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ON THE REPRESENTATION SPACE OF A FUNDAMENTAL GROUP

A thesis submitted for the Degree of
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Introduction

In a recent but already celebrated paper Atiyah [A] has brightly illustrated the remarkable link that mathematicians and theoretical physicists have created between low-dimensional topology and gauge theory.Here is his picture:

<u>dim2</u>	<u>dim3</u>	<u>dim4</u>
$\pi_1(M)$	Casson's invariant	Donaldson invariants
	Floer homology	

If we consider compact 2-dimensional manifolds, the fundamental group is a complete invariant, that is, it classifies all homomorphism classes.
$\pi_1(M)$ is still a strong invariant for compact 3-manifolds, but the question of classification of manifolds with isomorphic fundamental group is still unanswered as it depends on the well-known and unsettled Poincare' conjecture.
In defining alternative invariants for 3-manifolds one turns to study representations of $\pi_1(M)$ into non-abelian groups G (e.g. the Reidemeister torsion, chapter 4).

It is exactly on this line that Casson has recently (1985) discovered a new invariant $\lambda(M)$ for homology spheres with a peculiar behaviour with respect to surgery operations on knots.
Casson's invariant is really a powerful one for some old-dated questions of low-dimensional topology: for example, it gives a negative answer to the attempts of building counterexamples to Poincare'conjecture, with non-zero Rohlin invariant.
At this time of the story the differential geometry of gauge theory begins to appear.
Taubes reinterprets Casson's invariant (1986) in terms of spectral data on the quotient space of connections as an Euler characteristic, being based on the classical correspondence betwen conjugacy classes of representations of the fundamental group of a manifold M in G and gauge equivalence classes of flat connections on MxG.
It's a non-trivial metamorphosis of language and tools, passing from algebraic and geometric topology to global analysis!
The huge gap between the two subjects is indeed the origin of its difficulty and its interest.
But the idea is fruitful and after two years (1988) a new refinement of Casson's invariant is ready: by using a natural gauge functional Floer finds out an abelian group $I_*(M)$ with a natural $\mathbb{Z}_8$-grading s.t. $\lambda(M)={}^1/_2\Sigma_{i=0}{}^7(-1)^i \mathrm{rank}_{\mathbb{R}} I_i$.
The impressive feature of the matter is that a formal language which mathematical physicists have set up for a completely different purpose: tyding up gauge theory physics, looks a primary tool of analysis of basic topological questions. It is a phylosophically inspiring issue.

This remarkable connection between topology and physics is confirmed in toto by the 4-dimensional case.
Here the situation is much more involved as 4 is for certain aspects a critical number.
For n<4 any topological manifold admits a unique smooth structure (up to diffeomorphism) and for n>4 differentiable structures corresponding to a topological one can be classified.

For n=4 nothing can be said; for example we might have topological structures which don't admit differentiable one, on the other hand many diff.structures may correspond to a topological one.
What gauge theory provides in connection with is a device for distinguishing smooth structures associated to a simply-connected 4-manifold or even for giving negative results of existence of smooth structures: Donaldson polynomials which are constructed making an essential use of gauge theoretical structures.
I'll not deal with 4-manifolds in the following, but it's worthwhile telling that big developments are expected in the interaction between 3 and 4-manifold topology from the relations between Floer homology and Donaldson polynomials.
Moreover mathematics may gain a big deal of suggestions if one were able to translate from the physical level of rigor Witten's intuitions (e.g. [W1],[W2]).

This thesis is a mathematically-minded attempt of clarifying (for 3-manifolds) this connection between 3-manifolds and the differential geometry of gauge theory and framing it into a more general setting to connect it to some other recent developments of 3-dimensional theory.
Actually the mathematical object which is at the basis of the relations between Casson's invariant and Floer homology is the representation space of the fundamental group of a 3-manifold in SU(2).
My strategy is then to exhibit these invariants as natural constructions on R, representation space of $\pi_1(M)$ for M compact, connected, oriented into a non-abelian group G in the framing of a wider study of the structures of R.
This general point of view looks particularly useful as it becomes evident that other recent advances in the study of 3-manifolds have common roots in R and so it might be interesting to investigate their affinity.
In this sense Casson's invariant is a natural basic invariant (intersection number) of the canonical orientation of R_Σ (chapter 3) for a surface Σ using the Heegard decomposition of a

3-manifold.
But R_Σ possesses also a canonical volume form and a symplectic structure ([Jo] (1989); [GO1] (1984)) and so we're lead to Johnson invariants (chapter 4) and the possibility of a hamiltonian study of this space.
The symplectic structure on R_Σ is a tool which looks very promising and not yet completely understood. R_Σ is a configuration space worth of studying thanks to its role in the gauge theoretical context and what is striking ([T] (1989)) is that one can quantize (algebrically) a certain subalgebra of the algebra of observables and this quantization is the skein-module of an oriented 3-manifold.
This result (chapter 5) opens a completely new perspective, still to be investigated, on the bonds between our representation space and the invariants of knot theory.
On the other side ([R]), taking into account the Heegard decomposition for a closed orientable 3-manifold one finds that the representation spaces of the fundamental groups of handlebodies are lagrangian submanifolds of R.
This property is the key for reinterpreting (chapter 6) Casson's invariant in terms of spectral data for the symplectic action and relating it to the Euler characteristic of Floer complex for lagrangian intersections.

We're not really interested at this stage in a careful analysis of the topological meaning of the various invariants, as the main goal of the work is trying to focus how many constructions have a common origin in the natural structures of R, collecting them around this central theme.
Actually after clarifying the affinity among these different subjects of research, the following step might be to investigate the specific properties of each subject and to look if the comparison sheds some reciprocal light on the matter.
In the following, many chapters are provided with a section devoted to some forethoughts about lines of research that might be developed starting from this point.

1 Representation spaces

In this chapter we review some more or less known material about the differentiable structure of the representation space of a finitely generated discrete group π into a Lie group G that is the set Hom(π,G).

We'll analyze also the set of conjugacy classes of Hom(π,G) for the action of G (section 1.3).
A more detailed analysis is devoted to the representation space of the fundamental group of a surface (section 1.4) as this is the object we'll be primarily concerned from section 2 to 6.

1.1 HOM(π,G) as an algebraic set

Let π a finitely generated discrete group and G a Lie group.
Let R_π denote the space Hom(π,G) of homomorphisms of π into G with the compact-open topology.
$\forall\ x \in \pi$ we've a natural map

$$\textbf{(1.1.1)} \qquad \begin{aligned} X :\ & R_\pi \rightarrow G \\ & \rho \rightarrow X(\rho) = \rho(x) \end{aligned}$$

If F is a free group on $\{x_i\}_{i=1....m}$ we may identify R_F with G^m (according with the topology) by the map

$$\textbf{(1.1.2)} \qquad \begin{aligned} f_{\{x\}} :\ & R_F \rightarrow G^m \\ & \rho \rightarrow (X_1(\rho) \ldots\ldots X_m(\rho)) \end{aligned}$$

If R is a set of relations for F so that

$$\textbf{(1.1.3)} \qquad F = < x_1 \ldots\ldots x_m | r(x_1 \ldots\ldots x_m) = 1 \quad r \in R >$$

is a presentation for F, then we may identify R_F with $f_{\{x\}}(R_F) = V\{x_i\}$ where

(1.1.4) $V\{x_i\} = < (g_1 \ldots g_m) \mid r(g_1 \ldots g_m) = 1 >$

We can also describe this natural inclusion $R_F \subset G^m$ in a more formal way. Let

(1.1.5) $R \longrightarrow F \longrightarrow \pi \longrightarrow 0$

a presentation for the group π. F is free on $\{x_i\}_{i=1\ldots m}$ and R is free on $\{R_j\}_{j=1\ldots n}$. We've an induced sequence

(1.1.6) $R_r \xleftarrow{\alpha^*} F \xleftarrow{\beta^*} \pi \longleftarrow 0$

which is exact in the sense that the image $\beta^*(R_\pi)$ exactly equals α^{*-1}(trivial homomorphisms in R_R) .$\beta^*(R_\pi)$ is our natural identification of R_π with a subset of R_F.

1.2 The tangent space

If F is a free group , by (1.1.2), R_F may be given the structure of a manifold and this structure doesn't depend on the chosen set of generators. If π is a finitely generated group with a presentation (1.1.5), by (1.1.6) we can give it the structure of an algebraic set. Also this structure doesn't depend on the chosen set of generators.
So R_π is a subvariety of R_F. In general it isn't a submanifold, but exhibits singular points. Here we're not interested in analyzing these singularities (see [GO1], [GO2]) but only in getting rid of them for obtaining a smooth manifold. We can use the following classical result of differential geometry:

Theor. (1.2.1) Let X,Y be two smooth manifolds, $f: X \longrightarrow Y$ a subimmersion (a smooth map is said to be a subimmersion at a point x if there exists a neighborhood U of x in X s.t. the rank of f is constant on U), x a point of X, y = f(x). Then

i) $f^{-1}(y)$ is a closed submanifold of X.
ii) we've an exact sequence

$$0 \longrightarrow T_x(f^{-1}(y)) \longrightarrow T_xX \longrightarrow T_{f(x)}Y$$

iii) there exists an open neighborhood U of x in X s.t. f(U) is a submanifold of Y and

$$\dim_x X = \dim_{f(x)} f(U) + \dim_x(f^{-1}(y)) = \operatorname{rank}_x f + \dim_x(f^{-1}(y))$$

So if we set

$$\textbf{(1.2.2)} \quad \mathbf{R}_\pi = \{ \rho \in R_\pi \subset R_F : \alpha^* \text{ is a subimmersion at } \rho \}$$

we've (as the set of points at which α^* is a subimmersion in R_F is an open set and so a submanifold of R_F)

Coroll. (1.2.3) $\mathbf{R}_\pi$ is a smooth submanifold of R_F.

To study the tangent space to R_F, R_R, R_π there're various approaches. I'll follow the more algebraic one.

We need the following definitions.

Def. (1.2.4) Let π be a group and M a π-module (in fact one should say a $\mathbb{Z}\pi$-module). A crossed homomorphism of π in M is a map $\phi: \pi \longrightarrow M$ s.t.

$$\textbf{(1.2.5)} \qquad \phi(xy) = \phi(x) + x\phi(y)$$

XHom(π,M), the set of crossed homomorphisms between π and M is an abelian group under pointwise addition.

Def.(1.2.6) Let π be a group. We define $d\pi$ the π-module with a generator dx $\forall\ x \in \pi$ and relations

$$d(xy) = dx + xdy \qquad \forall\ x,y \in \pi$$

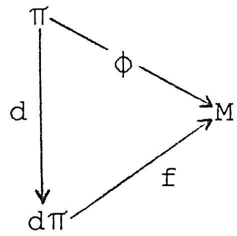
The map $d: \pi \longrightarrow d\pi$ given by $x \longrightarrow dx$ is a crossed homomorphism.

Prop.(1.2.7) There is a natural isomorphism

$$Hom_\pi(d\pi,M) \longrightarrow XHom(\pi,M)$$

where $Hom_\pi(d\pi,M)$ is the group of π-homomorphisms of $d\pi$ into M.

proof: given any crossed homomorphism $\phi: \pi \longrightarrow M$, there is a unique π-homomorphism $f: d\pi \longrightarrow M$ s.t. the following diagram commutes



In fact we must define $f(dx) = \phi(x)$. So we've only to check that f is a π-homomorphism. But $f(xdy) = f(d(xy)) - f(dx) = \phi(xy) - \phi(x) = x\phi(y) = xf(dy)$. The associations $\phi \longrightarrow f$, $f \longrightarrow f \circ d$ are inverse to each other and give the desired isomorphism.

Q.E.D.

Remark (1.2.8) M, G as above. The set of crossed homomorphisms from G to M coincides, by definition, with the set $Z^1(G,M)$ of Eilenberg-Mac Lane 1-cocycles of G with values in M. This topological point of view will be very useful later.

Let now $x \in \pi$. The map X (1.1.1) induces a map

$$TX: TR_{\pi} \longrightarrow TG \qquad \textbf{(1.2.9)}$$

Here we mean the tangent space and the tangent map for a variety and so the concept is well defined also for singular points.
We can also define for $\rho \in R_F$

$$\omega_x = T_\rho X \circ X(\rho)^{-1} : T_\rho R_\pi \longrightarrow \mathrm{Lie}G \qquad \textbf{(1.2.10)}$$

At any point ρ of R_π, LieG can be given the status of a left π-module with the action of $x \in \pi$ on $h \in \mathrm{Lie}G$ given by

$$x \circ h = \rho(x) h \rho(x)^{-1} \qquad \textbf{(1.2.11)}$$

We'll write $\mathrm{Lie}_\rho G$ to make this action evident.
Now we've

Prop. (1.2.12) The assignment (for a fixed $\rho \in R_\pi$)

$$\pi \to Lie_\rho G \otimes T_\rho^* R_\pi$$
$$x \to \omega_x$$

is a crossed homomorphism.

Remark (1.2.13) $T_\rho R_\pi$ has the structure of a left π-module (adjoint action). Then (as a vector space) $T_\rho^* R_\pi$ has a natural induced right-module structure. $Lie_\rho G \otimes T_\rho^* R_\pi$ is the tensor product of the left π-module $T_\rho^* R_\pi$ and the right $Lie_\rho G$.

As $x \to \omega_x$ is a crossed homomorphism between π and $Lie_\rho G \otimes T_\rho^* R_\pi$ (prop.(1.2.7)) we've an induced π-homomorphism

$$d\pi \to Lie_\rho G \otimes T_\rho^* R_\pi$$

If $h^* \in (LieG)^*$ we can apply h^* to ω_x obtaining an ordinary cotangent vector at $\rho \in R_\pi$. So we've built a map (for a fixed $\rho \in R_\pi$)

$$i_\rho: Lie_\rho^* G \otimes d\pi \to T_\rho^* R_\pi$$
$$h^* \otimes dx \quad \to h^*(\omega_x)$$

Theor. (1.2.14) (see [Jo] page 21) At $\rho \in \mathbf{R}_\pi$ i_ρ is an isomorphism (of vector spaces). In the case π free this means at any ρ.

proof: We first consider the case that π is a free group F on $\{x_i\}_{i=1....m}$.

CLAIM: dF is a free group on $\{dx_i\}_{i=1....m}$.
This is an application of Fox free differential calculus

First of all dF is generated by $\{dx_i\}_{i=1....m}$. Define then a map (Fox free derivative)

$$\text{(1.2.15)} \qquad \partial/\partial x_j : \quad F \longrightarrow ZF, \qquad x_i \longrightarrow \delta_{ij}$$

(it is sufficient to define it on generators of F).
$\partial/\partial x_j$ is a crossed homomorphism and so (prop.(1.2.7)) we've an induced π-homomorphism $\partial/\partial x_j$: $dF \longrightarrow ZF$.
If we've any relation $\Sigma_i r_i dx_i=0$, by applying $\partial/\partial x_j$ we get $r_j=0$ in ZF $\forall$ j and so the only relation among the dx_i's is the trivial one. This concludes the proof of the claim.

If $\{h_k^*\}$ is a basis for Lie*G, then $Lie_\rho{}^*G \otimes dF$ has a basis (as a vector space) $l_j^* \otimes dx_i$, which is mapped, by definition, to the cotangents $l_j^*(dX_i \circ X_i(\rho)^{-1})$. This latter set form a basis of $T_\rho{}^*R_F$ in view of (1.1.2).
Let's consider the case of a finitely presented group π

$$\text{(1.2.16)} \qquad R \longrightarrow F \longrightarrow \pi \longrightarrow 0$$

where F is free on $\{x_i\}_{i=1....m}$ and R is free on $\{R_j\}_{j=1....n}$. We've an induced sequence

$$\text{(1.2.17)} \qquad R_R \xleftarrow{\alpha^*} R_F \xleftarrow{\beta^*} R_\pi \longleftarrow 0$$

which is exact in the sense that the image $\beta^*(R_\pi)$ exactly equals α^{*-1}(trivial homomorphism in R_R).
Finally we've a sequence

$$\text{(1.2.18)} \qquad T^*R_R \longrightarrow T^*R_F \longrightarrow T^*R_\pi \longrightarrow 0$$

But R_R and R_F are free groups and so we can build the diagram

$$
(1.2.19) \qquad \begin{array}{ccccccc}
\mathrm{Lie}^*G \otimes dR & \xrightarrow{A} & \mathrm{Lie}^*G \otimes dF & \longrightarrow & \mathrm{Lie}^*G \otimes d\pi & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
T^*R_R & \longrightarrow & T^*R_F & \longrightarrow & T^*R_\pi & \longrightarrow & 0
\end{array}
$$

where the upper arrows are defined on generators (e.g. for the first one)

$$
(1.2.20) \qquad A(h_k{}^* \otimes dR_j) = h_k{}^* \otimes d(\alpha(R_j))
$$

The diagram is commutative and then also the last vertical arrow is an isomorphism.

Q.E.D.

If we keep in mind the following algebraic lemma

Lemma (1.2.21) let M be a vector space over F on which a group π acts on the left. Then M^* is a right π-module and so we can define $M^* \otimes d\pi$. The pairing

$$
(1.2.22) \qquad \begin{array}{l}
M^* \otimes d\pi \quad \times \quad \mathrm{Hom}_\pi(d\pi, M) \longrightarrow F \\
(m^* \otimes dx, f) \longrightarrow \langle m^* \otimes dx, f \rangle = \langle m^*, f(dx) \rangle
\end{array}
$$

is F-bilinear and nonsingular.

proof: [Jo].

We can also take the dual of (1.2.19)

$$
(1.2.23) \qquad \begin{array}{ccccccc}
TR_R & \xleftarrow{T\alpha^*} & TR_F & \xleftarrow{T\beta^*} & TR_\pi & \longleftarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathrm{Hom}_\pi(dR, \mathrm{Lie}G) & \xleftarrow{A^*} & \mathrm{Hom}_\pi(dF, \mathrm{Lie}G) & \longleftarrow & \mathrm{Hom}_\pi(d\pi, \mathrm{Lie}G) & \longleftarrow & 0
\end{array}
$$

Remark (1.2.24) According to the remark (1.2.8) it is also possible to identify $T_\rho R_\pi$ with $Z^1(\pi, Lie_\rho G)$. At last

$$(1.2.25) \quad T_\rho R_\pi \cong Hom_\pi(d\pi, LieG) \cong XHom(\pi, LieG) \cong Z^1(\pi, Lie_\rho G)$$

The space R_π may have different connected components. For example

Coroll. (1.2.26) Let R_π^* be the set of points $\rho \in R_\pi$ s.t. the corresponding map A^* is onto (it may be empty). R_π^* is a smooth manifold of dimension (m-n)dimG (where m is the number of free generators for F and n for R).

proof: Given $\rho \in R_\pi \subset R_F$ s.t. A is onto, we've that $T\alpha^*$ is surjective at $T_\rho R_F$. As R_π is exactly α^{*-1}(trivial homomorphisms in R_R) by classical results of differential geometry (R_F and R_R are manifolds and α^* is a smooth map), there exists a neighborhood U_ρ of ρ in R_F s.t. $R_\pi \cap U_\rho$ is a smooth submanifold of dimension $\dim_\rho R_F - \dim_\rho R_R =$ (m-n)dimG. The result follows from noticing that $R_\pi \cap U_\rho = R_\pi^* \cap U_\rho$.

Q.E.D.

