SISSA - SCUOLA
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SUPERIORE
DI STUDI AVANZATI
TRIESTE
Strada Costiera 11

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Wess-Zumino-Witten models from coadjoint orbits

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Candidate:
Ettore Aldrovandi

Supervisor:
Prof. Loriano Bonora

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Introduction

Very recently, there has been a renewed interest in the method of coadjoint orbits both in Theoretical and Mathematical Physics. It became apparent that actions for relevant $2D$-dimensional Quantum Field Theories can be constructed generalizing the Kostant-Kirillov construction [49,47] to infinite dimensional groups.

The starting point can be probably found in the representation theory of Kac-Moody and Virasoro algebras. As a matter of fact, these infinite dimensional Lie algebras (together with some of their generalizations) have come out to play a central role in the study of Conformal Field Theories, String Theory, Integrable Systems. Their representations proved indeed to be very useful in determining interesting physical properties such as the operator content of a field theory, values of the central charges and so on.

The method of coadjoint orbits is a powerful tool to deal also with representation theory of infinite dimensional groups. In the finite dimensional case of a compact Lie group, coadjoint orbits can be realized as certain complex homogeneous spaces where the group acts transitively. The geometric quantization [49,40,41] yields irreducible unitary representations as spaces of sections of holomorphic line bundles over the orbits. This is, roughly speaking, the contents of the Borel-Weil-Bott theorem [14,69]. On the infinite dimensional side, one has to consider the underlying groups of the above mentioned Lie algebras, that is the Kac-Moody and Virasoro groups, and try to apply the method of orbits to obtain their representations. Rigorous results are only partially available at the moment [57,58].

From the non rigorous point of view, a very interesting progress has been made by the Leningrad school in term of (formal) functional integral formulation [3,4,5]. Instead of canonically quantizing the orbits by means of usual Hamiltonian formalism on the phase space, they suggest to use a Lagrangian formalism by introducing what is now called the "geometric action". This is done in a purely formal way introducing the "primitive" α of the natural Kostant-Kirillov symplectic form ω on the orbit. They write this form as $\alpha = d^{-1}\omega$, where the "inverse" of the exterior differential should mean the operation of taking the primitive of ω, and α should play the role of $p\,dq$. Then they form an action functional by integrating this form α over a path on the orbit. Since in general the 2-form ω will not happen to be exact, it is clear that this operation could be valid only locally.

Surprisingly, when applying this procedure to the case of Kac-Moody and Virasoro algebras, the geometric action turns out to be the Wess-Zumino-Witten action in the first case, and the action for the Polyakov's $2D$-dimensional induced quantum gravity [56] in the second one.

This is of course very interesting, since the WZW model and $2D$-gravity are examples of field theories exhibiting remarkable properties such as complete integrability, conformal invariance, etc. Moreover a quite intriguing phenomenon seems to connect the two theories: in [56] Polyakov showed that there is a hidden $SL\widehat{(2,\mathbb{R})}$ symmetry in $2D$ quantum gravity. There is also some evidence that the geometrical action for $2D$-gravity can be derived from an $SL(2,\mathbb{R})$ WZW model [4]. Therefore it is very appealing to derive both models from the same general principle, since this could also reveal itself as a tool to understand part of the whole matter at a deeper level.

Clearly, from the point of view of a mathematical physicist the procedure outlined before is highly unsatisfactory. Beside the lack of rigour in itself, the local approach we have described does not take into account what happens if the symplectic form ω is not exact, and prevents one from having under control the global geometry.

This thesis deals with the problem of setting the whole machinery on rigorous global geometric grounds. Of course, we have not faced the problem in its completeness: here the case of the WZW model is treated, that is the case of coadjoint orbits in a Kac-Moody algebra.

It turns out that a rather complicated structure must be set up in order to overcome the problem of non-exactness of ω. For a compact semisimple Lie group G with Lie algebra $\mathfrak{g}$, consider the Kac-Moody algebra $\widehat{L\mathfrak{g}}$ and a coadjoint orbit $\mathcal{O} \subset \widehat{L\mathfrak{g}}^*$. The Kostant-Kirillov symplectic form ω is not exact on $\mathcal{O}$, thus we consider the path space $\mathcal{PO}$, which is always a contractible space. There, using a well known homotopy operator, it is possible to find a 1-form $\tilde{\alpha}$ such that $d\tilde{\alpha} = \tilde{\omega}$, where $\tilde{\omega}$ is the lift of ω to $\mathcal{PO}$. Therefore, for a path γ on $\mathcal{PO}$, we can form the functional

$$S(\gamma) = \int_I \gamma^* \tilde{\alpha}$$

which classically can be taken as genuine action. We show that it yields precisely the WZW model.

Of course, climbing on the path space is simply a device and we should not find track of it in any physical step of the theory. From a classical point of view, we have no problem, since the equations of motion are unaffected by this procedure. At the quantum level we are really interested in $\exp iS$ and a careful analysis of the multivaluedness of the action functional, together with the requirement on $\exp iS$ to be insensitive to the presence of the path space, yields the quantization for the central charge of the Kac-Moody algebra. It is very interesting to see that this result can be deduced directly in the framework of coadjoint orbits.

This thesis is organized as follows. In Chapter 1 we recall the basic ingredients of the method of coadjoint orbits in the finite dimensional case, relating it to the bordering subjects of Hamiltonian G-spaces and flag manifolds. In Chapter 2, we set up the basics of the theory of Loop Groups and Kac-Moody algebras. In particular we deal with central extensions and affine actions of Loop Groups, used to describe coadjoint actions in the dual of a Kac-Moody algebra. We also show how the based loop group ΩG can be seen as a coadjoint orbit. Chapter 3 is the core of this thesis. There we focus on ΩG and we describe in detail the Kostant-Kirillov symplectic form. Then, it is described the construction of the geometric action for a general symplectic manifold M and it is shown that in the case $M = \Omega G$ we obtain the WZW action on a sphere. Finally, the quantization of the central charge is discussed. Some auxiliary facts are collected in the appendices.

Chapter 1

The coadjoint orbit method

1.1 Coadjoint orbits

The coadjoint orbit method has been developed mainly by Kirillov [47], Kostant [49] and Soriau [61], see also [6,1], in connection with the representation theory of Lie groups and appears as a powerful tool in producing examples of symplectic manifolds: the orbits of the action of a Lie Group G on the dual of its Lie algebra $\mathfrak{g}$ are shown to carry a closed non-degenerate 2-form, which is now called Kostant-Kirillov form.

