega, \mathcal{S}, (\mathcal{P}_t), (X_t), \Pr)$ be a $(\mathfrak{X}, \mathcal{B})$ –valued stochastic process. X_t is said to be a *Markovian process* if the probability of a future state of the system at any time $t > s$ is independent of the past behaviour of the system at times $t' < s$, given the present state at time s, that is

$$\Pr(X_t \in A \mid \mathcal{P}_s) = \Pr(X_t \in A \mid X_s) \qquad s \le t \quad .$$

This condition is equivalent to

$$\Pr(A \cap B \mid X_t) = \Pr(A \mid X_t) \Pr(B \mid X_t) \qquad A \in \mathcal{P}_{t_1}, B \in \mathcal{P}_{t_2}, t_1 \le t \le t_2$$

We may think of Markov property as a version of the causality principle. From the symmetry of the above condition it follows that if X_t is Markovian then X_{-t} is Markovian also.

2.2 Markov transition functions

For a Markov process X_t the conditional probability

$$\Pr(X_t \in A \mid X_s) \equiv p(s, X_s, t, A) \qquad s \le t$$

satisfies the following properties :

i) $p(s,x,t,A)$ *is* $\mathcal{B}$ *–measurable in* x *, for fixed* s,t,A

ii) $p(s,x,t,A)$ *is a probability measure in* A *, for fixed* s,x,t

iii) p *satisfies the Chapman–Kolmogorov equation*

$$p(s,x,t,A) = \int_E p(\lambda,y,t,A)\, p(s,x,\lambda,dy) \qquad s<\lambda<t$$

for all x *except possibly for a* μ_X*–null set.*

In fact $p(s,x,t,A)$ can be modified on a set of measure zero in such a way that the Chapman–Kolmogorov equation is satisfied for all x and such that

$$\mathcal{P}r\,(s,x,s,A) = \chi_A(x) \quad .$$

In this case a function satisfying i), ii), iii) is called a *Markov transition* function and the notation

$$p(s,x,t,A) = \mathcal{P}r\,(X_t \in A \mid X_s = x)$$

is employed.

For a Markov process all finite dimensional distributions can be obtained from the transition function and the initial distribution :

$$\mathcal{P}r\,(X_{t_1} \in A_1, \dots, X_{t_n} \in A_n) =$$

$$= \int_E \int_{A_1} \cdots \int_{A_{n-1}} p(t_{n-1}, x_{n-1}, t_n, A_n) \cdots p(t_0, x_0, t_1, dx_1)\, d\mu_{X_{t_0}}(x_0) \quad .$$

By Kolmogorov's construction theorem, if E is a metric space which is σ–compact, for any Markov transition function p and any probability distribution μ on E there exists a $(E,\mathcal{B})$–valued Markov process with transition function p and initial distribution μ :

Theorem 2.1: *Let* E *be a* σ*–compact metric space and let* p *be a Markovian transition function. Then there exists*

i) a measurable space $(\Omega,\mathcal{S})$ *with a family of sub* σ*–algebras* $(\mathcal{P}_t^s)_{t\geq s}$ *such that* $\mathcal{P}_{t'}^{s'} \subset \mathcal{P}_t^s$ *if* $s \leq s'$, $t \leq t'$

ii) a family of random variables $X_t : \Omega \to E$ *such that* $\omega \mapsto X_t(\omega)$ *is* $\mathcal{P}_t^s$*–measurable for all* $s \leq t$

iii) a family of probability laws $(Pr^{x,s})_{x\in E}$ *on* $(\Omega, \bigcup_t \mathcal{P}^s_t)$ *such that*

$$Pr^{x,s}(X_s(w) = x) = 1 \quad ,$$

$$Pr^{x,s}(X_{t+h} \in A \mid X_t) = p(t, X_t, t+h, A) \qquad Pr^{x,s}\text{-a.s.} \quad .$$

The stochastic process $(\Omega, S, (\mathcal{P}^s_t)_{t\geq s}, (X_t), Pr^{x,s})$ *will be called a realization of the Markov process associated to* p.

2.3 Markovian semigroups

Let $M_b(E)$ be the space of all bounded, $\mathcal{B}$ –measurable, $\mathbb{R}$–valued functions on E with the norm $\|f\|_\infty := \sup_{x\in E} |f(x)|$. Given a Markov transition function p we define the mappings

$$T_{s,t} : M_b(E) \to M_b(E) \qquad T_{s,t}f(x) := \int_E f(y)\, p(s,x,t,dy) \quad .$$

It is easy to verify that $T_{s,t}$ is a *Markovian evolution operator*, i.e. :

i) the $T_{s,t}$ *are bounded contraction*

ii) $T_{s,t}f \geq 0$ *if* $f \geq 0$

iii) $T_{s,t}1 = 1$

iv) $T_{s,s} = \mathbb{1}$

v) $T_{s,t} = T_{s,r}T_{r,s}$.

The family $\{T_{s,t}\}$ is called the semigroup associated with the Markov transition function p , or with the corresponding Markov process. The following problem naturally arises: given a Markovian semigroup does there exist a Markov process generating such semigroup ? This question leads to the definition of Markov processes with the Feller property.

2.4 Feller Markov processes

Let $C_b(E)$ denote the space of all bounded, continuous, $\mathbb{R}$ –valued functions on E equipped with the sup norm. A Markov process is said to satisfy the *Feller property* if $T_{s,t} C_b(E) \subseteq C_b(E)$ for all $s \leq t$, i.e. the Feller property assures the w^*–continuity of the family of probability measures $x \mapsto p(s,x,t,\cdot)$. One has the following

Theorem 2.2: *Let E be a compact metric space and let $T_{s,t}$ be a Feller Markovian semigroup. Then there exists a E –valued Markov process to which the given family $\{T_{s,t}\}_{s \leq t}$ corresponds.*

2.5 Strong Markov processes

In the study of Markov processes there are situations in which it is important to know if the property of the independence of the future from the past, given the present, is preserved if one identifies the 'present' not with a fixed value of t (a 'sure' variable) but with a random variable. This leads to the definition of Markov time and of strong Markov process.

Let $(\Omega, \mathcal{S}, (\mathcal{P}_t)_{t \in [t_0,T]}, (X_t)_{t \in [t_0,T]}, \mathcal{P}r)$ be a Markov process. A (possibly extended) random variable $\tau : \Omega \to [t_0,T]$ is called a Markov or *stopping time* with respect to the Markov process X_t , if the sets $\{\tau \geq t_0\}$ and $\{\tau \leq T\}$ are $\mathcal{P}_t$ –measurable, for each $t \in [t_0,T]$. For a Markov time τ , the σ–algebra $\mathcal{P}_\tau$ consists of all events $A \in \mathcal{P}_T$ such that $A \cap \{\tau \leq t\}$ is $\mathcal{P}_t$ –measurable, for each $t \in [t_0,T]$. It is clear that $\tau \wedge s = \min\{\tau, s\}$ is a Markov time. For a continuous Markov process X_t and an open set D

$$\tau := \inf \{ t : X_t \in D^c \}$$

is a Markov time, called the *first exit time* of D or the *first hitting time* of D^c . A Markov process X_t is called a *strong Markov process* if the Markov property holds for Markov times, that is, for any Markov time τ , and any Borel set A ,

$$\mathcal{P}r\,(X_{t+\tau} \in A \mid \mathcal{P}_{[t_0,\tau]}) = \mathcal{P}r\,(X_{t+\tau} \mid X_\tau) \qquad t > 0 \quad .$$

The relation between the Feller and the strong properties is given in the following

Theorem 2.3: *Let X_t be a continuous Markov process with the Feller property.*

Then it satisfies the strong Markov property.

2.6 Diffusion processes

Roughly speaking the diffusion processes are continuous Markov processes for which the associated Markovian semigroup has a differential operator as infinitesimal generator. The correct definition is the following:

a continuous D–valued (D is an open set in $\mathbb{R}^n$) Markov process X_t with transition function p is called a *diffusion process* if for any $R>0$ and $s \geq 0$ the following limits exist :

i)

$$\lim_{h \to 0+} \frac{1}{h} \, p(s,x,s+h, B_R^c(x)) = 0$$

ii)

$$\lim_{h \to 0+} \frac{1}{h} \int_{B_R(x)} (y-x)\, p(s,x,s+h,dy) \equiv b(s,x)$$

iii)

$$\lim_{h \to 0+} \frac{1}{h} \int_{B_R(x)} (y-x) \otimes (y-x)\, p(s,x,s+h,dy) \equiv a(s,x) \quad .$$

The vector b is called the *drift coefficient* and the matrix a is called the *diffusion matrix*, it is positive definite by construction. Properties ii) and iii) may be written, similarly,

ii')

$$E(X_t - X_s \mid X_s = x) = b(s,x)(t-s) + o(t-s)$$

iii')

$$E((X_t - X_s) \otimes (X_t - X_s) \mid X_s = x) = a(s,x)(t-s) + o(t-s) \quad .$$

The drift coefficient gives the time of rate change of the conditional mean of the increment of the process. Since

$$E(X_t - X_s \mid X_s = x) \otimes E(X_t - X_s \mid X_s = x) = o(t-s)$$

by ii'), the diffusion matrix represents the rate of change of the conditional covariance of the increment.

It is easy to compute the infinitesimal generator A of the Markovian semigroup associated to the diffusion process X_t. By definition

$$A_s f(x) := \lim_{h \to 0+} \frac{T_{s,s+h} f(x) - f(x)}{h} .$$

The domain of A consists of all the functions in $M_b(D)$ for which the above limit exists for almost every x . Suppose $f \in C_b \cap C^2$. Then using i) – iii) and Taylor formula one obtains

$$A_s f = \frac{1}{2} \sum_{i,j} a_{ij} \partial^2_{ij} f + \sum_i b_i \partial_i f .$$

Under certain conditions on the coefficients it is possible to associate a diffusion process to any differential operator of the form above:

Theorem 2.4: *Let $b : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ be locally Lipschitz with respect to spatial variables and suppose that $\| b(x,t) \| \leq M(1 + \|x\|)$. Suppose that $\|a(x,t)\| \leq M^2(1+\|x\|$ and that a is of class C^2 and positively semi–definite or positively definite and locally Lipschitz. Then there exists a Feller diffusion process with generator*

$$L = \frac{1}{2} \sum_{i,j} a_{ij} \partial^2_{ij} + \sum_i b_i \partial_i .$$

More precisely the above diffusion process will be a realization of the Markov process associated to the transition function $p(s,x,t,y)dy$, where $p(s,x,t,y)$ is the fundamental solution of the anti–parabolic differential operator $L + \partial_t$.

3

STOCHASTIC INTEGRALS AND THE ITO'S FORMULA

3.1 *The definition of stochastic integral*

The scope of this section is to define the integral

$$\int_0^t X_s \, dB_s$$

where X_s is a stochastic process and B_s is a Brownian motion. Since the sample paths of B_s are not of bounded variation, the above integral cannot be defined in the usual Lebesgue–Stiltjes sense. The integral will be defined by an approximation procedure using a certain class of step functions.

If $(\Omega, \mathcal{S}, (\mathcal{F}_t)_{t\geq 0}, (B_t)_{t\geq 0}, P)$ is a $\mathbb{R}$–valued Brownian motion, a $\mathbb{R}$–valued stochastic process $(X_t)_{t\in[\alpha,\beta]}$ is called a *nonanticipative function* with respect to $(\mathcal{F}_t)_{t\geq 0}$ if

i) $(t,\omega) \longmapsto X_t(\omega)$ *is measurable from* $[\alpha,\beta]\times\Omega$ *into* $\mathbb{R}$

ii) X_t *is* $\mathcal{F}_t$*–measurable* .

By $L^p_B[\alpha,\beta]$ we denote the space of nonanticipative functions such that

$$Pr\left\{\omega : \int_\alpha^\beta |X_t(\omega)|^p dt < +\infty\right\} = 1$$

and

$$M^p_B[\alpha,\beta] := \left\{X_t \in L^p_B[\alpha,\beta] \ : \ E\int_\alpha^\beta |X_t|^p dt < +\infty\right\} .$$

A stochastic process $(X_t)_{t\in[\alpha,\beta]}$ is called a step function if

$$X_t(\omega) = \sum_{0\leq k\leq n-1} X_k(\omega)\, \chi_{[t_k, t_{k+1}]}(t)$$

where X_K are random variables and $\alpha = t_0 < t_1 < \cdots < t_n = \beta$ is a partition of $[\alpha, \beta]$. If X_t is a step function in $L^2_B[\alpha,\beta]$, we define the random variable

$$\int_\alpha^\beta X_s \, dB_s := \sum_{0 \le k \le n-1} X_{t_k} [B_{t_{k+1}} - B_{t_k}] \quad .$$

Since it may be shown that for every $X_t \in L^2_B[\alpha,\beta]$ there exists a sequence $\{(X^n_t)\}_n$ of step functions convergent in probability to X_t, i.e.

$$Pr\{\omega : \lim_{n \to +\infty} \int_\alpha^\beta |X^n_t(\omega) - X_t(\omega)| \, dt = 0\} = 1 \quad ,$$

and the sequence

$$\{ \int_\alpha^\beta X^n_s \, dB_s \}_n$$

is a Cauchy sequence in probability, i.e.

$$\forall \varepsilon > 0 , \lambda > 0 \quad \exists M \in \mathbb{N} \quad s.t.$$

$$Pr\{\omega : | \int_\alpha^\beta X^n_s(\omega) \, dB_s(\omega) - \int_\alpha^\beta X^m_s(\omega) \, dB_s(\omega) | > \lambda \} < \varepsilon \qquad n,m > M$$

then

$$\{ \int_\alpha^\beta X^n_s \, dB_s \}_n$$

converges in probability to a random variable that we denote by

$$\int_\alpha^\beta X_s \, dB_s$$

and call it the *Ito' integral* of X_t .

