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PERIODIC SOLUTIONS FOR SOME CLASSES OF NONLINEAR RETARDED FUNCTIONAL DIFFERENTIAL EQUATIONS

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Table of Contents

Introduction.

The main purpose of this thesis work is to present some results concerning existence of periodic solutions of retarded functional differential equations.
One of the main assumptions which lead to describe the state and evolution of a system by means of ordinary or partial differential equations, is that the system is governed by a principle of causality; that is, the future state of the system is determined solely by the present values of the state and of the rate of change of the state. However, after closer inspection, one realizes that this assumption is often only a first approximation to the true situation and that a more realistic model would include also some of the past states of the system. There are even examples in which it is meaningless not to have dependence on the past.
Although these facts have been realized a long time ago, until the pioneering works of Volterra there seems to have been little interest in a general theory of differential equations with dependence on the past: most results were concerned only with special properties of very special equations. The first non specific formulation for such problems is thus due to Volterra (in [V1], [V2], [V3]) who wrote some rather general differential equations incorporating the past states of the system, in his research on predator-prey models and viscoelasticity.
It is undoubtedly right from the applications that came the great rise of interest for the mathematical theory of these problems. Indeed, in the late thirties and early forties, Minorsky, in his study of ship stabilization and automatic steering, pointed out very clearly the importance of the consideration of the delay in the feedback mechanism (see for example [MIN]).
It was in the fifties that the first books completely devoted to this theory appeared, as well as the first attempts to organize the general theory of linear systems (see Mishkis, [MIS]). Moreover, the great interest and evolution of control problems that started just about that time influenced significantly the rapid development of retarded functional differential equations. Successively, Bellman and Danskin in [BD] and Bellman and Cooke in [BC] presented a well organized theory of linear equations with constant coefficients and the beginning of stability theory. They also pointed out the very wide range of applications of differential equations with deviating arguments, from biology to economics. It soon became evident that many problems in this field are more meaningful if one considers the motion in a function space, even though the state variable is a finite-dimensional vector. In the last thirty years or so the subject has undergone a rapid development, with new applications arising from most different fields, and new mathematical problems which require modification of even the definitions of the basic equations.

Before turning to the specific problems we have considered, we wish to give an idea of the very different types of equations which go under the name of functional differential equations. The simplest example of past dependence in a differential equation is through the state variable and not the derivatives of the state variable, that is, the so-called retarded functional differential equation, whose general form (in the case of a single constant delay τ) is given by

$$x'(t) = F(t, x(t), x(t-\tau)), \tag{0.1}$$

where $F : \Omega \subset \mathbb{R}^3 \to \mathbb{R}$ is for example a continuous function.

Particular cases of this equation can be found in the literature (e.g. [CU],[WR]) in the theory of the growth of a single species and in the study of the distribution of primes, just to quote two very different applications.

Very interesting features can also be found in equations in which the delayed argument also occurs in the derivatives of the state variable, the so called neutral functional differential equations. These problems are somewhat more difficult to motivate, but often arise in the study of two or more oscillatory systems with some interconnections between them. It is from this kind of application that the importance of periodic solutions becomes relevant, as it carries a precise physical meaning.

Finally, we would like to mention some variational problems that are strictly related to functional differential equations. El'sgol'ts in [EL] and other authors have considered the problem of minimizing the functional

$$I(x) = \int_0^1 F(t, x(t), x(t-\tau), x'(t), x'(t-\tau))dt \tag{0.2}$$

over some class of functions with suitable boundary conditions. Generally the Euler equations for functionals of the type (0.2) take the form

$$x''(t) = f(t, x(t), x(t-\tau), x'(t), x'(t-\tau), x''(t-\tau)), \tag{0.3}$$

which leads to neutral differential equations.

Many more examples could be produced to illustrate the importance and frequency of occurrence of equations which depend on past history; for a good collection of these, we refer to the book of Hale [H1].

From this great variety of situations it becomes clear at first glance that it is almost impossible to find a class of equations which contains all of these and is still mathematically tractable and interesting. In that case one could write an equivalent integral equation and then consider general operator equations to obtain existence, uniqueness and other properties. Indeed there have been in the literature some attempts in this direction (e.g. [NE]); the main difficulty in this approach is to carry into the resulting functional equation all the properties contained in the original differential equation. One could in this way hope to get a general existence result, but then it becomes a major task to verify that one of the special equations satisfies all of the required hypotheses.

This is one of the reasons why we have chosen a different approach for this thesis work. Our goal is to study the existence of periodic solutions of some delayed equations linked by a common feature, which we are going to illustrate. Various methods known in the literature rely upon the fact that in some cases the delayed differential equation can be seen as a perturbation of an ordinary differential equation. This occurs for example when the delayed term can be considered smaller in some sense or "dominated" by the ordinary one. To fix ideas, one can think of the model problem we have chosen for the first part of this work, namely the existence of solutions for

$$(0.4) \qquad \begin{cases} x'(t) + ax'(t-\tau) = f(t,x(t)) \\ x(0) = x(2\pi) \end{cases}$$

where $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function 2π-periodic in the first variable, $a \in \mathbb{R}$ and $\tau \in]0,2\pi[$. This equation, when $a \neq 0$, is a neutral equation. Here the "domination" of the term $x'(t)$ over the delayed one $x'(t-\tau)$ can be well understood when we take the parameter a such that $|a| < 1$. In this case we have in fact many good properties concerning the differential operator, such as invertibility and continuity of the inverse. These are the properties which enable us to transform the original problem into a fixed point equation that can be studied by means of some topological degree argument. It is moreover worth mentioning that the domination we have described allows also to obtain some a priori estimates that can be very hard to prove or even false when no domination occurs. The study of Problem (0.4) is carried out letting the parameter a take values in $[-1,1]$ and looking for sufficient conditions on f to ensure the existence of solutions, for every delay τ. Restrictions on τ are only imposed in the analysis of the critical case $a = 1$. There we can still get an existence result using techniques which were first introduced to study partial differential equations, and in particular for the wave operator with periodic boundary conditions, a problem which has, rather surprisingly, many points in common with ours.

Following the same idea that led us to study the case in which domination is not present, we examine in Chapter 2 a different equation, this time with the delay in the nonlinear terms, but which gives rise to similar problems. We again look for sufficient conditions on the nonlinearity to obtain a solution of

$$(0.5)\qquad \begin{cases} x'(t) = af(t,x(t)) - bf(t,x(t-\tau)) + p(t) \\ x(0) = x(2\pi) \end{cases}$$

both in the "good" case $|a| > |b|$ and in the critical case $|a| = |b| = 1$.

The choice of Equation (0.5) was motivated, among other reasons, also because it is, along with some of its variants, a very important model in the theory of diffusion of infectious diseases and epidemics (see e.g. [CK],[DL]).

Many results are known in the literature concerning Problem (0.5), especially for what concerns the asymptotic behaviour of solutions. One of the basic hypotheses in many papers is that the nonlinearity f be monotone. Our aim is to show that this assumption can be completely abandoned if it is replaced by some growth condition, as far as periodic solutions are concerned.

The thesis is subdivided in two chapters: the first one is devoted to the study of Problem (0.4), while in the second we examine Problem (0.5).

In Section 1.2. we make a complete analysis of the properties of the differential operator when the parameter a is allowed to vary, and in Section 1.3. we give a first existence result in the case $|a| < 1$. The case without domination is treated in Section 1.4., while in Section 1.5. we re-examine all the results obtained in the previous parts,when the nonlinearity is allowed to take the special form $f(t,x) = g(x) + h(t)$. We are able to show that all the results obtained still hold true when the hypotheses are very much weakened.

In Section 2.1. we briefly expose some motivations and the "physical background" in which Equation (0.5) is situated, and we make some preliminary observations; in Section 2.2. we give a first existence result. Section 2.3. is devoted to the dominated case $|a| > |b|$. Finally, the critical case $|a| = |b| = 1$ is treated in Section 2.4.

Preliminaries.

Throughout this thesis we will denote by $\mathbb{N}$ the set of positive integers, $\{1,2,...\}$.

All the function spaces we use are spaces of 2π-periodic functions, except where expressly stated. We denote therefore simply by C^0 the space of continuous 2π-periodic functions, equipped with the norm $\|u\|_\infty = \max\{|u(t)| \,/\, t\in[0,2\pi]\}$.We denote moreover by $L^p$the completion of C^0 under the norm $\|u\|_p = \left(\int_0^{2\pi} |u(t)|^p dt\right)^{1/p}$, $1 \le p < +\infty$.

H^1 is the usual Sobolev space of L^2 (2π-periodic) functions whose distributional derivatives are still representable by an L^2 function. We denote by $\|\cdot\|_{1,2}$ its norm and we recall that it is compactly embedded in L^p and C^0, for every $p \ge 1$.
If Ω is a bounded open set in $\mathbb{R}^n$, $p \in \mathbb{R}^n$ and if $g : \overline{\Omega} \to \mathbb{R}^n$ is continuous, we will denote by $\deg(g,\Omega,p)$ the Brouwer degree of g with respect to Ω and p.
Similarly, if X is a Banach space, $\Omega \subset X$ is open and bounded, $p \in X$ and $f : \overline{\Omega} \to X$ is a compact perturbation of the identity, $\deg(f,\Omega,p)$ is the Leray-Schauder degree of f with respect to Ω and p.

CHAPTER 1

NONLINEAR EQUATIONS OF NEUTRAL TYPE

1.1. Neutral equations.

The precise definition of what a neutral equation is, is not at all simple, due to many technical subtleties. Roughly speaking, one can think of a neutral equation as a differential equation where the value of the unknown function depends not only on the present values of the function itself and of its derivatives, but also on past values, and in particular past values of the derivatives. That is to say, for first order equations, delays are present in the derivatives of the unknown function. To fix ideas, one can think to the model nonhomogeneous linear equation of the first order, which is

$$x'(t) + ax'(t-\tau) = bx(t) + cx(t-\tau) + f(t) \tag{1.1.1}$$

where a,b,c, τ are constants, $a \neq 0$, and f is continuous.
Clearly the general definition of neutral functional differential equation is much more complicated, since it has to contain a much wider class of equations than those represented by (1.1.1).
In order to make clear what is meant by neutral equation, we need some preliminary definitions from the classical theory for which we follow the book of J. Hale [H1].
Suppose $\tau \geq 0$ is a given real number, $C = C^0([-\tau,0];\mathbb{R}^n)$ is the Banach space of continuous mappings from $[-\tau,0]$ to $\mathbb{R}^n$ with the topology of the uniform convergence; if $\sigma \in \mathbb{R}$, $A \geq 0$ and $x \in C([\sigma-\tau, \sigma+A]; \mathbb{R}^n)$, then for any $t \in [\sigma,\sigma+A]$ we let $x_t \in C$ be defined by $x_t(\theta) = x(t+\theta)$ for all $\theta \in [-\tau, 0]$.
If D is a subset of $\mathbb{R} \times C$ and $f:D \to \mathbb{R}^n$, the relation

$$x'(t) = f(t,x_t) \tag{1.1.2}$$

is called a retarded functional differential equation.
A function x is said to be a solution of (1.1.2) on $[\sigma-\tau, \sigma+A[$ if there are $\sigma \in \mathbb{R}$ and $A > 0$ such that $x \in C([\sigma-\tau, \sigma+A[; \mathbb{R}^n)$, $(t,x_t) \in D$ and x(t) satisfies (1.1.2) for $t \in [\sigma,\sigma+A[$.
For given $\sigma \in \mathbb{R}$, $\phi \in C$, we say that $x = x(\sigma,\phi,f)$ is a solution of (1.1.2) with initial value ϕ

at σ or simply a solution through (σ,ϕ), if there is an $A>0$ such that $x(\sigma,\phi,f)$ is a solution of (1.1.2) on $[\sigma-\tau,\sigma+A[$ and $x_\sigma(\sigma,\phi,f)=\phi$.
We remark that (1.1.2) is a very general type of equation, and includes ordinary differential equations(τ=0), differential difference equations, as well as some integro-differential equations and much more general equations.
For various technical reasons, for which we refer to [H1], we introduce some more definitions. If X and Y are Banach spaces, we denote by $\mathcal{L}(X,Y)$ the usual Banach space of bounded linear mappings from X to Y. When $L\in \mathcal{L}(C, \mathbb{R}^n)$, by the Riesz representation theorem there exists an $n\times n$ matrix η on $[-\tau,0]$ of bounded variation such that

$$L\phi = \int_{-\tau}^{0}[d\eta(\theta)]\phi(\theta)\,, \quad \forall\ \phi\in C.$$

For any such η we always understand that we have extended the definition to $\mathbb{R}$ so that $\eta(\theta)=\eta(-\tau)$ for $\theta\le -\tau$ and $\eta(\theta)=\eta(0)$ for $\theta\ge 0$.