In order to illustrate the general features of the method, avoiding complications due to the presence of infinite dimensional objects, let G be a finite dimensional Lie group, and $\mathfrak{g} \equiv T_e(G)$ its Lie algebra. However, the aim of the subsequent sections will be to apply what we are going to say to the special infinite dimensional case of the Loop Groups and their Loop Algebras.

For each $a \in G$ consider the inner automorphism

$$\tau_a : G \longrightarrow G \quad g \longmapsto a\,g\,a^{-1}$$

of G into itself: it leaves the identity element fixed, therefore its derivative (= tangent map) at the identity maps the Lie algebra into itself. The *Adjoint representation* of G on $\mathfrak{g}$ is defined to be the map

$$a \longmapsto Ad_a \equiv T_e(\tau_a) \quad \forall a \in G \tag{1.1}$$

of G into $GL(\mathfrak{g})$. It easy to verify [1] that Ad_a is a Lie algebra homomorphism, i.e.

$$Ad_a([\xi, \eta]) = [Ad_a\xi, Ad_a\eta] \quad \xi, \eta \in \mathfrak{g}.$$

Also it follows from $\tau_{ab} = \tau_a \circ \tau_b$ that $Ad.$ is indeed a representation, i.e. $Ad_{ab} = Ad_a \circ Ad_b$.

The map 1.1 is differentiable. Its derivative at the identity of the group is a linear map from $\mathfrak{g}$ to the space $End(\mathfrak{g})$ of linear operators on g. This map is denoted by ad and its value on an element ξ in the algebra by ad_ξ, so that

$$g \longrightarrow End(\mathfrak{g}) \quad \xi \longmapsto ad_\xi = \left.\frac{d}{dt}\right|_{t=0} Ad_{\exp t\xi} \tag{1.2}$$

defines the *adjoint representation* of the Lie algebra on itself. From 1.2 one finds immediately that

$$ad_\xi \eta = [\xi, \eta].$$

Now consider the dual vector space $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$: the coadjoint action of G on $\mathfrak{g}^*$ is defined transposing the Adjoint action, that is for $a \in G$ the map $Ad^*_a : \mathfrak{g}^* \to \mathfrak{g}^*$ is the transpose of Ad_a, defined by

$$< Ad^*_a \alpha, \xi > = < \alpha, Ad_a \xi > \quad \alpha \in g^*,\ \xi \in \mathfrak{g}$$

and the map

$$Ad^* : G \longrightarrow GL(\mathfrak{g}^*) \quad a \longmapsto Ad^*_a \tag{1.3}$$

defines the *Coadjoint representation* of the group on $\mathfrak{g}^*$ (note, however, that we have $Ad^*_{ab} = Ad^*_b \circ Ad^*_a$ so strictly speaking the representation should be given by $a \longmapsto Ad^*_{a^{-1}}$. Anyway, we follow the conventions in [6]). Taking the derivative of 1.3 at the identity we obtain a linear map, to be denoted by ad^*, which sends the Lie algebra into the space of linear operators on its dual space, that is

$$\mathfrak{g} \longrightarrow End(\mathfrak{g}^*) \quad \xi \longmapsto ad^*_\xi = \left.\frac{d}{dt}\right|_{t=0} Ad^*_{\exp t\xi}. \tag{1.4}$$

and again it's easy to see that ad^* is the dual of ad:

$$< ad^*_\xi \alpha, \eta > = < \alpha, ad_\xi \eta > .$$

We now consider the orbits of the coadjoint representation of G on $\mathfrak{g}^*$: at each point of a given orbit there is a natural symplectic form firstly used by Kirillov to investigate representation of nilpotent Lie groups. We point out that a series of examples of symplectic manifolds is obtained by looking at different Lie groups and all possible orbits. For $\alpha \in \mathfrak{g}^*$ the *coadjoint orbit* $\mathcal{O} = \mathcal{O}_\alpha$ through α is

$$\mathcal{O} = \{Ad^*_g \alpha | g \in G\}$$

and if $\beta \in \mathcal{O}$ is any point of the orbit, its isotropy group in G is $G_\beta = \{g \in G | Ad^*_g \beta = \beta\}$, while the Lie algebra of G_β is $\mathfrak{g}_\beta = \{\xi \in \mathfrak{g} | ad^*_\xi \beta = 0\}$. The tangent space $T_\beta(\mathcal{O})$ at any point $\beta \in \mathcal{O}$ is contained into $\mathfrak{g}^*$ and we can make the identifications $\mathcal{O} \cong G/G_\beta$ and $T_\beta(\mathcal{O}) \cong \mathfrak{g}/\mathfrak{g}_\beta$; more precisely, we have the

Proposition 1.1.1 *There is an exact sequence*

$$0 \longrightarrow \mathfrak{g}_\beta \stackrel{\imath}{\longrightarrow} \mathfrak{g} \stackrel{\sigma}{\longrightarrow} T_\beta(\mathcal{O}) \longrightarrow 0 \tag{1.5}$$

where the map σ sends the element $\xi \in \mathfrak{g}$ to the vector $\sigma_\xi(\beta) = ad^*_\xi\beta$ and $\imath$ is the inclusion.

Proof. If $\beta \in \mathcal{O}$ and $\xi \in \mathfrak{g}$ we define a curve through β by

$$t \longmapsto Ad^*_{\exp t\xi}\beta.$$

The tangent vector at β along this curve will be identified calculating

$$\left.\frac{d}{dt}\right|_{t=0} f(Ad^*_{\exp t\xi}\beta)$$

for any function $f : \mathcal{O} \to \mathbb{R}$. It's sufficient to consider only those functions f_η defined by $f_\eta(\beta) =< \beta, \eta >$, for $\eta \in \mathfrak{g}$, thus we have

$$\begin{aligned} \left.\frac{d}{dt}\right|_{t=0} < Ad^*_{\exp t\xi}\beta, \eta > &= \left.\frac{d}{dt}\right|_{t=0} < \beta, Ad_{\exp t\xi}\eta > \\ &= < \beta, \left.\frac{d}{dt}\right|_{t=0} Ad_{\exp t\xi}\eta > \\ &= < \beta, [\xi, \eta] >=< ad^*_\xi\beta, \eta > \end{aligned}$$

and we see that the tangent vector in question is precisely $ad^*_\xi\beta$. Now the exact sequence in the proposition becomes obvious. □

We can state the

Definition 1.1.2 (Kostant-Kirillov form) *Given two tangent vectors X, Y belonging to $T_\beta(\mathcal{O})$ the Kostant-Kirillov form ω is defined by*

$$\omega_\beta(X, Y) =< \beta, [\xi, \eta] > \tag{1.6}$$

where $\xi, \eta \in \mathfrak{g}$ are such that $X = \sigma_\xi(\beta)$ and $Y = \sigma_\eta(\beta)$.