Ito's integrals do not follow the rules of usual calculus. For example

$$\int_0^t B_s \, dB_s = \frac{1}{2} B_t^2 - \frac{1}{2} t \quad .$$

Ito's integral has the following properties :

Theorem 3.1: *Let* $X_t \in L^2_B[\alpha,\beta]$ *. Then*

$$E \int_\alpha^\beta X_s dB_s = 0$$

$$E \left|\int_\alpha^\beta X_s dB_s\right|^2 = E \int_\alpha^\beta |X_s|^2 dB_s$$

$$E \left(\int_\alpha^\beta X_s dB_s \mid \mathcal{F}_\alpha \right) = 0$$

$$E \left(\left|\int_\alpha^\beta X_s dB_s\right|^2 \mid \mathcal{F}_\alpha \right) = E \left(\int_\alpha^\beta |X_s|^2 dB_s \mid \mathcal{F}_\alpha \right) = \int_\alpha^\beta E(X_s^2 \mid \mathcal{F}_\alpha)\, dB_s \quad .$$

3.2 *Indefinite integrals and the martingale property*

If $X_t \in L^2_B[0,T]$ we define the indefinite integral

$$I_t := \int_0^t X_s dB_s \qquad 0 \leq t \leq T$$

where, by definition

$$I_0 = \int_0^0 X_s dB_s = 0 \quad .$$

Such family $(I_t)_{t\in[0,T]}$ has fundamental properties :

Theorem 3.2:

$$\int_0^t X_s dB_s$$

is stochastically equivalent to a continuous process, and ,if $X_t \in M^2_B[0,T]$ *then it is a martingale.*

The martingale property is maintained if the extremum of integration is replaced by a random variable :

Theorem 3.3: *Let* $X_t \in M^2_B[0,\tau]$ *, where* $0 \leq \tau \leq T$ *is a stopping time with respect to* $\mathcal{F}_t$ *, i.e.* $\{\tau \leq t\} \in \mathcal{F}_t$ *for all* $t \in [0,T]$ *. Then the process*

$$\int_0^{\tau\wedge t} X_s dB_s\,(\omega) := \int_0^{\tau(\omega)\wedge t} X_s(\omega)\, dB_s(\omega)$$

is a martingale, and

$$E\int_0^{\tau\wedge t} X_s\,dB_s = 0 \quad .$$

3.3 The Ito's formula

Let B_t be a $\mathbb{R}^n$–valued Brownian motion, and let A_s be a $L(\mathbb{R}^n,\mathbb{R}^m)$–valued stochastic process, such that each component $(A_s)_{kj} \in L^2_B[\alpha,\beta]$. The stochastic integral

$$\int_\alpha^\beta A_s\cdot dB_s$$

is a $\mathbb{R}^m$–valued random variable defined by

$$\left(\int_\alpha^\beta A_s\cdot dB_s\right)_k := \sum_j\int_\alpha^\beta (A_s)_{kj}\, dB^j_s \quad ,$$

where each integral on the right is an usual Ito's integral.

Let X_t be a $\mathbb{R}^m$–valued stochastic process, and let $A_t\in L^2_B[\alpha,\beta]$, and $b_t\in L^1_B[\alpha,\beta]$ with A_t $L(\mathbb{R}^n,\mathbb{R}^m)$–valued and b_t $\mathbb{R}^n$–valued. We say that X_t has a stochastic differential given by

$$dX_t = b_t\,dt + A_t\cdot dB_t$$

if

$$X_{t_2}-X_{t_1} = \int_{t_1}^{t_2} b_s\,ds + \int_{t_1}^{t_2} A_s\cdot dB_s \qquad \alpha\le t_1<t_2\le\beta$$

Ito's formula tell us as stochastic differentials change under composition with regular functions :

Theorem 3.4: *Let $f\in C^{1,2}(\mathbb{R}_+\times\mathbb{R}^n)$, i.e. f has continuous first derivative with respect to $t\in\mathbb{R}_+$, and continuous first and second derivatives with respect to $x\in\mathbb{R}^n$. Let X_t have stochastic differential*

$$dX_t = b_t\,dt + A_t\cdot dB_t \quad .$$

Then $f(t,X_t)$ *has stochastic differential*

$$df(t,X_t) = \mathcal{L}f(t,X_t) + (A_t^* \nabla_x f(t,X_t) \cdot dB_t$$

with

$$\mathcal{L} := \partial_t + b_t \cdot \nabla_x + \frac{1}{2}(A_t A_t^*) \cdot D_x^2$$

*(* denotes transposition).*

By properties of Ito's integrals, from Ito's formula it follows, with X_t, f, b_t, and A_t as above,

$$E f(\tau, X_\tau) - E f(0, X_0) = E \int_0^\tau \mathcal{L} f(s, X_s)\, ds$$

provided

$$\mathcal{L} f(t, X_t) \in M_B^1[0,T] \quad , \quad A_t^* \cdot \nabla_x f(t, X_t) \in M_B^2[0,T] \, ,$$

and τ is a stopping time.

3.4 The Feynman–Kac formula

As application of Ito's formula we give the celebrated Feynman–Kac formula

Theorem 3.5: *Let the stochastic process* $(\Omega, \mathcal{S}, (P_t^x), (X_t^x), \mathcal{F}^x)$ *have the stochastic differential*

$$X_t = x + \int_0^t b(s, X_s)\, ds + \int_0^t \sigma(s, X_s)\, dB_s \quad ,$$

with $b(s, X_s) \in L_B^1(\mathbb{R}_+)$, $\sigma(s, X_s) \in L_B^2(\mathbb{R}_+)$.

Let $f \in C_b^2(\mathbb{R}^n)$ *and* $V \in C_b(\mathbb{R}^n)$. *Put*

$$v(t,x) := E^x [\exp(-\int_0^t V(X_s^x)ds) f(X_t^x)] \equiv$$

$$\equiv \int_\Omega (\exp - \int_0^t V(X_s^x(\omega))ds) f(X_t^x(\omega))\, dPr^x(\omega) .$$

Then

$$\partial_t v = \frac{1}{2} \sigma_t \sigma_t^* . D_x^2 v + b_t \cdot \nabla_x v - Vv .$$

In particular if X_t^x *is the Wiener process staring at* x, *then*

$$v(t,x) = \int_{C_x(\mathbb{R}_+,\mathbb{R}^n)} (\exp - \int_0^t V(\gamma(s))ds) f(\gamma(t))\, dPr^x(\gamma) ,$$

with

$$C_x(\mathbb{R}_+,\mathbb{R}^n) = \{\gamma \in C(\mathbb{R}_+,\mathbb{R}^n) , \gamma(0) = x\} ,$$

and Pr^x *the Wiener measure on* $C_x(\mathbb{R}_+,\mathbb{R}^n)$, *is a solution of*

$$\partial_t v = \frac{1}{2} \Delta_x v - Vv .$$

The proof follows from

$$E\, F(Z_t) = E\, F(Z_0) + E \int_0^t \mathcal{L} F(Z_s) ds$$

with

$$Z_t := (z_t^1, z_t^2) , \quad F(x,y) := xy ,$$

and

$$z_t^1 := f(x_t) , \quad z_t^2 := \exp - \int_0^t V(X_s) ds .$$

3.5 The Girsanov's Theorem

As another consequence of Ito's formula we will give the *Girsanov's theorem*. Let B_t be a $\mathbb{R}^n$–valued Brownian motion ,and suppose $\phi_t \in L^2_B[0,T]$. Define a $\mathbb{R}$ –valued process Z_t by

$$Z_t := \exp\left(\int_0^t \phi_s dB_s - \frac{1}{2} \int_0^t \|\phi_s\|^2 ds \right)$$

i.e. $Z_t = \exp X_t$, where X_t has stochastic differential

$$dX_t = -\frac{1}{2} \|\phi_t\|^2 dt + \phi_t \cdot dB_t \quad .$$

By Ito's formula, Z_t will have stochastic differential

$$dZ_t = Z_t \phi_t dB_t \quad .$$

If $Z_t \phi_t \in M^2_B[0,T]$ then Z_t is a martingale and $E(Z_t) = 1$. In general it may be shown that Z_t is only a positive sub–martingale and $E(Z_t) \leq 1$. In case Z_t is a martingale we may define the probability measure $\hat{Pr}$ on $(\Omega, \mathcal{F}_T)$ by

$$\left.\frac{d\hat{Pr}}{dPr}\right|_{\mathcal{F}_T} = Z_T \quad .$$

We have the following fundamental result:

Theorem 3.6: *Let*

$$Z_t := \exp\left(\int_0^t \phi_s dB_s - \frac{1}{2} \int_0^t \|\phi_s\|^2 ds \right)$$

with $\phi_s \in L^2_B[0,T]$ *, be a martingale. Then the process*

$$\left(\Omega, \mathcal{F}_T, (\mathcal{F}_t)_{t\in[0,T]}, (\hat{B}_t)_{t\in[0,T]}, \hat{Pr}\right)$$

with

$$\hat{B}_t := B_t - \int_0^t \phi_s ds$$

is a Brownian motion.

The following theorem gives sufficient conditions on ϕ_s in order Z_t be a martingale :

Theorem 3.7: *If*

$$E\left(\exp \lambda \int_0^T \|\phi_s\|^2 ds\right) < +\infty \qquad \lambda > 1/2 \quad ,$$

or

$$E\left(\exp \mu \|\phi_s\|^2\right) < c < +\infty \qquad \forall s \in [0,T] \quad , \quad \mu > 0 \quad ,$$

then Z_t *is a martingale.*

4

STOCHASTIC DIFFERENTIAL EQUATIONS

4.1 Strong solutions

Let $b: [0,T] \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $\sigma: [0,T] \times \mathbb{R}^n \longrightarrow M^{n\times n}(\mathbb{R})$ be measurable functions, let $(\Omega, \mathcal{S}, (\mathcal{F}_t)_{t\in[0,T]}, (X_t)_{t\in[0,T]}, P_r)$ be be a $\mathbb{R}^n$–valued stochastic process, and let $(\Omega, \mathcal{S}, (\mathcal{F}_t), (B_t), P_r)$ a $\mathbb{R}^n$–valued Brownian motion such that $b(t,X_t) \in L^1_B[0,T]$ and $\sigma(t,X_t) \in L^2_B[0,T]$. X_t is said to be a *strong solution* of the *stochastic differential equation* (s.d.e.)

$$\begin{cases} dX_t = b(t,X_t)dt + \sigma(t,X_t)dB_t \\ X_0 = \eta \end{cases} \tag{4.1}$$

if

$$X_t = X_0 + \int_0^t b(s,X_s)ds + \int_0^t \sigma(s,X_s)\,dB_s .$$

Obviously if $\sigma = 0$, then $X_t(\omega) = \phi_t(\eta(\omega))$, where ϕ_t denotes the flow (we suppose it exists) of the vector field b, and s.d.e.'s reduce to ordinary differential equations. Suppose σ is constant. Then

$$X_t - \sigma B_t = \eta + \int_0^t b(s,X_s)ds = \eta + \int_0^t b(s,(X_s - \sigma B_s) + \sigma B_s)\,ds$$

therefore

$$X_t(\omega) = \phi_t^{\omega}(\eta(\omega))$$

where ϕ_t^ω denotes the flow of the ω-dependent vector field

$$b_\omega(t,x) := b(t, x + \sigma B_t(\omega)) \quad .$$

4.2 Existence and uniqueness

Theorem 4.1: *Suppose* $b : [0,T] \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $\sigma : [0,T] \times \mathbb{R}^n \longrightarrow M^{n\times n}(\mathbb{R})$ *are measurable and*

$$\| b(t,x) \| \leq K(1 + \|x\|) \quad , \quad \| \sigma(t,x) \| \leq K(1 + \|x\|) \quad . \tag{4.2}$$

Suppose further that for any $R > 0$ *there is a positive constant* K_R *such that*

$$\| b(t,x) - b(t,y) \| \leq K_R \|x-y\| \quad , \quad \| \sigma(t,x) - \sigma(t,y) \| \leq K_R \|x-y\| \tag{4.3}$$

if $\|x\| \leq R$, $\|y\| \leq R$, $t \in [0,T]$. *Let* η *be any n–dimensional random vector independent of* B_t . *Then there exists a unique, continuous (up tho stochastic equivalence), strong solution* X_t *of (4.1). If* $E\|\eta\|^2 < +\infty$ *then* $X_t \in M_B^2[0,T]$ *and*

$$E\left(\sup_{t\in[0,T]} \|X_t\|^2 \right) \leq C_{T,K} \left(1 + E\|\eta\|^2\right) \quad .$$

The assertion of uniqueness means that if X_t , Y_t *are two solutions of (4.1), then*

$$\Pr\{ w : X_t(w) = Y_t(w) \quad \forall t \in [0,T] \} = 1 \quad .$$

The assumptions of the above theorem permit an elementary proof of the existence and uniqueness of solutions of s.d.e.'s which is analogous to the classical one for o.d.e.'s: the solution will be the fixed point of the continuous map

$$S : M_B^2[0,T] \to M_B^2[0,T] \qquad S(\eta_t) := \eta_0 + \int_0^t b(s,\eta_s)ds + \int_0^t \sigma(s,\eta_s)dB_s \; .$$

4.3 Solutions of s.d.e.'s as diffusion processes

The solutions of s.d.e.'s are not arbitrary processes. In fact we have the following :

Theorem 4.2: *Let b, σ satisfy (4.2) and (4.3), let η be independent of $\mathcal{P}_{[s,T]}$, and define $p(s,\eta,t,A) := P(X_t \in A)$, where X_t is the solution of the s.d.e.*

$$\begin{cases} dX_t = b\,dt + \sigma\, dB_t \\ X_s = \eta \end{cases} .$$

Then X_t is a diffusion process with Markovian transition function p ,initial distribution μ_η , and generator

$$A_{\restriction C^2 \cap C_b} = \frac{1}{2} \sum_{i,j} (\sigma\sigma^*)_{ij}\, \partial^2_{ij} + \sum_i b_i \partial_i \quad .$$

Moreover X_t is a Markov process with the Feller property, so, since it is continuous, it is a strong Markov process .