Definition 1.1.1. (Hale) Let Λ be an open subset of a metric space. We say that

$$L:\Lambda\to \mathcal{L}(C;\mathbb{R}^n)$$

has smoothness on the measure if, for any $\beta\in\mathbb{R}$ there is a scalar continuous function

$$\gamma:\Lambda\times\mathbb{R}\to\mathbb{R},$$

$\gamma(\lambda,0)=0$, such that, if $L(\lambda)\phi = \int_{-\tau}^{0}[d\eta(\lambda,\theta)]\phi(\theta)$, $\lambda\in\Lambda$, $s>0$, then

$$(1.1.3)\qquad \left|\lim_{h\to 0^+}\left(\int_{\beta+h}^{\beta+s}[d\eta(\lambda,\theta)]\phi(\theta)+\int_{\beta-s}^{\beta-h}[d\eta(\lambda,\theta)]\phi(\theta)\right)\right| \le \gamma(\lambda,s)\,\|\phi\|.$$

If $\beta\in\mathbb{R}$ and the matrix $A(\lambda;\beta,L)=\eta(\lambda,\beta^+)-\eta(\lambda,\beta^-)$ is nonsingular at $\lambda=\lambda_0$, we say that $L(\lambda)$ is *atomic at* β at λ_0. If $A(\lambda;\beta,L)$ is nonsingular on a set $K\subset\Lambda$, we say that $L(\lambda)$ is *atomic at* β on K. As far as we are concerned the case we are interested in is $\Lambda=\Omega\subset\mathbb{R}\times C$, that is $L\in C(\Omega;\mathcal{L}(C,\mathbb{R}^n))$.

If $D : \Omega \to \mathbb{R}^n$ has a continuous first (Frechet) derivative with respect to ϕ, then it is known that D_ϕ has smoothness on the measure.
Then we can give the following

Definition 1.1.2. (Hale) Suppose $\Omega \subset \mathbb{R} \times C$ is open. A mapping $D : \Omega \to \mathbb{R}^n$ is said to be *atomic at β* on Ω if D is continuous together with its first and second Frechet derivatives with respect to ϕ, and D_ϕ , the derivative with respect to ϕ is atomic at β on Ω.

For examples illustrating this definition we refer again to [H1].
We are now ready to give the general definition of neutral functional differential equation:

Definition 1.1.3. Suppose $\Omega \subset \mathbb{R} \times C$ is open, $f : \Omega \to \mathbb{R}^n$, $D : \Omega \to \mathbb{R}^n$ are continuous functions with D atomic at zero. The relation

(1.1.4) $$\frac{d}{dt} D(t,x_t) = f(t,x_t)$$

is called a neutral functional differential equation. The function D is called the difference operator.
A function x is said to be a solution of (1.1.4.) on $[\sigma-\tau, \sigma+A[$ if there are $\sigma \in \mathbb{R}$ and $A > 0$ such that

$$x \in C([\sigma-\tau, \sigma+A[; \mathbb{R}^n),$$
$$(t,x_t) \in \Omega,$$

$D(t,x_t)$ is continuously differentiable and satisfies (1.1.4) for $t \in [\sigma,\sigma+A[$. For given $\sigma \in \mathbb{R}$, $\phi \in C$ and $(\sigma,\phi) \in \Omega$ we say that $x(\sigma,\phi,D,f)$ is a solution of (1.1.4) through (σ,ϕ) if there is an $A > 0$ such that $x(\sigma,\phi,D,f)$ is a solution of (1.1.4) on $[\sigma-\tau,\sigma+A[$ and $x_\sigma(\sigma,\phi,D,f) = \phi$.
If $D(t,\phi) = D_0(t,\phi) - g(t)$ and $f(t,\phi) = L(t,\phi) + h(t)$ with D_0 and L linear in ϕ, the equation is called linear. It is linear homogeneous if $g = h = 0$; it is autonomous if D and f do not depend on t.
We now give some examples of neutral functional differential equations.

Example 1.1.1. If $D(\phi) = \phi(0)$ for all ϕ, then D is atomic at zero. Consequently, for any continuous $f : \Omega \to \mathbb{R}^n$, the pair (D,f) defines a neutral functional differential equation. therefore retarded functional differential equations are neutral functional differential equations.

Example 1.1.2. If $\tau > 0$, B is an $n \times n$ constant matrix, $D(\phi) = \phi(0) - B\phi(-\tau)$ and $f : \Omega \to \mathbb{R}^n$ is continuous, the pair (D,f) defines the neutral functional differential equation

$$\frac{d}{dt}[x(t) - Bx(t-\tau)] = f(t,x_t) \tag{1.1.5}$$

Many results concerning existence, uniqueness and continuous dependence are known for general neutral functional differential equations. For a good collection of this kind of results we refer to [H1].

It is clear however that general results for such a vast class of equations are very difficult to prove. In the first part of this thesis we are concerned with the search for periodic solutions for a very restricted class of neutral functional differential equations. In this framework it is natural to ask if some kind of Fredholm alternative theory can be made for such equations. The answer (see [H2]) is that there are results in this direction but only for a rather small subclass of neutral functional differential equations, namely for those equations whose difference operator D satisfies a property called D-stability, [H1]. Existence theorems for periodic solutions of D-stable equations are generally obtained via the application of the Leray-Schauder theory or the coincidence degree. One of the main points of interest in considering non D-stable operators is that in this case not only there is no known Fredholm alternative result, but also many rather unusual problems arise. These problems can be roughly classified at two different levels. First, at a "linear" level the injectivity of the differential operator does not necessarily imply its surjectivity; second and especially important in the applications, even when some kind of generalized inverse of the differential operator can be defined, it turns out that it is not continuous, let alone completely continuous. So, when one is interested in studying the existence of periodic solutions of an equation like

$$\frac{d}{dt} D(t,x_t) = f(t,x_t) \tag{1.1.6}$$

one realizes that, contrary to a great number of other similar problems, this equation cannot be transformed into a fixed point problem for a (at least) continuous mapping in Banach spaces. It is clear then, that some supplementary work has to be done even to choose for equation (1.1.6) a proper functional setting.

We wish to point out that neutral functional differential equations are not the only example of such a difficulty. Indeed, it is well known that in the theory of partial differential equations, and particularly for hyperbolic equations, this kind of problems are very common.

Just to make an example, we recall that when one studies the existence of periodic solutions for the semilinear wave equation, that is when one looks for a function $u : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ of

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = g(u) + h$$
$$u(0,x) = u(2\pi,x)$$
$$u(t,0) = u(t,2\pi)$$

one is naturally led to consider nonlinear perturbations of a linear operator whose kernel has infinite dimension (see for example [KA],[M2]).
Various techniques have been developed by many authors to overcome this kind of difficulty. The main motivation of this work is to investigate which among these techniques are suitable to study neutral functional differential equations.
Obviously, the difficulty of this task has forced us to consider not a generic neutral equation, for which, as many authors have pointed out, a general theory is still far to come, but a model problem depending on some parameters. When we allow these parameters to vary the equation under study takes a form which underlines the different situations and difficulties that we have described above. The particular equation we have chosen also allows us to compare our results with the ones known in the literature: in some cases we have weakened the hypotheses of existence theorems already proved, while in others our approach seems to be new, as far as neutral equations are concerned.
The plan of the Chapter goes as follows: in Section 1.2. we develop the linear theory for the differential operator and we establish the main properties that we will need in the applications. In Section 1.3. we prove an existence theorem for a relatively easy case; Section 1.4. is devoted to the "critical" case in which the kernel of the differential operator has infinite dimension. Finally, in Section 1.5. we examine the particular situation which arises when the nonlinearity has a special form: it is proved that the hypotheses of the theorems from the previous sections can be weakened very much without loosing the existence of solutions.

1.2. Study of the differential operator.

The aim of this section is to investigate the structure of a family of differential operators in order to establish the properties which we are going to use throughout this chapter. The properties we are concerned with are the invertibility and the continuity and compactness of the inverse or of some generalized inverse of a differential operator. This program is carried out by means of the study of an eigenvalue problem depending on a real parameter which describes all the cases that we are going to treat.

More precisely we consider the following family of operators:

(1.2.1) $$L_\lambda : \text{dom}\, L_\lambda \subset L^2 \to L^2$$

defined by

(1.2.2) $$\text{dom}\, L_\lambda = H^1$$

(1.2.3) $$(L_\lambda u)(t) = u'(t) + au'(t-\tau) + \lambda u(t)$$

where $\lambda \in \mathbb{R}$, $\tau \in]0, 2\pi[$ and, since we are interested in the delay equation, $a \in \mathbb{R} \backslash \{0\}$. We will study the eigenvalue problems

(1.2.4) $$L_\lambda u = 0 \text{ in } L^2$$

when a varies in its domain of definition with the aim of establishing for which values of a and λ the operator L_λ is invertible. We will achieve this by means of Fourier expansions, subdividing the problem into three cases depending on the range of values a is allowed to take.

For the properties on Fourier series we are going to use we refer to [ED].

First of all we introduce the complete orthonormal system $\mathcal{L}$ in L^2 given by

$$\mathcal{L} = \left\{ \frac{1}{\sqrt{2\pi}}, \frac{\cos kt}{\sqrt{\pi}}, \frac{\sin kt}{\sqrt{\pi}} \ / \ k \in \mathbb{N} \right\} .$$

Then, to every $u \in L^2$ we associate its expansion with respect to $\mathcal{L}$, that is

(1.2.5) $$u(t) \sim \frac{u_0}{\sqrt{2\pi}} + \frac{1}{\sqrt{\pi}} \sum_{k=1}^{\infty} u_k \cos kt + v_k \sin kt \ .$$

In the same way, if u, given by (1.2.5) belongs to H^1, we associate to $u'(t)$ and $u'(t-\tau)$ the expressions

$$u'(t) \sim \frac{1}{\sqrt{\pi}} \sum_{k=1}^{\infty} kv_k \cos kt \; - \; ku_k \sin kt \tag{1.2.6}$$

and

$$\begin{aligned} u'(t-\tau) \sim \frac{1}{\sqrt{\pi}} \sum_{k=1}^{\infty} & (kv_k \cos k\tau + ku_k \sin k\tau) \cos kt + \\ & + (kv_k \sin kt - ku_k \sin k\tau) \sin kt \, . \end{aligned} \tag{1.2.7}$$

It is now possible to write the differential operator L_λ as

$$\begin{aligned} (L_\lambda u)(t) \sim \lambda \frac{u_0}{\sqrt{2\pi}} + \frac{1}{\sqrt{\pi}} \sum_{k=1}^{\infty} & (kv_k + akv_k \cos k\tau + aku_k \sin k\tau + \lambda u_k) \cdot \\ & \cdot \cos kt + (-ku_k + akv_k \sin k\tau - aku_k \cos k\tau + \lambda v_k) \sin kt \, . \end{aligned} \tag{1.2.8}$$

We can study the invertibility of L_λ trying to solve, for a fixed $v \in L^2$ the equation

$$L_\lambda u = v \, . \tag{1.2.9}$$

If we call respectively y_k and z_k the Fourier coefficients of v with respect to $\mathcal{L}$, we are led to solve for (1.2.9) the following system of linear algebraic equations:

$$\begin{cases} \lambda u_0 = y_0 \\ u_k(ak\sin kt + \lambda) + v_k(k + ak \cos k\tau) = y_k \, , \; k \in \mathbb{N} \\ -u_k(k + ak \cos k\tau) + v_k(ak \sin k\tau + \lambda) = z_k \, , \; k \in \mathbb{N} \, . \end{cases} \tag{1.2.10}$$

With the notation

$$\begin{cases} R_k = ak \sin k\tau + \lambda \, , \; k \in \mathbb{N} \\ S_k = k + ak \cos k\tau \, , \; k \in \mathbb{N} \end{cases} \tag{1.2.11}$$

we can write (1.2.10) in the more convenient form

(1.2.12)
$$\begin{cases} \lambda u_0 = y_0 \\ R_k u_k + S_k v_k = y_k \ , \ k \in \mathbb{N} \\ -S_k u_k + R_k v_k = z_k \ , \ k \in \mathbb{N} \end{cases}$$

and finally, calling M_k the matrix $\begin{bmatrix} R_k & S_k \\ -S_k & R_k \end{bmatrix}$ we obtain

(1.2.13)
$$\begin{cases} \lambda u_0 = y_0 \\ M_k \begin{pmatrix} u_k \\ v_k \end{pmatrix} = \begin{pmatrix} y_k \\ z_k \end{pmatrix} \ , \ k \in \mathbb{N} \ . \end{cases}$$

At this point we begin by verifying for which values of a, τ and λ, L_λ is injective. We take therefore $v \equiv 0$ and we determine a, τ and λ in order that (1.2.13) has only the trivial solution. First of all observe that for every a and τ, if $\lambda = 0$, then L_0 is never injective since from (1.2.13),

$$\text{Ker } L_0 \supset \text{span } \{1\} \ ,$$

that is, all the constant functions are in the kernel of L_0.

We then suppose $\lambda \neq 0$ and we come to examine the injectivity of L_λ when a varies. The reason of this choice comes from the fact that the behavior of L_λ is determined by the values of a in such a way that, as far as injectivity is concerned, the three cases $|a| < 1$, $|a| = 1$, and $|a| > 1$ give rise to very different situations, and, as we shall see, the case $|a| = 1$ can be viewed as critical.