Now we have the

Theorem 1.1.3 (Kirillov, Arnol'd, Kostant and Soriau) *The form ω in 1.1.2 is closed and nondegenerate, hence it turns the orbit $\mathcal{O}$ into a symplectic manifold.*

Proof. First of all, we have to check that the definition 1.1.2 is well posed. To this end let ξ' be an element in $\mathfrak{g}$ such that $\sigma_{\xi'} = \sigma_\xi$. Therefore we have

$$\begin{aligned}\omega_\beta(X,Y) &= <\beta,[\xi,\eta]> \\ &= <ad^*_\xi\beta,\eta> \\ &= <ad^*_{\xi'}\beta,\eta> \\ &= <\beta,[\xi',\eta]>\end{aligned}$$

and the same argument apply also to η.
Secondly, we show that ω is nondegenerate. Suppose in fact that $\omega_\beta(X,Y) = 0$ for all $Y \in T_\beta(\mathcal{O})$. This means that $<\beta,[\xi,\eta]>= 0$ for all $\eta \in \mathfrak{g}$ and therefore

$$<ad^*_\xi\beta,\eta>= 0 \quad \forall\eta \in g$$

so that $ad^*_\xi\beta$ is the zero functional in $\mathfrak{g}^*$, which is the same to say that X is the zero vector in $T_\beta(\mathcal{O})$.
Finally, we show that the form is closed, that is $d\omega = 0$. Observe that if let β vary in $\mathcal{O}$, $ad^*_\xi\beta$ obviously defines a vector field over it, so that we can use the following formula to express $d\omega$[46,2]:

$$d\omega(\sigma_\xi,\sigma_\eta,\sigma_\zeta) = \sigma_\xi\cdot\omega(\sigma_\eta,\sigma_\zeta) - \omega([\sigma_\xi,\sigma_\eta],\sigma_\zeta) + cyclic\ perm.$$

where σ_ξ denotes the vector field $\beta \to \sigma_\xi(\beta) = ad^*_\xi\beta$ and the first term the action of σ_ξ on the function $\beta \to \omega_\beta(\sigma_\xi(\beta),\sigma_\eta(\beta))$. Now for generic $f : \mathcal{O} \to \mathbb{R}$ we have

$$(\sigma_\xi\cdot f)(\beta) = \frac{d}{dt}\bigg|_{t=0} f(Ad^*_{\exp t\xi}\beta)$$

and therefore

$$\begin{aligned}\sigma_\xi\cdot\omega(\sigma_\eta,\sigma_\zeta)(\beta) &= \frac{d}{dt}\bigg|_{t=0}\omega(\sigma_\eta,\sigma_\zeta)(Ad^*_{\exp t\xi}\beta) \\ &= \frac{d}{dt}\bigg|_{t=0} <Ad^*_{\exp t\xi}\beta,[\eta,\zeta]> \\ &= <ad^*_\xi\beta,[\eta,\zeta]> \\ &= <\beta,[\xi,[\eta,\zeta]]>\end{aligned}$$

having used the definition of ω and σ (namely, $\sigma_\eta(Ad^*_{\exp t\xi}\beta) = ad^*_\eta(Ad^*_{\exp t\xi}\beta)$). The commutator is treated by means of the following

Lemma 1.1.4 $[\sigma_\xi,\sigma_\eta] = \sigma_{[\xi,\eta]}$.

Proof of the lemma. First note that for $f : \mathcal{O} \to \mathbb{R}$ we have

$$\begin{aligned} (\sigma_\xi \cdot f)(\beta) &= \left.\frac{d}{dt}\right|_{t=0} f(Ad^*_{\exp t\xi}\beta) \\ &= df_\beta(\sigma_\xi(\beta)) \\ &= < \sigma_\xi(\beta), df_\beta > \\ &= < \beta, [\xi, df_\beta] > \end{aligned}$$

where in the third equality we have considered $\sigma_\xi(\beta)$ as lying in $\mathfrak{g}^*$ and df_β as lying in $\mathfrak{g}$. Using the above expressions

$$\begin{aligned} [\sigma_\xi, \sigma_\eta] \cdot f &= (\sigma_\xi\sigma_\eta - \sigma_\eta\sigma_\xi) \cdot f \qquad \textit{by def.} \\ &= \left.\frac{d}{dt}\right|_{t=0} < Ad^*_{\exp t\xi}\beta, [\eta, df_{(Ad^*_{\exp t\xi}\beta)}] > - (\xi \leftrightarrow \eta) \qquad \textit{by the previous eqns} \\ &= < ad^*_\beta, [\eta, df_\beta] > + < \beta, [\eta, \left(\left.\frac{d}{dt}\right|_{t=0}\right) df_{(Ad^*_{\exp t\xi}\beta)}] > -(\xi \leftrightarrow \eta) \\ &= < \beta, [\xi, [\eta, df_\beta]] > - < \beta, [\eta, [\xi, df_\beta]] > \\ &+ \; Hess\, f_\beta(\xi, \eta) - Hess\, f_\beta(\eta, \xi) \\ &= < \beta, [[\xi, \eta], df_\beta] > \quad \textit{using Jacoby id. and the symmetry of Hess} \\ &= (\sigma_{[\xi,\eta]} \cdot f)(\beta) \end{aligned}$$

which proves the lemma. ▽

Collecting all terms together

$$\begin{aligned} d\omega(\sigma_\xi, \sigma_\eta, \sigma_\zeta) &= < \beta, [\xi, [\eta, \zeta]] > - < \omega(\sigma_{[\xi,\eta]}, \sigma_\zeta) + \textit{cyclic perm.} \\ &= < \beta, [\xi, [\eta, \zeta]] > - < \beta, [[\xi, \eta,]\zeta] > + \textit{cyclic perm.} \\ &= 0 \qquad \textit{by Jacobi identity} \end{aligned}$$

and the theorem is proved. □

Corollary 1.1.5 *Coadjoint orbits of finite dimensional Lie groups are even dimensional.* □