4.4 Solutions of s.d.e.'s as Markov processes on path space

From the above theorem and and the continuity of the solution of a s.d.e. it follows that such processes can be realized on the path space $C([0,T], \mathbb{R}^n)$:

denote by $\mathcal{M}^s_t$ the smallest σ–algebra generated by the sets

$$\{\gamma \in C([0,T], \mathbb{R}^n) : \gamma(u) \in A \ , u \in [s,t]\}$$

where A is any Borel set in $\mathbb{R}^n$. Denote by $\mathcal{M}^s$ the smallest σ–algebra generated by

$$\{\gamma \in C([0,T], \mathbb{R}^n) : \gamma(u) \in A \ , \ u \in [s,T]\} \quad .$$

Define a continuous process by

$$X_t(\gamma) := \gamma(t) .$$

Finally let

$$Pr^{x,s}\{\gamma \in B\} := Pr\{\omega : X^x_{(\cdot)}(\omega) \in B\}$$

where B is any set in $\mathcal{M}^s_t$ and where X^x_t is a solution of the s.d.e.

$$\begin{cases} dX_t = b\,dt + \sigma\,dB_t \\ X_s = x \end{cases} \tag{4.4.}$$

with b and σ satisfying (4.2) and (4.3). Notice that for each $\omega \in \Omega$, $t \mapsto X^x_t(\omega)$ is a continuous path.

Theorem 4.3: *The process* $(C([0,T],\mathbb{R}^n), \mathcal{M}^s, (\mathcal{M}^s_t)_{t\geq s}, (X_t), Pr^{x,s})$ *with* $\mathcal{M}^s_t$, X_t, *and* $Pr^{x,s}$ *as above, is a Markov process with the Feller property, and therefore is a strong Markov process.*

From the above construction it is clear that $Pr^{x,s}$ depends only on the law of the process X^x_t. The following theorem tell us that the process X^x_t does not depends (in law) on the choice of the Brownian motion B_t in (4.4).

Theorem 4.4: *Let* $(\Omega_j, S_j, (\mathcal{F}^j_t), (B^j_t), Pr^j)$, *be two* $\mathbb{R}^n$*–valued Brownian motions and let* η_j , $j=1,2$, *be two random variables in* $L^2(\Omega_j, S_j, Pr^j)$ *independent of* B^j_t . *If* b *and* σ *satisfy (4.2) and (4.3), and* X^j_t *is the solution of the s.d.e.'s*

$$\begin{cases} dX^j_t = b\,dt + \sigma\,dB^j_t \\ X^j_0 = \eta_j \end{cases}$$

then the processes X^j_t *have the same law.*

4.5 S.d.e.'s on open domains

The following localization theorem permits us to define s.d.e. on domains of $[0,T]\times\mathbb{R}^n$. A subset D of $[0,T]\times\mathbb{R}^n$ will be called a *domain* if $D\cap\mathbb{R}^n\neq\phi$ and $D=D'\cap[0,T]\times\mathbb{R}^n$, where D' is open in $\mathbb{R}^{n+1}$.

Theorem 4.5: *Let* $b_j:[0,T]\times\mathbb{R}^n\to\mathbb{R}^n$, $\sigma_j:[0,T]\times\mathbb{R}^n\to M^{n\times n}(\mathbb{R})$ *satisfy (4.2) and (4.3), and let* D *be a domain of* $[0,T]\times\mathbb{R}^n$. *Suppose that* $b_{1\restriction D}=b_{2\restriction D}$ *and* $\sigma_{1\restriction D}=\sigma_{2\restriction D}$ *and let* X_t^j *be the solution of*

$$\begin{cases} dX_t^j = b_j\,dt + \sigma_j\,dB_t \\ X_0^j = \eta_j \end{cases}.$$

Assume finally that $\eta_1=\eta_2$ *for a.a.* ω *for which either* $\eta^1(\omega)\in D$ *or* $\eta_2(\omega)\in D$. *Denote by* τ_j *the exit time of* (t,X_t^j) *from* D *if* $\tau_j\leq T$, *and* $\tau_j=T$ *otherwise. then*

$$Pr\,(\tau_1=\tau_2)=1$$

$$Pr\left\{\sup_{t\in[0,T]}\|X_t^1-X_t^2\|=0\right\}=1\ .$$

Let us now consider two continuous functions

$$b: D\longrightarrow\mathbb{R}^n\quad,\quad\sigma: D\longrightarrow M^{n\times n}(\mathbb{R})$$

where D is a bounded domain in $\mathbb{R}_+\times\mathbb{R}^n$. Suppose that b and σ satisfy (4.3) on every closed subdomain of D. Let $\{D_n\}$ be an increasing sequence of domains such that $\bar{D}_n\subset D$, $\bigcup_n D_n=D$, and consider the sequence of functions b_n and σ_n satisfying the conditions of the existence and uniqueness theorem, and such that $b_{n\restriction\bar{D}_n}=b$ and $\sigma_{n\restriction\bar{D}_n}=\sigma$. Let X_t^n the solution of the s.d.e.

$$\begin{cases} dX_t^n = b_n\,dt + \sigma_n\,dB_t \\ X_0^n=\eta \end{cases}$$

with η independent of B_t and $Pr\{\eta \in D\}=1$. If τ_n denotes the exit time of (t, X^n_t) from D_n, then, from theorem 4.5 for $m>n$, we see that the processes X^m_t and X^n_t coincide on D_n and $\tau_m > \tau_n$. Furthermore, for $t<\tau_n$, we have

$$b_n(t,X^n_t) = b(t,X^n_t) \quad , \quad \sigma_n(t,X^n_t) = \sigma(t,X^n_t) \; .$$

Define $X_t := X^n_t$ if $t<\tau_n$. It is clear that X_t satisfies the s.d.e. on D

$$\begin{cases} dX_t = b\,dt + \sigma\, dB_t \\ X_0 = \eta \end{cases}$$

for $\quad t < \tau_D := \sup_n \tau_n \quad .$

4.6 Weak solutions and transformation of the drift

In the first section we defined strong solutions of s.d.e.'s. In such case the probability space and the Brownian motion with respect to which one performs stochastic integration were a priori assigned. Otherwise we will say that $(\Omega, \mathcal{S}, (\mathcal{F}_t)_{t\in[0,T]}, (X_t)_{t\in[0,T]}, (B_t), Pr)$ is a *weak solution* of the s.d.e.

$$\begin{cases} dX_t = b(t,X_t)dt + \sigma(t,X_t)dB_t \\ X_0 = \eta \end{cases}$$

if $(\Omega, \mathcal{S}, (\mathcal{F}_t), (B_t), Pr)$ is a Brownian motion,

$$b(t,X_t) \in L^1_B[0,T] \quad , \quad \sigma(t,X_t) \in L^2_B[0,T] \quad ,$$

and

$$X_t = \eta + \int_0^t b(s,X_s)\,ds + \int_0^t \sigma(s,X_s)\,dB_s \quad .$$

The existence of weak solutions of s.d.e.'s follows from Girsanov's theorem. In fact, suppose σ satisfy (4.2) and (4.3). Then, if $(\Omega, \mathcal{S}, (\mathcal{F}_t), (B_t), Pr)$ is a

Brownian motion, then there exists a solution of the s.d.e.

$$\begin{cases} dX_t = \sigma(t, X_t)\, dB_t \\ X_0 = \eta \end{cases} \tag{4.5}$$

with η independent of B_t. Suppose now that σ is invertible and b is such that

$$\phi_t := \sigma^{-1}(t, X_t)\, b(t, X_t) \in L^2_B[0,T] ,$$

and

$$Z_t := \exp\left(\int_0^t \phi_s\, dB_s - \frac{1}{2} \int_0^t \|\phi_s\|^2 ds \right)$$

is a martingale. Then, by Girsanov's theorem,

$$\hat{B}_t := B_t - \int_0^t \sigma^{-1}(s, X_s)\, b(s, X_s)\, ds \tag{4.6}$$

is a Brownian motion with respect to the measure $\hat{Pr}$, with

$$\left.\frac{d\hat{Pr}}{dPr}\right|_{\mathcal{F}_t} = Z_t .$$

Putting (4.6) into (4.5), one obtains

$$\begin{cases} dX_t = b(t, X_t) + \sigma(t, X_t)\, d\hat{B}_t \\ X_0 = \eta \end{cases} \tag{4.7}$$

i.e. $(\Omega, \mathcal{S}, (\mathcal{F}_t), (X_t), (\hat{B}_t), \hat{Pr})$ is a weak solution of (4.7). Moreover one can show that , for weak solutions, under the above assumptions on σ and b , there is unicity in law.

5

LYAPUNOV–TYPE THEOREMS FOR REGULARITY AND NONATTAINABILITY

5.1 *Regularity*

It follows from theorem 4.1 that if (4.2) and (4.3) hold for all $t \geq 0$ then the solution of equation (4.1) is defined for all $t \geq 0$ and the process X_t is said to be *regular*. Analogously for the solution X_t of a s.d.e. on a domain D , the process X_t is said to be regular if $\Pr\{\tau_D = \infty\} = 1$, where τ_D is the first exit time of the process (t, X_t) from the domain D . We note that if $D = \mathbb{R}_+ \times \mathbb{R}^n$ then the two definitions of regular process coincide. This follows from the continuity of X_t , defining τ_D to be the limit of the increasing sequence of random variables τ_n , where τ_n denotes the exit time of (t, X_t) from $\mathbb{R}_+ \times \{\|x\| < n\}$. The conditions under which theorem 4.1 assures the regularity of the solution process are rather restrictive and it is of paramount importance to find others, less stringent conditions (allowing, for instance, unbounded drifts) for the existence of the solution of a s.d.e. . The following theorem gives a more general sufficient condition for regularity through the Lyapunov function approach.

Theorem 5.1: *Let D be a domain in $\mathbb{R}_+ \times \mathbb{R}^n$ and let*

$$b : D \to \mathbb{R}^n \quad , \quad \sigma : D \to M^{n \times n}(\mathbb{R})$$

be the coefficients of the s.d.e. on D

$$\begin{cases} dX_t = b\,dt + \sigma\, dB_t \\ X_0 = \eta \end{cases}$$

with η *independent of* B_t *, and* $Pr\{(0,\eta)\in D\}=1$*. Let* $\{D_n\}_{n\geq 0}$ *be an increasing sequence of bounded domains such that* $\bar{D}_n \subset D$, $\bigcup_n D_n = D$*. Suppose that in every* D_n b, σ *satisfy conditions (4.2) and (4.3), and there exist a non negative function* $V \in C^{1,2}(D)$ *such that for some* $c \geq 0$, $\bar{n} \geq 0$

$$\mathcal{L} V \leq c V \quad \text{on } D \setminus D_{\bar{n}} \qquad \mathcal{L} := \partial_t + b\cdot\nabla_x + \frac{1}{2}\operatorname{tr}(\sigma\sigma^* D_x^2) \tag{5.1}$$

$$\lim_{n\to+\infty} \inf_{(t,x)\in\partial D_n} V(t,x) = +\infty \quad . \tag{5.2}$$

Then the solution process exists and it is regular, i.e. $Pr\{\omega : \tau_D(\omega)=\infty\}=1$*. Moreover if* $E\,\|\eta\|^2<+\infty$ *and* $E(V(0,\eta))<+\infty$ *then* $X_t \in M_B^2(\mathbb{R}_+)$ *.*

Proof. We at first note that from (5.1) it follows that the function

$$W(t,x) := e^{-ct}\, V(t,x)$$

satisfies $\mathcal{L}W \leq 0$. Now, let X_t^n be the solution of the s.d.e. on D_n with coefficients $b_{\restriction D_n}, \sigma_{\restriction D_n}$. Since $b_{\restriction D_n}, \sigma_{\restriction D_n}$ satisfy (4.2) and (4.3), X_t^n exists, and let τ_n denote the exit time of (t, X_t^n) from D_n . Define

$$Y_t^n := (t\wedge\tau_n , X_{t\wedge\tau_n}^n) \quad .$$

Since D_n is bounded, b and σ are continuous on D_n by (4.3) and $W \in C^{1,2}(D)$, one has

$$(\sigma^*\nabla_x W)(Y_t^n) \in M_B^2(\mathbb{R}_+) \; , \; \mathcal{L}W(Y_t^n) \in M_B^1(\mathbb{R}_+) \; .$$

So by Ito's formula applied to W , it follows

$$E\, W(Y_t^n) = E\,(V(0,\eta)) + E \int_0^t \mathcal{L}W(Y_s^n)\, ds \quad .$$

Suppose now that $n > \bar{n}$. Then

$$
\begin{aligned}
E \int_0^t \mathcal{L} W(Y_s^n)\,ds &= \int_{D_n} \mathcal{L} W(s,x)\,d\mu_s^n(x)\,ds = \\
&= \int_{D_n \setminus D_{\bar{n}}} \mathcal{L} W(s,x)\,d\mu_s^n(x)\,ds + \int_{D_{\bar{n}}} \mathcal{L} W(s,x)\,d\mu_s^{\bar{n}}(x)\,ds \le \\
&\le \int_{D_{\bar{n}}} \mathcal{L} W(s,x)\,d\mu_s^{\bar{n}}(x)\,ds \equiv A_{\bar{n}} .
\end{aligned}
$$

by (5.1) and the fact that on $D_{\bar{n}}$ the distributions μ_s^n and $\mu_s^{\bar{n}}$ coincide by theorem 4.5. Therefore

$$
Pr\{w : \tau_n(w) \le t\} \inf_{(t,x)\in\partial D_n} V(t,x) \le E\,V(Y_t^n) \le e^{ct}\,(E\,V(0,\eta) + A_{\bar{n}}) .
$$

Now, $A_{\bar{n}}$ is finite by

$$
\mathcal{L} W(Y_t^{\bar{n}}) \in M_B^1(\mathbb{R}_+)
$$

and suppose that the range of η is bounded with probability one, so that $E\,V(0,\eta) < +\infty$. Then by (5.2) one has $Pr\{\tau_D \le t\} = 0$ for all finite t, i.e. $Pr(\tau_D = +\infty) = 1$. The case of an arbitrary initial condition η may be dealt with in the way described in [10] pp. 465–466.