Remark(1.2.27) The identification of $T_\rho R_\pi^*$ with $Hom_\pi(d\pi, Lie_\rho G)$ can be given an explicit form. A tangent vector at $\rho \in R_\pi^*$ is given by the derivative at t=0 of a smooth curve of representations ρ_t with $\rho_0 = \rho$. To each $d\rho_t/dt|_{t=0}$ we associate a crossed homomorphism

$$(1.2.28) \quad \begin{aligned} \phi: \pi &\longrightarrow Lie_\rho G \\ x &\longrightarrow d\rho_t(x)/dt|_{t=0} \circ \rho(x)^{-1} \end{aligned}$$

In fact

$$\phi(xy) = d\rho_t(xy)/dt|_{t=0} \circ \rho(xy)^{-1} = d\rho_t(x)/dt|_{t=0} \circ \rho(x)^{-1} + \rho(x)\, d\rho_t(y)/dt|_{t=0} \circ \rho(y)^{-1}\rho(x)^{-1}$$

(the action of π on LieG is set up through conjugation). Finally remember that by (1.2.7) there is a natural isomorphism between

$Hom_{\pi}(d\pi, Lie_{\rho}G)$ and $XHom(\pi, Lie_{\rho}G)$. This association

$$T_{\rho}^{*}R_{\pi} \longrightarrow XHom(\pi, Lie_{\rho}G)$$

is exactly what is induced by (1.2.14).
By this construction we can also conclude that this isomorphism is canonical.

1.3 Conjugacy class spaces

Let π be a discrete group and G a Lie group.

G acts on Hom(π,G) by inner automorphisms and we'd like to make the set of conjugacy classes of representations a manifold with singularities.
Unfortunately when G is non-compact it is not generally true that $R_\pi{}^*/G$ is Hausdorff (R_π has the compact-open topology)

Example (1.3.1) ([Jo] page 24) $\pi=\mathbb{Z}$ $G=SL(2,\mathbb{R})$
Then $R_\pi \cong SL(2,\mathbb{R})$ (1.1.2).
Now $\begin{vmatrix}1 & 0\\0 & 1\end{vmatrix}$ and $\begin{vmatrix}1 & 1\\0 & 1\end{vmatrix}$ are not conjugate, but, for any $x\neq 1$ $\begin{vmatrix}x & 0\\0 & x^{-1}\end{vmatrix}$ and $\begin{vmatrix}x & 1\\0 & x^{-1}\end{vmatrix}$ are conjugate.
$\begin{vmatrix}x & 0\\0 & x^{-1}\end{vmatrix} \to \begin{vmatrix}1 & 0\\0 & 1\end{vmatrix}$ and $\begin{vmatrix}x & 1\\0 & x^{-1}\end{vmatrix} \to \begin{vmatrix}1 & 1\\0 & 1\end{vmatrix}$ for $x\to 1$ (in the compact-open topology for R_π) and so there're not two disjoint open sets containing respectively $\begin{vmatrix}1 & 0\\0 & 1\end{vmatrix}$ and $\begin{vmatrix}1 & 1\\0 & 1\end{vmatrix}$.
The way for avoiding this patology is to remove again some representations.
Suppose that

i) π is a finitely generated group
ii) G is a simple linear (i.e. it consists of matrixes) Lie group which acts irreducibly on a complex vector space V for each $x\in\pi$.

We can define a basic invariant (under conjugation) function on R_π, through the trace Tr:

(1.3.2) $\qquad Trx=Tr\bullet \mathbf{x}$

(Tr stands for the matrix trace and $\mathbf{x}$ is defined in (1.1.1))

If we set

(1.3.3) R_{π}^{**} = {set of irr.repr.in R_{π}}

we've

Theor.(1.3.4) [Jo page 25]

i) There exists a finite set $x_1 \ldots x_k \in \pi$ s.t. $\rho_1, \rho_2 \in R_{\pi}^{**}$ are conjugate repr. iff $Trx_i(\rho_1) = Trx_i(\rho_2)$ $1 \leq i \leq k$

ii) If $T = (Trx_1 \ldots Trx_k) : R_{\pi} \rightarrow \mathbb{C}^k$ then $T(R_{\pi}^{**})$ is a smooth real submanifold of $\mathbb{C}^k$, which we call **R**

iii) T restricted to R_{π}^{**} is a principal bundle projection with group structure $G/_{Z(G)}$ where Z(G) is the center of G.

It is also possible to obtain a more general result for a generic simple linear algebraic group defined over $\mathbb{R}$ [J-M page 55/56].
We'll use the same notation also for $R_{\pi}^{**} \cap \mathbf{R}$.
We can compute T**R** from (1.2.23),(1.2.25),(1.2.27)

Prop.(1.3.5) $T_{[\rho]}\mathbf{R} \cong H^1(\pi, Lie_{\rho}G)$

proof: Let's first compute the tangent space of a fiber through (1.2.27). It is the subspace in $T_{\rho}R_{\pi}*$ consisting of derivatives at t=0 of smooth curves ρ_t of representations satisfying:

i) $\rho_0 = \rho$

ii) $g(t)\rho g(t)^{-1} = \rho_t$ $\quad g:\mathbb{R} \rightarrow G$ smooth

Then we can compute its image in $XHom(\pi, Lie_{\rho}G)$.
We've

$$d\rho_t(x)/_{dt}\big|_{t=0} \bullet \rho(x)^{-1} = dg(t)/_{dt}\big|_{t=0} - \rho(x)\ dg(t)/_{dt}\big|_{t=0}\ \rho(x)^{-1} = h - x \bullet h$$

where we've set $h = dg(t)/_{dt}\big|_{t=0} \in LieG$ and used the notation"•" for the action of π on $Lie_{\rho}G$.
One checks that T_{ρ}(fiber) in the identification $T_{\rho}R_{\pi} \cong XHom(\pi, Lie_{\rho}G)$ is the set

(1.3.6) $T_\rho(\text{fiber})=\{f:\pi \to Lie_\rho G \mid f(x)=h-x\cdot h \quad \text{for } h\in LieG\}$

In the identification $T_\rho R_\pi \cong Z^1(\pi, Lie_\rho G)$

(1.3.7) $T_\rho(\text{fiber})= B^1(\pi, Lie_\rho G)$ (1-coboundaries)

As $R_\pi^{**} \to \mathbf{R}$ is a principal bundle we've the diagram

$$\begin{array}{ccccccccc} 0 & \to & T_\rho(\text{fiber}) & \to & T_\rho R_\pi & \to & T_{[\rho]}\mathbf{R} & \to & 0 \\ \textbf{(1.3.8)} & & & & & & & & \\ 0 & \to & B^1(\pi, Lie_\rho G) & \to & Z^1(\pi, Lie_\rho G) & \to & H^1(\pi, Lie_\rho G) & \to & 0 \end{array}$$

Remark (1.3.8)bis From (1.3.6) we've direct evidence that at ρ $\dim(\text{fiber})=\dim G-\dim\zeta(\rho)$ where $\zeta(\rho)$ is the centralizer of $\rho(\pi)$ in LieG.

Remark(1.3.9) If G has a finite center we've $LieG={}^{LieG}/_{Z(G)}$. By (1.3.4)iii), we've an identification

(1.3.10) $T_\rho(\text{fiber})\cong LieG$

and so an exact sequence

(1.3.11) $0 \to LieG \to Z^1(\pi, Lie_\rho G) \to H^1(\pi, Lie_\rho G) \to 0$

where α is the composite of the identification (1.3.10),(1.3.7) and the inclusion $B^1(\pi, Lie_\rho G) \to Z^1(\pi, Lie_\rho G)$.
This can also be seen by using the cohomology of groups.
We've

Prop.(1.3.12) [Jo page 27] π,G satisfying the hypotheses of this section $\rho\in R_\pi$. If ρ is irreducible and Z(G) is finite, then $H^0(\pi, Lie_\rho G)=0$.

So, from the cohomology theory we've an exact sequence

(1.3.13) $0 \rightarrow \text{LieG} \rightarrow Z^1(\pi, \text{Lie}_\rho G) \rightarrow H^1(\pi, \text{Lie}_\rho G) \rightarrow 0$

It turns out that $\alpha' = \alpha$ as

$$\alpha(h)\cdot(x) = h - x\cdot h$$

which is just δ_0 (1.2.5).

1.4 R for surfaces

In this section we consider in more detail the representation space of the fundamental group of a surface, as this is the object we'll be primarily concerned in the following.

Let M be a closed oriented surface (connected two-manifold) of genus g>1, G a Lie group on which there exists a symmetric bilinear non-degenerate form invariant *B*. Let $\pi=\pi_1(M)$. In this particular case we can say more about R_π, **R**.

Let D be an embedded two-disk in M, $p\in D$ a basepoint. We've the following diagram of natural inclusions

(1.4.1) $$(\partial(M-D),p) \longrightarrow (M-D,p) \longrightarrow (M,p)$$

and we can build the corresponding diagram for fundamental groups

(1.4.2) $$\pi_1(\partial(M-D),p) \xrightarrow{\alpha} \pi_1(M-D,p) \longrightarrow \pi_1(M,p) \longrightarrow 0$$

The surjectivity of the second map is due to the fact that M is obtained by M-D by adding a two-cell. Now $\pi_1(\partial(M-D),p)$ is a free group with one generator, $\pi_1(M-D,p)$ is a free group with 2g generators. So (1.4.2) is a presentation of $\pi_1(M,p)$. I can always choose 2g curves based at p $a_1 \ldots b_g$ and a generator x in M-D

(1.4.3) $$\alpha(x) = [a_1,b_1]\ldots[a_g,b_g]$$

and s.t. $\pi_1(M,p)$ is generated by $a_1 \ldots b_g$ with the condition $[a_1,b_1]\ldots[a_g,b_g] = 1$. We use the following notation for the sequence (1.4.2) in this realization

(1.4.4) $R \longrightarrow F \longrightarrow \pi \longrightarrow 0$

The map A^* (1.2.23) for a fixed $\rho \in R_F$ has the following eplicit expression

(1.4.5)
$$A^* : \mathrm{Hom}_\pi(dF, \mathrm{Lie}G) \longrightarrow \mathrm{Hom}_\pi(dR, \mathrm{Lie}G)$$
$$\phi \longrightarrow A^*(\phi)(dx) = \phi(d[a_1]\ldots[\quad b_g])$$

where we've to keep in mind that LieG is a F-module according to the representation ρ. Pose $\phi(da_1)=A_1 \ldots \phi(db_g)=B_g \in \mathrm{Lie}G$. Then

(1.4.6)
$$\phi(d[a_1b_1]\ldots[a_gb_g]) = A_1 + \rho(a_1)B_1 - \rho(a_1b_1a_1^{-1})A_1 - \rho(a_1b_1a_1^{-1}b_1^{-1})B_1 + \ldots$$

So that the orthogonal complement (in the sense of B) of the image of A^* contains a non-zero element $\gamma_h(dx) = h \quad h \in \mathrm{Lie}G$ iff h satisfies:

(1.4.7)
$$(h_1 - (a_1b_1a_1^{-1})A_1) \perp h \quad \forall A_1 \in \mathrm{Lie}G$$
$$(\rho(a_1)B_1 - \rho(a_1b_1a_1^{-1}b_1^{-1})B_1) \perp h \quad \forall B_1 \in \mathrm{Lie}G$$
$$\ldots$$

The map $\mathrm{Lie}G \longrightarrow \mathrm{Lie}G \quad k \longrightarrow k - \rho(a_1b_1a_1^{-1})k$ (corresponding to the first line) is of the form I-T where T is an orthogonal transformation of LieG. So $h \in \mathrm{Im}(I-T)^\perp = \mathrm{Ker}(I-T)^* = \mathrm{Ker}(I-T)$. Proceeding analogously for the other equations we've that h must belong to the intersection of the kernels of 2g operators and the only solution [GO1] is that h belongs to the centralizer in LieG of the elements $\rho(a_1)\ldots\rho(b_g)$. So we can conclude

<u>**Prop. (1.4.8)**</u> i) $\dim Z^1(\pi, \mathrm{Lie}_\rho G) = (2g-1)\dim G + \dim\zeta(\rho)$

($\zeta(\rho)$ stands for the centralizer of $\rho(\pi)$ in G)

ii) The set of points $Q=\{\ \rho\in Hom(\pi,G)\mid \dim^{\zeta(\rho)}/_{\zeta(G)}=0\}$ is a smooth submanifold of R_F

($\zeta(G)$ stands for the center of G)

Remark(1.4.9) We've selected a connected component of $\mathbf{R}_\pi$. In the particular case that G has a finite center, then $Q=R_\pi{}^{*}$.

I quote also the following result for surfaces.

Prop.(1.4.10) [Jo page 27] Let G as in (1.3). $R_\pi{}^{**}\subset\mathbf{R}_\pi$.

Another interesting feature of this particular case is that a surface M is an Eilenberg-Mac Lane space of type $K(\pi_1(M),1)$ (see e.g. [B] page 37). But for an Eilenberg-Mac Lane space the cohomology of the fundamental group is intimately related to the singular cohomology of the surface.

Notation. If F is a free $\mathbb{Z}G$-module with basis $\{e_i\}$, then we denote

F_G the free $\mathbb{Z}$-module F with basis $\{G\circ e_i\}$.

Def.(1.4.11) Given a group π by a π-complex we'll mean a complex K together with an action of π on the universal cover K which permutes the cells. K is a free π-complex if the action of π freely permutes the cells of K.

If K is a π-complex then the action of π on K induces an action of π on the cellular chain complex $C_*(K)$ which thereby becomes a chain-complex of π-modules.If K is a free π-complex $C_n(K)$ has a $\mathbb{Z}$-basis which is freely permuted by π,hence $C_n(K)$ is a free $\mathbb{Z}\pi$-module. To select a specific basis we must choose a representative cell from each orbit and we must choose an orientation of each representative.

We've

Theor.(1.4.12) If Y is a K(G,1)-complex with universal cover X,then

i) $C_*(X)_G=C_*(Y)$

ii) $C_*(X)$ is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$.

In particular this means:

(1.4.13) $\quad H_*(G) \cong H_*(Y) \quad$ (i and ii)

(1.4.14) $\quad H_*(G,M) \cong H_*(C_*(X) \otimes_G M) \quad$ (i)

(1.4.15) $\quad H^*(G,M) \cong H^*(\,Hom_G(C_*(X),M)) \quad$ (i)

for any G-module M (see[B] ch.3).
As an application of (1.4.20) we prove an application which will be useful later.
We know that for a finitely presented group like (1.1.6) we've an exact sequence (compare (1.2.23))

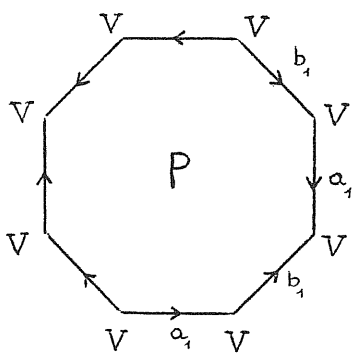
$$(1.4.16) \quad Hom_\pi(dR,LieG) \xleftarrow{A^*} Hom_\pi(dF,LieG) \longleftarrow Hom_\pi(d\pi,LieG) \longleftarrow 0$$

at any $\rho \in R_\pi$, for G Lie group. Now we've

Theor.(1.4.17) Let M be a closed oriented surface, G a Lie group satisfying the hypotheses of section 1.3 and with a finite center. Then for ρ irreducible A^* is onto.

proof: Let **M** be the universal cover of M. M is triangulizable as a finite cell-complex K with one 0-cell V, 2g 1-cells, one 2-cell P (we take the same generators $a_1 \ldots b_g$ as above)

(1.4.18)



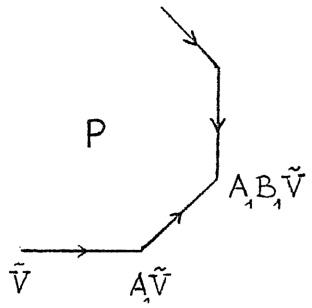
In view of (1.4.11) $C_*(M)$ is a complex of free $Z\pi_1(M)$-modules with as many generators as distinct cells in K. That is

$$(1.4.19) \quad 0 \longrightarrow \mathbb{Z}\pi \xrightarrow{\alpha} \mathbb{Z}\pi \otimes dF \longrightarrow \mathbb{Z}\pi \longrightarrow 0$$

(remind that dF is the free $\mathbb{Z}\pi$-module with 2g generators $da_1 \ldots\ldots db_g$)

For our purposes we need to describe only the behaviour of the map α.Choose as a generator of $C_2(\mathbf{K})$ the following lift of the 2-cell P

(1.4.20)



where A_1 (B_1 and so on) is the element of $\pi_1(K,V)$ corresponding to a_1(b_1 and so on).

Then $\quad \alpha(\mathbf{P}) = da_1 + A_1db_1 - A_1B_1A_1^{-1}da_1 + \ldots.$

But we can use (1.4.15) to compute $H^*(\pi, Lie_\rho G)$ for a fixed $\rho \in R_\pi$. Taking the π-homomorphisms of the previous chain into $Lie_\rho G$ we've

$$\textbf{(1.4.21)} \qquad Lie_\rho G \xleftarrow{\alpha^*} Z^1(F, Lie_\rho G) \longleftarrow Lie_\rho G$$

where

$$\textbf{(1.4.22)} \qquad \alpha^*(f) = f([a_1,b_1]\ldots.[a_g,b_g])$$

for each f crossed homomorphism from F to $Lie_\rho G$ (one has to make the reverse computation of (1.4.6)).

If M,G satisfy the hypotheses of the theorem then [Jo] $H^2(\pi_1(M), Lie_\rho G) = 0$ at $\rho \in R_\pi$ irreducible. So α^* is onto. But the left part of the sequence (1.4.21) coincide with the left part of the sequence (1.4.16) and so also A^* is onto.

<u>**Q.E.D.**</u>

<u>Remaek (1.4.23)</u> In the case G=SU(2) there is a tailored short proof of theor.(1.4.17) in [A-M] page 62.

Coroll.(1.4.24) M,G as in (1.4.25). For ρ irreducible the following two sequences are exact

(1.4.25)
$$0 \longleftarrow \mathrm{Lie}G \longleftarrow Z^1(F,\mathrm{Lie}_\rho G) \longleftarrow Z^1(\pi,\mathrm{Lie}_\rho G) \longleftarrow 0$$

(1.4.26)
$$0 \text{———} TR_\pi(\partial(\Sigma-D),p) \longleftarrow TR_\pi(\Sigma-D,p) \longleftarrow TR_\pi \longleftarrow 0$$

2 Natural structures on R

If we specialize to the case π=fundamental group of M closed oriented surface and G a Lie group satisfying the hypotheses of section (1.3) and Z(G) (the center of G) finite, it turns out that R possesses some natural structures which are deeply rooted in the topology of M.

In the first section we'll describe two technical tools which will be essential for the successive constructions. The first one is just the notion of compatible volume forms for an exact sequence of vector spaces. The second one is the procedure by which we can set up a canonical volume form on chain groups of free π-cell complexes.

In the second we use this construction for giving natural volume form and orientation to R_π^{**} and **R**. This interesting procedure is due to Dennis Johnson [Jo] (1989). The main idea is that the problem of giving a volume form to a representation space, as M is a $K(\pi_1(M),1)$-space, can be reduced to the problem of giving a volume form on chain groups of the universal covering of M.

Finally (third section) we endow **R** with a natural symplectic structure. This is the work of Goldman [GO1] (1984).

As **R** is oriented we can define Casson's invariant (Chapter 3).

The existence of a volume form allows us to give a generalization of it (Chapter 4).

The symplectic nature of **R** opens the way to a hamiltonian study of this space (Chapter 5).

2.1 Some technical tools

A) Compatible volume forms

Def.(2.1.1) Let V be a vector space of dimension n over a field F. A volume form on V is a choice of a non-zero element τ in $\wedge^n V$.