Now when $v \equiv 0$ and $\lambda \neq 0$, (1.2.13) has nontrivial solutions if and only if, for some $k \in \mathbb{N}$,

(1.2.14)
$$\Delta_k \equiv \det M_k = 0 \ .$$

But $\Delta_k = 0$ if and only if

(1.2.15)
$$\begin{cases} R_k = 0 \\ S_k = 0 \end{cases}$$

and this is precisely the equation that we are going to study when a varies.
We begin by supposing $|a| < 1$. In this case equation (1.2.15) means

$$\begin{cases} ak \sin k\tau + \lambda = 0 \\ a \cos k\tau + 1 = 0 \end{cases} \tag{1.2.16}$$

Since $|a| < 1$ one has $a \cos k\tau + 1 \neq 0$ for every $k \in \mathbb{N}$.
We can summarize this into

Proposition 1.2.1. Let $|a| < 1$.
If $\lambda = 0$ then $\operatorname{Ker} L_0 = \operatorname{span}\{1\}$.
if $\lambda \neq 0$ then L_λ is injective.

We now turn to the case $|a| > 1$. We have

Proposition 1.2.2. Let $|a| > 1$.
If $\lambda = 0$ then $\operatorname{Ker} L_0 = \operatorname{span}\{1\}$.
If $\lambda \neq 0$ and $|\lambda| \neq k\sqrt{a^2 - 1}$ for every $k \in \mathbb{N}$
then L_λ is injective.
If $\lambda \neq 0$ and $|\lambda| = k\sqrt{a^2 - 1}$ for some $k \in \mathbb{N}$
then $\operatorname{Ker} L_\lambda = \operatorname{span}\left\{\cos kt, \sin kt \,/\, k = \dfrac{|\lambda|}{\sqrt{a^2-1}}\right\}$.

Proof. Let $\lambda = 0$; then obviously $\operatorname{Ker} L_0 \supset \operatorname{span}\{1\}$. Moreover $\operatorname{Ker} L_0$ contains some nonconstant function if and only if $\Delta_k = 0$ for some $k \in \mathbb{N}$. But $\Delta_k = 0$ if and only if

$$\begin{cases} \sin k\tau = 0 \\ 1 + a \cos k\tau = 0 \end{cases}$$

and this is impossible since $|a| \neq 1$.
Let $\lambda \neq 0$; then, if $\cos k\tau = -1/a$, $|\sin k\tau| = 1/|a| \sqrt{a^2 - 1}$. But then $\lambda^2 = a^2 k^2 \sin^2 k\tau = k^2(a^2 - 1)$. Therefore, if $\forall k \in \mathbb{N}$, $|\lambda| \neq k\sqrt{a^2 - 1}$, the equation $a k \sin k\tau + \lambda = 0$ is never satisfied and L_λ is injective.
Finally, if $|\lambda| = k\sqrt{a^2 - 1}$ for some $k \in \mathbb{N}$, then $\operatorname{Ker} L_\lambda = \operatorname{span}\{\cos kt, \sin kt\}$. ♦

We end the discussion of the injectivity of L_λ by considering the case $|a| = 1$. As we have said before this case can be considered critical with respect to the others because the behavior of L_λ depends not only on λ and a , but also on τ .
It is convenient to distinguish between the two cases $a = 1$ and $a = -1$. More precisely, we have the following

Proposition 1.2.3. Let $a = 1$.

If $\lambda \neq 0$, then L_λ is injective.

If $\lambda = 0$ and either $\tau = \frac{q\pi}{p}$, with q, p coprime integers and q even , or $\frac{\tau}{\pi} \in \mathbb{R} \setminus \mathbb{Q}$,

then $\mathrm{Ker}\, L_0 = \mathrm{span}\,\{1\}$.

If $\lambda = 0$ and $\tau = \frac{q\pi}{p}$, with q, p coprime integers and q odd,

then $\mathrm{Ker}\, L_0 \supset \mathrm{span}\,\{1, \cos kt , \sin k\tau \,/\, k = (2n-1)\, p , n \in \mathbb{N}\}$.

Proof. The first part is immediate, reasoning as in the preceding propositions.
For the second case notice that if $\tau/\pi \in \mathbb{R} \setminus \mathbb{Q}$ then $1+\cos k\tau \neq 0$ $\forall k$ and so $\mathrm{Ker}\, L_0 = \mathrm{span}\{1\}$. Similarly, if $\tau = q\pi/p$ with q even, then $k\tau$ can never be an odd multiple of π and once again $1 + \cos k\tau \neq 0$ $\forall k$, which leads to $\mathrm{Ker}\, L_0 = \mathrm{span}\,\{1\}$.
Finally, in the last case we have $\Delta_k = 0$ if and only if k is an odd multiple of p , so that $\mathrm{Ker}\, L_0 \supset \mathrm{span}\,\{1, \cos kt, \sin kt \,/\, k = (2n-1)\, p , n \in \mathbb{N}\}$. ♦

The same kind of argument allows to prove the following proposition

Proposition 1.2.4. Let $a = -1$.

If $\lambda \neq 0$ then L_λ is injective.

If $\lambda = 0$ and $\frac{\tau}{\pi} \in \mathbb{R} \setminus \mathbb{Q}$ then $\mathrm{Ker}\, L_0 = \mathrm{span}\,\{1\}$.

If $\lambda = 0$ and $\tau = \frac{q\pi}{p}$, with q, p coprime integers,

then $\mathrm{Ker}\, L_0 \supset \mathrm{span}\,\{1, \cos kt, \sin kt \,/\, k = 2np , n \in \mathbb{N}\}$.

We now turn to the more interesting problem of the invertibility of the operators L_λ and of the continuity and compactness of their inverses.

We begin with the "regular" case $|a| < 1$. As we have seen, in this case (when $\lambda \neq 0$) L_λ is injective and so L_λ^{-1} is defined from $\operatorname{Im} L_\lambda$ to H^1.
We can actually say more, and precisely

Proposition 1.2.5. Let $|a| < 1$ and $\lambda \neq 0$.
Then L_λ is surjective onto L^2 and
$L_\lambda^{-1} : L^2 \to H^1$ is continuous.

Proof. Taken y in L^2 we show that there exists x in H^1 such that $L_\lambda x = y$ and that this x depends continuously on y.
If we call $\begin{pmatrix} u_k \\ v_k \end{pmatrix}$ the Fourier coefficients of x and $\begin{pmatrix} y_k \\ z_k \end{pmatrix}$ those of y, we have $L_\lambda x = y$ if and only if

$$(1.2.17) \qquad \begin{cases} u_0 = y_0/\lambda \\ M_k \begin{pmatrix} u_k \\ v_k \end{pmatrix} = \begin{pmatrix} y_k \\ z_k \end{pmatrix} . \end{cases}$$

Since L_λ is injective, M_k is (algebraically) invertible and it is easy to see that

$$(1.2.18) \qquad \begin{cases} u_k = \dfrac{1}{\Delta_k}(R_k y_k - S_k z_k) \\ v_k = \dfrac{1}{\Delta_k}(R_k z_k + S_k y_k) . \end{cases}$$

We have first of all to show that if x is defined by means of (1.2.18) then $x \in H^1$.
Now by Parseval's equality we have

$$(1.2.19) \qquad \|x\|_2^2 = \frac{|y_0|^2}{\lambda^2} + \sum_{k=1}^{\infty} \frac{y_k^2 + z_k^2}{\Delta_k}$$

and

$$(1.2.20) \qquad \|x'\|_2^2 = \sum_{k=1}^{\infty} \frac{k^2}{\Delta_k}(y_k^2 + z_k^2) .$$

So $\|x\|_2$ and $\|x'\|_2$ are finite if

(1.2.21)
$$\sup_{k\in\mathbb{N}} \frac{k^2}{\Delta_k} < +\infty .$$

But

(1.2.22)
$$\begin{aligned}\Delta_k &= a^2k^2\sin k\tau + \lambda^2 + 2a\lambda k\sin k\tau + a^2k^2\cos^2 k\tau + \\ &+ k^2 + 2ak^2\cos k\tau = a^2k^2 + \lambda^2 + k^2 + \\ &+ 2ak(\lambda \sin k\tau + k\cos k\tau) \geq \\ &\geq a^2k^2 + \lambda^2 + k^2 - 2|a|k\sqrt{\lambda^2+k^2} = (\sqrt{\lambda^2+k^2} - |a|k)^2 .\end{aligned}$$

Therefore

$$\frac{k^2}{\Delta_k} \leq \frac{k^2}{(\sqrt{\lambda^2+k^2} - |a|k)^2}$$

and

(1.2.23)
$$\limsup_{k\to\infty} \frac{k^2}{\Delta_k} \leq \lim_k \frac{k^2}{(\sqrt{\lambda^2+k^2} - |a|k)^2} = \frac{1}{(1-|a|)^2} < +\infty .$$

This implies that both $\|x\|_2$ and $\|x'\|_2$ are finite and consequently L_λ is surjective. Moreover, since

$$\|x\|_2^2 \leq \max\left(\frac{1}{\lambda^2}, \sup_k \frac{1}{\Delta_k}\right) \|y\|_2^2$$

and

$$\|x'\|_2^2 \leq \sup_k \frac{1}{\Delta_k} \|y\|_2^2 ,$$

we have that L_λ^{-1} is continuous from L^2 into H^1. ♦

In the following section we shall need an estimate of the norm of L_λ^{-1}, considered as an operator from L^2 into itself. This is exactly what states the following

Proposition 1.2.6. $\|L_\lambda^{-1}\|_{\mathcal{L}(L^2)} \leq \dfrac{1}{|\lambda| \sqrt{1-a^2}}$.

Proof. From (1.2.19) it follows that

$$\frac{\|x\|_2^2}{\|y\|_2^2} \leq \max\left(\frac{1}{\lambda^2} \ , \ \sup_k \ \frac{1}{\Delta_k}\right) . \tag{1.2.24}$$

Moreover, from (1.2.23),

$$\lim_{k\to\infty} \frac{1}{\Delta_k} = 0 \ ,$$

so that $\sup_k \dfrac{1}{\Delta_k}$ is achieved. We are going to estimate it as follows:

let $F : [0, +\infty[\ \to \mathbb{R}$ be defined by

$$F(t) \equiv (\sqrt{\lambda^2 + t^2} - |a|\, t)^2 \ .$$

We have $F'(t) = 0$ if and only if $t = \bar{t} \equiv \dfrac{|a\lambda|}{\sqrt{1-a^2}}$ and $\bar{t}$ is necessarily a minimum point. Then

$$\forall t \in [0, +\infty[\ , \ F(t) \geq F(\bar{t}) = \lambda^2(1 - a^2)$$

and

$$\frac{1}{\Delta_k} = \frac{1}{F(k)} \leq \frac{1}{F(\bar{t})} = \frac{1}{\lambda^2(1 - a^2)} \ .$$

If we carry this information into (1.2.24) we get

$$\max\left(\frac{1}{\lambda^2} \ , \ \frac{1}{\lambda^2(1 - a^2)}\right) = \frac{1}{\lambda^2(1 - a^2)}$$

and

$$\|L_\lambda^{-1}\|_{\mathcal{L}(L^2)} \leq \frac{1}{|\lambda| \sqrt{1-a^2}} .$$

♦

Carrying on with our program we now examine the case $|a| > 1$. We recall that when $|a| > 1$, L_λ is injective only for some particular values of λ and precisely for $|\lambda| \neq k\sqrt{a^2-1}$. The analogous of Proposition 1.2.5. is given by

Proposition 1.2.7. Let $|a| > 1$ and $|\lambda| \neq k\sqrt{a^2-1}$ $\forall k \in \mathbb{N}$.
Then L_λ is surjective onto L^2 and
$L_\lambda^{-1} : L^2 \to H^1$ is continuous.

Proof. The proof is similar to that of Proposition 1.2.5:
the proposition is true if $\sup_k \frac{k^2}{\Delta_k} < +\infty$.
But $\Delta_k \geq (\sqrt{\lambda^2+k^2} - |a|k)^2$ and we observe that $\Delta_k > 0$. Indeed, if for some k we have $\sqrt{\lambda^2+k^2} - |a|k = 0$ then $|\lambda| = k\sqrt{a^2-1}$, which is impossible by assumption. We end like in the proof of Proposition 1.2.5. ♦

We close this section with the analysis of the critical case $|a| = 1$. We point out that the criticity of this case is well understood when it comes to consider the continuity of L_λ^{-1} . In fact here the arguments used in Proposition 1.2.5 and 1.2.7 fail, because the estimate (1.2.22) gives $\Delta_k \geq 2k^2 + \lambda^2 - 2k\sqrt{\lambda^2+k^2}$ and $\lim_{k\to\infty} \Delta_k = 0$ which is of no use in trying to prove the continuity like we have done above for $|a| \neq 1$.
The line we follow here is to impose more conditions on the parameters (in particular on the delay) in order to guarantee some "invertibility" properties which can be used to obtain some results similar to those of Propositions 1.2.5 and 1.2.7.
We suppose therefore that $\tau = q\pi/p$ with p and q coprime integers and $p \neq 1$. We call Z the subspace of L^2 defined by

$$Z = \text{cl span} \{1 , \cos kt, \sin k\tau \,/\, \tfrac{k}{p} \notin \mathbb{N}\} .$$

The restriction of L_λ to $Z \cap H^1$ is still injective, and the following proposition holds:

Proposition 1.2.8. Under the above hypotheses,

L_λ^{-1} is surjective onto $L^2 \cap Z$ and

$L_\lambda^{-1} : L^2 \cap Z \to H^1 \cap Z$ is continuous.