5.2 Nonattainability

If in the above proof one works with hitting times instead of stopping ones one obtain the following nonattainability result:

Theorem 5.2: *Let D be a bounded domain in $\mathbb{R}_+ \times \mathbb{R}^n$, let Γ be a closed subset of $\mathbb{R}^{n+1}$, and let*

$$
b : D\setminus\Gamma \to \mathbb{R}^n \quad , \quad \sigma : D\setminus\Gamma \to M^{n\times n}(\mathbb{R})
$$

be the coefficients of the s.d.e. on $D \setminus \Gamma$

$$\begin{cases} dX_t = b\,dt + \sigma\, dB_t \\ X_0 = \eta \end{cases}$$

with η *independent of* B_t *and* $Pr\{(0,\eta) \in D\} = 1$. *Define* $\{D_n\}_{n \geq 1}$ *to be the increasing sequence of bounded domains*

$$D_n := D \cap \{(t,x) : d(t,x) > 1/n\}$$

where d *denotes the distance function of* Γ. *Suppose that in every* D_n b *and* σ *satisfy (4.2) and (4.3) and there exists a non negative function* $V \in C^{1,2}(D \setminus \Gamma)$ *such that for some* $c \geq 0$, $\bar{n} \geq 0$

$$\mathcal{L} V \leq c V \quad \text{on } (D \setminus \Gamma) \setminus D_{\bar{n}} \tag{5.3}$$

$$\lim_{n \to +\infty} \inf_{d(t,x) = 1/n} V(t,x) = +\infty \tag{5.4}$$

$$E\, V(0,\eta) < +\infty \tag{5.5}$$

Then the process (t, X_t) *never hits* Γ *with probability one, i.e.* $Pr\{\tau_\Gamma = +\infty\} = 1$, *where* τ_Γ *denotes the first hitting time of* Γ.

5.3 *Nonattainability of regular submanifolds*

If in the above theorem Γ is a C^2–submanifold of $\mathbb{R}^n$ then one can use the logarithm of the inverse of the distance function as a Lyapunov function.

Theorem 5.3: *Let* D *be a bounded open subset of* $\mathbb{R}^n$, *let* Γ *be a* C^2–*m–codimensional submanifold of* $\mathbb{R}^n$, *and let* b *be the drift of the s.d.e. on* $D \setminus \Gamma$

$$\begin{cases} dX_t = b\,dt + dB_t \\ X_0 = \eta \end{cases}$$

with $Pr\{\eta \in D\setminus\Gamma, d(\eta) \geq c > 0\} = 1$ *and* η *independent of* B_t . *Suppose that in every bounded domain*

$$D_n := D \cap \{x : d(x) > 1/n\} \quad ,$$

where d *denotes the distance function of* Γ , b *satisfy (4.2) and (4.3), and suppose there exists a nbh.* U *of* Γ *such that*

$$b \cdot d\nabla d > (2-m)/2 \qquad \text{on} \quad U\setminus\Gamma \quad . \tag{5.6}$$

Then Γ *is nonattainable.*

Proof. Let V be a C^2 nonnegative function on $D\setminus\Gamma$ such that

$$V\restriction_{U\setminus\Gamma} = -\log d \quad .$$

One has, on $U\setminus\Gamma$,

$$\mathcal{L}V = -\{\, b\cdot d\nabla d - \|\nabla d\|^2/2 + d\Delta d/2 \,\}/d^2 \quad .$$

From $d\Delta d = m-1+o(d)$ (see the appendix), $\|\nabla d\| = 1$, and (5.6),it follows $\mathcal{L}V \leq 0$ on a sufficiently small nbh. of Γ . Obviously V satisfy (5.4), so nonattainability follows from theorem 5.2.

5.4 Appendix. The distance function

Let Γ be a non–empty set in a metric space X with metric function m . The *distance function* d is defined by

$$d(x) := \inf_{y\in\Gamma} m(x,y) \quad .$$

Let us now suppose that Γ is a m–dimensional C^k–submanifold of $\mathbb{R}^n$. It is readily shown that there exists a nbh. $\mathcal{U}$ of Γ such that d is a C^k function on $\mathcal{U}\setminus\Gamma$. Indeed, if $f: \mathcal{V} \to \mathbb{R}^n$ is a local coordinate system in $\mathbb{R}^n$ such that $f(\mathcal{V}\cap\Gamma) \subset \{0\}\times\mathbb{R}^m$, one has $d(x) = \|(x_1,\dots,x_{n-m})\|$ with respect to this coordinate system. From this also it follows that $\exists \varphi \in C^k: d(x) = \|x - \varphi(x)\|$. For sake of future convenience we show its existence by making use of the Lagrange's multipliers theory and implicit function theorem.

Theorem 5.4: *Let $y \in \Gamma$ and let $F = (F_1,\dots,F_m)$ be a $\mathbb{R}^m$–valued C^k function such that $dF_1(y)\wedge\dots\wedge dF_m(y) \neq 0$, and Γ is given by $F(x) = 0$ on a nbh. of y. Then there exist a nbh. of y, $\mathcal{U}_y$ and C^{k-1} functions*

$$\varphi: \mathcal{U} \to \Gamma \quad , \quad \Lambda : \mathcal{U} \to \mathbb{R}^m$$

such that

$$G(x, \varphi(x), \Lambda(x)) := (2(\varphi(x) - x) + DF^*(\varphi(x))\cdot\Lambda(x), F(\varphi(x))) = 0$$

for all $x \in \mathcal{U}$. Moreover $d(x) = \|x - \varphi(x)\|$.

Proof. We prove the theorem making use of the implicit function theorem applied to the function

$$G(x, y, \Lambda) = (2(x-y) + DF^*(y)\cdot\Lambda, F(y)) \quad .$$

To this end we must prove the invertibility of the derivative of G with respect to the 2–nd and 3–rd variables evaluated in $(y,y,0)$, $y \in \Gamma$. Since

$$D_{2,3}G(x,y,\Lambda)(h,M) = (2h + D(DF^*(y)\cdot\Lambda)\cdot h + DF^*(y)\cdot M, DF(y)\cdot h)$$

we must resolve the linear system

$$\begin{cases} 2h + DF^* M = v \\ DF\cdot h = N \end{cases} \quad .$$

From the first equation we obtain

$$h = \frac{1}{2} \, (v - DF^{*} \cdot M)$$

and making use of the second one we have

$$N = \frac{1}{2} \, DF \cdot v - DF \cdot DF^{*} \cdot M \, .$$

Since

$$\det DF \cdot DF^{*} = dF_1 \wedge \ldots \wedge dF_m \, (\nabla F_1, \ldots, \nabla F_m) \neq 0 \quad ,$$

the above system has one and only one solution for any given v and N , so $D_{2,3}G\,(y, y, 0)$ is invertible. We end the proof observing that, by Lagrange's theory of multipliers, $\varphi(x)$ is the minimum of the function $y \mapsto \|x - y\|^2$ restricted to Γ .

Corollary 5.1:

$$d \nabla d = x - \varphi(x) \quad .$$

Proof.

$$\nabla d^2 = 2d \nabla d = \nabla \, \|x - \varphi(x)\|^2 = 2\,(\mathbb{1} - D\varphi^{*}(x)) \cdot (x - \varphi(x)) =$$

$$= 2\,(x - \varphi(x) - D\varphi^{*}(x) \cdot (x - \varphi(x)) \, .$$

From $F(\varphi(x)) = 0$ it follows

$$DF\,(\varphi(x)) \cdot D\varphi(x) = 0$$

and, since

$$x - \varphi(x) = \frac{1}{2} \, DF^{*}(\varphi(x)) \cdot \Lambda(x) \quad ,$$

one has

$$D\varphi^{*}(x)\,(x - \varphi(x)) = \frac{1}{2} \, D\varphi^{*}(x) \, DF^{*}(\varphi(x)) \, \Lambda(x) = 0 \, .$$

From the the regularity of d^2 and from the equation

$$\Delta d^2 = 2\|\nabla d\|^2 + 2d\Delta d = 2\,(1 + d\Delta d)$$

we see that $d\Delta d$ is a C^{k-1} function. We are now interested in the value of the function $d\Delta d$ on a nbh. of Γ .

Theorem 5.5:

$$d\Delta d = (\operatorname{codim}\Gamma - 1) + O(d)\,.$$

Proof. From $\nabla d^2 = 2(x-\varphi(x))$ it follows $\Delta d^2 = 2(n - \operatorname{div}\varphi(x))$ and, since $x = \varphi(x) + d\nabla d$ for x in a sufficiently small nbh. of Γ , one has

$$\operatorname{div}\ \varphi(x) = \operatorname{div}\varphi\ (\varphi(x)) + O(d)\ .$$

So, we have to evaluate $D\varphi(\varphi(x))$. From

$$2\,(x-\varphi(x)) = DF^*(\varphi(x))\cdot\Lambda(x)$$

it follows

$$2\,(\mathbb{1} - D\varphi(\varphi(x))) = DF^*(\varphi(x))\cdot D\Lambda(\varphi(x))$$

and

$$2\,DF(\varphi(x))(x-\varphi(x)) = DF(\varphi(x))\cdot DF^*(\varphi(x))\,\Lambda(x)$$

therefore $\qquad \Lambda(x) = 2\,(DF\cdot DF^*(\varphi(x)))^{-1}\,DF(\varphi(x))\,(x-\varphi(x))$

and $\qquad D\Lambda(\varphi(x)) = 2\,(DF\cdot DF^*(\varphi(x)))^{-1}\,DF(\varphi(x))\,(\mathbb{1} - D\varphi(\varphi(x)))\,.$

Since $DF(\varphi(x))\cdot D\varphi(\varphi(x)) = 0$, in conclusion we have

$$D\varphi(\varphi(x)) = \mathbb{1} - DF^*(\varphi(x))\,(DF\cdot DF^*(\varphi(x)))^{-1}\,DF(\varphi(x))\,.$$

We conclude the proof evaluating the trace of $DF^*(DF\cdot DF^*)^{-1}DF$:

$$\sum_k (DF^*(DF\cdot DF^*)\,DF)_{kk} = \sum_{k,j,n} \partial_k F_j\,(DF\cdot DF^*)_{jn}\,\partial_k F_n =$$

$$(\det\ DF\cdot DF^*)^{-1}\sum_{j,n} dF_j(\nabla F_n)\,(-1)^{j+n}\,dF_1\wedge\ldots\wedge\widehat{dF_j}\wedge\ldots\wedge dF_m\,(\nabla F_1,\ldots,\widehat{\nabla F_n},\ldots\nabla F_m)$$

$$= (\det\ DF\cdot DF^*)^{-1}\ \sum_j(\det\ DF\cdot DF^*) = m\,.$$

6

STOCHASTIC QUANTIZATION

6.1 Stochastic mechanics

Let X_t be a stochastic process. It is well know that for many important processes X_t is not differentiable. This is the case for the Wiener process. To discuss the kinematics of stochastic processes therefore one needs a substitute for the derivative. Let $(\Omega, \mathcal{S}, (\mathcal{P}_t)_{t\in I}, (X_t)_{t\in I}, Pr)$ be a continuous $\mathbb{R}^n$–valued stochastic process such that

i) $t \mapsto X_t$ *is a continuous map from* I *into* $L^1(\Omega, \mathcal{S}, Pr)$

ii)
$$D_+ X_t := \lim_{h\to 0+} (1/h)\, E\,(X_{t+h} - X_t \mid \mathcal{P}_t)$$
exists as a limit in $L^1(\Omega, \mathcal{S}, Pr)$ *and* $t \mapsto D_+X_t$ *is continuous from* I *into* $L^1(\Omega, \mathcal{S}, Pr)$

iii)
$$a_{+t} := \lim_{h\to 0+} \frac{1}{h} E\left((X_{t+h} - X_t)\otimes(X_{t+h} - X_t) \mid \mathcal{P}_t\right)$$

exists as a limit in $L^1(\Omega, \mathcal{S}, Pr)$, *and* $t \mapsto a_{+t}$ *is continuous from* I *into* $L^1(\Omega, \mathcal{S}, Pr)$, *and* $\det a_{+t} > 0$ *a.e.* .