Let $$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

be a short exact sequence of vector spaces and suppose A,C have assigned volume forms τ_A, τ_C represented as $a_1 \wedge \ldots . a_n$, $c_1 \wedge \ldots . c_n$ for some basis $\{a_1 \ldots . a_n\}$ in A, $\{c_1 \ldots . c_m\}$ in C. Then we can naturally induce from τ_A, τ_C a volume form on B τ_B as

$$\textbf{(2.1.2)} \qquad \tau_B = a_1 \wedge \ldots . a_n \wedge b_1 \wedge \ldots . b_m$$

where $b_1 \ldots . b_m \in B$ project respectively on $c_1 \ldots . c_m$. In fact (2.2.1) doesn't depend on the choice of $b_1 \ldots . b_m$. Usually such an induced volume form is denoted as

$$\textbf{(2.1.3)} \qquad \tau_B = \tau_C / \tau_A$$

Similarly if τ_A, τ_B are given, there is a unique volume form on $\tau_C = \tau_A \tau_B$ and likewise for $\tau_A = \tau_B / \tau_C$. The construction may be extended with a little more effort to a generic exact sequence of vector spaces

$$0 \longrightarrow C_n \xrightarrow{\partial} C_{n-1} \ \partial \ \ldots . \ \partial \ \ C_0 \longrightarrow 0$$

For example, we consider the case that each vector space C_i $i=1....n$ has a given volume form v_i and we want to obtain a volume form v_0 for C_0. Now if we consider the short exact sequences

$$
\begin{array}{ll}
(2.1.4) & 0 \longrightarrow C_n \longrightarrow C_{n-1} \longrightarrow \partial C_{n-1} \longrightarrow 0 \\
& 0 \longrightarrow \partial C_{n-1} \longrightarrow C_{n-2} \longrightarrow \partial C_{n-2} \longrightarrow 0 \\
& \cdots\cdot \\
& 0 \longrightarrow \partial C_2 \longrightarrow C_1 \longrightarrow C_0 \longrightarrow 0
\end{array}
$$

we see that in each one we can build as above a volume form on the third element of the sequence, by using the assigned volume form on the central element and obtaining a volume form for the first element from the upperlying sequence. Finally we get a volume form for C_0. The same procedure can be extended to an exact sequence of vector bundles.

B) A canonical volume form for free π-cell complexes

Let now K be a finite cell complex and **K** the universal covering, with $\pi=\pi_1(K)$. Then K is a free π-complex and we can formally write (1.4)

$$(2.1.5) \qquad C_*(\mathbf{K}) = \mathbb{Z}\pi \otimes C_*(K)$$

that is $C_*(\mathbf{K})$ consists of free $\mathbb{Z}\pi$-modules with basis given by ordering and orienting the cells of K and choosing lifts of these cells to **K.**

To define coherently a concept of volume form or torsion (see later) for $C_*(\mathbf{K})$, we've to choose an $F\pi$-module M for some field F and consider the chain complexes $M^*\otimes_\pi C_*(\mathbf{K})$ (M^* has a natural structure of right $F\pi$-module). Suppose that we've chosen

i) orientation and ordering of the cells of K.

ii) lifts of the cells of K to a $\mathbb{Z}\pi$-basis of $C_*(\mathbf{K})$

iii) a basis $\{m_i\}_{i=1....d}$ for M.

Then we can select a basis in $M^*\otimes_\pi C_i(\mathbf{K})$

(2.1.6) $m_1^*\otimes\sigma_1, m_2^*\otimes\sigma_1 \ldots . m_d^*\otimes\sigma_1, m_d^*\otimes\sigma_2 \ldots . m_d^*\otimes\sigma$

(n is the number of i-cells in K; $\{m_j^*\}$ is the basis dual to $\{m_i\}_{i=1\ldots d}$)
The basis (2.1.6) gives rise to a volume form $m_1^*\otimes\sigma_1\wedge\ldots . m_d^*\wedge\sigma_n$ on $M^*\otimes_{\pi}C_i(\mathbf{K})$ which depends on i) ii) iii). But an algebraic analysis shows that this dependence is in fact less heavy.

Def.(2.1.7) An $F\pi$-module M is unimodular if π acts trivially on $\wedge^d M$ ($d=\dim_F M$).

Def.(2.1.8) Let G be a free abelian group of rank d; an orientation of A is a choice of a generator of $\wedge^d A$.

We've

Prop.(2.1.9) [Jo] If M is unimodular, the volume form associated to (2.1.6) depends only on the choice of a volume γ on M and an orientation t_i on $C_i(K)$.

This means that if we choose a basis $\{m_i\}_{i=1\ldots d}$ on M compatible with γ and order and orient i-cells in K according to t_i, the volume form (2.1.6) is well defined.

2.2 Canonical volume form and orientation for R

M,G as specified. Let Π be a free group. If we choose a basis of Π $x_1 \ldots x_n$, we know (Theor.(1.2.14)) that $d\Pi$ is free on $dx_1 \ldots dx_n$. So we can identify $d\Pi$ with $C_1(\mathbf{K})$ where $\mathbf{K}$ is the universal covering of a 1-complex K with one vertex and n edges. If M is a unimodular $F\pi$-module (F a field) we can construct a natural volume and orientation on $M^* \otimes_\pi d\Pi$, which depends only on the choice of a volume on M and an orientation t of $\Pi/_{[\Pi,\Pi]}$ (the abelianization of Π). From Prop.(2.1.9) we get the result. The identification of $d\pi$ with $C_1(\mathbf{K})$ depends on the choice of the basis in $d\pi$, but the volume form on $M^* \otimes_\pi d\Pi$ does not.

Prop.(2.2.1) π,M as above. There exists a canonical volume form and orientation on $M^* \otimes_\pi d\pi$ depending only on the choice of a volume form on M and an orientation on the abelianization of π.

By dualization (Lemma(1.2.22)) and previous results (prop.(1.2.7) and the following remark) we obtain a natural volume form and orientation on

$$Hom_\pi(d\pi,M) \cong XHom(\pi,M) \cong Z^1(\pi,M)$$

We'll turn now to the representation space R_π of a generic discrete group π finitely generated. Let's consider first the case of a free group π. In this case R_π is a manifold with tangent space $T_\rho R_\pi$

$\cong Z^1(\pi, Lie_\rho G)$ (1.2.25). So we can define pointwise a nowhere zero top degree form on R_π by the procedure seen above once we've fixed a volume form on LieG and an orientation t of the abelianization of π.

Consider now the case of the fundamental group of a closed oriented surface M., G a Lie group like in (1.3) with finite center. In this case it is possible to get a natural volume form on R_π^{**} ($\pi=\pi_1(M)$), using the volume form we've constructed on R_F for F free and the existence of a natural orientation on the abelianization of π.

Choosing an orientation for $H_1(M,\mathbb{Z})$ by Poincare' duality and (2.1.8) is equivalent to a choice of orientation for $H_1(M,\mathbb{R})$. But the pairing

$$\textbf{(2.2.2)} \qquad \omega: H_1(M,\mathbb{R}) \otimes H_1(M,\mathbb{R}) \longrightarrow \mathbb{R}$$
$$(u,v) \longrightarrow \langle u \cup v, \mu \rangle$$

where μ is the fundamental class of M, is non-degenerate and antisymmetric (i.e. it is symplectic). Then ω_g (g=genus of M) selects a natural orientation on $H_1(M,\mathbb{R})$. We know that π has the presentation (1.4.4) and F is a free group with 2g generators. According to the above construction we've a volume form on R_F which depends only on a chosen volume form on LieG, as ${}^F/_{[F,F]} \cong H^1(M,\mathbb{Z})$ has a natural orientation.
We've thus built a volume form on $Z^1(F,LieG)$ and by the canonical isomorphism (1.3.8) we've a volume form on TR_F.
But consider the following choices of orientation (notation of 1.4)

M−D   induced orientation from M

∂(M−D)   induced orientation from M−D

The orientation on ∂(M−D) determines a unique generator of $\pi_1(\partial(M-D),p)$ and so through (1.1.2) a well defined orientation on $R_{\pi(\partial(M-D),p)}$ as we've given to LieG a fixed orientation. By the sequence (1.4.26) and the method outlined in the previous section, we've a volume form on TR_π at irreducible ρ. Finally through the exact sequence (1.3.8) and the same procedure (consider (1.3.10)) we've a natural volume form and orientation on **R**.

2.3 A natural symplectic form on R

In addition to our hypotheses we require that G preserves a non-degenerate symmetric bilinear form *B* on LieG (in this case it is possible to deal with groups acting on real vector spaces).
According to (1.3.5), to give **R** a symplectic structure is equivalent to find a smoothly varying and closed antysimmetric non-degenerate bilinear form on $H^1(\pi_1(M), Lie_\rho G)$ for ρ irreducible.
But, pointwise, from cohomology theory we've the following dual pairing (dual by Poincare' duality and non-degeneracy of *B*)

(2.3.1) $\omega^{(B)}: H^1(\pi_1(M), Lie_\rho G) \times H^1(\pi_1(M), Lie_\rho G) \longrightarrow H^2(\pi_1(M), \mathbb{R}) = \mathbb{R}$

defined by the cup-product and using *B* as a coefficient homomorphism.
(2.3.1) defines a non-degenerate bilinear form on $H^1(\pi_1(M), Lie_\rho G)$.
The fact that $\omega^{(B)}$ is antisymmetric follow from the fact that the antisymmetry of the cup product (dimension 1) and the symmetry of *B*.
Finally we've

Theor. (2.3.2) $\omega^{(B)}$ is closed.

The proof of this fact will be given only in chapter 7 according to [A-B], [GO1].

Remark (2.3.3) If we take G=PSL(2,$\mathbb{R}$) $\omega^{(B)}$ retsricts to the Weyl-Petersson Kaehler form on Teichmueller space [GO1].

3 Casson's invariant

The natural orientation of **R** for the fundamental group of a surface and the Heegard decomposition of a 3-manifold allows us to define Casson's invariant for oriented homology 3-sphere (1985). This invariant is a generalization of Rohlin's invariant and gives surprising corollaries in low dimensional topology. We're not really interested in a careful analysis of the topological meaning of this invariant as the main goal of this thesis is trying to focus how many unconnected constructions have common roots in the natural structures of **R**. Actually, after clarifying the affinity among these different subjects of research, the following step might be to investigate the specific properties of each subject, and to look if this comparison sheds some reciprocal light on the matter. However I'll tell here a few words on the essential properties of this invariant and explain why topologists are deeply interested in them.

The most interesting feature of Casson's invariant is its behaviour with respect to surgery on knots.
There is a unique map

$$\lambda : \mathcal{M} \longrightarrow \mathbb{Z}$$

where $\mathcal{M}$ is the set of orientation preserving diffeomorphisms classes of oriented homology 3-spheres, s.t.

1) $\lambda(M,K_{n+1}) - \lambda(M,K_n) = \frac{1}{2}\Delta_K''(1)$ for any knot $K \subset M$, for any $M \in \mathcal{M}$. By (M,K_{n+1}) we denote the manifold obtained from M by $^1/_{n+1}$ surgery on K. $\Delta_K(t)$ represents the symmetrized Alexander polynomial of K.

2) $\lambda(S^3)=0$

Such a map (Casson's invariant) is unique as, in view of 1) and 2) we can compute $\lambda(M)$ for any oriented homology 3-spheres M: in fact any M is given by surgery on a framed link in S^3 with a diagonal linking matrix whose diagonal entries are all ± 1. The map λ necessarily satisfies some additional properties:

3) $\lambda(-M) = -\lambda(M)$ (-M denotes M with the opposite orientation)

4) $\lambda(M_1 \# M_2) = \lambda(M_1) + \lambda(M_2)$ (# denotes the connected sum)

5) $\lambda(M) = \mu(M) \bmod 2$ where $\mu(M)$ is the Rohlin's invariant.

The importance of the existence of $\lambda(M)$ lies in the outcomes it is possible to gain in two old-dated questions of low dimensional topology. First of all we've a negative answer to the attempts of building counterexamples to Poincare's conjecture with non-zero Rohlin's invariant:

Corollary A Any homotopy 3-sphere has zero Rohlin's invariant.

On the other side a new field of exploration has been opened in the sphere of triangulization of 4-manifolds

Corollary B There exist non triangulable 4-manifolds.

I close this flash exposition by listing some references which are now available on the subject.
For a general introduction see [A-M],{Ma1],[Ma2].
Recently Casson's invariant has been extended to homology lens spaces [B-L]. In [Mo1],[Mo2] Morita exhibits some singular bonds of this invariant with characteristic classes of surface bundles. The deep relations of Casson's invariant and the gauge theoretical context will be discussed later (section 8).

3.1 Casson's invariant

In this section we describe the construction of Casson's invariant according to the original outline of Casson.

Let W be a handlebody of genus g, $\Sigma = \partial W$, D an embedded 2-disk in Σ, $p \in D$ a basepoint.

Let $h: (\Sigma, D, p) \longrightarrow (-\Sigma, -D, p)$ an orientation preserving homeomorphism ("-" stands for the opposite orientation).

Def.(3.1.1) The Heegard model for the pair (W,h) is the following closed orientable 3-manifold:

$$M = \frac{W \times \{1\} \cup -W \times \{2\}}{<(x,1) \sim (h(x),2) \mid x \in S>}$$

It is a classical result of 3-manifolds theory:

Theor.(3.1.2) Every closed orientable connected 3-manifold is diffeomorphic to a Heegard model for some (W,h).

Given a Heegard model M we've the following diagram of natural inclusions:

(3.1.3)
$$(\Sigma - D, p) \longrightarrow (\Sigma, p) \rightrightarrows \begin{matrix} (W \times \{1\}, p) \\ (W \times \{2\}, p) \end{matrix} \rightrightarrows (M, p)$$

Then we can build the correspondent diagram for fundamental groups

(3.1.4)

$$\begin{array}{ccccc} & & \pi_1(W \times \{1\},p) & & \\ & \overset{i_2}{\nearrow} & & \overset{i_4}{\searrow} & \\ \pi_1(\Sigma - D,p) \xrightarrow{i_1} \pi_1(\Sigma,p) & & & & \pi_1(M,p) \\ & \underset{i_3}{\searrow} & & \underset{i_5}{\nearrow} & \\ & & \pi_1(W \times \{2\},p) & & \end{array}$$

Some remarks about this diagram

(3.1.4) i) $\pi_1(\Sigma - D,p)$ is a free group with 2g generators

$\pi_1(\Sigma,p)$ can be realized as a group with 2g generators $a_1 \ldots . b_g$ satisfying the single relation $[a_1,b_1] \ldots . [a_g,b_g]=1$

$\pi_1(W \times \{1\},p)$ is a free group with g generators

$\pi_1(-W \times \{2\},p)$

$\pi_1(M,p)$ can be obtained from $\pi_1(\Sigma,p), \pi_1(W \times \{1\},p), \pi_1(-W \times \{2\},p)$ by van Kampen's theorem.

(3.1.4) ii) All the maps are surjections.
For i_1, i_2, i_3 it follows from the fact the successive space is obtained from the previous one by attaching a 2 or a 3-cell; for i_4, i_5 it follows from van Kampen's theorem and the fact that i_2 and i_3 are surjective.

At last we can consider the dual diagram for representation spaces of these groups into SU(2) (the choice of this particular Lie group will be discussed later. SU(2) satisfies the hypotheses of section 1.3 and has discrete center)

$$(3.1.5) \qquad \begin{array}{ccccc} & & Q_1 & & \\ & \swarrow & & \nwarrow & \\ R^* \longleftarrow R & & & & R(M) \\ & \nwarrow & & \swarrow & \\ & & Q_2 & & \end{array}$$

where $R^*= R_{\pi(\Sigma-D,p)}$, $R_{\pi}=R_{\pi(\Sigma,p)}$, $Q_1=R_{\pi(W \times \{1\},p)}$, $Q_2=R_{\pi(-W \times \{2\},p)}$, $R(M) = R_{\pi(M,p)}$.
As all the maps in (3.1.4) are surjections , all the maps in (3.1.5) are inclusions. So we can naturally consider $R,Q_1,Q_2,R(M)$ as subsets of R^* and, in this sense we've

$$(3.1.6) \qquad R(M) = Q_1 \cap Q_2$$

From the remark i) following (3.1.4) and (1.1.2), we've that

$$(3.1.7) \qquad \begin{array}{l} R^* \cong SU(2)^g \cong \underbrace{S^3 \times \ldots S^3}_{2g \text{ times}} \\ Q_{1,2} \cong SU(2)^g \cong \underbrace{S^3 \times \ldots S^3}_{g \text{ times}} \end{array}$$

In particular R^* is a compact, connected , 6g dimensional, orientable manifold and Q_1,Q_2 are embedded compact, connected 3g-dimensional orientable submanifolds of R^*. So we can associate to $R(M)$ a natural topological invariant: the intersection number of Q_1 and Q_2 in R^*. Really this intersection number isn't very interesting: for getting something different from the topological information of $\pi_1(M)$, we've introduced representation spaces of $\pi_1(M)$ in some Lie group G, but the intersection number of Q_1 and Q_2 completely forgets the fact that we're dealing with representation spaces. In fact it depends only on the first homology group of M

<u>**Prop.(3.1.8)**</u> (We denote $\langle Q_1,Q_2\rangle_{R^*}$ the intersection number of Q_1 and Q_2 in R^*)

i) $\langle Q_1,Q_2\rangle_{R^*} = 0 \Leftrightarrow H_1(M,\mathbb{R})\neq 0$

ii) $|\langle Q_1, Q_2\rangle_{R^*}| = |H_1(M,\mathbb{Z})| \iff H_1(M,\mathbb{R})=0$

proof: it's only a computation [A-M page 58].

To make use of the new structure we consider the manifolds $\mathbf{Q}_1,\mathbf{Q}_2,\mathbf{R}$ (see 1.3). Notice that

i) $\mathbf{R}$ is a (6g-6)-manifold orientable by (3.1.4)ii, (1.1.2), (1.4.17), (1.3.4)

ii) $\mathbf{Q}_1,\mathbf{Q}_2$ are embedded (3g-3)-submanifolds orientable (3.1.4)ii, (1.1.2), (1.3.4)

Also in this case it would be natural to consider the intersection number of $\mathbf{Q}_1$ and $\mathbf{Q}_2$ in $\mathbf{R}$.

Unfortunately we can't guarantee that $\mathbf{Q}_1\cap\mathbf{Q}_2$ is a compact set, unless we require some additional hypotheses. I mean that, in general, it isn't possible to develop a good intersection theory. Consider the following two facts:

Prop. (3.1.9) Q_1 and Q_2 are transversal (as submanifolds of R^*) at the trivial representation iff $H_1(M,\mathbb{R})=0$.

proof: [A-M] page 58.

Prop. (3.1.10) Let G be a group. Every reducible representation of G in SU(2) is conjugate to a diagonal representation.

Remark (3.1.11) S^1 is naturally embedded in SU(2) by the map

$$i: S^1 \longrightarrow SU(2)$$
$$e^{i\theta} \longrightarrow \begin{matrix} e^{i\theta} & \\ & e^{-i\theta} \end{matrix}$$

By a diagonal representation ρ of G in SU(2), we mean that ρ can be written as $\rho=i\circ\sigma$ where σ is a representation of G in S^1.
From (3.1.9) and (3.1.10) we can deduce some interesting properties for the case in which M is a homology sphere, that is a connected, closed, orientable manifold with $H_1(M,\mathbb{Z})=0$. Given a Heegard model

for M, we've that Q_1 and Q_2 are transversal at the identical representation. Moreover as any reducible representation (in Q_1, Q_2 or R) is conjugate to a diagonal one (in particular it is abelian) it is also a representation of $H_1(M,\mathbb{Z})$ because $H_1(M,\mathbb{Z})$ is the abelianization of $\pi_1(M)$.
But, by hypothesis, $H_1(M,\mathbb{Z})=\{e\}$ and so the only reducible representation of R and that Q_1 and Q_2 are transversal at it, we deduce that $\mathbf{Q}_1\cap\mathbf{Q}_2$ is compact: $Q_1 \cap Q_2$ is compact and we get rid of the isolated point {e}, then quotienting by the action of G.
Under these hypotheses the intersection number of $\mathbf{Q}_1$ and $\mathbf{Q}_2$ in **R** is well defined and we can set:

<u>**Def.(3.1.12)**</u> Let M be an oriented homology 3-sphere and (W,h) a Heegard model for M. We define

$$\lambda(M,W,h) = \frac{(-1)^g\langle \mathbf{Q}_1,\mathbf{Q}_2\rangle_{\mathbf{R}}}{2\langle Q_1,Q_2\rangle_{R^*}}$$

where g is the genus of W.