Proof. We begin like in the proof of Proposition 1.2.5. With the same notations we take $y \in L^2 \cap Z$ and we look for an $x \in H^1 \cap Z$ such that $L_\lambda x = y$. We show that x is given by $L_\lambda^{-1} y$, where L_λ^{-1} is the algebraic inverse of L_λ. The Fourier coefficients of x are therefore given by

$$\text{(1.2.28)} \qquad \begin{cases} u_0 = y_0/\lambda \\ u_k = \dfrac{1}{\Delta_k}(R_k y_k - S_k z_k) \\ v_k = \dfrac{1}{\Delta_k}(R_k z_k + S_k y_k) \ . \end{cases}$$

Once again we have to show that $x \in H^1$, that is that

$$\text{(1.2.29)} \qquad \sup_{\substack{k \in \mathbb{N} \\ k/p \notin \mathbb{N}}} \frac{k^2}{\Delta_k} < +\infty \, .$$

Expliciting Δ_k we obtain

$$\Delta_k = 2k^2 + \lambda^2 + 2ak\sqrt{\lambda^2 + k^2}\left(\frac{\lambda}{\sqrt{\lambda^2+k^2}}\sin k\tau + \frac{k}{\sqrt{\lambda^2+k^2}}\cos k\tau\right) \geq$$

$$\geq 2k^2 + \lambda^2 - 2k\sqrt{\lambda^2 + k^2}\,|\cos(k\tau + \theta_k)|$$

where $\theta_k \in [0, 2\pi]$ is defined by

$$\text{(1.2.30)} \qquad \cos\theta_k = \frac{k}{\sqrt{\lambda^2+k^2}}$$

$$\sin\theta_k = \frac{-\lambda}{\sqrt{\lambda^2+k^2}} \,. \tag{1.2.31}$$

Let us suppose that there exists a $\mu > 0$ such that

$$|\cos(k\tau + \theta_k)| \leq \mu < 1 \qquad \forall k \in \mathbb{N}\,,\ k/p \notin \mathbb{N}\,. \tag{1.2.32}$$

In this case we have

$$\Delta_k \geq 2k^2 + \lambda^2 - 2\mu k\sqrt{\lambda^2 + k^2}$$

and then

$$\lim_{\substack{k\to\infty \\ k/p\notin \mathbb{N}}} \frac{\Delta_k}{k^2} = 2(1-\mu) \neq 0\,.$$

Since $\Delta_k \neq 0$, $\forall k \in \mathbb{N}$, $k/p \notin \mathbb{N}$ we get

$$\sup_{\substack{k\in \mathbb{N} \\ k/p\notin \mathbb{N}}} \frac{k^2}{\Delta_k} < +\infty$$

and we can use the same argument as in Proposition 1.2.5.
It remains to show that there exists a μ verifying (1.2.32). Since $\tau = q\pi/p$, the set $\{k\tau \,/\, k/p \notin \mathbb{N}\,,\ k \in \mathbb{N}\}$ is finite (mod π) and as k is never a multiple of p , it does not contain π . Therefore

$$\sigma \equiv \min_{\substack{k/p\notin \mathbb{N} \\ k\in \mathbb{N}}} (|k\tau - \pi| \bmod \pi) > 0\,.$$

Now we fix $\varepsilon > 0$ such that $\varepsilon < \sigma$; since $\theta_k \to 0$ for $k \to \infty$, there exists $k_0 = k_0(\varepsilon)$ such that $\forall k \geq k_0$, $|\theta_k| < \varepsilon$. But then we have

$$|k\tau + \theta_k - \pi| \bmod \pi > \sigma - \varepsilon > 0 \qquad \forall k \in \mathbb{N}\,,\ \frac{k}{p} \notin \mathbb{N}\,.$$

This means that the distance between $\{k\tau + \theta_k \,/\, k \in \mathbb{N} \,,\, k/p \notin \mathbb{N}\}$ and the multiples of π is bounded from below by a positive constant, so that there exists a μ such that $|\cos(k\tau + \theta_k)| \leq \mu < 1$, and the proof is complete. ♦

Remark 1.2.1. Under the hypotheses of Propositions 1.2.5, 1.2.7 and 1.2.8, that is when L_λ^{-1} is continuous between L^2 and H^1 ($L^2 \cap Z$ and $H^1 \cap Z$ in the case of Proposition 1.2.8), L_λ^{-1} is also completely continuous between L^2 and C^0 or L^2 (respectively between $L^2 \cap Z$ and $L^2 \cap Z$ or $C^0 \cap Z$) , thanks to the compact embeddings of H^1 into L^2 and C^0 .

1.3. Existence of periodic solutions.

This section is devoted to the following problem:
let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a 2π-periodic function in the first variable. State under which conditions on f there exist 2π-periodic solutions of the equation

$$x'(t) + ax'(t - \tau) = f(t, x(t)) \quad . \tag{1.3.1}$$

We begin by choosing a functional setting in order to apply the results of Section 1.2.
We set

$$X = L^2 \tag{1.3.2}$$

$$L_0 : \operatorname{dom} L_0 \subset X \to X \tag{1.3.3}$$

$$\operatorname{dom} L_0 = H^1 \tag{1.3.4}$$

$$(L_0 x)(t) = x'(t) + ax'(t - \tau) \quad . \tag{1.3.5}$$

From Proposition 1.2.1 we know that $\operatorname{Ker} L_0 = \operatorname{span}\{1\}$; if we write (1.3.1) in the equivalent form

$$x'(t) + ax'(t - \tau) - \lambda x(t) = f(t, x(t)) - \lambda x(t) \quad , \quad \lambda \in \mathbb{R} \tag{1.3.6}$$

and, with the same notations as in Section 1.2, we set

(1.3.7) $$(L_{-\lambda}x)(t) = x'(t) + ax'(t-\tau) - \lambda x(t) \ ,$$

we can write (1.3.1) as

(1.3.8) $$(L_{-\lambda}x)(t) = f(t, x(t)) - \lambda x(t)$$

where $L_{-\lambda}$ is a bijection between H^1 and L^2, and $L_{-\lambda}^{-1} : L^2 \to L^2$ is completely continuous (Remark 1.2.1). Then, if we call N the Nemytskii operator of f (we suppose, to begin with, that $N : L^2 \to L^2$ is continuous, we shall see later which conditions on f guarantee this continuity), we can write (1.3.1) as

$$L_{-\lambda}\, x = Nx - \lambda x \ , \text{ or}$$

(1.3.9) $$x = L_{-\lambda}^{-1}(Nx - \lambda x) \equiv T_\lambda x$$

where $T_\lambda : L^2 \to L^2$ is a completely continuous mapping. The originary problem is now reduced to a fixed point problem, which will be solved via the Leray Schauder continuation theorem (A.2.).

The argument used in this section is essentially the one introduced by Metzen in [ME], combined with the properties of the differential operator established in Section 1.2. We actually need something more than what we have done in Section 1.2, that is the spectral properties of an operator closely related to $L_{-\lambda}$.

We denote with K the complex extension of L_0, that is the operator whose domain is

$$\text{dom}\, K = \{u + iv \,/\, u\, , v \in \text{dom}\, L_0\}$$

and whose definition is

$$K(u + iv) = L_0\, u + iL_0\, v$$

and we recall (see [WE] for details) the following definition:

Definition 1.3.1. let H be an Hilbert space and $T : \text{dom}T \subset H \to H$ a linear densely defined operator.

T is said normal if:

i) $\text{dom}T = \text{dom}T^*$ (T^* is the adjoint of T)

ii) $\|T^*u\|_H = \|Tu\|_H$ $\forall u \in \text{dom}T$.

Concerning K the following proposition holds (we state it without proof):

Proposition 1.3.1. K is a normal operator.

Now we come to the analysis of some spectral properties of K , studying the eigenvalue equation

(1.3.10) $$Ku_n = \lambda_n u_n \,, \quad \lambda \in \mathbb{C} \,.$$

By taking the Fourier expansion of u one easily gets for the eigenvalues of k

(1.3.11) $$\lambda_n = in + aine^{-in\tau}$$

and for the corresponding eigenfunctions

(1.3.12) $$\phi_n(t) = e^{int} \,.$$

We want to estimate, for $\omega \in \mathbb{R}$, the distance between ω and the spectrum $\sigma(K)$ of K . Since $\sigma(K)$ coincides with the set of the eigenvalues of K , we have

$$\text{dist}(\omega, \sigma(K))^2 = \inf_{k \in Z} \left[(\omega - ak \sin k\tau)^2 + (k + ak\cos k\tau)^2\right] \,.$$

Using the same calculations as in Proposition 1.2.6 we get

(1.3.13) $$\text{dist}(\omega, \sigma(K)) \geq |\omega| \sqrt{1 - a^2} \,.$$

We are going to use the quantity on the right-hand-side of (1.3.13) to state the condition to impose on f in order to solve equation (1.3.1).
More precisely, if λ is the number introduced in (1.3.6), we set

(1.3.14) $$r = |\lambda| \sqrt{1 - a^2} \leq \text{dist}(\lambda, \sigma(K))$$

and we suppose that $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function 2π-periodic in the first variable such that

G1) $\forall R > 0$, $\exists f_R \in L^2$ such that $|f(t, s)| \leq f_R(t)$ $\forall |s| \leq R$ and a.e.t

G2) There exist two functions $\alpha, \beta \in L^\infty$ and a set $P \subset [0, 2\pi]$ of positive measure such that

(1.3.15) $$\lambda - r \leq \alpha(t) \leq \liminf_{|s|\to\infty} \frac{f(t,s)}{s} \leq \limsup_{|s|\to\infty} \frac{f(t,s)}{s} \leq \beta(t) \leq \lambda + r$$

uniformly in t and $\lambda - r < \alpha(t) \leq \beta(t) < \lambda + r$ for a.e. $t \in P$.

First of all we remark that conditions G1 and G2 are sufficient to obtain the continuity of the Nemytskii operator of f , from L^2 into itself. To make use of the Leray-Schauder continuation theorem we shall need the following two lemmas, the first of which goes under the name of "property p" (see [FO]).

Lemma 1.3.1. Let $p \in L^\infty$ be such that

$$\alpha(t) \leq p(t) \leq \beta(t) \qquad \text{for a.e. } t .$$

Then $$L_0 x = px \text{ implies } x \equiv 0 .$$

Proof. Suppose by contradiction that there exists $x_0 \in \text{dom } L_0$ such that $L_0 x_0 = p x_0$ and $x_0 \not\equiv 0$.

Let $\lambda_1, \ldots, \lambda_k$ be the eigenvalues of K which lie on the circle $|z - \lambda| = r$; since every λ_j has finite multiplicity, if we set

$$V = \bigoplus_{j=1}^{k} \text{Ker}(K - \lambda_j) ,$$

then $\dim V < +\infty$ and all the nonzero functions in V, being analytical, have only a finite number of zeros in the interval $[0, 2\pi]$.

It is easy to see that

$$x \in V \text{ if and only if } \|Kx - \lambda x\| = r \, \|x\| .$$

Now, from Proposition 1.2.6 it follows that

(1.3.18) $$\forall x \in \mathrm{dom} L_0 \,,\ \|Kx - \lambda x\| \geq r\|x\| \ .$$

On the other hand, if $L_0 x_0 = p x_0$, then

$$\|Kx_0 - \lambda x_0\| = \|L_0 x_0 - \lambda x_0\| \leq r\|x_0\| \,,$$

so that $x_0 \in V$, and x_0 has only a finite number of zeros. Then being $\lambda - r < p(t) < \lambda + r$ on P and $\mathrm{meas}(P) > 0$ we get

$$\|L_0 x_0 - p x_0\| < r \, \|x_0\|$$

which is absurd. ♦

The next lemma shows that the hypotheses of Lemma 1.3.1 can be made a little less restrictive.