Corollary 1.1.6 *If $\beta \in \mathcal{O}$, then via the identification $\mathcal{O} \cong G/G_\beta$ we have that G/G_β becomes a symplectic manifold.* □

Proposition 1.1.7 *The action of the Lie group G leaves invariant the Kostant Kirillov form, namely we have*

$$(Ad^*_{g} \cdot)^* \omega = \omega$$

Proof. In fact calling Φ_g the action $\beta \to Ad^*_g \beta$ we have

$$\begin{aligned}
T_\beta \Phi_g(\sigma_\xi(\beta)) &= \left.\frac{d}{dt}\right|_{t=0} Ad^*_g Ad^*_{\exp t\xi}\beta = \left.\frac{d}{dt}\right|_{t=0} Ad^*_{(\exp t\xi)g}\beta \\
&= \left.\frac{d}{dt}\right|_{t=0} Ad^*_{gg^{-1}(\exp t\xi)g}\beta = \left.\frac{d}{dt}\right|_{t=0} Ad^*_{g\exp t(Ad_{g^{-1}}\xi)}\beta \\
&= \left.\frac{d}{dt}\right|_{t=0} Ad^*_{\exp t(Ad_{g^{-1}}\xi)} Ad^*_g\beta \\
&= ad^*_{(Ad_{g^{-1}}\xi)}(Ad^*_g\beta) \\
&= \sigma_{(Ad_{g^{-1}}\xi)}(Ad^*_g\beta)
\end{aligned}$$

and therefore

$$\begin{aligned}
(\Phi^*_g\omega)_\beta(\sigma_\xi(\beta), \sigma_\eta(\beta)) &= < Ad^*_g\beta, [Ad_{g^{-1}}\xi, Ad_{g^{-1}}\eta] > \\
&= < \beta, Ad_g Ad_{g^{-1}}[\xi, \eta] > \\
&= \omega_\beta(\sigma_\xi(\beta), \sigma_\eta(\beta))
\end{aligned}$$

i.e. the invariance $\Phi^*_g\omega = \omega$. □

Remark 1.1.8 Another proof of theorem 1.1.3 could have been given by means of the notion of Lie-Poisson structure [6,8,22,23,60]: we say that a manifold M has a *Poisson structure* when there is a bilinear operation

$$\{,\} : C^\infty(M) \times C^\infty(M) \longrightarrow \mathbb{R}$$

that turns $(C^\infty, \{,\})$ into a Lie algebra and acts as a derivation on $C^\infty(M)$:

$$\{f, gh\} = \{f, g\}h + \{f, h\}g.$$

We speak of a *Lie-Poisson structure* on a Lie group G when it carries a Poisson structure $\{,\}$ which is compatible with the group multiplication in the following sense. If $\mu : G \times G \to G$ is the multiplication and $\mu^* : C^\infty(G) \to C^\infty(G \times G)$ the induced mapping on functions, the compatibility of the Poisson structure with μ means that

$$\{\mu^* f, \mu^* g\}_{G\times G} = \mu^*(\{f, g\}_G)$$

where under the natural identification $C^\infty(G \times G) \cong C^\infty(G) \otimes C^\infty(G)$ the Poisson structure on $G \times G$ is given by [8]

$$\{a_1 \otimes b_1, a_2 \otimes b_2\}_{G\times G} = \{a_1, a_2\}_G \otimes b_1\, b_2 + a_1\, a_2 \otimes \{b_1, b_2\}_G.$$

The dual $\mathfrak{g}^*$ of a Lie algebra $\mathfrak{g}$ has an obvious Lie-Poisson structure, with respect to the abelian Lie group structure defined by the vector space addition, given by

$$\{f, g\}(\alpha) =< \alpha, [df_\alpha, dg_\alpha] >$$

for $f, g \in C^\infty(\mathfrak{g}^*)$. Now another way to state (and prove) theorem 1.1.3 is to show that

$$\{f, g\}(\alpha) = \{f|_{\mathcal{O}}, gf|_{\mathcal{O}}\}(\alpha)$$

where the bracket on the left is the Lie-Poisson bracket, while the RHS bracket is the Poisson bracket defined by the Kostant-Kirillov form on $\mathcal{O}$.

1.2 Miscellaneous results

We now collect some interesting facts related to the subject of coadjoint orbits. We mainly refer to [49] and [40] (see also [1]).

If (M, ω) is a symplectic manifold we say that the vector field $X : M \to TM$ is *hamiltonian* if it exists a function $h_X : M \to \mathbb{R}$ such that

$$i(X)\omega = dh_X$$

where $i(\cdot)$ is the interior product. Denote by $ham(M)$ the class of hamiltonian vector fields. (M, ω) will be called a *G- symplectic space* if there is an action

$$\Phi : G \times M \longrightarrow M$$

of G on M through symplectic diffeomorphisms, while it will be called a *G-strongly symplectic space* if the vector field σ_ξ defined by

$$\sigma_\xi(x) = \frac{d}{dt}\bigg|_{t=0} \Phi(\exp t\xi, x)$$

is hamiltonian $\forall \xi \in \mathfrak{g}$, i.e. $\mathfrak{g}$ is sent into $ham(M)$ (note that this is a homomorphism of Lie algebras, since $\sigma_{[\xi,\eta]} = [\sigma_\xi, \sigma_\eta]$).

We have that strongly symplectic is equivalent to symplectic in either one of the following two cases [49]:

1. M is simply connected

2. $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$ i.e. $H^1(\mathfrak{g}, \mathbb{R}) = \{0\}$ (this happens e.g. if $\mathfrak{g}$ is semisimple).