We'll discuss in some detail the problem of orientation in (3.1.12) as it is important for our further developments.
Consider the following choices of orientation:

W	induced orientation from M
Σ	induced orientation from W
Σ-D	induced orientation from Σ
$\partial(\Sigma-D)$	induced orientation from Σ-D

The orientation on $\partial(\Sigma-D)$ determines a unique generator of $\pi_1(\partial(\Sigma-D),p)$ and so, through (1.1.2), a well defined orientation on $R_{\pi(\partial(\Sigma-D),p)}$, if we give $SU(2)\cong S^3$ a fixed orientation.
Finally we've an analogous exact sequence for $Q_1(Q_2)$ (1.3.8) and so, from an arbitrary orientation on Q_1 and (1.3.10) we can build a well defined orientation on $\mathbf{Q}_1$.
We can summarize all these constructions in this way: if we fix an orientation on S^3 we've

i) a well defined procedure for getting from an arbitrary orientation t on $Q_1 (Q_2)$ an orientation on $\mathbf{Q}_1 (\mathbf{Q}_2)$ (depending on t)

ii) a well defined procedure (depending on the orientation of M) for getting from an arbitrary orientation t on R* an orientation on **R** (depending on t and through the procedure also on M).

At this point we can go back to (3.1.12). What we mean by this formula is:
compute the intersection numbers $<\mathbf{Q}_1,\mathbf{Q}_2>_{\mathbf{R}}$, $<Q_1,Q_2>_{R*}$ with respect to any assigned triples of orientations t_1,t_2,t_3 for $\mathbf{Q}_1,\mathbf{Q}_2,\mathbf{R}$, $\underline{t}_1,\underline{t}_2,\underline{t}_3$ for $Q_1,Q_2,R*$ provided that $\underline{t}_1$ and t_1, $\underline{t}_2$ and t_2, $\underline{t}_3$ and t_3 are compatible in the sense just seen.
Then (3.1.12) is well defined.

To make the number $\lambda(M,W,h)$ an invariant for M we've to check that it is independent from the chosen Heegard model for M. In fact this is true:

Theor.(3.1.13) Let M be an oriented homology 3-sphere and (W,h) (W',h') two Heegard models for M. Then $\lambda(M,W,h) = \lambda(M,W',h')$

The proof of this theorem ([A-M] page 75) is only a careful computation which depends on the basic fact that any two Heegard models of an orientable closed 3-manifold are stably equivalent.

So we can set

Def.(3.1.14) If M is an oriented homology 3-sphere we define the Casson's invariant for M $\lambda(M)$ to be

$$\lambda(M) = \lambda(M,W,h)$$

for any Heegard model (W,h) for M.

In view of the construction $\lambda(M)$ is an invariant for orientation preserving homeomorphisms.

3.2 Casson's invariant and the canonical orientation form

We substantiate our claim that Casson's invariant is an outcome of the existence of a natural orientation on **R**.
Let M be a closed oriented connected 3-manifold and (W,h) a Heegard model for M (3.1.2). We know that $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are embedded (3g-3) orientable submanifolds of **R**. By the methods of section (2.2) we've natural orientations t_1, t_2, t on Q_1, Q_2, R^* $\mathbf{Q}_1, \mathbf{Q}_2, \mathbf{R}$ if we've fixed a volume form on LieG. But while t doesn't depend on any other choice, t_1, t_2 depend on the choice of an orientation respectively in the abelianizations of $\pi_1(W \times \{1\}, p)$, $\pi_1(W \times \{2\}, p)$. However it is possible [Jo] for a chosen orientation t_1 to give a "naturally" paired orientation t_2.
From the reduced Mayer-Vietoris sequence for $W \times \{1\} \cup W \times \{2\} = M$ we get the following exact sequence

(3.2.1) $$0 \longrightarrow H_2(M,\mathbb{R}) \longrightarrow H_1(\Sigma,\mathbb{R}) \longrightarrow H_1(W \times \{1\},\mathbb{R}) \oplus H_1(W \times \{2\},\mathbb{R}) \longrightarrow H_1(M,\mathbb{R}) \longrightarrow 0$$

But $H_1(M,\mathbb{R})$ and $H_2(M,\mathbb{R})$ are dual and if V is a vector space with a definite orientation (or volume form) described by a basis $a_1 \wedge \ldots . a_n$, V^* has a well defined volume form (orientation) $a_1^* \wedge \ldots . a_n^*$ (in fact one verifies that there is not dependence on the choice of the representation basis). So $H_1(M,\mathbb{R})$, $H_2(M,\mathbb{R})$ may be given dual orientations. We know (2.2) that $H_1(\Sigma,\mathbb{R})$ has a natural orientation and so (2.1) we've an induced orientation on H_1(W x

$\{1\},\mathbb{R})\oplus H_1(W \times \{2\},\mathbb{R})$. This orientation doesn't depend on the choice of an arbitrary orientation in $H_1(M,\mathbb{R})$ (or $H_2(M,\mathbb{R})$). This shows how to get a paired orientation t_2 for $\mathbf{Q_2}$ given t_1 on $\mathbf{Q_1}$ (Q_1 and Q_2).

__Prop.(3.2.2)__ Suppose Q_1, Q_2 are endowed with paired orientations, R* has the natural orientation described in (2.2) (we've a fixed volumeform on LieG). Then

$$\langle Q_1,Q_2\rangle_{R^*} = (-1)^g \left| H_1(M,\mathbb{Z}) \right| \quad \text{(cardinality of } H_1(M,\mathbb{Z}))$$

proof: Choose 2g curves based at p in $\Sigma-D$ $a_1....a_g b_1....b_g$ that generate $\pi_1(\Sigma,p)$. As Σ is oriented (see 3.1) we've a a well defined intersection pairing on $H_1(\Sigma,\mathbb{Z})$ (compare with (3.2.2))

$$\textbf{(3.2.3)} \qquad (\ ,\ ):\ H_1(\Sigma,\mathbb{Z}) \times H_1(\Sigma,\mathbb{Z}) \longrightarrow \mathbb{Z}$$

given by cup product and Poincare' duality. We assume that (,) has the following matrix with respect to the basis $a_1....a_g b_1....b_g$

$$\textbf{(3.2.4)} \qquad \begin{vmatrix} 0 & I \\ -I & 0 \end{vmatrix}$$

By our choices (2.2) $a_1\wedge....a_g\wedge b_1....b_g$ defines the positive orientation in $H_1(M,\mathbb{R})$. So if $m_1^*\wedge....\wedge m_g^*$ is a volume form for LieG*, the positive orientation in LieG*$\otimes$dF (where $F=\pi_1(\Sigma-D,p)$ and we've taken any $\rho\in R_\pi$) is determined by

$$\textbf{(3.2.5)} \quad (m_1{}^*\otimes a_1)\wedge (m_1{}^*\otimes a_1)\wedge \ldots.\ (m_1{}^*\otimes a_2)\wedge \ldots.\ (m_g{}^*\otimes b_g)$$

Through the pairing (1.2.21), (3.2.5) becomes

$$\textbf{(3.2.6)} \qquad f_{a_1 m_1}\wedge\ldots..f_{a_g m_g}$$

where $f_{a_1 m_1} \in Hom_{\pi}(dF, LieG)$ sends a_1 to $m_1 \in LieG$ (and the other elements of the basis to zero). Keeping in mind (1.2.27) we've that the orientation specified by (3.2.6) on R* is the same as the one given by identifying R* with G^{2g} by (1.1.2) for the basis $a_1 \ldots a_g b_1 \ldots b_g$.

So if we choose 2g curves $a_1 \ldots b_g$ based at p in Σ-D generating $\pi_1(\Sigma, p)$ and with the intersection matrix (3.2.4) R* can be identified with G^{2g} through these curves compatibly with the natural orientation.
Analogously if we make any choice of g curves $c_1 \ldots c_g$ in Σ-D based at p generating $\pi_1(W \times \{1\}, p)$ and we adopt as a basis of $\pi_1(W \times \{2\}, p)$ the curves $h(c_1) \ldots h(c_g)$, the orientations that Q_1 Q_2 get from the identification $Q_1 \cong G^g$ through $c_1 \ldots c_g$, $Q_2 \cong G^g$ through $h(c_1) \ldots h(c_g)$ are paired in the sense described at the beginning of the section.
So to compute $<Q_1, Q_2>_{R*}$ with prescribed orientations it is sufficient to select 2g curves $a_1 \ldots b_g$ based at p with properties

i) $b_1 \ldots b_g$ generate Q_1

ii) $h(b_1) \ldots h(b_g)$ generate Q_2

iii) (,) in (3.2.3) with respect to the basis $a_1 \ldots a_g b_1 \ldots b_g$ has the matrix (3.2.4)

For the identifications achieved through i) ii) iii) it is known that ([A-M]) $<Q_1, Q_2>_{R*} = (-1)^g \,|H_1(M, \mathbb{Z})|$

Q.E.D.

In view of (3.2.2) we can state

Coroll. (3.2.7) Let M be an oriented homology 3-sphere. Then $\lambda(M) = 1/2 <\mathbf{Q}_1, \mathbf{Q}_2>_{R*}$ for any Heegard model (W,h) for M, where the intersection number is computed with respect to the natural orientation of **R** and paired orientations for $\mathbf{Q}_1, \mathbf{Q}_2$.

The coroll.(3.2.7) explains that $\lambda(M)$ is simply an intersection number on representation spaces associated to a Heegard decomposition of M. Of course it is always possible to find for (W,h) arbitrary orientations for $Q_1, Q_2, R*$ s.t. $<Q_1, Q_2>_{R*} = (-1)^g$: what prop.(3.2.2) asserts is that there is a canonical way of selecting such a triad of orientations.

3.3 Some forethoughts

From our comparative point of view, it is somewhat unpleasant that Casson's invariant relies on the specific choice of the group SU(2). A natural question arises as a first subject to be investigated: is it possible to generalize this construction to representation spaces in SU(n) or, more widely, to compact simple groups?

The characteristic properties of SU(2) which are appealed to for setting up the invariant are the following:

i) the reducible SU(2)-valued representation are abelian.

ii) SU(2)-{1} is a contractible space.

i) is requested to found on a reasonable basis the intersection theory of sets of irreducible repr. of handlebodies in R_{π}. This doesn't seem to be a formidable obstacle to surpass.

ii) plays a more delicate role in proving property 1) of Casson invariant (see introduction) which is its peculiar feature (see [A-M] or [Ma1],[Ma2]). Here one has to be more careful and probably to find out a different demonstration path for bypassing the use of this contractibility property.

In any case the problem is an intriguing one.

While ending this thesis (august 1989) I've received some notes [CLM] announcing a generalization of Casson invariant to SU(n) and claiming even further extensions to any compact semisimple Lie group. The authors disclose to make use of symplectic geometry techniques in their work, but detailed proofs are not available and I'm not able to discuss it.

However such developments give rise, as it will be clear later, to new interesting questions about the connections between this construction and the other subject we'll deal with in the following.

4 Johnson's invariants

Trying to generalize Casson's invariant, D.Johnson (1989) has introduced a series of new invariants which still ask for a complete settlement and of which the importance is not yet fully understood.
The key tool for this construction is the existence of a natural volume form on **R** and the possibility of pairing orientations for $\mathbf{Q}_1$ and $\mathbf{Q}_2$. The aim of this chapter is to have a look at what is doing a working topologist.

4.1 Reidemeister torsion

This section is closely related to (2.1) and shows how to associate an integer (the torsion) to a finite exact sequence of finite dimensional vector spaces over F. By mimicing this procedure we can produce, following [Jo], a generalization of Casson invariant (section 4.3).
We've seen that, given an exact sequence of vector spaces over a field F

$$0 \longrightarrow C_n \longrightarrow \qquad C_0 \longrightarrow 0$$

we can produce canonically a volume form v_0 on C_0 if each of the vector spaces C_i $i=1....n$ has a given volume form v_i. On the other side if C_0 also has an assigned volume form θ_0 we can define a non-zero element in F (torsion of C*)

(4.1.1) $$\tau(C^*) = v_0 / \theta_0$$

$\tau(C^*)$ depends on the choice of the volume forms. Changing v_i to $f_i v_i$ $f_i \in F \setminus \{0\}$, τ changes to

(4.1.2) $$f_0 f_1^{-1} f_2 .... f_n^{\pm 1} \tau$$

Let's consider now the case of a cell complex **K** on which a group π acts freely, s.t. K=**K**/π is a finite complex (like in 3.1). We've seen that, if M is an unimodular Fπ-module for F a field, we can canonically construct a volume form on $M^* \otimes_\pi C_i(\mathbf{K})$ and this volume form depends only on the choice of a volume form θ in M and an orientation t_i on $C_i(K)$ (prop.(3.1.6)).

Now we want to study the torsion of the sequence of chain complexes

$$\textbf{(4.1.3)} \qquad M^*\otimes C_1(\mathbf{K}) \longrightarrow \qquad \longrightarrow M^*\otimes C_n(\mathbf{K})$$

in the hypothesis it is exact (note that we're not requiring $C^*(K,\mathbb{Z})$ be acyclic).

We've

Def.(4.1.4) An orientation of $C^*(K)$ is an orientation of $\oplus C_i = C_0\oplus \dots . C_n$.

Prop.(4.1.5) The torsion of (4.1.3) depends only on the choice of an orientation on $C^*(K)$.

In the case that K is the cell complex associated to an oriented manifold M, we've a canonical orientation of $C^*(M)$, thus making the sign of torsion well defined. It is equivalent to study orientation on $C^*(K,\mathbb{Z})$ or $C^*(K,\mathbb{R})$. Now an orientation on $C^*(K,\mathbb{R})$ is completely determined by an orientation h on $H^*(K,\mathbb{R}) = \oplus H_i(K,\mathbb{R})$.
First of all, one chooses orientations h_i on $H_i(K,\mathbb{R})$ s.t. they represent h. Then use the following short exact sequences (exactly in the same way as for volumes in (3.1))

$$\textbf{(4.1.6)} \qquad \begin{array}{ccccccccc} 0 & \longrightarrow & H_n & \longrightarrow & C_n & \longrightarrow & \partial C_n & \longrightarrow & 0 \\ 0 & \longrightarrow & \partial C_n & \longrightarrow & Z_{n-1} & \longrightarrow & H_{n-1} & \longrightarrow & 0 \\ 0 & \longrightarrow & Z_{n-1} & \longrightarrow & C_{n-1} & \longrightarrow & \partial C_{n-1} & \longrightarrow & 0 \\ & & \dots\dots & & & & & & \\ 0 & \longrightarrow & \partial C_1 & \longrightarrow & C_0 & \longrightarrow & H_0 & \longrightarrow & 0 \end{array}$$

for getting an orientation t on C^* (it is possible to fix arbitrary orientations b_i on ∂C_{i+1} $0\le i\le n-1$ as t doesn't depend on this choice).
Finally notice that the orientation of a manifold induces a natural orientation on $H^*(K,\mathbb{R})$. Suppose dimX=n isn't a multiple of 4. Then choose arbitrary orientations for $H_0, H_1 \dots . H_s(K,\mathbb{R})$ for $s=[^{n-1}/_2]$ (that is the homology groups below the middle dimension). The groups H_n, $H_{n-1}, \dots . H_{n-k}(K,\mathbb{R})$ acquire canonical orientation from their dual group. The total orientation is well defined, that is it doesn't depend on the choice of orientations on $H_0, H_1 \dots . H_s$. It may

happen that n is even and we've to deal with $H_{n/2}$. But as $n \neq 4k$ $n/2$ is odd and so the self-duality on $H_{n/2}$ is an antisymmetric form ω. But then we've a canonical orientation $\wedge^{n/2}\omega$ on $H_{n/2}{}^*$ (and so on $H_{n/2}$).

We can collect all these constructions in one proposition.

<u>**Prop.(4.1.7)**</u> [Jo] Let X be a closed oriented manifold of dimension n not divisible by 4, K a cell complex for X. Then there is a well defined torsion on $M^* \otimes_{\pi} C_*(\mathbf{K})$ for M $F\pi$-unimodular (M-valued Reidemeister torsion of X).

4.2 Johnson's invariants

Consider the framework of chapter 3. We've seen (3.2) that on **R** there exists a natural volume form τ and it is possible to define canonically paired orientations τ_1, τ_2 for $\mathbf{Q}_1, \mathbf{Q}_2$. Actually the intersection number $<\mathbf{Q}_1, \mathbf{Q}_2>_{\mathbf{R}}$ is twice Casson's invariant. To make the most of the existence of natural volume forms one defines a torsion number on analogous lines.
First of all, remind that the particular choice of SU(2) is a technical point related to a good definition of intersection theory and the typical behaviour of λ with respect to surgery on knots. in this case we've not to ask for some particular properties of the Lie group G and we can go back to the general hypotheses of section (1.3). Then

Def.(4.2.1) Let M be a closed oriented 3-manifold with a Heegard model (W,h), G a Lie group like in (1.3). Then at each $\rho \in Q_1 \cap Q_2$ we define the torsion number

$$t_\rho = \frac{\tau_1 \wedge \tau_2}{\tau}$$

(in fact $t_\rho = 0$ if $\mathbf{Q}_1, \mathbf{Q}_2$ are not transversal at ρ).

From the previous discussion (3.2) we know that the value of t_ρ only depends on the choice of a volume form ξ on $\text{Lie}_\rho G$. Actually by looking at the definition itself we conclude that it doesn't depend on ξ.

So the value of t_ρ is well defined at any $\rho \in Q_1 \cap Q_2$.

The main result of Johnson [Jo] connects t_ρ at transverse ρ with the $Lie_\rho G$-Reidemeister torsion of M.
In fact there exists an adapted cell-complex representation for an orientable closed 3-manifold M for each Heegard model (W,h). It is given as a handle-decomposition with one 0-handle V, g 1-handles A_i, g 2-handles P_k and one 3-handle U. If we denote $V \cup \{A_i\} = H$ (a genus g-handlebody) and $K = \partial H$ it is possible to choose a geometric basis p_k g_k of curves based at V on K s.t. the 2-handle P_k is glued to p_k. Let r_k, s_k the images of p_k, g_k in $\pi_1(H)$. Then $C_*(\mathbf{M})$ looks like $(\pi = \pi_1(M))$

$$\textbf{(4.2.2)} \qquad 0 \longrightarrow \mathbb{Z}\pi \xrightarrow{\;1 \to \Sigma(s_k - 1)dP_k\;} {}^{\pi}dR \xrightarrow{\;dP_k \to d\tau_k\;} {}^{\pi}dF \longrightarrow \mathbb{Z}\pi \longrightarrow 0$$

where ${}^{\pi}dR$, ${}^{\pi}dF$ are free $\mathbb{Z}\pi$-modules with 2g generators.
Using (4.2.2) and an exceptionally detailed computation Johnson proves

<u>Theor.(4.2.3)</u> At a transverse representation $\rho \in Q_1 \cap Q_2$

$$t_\rho = (-1)^d \tau_\rho$$

where τ_ρ is the Reidemeister torsion of M with $Lie_\rho G$ coefficients and d is the dimension of G.
(In fact at transverse ρ $Lie_\rho G \otimes_\pi C^*(\mathbf{K})$ is acyclic).

If $\mathbf{Q}_1, \mathbf{Q}_2$ were transversal we'd have (for M homology 3-sphere) $\lambda(M) = \Sigma_\rho \mathrm{sign} t_\rho$. This motivates the discussion of expressions like $\Sigma_\rho t_\rho$ or, more in general, polynomials

$$\textbf{(4.2.4)} \qquad T(t) = \Pi (t - t_\rho)$$

where the product is taken over all transversal ρ.
In fact if $\rho \in R$ isn't transversal t_ρ (so there is an intrinsic way of selecting transversal ρ).
(4.2.4) doesn't depend on the Heegard model we've adopted for M as

a consequence of theor.(4.2.3) and the fact that transversal ρ are exactly $\rho\in\Pi$ s.t. the $Lie_\rho G$-torsion is zero.
We're here still in the area of research and study.
Various remarks can be done.