The stochastic process $(\Omega, \mathcal{S}, (\mathcal{P}_t), (D_+X_t), Pr)$ is called the *mean forward derivative* and it represent the best prediction one can make, given any relevant information at time t , of

$$(X_{t+h} - X_t)/h$$

for infinitely small positive h . If $t \mapsto X_t$ has a continuous strong derivative in $L^1(\Omega, \mathcal{S}, Pr)$ then $D_+X_t = dX_t/dt$. As an example of a stochastic process satisfying i)–iii) consider a solution of a s.d.e.

$$dX_t = b\,dt + \sigma\, dB_t\ .$$

In this case $D_+X_t = b(t,X_t)$ and $a_{+t} = \sigma(t,X_t)\sigma^*(t,X_t)$. This situation, in some sense, exhausts all the examples of processes satisfying i)–iii) :

Theorem 6.1: *Let $(\Omega, S, (P_t), (X_t), Pr)$ be a continuous stochastic process satisfying i)–iii). Then there exists a Brownian motion $(\Omega, S, (P_t), (B_{+t}), Pr)$ such that*

$$X_t = X_0 + \int_0^t D_+X_s\,ds + \int_0^t \sqrt{a_{+s}}\,dB_{+s} \quad .$$

Let us now suppose there exists a decreasing family of σ–algebras (F_t) such that the continuous stochastic process $(\Omega, S, (P_t), (X_t), Pr)$ satisfying i)–iii) is F_t –measurable (P_t represents the past, F_t the future). In that case one may define the time–symmetric analogue of i)–iii) :

iv)

$$D_-X_t := \lim_{h\to 0+} \frac{1}{h} E(X_t - X_{t-h} \mid F_t)$$

exists as a limit in $L^1(\Omega,S,Pr)$ and $t \mapsto D_-X_t$ is continuous into $L^1(\Omega,S,Pr)$

v)

$$a_{-t} := \lim_{h\to 0+} \frac{1}{h} E((X_t - X_{t-h}) \otimes (X_t - X_{t-h}) \mid F_t)$$

exists as a limit in $L^1(\Omega,S,Pr)$, $t \mapsto a_{-t}$ is continuous, and $\det a_{-t} > 0$ a.e. .

The stochastic process D_-X_t is called the *mean backward derivative* and is in general different from D_+X_t. If $t \mapsto X_t$ is strongly differentiable in $L^1(\Omega,S,Pr)$ then $D_+X_t = D_-X_t = dX_t/dt$. It may be shown that

$$E(D_+X_t) = E(D_-X_t) \quad , E(a_{+t}) = E(a_{-t}) \quad .$$

Moreover X_t is a constant process if and only if $D_+X_t = D_-X_t = 0$. There is an analogue of theorem 6.1:

Theorem 6.2: *Let $(\Omega, S, (F_t), (X_t), Pr)$ be a continuous stochastic process satisfying i),iv),v). Then there exists a Brownian motion $(\Omega, S, (F_t), (B_{-t}), Pr)$ such that*

$$X_t = X_0 + \int_0^t D_-X_s\,ds + \int_0^t \sqrt{a_{-s}}\,dB_s \quad .$$

The above theorem needs some explanation since (F_t) is a decreasing σ–algebra. $(\Omega, S, (F_t), (B_{-t}), Pr)$ is called a Brownian motion since $B_{-0} = 0$, $B_{-t} - B_{-s}$ is normally distributed with variance $t-s$ and zero mean, and $B_{-t} - B_{-s}$ is independent of $\mathcal{F}_t$. The stochastic integral with respect to such processes is defined in a manner analogous to the Ito's integral with respect to the usual Brownian motion where the only difference is that the limit of Riemann sums is performed using backward increments. Let us now assume that the past $\mathcal{P}_t$ and the future $\mathcal{F}_t$ are conditionally independent given the present $\mathcal{P}_t \cap \mathcal{F}_t$. If X_t is a Markov process and

$$\mathcal{P}_t = \sigma(X_s, s \leq t) \quad , \quad \mathcal{F}_t = \sigma(X_s, s \geq t)$$

this is certainly the case. With the above assumption on the $\mathcal{P}_t$ and $\mathcal{F}_t$, if X_t satisfies i) and iv) then D_+X_t and D_-X_t are $\mathcal{P}_t \cap \mathcal{F}_t$ –measurable, and we may define $D_+D_-X_t$ and $D_-D_+X_t$ if the corresponding limits exist. Assuming they exist, we define

$$A_t := \frac{1}{2} \left(D_+D_-X_t + D_-D_+X_t \right)$$

and call it the *mean second derivative* or *mean acceleration*. If $t \mapsto X_t$ is smooth then $A_t = d^2X_t/dt^2$.

If the process X_t is supposed to represent the motion of a particle of mass m in the presence of an external force F and an unanalized dynamical mechanism causing random fluctuations one is lead to write down a stochastic analogue of Newton's second law setting

$$F(X_t(\omega)) = m\, A_t(\omega) \quad .$$

6.2 Smooth diffusions

Let us now suppose that $(\Omega, \mathcal{S}, (\mathcal{P}_t)_{t\in I}, (\mathcal{F}_t)_{t\in I}, (X_t)_{t\in I}, P)$ is a *smooth diffusion*, that is

i) X_t *is a continuous* $\mathbb{R}^n$*–valued Markov process*

ii) there exist smooth bounded functions

$$b_+, b_- : I \times \mathbb{R}^n \to \mathbb{R}^n \quad , \quad \| b_\pm(t,x) \| \leq C(1 + \|x\|) \quad , \quad \sigma : \mathbb{R}^n \to M^{n\times n}(\mathbb{R}) \quad ,$$

σ *positive definite, such that*

$\forall f \in C_0^\infty(I\times\mathbb{R}^n)$, $D_+f(t,X_t)$ *and* $D_-f(t,X_t)$ *exists and*

$$D_+f(t,X_t) = (\partial_t f + \sum_{ij} \sigma_{ij}\partial^2_{ij} f + b_+\cdot\nabla f)(t,X_t)$$

$$D_-f(t,X_t) = (\partial_t f - \sum_{ij} \sigma_{ij}\partial^2_{ij} f + b_-\cdot\nabla f)(t,X_t) \quad .$$

The definition emphasizes the fact that diffusions are time–symmetric. The differential operators appearing in ii) are called the *forward* and *backward generators*. Analogously b_+ and b_- are called the *forward* and *backward drift* respectively.

We will now derive a relation between b_+ and b_- . From now we will suppose $\sigma=\nu\mathbb{1}$, $\nu>0$ for simplicity.

Let $f\in C_0^\infty(I\times\mathbb{R}^n)$ and let ρ be a measure such that

$$\int_{I\times\mathbb{R}^n} f\,d\rho = \int_I E f(t,X_t)\,dt \quad .$$

Since

$$\int_I \frac{d}{dt} E f(t,X_t)\,dt = 0 = \int_I E\, D_+f(t,X_t)dt = \int_{I\times\mathbb{R}^n} (\partial_t f + \nu\Delta f + b_+\cdot\nabla f)\,d\rho \quad ,$$

the measure ρ will be a weak solution of the *forward Fokker–Planck equation*

$$\partial_t \rho = \nu\Delta\rho - \mathrm{div}(\rho b_+) \quad . \qquad (6.1)$$

This equation is parabolic with smooth coefficients, so ρ will be smooth and strictly positive. Since

$$E\, D_+f(t,X_t) = E\, D_-f(t,X_t) \quad ,$$

ρ will be also a solution of the *backward Fokker–Planck equation*

$$\partial_t \rho = -\nu\Delta\rho - \mathrm{div}(\rho b_-) \quad . \qquad (6.2)$$

Averaging (6.1) and (6.2) one obtains the *current equation* (also called the *equation of continuity*)

$$\partial_t \rho = -\mathrm{div}\ \rho v$$

where the *current velocity vector* v is defined by

$$v := \frac{1}{2}(b_+ + b_-) \quad .$$

Taking the difference of (6.1) and (6.2) one obtains

$$\operatorname{div}\left(2\nu \nabla \rho - \rho(b_+ + b_-)\right) = 0$$

It may be shown that $2\nu\nabla\rho - \rho(b_+ + b_-)$ is not only divergence free but it is actually zero, so that

$$b_- = b_+ - 2\nu\nabla \log \rho$$

and one has the *osmotic equation*

$$u = \nu \nabla \log \rho$$

where the *osmotic velocity vector* u is defined by

$$u := \frac{1}{2}(b_+ - b_-) \quad .$$

Since X_t is supposed to be a Markov process one has, given $f \in C_0^\infty(\mathbb{R}^n)$,

$$E(f(X_t) \mid X_s) = \int_{\mathbb{R}^n} f(y)\, p_+(s, X_s, t, dy) \qquad s \leq t$$

and

$$E(f(X_s) \mid X_t) = \int_{\mathbb{R}^n} f(x)\, p_-(s, dx, t, X_t) \qquad s \leq t$$

where

$$p_+(s, X_s, t, A) := \Pr(X_t \in A \mid X_s) \qquad s \leq t$$

$$p_-(s, A, t, X_t) := \Pr(X_s \in A \mid X_t) \qquad s \leq t$$

are the *forward* and *backward transition probability* respectively. Note that p_+ is the usual Markovian transition function. p_- is also a transition function and its definition emphasizes the time symmetry in the definition of Markov process.

Let F, G be two functions such that $F(t, X_t)$ and $G(t, X_t)$ are forward and backward martingales respectively. A stochastic process X_t is a *backward martingale* in case each X_t is $\mathcal{F}_t$–measurable and $E(X_s | \mathcal{F}_t) = X_t$ for all $s \le t$. A *forward martingale* corresponds to the usual definition of martingale. Since X_t is Markovian we have that

$$F(s, X_s) := E(f(X_t) \mid X_s) =$$

$$= \int_{\mathbb{R}^n} f(y)\, p_+(s, X_s, t, dy)$$

and

$$G(t, X_t) := E(g(X_s) \mid X_t) =$$

$$= \int_{\mathbb{R}^n} g(x)\, p_-(s, dx, t, X_t)$$

are forward and backward martingales respectively for all $f, g \in C_0^\infty(\mathbb{R}^n)$

Since it is easy to prove that if a process X_t is a forward (backward) martingale, then $D_+ X_t = 0$ ($D_- X_t = 0$), by ii) it follows that F and G are solutions of the *forward diffusion equation* and the *backward diffusion* equation respectively :

$$(\partial_s + \nu\Delta + b_+ \cdot \nabla)\, F = 0$$

$$(\partial_t - \nu\Delta + b_- \cdot \nabla)\, G = 0 \quad .$$

From our regularity assumptions on the coefficients σ , b_+ , b_- and from the standard theory of parabolic partial differential equations it follows that the transition probabilities there are strictly positive smooth functions p_+, p_- , the fundamental solutions of the forward and backward operators respectively, such that

$$p_+(s, x, t, dy) = p_+(s, x, t, y)\, dy$$

$$p_-(s, dx, t, y) = p_-(s, x, t, y)\, dx \quad .$$

From this there also follows that p_+ satisfies the forward Fokker–Plank equation in t and y , and the forward diffusion equation in s and x :

$$\partial_t p_+ = \nu \Delta_y p_+ - \mathrm{div}_y (p_+ b_+)$$

$$\partial_s p_+ = -\nu \Delta_x p_+ - b_+ \cdot \nabla_x p_+ \quad .$$

Analogously for p_- one has

$$\partial_s p_- = -\nu \Delta_x p_- - \mathrm{div}_x (p_- b_-)$$

$$\partial_t p_- = \nu \Delta_y p_- - b_- \cdot \nabla_y p_- \quad .$$

Moreover, since for all $f, g \in C_0^\infty(\mathbb{R}^n)$

$$E\, f(X_s) g(X_t) = E\, E(f(X_s) g(X_t) | X_s) = E\, E(f(X_s) g(X_t) | X_t) =$$

$$= f(X_s) E(g(X_t) | X_s) = g(X_t) E(f(X_s) | X_t) =$$

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} f(x) g(y) p_+(s,x,t,y) \rho(s,x) dx dy = \int_{\mathbb{R}^n \times \mathbb{R}^n} f(x) g(y) p_-(s,x,t,y) \rho(t,y) dx dy$$

one has

$$\rho(s,x) p_+(s,x,t,y) = p_-(s,x,t,y) \rho(t,y) \quad .$$

6.3 Smooth diffusions and the Schrödinger equation

Let us now return to the stochastic analogue of Newton's law. Let X_t be a smooth diffusion process. By ii) it follows $D_+ X_t = b_+(t, X_t)$ and $D_- X_t = b_-(t, X_t)$.