This is easy to understand, since symplectic means $\mathcal{L}_{\sigma_\xi}\omega = 0 = di(\sigma_\xi)\omega$ (recall that ω is closed) and in the first case we have $\pi_1(M) = \{0\} \Rightarrow H^1(M, \mathbb{R}) = \{0\}$, while in the case 2. we have the general relation $[\mathcal{L}_X, i(Y)] = i([X, Y])$ and

$$\begin{aligned} i(\sigma_{[\xi,\eta]})\omega &= i([\sigma_\xi, \sigma_\eta])\omega = [\mathcal{L}_{\sigma_\xi}, i(\sigma_\eta)]\omega \\ &= \mathcal{L}_{\sigma_\xi} i(\sigma_\eta)\omega = d(i(\sigma_\xi)i(\sigma_\eta)\omega. \end{aligned}$$

If a strongly symplectic action is given, the map that associates to every $\xi \in \mathfrak{g}$ its hamiltonian function f_ξ such that

$$i(\sigma_\xi)\omega = df_\xi$$

can always be made linear. In the case the commutative diagram of Lie algebras

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{R} & \longrightarrow & C^\infty(M) & \longrightarrow & ham(M) & \longrightarrow & 0 \\ & & & & & \nwarrow^{\lambda} & \uparrow & & \\ & & & & & & \mathfrak{g} & & \end{array} \tag{1.7}$$

exists, λ is called a *lift* of σ. λ exists if and only if the class $[\mu] \in H^2(\mathfrak{g}, \mathbb{R})$ vanishes, where μ is the cocycle

$$\mu(\xi, \eta) = \{\mu_0(\xi), \mu_0(\eta)\} - \mu_0([\xi, \eta])$$

for $\mu_0 : \mathfrak{g} \to C^\infty(M)$ any linear map making the above diagram commutative. In particular λ always exists if $H^2(\mathfrak{g}, \mathbb{R}) = \{0\}$ (e.g if $\mathfrak{g}$ is semisimple). Note also that if λ is a lift then so is $\lambda + \alpha$ for $\alpha \in \mathfrak{g}^*$ such that $\alpha|_{[\mathfrak{g},\mathfrak{g}]} = 0$ (i.e. if α is a cocycle on $\mathfrak{g}$). Therefore $H^1(\mathfrak{g}, \mathbb{R})$ measures the non-unicity of the lift λ in the case it exists.

In [49] Kostant also introduces the category $\mathcal{H}(G)$ of *Hamiltonian G-spaces*, in the sense of the following

Definition 1.2.1 *Hamiltonian G-spaces are those strongly symplectic G-spaces M with symplectic form ω satisfying the condition that a lift $\lambda : \mathfrak{g} \to C^\infty(M)$ exists with the properties that*

1. $d\lambda(\xi)_x$ *spans* $T^*_x(M) \quad \forall x \in M$

2. the hamiltonian vector field $X_{\lambda(\xi)} \equiv \sigma_\xi$ *is globally integrable* $\forall \xi \in \mathfrak{g}$.

Morphisms in $\mathcal{H}(G)$ are maps $f : M \to N$ such that

$$f^*\omega_N = \omega_M \quad \text{and} \quad \lambda_M(\xi) = \lambda_N(\xi) \circ f$$

It's easy to see that for a coadjoint orbit $\mathcal{O}$ a lift $\lambda : \mathfrak{g} \to C^\infty(\mathcal{O})$ is given by

$$\xi \longmapsto \lambda_{\mathcal{O}}(\xi) = -f_\xi$$

where f_ξ is the function $f_\xi(\beta) =< \beta, \xi >$, $\beta \in \mathcal{O}$, so that collecting what we have proved so far we get

Proposition 1.2.2 *Every coadjoint orbit is a Hamiltonian G-space.* □

Now the importance of coadjoint orbits relies on the following results, the proof of which can be found in [49,40]:

Theorem 1.2.3 *The map* $f : (M, \omega_M, \lambda_M) \to (N, \omega_N, \lambda_N)$ *between hamiltonian G-spaces is G-equivariant and such that*

$$T_x f(X_{\lambda_M(\xi)}(x)) = X_{\lambda_M(\xi)}(f(x)) \qquad \forall \xi \in \mathfrak{g}$$

Moreover for any $x \in M$

$$T_x f : T_x(M) \longrightarrow T_{f(x)}(N)$$

is an isomorphism and in fact f *is a covering map of manifolds.* □

Theorem 1.2.4 *Let* (M, ω_M, λ_M) *be any hamiltonian G-space. Then there exists a unique orbit* $\mathcal{O}$ *such that a map* $f : M \to \mathcal{O}$ *of hamiltonian G-spaces exists. Moreover* f *is unique.* □

An important tool in proving theorem 1.2.4 is the *momentum mapping*

$$J : M \longrightarrow \mathfrak{g}^*$$

defined by

$$< J(x), \xi >= \lambda(\xi)(x) \qquad \xi \in \mathfrak{g}, x \in M$$

Due to theorem 1.2.3 the momentum mapping is equivariant. Note also that for a coadjoint orbit the moment map is nothing else that (minus) the inclusion into $\mathfrak{g}^*$.

The above results should clarify the importance of coadjoint orbits besides the fact they are examples of symplectic manifolds. Noticing that they actually are *homogeneous* symplectic manifolds (corollary after theorem 1.1.3) and also that Hamiltonian G-spaces too are homogeneous (as condition 1. in 1.2.1 implies that the tangent space at point x is spanned by $X_{\lambda(\xi)}(x)$, $\xi \in \mathfrak{g}$, since ω is nondegenerate – see [49]), the previous theorems show that in some sense each symplectic manifold carrying a strong liftable action is a coadjoint orbit (up to covering). However, if one drops the lifting and retains only the notion of G-symplectic space, if a covering

$$f : M \longrightarrow \mathcal{O}$$

exists then either f or $\mathcal{O}$ need not be unique. Nevertheless, if we consider G-symplectic *homogeneous* spaces we have that if such a covering exists it is unique up to the addition of a 1-cocycle on $\mathfrak{g}$. This in turn implies, together with theorem 1.2.4, that [49]

Proposition 1.2.5 *If $H^1(\mathfrak{g}, \mathbb{R}) = H^2(\mathfrak{g}, \mathbb{R}) = \{0\}$ the most general G-symplectic homogeneous space covers an orbit $\mathcal{O} \subset \mathfrak{g}^*$. Moreover $\mathcal{O}$ and the covering map are unique.* □

These last results from [49] tend to a classification of G-symplectic (homogeneous) spaces and bring into the highly developed subject of homogeneous complex manifolds, representation theory, etc.