1) There is no reason for being reduced to $Lie_\rho G$ coefficients. For any representation of G on a complex vector space V, we can get a Π-module V_ρ at any $\rho\in\Pi$, computethe Reidemeister torsion of M with V_ρ-coefficients and assembling the polynomial (4.2.4) at ρ with non-zero torsion.

2) Actually there might be an infinite number of $\rho\in\Pi$ with non-zero torsion and (4.2.4) be not well defined. The idea of Johnson is to restrict our attention to the 0-dimensional subset $\mathbf{R}_0\subset\mathbf{R}$ of the representation space. It is not yet clear how much unnatural is this restriction.

So we're faced with a whole family of new invariants and it is interesting to know if they keep any track of their origin by exhibiting some characteristic behaviour with respect to surgery on knots like Casson invariant.
I quote the following result for the case $G=SL(2,\mathbb{C})$ and the natural representation of G on $\mathbb{C}^2$.

Theor.(4.2.5) Let $s_\rho = {}^1/_2 t_\rho$ and S the polynomial

$$S=\Pi(s-{}^1/s_\rho)$$

Then we've the following recursive formula for surgery on the trefoil knot:

$$S_{-1} = -s^2 + 3s - 1$$
$$S_0 = 1$$
$$\cdots$$
$$S_{n+1} = DS_n - S_{n-1} \qquad \text{where } D = s^3 - 6s^2 + 9s - 2$$

4.3 Some forethhoughts

The reference on which the chapter is based [Jo], is a manuscript in circulation still under examination. Many features of the matter stimulate questions or reflections: the role of transversality in the construction of these invariants, the meaning of changing coefficients, the reasons of the presence of the curious factor 1/2 in the definition of these and Casson invariant.

I think that it might be interesting to reinterpret these invariant in the gauge-theoretical context emulating the work of Taubes [T] for Casson invariant (see chapter 8). It is not a purely academic exercise because, with the tools available in Fredholm theory of operators, the machinery of transversality is much better managed.

5 Towards knot theory invariants

The symplectic structure on **R** is a tool which looks very promising and not yet completely understood. In this and the following section, we pursue two completely different paths for gaining fruitful developments from symplectic geometry.

Here we adopt a heuristic dual point of view and try to understand **R** by studying the algebra of functions on it.
In a vague sense there is in fact an equivalence between the space M and the algebra of functions defined on M.
In the physical language we're speaking of classical observables on M.
That **R** is a configuration space worth of studying will appear evident in section 8, after understanding its role in gauge theoretical context ant the analysis of instantons.
But what is striking (Turaev 1989) is that one can quantize (algebrically) a certain subalgebra of the algebra of observables and this quantization is the skein module of an oriented 3-manifold.
This result opens a completely new perspective, still to be investigated, on the bonds of orur representation space and these invariants of knot theory.

5.1 Symplectic geometry on R

Let M be a closed oriented surface of genus g>1, G a Lie group (as in 2.3). We've seen that, if LieG is endowed with a non-degenerate symmetric bilinear form *B* invariant under AdG, then we can define a natural symplectic structure on **R** ($\pi=\pi_1(M)$).

We want to study [GO2] the Hamiltonian flow associated to this family of functions: for each $x\in\pi$, $f:G \longrightarrow \mathbb{R}$ a conjugation invariant smooth function, define

(5.1.1)

$$f_x:\mathbf{R} \longrightarrow \mathbb{R}$$

$$[\Phi] \longrightarrow f[\Phi(x)]$$

Really f_x depends only on the conjugacy class of $x\in\pi_1(M)$.

The set of conjugacy classes of $\pi_1(M)$ is in a natural bijective correspondence with the set π of free homotopy classes of oriented closed curves in M. So it is also well defined a correspondence

(5.1.2) $$f_\alpha:\ \mathbf{R} \longrightarrow \mathbb{R}$$

taking α a free homotopy class of closed oriented curves in the surface.

When α is a simple closed curve, then the Hamiltonian flows of a function f_α admit an explicit description. This is in fact a natural family of vector fields on $R_\pi{}^{**}$ covering the hamiltonian vector fields on **R** (on Teichmueller space they correspond to Fenchel-Nielsen twist flows [W1]).

We can compute the Poisson bracket of two observables f_α, f'_β for $f,f':G \longrightarrow \mathbb{R}$ conjugation invariant smooth functions. The computation can be found in [GO2]

Prop. (5.1.3)

$$\{ f_\alpha, f'_\beta\}: \mathbf{R} \longrightarrow \mathbb{R}$$

$$[\Phi] \longrightarrow \Sigma_{p\in a\#b}\varepsilon(p)B(F(\Phi(\alpha_p)), F'(\Phi(\beta_p)))$$

I explain the notation used in (5.1.3).

i) α,β are free homotopy classes of oriented closed curves.
- choose two transversely intersecting embedded representatives with transverse double points α,β for these two classes
- for each intersection point p ($p\in a\#b$) select closed curves $\alpha_p, \beta_p \in \pi_1(M,p)$ representing α,β.
- pick up a representative $\Phi:\pi_1(M,p) \longrightarrow G$ for $[\Phi]$.

ii) $F:G \longrightarrow \mathrm{Lie}G$ is the composition $F=\mathbf{B}^{-1}\bullet\mathbf{F}$ of the following two maps:

(5.1.4)

$$\mathbf{F}:G \longrightarrow \mathrm{Lie}G^*$$

$$g \longrightarrow \mathbf{F}(g)h = {}^d/dt\,|_{t=0}f(g\exp th)$$

(5.1.5)

$$B:\mathrm{Lie}G \longrightarrow \mathrm{Lie}G^*$$

$$h \longrightarrow B(h,\ ) \qquad \text{(analogously for F')}$$

iii) $\varepsilon(p)$ stands for the intersection number of a,b at p.

In view of equivariance of F and invariance of B the expression in (5.1.3) doesn't depend on the choices in i); the reason by which the formula looks like that is that we've used the standard geometric description of Poincare' duality in terms of intersection of cycles.

Remark (5.1.6) On Teichmueller space (5.1.3) reduces to known basic formulas ([GO2], [W2]).

It is inspiring to specialize (5.1.3) to some particular cases. The most "natural" invariant function on G to be considered is the character (for the natural repr. $g \to Trg$) and the most "natural" form on LieG is the trace (I remark we're considering linear groups).

For example an algebraic computation [GO2] gives for $G=GL(n,\mathbb{R})$

$$\{ f_\alpha, f_\beta\} = \Sigma_{p\in a\# b}\varepsilon(p) f_{\alpha p\beta p} \tag{5.1.7}$$

where $\alpha_p\ \beta_p$ denotes the product in $\pi_1(M,p)$ of the elements $\alpha_p, \beta_p \in \pi_1(M,p)$. In other words the Poisson bracket of trace functions f_α is a linear combination of trace functions.
Unfortunately this linearity isn't general: it works for $GL(n,\mathbb{C})$, $GL(n,\mathbb{H})$ but not for $SL(n,\mathbb{R})$, $SU(p,q)$, $SL(n,\mathbb{C})$. But, at least for $GL(n,\mathbb{R})$, the formula suggests to look for an abstract Lie algebra structure on the space of closed curves. Let's state the following

Def. (5.1.8) Let $\mathbb{Z}\pi$ denote the free abelian group with basis π. We define the bracket

$$\mathbb{Z}\pi \times \mathbb{Z}\pi \longrightarrow \mathbb{Z}\pi$$

$$[a,b] = \Sigma_{p\in a\# b}\varepsilon(p) f_{\alpha p\beta p}$$ (it is sufficient to define it on generators)

where $\alpha_p\beta_p$ is the free homotopy class of oriented curves corresponding to $\alpha_p\beta_p \in \pi_1(M,p)$.

In fact this definition is well posed as it doesn't depend on the choices of a,b ([GO2]). The business goes how one should expect:

Theor. (5.1.9) i) $\mathbb{Z}\pi$ is a Lie algebra under (5.1.8)

ii) the map $\mathbb{Z}\pi \longrightarrow C^\infty(\mathbf{R},\mathbb{R})$

$$\alpha \longrightarrow f_\alpha$$

is an algebra homomorphism.

proof:(of i) [GO2] chapter5.

A preliminary study of the map ii) in toy examples shows that it isn't injective.

Conjecture (5.1.10) [GO2] The map ii) is never injective.

This construction, as we've seen, depends strongly on the expression of the Poisson bracket in (5.1.7) and so it isn't generalizable in a straightforward way to general linear groups. For $G=O(p,q)$, $O(n,\mathbb{C})$, $O(n,\mathbb{H})$, $U(p,q)$, $Sp(n,\mathbb{R})$, $Sp(p,q)$ there exists [GO2] an analogous homeomorphism with domain a Lie subalgebra $\mathbb{Z}\underline{\pi}$ of $\mathbb{Z}\pi$ based on unoriented curves.

The study of the symplectic geometry of **R** is a stimulating subject: for example it is possible to see theor.(5.1.9) by duality in terms of structures on $\mathbf{R}_\pi$ and prove that the character map on

$\mathbf{R}_\pi \longrightarrow \mathbb{Z}\pi^*$ $\quad \rho \rightarrow Tr\rho(\ )$ is a covering map of a coadjoint orbit in $\mathbb{Z}\pi^*$. But the main reason why we've taken care of it is to trace the path towards the construction of other 3-manifolds invariants.

5.2 Glancing at link invariants

The intent of this section is to clarify the relationship between the skein invariants of links in 3-manifolds and the algebras of curves introduced in (5.1). The introduction of this connection is due to Turaev and I'm going to report about his recent preprint [T].

From a physical point of view this connection might be referred to as "quantization": in fact Turaev has shown that skein algebras of links lying in the cylinder over M "quantize" algebras of the form $\mathbb{Z}\pi$, $\mathbb{Z}\underline{\pi}$.
The inspiration for such a quantum nature comes out from the discovery of deep bonds between knot theory and quantum R-matrices. From an algebraic point of view quantization is a non-commutative extension (or h-deformation) of a Poisson algebra so that the first approximation to non-commutativity is determined by the Lie bracket.
Let's specify the formal framework. $\mathbb{K}$ denotes a commutative associative ring containing the field of rationals $\mathbb{Q}$.

<u>**Def. (5.2.1)**</u> A Poisson algebra is a commutative associative algebra S equipped with a Lie bracket which satisfies the Leibniz rule

(5.2.2) $[ab,c]=a[b,c] + [a,c]b$ for each $a,b,c\in S$

We can construct a Poisson algebra from a $\mathbb{K}$-module g with a Lie bracket. To g we associate

(5.2.3) $S(g)=\oplus_{i\geq 0}S^i(g)$

$S^0(g)=\mathbb{K}$, $S^i(g)$ is the i-th symmetric tensor power of g for i>0. The Lie bracket in G uniquely extends by the Leibniz rule to a Lie bracket in S(g) and this makes S(g) a Poisson algebra.
We introduce now the algebraic notion of quantization.
Let Q be a commutative associative $\mathbb{K}$-algebra with unit. Let F: Q $\rightarrow$ $\mathbb{K}$ be a unit-preserving $\mathbb{K}$-algebra homomorphism. An additive homomorphism $p:A \rightarrow S$ of a Q-module A into a $\mathbb{K}$-module S is called ϕ-linear if $p(q,a) = \phi(q)p(a)$ for each $q\in Q$, $a\in A$. Let $h\in \mathrm{Ker}\phi$.

Def. (5.2.4) A quantization over (Q,ϕ,h) of a Poisson $\mathbb{K}$-algebra S is a pair (A,p) where A is a Q-algebra, $p:A \rightarrow S$ a ϕ-linear ring homomorphism s.t. for each $a,b\in A$

$$ab-ba= hp^{-1}([p(a),p(b)]) \quad \mathrm{mod} hkerp$$

The main point is that the quantization of $S(\mathbb{Z}\pi)$ has a nice description in terms of a skein algebra of links. We'll denote by Z the free $\mathbb{R}$-module with basis π and extend in ordinary way the map (5.1.9)ii to a Poisson algebra homomorphism $S(Z) \rightarrow C^{\infty}(\mathbf{R},\mathbb{R})$.
Our target is the skein algebra of links of an oriented 3-manifold N.

Def. (5.2.5) A triple of non-empty oriented links L_+, L_-, L_0 in N is called a Conway triple if L_+, L_-, L_0 are identical outside some ball $B\subset M$ and look as in the picture inside B.

We define the type of the Conway triple to be 1 if $|L_+|=|L_0|+1$ ($|L|$ denotes the number of components of a link) and to be -1 if $|L_+|=|L_0|-1$.

We're now ready to introduce the skein-module $A(N)$.

Def. (5.2.6) Let L be the set of isotopy classes of oriented links in N. We define the skein-module $A(N)$ as the quotient of the free $K[x,x^{-1},h,\underline{h}]$-module with basis L by the submodule generated by elements of two types

i) the elements $xL_+ - x^{-1}L_- - hL_0$ corresponding to arbitrary Conway triples of type 1

ii) the elements $xL_+ - x^{-1}L_- - \underline{h}L_0$ corresponding to arbitrary Conway triples of type 2.

The elements of this quotient module represented by an oriented link L, denoted by $|L|$ is the Jones-Conway invariant of the link.

The skein module of N is related to $\pi_1(N)$ as follows.

Each oriented link $L \subset N$ with components $L_1 \dots L_n$ gives rise to the element $\langle L\rangle = \prod_{i=1}^{n} \langle L_i \rangle \in S(Z)$ where $\langle L_i\rangle$ is the class of L_i in p. The formula

$$\textbf{(5.2.7)} \qquad p: A(M) \longrightarrow S(Z), \qquad [L] \longrightarrow \langle L\rangle$$

is an additive homomorphism linear over the coefficient ring homomorphism.

On the other side if we consider a manifold of the form $M\times[0,1]$ (with the product orientation) it is possible to give $A(M)$ the structure of an algebra. In fact we can define the product of two links $LL' \subset M\times[0,1]$ by the formula

$$\textbf{(5.2.8)} \qquad LL' = \{(a,t) \in M\times[0,1] \mid t \geq 1/2 \text{ and } (a,2t-1) \in L \text{ or } t \leq 1/2 \text{ and } (a,2t) \in L'\}$$

(5.2.8) defines a structure of associative algebra in $A(M\times[0,1])$ with unit $[\phi]$.

Finally here is the claimed result

Theor.(5.2.9) ($A(M\times[0,1],p)$) is a quantization of $S(Z)$.

Goldman's result show that loops on M can be treated as functions on the representation spaces of $\pi_1(M)$. In the language of classical mechanics these functions are classical observables. But links in $M\times[0,1]$ can be treated as quantum observables.

Projection to the surface or, what is the same, forgetting the under/over crossing information is the usual degeneration of quantum objects into the classical ones.

Heuristically, a space is equivalent to the algebra of functions on it. Thus, on the heuristic level, the skein algebras quantize the representation spaces of $\pi_1(M)$.

6 Lagrangian intersections

We remain in the sphere of symplectic geometry but, in analogy with chapters 3 and 4 we study the Heegard model for a closed orientable 3-manifold and find that the representation spaces of the fundamental groups of handlebodies are lagrangian submanifolds of **R**. This property is the key for reinterpreting Casson's invariant in terms of spectral data for the symplectic action and relating it to the Euler characteristic of Floer's complex for lagrangian intersections.
This chapter reflects the content of a personal work.

6.1 Lagrangian intersections

For this section we follow closely [F3].Let P a symplectic manifold with symplectic form ω and let L_1, L_2 be lagrangian submanifolds of P.We may define the configuration space:

(6.1.1) $$\Omega(L_1,L_2) = \{z \in C^\infty([0,1],P) \mid z(0) \in L_1 \quad z(1) \in L_2\}$$

Consider the 1-form on $\Omega(L_1,L_2)$

(6.1.2) $$f(z)\xi = \int_0^1 \omega(z'(t),\xi(t))\,dt$$

where $\xi(t) \in T_{z(t)}P$ and $\xi(0) \in T_{z(0)}L_1 \quad \xi(1) \in T_{z(1)}L_2.$

Prop.(6.1.3) The critical points of (6.1.2) are constant paths in the intersection of L_1 and L_2.

So we've at disposal an infinite dimensional manifold with a smooth 1-form and if we suppose that L_1 and L_2 are transversal, the critical points of our vector field are isolated. In general it is not possible to develop a naive infinite dimensional Morse theory unless we require some additional properties for our functional (e.g. Palais-Smale conditions). On the other hand, as Floer has shown, one can draw an infinite dimensional generalization of Witten's approach [W] to Morse theory in this setting, due to specific properties of the vector field (these properties have yet to be fully investigated).

At present, only one other case is known [F1] where such procedure works). Let's review this nice idea.

First of all, we need a metric structure on Ω for obtaining a gradient vector field. This is easily done, as there always exists ([F3] page 780) an almost Kaehler structure on P, $J\in End(TP)$ s.t. the bilinear form $g(.,.)=\omega(J,.)$ is a metric.

Then we comsider on $T\Omega$ the L_2-metric

$$\textbf{(6.1.4)} \quad \langle\xi,\zeta\rangle = \int_0^1 g(\xi(t),\zeta(t))\,dt$$

(on the fiber of $T\Omega$ lying on $p(\xi(t))=p(\zeta(t))\subset P \quad p:TP \rightarrow P$).
For technical reasons (this metric doesn't define a Hilbert-space structure on the fibers of $T\Omega$ and the L^2-gradient vector field corresponding to f is not a section of $T\Omega$) it is better to rephrase the problem in a different language.
By the identification

$$\textbf{(6.1.5)} \qquad \theta= \mathbb{R}x[0,1]= \{z\in\mathbb{C}\,|\,0\le Imz\le 1\}$$

the solutions of the gradient flow are exactly holomorphic maps between θ and the almost complex manifold (P,J)

$$\textbf{(6.1.6)} \qquad \partial_J u(\tau,t) = {\partial u(\tau,t)}/{\partial t} + J(u(\tau,t))\,{\partial u(\tau,t)}/{\partial t}=0$$

where $u:\mathbb{R} \rightarrow \Omega(L_1,L_2)$ has been written as a map $u:\mathbb{R}x[0,1] \rightarrow P$.
Then we formally define the gradient flow of f with respect to J as the set

$$\textbf{(6.1.7)} \quad \mathcal{F}_J = \{u\in C^\infty(\theta,P)\,|\, u(\mathbb{R}x\{0\})\subset L_1 \quad u(\mathbb{R}x\{1\})\subset L_2 \quad \partial_J u=0\}$$

The crucial ingredient which allows us to set up a Morse theory in the finite-dimensional case, is the existence of an index (Morse index) by which we may weight the topological meaning of critical points. In the general situation (as in this one) the negative and positive eigenspaces of the Hessian of the gradient vector field at critical points are infinite-dimensional themselves. But Floer has pointed out that, for developing Witten's approach we don't need to know the absolute value of the Morse indexes of our critical points, but only the "difference" between the dimension of the negative eigenspace of the Hessian at any pair of critical points. This strategy here goes throrugh, because our flow is of Morse-Smale type and the set of trajectories connecting two critical points is a finite-dimensional manifold. In fact consider the set of bounded trajectories

$$\textbf{(6.1.8)} \qquad \mathcal{M}_J = \{u \in \mathcal{F}_J \mid \ |\nabla u| < \infty\}$$

(for instance ∇ is the Levi-Civita connection)

Prop. (6.1.9) For $x_+, x_- \in L_1 \cap L_2$ define

$$\mathcal{M}_J(x_+, x_-) = \{u \in \mathcal{M}_J \mid \lim_{t \to \mp\infty} u(\tau, t) = x_\pm \text{ for all } t\}.$$

Then there is a dense set of smooth almost Kaehler structures on P s.t. all sets $\mathcal{M}_J(x_+, x_-)$ are smooth finite-dimensional manifolds.

proof: [F3] page778.