By ii) again,

$$D_+ D_- X_t = \partial_t b_-(t,X_t) + b_+ \cdot \nabla b_-(t,X_t) + \nu \Delta b_-(t,X_t)$$

$$D_- D_+ X_t = \partial_t b_+(t,X_t) + b_- \cdot \nabla b_+(t,X_t) - \nu \Delta b_+(t,X_t)$$

so that the mean acceleration is given by $A(t,X_t)$, where

$$A = \partial_t \frac{b_+ + b_-}{2} + \frac{1}{2} b_+ \cdot \nabla b_- + \frac{1}{2} b_- \cdot \nabla b_+ + \nu \Delta \frac{b_+ - b_-}{2}$$

that is

$$\partial_t v = A + u \cdot \nabla u - v \cdot \nabla v + \nu \Delta u \quad .$$

Suppose now that $mA(t,X_t) = F(t,X_t)$ holds. The above equations becomes

$$\partial_t v = F/m + u \cdot \nabla u - v \cdot \nabla v + \nu \Delta u$$

and from the osmotic equation and the continuity equation one obtains

$$\partial_t u = \nu \nabla \partial_t \log \rho = \nu \nabla(-\operatorname{div} \rho v) =$$

$$= -\nu \nabla(\operatorname{div} v + v \cdot \nabla\rho/\rho) = -\nu \nabla \operatorname{div} v - \nabla v . u \quad .$$

In conclusion, if u_0 and v_0 are given, one has an initial value problem for a system of non–linear partial differential equations

$$\begin{cases} \partial_t u = -\nu \nabla \operatorname{div} v - \nabla u \cdot v \\ \partial_t v = F/m - v \cdot \nabla v + u \cdot \nabla u + \nu \Delta u \quad . \end{cases} \tag{6.3}$$

Such equations, once resolved, give b_+, b_-, and so the Markov process the particle obeys is known.

Suppose now that F and v are gradients, pose

$$F = -\nabla V \quad , \quad v = 2\nu \nabla S \quad ,$$

and define

$$R := \frac{1}{2} \log \rho \ .$$

Consider now the complex function $\psi := \exp R + iS$, and put $\nu = \hbar/2m$. The remarkable fact is that from (6.3) easily follows that ψ satisfy the Schrödinger equation

$$i\hbar \partial_t \psi = -\frac{\hbar^2}{2m} \Delta \psi + V\psi + \alpha(t)\psi \ . \tag{6.4}$$

Since $|\psi|^2 = \rho$, if we multiply by ψ^*, integrate over space, and take real parts, we see that if (6.4) is satisfied, then $\alpha(t)$ is real. We can always choose the potential S so that $\alpha(t) = 0$.

Conversely, if ψ is any smooth, nowhere zero, solution of the Schrödinger equation, normalized so that $\|\psi\|^2_{L^2} = 1$, we may write

$$\psi = \exp R + iS \ , \ u = \frac{\hbar}{m} \nabla R \ , \ v = \frac{\hbar}{m} \nabla S \ ,$$

and the diffusion process with diffusion matrix $\hbar \mathbb{1}/m$, forward drift $b_+ = u + v$, backward drift $b_- = v - u$, has probability density $|\psi|^2$, and mean acceleration $A = -\nabla V/m$. The same considerations apply to systems of n particles considering $\mathbb{R}^{3n}$–valued diffusion processes with diffusion matrix $(\hbar \, \delta_{ij}/m_j)$.

In conclusion there is a bijective correspondence between smooth diffusion with diffusion matrix $\hbar \mathbb{1}/m$, and drift satisfying (6.3), and smooth nowhere zero solution ψ of the Schrödinger equation, the correspondence being determined by

$$\rho = |\psi|^2 \ , \quad b_\pm = \frac{\hbar}{m} (\mathrm{Re} \pm \mathrm{Im}) \nabla \log \psi \ .$$

7

NONATTAINABILITY OF NODES AND EXISTENCE OF NELSON DIFFUSIONS

7.1 *Nelson diffusions*

Let ρ_t be a family of probability densities on $\mathbb{R}^n$, let b be a vector field on Z_ρ^c, where

$$Z_\rho := \{(t,x) \in \mathbb{R} \times \mathbb{R}^n : \rho(t,x) = 0\},$$

and suppose that, on Z_ρ^c, ρ and b satisfy, in some weak sense, the Fokker–Planck equation

$$\partial_t \rho = -\operatorname{div}(\rho b) + \tfrac{1}{2}\Delta \rho .$$

A solution, if it exists, of the s.d.e. on Z_ρ^c

$$\begin{cases} dX_t = b\,dt + dB_t \\ X_0 = \eta \end{cases} \tag{7.1}$$

with $\mu_\eta = \rho_0 dx$, will be called a *Nelson diffusion.* If

$$\rho = |\psi|^2 , \quad b = (\mathrm{Re} + \mathrm{Im})\, \nabla \log \psi ,$$

where ψ is a solution of the Schrödinger equation, then X_t will be called a *Schrödinger diffusion.* Under suitable conditions $D_+ X_t$ exists and it is equal to b, and moreover

$$\Pr(f(X_t)\in A)=\int_A f(x)\,\rho(t,x)\,dx \quad ,$$

for all bounded measurable function f and all measurable Borel sets A. As regards the equality $D_+X_t=b(t,X_t)$ Föllmer proved that if X_t is a solution of (7.1) with

$$E\int_0^T \|b(t,X_t)\|^p dt < +\infty \qquad p\geq 1 \quad ,$$

then, for almost all $t\in[0,T]$, D_+X_t exists as a limit in $L^p(\Omega,\mathcal{S},\Pr)$, and $D_+X_t=b(t,X_t)$.

Let us suppose that b is sufficiently regular on Z_ρ^c. Then, from the existence theorem for s.d.e.'s, there exists a local Markovian diffusion in some bounded domain in Z_ρ^c. The process will be defined up to some stopping time indicating that the process will reach the nodes or will escape to infinity in a finite time. Since the paradigm of stochastic quantization tells us that X_t represents the trajectory of the particle, and since quantum mechanics tells us that $|\psi(x)|^2$ represents the probability density to find the particle in x, it seems very unphysical that the process may reach the nodes. It turns that the solution of this problem is given by its cause, i.e. the singularity of the drift on Z_ρ. In fact, if the singular drift field points away from the nodes, it will give rise to a strong repulsion, and prevent the particle from approaching the nodes.

Let us now check that, at least for stationary solutions of the Schrödinger equation, the corresponding drifts are repulsive at the nodes.

7.2 *Bound states*

Let $\psi_0=\alpha+i\beta$ be a C^1 eigenfunction of the Schrödinger on $\mathbb{R}^n$. The drift field will be a time-independent vector field on $Z^c{}_\rho$

$$b(x) = [\,(\alpha-\beta)\nabla\alpha + (\alpha+\beta)\,\nabla\beta\,](x)\,/\,(\alpha^2(x)+\beta^2(x)) \quad . \tag{7.2}$$

Suppose now $d\alpha(x_0)\wedge d\beta(x_0)\neq 0$, with $x_0\in Z$, so that Z will be a (n-2)–dimensional C^1 local manifold in a nbh. of x_0 in Z, and consider the local change of coordinate

$$f: \ x \longmapsto (\alpha(x), \ \beta(x), \ \varphi(x))$$

defined on a nbh. of x_0 in $\mathbb{R}^n$, where φ is a local chart of Z_ρ in a nbh. of x_0 in Z_ρ. From (7.2) it follows

$$f_* b\,(x_1,..,x_n) = \frac{1}{x_1^2+x_2^2}\,[(x_1-x_2)\,e_1 + (x_1+x_2)\,e_2\,] \tag{7.3}$$

where $\{e_j\}$ is the standard basis in $\mathbb{R}^n$. From (7.3) it follows that, locally, b is a vector field that 'lives' on the fibers of the normal bundle of Z_ρ. It is easy to study the phase portrait of the vector field

$$X = \left(\frac{x-y}{x^2+y^2}\,,\ \frac{x+y}{x^2+y^2} \right)$$

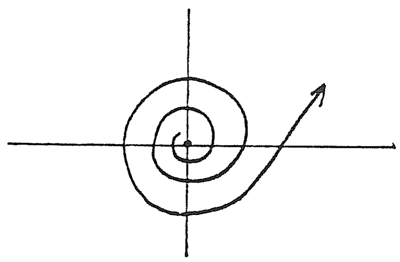
on $\mathbb{R}^2 \setminus \{0\}$. With respect to polar coordinates one has $X = (2, 1/r^2)$, and the solution of the differential equation

$$\begin{cases} \dot r\,(t) = 2 \\ \dot\theta\,(t) = 1/r^2(t) \end{cases}$$

is

$$r(t) = 2t + r_0 \qquad \theta(t) = \theta_0 + 1/2r_0 - 1/(2(t+r_0)) \ \ (\mathrm{mod}\ 2\pi)\,,$$

so that the origin (i.e. Z_ρ) is a repellor with basin equal to $\mathbb{R}^2$, and the orbits turn round the origin infinitely many times in a finite time.



From the above analysis, if d denotes the distance function of Z, then d is certainly increasing under the flow of b, so that one can hope to use theorem 5.3 to prove the nonattainability of nodes. To this end let us now give an estimate of $b \cdot d\nabla d$ on a nbh. of Z_ρ. From (7.2) and from

$$\alpha(x) = d(x)\nabla d(x) \cdot \nabla\alpha(x_0) + O(d^2)$$

$$\beta(x) = d(x)\nabla d(x) \cdot \nabla\beta(x_0) + O(d^2)$$

there follows

$$b \cdot d\nabla d = 1 + O(d)$$

on a nbh. of Z_ρ. The above estimate permits us to use theorem 5.3 :

Theorem 7.1: *Let $\psi_0 = \alpha + i\beta$ be a C^1 eigenfunction of the Schrödinger equation on $\mathbb{R}^n$, such that $\psi_0^{-1}(0)$ is a C^2 submanifold, and let define b on Z_ρ^c by*

$$b = (\mathrm{Re} + \dot{\mathrm{Im}})\nabla \log \psi_0 \quad .$$

Then, for every bounded open domain D in $\mathbb{R}^n$, the solution of the s.d.e. on $D \setminus Z_\rho$

$$\begin{cases} dX_t = b\,dt + dB_t \\ X_0 = \eta \end{cases}$$

with $\Pr\{\eta \in D \setminus Z\} = 1$, η independent of B_t, does not reach the nodes with probability one.

7.3 A Lyapunov–type theorem for Nelson diffusions

The above analysis seems to fail in the time–dependent case. In fact, let us try to use theorem 5.2 with $V = -\log d$. We have

$$\mathcal{L}V = -\{d\dot{d} + b \cdot d\nabla_x d - \|\nabla_x d\|^2/2 + d\Delta_x d/2\}/d^2$$

Since (see the proof of theorem 5.5, we pose $F=(\alpha,\beta)$)

$$d\Delta_x d = 1 + \dot{d}^2 - \frac{\|\dot{\alpha}\nabla_x\beta - \dot{\beta}\nabla_x\alpha\|^2}{\|\dot{\alpha}\nabla_x\beta - \dot{\beta}\nabla_x\alpha\|^2 + \|\nabla_x\alpha\|^2 + \|\nabla_x\beta\|^2 - |\nabla_x\alpha\cdot\nabla_x\beta|} + O(d) \equiv$$

$$\equiv 1 + \dot{d}^2 + f + O(d)$$

on a nbh. of Z_ρ, $\mathcal{L}V \leq 0$ implies

$$b.\, d\nabla_x d + \dot{d}^2 > \frac{1}{2} f \tag{7.4}$$

on a sufficiently small nbh. of Z_ρ. Since

$$b = \frac{(\alpha-\beta)\nabla_x\alpha + (\alpha+\beta)\nabla_x\beta}{\alpha^2+\beta^2}$$

and

$$\alpha = d\,(\dot{\alpha}\,\dot{d} + \nabla_x\alpha\cdot\nabla_x d) + O(d^2)$$

$$\beta = d\,(\dot{\beta}\,\dot{d} + \nabla_x\beta\cdot\nabla_x d) + O(d^2)$$

(7.4) is equivalent to

$$\frac{(\nabla_x\alpha\cdot\nabla_x d)^2 + (\nabla_x\beta\cdot\nabla_x d)^2 + \dot{d}\,\nabla_x d\cdot(\dot{\alpha}\nabla_x\alpha + \dot{\beta}\nabla_x\beta + \dot{\alpha}\nabla_x\beta - \dot{\beta}\nabla_x\alpha)}{(\nabla_x\alpha\cdot\nabla_x d + \dot{\alpha}\dot{d})^2 + (\nabla_x\beta\cdot\nabla_x d + \dot{\beta}\dot{d})^2} + \dot{d}^2 > \frac{1}{2} f. \tag{7.5}$$

In the one dimensional case Z_ρ will be a discrete set in $\mathbb{R}^2$, and (7.5) will reduce to

$$\frac{(\alpha'^2+\beta'^2)X^2 + TX(\dot{\alpha}\alpha' + \dot{\beta}\beta' + \dot{\alpha}\beta' - \dot{\beta}\alpha')}{(\alpha' X + \dot{\alpha} T)^2 + (\beta' X + \dot{\beta} T)^2} + \frac{T^2}{X^2+T^2} > \frac{1}{2} \tag{7.6}$$

with $X=x-x_0$, $T=t-t_0$, $(x_0,t_0)\in Z_\rho$. Moreover, on a nbh. of its singularities

$$b(T,X) = \frac{(\alpha'^2+\beta'^2)X + T(\dot{\alpha}\alpha' + \dot{\beta}\beta' + \dot{\alpha}\beta' - \dot{\beta}\alpha')}{(\alpha' X + \dot{\alpha} T)^2 + (\beta' X + \dot{\beta} T)^2} + O(1) \tag{7.7}$$

Formula (7.7) shows that in the general case b does not always point away from the node. Let us see it with an example. Let ψ_t be a solution of the Schrödinger equation for the harmonic oscillator with initial condition

$$\psi_0 = \lambda\phi_0 + \mu\phi_1 \ , \quad \phi_0 = e^{-x^2/2}/\sqrt[4]{\pi} \ , \quad \phi_1 = \sqrt{2}\, x e^{-x^2/2}/\sqrt[4]{\pi},\ \lambda^2+\mu^2=1,\ \lambda,\mu\in\mathbb{R}.$$