For G a compact connected semisimple Lie group, it is immediate to see that the stabilizer subgroup G_μ of $\mu \in \mathfrak{g}^*$ under coadjoint action is the centralizer of a torus T in G. More precisely, if $X_\mu \in \mathfrak{g}$ is the vector corresponding to μ via the Killing form, G_μ is the centralizer $C(T)$ of the torus T generated by $\exp tX_\mu$, that is

$$G_\mu = \{g \in G \mid g\,(\exp tX_\mu)g^{-1} = \exp tX_\mu\} \equiv C(T)$$

Therefore, for what we have seen in section 1.1, a coadjoint orbit is of the form $G/C(T)$ that is, by definition, a *generalized flag manifold* see [14,69,13]. A flag manifold $M = G/C(T)$ can be endowed with a complex structure [13,68,69] such that G acts on X as a group of holomorphic diffeomorphisms. This complex structure is not unique, depending on the choice of a set of positive roots for $\mathfrak{g}_{\mathbb{C}}$, the complexification of $\mathfrak{g}$ [13]. A flag manifold can be also represented as the homogeneous space

$$M \cong G_{\mathbb{C}}/P$$

for $G_{\mathbb{C}}$ the complexification of G and P a suitable parabolic subgroup of $G_{\mathbb{C}}$ [69]. The complex structure inherited from this representation is also known to be Kähler [68]. Therefore it follows that coadjoint orbits are Kähler manifolds.

They are also readily seen to be simply connected, since in the case $\pi_1(G) = 0$ an easy application of the homotopy sequence of a fibration [15] shows that

$$\pi_1(\mathcal{O}) \cong \pi_0(G_\mu) \cong G_\mu/G_\mu^0$$

where G_μ^0 is the connected component of the identity of G_μ. The centralizer of a torus in a compact group is connected [69], therefore $\pi_1(\mathcal{O}) = 0$. In the case $\pi_1(G) \neq 0$ we can pass to the universal covering $\tilde{G} \xrightarrow{p} G$ to find

$$\mathcal{O} \cong G/G_\mu \cong \tilde{G}/\tilde{G}_\mu$$

for $\tilde{G}_\mu = p^{-1}(G_\mu)$, and apply again the above mechanism.

To summarize, coadjoint orbits of compact groups are simply connected homogeneous compact Kähler manifolds. They are the only manifolds of this type, in some sense, since Wang has shown in [68], among other things, that a simply connected compact homogeneous complex manifold is Kähler if and only if it is a flag manifold.

This result on the classification of this type of manifolds is implied by the results of Kostant we quoted above, since if M is such a manifold, Kähler implies symplectic and, by the proposition 1.2.5, it covers a unique coadjoint orbit $\mathcal{O}$. As homogeneous spaces

$$M \cong G/G_x \qquad \mathcal{O} \cong G/G_\mu,$$

where $G_x \subset G$ is the stabilizer subgroup of $x \in M$, and $G_\mu^0 \subset G_x \subset G_\mu$ [49]. But since the coadjoint orbits are simply connected, one obtains Wang's result in an improved form, that is with only the assumption of being symplectic [49].

Chapter 2

Loop Groups and Kac-Moody Algebras

2.1 Kac-Moody Algebras

2.1.1 General definitions

Kac-Moody algebras are a subclass of infinite dimensional Lie algebras intensively studied both in Mathematics and Physics. In general in the spread literature concerning Lie algebras and their applications the name "Kac-Moody algebra" is not used to always mean the same object. In particular it seems that in Physics the name is used mainly for (central extension of) current algebras, while in Mathematics a slight more general object is meant. We choose to agree with the definition in [45], which is now a standard reference, even though we'll actually use a restricted subclass.

In [45] the a Kac-Moody algebra is constructed somewhat axiomatically starting from the notion of *Generalized Cartan Matrix*(GCM for brevity):

Definition 2.1.1 *A Generalized Cartan Matrix A is an $n \times n$ matrix $A = \{a_{ij}\}$ of rank l subject to the conditions*

1. $a_{ii} = 2 \qquad i = 1, \ldots n$
2. a_{ij} *are non-positive integers for* $i \neq j$
3. $a_{ij} = 0 \;\Rightarrow\; a_{ji} = 0$

An even more important concept is that of *realization* of a Generalized Cartan Matrix, in the sense of the following

Definition 2.1.2 *A realization of a Generalized Cartan Matrix A is a triple $(\mathfrak{h}, \Pi, \check{\Pi})$ where $\mathfrak{h}$ is a complex vector space, $\Pi = \{\alpha_1, \ldots, \alpha_n\} \subset \mathfrak{h}^*$ and $\check{\Pi} = \{\check{\alpha}_1, \ldots, \check{\alpha}_n\} \subset \mathfrak{h}$ obeying the following conditions:*

1. *both Π and $\check{\Pi}$ are linearly independent*
2. $< \alpha_j, \alpha_i >= a_{ij}$
3. $n - l = dim\ \mathfrak{h} - n$

We note explicitly that the condition *3.* let $dim\ \mathfrak{h}$ vary between the two extreme cases of maximal rank $l = n = dim\ \mathfrak{h}$ and $l = 1$ $dim\mathfrak{h} = 2n - 1$ (the most degenerate one) and the first one corresponds to the finite dimensional simple Lie algebras. The vector space $\mathfrak{h}$ is the Cartan subalgebra and the sets Π, $\check{\Pi}$ correspond respectively to the roots and the coroots.

Given a GCM A with realization $(\mathfrak{h}, \Pi, \check{\Pi})$ form the auxiliary Lie algebra $\tilde{\mathfrak{g}}(A)$ by generators $\{e_i, f_i\}_{i=1}^n$ and $\mathfrak{h}$ with relations

$$\begin{aligned} [e_i, f_j] &= \delta_{ij}\check{\alpha}_i \\ [h, h'] &= 0 \quad \text{for} \quad h, h' \in \mathfrak{h} \\ [h, e_i] &= < \alpha_i, h > e_i \\ [h, f_i] &= - < \alpha_i, h > f_i \end{aligned}$$

and in [45] is proved the following

Theorem 2.1.3 *There is a unique maximal ideal $\mathfrak{r}$ in $\tilde{\mathfrak{g}}(A)$ among the ideals intersecting $\mathfrak{h}$ trivially. Furthermore*

$$\mathfrak{r} = (\mathfrak{r} \cap \tilde{\mathfrak{n}}_-) \oplus (\mathfrak{r} \cap \tilde{\mathfrak{n}}_+)$$

as a direct sum of ideals, where $\tilde{\mathfrak{n}}_-(\tilde{\mathfrak{n}}_+)$ is the subalgebra in $\tilde{\mathfrak{g}}(A)$ generated by the set $\{e_1, \ldots, e_n\}$ $(\{f_1, \ldots, f_n\})$. □

Now we are ready to state what a Kac-Moody algebra is.