Lemma 1.3.2. There exists an $\varepsilon > 0$ such that if

$$p \in L^\infty \text{ verifies } \alpha(t) - \varepsilon \leq p(t) \leq \beta(t) + \varepsilon \qquad \text{for a.e. } t$$
$$\text{and } L_0 x = px \text{ , then } x_0 \equiv 0 \,.$$

Proof. Suppose by contradiction that there exists a sequence $p_n \in L^\infty$ with

$$\alpha(t) - \frac{1}{n} \leq p_n(t) \leq \beta(t) + \frac{1}{n} \qquad \text{for a.e. } t$$

and a sequence $x_n \in \mathrm{dom}\, L_0$, $x_n \not\equiv 0$ $\forall n$ such that

(1.3.20) $$L_0 x_n = p_n x_n \,.$$

It is not restrictive to suppose $\|x_n\|_2 = 1$.
Then there exist subsequences (still denoted p_n and x_n) such that

(1.3.21) $$x_n \to x \text{ in } L^2 \text{ weakly}$$

(1.3.22) $$p_n \to p \text{ in } L^\infty \text{ weakly } *.$$

If we set $v_n \equiv (L_0 - \lambda)\, x_n$, we get

$$\|v_n\|_2 = \|(L_0 - \lambda)x_n\|_2 = \|(p_n - \lambda)x_n\|_2 \leq \text{const}$$

and then by extracting a subsequence we can suppose

$$v_n \to v \text{ in } L^2 \text{ weakly}.$$

On the other hand, $(L_0 - \lambda)^{-1} : L^2 \to L^2$ is completely continuous so that actually the convergence in (1.3.21) holds strongly. But then $p_n x_n \to px$ in L^2 weakly and passing to the limit in (1.3.20) we obtain

$$L_0 x = px$$

where $\alpha(t) \leq p(t) \leq \beta(t)$ for a.e. t and $x \neq 0$ because $\|x_n\|_2 = 1$ and $x_n \to x$ in L^2 strongly. This is against Lemma 1.3.1, and the proof is complete. ♦

Now we have all the tools to state and prove the main theorem.

Theorem 1.3.1. Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function 2π-periodic in the first variable such that (G1) and (G2) hold for some $\lambda \neq 0$.
Then the problem

$$\begin{cases} x'(t) + ax'(t - \tau) = f(t, x(t)) \\ x(0) = x(2\pi) \end{cases}$$

has at least one solution.

Proof. We want to apply the Leray-Schauder continuation theorem (A.2.) to equation (1.3.9). All we have to do is to show that the homotopy

$$x = \mu T_\lambda x \ , \ \mu \in [0, 1] \tag{1.3.24}$$

is admissible, that is that solutions of $x = \mu T_\lambda x$ are uniformly bounded in L^2 with respect to μ. We choose an ε which satisfies Lemma (1.3.2) and an $R > 0$ such that if $|s| > R$, then by (G2)

$$\alpha(t) - \varepsilon \leq \frac{f(t,s)}{s} \leq \beta(t) + \varepsilon \qquad \text{for a.e. } t .$$

We define the auxiliary functions g and h by

$$g(t, s) = \begin{cases} \dfrac{f(t,s)}{s} & \text{if } |s| > R \\ \dfrac{f(t,s)s}{R^2} & \text{if } |s| \leq R \end{cases}$$

and $h(t, s) = \max\{\alpha(t) - \varepsilon , \min \{\beta(t) + \varepsilon , g(t, s)\}\}$.
It is clear that $\alpha(t) - \varepsilon \leq h(t, s) \leq \beta(t) + \varepsilon$ and that

(1.3.25) $$|f(t, s) - h(t, s)s| \leq \eta(t) \qquad \text{for a.e. } t$$

where η is some function in L^2 .

Next we denote by H the (continuous) Nemytskii operator associated to $h(t, s) \cdot s$ and we notice that, thanks to (1.3.25)

(1.3.26) $$\|Nx - Hx\|_2 \leq \text{const} , \qquad \forall x \in L^2 .$$

With all the notations introduced we can write the homotopy (1.3.24) in the equivalent form

(1.3.27) $$x = \mu L_{\lambda}^{-1} (N - H)x + \mu L_{-\lambda}^{-1} (H - \lambda)x , \qquad \mu \in [0, 1] .$$

We now suppose that an a priori bound on the L^2 norm of the solutions of (1.3.27) does not exist: in this case we can find two sequences $\mu_n \in [0, 1]$ and $x_n \in L^2$ such that

(1.3.28) $$\mu_n \to \mu^*$$

(1.3.29) $$\|x_n\|_2 \to \infty$$

(1.3.30) $$x_n = \mu_n L_{-\lambda}^{-1} (N - H)x_n + \mu_n L_{-\lambda}^{-1} (N - \lambda)x_n .$$

If we divide (1.3.30) by $\|x_n\|_2$ and we notice that

(1.3.31) $$\|\mu_n L_{-\lambda}^{-1}(N-H)x_n\| \le \|L_{-\lambda}^{-1}\|\ \|(N-H)x_n\| \le \text{const.}$$

we can write

(1.3.32) $$\frac{x_n}{\|x_n\|_2} = r_n + \frac{\mu_n}{\|x_n\|_2} L_{-\lambda}^{-1}(H-\lambda)x_n$$

with $r_n \to 0$ in L^2.
If we set $v_n = \frac{x_n}{\|x_n\|_2}$ and $p_n(t) = h(t, x_n(t))$, (1.3.32) is equivalent to

(1.3.33) $$v_n = r_n + \mu_n L_{-\lambda}^{-1}(p_n - \lambda)v_n$$

with $\|v_n\|_2 = 1$ and $\alpha(t)\ \varepsilon \le p_n(t) \le \beta(t) + \varepsilon$ for a.e. t.
Now we can pass to subsequences and suppose that

(1.3.34) $$v_n \to v \text{ in } L^2 \text{ weakly}$$

(1.3.35) $$p_n \to p \text{ in } L^\infty \text{ weakly* }.$$

But since $L_{-\lambda}^{-1}$ is completely continuous and $\|p_n v_n\|_2 \le \text{const}$, we have that (1.3.34) holds strongly, which in turn implies that

(1.3.36) $$p_n v_n \to pv \text{ in } L^2 \text{ weakly }.$$

Now we can pass to the weak L^2 limit in (1.3.33) to get

(1.3.37) $$v = \mu^* L_{-\lambda}^{-1}(p-\lambda)v\ ,$$

with $\alpha(t) - \varepsilon \le p(t) \le \beta(t) + \varepsilon$.
In other words (1.3.37) means

(1.3.38) $$L_0 v = [\mu^* p + (1-\mu^*)\lambda]v$$

and $v \not\equiv 0$, since the convergence of v_n holds strongly. But this, when λ is suitably chosen contradicts Lemma 1.3.2, and the proof is complete. ♦

Remark. Strengthening a bit the hypotheses of Theorem 1.3.1. on f, it is possible to prove, with the aid of Lemma 1.3.1., that the solution provided by that theorem is unique.
More precisely, suppose that $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies all the hypotheses of Theorem 1.3.1. and suppose in addition that it is differentiable in the second variable with

$$\alpha(t) \le \frac{\partial f}{\partial s}(t,s) \le \beta(t)/2\pi \qquad \text{for all s and a.e. t.} \tag{1.3.39}$$

Then there exists at most one solution to problem

$$\begin{cases} x'(t) + ax'(t-\tau) = f(t,x(t)) \\ x(0) = x(2\pi). \end{cases} \tag{1.3.40}$$

Indeed, suppose by contradiction that y is another solution of (1.3.40); then, by the linearity of L_0, we have

$$L_0(x-y) = f(t,x(t)) - f(t,y(t)) = (x-y)\int_0^{2\pi} \frac{\partial f}{\partial s}(t,y+s(x-y))ds. \tag{1.3.41}$$

If we set $p(t) = \int_0^{2\pi} \frac{\partial f}{\partial s}(t,y+s(x-y))ds$, with $\alpha(t) \le p(t) \le \beta(t)$ for a.e. t, (1.3.41) becomes

$$L_0(x-y) = p(x-y), \tag{1.3.42}$$

and invoking Lemma 1.1.1 we get $x - y \equiv 0$.

1.4. Infinite dimensional kernels.

In Section 1.3. we have seen how to deal with the "easy" case $|a| < 1$, that is when invertibility and compactness of the inverse of the differential operator can be proved.
Now we want to investigate what happens in the critical case $|a| = 1$, and precisely when the dimension of the kernel of the differential operator is infinite.
Differential problems in which the kernel of the differential operator has infinite dimension have frequently appeared in the literature (see e.g. [M2], [MW1], [KA], [CO], [BN]) and different approaches have been developed in order to deal with them.
One of the most useful ones, which goes back to [DST], [MAN], [CO], is particularly suitable to treat the wave equation with periodic boundary conditions and monotone nonlinearities, a problem which can be considered as the starting point for all such techniques. Many papers are devoted to the study of the semilinear wave equation with periodic boundary conditions and techniques have been refined in order to increase their range of applicability.
When we try to modify such techniques to apply them to our problem, we immediately meet with a very hard difficulty. In fact, all these techniques which have been developed for the wave equation can only deal with differential operators which have some good properties such as self-adjointness: all these results require that the differential operator be at least closed.
So if we try to apply them we have to start by verifying the closedness of our operator.
More precisely, we are going to study the closedness of the operator

(1.4.1) $$L : \mathrm{dom}\, L \subset L^2 \to L^2$$

defined by

(1.4.2) $$\mathrm{dom}\, L = H^1$$

(1.4.3) $$(Lu)(t) = \dot{u}(t) + \dot{u}(t - \tau)$$

where $\tau = \frac{q\pi}{p}$ with q and p odd coprime integers. We know from Proposition (1.2.3) that this is the case in which

(1.4.4) $$\mathrm{Ker}\, L \supset \mathrm{span}\ \{1, \cos kt, \sin kt \,/\, k = (2n-1)p,\ n \in \mathbb{N}\}.$$

Now we show that for every p, q coprime odd integers there exists a sequence $u_\varepsilon \in \mathrm{dom}\, L$ such that

(1.4.5) $$u_\varepsilon \to u \text{ in } L^2 \text{ for } \varepsilon \downarrow 0$$

(1.4.6) $$Lu_\varepsilon \to w \text{ in } L^2 \text{ for } \varepsilon \downarrow 0$$

but $u \notin \mathrm{dom}\, L$, that is, the operator L is not closed.
To this end we first recall that $\mathrm{Ker}\, L \cap \mathrm{dom}\, L$ can be characterized as a set of functions which have a certain symmetry, and precisely we have that

(1.4.7) $$\mathrm{Ker}\, L = \{u \in \mathrm{dom}\, L \,/\, \dot{u}(t-\tau) = -\dot{u}(t) \text{ for a.e. } t \in \mathbb{R}\} .$$

Then we fix in the interval $[0, 2\pi]$ the points

(1.4.8) $$t_k = k\frac{\pi}{p} \text{ for } k = 0, 1, \ldots, 2p$$

and we define, for every $\varepsilon > 0$, the function

(1.4.9) $$u_\varepsilon(t) = \begin{cases} (t - t_{2j})^{1/2+\varepsilon} & \text{if } t \in [t_{2j}, t_{2j+1}] \\ (t_{2j} - t)^{1/2+\varepsilon} & \text{if } t \in [t_{2j-1}, t_{2j}] \end{cases} .$$

We notice that u_ε is continuous and that

(1.4.10) $$\dot{u}_\varepsilon(t) = \begin{cases} (\frac{1}{2}+\varepsilon)(t - t_{2j})^{-1/2+\varepsilon} & \text{if } t \in]t_{2j}, t_{2j+1}[\\ -(\frac{1}{2}+\varepsilon)(t_{2j} - t)^{-1/2+\varepsilon} & \text{if } t \in]t_{2j}, t_{2j-1}[\end{cases}$$

so that $\dot{u}_\varepsilon \in L^2$ and therefore $u_\varepsilon \in \mathrm{dom}\, L$.
Moreover, by (1.4.10) $\dot{u}_\varepsilon(t - \frac{q\pi}{p}) = -\dot{u}_\varepsilon(t)$ so that $u_\varepsilon \in \mathrm{Ker}\, L$ $\forall \varepsilon > 0$.

But now if u_0 is defined by $u_0(t) = \begin{cases} (t - t_{2j})^{1/2} & \text{if } t \in [t_{2j}, t_{2j+1}] \\ (t_{2j} - t)^{1/2} & \text{if } t \in [t_{2j-1}, t_{2j}] \end{cases}$ we have that

$u_\varepsilon \to u_0$ in , C^0 and a fortiori in L^2, but $u_0 \notin \text{dom}\, L$ since $\dot{u}_0 \notin L^2$.

Thus the operator L is not closed and the approach which we have discussed above is not useful.

Even if we replace L by $L - \lambda I$, an operator which is injective (Proposition 1.2.3) we cannot go any further because, recalling the proof of Proposition 1.2.5 one sees that $L - \lambda I$ might fail to be surjective, let alone $(L - \lambda I)^{-1}$ to be continuous.

A different tool is needed to study the existence of periodic solution of the equation

(1.4.12) $$x'(t) + x'(t - \tau) = f(t, x(t))$$

and this is provided by some ideas which can be found in the work of Coron [CO].

The main feature of this new method consists in restricting problem (1.4.12) to a subspace of L^2 which is invariant under the action of L, of f and such that the restriction of Ker L to this subspace has finite dimension. One can then hope to prove continuity and compactness of some generalized inverse of L.

One possible choice for such a subspace is provided by Proposition 1.2.8: if we set

(1.4.13) $$V = \text{cl span}\, \{1, \cos kt\, , \sin kt \,/\, \tfrac{k}{p} \notin \mathbb{N}\}$$

than $\text{Ker}\, L \cap V = \{0\}$. However this choice turns out rather awkward when it comes to impose conditions in order that f map V into itself.