After some calculations one obtains

$$Z_\rho = \{(t,x)\in\mathbb{R}^2 : t=k\pi \ , k\in\mathbb{Z} \ , x=\pm\lambda/\sqrt{2}\mu\} \ ,$$

and, in $(2\pi, -\lambda/\sqrt{2}\mu)$

$$\dot{\alpha}=0 \ , \quad \dot{\beta} = -\lambda e^{-\lambda^2/4\mu^2}/\sqrt[4]{\pi} \ , \quad \alpha' = -\mu\sqrt{2}\, e^{-\lambda^2/4\mu^2}/\sqrt[4]{\pi} \ , \quad \beta'=0 \ ,$$

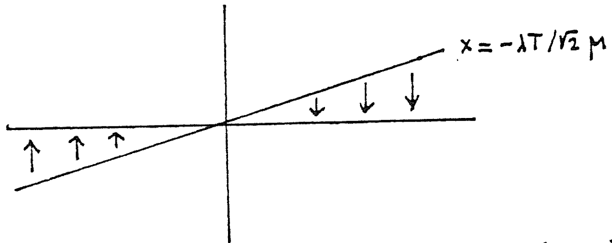
so that

$$b(T,X) = \frac{2\mu^2 X + \sqrt{2}\lambda\mu T}{2\mu^2X^2+\lambda^2T^2} + O(1)$$

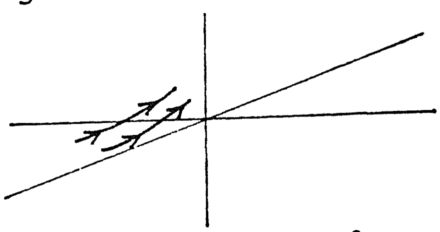
on a nbh. of $(0,0)$. This shows that on the sectors

$$\{T<0 \ , -\lambda T/\sqrt{2}\mu < x < 0\} \ , \{T>0, \ 0<x<-\lambda T/\sqrt{2}\mu\}$$

(we are supposing $\lambda/\mu>0$) b points in the wrong direction



and (x^2+T^2) will not be increasing under the flow of $(1,b)$ in the sector $\{T<0 \ , -\lambda T/\sqrt{2}\mu < x < 0\}$.



Moreover, for instance for λ,μ such that $\sqrt{2}\mu^2+\mu\lambda<0$, (7.6) will be violated on the line $x=T$. This example shows us that there is no hope to apply Lyapunov theorem, at least as it stands, to Nelson diffusions. The key fact for resolving this problem is that for Nelson diffusions the density of the process is a priori known. This will allow us to replace the pointwise condition $\mathcal{L}V \leq 0$ with a more manageable condition in the mean. The key technical ingredient is the following result of Zheng :

Theorem 7.2: *Let ρ be a function on $\mathbb{R}_+ \times \mathbb{R}^n$ such that $\rho(t,\cdot)$ is a probability density $\forall t \in \mathbb{R}_+$ and $\sqrt{\rho}$ satisfy the following Holder condition:*

$$|\sqrt{\rho(s,x)} - \sqrt{\rho(t,y)}| \leq C_K (\|x-y\| + |t-s|^{1/2}) \qquad x,y \in K \text{ compact} .$$

Let b a locally Lipschitz vector field on Z_ρ^c with $Z_\rho := \{\rho = 0\}$. Suppose ρ and b satisfy the Fokker–Planck equation in the following weak sense:

$$< f, \rho_t - \rho_s > = \int_s^t < \frac{1}{2}\Delta f + b_r.\nabla f, \rho_r > dr \qquad \forall f \in C_0^\infty(\mathbb{R}^n) ,$$

where $<\cdot,\cdot>$ is the usual L^2 scalar product with respect to the Lebesgue measure.If X_t is the Nelson diffusion defined by ρ and b , then, for every random variable ζ satisfying $0 \leq \zeta \leq \tau \wedge T, T > 0$ (τ being the stopping time of X_t), and $f \geq 0$,

$$\mathbb{E} \int_0^\zeta f(t, X_t)\, dt \leq \int \int_0^T f(t,x)\, \rho(t,x)\, dt\, dx .$$

Let us now explain as the above results enters in the proof of the Lyapunov's theorem. In the usual proof, see §5.1, one has

$$\mathbb{E}\, V(Y_t^n) - \mathbb{E}\, V(Y_0^n) = \mathbb{E} \int_0^t \mathcal{L}V(Y_s^n) ds = \int_{D_n} \mathcal{L}V(s,x) d\mu_s^n(x) ds ,$$

where μ_s^n is the distribution of the process X_s^n , and, since one has no knowledge on μ_s^n , the only way to obtain an uniform (with respect to n) upper estimate on

$$\mathbb{E} \int_0^t \mathcal{L}V(Y_s^n) ds$$

is to require $\mathcal{L}V \leq 0$. For Nelson diffusions, using theorem 7.2, we may write

$$\mathbb{E} \int_0^t \mathcal{L}V(Y_s^n) ds \leq \int_D (\mathcal{L}V)^+(s,x)\, \rho(s,x)\, ds dx ,$$

and it will suffice to require

$$(\mathcal{L}V)^+ \in L^1_{loc}(\mathbb{R}_+\times\mathbb{R}^n, \rho\, dt dx) .$$

So we have the following

Theorem 7.3: *Let ρ and b as in theorem 7.2 and let D be a bounded domain in $\mathbb{R}_+\times\mathbb{R}^n$. Suppose there exists a non negative function $V\in C^{1,2}(D)$ such that*

$$(\mathcal{L}V)^+ \in L^1(D, \rho\, dt dx) ,$$

and

$$V(0,\cdot) \in L^1(D\cap\mathbb{R}^n, \rho_0\, dx) ,$$

$$\lim_{n\to+\infty} \inf_{d(t,x)=1/n} V(t,x) = +\infty ,$$

where d denotes the distance function of Z_ρ, and

$$\mathcal{L} = \partial_t + b\cdot\nabla_x + \frac{1}{2}\Delta_x .$$

Then the solution of the s.d.e. on $D\setminus Z_\rho$

$$\begin{cases} dX_t = b\, dt + dB_t \\ X_0 = \eta \qquad\qquad \mu_\eta = \rho_0\, dx \end{cases}$$

does not reach the nodes with probability one.

Suppose that Z_ρ is such that $d\in C^{1,2}$, and put $V=-\log d$. One obtains

$$\mathcal{L}V = -\{d\dot{d} + b\cdot d\nabla_x d + \|\nabla_x d\|^2/2 + d\Delta_x d/2\}/d^2 .$$

Suppose now that $\rho = O(d^2)$ on a nbh. of Z_ρ, then

$$(\mathcal{L}V)^+ \in L^1(D,\rho) \quad \text{iff} \quad (b\cdot d\nabla_x d)^- \in L^1(D) .$$

This generalizes the pointwise condition $b\cdot d\nabla d \geq 0$ of theorem 5.3 (here Z has codimension two). If $\rho=|\psi|^2$, with ψ a C^1 solution of the Schrödinger equation, then $\rho = O(d^2)$, and from

$$b = [(\alpha-\beta)\nabla_x\alpha + (\alpha+\beta)\nabla_x\beta]/(\alpha^2+\beta^2)$$

it follows $b = O(1/d)$, so $(b.d\nabla d)^- \in L^1$ is always satisfied.

Observing that in the Lyapunov theorem is sufficient to require

$$V \in C^{1/2}(D_n) \quad , \quad \bigcup_n D_n = D \quad , \quad \int_{D_n} (\mathcal{L}V)^+ \rho\, dt dx \leq M < +\infty \quad \forall n \quad ,$$

we can prove the following

Theorem 7.4: *Let ρ, b be as in theorem 7.2 and suppose that Z_ρ is the union of a numerable family of C^2 submanifolds. Suppose moreover that*

$$\rho/d^2 \in L^1(D) \quad , \quad \rho\|b\|/d \in L^1(D) \tag{7.8}$$

with d the distance function of Z_ρ , and D bounded. Then the corresponding Nelson diffusion on $D \setminus Z_\rho$ never reaches Z_ρ .

Proof. Pose

$$Z_\rho = \bigcup_{n \in \mathbb{N}} \Gamma_n$$

where the Γ_j 's are C^2 submanifolds of $\mathbb{R}^{n+1}$. Let d_j denote the distance function of Γ_j . Let V_j be a non negative $C^{1/2}$ function on $D\setminus\Gamma_j$ such that $V_j = -\log d_j$ on a sufficiently small nbh. of Γ_j . Since V_j is regular,

$$|\dot{d}_j| \leq 1 \quad , \quad \|\nabla_x d_j\| \leq 1 \quad , \quad \|D_x^2 d_j\| \leq c/d_j + \|O_j(1)\| \quad ,$$

with $O_j(1)$ bounded (see theorem 5.5), $1/d < n$ on D_n , $d_j \geq d$, and D_n is bounded, for every j there exists a constant $M_j \geq 1$ such that

$$\|V_j\|_{D_n} \leq (1+\log n) M_j \quad , \quad \|\nabla V_j\|_{D_n} \leq n M_j \quad , \quad \|D_x^2\|_{D_n} \leq n^2 M_j \quad . \tag{7.9}$$

Let

$$V := \sum_{j \in \mathbb{N}} e^{-j} V_j / M_j \quad .$$

By (7.9) the above series is uniformly convergent on D_n together with the series

$$\sum_j e^{-j} \nabla V_j / M_j \quad , \quad \sum_j e^{-j} D_x^2 V_j / M_j \quad ,$$

so $V \in C^{1/2}(D_n) \quad \forall n \geq 1$.

By (7.8) we have

$$\int_{D_n} |\mathcal{L}V| \rho\, dt\, dx \le \sum_j \int_{D_n} e^{-j} |\mathcal{L}V_j| \rho\, dt\, dx \le \sum_j \int_D e^{-j} |\mathcal{L}V_j| \rho\, dt\, dx \le K + \int_{D\setminus D_{\bar n}} e^{-j} \Big\{ \frac{|\dot{d}_j|}{d_j} + \frac{|b\cdot \nabla_x d_j|}{d_j} + \frac{\|\nabla_x d_j\|^2}{2\, d_j^2} + \frac{|\Delta_x d_j|}{2 d_j} \Big\} \rho\, dt\, dx \le K + \sum_j \int_{D\setminus D_n} e^{-j} \Big\{ \frac{1}{d} + \frac{\|b\|}{d} + \frac{1}{2d^2} + \frac{N}{d^2} \Big\} \rho\, dt\, dx < +\infty$$

and so nonattainability follows from theorem 7.3.

If ρ is sufficiently regular one may use $V = -\log \rho$ as Lyapunov function. In this case one obtains the following result of Blanchard and Golin :

Theorem 7.5: *Let ρ, b, and Z be as in theorem 7.3, and suppose*

$$\rho \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n), \quad b \in C^{0,1}(Z), \quad \rho\, \mathrm{div}_x\, v \in L^1_{loc}(\mathbb{R}_+ \times \mathbb{R}^n)$$

with

$$v := \begin{cases} b - \frac{1}{2} \log \rho & \text{on } Z^c \\ 0 & \text{on } Z \end{cases}.$$

Then the corresponding Nelson diffusion on $D\setminus Z$, D bounded, never reaches Z.

Proof. Use theorem 7.3 with $V = -\log \rho$, and observe that, by the Fokker–Planck equation,

$$\mathcal{L}V = \frac{1}{2} \Delta_x \rho - \rho\, \mathrm{div}_x\, v .$$

Since ρ is regular, $\rho\, \mathrm{div}\, v \in L^1_{loc}$ implies $(\mathcal{L}V)^+ \in L^1$.

In order to apply the preceeding theorem to Schrödinger diffusions, one may use the following

Theorem 7.6: *Let ψ_t be a solution of the Schrödinger equation such that*

$$\psi \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n), \quad \nabla \psi \in L^2_{loc}(\mathbb{R}_+ \times \mathbb{R}^n) .$$

Then

$$\rho = |\psi|^2 \qquad b = (\mathrm{Re} + \mathrm{Im})\, \nabla \log \psi$$

satisfy the conditions of the theorem 7.5. Moreover suppose that

$$\psi_0 \in H^{2m}(\mathbb{R}^n), \quad V \in C_b \cap C^{2m-2} .$$

Then

$$\psi_t = (\exp -itH)\,\psi_0 \quad , \quad H = -\frac{1}{2}\Delta + V$$

satisfies

$$\psi_t \in \bigcap_{\ell=0}^{m} C^{\ell}(\mathbb{R}_+, H^{2(m-\ell)}(\mathbb{R}^n)) \ ,$$

so that the above conditions may be recovered by Sobolev embedding theorem.