Definition 2.1.4 *Given a GCM with realization $(\mathfrak{h}, \Pi, \check{\Pi})$ form the Lie algebra $\tilde{\mathfrak{g}}(A)$: the Lie algebra*

$$\mathfrak{g}(A) = \tilde{\mathfrak{g}}(A)/\mathfrak{r}$$

is the Kac-Moody algebra associated with A.

This seems to be the most general definition of a Kac-Moody algebra. At this level of generality, keeping the name e_i, f_i for the image in $\mathfrak{g}(A)$ of the generators of $\tilde{\mathfrak{g}}(A)$, we record the following facts:

1. there is the *triangular decomposition*

$$\mathfrak{g}(A) = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$$

where the subalgebras $\mathfrak{n}_+$, $\mathfrak{n}_-$ are generated by $\{e_1, \ldots, e_n\}$ and $\{f_1, \ldots, f_n\}$ respectively.

2. if $\mathfrak{g}^1(A) = [\mathfrak{g}(A), \mathfrak{g}(A)]$ is the derived algebra we have

$$\mathfrak{g}(A) = \mathfrak{g}^1(A) + \mathfrak{h}$$

with $\mathfrak{g}^1(A) \cap \mathfrak{h} = \mathfrak{h}'$ for $\mathfrak{h}' = \sum_{i=1}^n \mathbb{C}\check{\alpha}_i$

3. the centre $\mathfrak{z}$ of $\mathfrak{g}(A)$ or $\mathfrak{g}^1(A)$ is equal to

$$\mathfrak{z} = \{h \in \mathfrak{h} | < \alpha_i, h >= 0 \quad i = 1, \ldots, n\}$$

4. there is a $\mathbb{C}$-linear involution ω of $\mathfrak{g}(A)$ which acts on the generators as

$$e_i \mapsto -f_i, \quad f_i \mapsto -e_i \quad h \mapsto -h \quad h \in \mathfrak{h}.$$

The fixed point set of this involution is (the generalization of) the *compact form* and is a real Lie algebra.

Actually, restrictions are imposed on the allowable Cartan matrices in order to get more manageable definitions. The subclass of *symmetrizable* GCM is still vaste but at the same time it's sufficient to produce a more transparent definition of Kac-Moody algebra. A GCM A is said to be symmetrizable if it can be represented in the form $A = DB$, where D is a nondegenerate diagonal matrix and B is a symmetric matrix. In [45] is proved the highly non trivial result, which surprisingly is a consequence of the representation theory of Kac-Moody algebras, that a Kac-Moody algebra $\mathfrak{g}(A)$ associated with a symmetrizable GCM A defined as in 2.1.4, can be equivalently defined as the Lie algebra with generators $\{e_i, f_i\}_{i=1}^n$ and $\mathfrak{h}$ satisfying the *defining relations*[37]:

$$\begin{array}{rcl} [e_i, f_j] = \delta_{ij}\check{\alpha}_i & & [h, h'] = 0 \;\; h, h' \in \mathfrak{h} \\ [h, e_i] =< \alpha_i, h > e_i & & [h, f_i] = - < \alpha_i, h > f_i \\ (ade_i)^{1-a_{ij}} e_j = & 0 & = (adf_i)^{1-a_{ij}} f_j \quad i \neq j \end{array}$$

This is the definition usually found in the mathematical literature. Kac-Moody algebras associated to symmetrizable GCM posses remarkable properties, one of them being the existence of an *ad*-invariant $\mathbb{C}$-valued nondegenerate bilinear form $(\cdot, \cdot)$[1]. Moreover the derived algebra $\mathfrak{g}^1(A)$ is a *universal central extension* by its centre [45]:

$$0 \longrightarrow \mathfrak{z} \longrightarrow \mathfrak{g}^1(A) \longrightarrow \mathfrak{g}^1(A)/\mathfrak{z} \longrightarrow 0$$

Sometimes this derived algebra itself is called a Kac-Moody algebra.

[1]We prefer to recall its definition in a still more restrictive case, which will be the one of our practical interest, where it can be done even without referring to the Cartan matrix

2.1.2 The affine case

Cartan matrices can belong to only three mutually exclusive classes [45]:

Theorem 2.1.5 *Given a GCM A one and only one of the following possibilities holds for both A and ${}^tA$:*

Fin *$\det A \neq 0$; there exists $u > 0$ such that $Au > 0$; $Av \geq 0$ implies $v > 0$ or $v = 0$; this happens if and only if all its principal minors are positive*

Aff *$\operatorname{corank} A = 1$; there exists $u > 0$ such that $Au = 0$; $Av \geq 0$ implies $Av = 0$; this happens if and only if all its* proper *principal minors are positive and $\det A = 0$*

Ind *there exists $u > 0$ such that $Au < 0$; $Av \geq 0$, $v \geq 0$ imply $v = 0$.* □

Cartan matrices of finite or affine type are symmetrizable and they all are classified by the extended Dynkin diagrams [45]; those of finite type correspond to finite dimensional simple Lie algebras. In the affine case the Dynkin diagram carries $l+1$ labels $\{a_0, a_1, \ldots, a_n\}$ which are the coordinates of the unique vector δ such that $A\delta = 0$ and are positive relatively prime integers. Given a GCM of affine type we'll also have labels $\{\check{a}_0, \check{a}_1, \ldots, \check{a}_n\}$ relative to the transposed matrix which is of affine type too.