We therefore introduce the following subspace

(1.4.14) $$H = \text{cl span}\, \{1, \cos kt, \sin kt \,/\, k = 2n\, , n \in \mathbb{N}\}$$

noticing that it is nothing else than the space of π-periodic functions.

We now set

(1.4.15) $$X = C^0 \cap H$$

(1.4.16) $$Z = L^2 \cap H$$

so that L maps $X \cap \text{dom}\, L$ into Z.
If we call $\tilde{L}$ the restriction of L to H, that is the operator defined by

$$\tilde{L} : \text{dom}\, \tilde{L} \subset X \to Z, \tag{1.4.17}$$

$$\text{dom}\, \tilde{L} = H^1 \cap H, \tag{1.4.18}$$

$$\tilde{L}u = Lu, \tag{1.4.19}$$

we can prove the following

Proposition 1.4.1. $\tilde{L}$ is a Fredholm operator of index zero. In particular

$$\text{Ker}\, \tilde{L} = \text{span}\ \{1\} \tag{1.4.20}$$

$$\text{Im}\, \tilde{L} = \{x \in Z \,/\, \int_0^\pi x(s)\, ds = 0\} \equiv Z_0 . \tag{1.4.21}$$

Proof. Substituting for x its expansion in Fourier series and using the same calculation as in Proposition 1.2.3 we immediately see that $\text{Ker}\, \tilde{L} = \text{span}\ \{1\}$.
To characterize $\text{Im}\, \tilde{L}$ we notice first that

$$Z_0 \supset \text{Im}\, \tilde{L} ;$$

to verify the converse inclusion we take $y \in Z_0$ with Fourier coefficients (y_n, z_n) ; the equation $Lx = y$ has a solution if

a) $$\Delta_k \neq 0 \quad \forall k$$

b) $$x \equiv \begin{pmatrix} u_k \\ v_k \end{pmatrix} = M_k^{-1} \begin{pmatrix} y_k \\ z_k \end{pmatrix} \in H^1 \ \text{ for some } k \in \mathbb{N} .$$

Now a) is always true because

$$\Delta_k = 0 \leftrightarrow \begin{cases} \sin 2k\tau = 0 \\ 1 + \cos 2k\tau = 0 \end{cases}$$

and this is always false.
For what concerns b), using standard computations, we see that $x \in H^1$ because

$$\sup_k \frac{(2k)^2}{\Delta_{2k}} \leq \frac{1}{2(1+\cos 2k\tau)} < +\infty .$$

This proves that $Z_0 = \operatorname{Im} \tilde{L}$ and consequently $\tilde{L}$ is a Fredholm operator of index zero. ♦

Thanks to this proposition, we can use the standard theory (see for example [GM]) to ensure that there exist continuous projections

$$P : X \to X, \qquad Q : Z \to Z$$

such that $\operatorname{Im} P = \operatorname{Ker} \tilde{L}$ and $\operatorname{Ker} Q = \operatorname{Im} \tilde{L}$.
If we restrict $\tilde{L}$ to $\operatorname{Ker} P \cap \operatorname{dom} \tilde{L}$, $\tilde{L}$ is bijective and so it admits a unique (depending on P) inverse

$$K_P : \operatorname{Im} \tilde{L} \to \operatorname{dom} \tilde{L} \cap \operatorname{Ker} P .$$

It is then possible to define a generalized inverse of $\tilde{L}$, namely the operator

$$K_{P,Q} : Z \to \operatorname{dom} \tilde{L} \cap \operatorname{Ker} P$$
$$K_{P,Q} = K_P(I - Q) .$$

It is easy to see that $K_{P,Q}$ is continuous from Z into $\operatorname{dom} \tilde{L} \cap \operatorname{Ker} P$ and consequently that $K_{P,Q}$ is completely continuous from Z into X, thanks to the compactness of the embedding of H^1 into C^0. We are now almost ready to state the main theorem concerning the existence of periodic solutions to equation

(1.4.30) $$x'(t) + x'(t-\tau) = f(t, x(t)) .$$

To complete the general setting that we are going to use to apply Mawhin's Continuation Theorem we need some restriction on the nonlinearity f in order that it send continuously Z into itself. We shall then suppose that $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function π-periodic in the first variable and such that

(1.4.31) $$\forall R > 0 \ , \exists f_R \in Z \text{ such that}$$
$$|f(t, s)| \leq f_R(t) \quad \forall t \quad \forall |s| \leq R \ .$$

As we have seen above this implies that the Nemytskii operator associated to f is continuous from X into Z. We remark that in view of these properties the mapping

$$K_{P,Q}N : X \to X$$

is completely continuous.

The existence of (π-) periodic solutions to Equation (1.4.30) follows now from the existence of fixed points for the mapping $K_{P,Q}N$, and this is precisely what we are going to prove.

Theorem 1.4.1. Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function π-periodic in the first variable and such that (1.4.31) holds. Suppose moreover that there exists M such that

(1.4.32) $$f(t, x)x > 0 \text{ for } |x| > M$$

and

(1.4.33) $$\lim_{|s| \to \infty} \frac{f(t,s)}{s} = 0 \ .$$

Then there exists at least a π-periodic solution to Equation (1.4.30).

Proof. With the notations introduced in this section we know that the theorem is true if there exists a fixed point for the mapping $K_{P,Q}N$, that is a solution of the equation

(1.4.34) $$x = K_{P,Q}Nx \ .$$

We are going to show the existence of such a solution via Mawhin's Continuation Theorem (A.3.). All we have to do then is to show that the hypotheses of this theorem hold true, and precisely that there exists an open bounded set $\Omega \subset X$ such that

a) $QN : X \to Z$ is continuous and sends bounded sets into bounded sets.

b) If $g \equiv QN|_{\bar{\Omega}\cap \mathrm{Ker}\tilde{L}}$, then

$$g(x) \neq 0 \quad \forall x \in \partial\Omega \cap \mathrm{Ker}\tilde{L}$$

c) $\deg(g, \Omega\cap\mathrm{Ker}\tilde{L}, 0) \neq 0$

d) $\tilde{L}x \neq \lambda Nx \quad \forall \lambda \in]0, 1[\,, \forall x \in \partial\Omega\cap\mathrm{dom}\tilde{L}$.

We take $\Omega = B_r = \{x \in X \,/\, \|x\|_x < r\}$, with r sufficiently big.

Now QN is clearly continuous and sends bounded sets into bounded sets because if $B \subset X$ is such a bounded set, then $\forall x \in B$

$$\int_0^\pi |f(t, x)|^2 dt \leq \text{const}$$

and $\|QNx\| = \left| \frac{1}{\pi} \int_0^\pi f(t,x)dt \right| \leq \text{const}$ independent of x ;

a) is proved.

b) is trivial if we choose $r > M$ and we remark that $\partial\Omega\cap\mathrm{Ker}\,\tilde{L}$ consists of constant functions: if $c \in \partial\Omega\cap\mathrm{Ker}\tilde{L}$, then $g(c) = \frac{1}{\pi}\int_0^\pi f(t,c)dt \neq 0$ by (1.4.32).

c) is trivial too if we remark that

$$\deg(g, \Omega\cap\mathrm{Ker}\,\tilde{L}, 0) = 1$$

being equal to the degree of a linear map through $(-r, g(-r))$ and $(r, g(r))$, with $g(-r) < 0 < g(r)$.

Finally, we consider d): working like in the proof of Proposition 1.4.1. it is easy to show that there exists $k > 0$ such that

$$\|\tilde{L}u\|_Z \geq k\|\dot{u}\|_Z \qquad \forall u \in \mathrm{dom}\,\tilde{L} . \tag{1.4.36}$$

Now by (1.4.33) it follows that $\forall \varepsilon > 0 \,, \exists c > 0$ such that

$$\|Nx\|_Z \leq c + \varepsilon\,\|x\|_X; \tag{1.4.37}$$

by (1.4.32), for every solution x of $\tilde{L}x = \lambda Nx$ there exists $t_0 \in [0, 2\pi]$ such that

(1.4.38) $$|x(t_0)| \leq M \ ,$$

and consequently, for every solution of $Lx = \lambda Nx$ we get

(1.4.39) $$\|x\|_x \leq M + \sqrt{\pi}\, \|\dot{x}\|_2$$

which, substituted into (1.4.37) yields

(1.4.40) $$\|Nx\|_z \leq c + \varepsilon M + \varepsilon \sqrt{\pi}\, \|\dot{x}\|_2 \ .$$

Finally, using (1.4.36) we get

(1.4.41) $$k\|\dot{x}\|_z \leq \|\tilde{L}x\|_z \leq \|Nx\|_z \leq c + \varepsilon M + \varepsilon \sqrt{\pi}\, \|\dot{x}\|_2$$

which means, for ε small enough,

(1.4.42) $$\|\dot{x}\|_z \leq \text{const.}$$

Recalling that for every solution of $\tilde{L}x = \lambda Nx$ we have also $\inf_t |x(t)| \leq M$, we finally get

(1.4.43) $$\|x\|_{1,2} \leq \text{const}$$

which is the required estimate. ♦

1.5. A special form of the nonlinearity.

The purpose of this section is to show how the hypotheses of Theorems 1.3.1. and 1.4.1. can be considerably weakened when the nonlinearity has the special form $f(t, x) = g(x) + h(t)$. We start with the analogous of Theorem 1.3.1., that is with the "compact" case $|a| < 1$.

Theorem 1.5.1. let $|a| < 1$ and let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function such that there exists $M > 0$ with

$$(1.5.1) \qquad g(x)x \geq 0 \qquad \text{if } |x| \geq M .$$

Let $h \in L^2$ have zero mean value.
Then there exists at least one 2π-periodic solution of the equation

$$(1.5.2) \qquad x'(t) + ax'(t-\tau) = g(x) + h(t) .$$

Proof. We fix $\mu < 0$ and we write (1.5.2) in the equivalent form

$$(1.5.3) \qquad x'(t) + ax'(t-\tau) + \mu x(t) = g(x) + \mu x(t) + h(t) .$$

If we define the operator $L_\mu : \text{dom}\, L_\mu \subset C^0 \to L^2$ by $\text{dom}\, L_\mu = H^1$

$$(L_\mu x)(t) = x'(t) + ax'(t-\tau) + \mu x(t) ,$$

we have from the general theory of Section 1.2 that L_μ is invertible, and that L_μ^{-1} is completely continuous from L^2 into C^0.
If we call $N : C^0 \to L^2$ the (continuous) Nemytskii operator associated to $g + \mu I + h$ and T_μ the mapping $L_\mu^{-1} N$, we have that $T_\mu : C^0 \to C^0$ is completely continuous and its fixed points are the solutions of Equation (1.5.2). We look for fixed points of T_μ with the aid of Leray-Schauder's Continuation Theorem.
The only thing to verify is that

$$\forall \lambda \in]0, 1[, x \neq \lambda T_\mu x \ \ \forall x \in \partial B_R ,$$

where $B_R = \{x \in C^0 / \|x\| < R\}$ and R is to be chosen conveniently.
We begin by noticing that for every solution of $x = \lambda T_\mu x$ we have $\inf_t |x(t)| < M$. Indeed integrating $L_\mu x = \lambda N x$ between 0 and 2π we get

$$(1.5.6) \qquad \int_0^{2\pi} [\lambda g(x(t)) - \mu(1-\lambda)\, x(t)]dt \ = \ 0$$

and, by the mean value theorem, there exists t_0 such that

(1.5.7) $$\lambda g(x(t_0)) - \mu(1-\lambda)x(t_0) = 0 .$$

If $x(t_0) = 0$ there is nothing to prove; if not, we multiply (1.5.7) by $x(t_0)$ to get

$$\lambda g(x(t_0))\, x(t_0) = \mu(1-\lambda)\, |x(t_0)|^2 < 0$$

because $\mu < 0$ and then $x(t_0) < M$.
It remains to show that there exists an a priori estimate independent of λ on $\|\dot{x}\|_2$, where x is a solution of $x = \lambda T_\mu x$, to end like in the proof of Theorem 1.5.1.
We put for simplicity $x_\tau(t) = x(t-\tau)$ and we write $L_\mu x = \lambda N x$ in the form

(1.5.8) $$x' - \lambda g(x) + (1-\lambda)\mu x = -\, a x'(t-\tau) + \lambda h .$$

Taking the L^2 norm of both sides we get

$$\{\int_0^{2\pi} |\dot{x}|^2 + \lambda^2 \int_0^{2\pi} |g(x)|^2 + (1-\lambda)^2 \mu^2 \int_0^{2\pi} |x|^2 - 2\lambda(1-\lambda)\mu \int_0^{2\pi} g(x)x\}^{1/2} \le$$
$$\le |a|\, \|\dot{x}\|_2 + \lambda \|h\|_2 ;$$

therefore

$$\{\int_0^{2\pi} |\dot{x}|^2 - 2\lambda(1-\lambda)\mu) \int_0^{2\pi} g(x)x\}^{1/2} \le |a|\, \|\dot{x}\|_2 + \lambda \|h\|_2 .$$

Squaring gives

$$(1-a^2)\, \|\dot{x}\|_2^2 \le 2\lambda(1-\lambda)\mu \int_0^{2\pi} g(x)x + \lambda^2 \|h\|_2^2 + 2\lambda\, |a|\, \|\dot{x}\|_2\, \|h\|_2 \le$$

$$\le 2\lambda(1-\lambda)\mu[\int_{\{|x|\le M\}} g(x)x + \int_{\{|x|>M\}} g(x)x] + c + c'\, \|\dot{x}\|_2 .$$

Now if $|x| > M$, $g(x)x \ge 0$ and $2\lambda(1-\lambda)\mu \int_{\{|x|>M\}} g(x)x \le 0$ and then

$$(1 - a^2)\, \|\dot{x}\|_2^2 \le 2\pi|\mu|\, M \sup_{|s|\le M} |g(s)| + c + c'\, \|\dot{x}\|_2 \le c'' + c'\, \|\dot{x}\|_2$$

from which follows $\|\dot{x}\|_2 \le$ const independent of λ. ♦

We close this part with the analogous of Theorem 1.4.1.: also in this case the sign condition on g is sufficient to ensure the existence of a periodic solution when the forcing term has period π.