For evaluating the "difference" of Morse indexes at two critical points, there is a standard regularization which is known in the literature as spectral flow and it turns out that its value is exactly the dimension of the manifold of connecting trajectories $\mathcal{M}_J(x_+, x_-)$.

We recall the definition because it is crucial for further developments.

Define the operator field

$$\textbf{(6.1.10)} \qquad A_z : T_z\Omega \longrightarrow L_z, \qquad \xi \longrightarrow J\nabla_z \xi + (\nabla_\xi J) z'$$

where $L_z = L^2(z^*TP)$. The domain of A_z can be canonically extended to the Sobolev space $W_1^{\,2}(z^*TP)$. Then A_z is a closed operator on $L_z \subset W_1^{\,2}(z^*TP)$ and if $x \in L_1 \cap L_2$ is self-adjoint and its zero eigenspace is isomorphic to $T_xL_1 \cap T_xL_2$ (that is, for transversal lagrangian submanifolds, critical points are non-degenerate). We can define a spectral flow for A_z, owing to the following:

Prop.(6.1.11) Let $x,y \in L_1 \cap L_2$ be transverse intersections and let $u:[0,1] \rightarrow \Omega$ be a smooth path joining **x** and **y** (**x** stands for the constant path $z(t)=x$). There exists a matrix operator field $B \in \mathrm{End}\,u^*(TP)$ with the following properties: if B_τ denotes the continuous operator on $L_{u(\tau)}$ defined by $(B_\tau \xi)(t) = B(\tau,t)\xi(t)$ then $B_0=0$ and $B_1=0$ and for each $t \in [0,1]$ the operator $A_\tau = A_{u(\tau)} + B_{u(\tau)}$ is self-adjoint and has no double eigenvalues.

proof:[F4]

Due to standard perturbation theory we can distinguish smooth families of eigenvalues $a(\tau)$ for A_τ.

Remark(6.1.12) As for each τ A_τ is self-adjoint, eigenvalues are discrete. This fact and the property that A_τ has no double eigenvalues shows that zero eigenvalues are discrete in τ. The spectral flow of A_τ is independent of the perturbation B. In particular we may choose B in such a way that $a'(\tau) \neq 0$ for $a(\tau)=0$.

Def.(6.1.13) We denote by $\mu_u(x,y)$ (spectral flow for A_τ along μ) the number of eigenvalue families with $a(0)<0<a(1)$, minus the number of eigenvalue families with $a(0)>0>a(1)$.
This definition doesn't depend on the perturbation ([F4]).

We're now ready for defining Floer's complex ([F2]). A final obstruction is that $\mu_u(x,y)$ depends on the homotopy type of μ. To avoid problems we may assume that there're not homotopically distinct paths between critical points. More exactly if we suppose that $\pi_2(P,L_1)=0$ and that $L_2=\Phi(L_1)$ where $\Phi:P \rightarrow P$ is a symplectic diffeomorphism which is isotopic to the identity through symplectic diffeomorphisms, then there exists an integer $\mu(x)$ for each transverse intersection $x \in L_1 \cap L_2$ s.t.

$$\textbf{(6.1.14)} \qquad \mu_u(x,y) = \mu(x) - \mu(y)$$

(up to an additive constant).

The elements C^p of chain complex are the free $\mathbb{Z}_2$-modules over the set of $x \in I(L_1,\Phi)$ with $\mu(x)=p$ ($I(L_1,\Phi)=\{x \in L_1 \cap \Phi(L_1)$ s.t. $\{\Phi_t(x)\}$ defines the zero element in $\pi_1(P,L)\}$). The boundary operator $\partial: C^p \rightarrow C^{p-1}$ is so defined

$$\partial x = \sum_{\mu(y)=p-1} \gamma(x,y)\, y$$

where $\gamma(x,y)$ is 0,1 according if the number of trajectories joining x and y is even or odd (this number is finite for prop.6.1.9).
This complex provides a new powerful tool for extimating number of lagrangian intersections or of fixed points of exact diffeomorphisms on a symplectic manifold (see [F2] for applications).

6.2 Heegard decompositions and the symplectic structure

Let M be a closed 3-manifold and (W,h) a Heegard model for M. In chapter 3 we've seen that it is worthwhile studying the intersection number of $\mathbf{Q}_1$ $\mathbf{Q}_2$ according to the natural orientation of **R** (Casson's invariant) and in chapter 4 that an analogous work can also be done using volume forms. We've not yet evaluated these natural submanifolds ($\mathbf{Q}_1$ $\mathbf{Q}_2$) of **R** from the point of view of the symplectic structure. Actually their behaviour is characteristic

Theor.(6.2.1) M as above,G like in section (2.3). $\mathbf{Q}_1$ $\mathbf{Q}_2$ are lagrangian submanifolds of **R** according to the natural symplectic structure built in (2.3).

proof: section 7.3.

Theor.(6.2.1) is the key passage by which we can relate Casson's invariant to spectral data for the symplectic action.
In a recent preprint Taubes [Ta] has interpreted Casson's invariant in gauge theoretical terms as one half of the Euler characteristic of Floer's complex for flat su(2) connections on a homology 3-sphere [Fl] (see chapter 8). In a different context we'll prove that the intersection number of two lagrangian submanifolds in a symplectic manifold may be associated to the Euler characteristic of Floer's complex for lagrangian intersections if it does exist.

We adopt a fiber-bundle presentation of the intersection number (see [T]).
Consider two smooth oriented submanifolds L_1, L_2 in a third oriented manifold P, of complementary dimensions. Let x,y be distinct points where the intersection is transverse. Choose a triple (z,E,h) where

1) $z:[0,1] \longrightarrow P$ obeys $z(0) = x$, $z(1)=y$ and is C^∞

2) $E \rightarrow [0,1]$ is a smooth oriented vector bundle
3) $h: E \rightarrow z^*(TP)$ is a bundle homomorphism.

We'll suppose that the following two properties hold:

i) at t=0 and t=1 $h: E_{0,1} \rightarrow z(0,1)^*(T_{z(0,1)}L_1 \oplus T_{z(0,1)}L_2)$ is an orientation preserving isomorphism.

ii) Consider $h: E \rightarrow z^*(TP)$.
Pointwise we can define deth as $e_1 \wedge \ldots . e_n = \mathrm{deth}(h(e_1) \wedge \ldots . h(e_n))$ for each vector of $\wedge_n E_x$. So deth is a section of $\wedge^n V \otimes [\wedge^n z^*(TP)]^*$. We'll require that deth be transverse to the zero section.

Prop.(6.2.1) The relative orientation of $h(E)=z^*(TL_1 \oplus TL_2)$ and $z^*(TP)$ at t={0,1} differ by the mod2 cardinality of $\mathrm{deth}^{-1}(0)$.

proof: Let $e_1(t) \wedge \ldots . e_n(t)$ an oriented n-form for E and $g_1(t) \wedge \ldots . g_n(t)$ an oriented n-form for $z^*(TP)$. At t=0,1 h is an isomorphism and so $h^{*-1}(g_1(0) \wedge \ldots . g_n(0))$ is a non-zero n-form in E_0 (resp. $h^{*-1}(g_1(1) \wedge \ldots . g_n(1))$ in E_1). So there exists $c_0, c_1 \neq 0$ s.t.

$$h^{*-1}(g_1(0) \wedge \ldots . g_n(0)) = c_0 [e_1(0) \wedge \ldots .. e_n(0)]$$

$$(\text{resp. } h^{*-1}(g_1(1) \wedge \ldots . g_n(1)) = c_1 [e_1(1) \wedge \ldots . e_n(1)]).$$

Pose $\quad h^*[g_1(t) \wedge \ldots . g_n(t)] = c(t)[e_1(t) \wedge \ldots . e_n(t)]$.

If deth≠0 ∀t, h* is an isomorphism and so $h^*[g_1(t) \wedge \ldots . g_n(t)]$ may be written as $c(t)[e_1(t) \wedge \ldots . e_n(t)]$ with a continuous nowhere zero-function. But in this case $\mathrm{sgn} c_0 = \mathrm{sgn} c_1$.

Suppose there exists t_0 s.t. $\mathrm{deth}(t_0)=0$. Then

$$h^*[g_1(t_0) \wedge \ldots . g_n(t_0)] = 0.$$

In a neighborhood of t_0, by transversality, $h^*[g_1(t) \wedge \ldots . g_n(t)]$ can't be zero and c(t) must change sign in t_0 (as $c'(t)|_{t=0} \neq 0 \Leftrightarrow \mathrm{deth}'(t)|_t \neq 0$).

Through this presentation an intersection number may be translated into spectral data for the symplectic action. This happens because in the lagrangian context there exists a natural triple (z,E,h) with the properties we've required.

For $\alpha=0,1,2$ introduce the spaces Ω_α

$$\Omega_0=\{z\in C^\infty([0,1],P)\}$$

(6.2.2)
$$\Omega_1=\{z\in C^\infty([0,1],P)\mid z(0)\in L_1\}$$
$$\Omega_2=\{z\in C^\infty([0,1],P)\mid z(1)\in L_2\}$$

We've natural maps (up to a reparametrization)

$$\phi_{1(2,0)}:\Omega \longrightarrow \Omega_1(\text{resp.}\Omega_2,\Omega_0)$$
$$z \longrightarrow z|_{[0,2/3]} \quad (\text{resp.}z|_{[1/3,1]},z|_{[1/3,2/3]})$$

(6.2.3)
$$\psi_{1,2}\ :\Omega_1(\text{resp.}\Omega_2) \longrightarrow \Omega_0$$
$$z \longrightarrow z|_{[1/3,2/3]}$$

We may define on each Ω_α a 1-form f_α by using the symplectic form on P with the same formal expression as in (1.2) and consider again its critical points: by prop.(6.1.3) it turns out that if we denote by m_α the set of critical points of f_α on Ω_α, then

m_0 = {set of constant paths in P}
m_1 = {set of constant paths in L_1}
m_2 = {set of constant paths in L_2}

Let $x,y\in L_1\cap L_2$ be transverse intersections and let $u:[0,1] \to \Omega$ be a smooth path joining **x** and **y**. Let A_τ as in prop.(6.1.11) (see also remark (6.1.12)). We may define operator fields

(6.2.4)
$$A_\tau{}^\alpha:T_{\phi(u(\tau))}\Omega_\alpha \longrightarrow L_{\phi(u(\tau))}$$
$$\xi \longrightarrow J\nabla_{\zeta\alpha(\tau)}\xi+(\nabla_\xi J)z'_\alpha(t)+B_{z(t)}$$

where $z_\alpha(t)=\phi_\alpha(u(\tau))(t)$ and $L\phi(u(\tau))=L^2(z_\alpha{}^*TP)$.We can again extend $A_\tau{}^\alpha$ to $W_1{}^2(z_\alpha{}^*TP)$.
We've the following

<u>**Prop.(6.2.5)**</u> $\quad \text{Ker}A_0{}^1\cong T_xL_1$
(resp. $\text{Ker}A_0{}^2\cong T_xL_2$ $\quad \text{Ker}A_0{}^0\cong T_xP$ and corresponding relations for $\tau=1$).

We now come to the main theorem.
Suppose L_1 and L_2 are transversal and that we've chosen an almost Kaehler structure J on P for which $M_J(x,y)$ is a smooth finite-dimensional manifold for each pair of critical points (x,y) (Prop.6.1.9)).

Theor.(6.2.6) Given $x,y \in L_1 \cap L_2$, the relative intersection number of L_1 and L_2 at x,y is exactly the spectral flow mod2 of A_τ along any path joining **x** and **y**.

proof: Choose a path λ_0: $[0,1] \rightarrow m_0$ between **x** and **y** which are critical points for f on Ω. Since m_2 is path connected there is no loss of generality by assuming that λ_0 takes values in constant paths with values in L_2. λ_0 may be identified with a path in L_2(I'll denote it with λ_0). Build any path $\lambda:[0,1] \rightarrow \Omega$ s.t. $\phi_\alpha(\lambda)(\tau)=\lambda_\alpha(\tau)$ with $\lambda(0)=\mathbf{x}$ $\lambda(1)=\mathbf{y}$. Let $A_\lambda(\tau)=A_\tau$ ($A\lambda_{1,2}(\tau)^{1,2}=A_{\tau 1,2}$). We can build a triple(z,E,h) with the properties required at the beginning of the section.

i) $z=\lambda_0$

ii) $E \rightarrow [0,1]$ is the vector bundle $\mathrm{Ker}A_\tau^1 \oplus \mathrm{Ker}A_\tau^2$. In fact by prop.(6.2.5) we may identify $\lambda_2{}^*\mathrm{Ker}A_\tau^2$ with $\lambda_2{}^*T\lambda_2(\tau)m_2$ and so $\mathrm{Ker}A_\tau^2 \rightarrow [0,1]$ is a vector bundle.
On the other side $\mathrm{Ker}A_\tau^1$ for a fixed value τ_0 is the set of solutions $\xi:[0,1] \rightarrow TP$ of the equation

$$J\nabla\lambda_1(\tau_0)\xi + (\nabla_\xi J)\lambda'_1(\tau_0) + B|\lambda_1(\tau_0) = 0$$

in $W_1^2(\lambda_1(\tau_0)^*TP)$. In $C^\infty(\lambda_1(\tau_0)^*TP)$ this is a Cauchy problem on a compact set [0,1] with smooth coefficients and so it admits a complete set of solutions in correspondence of the initial conditions $\xi(0) \in T\lambda_1(\tau_0,0)L_1$ (functions in W_1^2 are continuous). For smooth dependence on the parameter t $\mathrm{Ker}A\lambda_1(\tau)^1$ defines a vector bundle on [0,1].

iii) $h=T\psi_1-T\psi_2$ (see 6.2.3)

Let's check this triple has the required properties.

1) We've to prove that h in t=0,1 is a preserving-orientation isomorphism of E on $T\lambda_0(\tau)m_0$ at τ=0,1. E.g., at t=0 $KerA_0{}^1 \cong T_xL_1$ and $KerA_0{}^2 \cong T_xL_2$, $KerA_0{}^0 \cong T_xP$. So h is a linear map of finite dimensional vector spaces $T_xL_1 \oplus T_xL_2 \rightarrow T_xP$ s.t. $dimL_1+dimL_2=dimP$. Moreover h has zero kernel. In fact if there exists $\xi_1,\xi_2 \in V_0$ s.t. $h(\xi_1,\xi_2)=0$ then, if we identify T_xL_1 T_xL_2 as subspaces of T_xP, $\xi_1=\xi_2$. But L_1 and L_2 are transversal at x and so $\xi_1=\xi_2=0$.
With regard to orientation we've only to be sure that h is orientation preserving on $KerA\lambda_1(\tau)^1$. At any point τ we've a natural identification of $KerA\lambda_1(\tau)^1$ with $T\lambda_1(\tau)L_1$ by taking the smooth solution of $A\lambda_1(\tau)^1\xi=0$ corresponding to a initial value $v\in T\lambda_1(\tau)L_1$. We give to $KerA\lambda_1(\tau)^1$ the orientation of L_1.

2) That deth is transversal to the zero section will appear clear in a moment (it depends on the choice made in (6.1.12)).

The relative intersection number of **x** and **y** is thus the number(mod2) of points in Ω_0 where h isn't an isomorphism. This is equivalent to say that there exists v_1,v_2 $v_1\in KerA_\tau{}^1$ $v_2\in KerA_\tau{}^2$ s.t. $h(v_1,,v_2)=0$. At any t we've a natural identification of $KerA_\tau{}^1$ with a subspace of $T\lambda_0(\tau)m_0$ by taking the restriction of $\xi\in KerA_\tau{}^1$ to $\xi|_{[1/3,2/3]}$. On the other side $KerA_\tau{}^2 \cong T\lambda_0(\tau)L_2$ may be naturally included in $T\lambda_0(\tau)P$. Through these identifications $h(v_1,v_2)=0$ implies $v_1=v_2$. But this is equivalent to say there exists $\xi\in T_{\lambda(\tau)}\Omega$ s.t. $A_{\lambda(\tau)}\xi(\tau)=0$. It is exactly

$$\xi(\tau)=\begin{cases} v_1(t) & t\in[0,2/3] \\ v_2(t) & t\in[2/3,1] \end{cases}$$

That is the relative intersection number computes the number mod2 of points τ s.t.

$$C:\ T_{\lambda(\tau)}\Omega \longrightarrow T_{\lambda(\tau)}\Omega$$
$$v \longrightarrow A_\tau v$$

has a non-trivial kernel. This is exactly the spectral flow mod2 along $\lambda:[0,1] \to \Omega$ for A_τ.

The proof of the theorem is concluded by noticing that we've computed the spectral flow along a path in Ω which is built over a path $\lambda_0:[0,1] \to m_0$. But any path in Ω between **x** and **y** may be obtained in this way (by reparametrization).

Q.E.D.

Coroll.(6.2.7) The spectral flow mod2 for A_τ between two constant paths in Ω is independent from the connecting path $\lambda:[0,1] \to \Omega$.

On the other hand, if we want to define the full homology complex, we've to require that $\pi_2(P,L_1)=0$ and that L_2 may be obtained from L_1 through an exact deformation, to exclude the possibility of homotopically inequivalent paths between critical points (in any case it is expected that this condition may be dropped [F3]).

Let μ as in (6.1.14) and define a chain complex where elements C^p are the free $\mathbb{Z}$-modules over the set $x \in L_1 \cap L_2$ with $\mu(x)=p$. We'll define the Euler characteristic of this complex as

$$\textbf{(6.2.28)} \qquad \Sigma_x (-1)^{\varepsilon(x)} = E$$

where $\varepsilon(x)=\pm 1$ according if μ is even or odd.
Then

Coroll.(6.2.9) The absolute value of the intersection number of L_1 and L_2 is equal to $|E|$.

Remark(6.2.10) Unfortunately it isn't possible to do better, dropping absolute values. In fact we've not a naturally selected way of choosing the additive constant which defines μ (we may not select in a natural way a particular lagrangian intersection).
in the gauge theoretical frame of flat connections, on the other hand, we've such a distinct object (the trivial flat su(2)-connection on a homology 3-sphere see [T]).

7 Flat structures on M

This chapter involves a non-trivial metamorphosis of our object of study. We can investigate **R** from a differential geometric point of view rather than a topological item.
It is worthwhile doing so for various reasons.
First of all, looking at **R** as an embedded subset of a larger space (the affine space of connections) often sheds light on the structures of the smaller one (examples in 7.3).
Also the new technical language of cohomology in flat vector bundles may be practically easier to use (7.3).
Moreover this is the gate to the physics flavoured world of gauge theory (chapter 8).

7.1 R and flat structures on M

In this section we exhibit the natural bijective correspondences existing between **R** and the gauge equivalence classes of flat connections on a principal G-bundle over M or the equivalence classes of flat G-bundles over M. We forget the natural structures we've built on **R** and think of it as a set.
M is a connected differentiable manifold and G a Lie group.

A) From flat connections to $Hom(\pi_1(M),G)/G$

Let M be a diff.manifold, connected and paracompact, (P,M,p,G) a principal G-bundle over M and Γ a flat connection. Fix a point $r\in P$ and set $p(r)=x$.Define the holonomy group of Γ at r $\Phi(r)$ and the restricted holonomy group at r $\Phi_0'(r)\subset G$ as the subgroup of the holonomy group arising from contractible loops based at x. $\Phi(r)$ is a Lie subgroup of G with identity component $\Phi_0(r)$ and it is known that

Theor.(7.1.1) (Ambrose-Singer) Let $P,M,\Phi(r)$ as above, ω a connection, with curvature form Ω and P(r) the holonomy bundle of ω through r. Then the Lie algebra of $\Phi(r)$ is equal to the subspace of LieG spanned by all elements of the form $\Omega_s(X,Y)$ where $s\in P(r)$ and X,Y are arbitrary horizontal vectors at s.