7.4 Existence by P.D.E. methods

The results of the previous sections, based on the Lyapunov function approach, although relatively easy to prove, rely on some regularity hypotheses on the density probability and on the drift. Now we will weaken these requirements and we will suppose that ρ and b satisfy the Fokker–Planck equation in a weak sense. E. Carlen proved that if the drift field satisfy a 'finite energy' condition, then there exists a weak solution of the s.d.e.

$$\begin{cases} dX_t = b\,dt + dB_t \\ X_0 = \eta \end{cases}$$

defined for times. The complete statement of the result is the following

Theorem 7.7: *Let* ρ, b *satisfy the weak continuity equation*

$$\frac{d}{dt}\int f \rho_t \,dx = \int v\cdot\nabla f\,\rho_t\,dx \qquad \text{a.e. in } t \quad \forall f \in C_b^1(\mathbb{R}^n) \quad (7.1$$

with

$$v := \begin{cases} b - \frac{1}{2}\nabla \log \rho & \text{on } Z_\rho^c \\ 0 & \text{on } Z_\rho \end{cases} \ .$$

Suppose

$$\int x^2 \rho_t\,dx < +\infty$$

and

$$\int_0^T \int (\|u\|^2 + \|v\|^2)\,\rho_t\,dt\,dx < +\infty \qquad (7.1\cdot$$

with $u := b - v$. *Then there exists a Borel probability measure* P_r *on* $C(\mathbb{R}_+, \mathbb{R}^n)$ *such that:*

i) *the evaluation process* $X_t(\gamma) := \gamma(t)$ *is a square integrable Markov process under* P_r *which as a jointly measurable version*

ii) *the distribution of* X_t *has density* ρ_t

iii) $(C(\mathbb{R}_+,\mathbb{R}^n), \mathcal{B}, (\mathcal{P}_t), (\mathcal{B}_t), P_r)$ *, with* $\mathcal{B}$ *the Borel* σ*–algebra,* $(\mathcal{P}_t)$ *the natural filtration, and*

$$B_t := X_t - X_0 - \int_0^t b_s(X_s)\,ds \quad ,$$

is a Brownian motion

iv) *for any* $f \in C_0^\infty(\mathbb{R}^n)$*, the following limits exist strongly in* $L^2(P_r)$ *:*

$$\begin{aligned} \lim_{h\to 0_+} \frac{1}{h} E\left(f(X_{t+h}) - f(X_t)\,|X_t\right) &= \left(\frac{1}{2}\Delta + b(t,X_t)\cdot\nabla\right) f(X_t) \\ \lim_{h\to 0_+} \frac{1}{h} E\left(f(X_t) - f(X_{t-h})\,|X_t\right) &= \left(-\frac{1}{2}\Delta + b_*(t,X_t)\cdot\nabla\right) f(X_t) \end{aligned} \tag{7.12}$$

with $b_* := b - \nabla \log \rho$.

We give a sketch of the proof. The first step consists of the construction of a Markovian propagator. Such propagator will map any function $f \in L^2(\rho_s dx)$ to a solution (in a sense that will be made precise below) $f_t \in L^2(\rho_t dx)$ of the backward diffusion equation

$$\partial_t f_t = \left(\frac{1}{2}\Delta - b_* \cdot \nabla\right) f_t \qquad f_s = f \quad . \tag{7.13}$$

The construction goes as follows. Let

$$\{ b_*^j \}_j \subset C_0^\infty([0,T]\times\mathbb{R}^n)$$

be a sequence such that

$$L^2(\rho\,dt\,dx) - \lim_{j\to+\infty} b_*^j = b_*$$

and let f_t^j be the solution of the initial value problem

$$\partial_t f_t^j(x) = \left(\frac{1}{2}\Delta - b_*^j\cdot\nabla\right) f_t^j(x) \qquad f_s^j(x) = f_s(x)$$

where $f_s \in C_0^\infty(\mathbb{R}^n)$. Such solution exists by the regularity hypotheses on b_*^j and f_s . Moreover one has, for the solution f^j , the maximum principle

$$\min_x f_s(x) \le f^j(t,x) \le \max_x f_s(x) \qquad \forall (t,x) \in [0,T] \times \mathbb{R}^n .$$

By the weak continuity equation (7.10), the hypothesis (7.11), the maximum principle, and repeated integration by parts, one shows that $\{f_t^j\}_{j\in\mathbb{N}}$ is a Cauchy sequence in $L^2(\rho_t dx)$. Let f_t denote the limit of such sequence. Define the family of operators $\{T_{s,t}\}$ by

$$T_{s,t} \; : C_0^\infty(\mathbb{R}^n) \longrightarrow L^2(\rho_t dx) \qquad T_{s,t} f_s := f_t \quad .$$

One has, denoting by $(\cdot\,,\cdot)_t$ the scalar product on $L^2(\rho_t dx)$,

$$(1\,, T_{s,t} f_s)_t = (1, f_s)_s$$

and $T_{s,t}$ extends by continuity to a Markovian propagator

$$T_{s,t} \; : \quad L^2(\rho_s dx) \longrightarrow L^2(\rho_t dx) \quad .$$

Moreover

$$(T_{s,t} f, g)_t - (f, g)_t = \int_s^t ((T_{r,s} f, O_r g)_r + (T_{r,s} f, v \cdot \nabla g)_r\, dr \qquad \forall g \in C_b^2(\mathbb{R}^n) \qquad (7.14)$$

with O_r being the osmotic operator

$$O_r := -\frac{1}{2}\Delta + u_r \cdot \nabla \qquad .$$

When $f \in C_b^2(\mathbb{R}^n)$ also, $t \mapsto (T_{t,s} f, g)_t$ is right differentiable at $t = s$, and

$$\frac{d}{dt}\Big|_{t=s} (T_{s,t} f, g)_t = (f, O_s g)_s + (f, v_s \cdot \nabla g)_s \quad . \qquad (7.15)$$

(7.14) tells that $f(t,x) := T_{t,s} f(x)$ is a solution of (7.13) in the sense that, formally, (7.14) gives, differentiating and using (7.10),

$$\int \partial_t f \, g \, \rho_t dx + \int (v \cdot \nabla f) g \rho_t dx + \int (v \cdot \nabla g) f \rho_t dx =$$

$$= \int \left(\left(\tfrac{1}{2}\Delta + u \cdot \nabla\right) f \right) g \, \rho_t dx + \int (f \; v \cdot \nabla g)\, \rho_t \, dx \quad .$$

Cancelling and rearranging one obtains

$$\int(\partial_t f - \tfrac{1}{2}\Delta f - b_*\cdot\nabla f)g\,\rho_t dx = 0 \qquad \forall g\in C_0^\infty(\mathbb{R}^n)\ .$$

The second step consists of the construction of a probability measure on $\Omega = (\dot{\mathbb{R}}^n)^T$. Such measure is defined making use of the same technique employed for the construction of the Wiener measure in §1.6. One defines a linear functional on $C_{fin}(\Omega)$ by

$$L\varphi := (f_n, T_{T,t_{n-1}} f_{n-1} T_{t_{n-1},t_{n-2}} f_{n-2} \cdots T_{t_2,t_1} f_1)_T$$

with

$$\varphi(\gamma) := \prod_{j=1}^{n} f_j(\gamma(t_j)) \quad , \quad f_j\in C(\dot{\mathbb{R}}^n)\ .$$

Extending L to $C(\Omega)$, and making use of Riez's representation theorem, one obtains a probability measure Pr on Ω . The evaluation process $X_t(\gamma) := \gamma(t)$ is automatically a Markov process by the construction of Pr . Again by the definition of Pr , one has

$$E\ g(X_t) = (1, T_{T,t}\,g)_T = (1, g)_t = \int g\,\rho_t dx \qquad \forall g\in C_0^\infty(\mathbb{R}^n)$$

so that $\mu_{X_t} = \rho_t dx$. Moreover, always making use of the definition of Pr , $\forall f,g \in C_b^2(\mathbb{R}^n)$

$$\frac{1}{h} E((f(X_t) - f(X_{t-h}))g(X_t)) = \frac{1}{h}((f,g)_t - (T_{t,t-h}f, g)_t) =$$

$$= \frac{1}{h}((f,g)_t - (f,g)_{t-h}) + \frac{1}{h}((f,g)_{t-h} - (T_{t,t-h}f,g)_t)\ .$$

Taking the limit $h\to 0+$, applying the weak continuity equation to the first term, and (7.15) to the second term, one has

$$(v\cdot\nabla f, g)_t + (f, v\cdot\nabla g)_t + \tfrac{1}{2}(\nabla f, \nabla g)_t - (f, v\cdot\nabla g)_t$$

so that, integrating by parts,

$$\lim_{h\to 0+} \frac{1}{h} E((f(X_t) - f(X_{t-h}))g(X_t)) = ((-\tfrac{1}{2}\Delta + b_*\cdot\nabla)f, g)_t\ .$$

In a similar way one obtains

$$\lim_{h\to 0+} \frac{1}{h} E((f(X_{t+h}) - f(X_t))g(X_t)) = ((\tfrac{1}{2}\Delta + b\cdot\nabla)f, g)_t\ .$$

The third step begin defining the process

$$B_t := X_t - X_0 - \int_0^t b(s, X_s)\,ds \,.$$

One shows that such process is a Brownian motion by explicitly evaluating its mean and covariance. Since

$$t \longmapsto \int_0^t b(s, X_s)\,ds$$

is a.s. continuous, and B_t is a.s. continuous being a Brownian motion, X_t is a.s. continuous, and so $Pr(C(\mathbb{R}_+, \mathbb{R}^n)) = 1$.

The following result of E.Carlen permits to apply the previous theorem to Schrödinger diffusions. Before stating the next theorem, we recall the definition of *Rellich class potential.* A potential V is called a Rellich class potential if, as a multiplication operator on $L^2(\mathbb{R}^n)$, the domain of V contains the domain of Δ, and for some $a, b < 1$,

$$\| V\varphi \| \le a \| \Delta\varphi/2 \| + b \| \varphi \| \,.$$

The Kato–Rellich theorem asserts that, if V is a Rellich class potential, then $-\frac{1}{2}\Delta + V$ is selfadjoint with domain $\mathcal{D}(\Delta)$.

Theorem 7.8: *Let V be a Rellich class potential, and let $\varphi_0 \in H^1(\mathbb{R}^n)$. Let*

$$\psi_t = \exp -it(-\Delta/2 + V)\;\varphi_0 \,.$$

Then :

i) for all t $\psi_t \in H^1(\mathbb{R}^n)$ and $t \mapsto \|\nabla\psi_t\|^2$ is continuous

ii) there are unique jointly measurable functions $\psi(t,x)$, $\nabla\psi(t,x)$ such that $\psi(t,\cdot) = \psi_t$ and $\nabla\psi(t,\cdot) = \nabla\psi_t$ a.e.

iii) if

$$u := \mathrm{Re}\,\nabla \log \psi \,, \qquad v := \mathrm{Im}\,\nabla \log \psi \,,$$

then for each finite interval $[0,T]$, there exists a constant $M_T < +\infty$ such that

$$\int (\|u\|^2 + \|v\|^2)(t,x)\,|\psi(t,x)|^2 dx < M_T \qquad \text{for a.e. } t \in [0,T]$$

iv) for all $f \in C_b^1(\mathbb{R}^n)$

$$t \mapsto \int f(x)\,|\psi(t,x)|^2 dx$$

is in $C^1[0,T]$ *and*

$$\frac{d}{dt}\int f(x)\,|\psi(t,x)|^2\,dx = \int v(t,x)\cdot\nabla f(x)\,|\psi(t,x)|^2\,dx$$

v) if $\int x^2 \psi_0(x)\,dx < +\infty$ *then* $\int x^2 \psi(t,x) < +\infty \quad \forall t \in [0,T]$.

BIBLIOGRAPHICAL REMARKS

CHAPTERS 1–4. For the proofs of the standard results of these chapters see [1], [7], [8], [9], [10]. For the construction of the Wiener measure given in §1.6 see [12]

CHAPTER 5. Theorem 5.1, in the case $D = I \times D'$, is given in [11].

CHAPTER 6. The results of this chapter are due to E.Nelson. See [13], [14], [15], [16], [2].

CHAPTER 7. Theorem 7.2 is due to W.Zheng [17], [18]. For theorems 7.5 and 7.6 see [2] and [3]. Theorems 7.7 and 7.8 are due to E.Carlen [4], [5].

REFERENCES

[1] P.Baldi, Equazioni differenziali stocastiche e applicazioni, Quaderni U.M.I. n.28, (1984)

[2] P.Blanchard, P.Combe, W.Zheng, Mathematical and Physical aspects of Stochastic Mechanics, L.N.P. n.281, (1987)

[3] P.Blanchard, S.Golin, Diffusion processes with singular drift fields, Comm. Math.Phys. 109 (1987), 421–435

[4] E.Carlen, Ph.D. Thesis, Princeton (1984)

[5] E.Carlen, Conservative diffusions, Comm.Math.Phys. 94 (1984), 293–315

[6] H.Fölmer, Time reversal on Wiener spaces, in: Stochastic processes–mathematics and physics, L.N.M. n.1158, (1985)

[7] A.Friedman, Stochastic differential equations and Applications, Academic Press (1975)

[8] T.Gard, Introduction to stochastic differential equations, Dekker (1988)

[9] I.Guikhman, A.Skorokhod, Stochastic differential equations, Springer (1972)

[10] I.Guikhman, A.Skorokhod, Introduction à la théorie des processus aléatoires, Mir (1980)

[11] R.Z.Has'minskii, Stochastic stability of differential equations, Sijthoff and Noordhoff (1980)

[12] E.Nelson, Feynman integrals and the Schrödinger equation, J.Math.Phys., 5 (1964), 332–343

[13] E.Nelson, Derivation of the Schrödinger equation from Newtonian mechanics, Phys.Rev. 150 (1966)

[14] E.Nelson, Dynamical theories of Brownian motion, Princeton Univ. Press, (1967)

[15] E.Nelson, Quantum fluctuations, Princeton Univ. Press, (1984)

[16] E.Nelson, Lectures at the Ecole d'Eté de probabilités de Saint–Flour, (1987)

[17] W.Zheng, Tightness results for laws of diffusion processes. Applications to stochastic mechanics, Ann.Inst.H.Poincaré, B21 (1985), 103–124

[18] W.Zheng, P.A.Meyer, Sur la construction de certain diffusions, in: Seminaire des probabilités XX, L.N.M. n.1204, (1986)