Theorem 1.5.2. Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function such that there exists $M > 0$ with

(1.5.10) $$g(x)x \ge 0 \text{ if } |x| \ge M .$$

Let $h \in L^2$ have period π and zero mean value.
Then there exists at least one (π-) periodic solution of

(1.5.11) $$x'(t) + x'(t - \tau) = g(x(t)) + h(t) .$$

Proof. Like in the previous theorem the only thing to prove is that

(1.5.12) $$\tilde{L}x \ne \lambda N x \qquad \forall x \in \partial B_R \cap \operatorname{dom} \tilde{L}.$$

With the same notation as in Theorem 1.4.1. there exists $k > 0$ such that

(1.5.13) $$k\, \|\dot{x}\|_2 \le \|\tilde{L}x\|_2 \le \lambda\, \|g(x)\|_2 + \lambda \|h\|_2$$

for every solution of $\tilde{L}x = \lambda N x$.
Taking the L^2 norm of $\dot{x} - \lambda g(x) = -\dot{x}_\tau + \lambda h$ we get

$$\|\dot{x}\|_2^2 + \lambda^2\, \|g(x)\|_2^2 \le \|\dot{x}\|_2^2 + \lambda^2\, \|h\|_2^2 - 2\lambda \int_0^\pi h \dot{x}_\tau .$$

Then, by (1.5.13)

$$\lambda^2 \|g(x)\|_2^2 \le \lambda^2\, \|h\|_2^2 + 2 \frac{\lambda}{k} \|h\|_2 \left(\lambda\, \|g(x)\|_2 + \lambda \|h\|_2\right) \le$$

$$\leq \lambda^2(1 + \frac{2}{k}) \|h\|_2^2 + 2\frac{\lambda^2}{k} \|h\|_2 \|g(x)\|_2$$

which gives $\|g(x)\|_2 \leq$ const independent of λ.
Finally substituting this estimate in (1.5.13) we get

$$\|\dot{x}\|_2 \leq \text{const.}$$

We then invoke the sign condition to get the required estimate, as in the proof of Theorem 1.4.1.
♦

Remark. In this section we have studied the case $a = 1$, imposing some conditions on f (π-periodicity) suggested by Proposition 1.2.3. It is perfectly clear that analogous results can be obtained in the case $a = -1$, with the help of Proposition 1.2.4., by requiring that f have some suitable "simmetry" properties.

CHAPTER 2

SOME DIFFERENTIAL EQUATIONS WITH DELAYS IN THE NONLINEARITIES

2.1. Motivations and general remarks.

The aim of this section is to introduce to the study of some nonlinear delay differential equations where the delay appears in the nonlinearity. As we have done in Chapter 1, we choose a particular type of equation depending on some parameters in order to present the various situations which occur when the parameters are allowed to vary.
As in the first part of the thesis we are interested in finding periodic solutions of these equations with prescribed periodicity, which we will always assume to be 2π.
The equations that we are going to consider are essentially of two kinds, namely

$$x'(t) = f(t,x(t)) - f(t-\tau,x(t-\tau)) \tag{2.1.1}$$

with its subcase

$$x'(t) = g(x(t)) - g(x(t-\tau)) + p(t) \tag{2.1.2}$$

and

$$x'(t) = f(t,x(t)) - f(t,x(t-\tau)). \tag{2.1.3}$$

The study of the existence of periodic solutions for these equations will take place throughout Chapter 2.
In this section we are going to make some introductory remarks and to present some negative results.
Equations (2.1.1), (2.1.2) and (2.1.3) have been studied extensively in the literature by many authors, see for example [AS], [CK], [DL], [JE], [LW1], [LW2]; in particular, much attention has been devoted to the analysis of the asymptotic behaviour of the solutions; for this and related matters we refer to the interesting survey paper by Haddock, [HD].

The study of the asymptotic behaviour of solutions of these equations turns out to be especially significative when one is interested in periodic solutions. As a matter of fact, a first (negative) result was provided for equation (2.1.3) by the work of Jehu, in [JE]. This work answered the following question posed by Haddock in [HD]:
given a functional differential equation for which each constant function is a solution, when do all solutions tend to constant as $t \to \infty$?
Noticing that for equation (2.1.3) all constant functions are solutions, we can take advantage of Jehu's result to remark that there are qualitative hypotheses on f such that for (2.1.3) the only periodic solutions are the constant ones. Indeed Jehu proved that if f is continuous in x, periodic in t and strictly decreasing, each solution of (2.1.3) tends to a constant as $t \to \infty$. We are able to say something more, for some particular values of the delay τ.
We have indeed

Proposition 2.1.1. let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be continuous, 2π periodic in the first variable and strictly monotone in the second.
If $\tau/\pi \notin \mathbb{Q}$, then the only 2π-periodic solutions of (2.1.3) are the constant functions.

Proof. Let x be a 2π-periodic solution of (2.1.3). Notice that, being x of class C^0, it is automatically (from the equation) of class C^1.
Let t_0 be a point of absolute maximum for x; then

$$0 = f(t_0, x(t_0)) - f(t_0, x(t_0 - \tau))$$

and, being f strictly monotone, $x(t_0) = x(t_0 - \tau)$, that is, $t_0 - \tau$ is a point of absolute maximum for x. Now, because $\tau/\pi \notin \mathbb{Q}$, x cannot be both τ-periodic and 2π-periodic; so we can apply the preceding argument to the point $t_0 - \tau$, to show that $t_0 - 2\tau$ is another absolute maximum for x, and so on. Then the points $(t_0 - k\tau)_{k \in \mathbb{N}}$ are all absolute maxima for x and, since they are dense (mod 2π) in $[0,2\pi]$, we conclude that $x(t) = x(t_0)$ for all t. ♦

Another negative result is immediately obtained for equation (2.1.3) when f does not depend explicitly on t, that is for the equation

$$x'(t) = f(x(t)) - f(x(t-\tau)). \tag{2.1.4}$$

Indeed for every continuous f, the periodic solutions of (2.1.4) are all constant, as one can easily see taking the L^2 norm of both sides of

$$x'(t) - f(x(t)) = f(x(t-\tau)),$$

and remarking that the mixed product in the left-hand-side vanishes because of periodicity: we get

$$\|x'\|_2^2 + \|f(x)\|_2^2 = \|f(x)\|_2^2 ,$$

that is, $x \equiv$ const.

This trivial remark settles the matter as far as equation (2.1.4) is concerned. A completely different situation arises when the right-hand-side of (2.1.4) is perturbed with a periodic forcing term $p(t)$, that is when we consider equation(2.1.2). The study of existence of periodic solutions of (2.1.2) will take place in the following sections.

We want to end this introduction with some motivations, from the point of view of the applications, for the choice of equations (2.1.1),(2.1.2), and (2.1.3).

Indeed, they represent very important models of various phenomena in mathematical biology. One of the first authors who started the study of these equations was Lotka, in his research on the growth of a population. To make an example, we would like to explain how to derive an equation of the type (2.1.2), ruling the growth of a population, from very general assumptions. Suppose for instance that the "length" of life of each individual is independent of the total number of individuals in the aggregate and of the distribution of the ages among them. Assume moreover that the number of births per unit time is a function only of the population size; that is, the birth rate is "density-dependent" but not age-dependent.

Now suppose that $x(t)$ denotes the number of individuals at time t. Then the number of births per unit time is some function of x, say $f(x(t))$.

Finally we introduce an assumption which is rather restrictive but nonetheless meaningful if we consider long time intervals and average data: we suppose that every individual has the same life span, τ. Therefore, if every individual dies at age τ, the number of deaths per unit time is given by $f(x(t-\tau))$. Since the difference $f(x(t)) - f(x(t-\tau))$ is the net change in the population per unit time, the growth of the population is governed by the equation

$$x'(t) = f(x(t)) - f(x(t-\tau)),$$

which is (2.1.2) with $p \equiv 0$.

We recall eventually that the same equations (2.1.1-3) are also very good models for the diffusion of infectious diseases. For these aspects we refer to [CK] and references therein.

2.2. A first existence result.

In this section we want to give a first and easy existence result concerning equation (2.1.1). This equation has been studied by Arino and Seguier in [AS], with the aim of investigating the asymptotic behaviour of its solutions. As a by-product of this study they were able to prove the following

Theorem 2.2.1. (Arino - Seguier) let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be continuous, T-periodic in the first variable, nonincreasing in the second and either strictly decreasing or lipschitzean in the second variable. Suppose moreover that $f(t,0) \equiv 0$. Then, for every real number α there exists a T-periodic solution of

$$x'(t) = f(t,x(t)) - f(t-1,x(t-1))$$

such that

$$x(t) - \int_{t-1}^{t} f(s,x(s))ds = \alpha.$$

We are going to show that if monotonicity is replaced by some growth assumption on f, then one can still get at least one periodic solution.
To this aim we suppose that f is a Caratheodory function, 2π-periodic in the first variable and such that for every $R > 0$, there exists an $f_R \in L^2$ with

$$|f(t,s)| \leq f_R(t), \qquad \forall\ |s| \leq R \text{ and a.e. } t. \tag{2.2.1}$$

Under this assumption the Nemytskii operator N associated to f is a continuous mapping from C^0 to L^2. Moreover, we observe that if we call T_τ the mapping defined by

$$(T_\tau x)(t) = x(t-\tau),$$

then the right-hand-side of (2.1.1) can be written $Nx - T_\tau Nx$. The mapping $N - T_\tau N$ takes functions with zero mean value into functions with zero mean value, so that if we denote by X and Z respectively the spaces of C^0 and L^2 functions with zero mean value, we can think of $N - T_\tau N$ as a continuous mapping from X to Z.
Since $L \equiv \frac{d}{dt}$ is continuously invertible between $X \cap H^1$ (endowed with the H^1 topology) and Z, L^{-1} is a completely continuous operator from Z to X and $T \equiv L^{-1}(N - T_\tau N) : X \to X$ is also completely continuous.
By Leray-Schauder continuation theorem there exists a fixed point of T, that is a periodic solution of (2.1.1), if one is able to prove that solutions of

(2.2.2) $$x = \lambda Tx, \quad 0 < \lambda < 1,$$

are uniformly bounded in X with respect to λ.
This is exactly what ensures the following

Proposition 2.2.1. There exists an $\varepsilon > 0$ such that if

(2.2.3) $$\limsup_{s \to \pm\infty} \left| \frac{f(t,s)}{s} \right| < \varepsilon \quad \text{for all } t,$$

then the mapping T has a fixed point.

Proof. It is enough to show that solutions of (2.2.2) are a priori bounded with respect to λ.
Now by (2.2.3) there exist constants $M = M(\varepsilon)$ and $\delta = \delta(\varepsilon)$, with $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$ such that

(2.2.4) $$\|Nx\|_2 \leq M + \delta \|x\|_2 \quad \text{for all } x \in X.$$

Then, if x satisfies (2.2.3) for some $\lambda \in \,]0,1[$, we have

(2.2.5) $$\|x'\|_2 \leq \|Nx\|_2 + \|T_\tau Nx\|_2 = 2\|Nx\|_2 \leq 2M + 2\delta \|x\|_2,$$

where we have also used the fact that T_τ is an isometry from L^2 into itself.
Now if we recall that x has zero mean value, we can apply Wirtinger's inequality to get from (2.2.5) the estimate

(2.2.6) $$\|x'\|_2 \leq 2M + 2C\delta\|x'\|_2,$$

for some constant C independent of x.
When ε is chosen small enough, we can assume $2C\delta < 1$, so that (2.2.6) yields

$$\|x'\|_2 \leq \text{const.}, \quad \text{for all } \lambda \in\ ]0,1[.$$

Using again Wirtinger's inequality we get the boundedness of the H^1 norm of solutions of (2.2.2), for every $\lambda \in\ ]0,1[$, which is enough to obtain a fixed point of T. ♦

Remark. As it is well-known, Wirtinger's constant for functions with zero mean value can be taken equal to one, so that in the hypotheses of Proposition 2.2.1. we just have to ask that ε be such that $\delta < 1/2$.