As Γ is flat $\Phi_0(r)=e$ and any two closed curves based at x, representing the same element of the first homotopy group $\pi_1(M,x)$ based at x, give rise to the same element of $\Phi(r)$.
So for a fixed $r\in P$ it is well defined an epimorphism

(7.1.2) $$\phi_r:\ \pi_1(M,p(r)) \longrightarrow \Phi(r)$$

by simply associating to a homotopy class the holonomy at r of any representative closed curve. If we take a different $r'\in P$ $r'=r\circ g$ $g\in G$ the holonomy $\Phi_u(r')$ at r' for a loop u based at x is

(7.1.3) $$\Phi_u(r')=g^{-1}\Phi_u(r)g$$

and so

(7.1.4) $$\phi_{r'}=g^{-1}\phi_r g$$

that is $\phi_{r'}$ is conjugate to ϕ_r as a homomorphism of $\pi_1(M,p(r))$ in G.

As M is connected i_γ: $\pi_1(M,x)\longrightarrow\pi_1(M,x')$ $x,x'\in M$ and we can always choose $s\in P$ $p(s)=x'$ s.t. s and r are joined by a horizontal curve γ, $p(\gamma)=\gamma$. In this case

(7.1.5) $$\phi_s\circ i_\gamma=\phi_r$$

So we can conclude that, given a flat connection Γ there exists a well-defined representation $\phi_\Gamma\in Hom(\pi_1(M),G)/G$. If Γ,Γ' are gauge-equivalent flat connections the holonomies $\Phi_u^{\Gamma}(r)$, $\Phi_u^{\Gamma'}(r)$ at r for a loop u based at x are conjugate for the action of an element $g\in G$ which doesn't depend on u.

On the other side if (P',M,π',G) is isomorphic to (P,M,π,G) we obtain the same conjugacy class of representations..

At last we've set up a corrispondence

(7.1.6) $$\alpha:\left\{\begin{array}{c}\text{eq.classes of flat}\\ \text{connections (on an}\\ \text{iso.class of pnp,G-bndl)}\end{array}\right\}\rightarrow Hom(\pi_1(M),G)/G$$

for a fixed principal G-bundle P over a connected, paracompact, diff.manifold M.

B) From Hom(π_1(M),G)/G to flat G-bundles

The isomorphism classes of flat G-bundles over M with structure group G and fibre F are in a natural one--one correspondence with the elements of the cohomology set $H^1(M, M \times G)$ where MxG is the constant sheaf of germs of locally constant diff.mappimgs from M to G once we've chosen an effective action of G on F, F=G and the left multiplication of G on G, obtaining isomorphism clsses of flat principal G-bundles.
An equivalent definition of flat G-bundles is the following

Def.(7.1.7) A G-bundle P is said to be flat if it is possible to find an open covering $\mathcal{U}$ of M and charts for P on $\mathcal{U}$ with constant transition functions in G (G acts on the fiber by a chosen action)

As M is connected we can build the universal covering **M** of M and this is a principal π_1(M)-bundle over M.
Define the associated bundle

$$P = \mathbf{M} \times G \tag{7.1.8}$$

(ρ acts on G by left multiplication).

Claim P is a flat G-bundle
(in fact a flat principal G-bundle as we choose the right multiplication of G as action of G on the fiber of P).

As **M** is a covering of M,we can find a family $(U, s_U)_{U \in \mathcal{U}}$ where $\mathcal{U}$ is a covering of M and s_U a constant section $s_U: U \longrightarrow \mathbf{M}$ $\forall$ $U \in \mathcal{U}$.
Each s_U induces a chart for P

$$\begin{aligned} \phi_U: U \times G &\longrightarrow P|_U \\ (x,g) &\longrightarrow (s_U(x),g) \end{aligned} \tag{7.1.9}$$

and the transition functions between ϕ_U and ϕ_V are

(7.1.10) $\phi_{UV} = \rho(h)$ (left multiplication)

where $h \in G$ is s.t. $s_U(x) \circ h = s_V(x) \quad \forall\ x \in U \cap V$

If ρ' is conjugate to ρ $\rho' = t\rho t^{-1}$ then the flat structure on $\mathbf{M}x_{\rho'}G$ induced from the same family $(U, s_U)_{U \in \mathcal{U}}$ is equivalent to the previous one in view of

(7.1.11) $t\Phi_{UV}t^{-1} = \Phi'_{UV}$

At last we've set up a corrispondence

(7.1.12) $$\beta : \mathrm{Hom}(\pi_1(M), G)/G) \longrightarrow \left\{ \begin{array}{c} \text{eq.classes of flat} \\ \text{principal bundles} \\ \text{on } M \end{array} \right\}$$

For equivalence class we mean a set of isomorphic principal bundles with equivalent flat structure.

C) From flat G-bundles to flat connections

M, G as before.
In this case it is easier to describe the correspondence from flat G-bundles to flat connections in the framework of flat vector G-bundles and so, suppose we've fixed any representation $G \longrightarrow GL(n, \mathbb{R})$.

Let $E \longrightarrow M$ be a flat vector G-bundle. By (7.1.7) a flat structure on E is given by an open cover $(U, s_U)_{U \in \mathcal{U}}$ with local frame fields $s_U : U \longrightarrow E$ s.t.

(7.1.13) $s_U(x) = s_V(x)\ \rho(\Phi_{UV}(x)) \quad \forall\ x \in U \cap V$
(s_U, s_V row vectors)

with $\phi_{UV} : U \cap V \longrightarrow G$ constant functions and ρ our chosen representation. Define a connection Γ s.t.

$$(7.1.14) \qquad \nabla s_U(x)=0 \quad \forall U\in \mathcal{U}$$

where ∇ is the covariant derivative associated to Γ.
Γ is well defined in the sense that

i) the two conditions that s_U and s_V are horizontal are compatible on $U\cap V$ by (7.1.13) and the fact that Φ_{UV} are constant functions.

ii) the connection is completely determined as it is determined its behaviour on each smooth section of E

iii) if we choose a different open cover $(U,s_U')_{U\in \mathcal{U}}$ (there is no loss of generality in taking the same $\mathcal{U}$) there must exist G-valued mappings $\{\Psi_U\}_{U\in \mathcal{U}}$ s.t.

$$(7.1.15) \qquad s_U'=s_U\circ\rho(\Psi_U) \qquad U\in \mathcal{U}$$

and so, repeating our construction for the new open cover, we obtain a gauge-equivalent connection.

Claim The connection Γ is flat.

Locally for the frame field s_U the curvature form Ω of Γ can be defined as

$$(7.1.16) \qquad s_U\Omega = \nabla^2 s_U \qquad (s_U \text{ row vector})$$

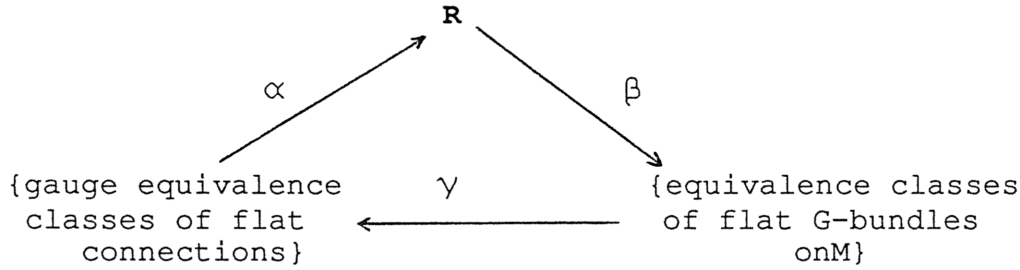
and,by (7.1.14), $\Omega=0$.

If $E'\longrightarrow M$ is an equivalent flat vector G-bundle we just repeat the procedure of iii) and obtain a gauge-equivalent connection.
We've set up a corrispondence

$$(7.1.17) \quad \gamma: \left\{ \begin{array}{c} \text{eq.classes of} \\ \text{flat G-bundles} \\ \text{overM} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{gauge equival.} \\ \text{classes of flat} \\ \text{connections} \end{array} \right\}$$

We can summarize the constructions of A,B,C in a diagram

(7.1.18)



We collect in one theorem the results existing in the literature about this diagram.

Theor.(7.1.19) i) the diagram is commutative

ii) β is a bijection

iii) α is a bijection (all isomorphism classes of principal G-bundles)

proof: [S],[G2].

It is worthwhile doing some remarks about (7.1.19). Actually the correspondence β is expressed in terms of purely topological objects and it has a well defined meaning even if M is a topological space. In this general case it is possible to prove that β is a bijection if M is path-connected, locally path-connected and semilocally 1-connected (in particular a topological manifold) [S].

Def.(7.1.20) A topological space X is said to be semilocally 1-connected if, for each point $x \in X$, there is a neighborhood V of x such that each closed curve in V is homotopic to a constant in X leaving its end points fixed.

On the other side the correspondence α lies on diff.geometric structures (as we need the notion of covariant derivative). It is interesting that we've no loss of information in taking into account this more sophisticated structure, even if we could have given a topological manifold different compatible diff.structures.
So it isn't a formal changement of language, but a really different point of view for studying the problem.
I underline the turning point at which we pass from purely topological considerations to the context of diff. geometry. If M is a topological space and G is a topological group we can describe the equivalence classes of flat G-bundles in terms of cohomology

sets with values in a presheaf. Let MxG the constant sheaf (that is the canonical presheaf of MxG associates to an open set $U \subset X$ the group of constant functions from U to G). Then the equivalence classes of flat G-bundles over X are in a natural one-to-one correspondence with the elements of the cohomology set $H^1(X, MxG)$. If M is a diff.manifold and G a Lie group the constant sheaf MxG can be viewed as a subsheaf of the sheaf $\mathcal{G}_d$ of germs of diff.mappings from M into G. The inclusion mapping of sheaves
$i: MxG \to \mathcal{G}_d$ induces a mapping of cohomology sets

$$\textbf{(7.1.21)} \qquad i^*: H^1(M, MxG) \to H^1(M, \mathcal{G}_d)$$

When we look at the elements of $H^1(M, MxG)$ from this larger set our purely topological object becomes a diff.one.

Remark(7.1.22) In the case G is an abelian group $Hom(\pi_1(M), G)/G = Hom(\pi_1(M), G)$. Since G is abelian $Hom(\pi_1(M), G) \cong Hom(H_1(M), G)$ as $H_1(M)$ is obtained from $\pi_1(M)$ by imposing commuting relations and an abelian representation of $\pi_1(M)$ automatically satisfy these relations. So theor.(7.1.19) shows a bijection between $Hom(H_1(M), G)$ and $H^1(M, G)$.

Remark (7.1.23) The map β allows us to use a different cohomological language instead of cohomology of groups for representing $T\mathbf{R}$. Suppose G, acting on V, satisfies the hypotheses of (1.3) and fix $[\rho] \in \mathbf{R}$. Let $E \to M$ a representation for $\beta[\rho]$, realized for the natural action of G on V. We denote by $\mathcal{F}(E)$ the sheaf of germs of flat sections of E. Then we've

Theor.(7.1.24) Suppose M is an Eilenberg-Mac Lane space of type $K(\pi, 1)$.

$$H^q(\pi_1(M), V) \cong H^q(M, \mathcal{F}(E)) \quad \text{for all } q$$

$H^*(M, \mathcal{F}(E))$ is called cohomology of M in the flat vector bundle E.

7.2 The problem of the quotient space

We've seen in section 1.3 that to give **R** the structure of a Hausdorff manifold, it is necessary to restrict to a subset of R_π before quotienting via the adjoint conjugation. For example in the hypothesis that G is a simple linear group acting irreducibly over a complex vector space, a good choice is the open subset R_π** of irreducible representations.
Analogous concept of irreducibility and analogous problems of quotienting have been considered also in the language of flat structures.
In this section we explain that the relation between the various notions of irreducibility is the most natural one.

M as in 7.1, G a simple linear group acting irreducibly over a complex vector space V. For defining the notion of irreducibility in the case of flat G-bundles we need to realize them via the particular action of the linear group G on V. In this case a flat G-bundle is a vector bundle $E \to M$ with the property (7.1.7)

Def. (7.2.1) E is called reducible if it is possible to find an open covering $\mathcal{U}$ of M and charts for E on $\mathcal{U}$ with constant transition matrixes in G of the form

$$\begin{vmatrix} A & B \\ 0 & C \end{vmatrix}$$

(A mxm matrix). Otherwise E is called irreducible

If E' is equivalent, as a flat G-bundle, to E irreducible, it is also irreducible; so it is well defined a notion of equivalence classes of irreducible flat G-bundles.

Of much wider use there is the concept of irreducible connection. In fact an analogous problem of quotienting the affine space of 1-forms of connections $\mathcal{C}$ on a principal bundle by the gauge action, has beeen deeply studied under the physical impulse because $\mathcal{C}$ represents the configuration space of gauge theories and one would like to get rid of the non-true degrees of freedom. It has turned out that under our hypotheses for G (see section 1.3; really physical requests regard only a few classical groups) the"good" subset of connections to be considered for obtaining a manifold structure on the quotient space is the open set of irreducible connections C_0.

In the spirit of our previous definitions (but in the gauge theoretical literature you'll not meet such a generality as one is usually concerned with SU(n) where irreducible connections have specific properties)

Def.(7.2.2) Let E $\rightarrow$ M a vector G-bundle for the natural action of G on V, G a connection for E with covariant derivative ∇. G is reducible if there exists proper vector subbundle E' of E s.t. D reduces to E'. Otherwise G is called irreducible.

In any case things behave exactly as one should expect for and the diagram (7.1.8) with the properties (7.1.9) goes through if we consider the subset of irreducible connections.

7.3 Two applications

In this section we show how the differentiable point of view can be used for a better understanding of the natural structures lying on **R**.

I On the moduli space of connections we can build easily a natural symplectic form ω and it turns out that the restriction of ω to the subset of gauge-equivalence classes of flat connections is exactly the symplectic structure of (2.3). On the enlarged set of equivalence classes of flat connections, its closedness is more transparent.

Let M be a closed oriented connected riemannian two-dimensional manifold and G a Lie group which preserves a non-degenerate symmetric bilinear form B on its algebra. Fix any principal G-bundle (P,p,M,G) and let C be the set of all 1-forms of connection smooth on P. C is an affine space with $\Omega_1(M,Px_G\text{LieG})$ as group of translations (G acts by conjugation on LieG). We define the pairing for each $A\in C$

(7.3.1)
$$\underline{\omega}^B : T_A C \times T_A C \longrightarrow \mathbb{R}$$
$$(\eta,\theta) \longrightarrow \int_M B_*(\eta\wedge\theta)$$

where

i) $B_*(\eta\wedge\theta)$ has the following meaning. Fix a basis $\{e_a\}_{a=1....n}$ in LieG and let $\eta=\eta_a e_a$ $\theta=\theta_a e_a$ $B(e_a,e_b)=B_{ab}$.

Then $B_*(\eta\wedge\theta)=\Sigma_{1\le a,b\le n}B_{ab}\eta_a\wedge\theta_b$ is a well-defined 2-form on M.

ii) the integral is computed with respect to the volume form induced by the metric and the orientation.

iii) the integral is well defined as M is compact.

In fact (7.3.1) is a 2-form on C. It is closed as it is invariant under translations.Now consider the subset of irreducible flat connections $C_0 \subset C$. It is a submanifold of the open subset of irreducible connections (see 7.1 and [G2]) with tangent space $Z^1(M,Lie_aG)$ at $a \in C_0$. This can be seen from 1.2 or by a direct computation: in fact C_0 is open in the set of connections a s.t. $d_a \bullet d_a=0$. So the tangent space T_aC_0 may be characterized as the subspace $L_a=\{\eta \in \Omega_1(M,Px_GLieG)$ s.t. $d_a\eta=0\}$ and this is the De-Rham way ([G2]) of expressing cohomology of M with values in the vector bundle Px_GLieG with flat structure induced by d_a. Finally one appeals to (7.1.15).
So the exterior 2-form (7.3.1) on T_aC restricts to a closed 2-form on $Z_1(\pi_1(M),LieG)$. $\underline{\omega}^B$ is degenerate on $Z_1(\pi_1(M),LieG)$ as if $\eta,\theta \in \Omega_1(M,Px_GLieG)$ s.t. $\eta=d_a\sigma$ $\sigma \in \Omega_0(M,Px_GLieG)$, then

$$\textbf{(7.3.2)} \qquad \underline{\omega}^B(\eta,\theta)=\int_M B_*(d_a\sigma\wedge\theta)= \int_M d(B_*(\sigma\wedge\theta))=0$$

At last the tangent space to the manifold of gauge equivalence classes of irr.connections can be identified with $H^1(M,LieG)$ (again this can be obtained from 1.3 or a direct differentiation of the gauge action and (7.1.15)). The symplectic structures ω^B and $\underline{\omega}^B$ coincide under the identification (7.1.15) (generalized De Rham theory) and, from the fact that the 2-form $\underline{\omega}^B$ is closed it follows that the reduced 2-form on the quotient space and so also ω^B are closed. Thus the proof of theor.(2.3.2) is complete.

II Another useful description of the symplectic structure (2.3) is given through theor.(7.1.15) and algebraic topological properties of cohomology in a flat vector bundle.
Let M,G as in (2.3). By theor.(7.1.15) we've an isomorphism

$$\textbf{(7.3.3)} \qquad H^1(\pi_1(M),Lie_\rho G) \cong H^1(M,\mathcal{F}(E))$$

where $E \to M$ is a flat vector bundle representing $\beta[\rho']$ $\rho':\pi_1(M) \to Aut(LieG)$ $\rho'(s)h=\rho(s)h\rho(s)^{-1}$ $h \in LieG$.
There exists a representation of cohomology in a flat vector bundle E in terms of singular chains on M. For instance a basis for the complex $C_k(M,E)$ of smooth k-chains with values in E consists of

smooth maps $\sigma:\Delta^k$ to M together with a flat section σ of σ^*E on Δ^k (denoted by $\sigma\otimes s$). Analogously a k-cochain with values in E is a function which assigns to each singular k-simplex σ a flat section of σ^*E over Δ^k. Most of cycle constructions of singular homology and cohomology work equally well with coefficients in a flat vector bundle.

As a useful example Poincare' duality isomorphism P_k: $H^k(M,E) \to H_{n-k}(M,E)$ may be interpreted on the chain level as follows.

If $A=\Sigma_{i=1}^m \Sigma\sigma_i\otimes a_i$ is a E-valued (n-k)-cycle, then $P_{k-1}[A]$ is the E-valued k-cocycle which assigns to each k-simplex τ: $\Delta^k \to M$ which intersects transversely the σ_i the flat section of $\tau^*(E)$ given by

$$\Sigma_{i=1}^m \Sigma_{p\in\sigma i\neq\tau}\varepsilon(p, \sigma_i, \tau)\tau^* a_i$$

Here $\varepsilon(p, \sigma_i, \tau)$ is the intersection number at p of σ_i and τ and $\sigma_i\neq\tau$ is the set of transverse intersections of σ_i and τ. For what concerns the value on the generic simplex β, as $P_{k-1}[A]$ is a cocycle, its value is the same as on any β' homotopic to β rel$\partial\beta$ and thus we may replace each k-simplex by one which is transverse to the σ_i.

We're mainly interested in the geometric interpretation of cup products in terms of intersection [D].