2.3. The dominated case.

In this section we present a result concerning the existence of periodic solutions for equations of the type

(2.3.1) $$x'(t) = af(x(t)) - b\bar{f}(x(t-\tau)) + p(t)$$

when the constants a and b allow the "ordinary" term $f(x(t))$ to dominate the "functional" term $f(t,x(t-\tau))$.
We recall that similar existence results have been obtained by Arino and Seguier in [AS], under the crucial hypothesis that the nonlinearity be monotone. It is the main purpose of this section to show that in the dominated case, monotonicity plays no essential role and can therefore be abandoned. We will show in Section 2.4. that this happens even if such a domination is not present, introducing some suitable growth conditions on f.
We remark that, although the form of the nonlinear term that we consider might look at first glance less general than that of Arino and Seguier, this is not the case: to write (2.3.1) as in the paper by Arino and Seguier one has to show that there exists a function α such that

$$p(t) = \alpha(t) - \alpha(t-\tau),$$

and we know from the general results of section 1.2. that this is in general false.We are nevertheless able to prove the existence for every delay τ, even if $\tau/\pi \notin \mathbb{Q}$.
We begin to precise the functional setting in which we want to study equation (2.3.1).
As usual we denote by L^2, H^1 and C^0 the spaces of 2π-periodic L^2, H^1 and C^0 functions; if f is a real-valued continuous function and if $p \in L^2$, then the Nemytskii operator associated to the right-hand-side of (2.3.1) maps C^0 into L^2 continuously. Our aim is to transform (2.3.1) into a fixed point problem which can be solved by means of the Leray-Schauder continuation theorem. We think of the operator $\frac{d}{dt}$ as a linear map

$$L : \mathrm{dom}\, L \subset C^0 \to L^2,$$

with $\mathrm{dom}\, L = H^1$. As L is not invertible, we write (2.3.1) in the equivalent form

(2.3.2) $$x'(t) - \mu x(t) = af(x(t)) - bf(x(t-\tau)) - \mu x(t) + p(t)$$

where μ is a constant whose sign will be conveniently chosen. If now we set

$$L_\mu : \mathrm{dom}\, L_\mu \subset C^0 \to L^2,$$

$$\mathrm{dom}\, L_\mu = H^1,$$

$$(L_\mu x)(t) = (Lx)(t) - \mu x(t),$$

it is well known that L_μ is invertible and that $L_\mu^{-1} : L^2 \to C^0$ is completely continuous.
The existence of a periodic solution to problem (2.3.1) is then proved, provided that the map

$$T_\mu \equiv L_\mu^{-1} N_\mu : C^0 \to C^0,$$

where $N_\mu : C^0 \to L^2$ is defined by $(N_\mu x)(t) = (Nx)(t) - \mu x(t)$, has a fixed point. We remark that T_μ is completely continuous.

We now come to the theorem which makes precise all the hypotheses needed in order for T_μ to have a fixed point.

Theorem 2.3.1. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and such that

(2.3.4) $$f(x)x \geq 0 \qquad \text{for all } x \in \mathbb{R}.$$

Let $a,b \in \mathbb{R}$ be such that $|a| > |b|$.
Then for every $p \in L^2$ with zero mean value there exists at least one 2π-periodic solution of

$$x'(t) = af(x(t)) - bf(x(t-\tau)) + p(t).$$

Proof. As we have already remarked, the thesis of the theorem is equivalent to the fixed point problem

$$\text{find } x \in C^0 \text{ such that}$$

(2.3.6) $$x = T_\mu x.$$

By Leray-Schauder continuation theorem, (2.3.6) holds true if one can show that the solutions of

(2.3.7) $$x = \lambda T_\mu x$$

are uniformly bounded with respect to $\lambda \in \,]0,1[$. We are going to check this property by establishing some a priori estimates.
Indeed, if x is a solution of (2.3.7), then

(2.3.8) $$x'(t) - \mu(1-\lambda)x(t) - \lambda af(x(t)) = -\lambda bf(x(t-\tau)) + \lambda p(t).$$

If we take the L^2 norm of both sides of (2.3.8) we get

(2.3.9) $$\|x'\|_2^2 + \mu^2(1-\lambda)^2\|x\|_2^2 + \lambda^2a^2\|f(x)\|_2^2 + 2\mu\lambda a(1-\lambda)\int_0^{2\pi} f(x)x\,dt \leq$$

$$\leq \lambda^2 b^2 \|f(x)\|_2^2 + \lambda^2 \|p\|_2^2 + 2\lambda^2 |b| \, \|f(x)\|_2 \|p\|_2.$$

Now if we choose μ such that $\mathrm{sgn}(\mu a) > 0$, we have

$$\lambda^2(a^2 - b^2) \, \|f(x)\|_2^2 \leq \lambda^2 \|p\|_2^2 + 2\lambda^2 |b| \, \|f(x)\|_2 \|p\|_2.$$

Dividing this last relation by λ^2 we obtain

(2.3.10) $$\|f(x)\|_2^2 \leq \text{const. (independent of } \lambda).$$

Taking advantage from this inequality we see from (2.3.9) that

(2.3.11) $$\|x'\|_2 \leq \text{const. (independent of } \lambda).$$

To complete the estimates we just have to show that every solution of (2.3.8) has a zero, in order to bound its C^0 norm by means of the inequality

(2.3.12) $$\|x\|_\infty \leq (2\pi)^{1/2} \|x'\|_2.$$

But this property follows easily integrating equation (2.3.8) and recalling that p has zero mean value:

(2.3.13) $$0 = \lambda(a\text{-}b) \int_0^{2\pi} f(x)dt \; + \mu(1 - \lambda) \int_0^{2\pi} x dt$$

Now if $a > 0$, then $a - b > 0$ and $\mu > 0$; if x does not change sign, then the two quantities in the right-hand-side of (2.3.13) have the same sign, and (2.3.13) is false. The case $a < 0$ is analogous. We have thus obtained that all the solutions of (2.3.7) are uniformly bounded with respect to λ, and the proof is complete. ♦

2.4. Sufficiency of the growth condition.

We now turn to the case in which the ordinary term $f(x(t))$ does not dominate the functional one $f(x(t-\tau))$, that is, when $|a| = |b|$. For simplicity we assume $|a| = |b| = 1$ and again we consider the problem of finding 2π-periodic solutions of

$$x'(t) = f(x(t)) - f(x(t-\tau)) + p(t)\,. \qquad (2.4.1)$$

As we have said at the beginning of the previous section, Arino and Seguier proved that the equation

$$x'(t) = f(t-1,x(t-1)) - f(t,x(t)) \qquad (2.4.2)$$

has infinitely many T-periodic solutions, each of which is completely determined by the value of the "first integral"

$$H(x) = x(t) + \int_{t-1}^{t} f(s,x(s))ds.$$

One of the basic assumptions which were made on f in [AS] was that it be nondecreasing in the state variable x and either strictly increasing or locally lipschitzean.
Our main goal is to show that these hypotheses can be completely abandoned if we replace them with some growth condition on f, to get the existence of at least one 2π-periodic solution.
We denote by X and Z respectively the spaces of C^0 and L^2 2π-periodic functions with zero mean value and we observe that, if f is continuous and $\int_0^{2\pi} p(s)ds = 0$, the Nemytskii operator N associated to the right-hand-side of (2.4.1) is continuous from X to Z.
If we define as usual the linear operator L by

$$L : \mathrm{dom}\, L \subset X \to Z,$$

$$\mathrm{dom}\, L = H^1,$$

$$(Lx)(t) = x'(t),$$

we have that L is continuously invertible between $X \cap H^1$ (endowed with the H^1 topology) and Z, so that $L^{-1}: Z \to X$ is completely continuous.
Once again, after defining $T = L^{-1}N$, the existence of a periodic solution for (2.4.1) will follow from the existence of a fixed point for the completely continuous map $T: X \to X$.
This is precisely what ensures the following

Theorem 2.4.1. Let $f: \mathbb{R} \to \mathbb{R}$ be continuous.
For every $p \in L^\infty$, with $\int_0^{2\pi} p(s)ds = 0$, there exists a positive constant $\varepsilon = \varepsilon(p)$ such that if

(2.4.3) $$\limsup_{s \to \pm\infty} \frac{|f(s)|}{s^2} < \varepsilon(p)$$

then there exists at least one 2π-periodic solution of (2.4.1).

Proof. We show the existence of a fixed point for the map T by means of the Leray-Schauder continuation theorem. To this aim, we have to show, like in the proof of theorem 2.3.1 that there exists a uniform bound for the C^0 norm of the solutions of

(2.4.4) $$x = \lambda T x,$$

for every $\lambda \in]0,1[$.
Since we know that solutions of (2.4.4) have zero mean value, by Wirtinger's inequality we only have to find a bound on the L^2 norm of their derivatives.
Indeed, let x be such a solution, for some $\lambda \in]0,1[$; then

(2.4.5) $$x'(t) - \lambda f(x(t)) = \lambda f(x(t-\tau)) - \lambda p(t).$$

Taking the L^2 norm yields

(2.4.6) $$\|x'\|_2^2 + \lambda^2\|f(x)\|_2^2 = \lambda^2\|f(x)\|_2^2 + \lambda^2\|p\|_2^2 - 2\lambda^2 \int_0^{2\pi} f(x(t-\tau))p(t)dt$$

that is, by Holder's inequality,

(2.4.7) $$\|x'\|_2^2 \leq \lambda^2\|p\|_2^2 + 2\lambda^2\|p\|_\infty\|f(x)\|_1.$$

Now by (2.4.3) there exists a constant $M > 0$ such that

(2.4.8) $$\|f(x)\|_1 \leq M + \varepsilon(p)\ \|x'\|_2^2.$$

If we carry this inequality in (2.4.7) we get

$$\|x'\|_2^2 \leq \|p\|_2^2 + 2\|p\|_\infty(M + \varepsilon(p)\ \|x\|_2^2) \leq$$

$$\leq \|p\|_2^2 + 2\|p\|_\infty(M + \varepsilon(p)\|x'\|_2^2),$$

from which we deduce, if $\varepsilon(p) < (2\|p\|_\infty)^{-1}$,

$$\|x'\|_2 \leq \text{const.},\quad (\text{independent of } \lambda),$$

which is the desired estimate. ♦

Appendix A. Fixed points and continuation theorems.

In this Appendix we recall briefly the main classical theorems which we use throughout the thesis. We omit their proofs, for which we refer to [DE] and [GM].
In what follows, whenever E is a normed space, B_R will always denote the open ball of radius R, that is, the set $\{x \in E \,/\, \|x\| < R\}$.
We begin with the classical Schauder Fixed Point Theorem:

Theorem A.1. (Schauder). Let E be a Banach space. Let $T : \overline{B}_R \to \overline{B}_R$ be a completely continuous map. Then there exists an $x \in \overline{B}_R$ such that $Tx = x$.

Strictly related to the previous theorem is the following Leray-Schauder Continuation Theorem:

Theorem A.2. (Leray-Schauder). Let E be a normed space and let $T : E \to E$ be a completely continuous map. Suppose there exists an $R > 0$ such that, for every $\lambda \in \,]0,1[$, the equation

$$x = \lambda Tx$$

has no solutions on ∂B_R. Then there exists $x \in \overline{B}_R$ such that $Tx = x$.

Let us suppose now that X, Z are normed spaces and that $L : \text{dom}\, L \subset X \to Z$ is a linear Fredholm operator of index zero. With the same notations used in Section 1.4., we denote by P and Q a pair of continuous projections (in X and Z respectively) such that $\text{Im}\, P = \text{Ker}\, L$ and $\text{Ker}\, Q = \text{Im}\, L$, and by $K_{P,Q} : Z \to \text{dom}\, L \cap \text{Ker}\, P$ the generalized inverse of L.
We recall that if $\Omega \subset X$ is open and bounded, a mapping $N : \overline{\Omega} \to Z$ is called L-completely continuous if QN is continuous, takes bounded sets into bounded sets, and if $K_{P,Q}N$ is completely continuous from X into itself.
Then we can now state Mawhin Continuation Theorem:

Theorem A.3. (Mawhin). Let X, Z be normed spaces and $L : \text{dom}\, L \subset X \to Z$ a linear Fredholm operator of index zero. Let $\Omega \subset X$ be an open bounded set and let $N : \overline{\Omega} \to Z$ be an L-completely continuous map.
Let $g \equiv QN|_{\overline{\Omega} \cap \text{Ker}\, L} : \overline{\Omega} \cap \text{Ker}\, L \to \text{Im}\, Q$ be such that $g(x) \neq 0, \forall x \in \partial\Omega \cap \text{Ker}\, L$.
Suppose moreover that

i) $Lx \neq \lambda Nx$, $\forall \lambda \in]0,1[$, $\forall x \in \partial\Omega \cap \text{dom } L$.

ii) $\deg(g, \Omega \cap \text{Ker } L, 0) \neq 0$.

Then there exists an $x \in \overline{\Omega} \cap \text{dom } L$ such that $Lx = Nx$.

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