Namely let E_1,E_2 be flat vector bundles over M and B: $E_1 x E_2 \to \mathbb{R}$ a pairing. Let $A=\Sigma_{i=1}^l\sigma_i\otimes a_i$ be a E_1-valued k-cycle and $B=\Sigma_{j=1}^m \tau_j\otimes b_j$ a E_2-valued (n-k)-cycle. Then the cup product of the Poincare' duals of [A] and [B] is Poincare' dual to the intersection

$$\textbf{(7.3.4)} \quad A\bullet B = \Sigma_{i=1}^l\Sigma_{j=1}^m \Sigma_{p\in\sigma i\neq\tau}\varepsilon(p, \sigma_i, \tau_j)B(a_i(p), b_j(p))$$

Through this interpretation we're able to prove theor.(6.2.1)

proof of theor.(6.2.1) $T_\rho R \cong H^1(M,\mathcal{F}(E))$ (7.3.3). Let's choose 2g curves $a_1 \ldots . b_g$ on Σ, generators of $\pi_1(\Sigma,p)$ s.t. $\pi_1(W \times \{1\},p)$ is the free group genearated by $a_1 \ldots . a_g$ and with an intersection matrix

(7.3.5)

$$\begin{vmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{vmatrix}$$

Then $T_\rho \mathbf{Q}_1$ is generated by maps which assigns to $b_1 \dots . b_g$ the zero section of $F(E)$ on $b_1 \dots . b_g$.
By (7.3.4) and (7.3.5) $\mathbf{Q}_1$ is a lagrangian submanifold of $\mathbf{R}$.
Analogously for $\mathbf{Q}_2$.

8 Floer homology

This chapter must be read in close connection with chapter 6 and 7. In the light of diff.geometry we can see **R** as an embedded subset in the larger space of equivalence classes of 1-forms of connection on a principal G-bundle. The interesting fact is that this subset is exactly the zero set of a natural vector field on A and so we're enabled to use Morse-theory tools for studying it.
This is the way to produce, in close analogy with the study of Lagrangian intersections, a honology theory for **R** as Floer has shown with a careful analysis (1988). Floer homology is strictly related to Casson's invariant and this latter can be reinterpreted (Taubes 1988) in terms of spectral data as the Euler characteristic of the Floer's complex (compare with chapter 6). So in some sense Floer homology is a refinement of Casson's invariant but its meaning is not yet clear owing to the strong difficulties one meets in computimg it. At the physical level of rigor there exist many suggestions that Floer homology is deeply related to corresponding gauge-theoretical objects on four-manifolds like Donaldson's polynomials. But this is a story that mathematicians have still to write down.

8.1 Floer homology

The correspondence exhibited in (7.1) between Hom(π_1(M),G)/G and the set of flat connections on principal G-bundles over M, is a bridge towards the world of gauge theory as its configuration space is the affine space C of 1-forms of connections for a fixed principal G-bundle. Moreover in the peculiar case of 3-manifolds the curvature is a natural vector field(!) with critical points at exactly the flat connections..
In a heuristic sense, like in (7.1), the merit of differential geometry is that we can see **R** as embedded in a larger space, precisely the zero set (with respect to curvature) of C. What we gain is that we can use Morse theoretical tools on C for studying the set of critical points. Let's precise the framework.
If M is a oriented homology 3-sphere there exists only one class of isomorphisms of principal SU(2)-bundles over M. In fact $H^2(M,\mathbb{C})=0$ implies, by Chern isomorphism that, for each n all complex vector bundles of dimension n over M are trivializable. This means $H^1(M,GL(2,\mathbb{C}))$ and so $H^1(M,SU(2))=0$. So if we take MxSU(2) the trivial SU(2)-principal bundle over M, we know that the correspondence α (7.1) is a bijection between gauge equivalence classes of flat connections on MxSU(2) and equivalence classes of representations. Consider the whole affine space of 1-forms of connection on MxSU(2). For tecnical reasons it is usual to enlarge the space of smooth 1-forms and consider a Sobolev space of type $L_k{}^p$ (remember that M is compact): actually C is only a Frechet space and we would miss the possibility of using metric structures. So let $C_k{}^p=L_k{}^p(\Omega^1(M)\otimes su(2))$ where $\Omega^1(M)$ stands for 1-form on M and su(2) is the Lie algebra of SU(2). Notice that, in general, $C_k{}^p$ is an affine space with underlying space of translations the vector space of sections $\Omega^1(M)\otimes SU(2)$, but in this case we've a canonical choice of origin in the affine space: the trivial flat connection on MxSU(2). Analogously we can, in view of the triviality of the principal bundle identify the gauge group G with $C^\infty(M,SU(2))$ and we prefer to work with the Sobolev extended $G_k{}^p=L_{k+1}{}^p(M,SU(2))$. We'll assume that k+1>3/p so that $G_k{}^p$ consists of continuous

transformations.
We denote by $\mathcal{B}(=\mathcal{B}^p)$ the quotient space of the open dense set of irreducible connections in $\mathcal{C}_k{}^p$ by the gauge action of $G_k{}^p$. The assignment of the curvature to each connection may be converted to the following vector field (as we're on three manifolds) for a fixed metric σ on M

(8.1.1) $$f_\sigma : \mathcal{B} \rightarrow T\mathcal{B}$$ $$[a] \rightarrow [*F_a]$$

I explain the notation of (8.1.1).
To each 1-form of connection $a \in \mathcal{C}_k{}^p$ we can associate the curvature form $F_a = d_a a$ where d_a is the covariant derivative on $\Omega^1(M) \otimes su(2)$ associated to a. So F_a is a su(2)-valued 2-form on M (of type $L_{k-1}{}^p$). As M is Riemannian (for σ) we've a Hodge operator $*: \Omega^p(M) \rightarrow \Omega^{3-p}(M)$ for smooth forms and this can be extended naturally to

(8.1.2) $$*: L_k{}^p(\Omega^p(M) \otimes SU(2)) \rightarrow L_k{}^p(\Omega^{3-p}(M) \otimes SU(2))$$

So in (8.1.1) $*F_a$ must be understood as an element of $L_{k-1}{}^p(\Omega^1(M) \otimes su(2))$. In particular $*F_a$ has the property of being closed with respect to the adjoint covariant derivative $d_a{}^*$

(8.1.3) $$d_a{}^*(*F_a) = *d_a{}^*(*F_a) = *d_a F_a = \text{(Bianchi identity)}\,0$$

Now through generalized Hodge theory for Sobolev-type differential operators we've an identification

(8.1.4) $$T_{[a]}\mathcal{B} = \{b \in L_k{}^p(\Omega^1(M) \otimes su(2)) \mid d_a{}^*b = 0\}$$

for each chosen $a \in [a]$. So $[*F_a]$ is a well defined element in $T_{[a]}\mathcal{B}_{k-1}{}^p$ if we use the identification induced by $a \in [a]$.
At last we've to verify that the map (8.1.1) is well defined. If $a' \in [a]$ $a' = g \bullet a$ for some $g \in \mathcal{G}_k{}^p$, $*F_a{}' = g(*F_a)g^{-1}$ and by the identification induced by a' we've the same element as before in $T_{[a]}\mathcal{B}_{k-1}{}^p$.

We're in a situation analogous (and the analogy is much less superficial than what I'm telling here) to section (6.1): we've an infinite dimensional manifold with a smooth gradient vector field and we're interested in a generalized Morse-theory for this functional.
Notice that, as in the symplectic case we can't integrate globally (8.1.1) to a real function over B. On the universal covering of B, $\mathbf{B}$, the Chern-Simons form (not depending on σ)

(8.1.5)
$$F: \mathbf{B} \longrightarrow \mathbb{R}$$
$$[a] \longrightarrow \int_M tr({}^1/_2 a \wedge da + {}^1/_3 a \wedge a \wedge a)$$

induces by exterior derivative the 1-form

(8.1.6)
$$\underline{f}: \mathbf{B} \longrightarrow T^*\mathbf{B}$$
$$[a] \longrightarrow F([a])\xi_{[a]} = \int_M tr(F_a \wedge \xi_{[a]})$$

(remember that $\mathbf{B}$ is C_k^p (the irreducible subset) quotiented by the gauge action of the gauge subgroup $\{g \in G_{k+1}^p \mid \deg g=0\}$). $\underline{f}$ is exactly the 1-form one can get from the vector field f_σ (on $\mathbf{B}$) using the metric σ. But, with respect to gauge transformations F behaves as (on C)

(8.1.7) $$F(g \circ a) = F(a) + 2\pi \deg g \qquad g \in G_k^p$$

and so it isn't a well defined function on B.
At this point it is important to underline a property of the flow induced by f which I don't explore in all its consequences but it's crucial for framing our analysis in the much more developed study of instantons on four-manifolds. The trajectories of f can be identified with instantons on $M\times\mathbb{R}$. In fact an instanton on $M\times\mathbb{R}$ is an equivalence class of su(2)-valued 1-forms [A] on $M\times\mathbb{R}$ satisfying the equation

(8.1.8) $$F_A = *F_A \qquad (A \in [A])$$

(where the Hodge operator is defined with respect to the product metric). We can always choose $A \in [A]$ s.t. (8.1.8) has the form

(8.1.9) $$\partial A(\tau)/\partial\tau + f(A(\tau)) = 0$$

with zero temporal component. So an instanton can be identified with the one-parameter family

(8.1.10) $$\begin{aligned} \mathbb{R} &\longrightarrow B \\ \tau &\longrightarrow [A(\tau)] \end{aligned}$$

The memorable feature of all this stuff is that one can draw also in this case an infinite dimensional generalization of Witten's approach [W] to Morse theory. There is a delicate technical problem to deal with, because we can't guarantee a priori tht the critical points are isolated.

Def.(8.1.11) A non degenerate zero of f on B is the equivalence class of a flat connection for which 0 isn't in the spectrum of Tf.

By Sobolev inequalities a non-degenerate zero is isolated.

It is also interesting that this method has been succesfully employed for the first time for an analogous problem on 4-dim.manifolds [Do].
Consider a smoothly embedded loop in M γ and a tubular neighborhood $v(\gamma)$ of γ. Choose any diffeomorphism

(8.1.12) $$\phi_\gamma : S^1 x D^2 \longrightarrow v(\gamma)$$

s.t. $\phi_\gamma(S^1,\{0\})=\gamma$. Fix a smooth, rotationally symmetric bump function $\eta: D^2 \longrightarrow [0,1]$ which is 1 on (0,0) and 0 on ∂D^2. Let a a su(2)-valued 1-form of connection on MxSU(2). As it is a conjugation invariant the trace of the parallel transport defined by a along $\phi_\gamma(.,y)$ is a well defined number (independently of a chosen point in the fiber SU(2) and a base point in $\phi_\gamma(.,y)$) $p_\gamma(y,a)$. Moreover $p_\gamma(y,a)$ doesn't depend on the chosen element $a \in [a]$ (see 7.1). Thus we can define a function

(8.1.13)

$$P(\gamma,\eta): B \longrightarrow \mathbb{R}$$

$$[a] \longrightarrow \int_{D^2} p_\gamma(y,a)\,\eta(y)\,dy$$

This construction can be extended to a family $\{\gamma_i\}_{i=1....N}$ of smoothly embedded loops if we use any smooth function $g: \mathbb{R}^N \longrightarrow \mathbb{R}$

(8.1.14) $$P(\{\gamma_i\},\{\eta_i\}) = g(P(\gamma_1,\eta_1)....P(\gamma_N,\eta_N))$$

Let Π be the set of all $P(\{\gamma_i\},\{\eta_i\})$.

Theor.(8.1.15) Let $h\in\Pi$.

i) h is a smooth function on B.

ii) For every smooth metric σ on M, there exists a smooth section $\mathrm{grad}_\sigma h: B \longrightarrow TB$ s.t. $\forall\gamma\in T_{[a]}B$

$$<\mathrm{grad}_\sigma h([a]),\xi> = T_{[a]}h(\xi)$$

iii) Denote by R_h the zero set of the section

$$f_{\sigma h}: B \longrightarrow TB$$

$$f_{\sigma h} = f_\sigma + \mathrm{grad}_\sigma h$$

For a dense set of parameters $\sigma,h \in \Sigma\times\Pi$ (Σ is the set of metrics), **R** is non-degenerate and the set M of trajectories for $f_{\sigma h}$ decomposes into smooth oriented manifolds M(a,b) of trajectories connecting a,b∈R. (compare prop.6.1.9).

Again this decomposition of the set of trajectories and the property of our flow of being of Morse-Smale type, we can regularize the "difference" of Morse indexes at two critical points and it turns out that its value is exactly the dimension of the manifold of connecting trajectories.
This regularization is the spectral flow of the "Hessian" on the bundle $(\Omega^0\oplus\Omega^1)\otimes su(2)$

$$D_{\sigma\pi}(a) : L_k{}^p \longrightarrow L_{k-1}{}^p$$

(8.1.16)

$$D_{\sigma\pi}(a)(\Phi,\alpha) = (d_a{}^*\alpha, d_a\Phi + Df_{\sigma\pi}(a)\alpha)$$

which is selfadjoint and elliptic up to a part which is compact on the domain. The relative Morse index μ is well defined mod8 as it depends on the homotopy type of the trajectory and the spectral flow around a closed loop is a multiple of 8 ($\pi_1(B) = \mathbb{Z}$). Compare with section 6 and [F1].
We're now ready for defining Floer's complex (**R** non degenerate).
The elements C^p of the chain complex are the free $\mathbb{Z}$-modules over the set $x \in R$ s.t. $\mu(x)=p$ $p \in \mathbb{Z}_8$. The boundary operator $\partial : C^p \rightarrow C^{p-1}$ is so defined

(8.1.17) $$\partial x = \Sigma_{\mu(y)=p-1} \gamma(x,y) y$$

where $\gamma(x,y)$ is the sum over the signs of trajectories joining x and y (it's a finite number). The homology of the complex is well defined and does not depend on the choice of parameters.

8.2 Floer homology and Casson's invariant

In the spirit of our comparative approach, the next step consists of looking for a natural diagram like (3.2.5) at the level of connection spaces for a Heegard model of a homology 3-sphere. Such a diagram comes out quite spontaneously [Ta].

Let (W,h) be a Heegard model for an oriented homology 3-sphere M, An equivalent description of (W,h) is given by a self-indexing Morse function $f:M \rightarrow [0,3]$ with $Wx\{1\} = f^{-1}[0,3/2]$, $Wx\{2\} = f^{-1}[3/2,3]$. For convenience we'll deal with enlarged sets of W_1 and W_2. Denote

(8.2.1)
$$M_1 = f^{-1}([0,7/4]) \qquad M_2=f^{-1}([5/4,3])$$
$$M_0=M_1 \cap M_2$$

Since f has no critical values in the interval (1,2) M_0 is diffeomorphic to $\Sigma x[5/4,7/4]$ $(\Sigma=\partial W)$.

For $\alpha=0,1,2$ let C_α denote the space of $L_k{}^p$ connections on the trivial principal bundle and R_α the subset of conjugacy classes of irreducible flat SU(2)-connections . The inclusions

(8.2.2)
$$M_0 \xrightarrow{j_{1,2}} M_1, M_2 \xrightarrow{i_{1,2}} M$$

induce by pull-back the maps

(8.2.3)
$$C \xrightarrow{I} C_1 x C_2 \xrightarrow{J} C_0 x C_0$$

where $I=i_1{}^* x i_2{}^*$ and $J= j_1{}^* x j_2{}^*$. The equivariant version of (8.2.3) is

(8.2.4) $B \xrightarrow{I} B_1 x B_2 \xrightarrow{J} B_0 x B_0$

where ve've introduced the quotients of the subsets of connections which restrict to irreducible connections on $([11/8,13/8] x\Sigma)$ x SU(2).

__Prop. (8.2.5)__ The sequence (8.2.4) is exact in the sense that I is an embedding, J an immersion and the image of I is the counterimage $J^{-1}(\Delta)$ of the diagonal Δ. R_α is a submanifold of B_α.

The assignment of curvature to a connection on B_α is a section of a vector bundle $L_\alpha \rightarrow B_\alpha$ (it is no longer the tangent bundle) with zero set R_α.
Like in (8.1) we can remove degeneracy in R_α in a dense set of perturbative parameters (the family of embedded loops and their tubular neighborhoods must lie in the interior of M_0). So the natural substitute of (3.2.5) is the following identity

(8.2.6) $\mathbf{R} = I^{-1}[(R_1 x R_2) \cap J^{-1}(\Delta)]$

It is the Fredholm properties of the maps which appear in eq.(8.2.6) which allow for the comparison of spectral flow data on B with intersection data on **R**.

We're now ready for introducing an analytical definition of Casson's invariant and explaining why Floer homology is a generalization of it ([Ta]).
The object which corresponds to Casson's invariant in the gauge theoretical framework is the Euler class of Floer homology. I want to clarify this point in some detail. Given a non-degenerate perturbation $f_{\sigma\pi}=f'$ of f on B (8.1) the mod-2 spectral flow of the family of operators $D_{\sigma\pi}$ (8.1.15) along a path in B between two zeros is a regularization of the relative sign between the determinants of the matrices at the two endpoints of the path.
The set **R'** is finite and, chosen a particular zero [a] we can associate to each zero [a'] of f' in B the mod2 spectral flow Δ(a,a') for Df' along paths in B between [a] and [a']. The choice of a basepoint was arbitrary in section 6, but here we've a characteristic flat connection orbit, the orbit of the product connection on MxSU(2). It is understood, from here onwards, that this is the choice we make.
Finally we set

Def. (8.2.7) $\chi(f) = \Sigma_{[a]\in R'} \Delta(a,a')$

$\chi(f)$ can be regarded as the Euler characteristic of Floer homology for M and it doesn't depend on the choice of the perturbation.

Roughly speaking Casson's invariant and $\chi(f)$ associate both a sign to each point in **R**. The whole job is in comparing these signs. The procedure is analogous to the one outlined in section 6 except for the additional obstacle of two technical difficulties.
First of all the operators involved have different properties and in this case one has to deal with the delicate tool of infinite-dimensional K-theory.
Moreover the following diagram

(8.2.8)
$$\begin{array}{ccccc} T\mathcal{B} & \longrightarrow & T\mathcal{B}_1 \mathrm{x} T\mathcal{B}_2 & \longrightarrow & T\mathcal{B}_0 \mathrm{x} T\mathcal{B}_0 \\ {\scriptstyle Df'}\downarrow & & {\scriptstyle Df_1'\mathrm{x}Df2'}\downarrow & & {\scriptstyle Df_0'\mathrm{x}Df_0'}\downarrow \\ L & \longrightarrow & L_1 \mathrm{x} L_2 & \longrightarrow & L_0 \mathrm{x} L_0 \end{array}$$

is <u>not</u> commutative and one has to adjust the operators Df with something else.
The final result is

Theor. (8.2.9) [Ta] Let M be an oriented homology 3-sphere. Then

$$\lambda(M) = 2\chi(f)$$

8.3 Some forethoughts

Unlike Casson's invariant which is relatively easy to compute as its behaviour under surgery is known, there're few examples of explicit computation for Floer homology. Fintushel and Stern ([F-S]) have computed it for Brieskorn homology spheres where cycles occur only in even or odd dimensions and it is expected that this phenomenon of having chains in even or odd dimensions is more general ([Bra]). Perhaps some results can be obtained in the evaluation of homology groups for connected sums of homology spheres. So it is still difficult to appreciate the topological meaning of Floer homology.
Various remarks are in order.

Regarding Floer homology as a refinement of Casson invariant one is pressed to look for surgery properties of these homology groups or an algorithm for describing them in terms of representation spaces (see the suggestions in [A] [Bra]).

On the other side one could use the tools of global analysis for characterizing Floer homology.
In finite dimensions the Witten-Morse approach leads to the usual homology of the underlying manifold, while in this case it seems they're not related to the ordinary homology groups of B and they're affected by the topology of B in a non-trivial way.
I think that a beautiful model to keep in mind for such an analysis is the study of the relevant ends of the Yang-Mills functional accomplished by Taubes [Ta2] in line with the current work of mathematicians on analogous problem in "infinite dimensional Morse theory".

Unending lists of open problems and conjectures about the relation of Floer homology and invariants of 3 and 4-manifolds can be found in [A].

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INDEX