From now on we shall consider only Kac-Moody algebras of affine type (i.e. those associated to affine Cartan matrices). Note that since $rank A = l = n-1$ then $\dim \mathfrak{h} = n+1 = l+2$, and being $\check{\Pi}$ a set of linearly independent coroots we have that the center is one-dimensional. Actually it is the span of the element

$$\hat{c} = \sum_{i=0}^{l} \check{a}_i \check{\alpha}_i$$

as follows from the fact that $\hat{c}$ must satisfy $< \alpha_i, \hat{c} >= 0 \quad \forall i = 0, 1, \ldots, n$. Moreover it is possible to fix an element $\hat{d} \in \mathfrak{h}$ such that

$$< \alpha_i, \hat{d} >= 0 \quad i = 1, \ldots, n \qquad < \alpha_0, \hat{d} >= 1$$

called the *scaling element*, which forms together with $\check{\Pi}$ a basis for $\mathfrak{h}$ and since we saw that $\mathfrak{g}^1(A) \cap \mathfrak{h} = \mathfrak{h}'$ we have the direct sum decomposition

$$\mathfrak{g}(A) = \mathfrak{g}^1(A) \oplus \mathbb{C}\hat{d}.$$

Affine Kac-Moody algebras are completely described in terms of an "underlying" finite dimensional simple Lie algebra $\mathring{\mathfrak{g}}$ [45]: this is the algebra generated by $\{e_i, f_i\}_{i=1}^{l}$ and clearly it is the Kac-Moody algebra associated to the Cartan matrix obtained from A deleting the $0^{\underline{th}}$ row and column. If $\overset{\circ}{\mathfrak{h}}$ is the span of $\check{\alpha}_1, \ldots, \check{\alpha}_n$, obviously $\overset{\circ}{\mathfrak{h}} = \overset{\circ}{\mathfrak{g}} \cap \mathfrak{g}$ and $\overset{\circ}{\mathfrak{h}}$ is the Cartan subalgebra of $\overset{\circ}{\mathfrak{g}}$. Recall that for a Cartan matrix of finite type the rank is equal to the order, so that $\check{\alpha}_1, \ldots, \check{\alpha}_n$ form a basis for $\overset{\circ}{\mathfrak{h}}$ and also we have $\overset{\circ}{\mathfrak{g}} = [\overset{\circ}{\mathfrak{g}}, \overset{\circ}{\mathfrak{g}}]$, i.e. the first (real) cohomology group vanishes. For the Cartan subalgebra of the affine Kac- Moody algebra it follows that

$$\mathfrak{h} = \overset{\circ}{\mathfrak{h}} \oplus (\mathbb{C}\hat{c} + \mathbb{C}\hat{d}).$$

We now describe a "concrete " realization of all *non-twisted* affine Kac-Moody algebras, where non twisted means all those affine algebras whose Dynkin diagrams are listed in table *Aff. 1* in [45][2]. Let's change a little bit the notation and let $\mathfrak{g}$ denote from now on a finite dimensional complex simple Lie algebra with (compact) real form $\mathfrak{g}_{\mathbb{R}}$ obtained with the usual Cartan involution [44]. Consider the *loop algebra* [36,35,45]

$$L\mathfrak{g} = \mathbb{C}[z, z^{-1}] \otimes_{\mathbb{C}} \mathfrak{g}$$

which is the Lie algebra of polynomial maps $\mathbb{C}^{\times} \longrightarrow \mathfrak{g}$ with the obvious commutator defined extending pointwise the commutator in the "small" Lie algebra $\mathfrak{g}$:

$$[P \otimes \xi, Q \otimes \eta] = PQ \otimes [\xi, \eta] \qquad P, Q \in \mathbb{C}[z, z^{-1}] \;\; \xi, \eta \in \mathfrak{g}. \tag{2.1}$$

If D is a derivation in $\mathbb{C}[z, z^{-1}]$ it obviously extends to a derivation on $L\mathfrak{g}$ by the rule

$$D(P \otimes \xi) = D(P) \otimes \xi$$

and if $<,>$ is a multiple of the Killing form it extends to a symmetric invariant bilinear form

$$< , >_z : \; L\mathfrak{g} \times L\mathfrak{g} \longrightarrow \mathbb{C}[z, z^{-1}]$$

by pointwise evaluation. Consider now the $\mathbb{C}$-valued bilinear form φ defined by

$$\varphi(X, Y) = Res\left(< \frac{dX}{dz}, Y >_z \right) \tag{2.2}$$

where

$$Res\ P = c_{-1} \quad \text{for} \quad P = \sum_k c_k z^k.$$

This form satisfies the two conditions

1. $\varphi(X, Y) = -\varphi(Y, X)$
2. $\varphi([X, Y], Z) + \varphi([Y, Z], X) + \varphi([Z, X], Y) = 0$

[2] a "constructive" definition will be given later on

i.e. it is a 2–cocycle, as can be easily proved on simple tensors; therefore we can form the corresponding one-dimensional central extension $\widehat{L\mathfrak{g}}$ which fits into the exact sequence of Lie algebras

$$0 \longrightarrow \mathbb{C} \longrightarrow \widehat{L\mathfrak{g}} \longrightarrow L\mathfrak{g} \longrightarrow 0$$

with commutator $[\ ,\ ]_\wedge$ given by

$$[X \oplus \alpha\hat{c}\ , Y \oplus \beta\hat{c}\,]_\wedge = [X,Y] \oplus \varphi(X,Y)\hat{c} \tag{2.3}$$

for $X, Y \in L\mathfrak{g}$ and $\alpha, \beta \in \mathbb{C}$, while $\hat{c}$ is any element that spans the one-dimensional center of $\widehat{L\mathfrak{g}}$. Explicitly, $\widehat{L\mathfrak{g}} = L\mathfrak{g} \otimes \mathbb{C}\hat{c}$.

Finally, denote with $\overline{L\mathfrak{g}}$ the Lie algebra obtained adding to $\widehat{L\mathfrak{g}}$ a derivation $\hat{d}$ that operates on $L\mathfrak{g}$ as $z\frac{d}{dz}$ and sends the center to zero. More precisely, $\overline{L\mathfrak{g}}$ is the vector space

$$\overline{L\mathfrak{g}} = L\mathfrak{g} \oplus \mathbb{C}\hat{c} \oplus \mathbb{C}\hat{d}$$

with the bracket

$$[X \oplus \alpha\hat{c} \oplus \beta\hat{d}\ , X_1 \oplus \alpha_1\hat{c} \oplus \beta_1\hat{d}\,]_\sim = \left([X, X_1] + \beta z\frac{dX_1}{dz} - \beta_1 z\frac{dX}{dz}\right) \oplus \varphi(X, X_1)\hat{c} \tag{2.4}$$

for $X, Y \in \mathfrak{g}$ and $\alpha, \alpha_1, \beta, \beta_1 \in \mathbb{C}$. The Lie algebra $\overline{L\mathfrak{g}}$ possesses a nondegenerate symmetric bilinear form, denoted with $<,>_\sim$, defined by

$$< X \oplus \alpha\hat{c} \oplus \beta\hat{d}\ , X_1 \oplus \alpha_1\hat{c} \oplus \beta_1\hat{d}\ >_\sim = Res(z^{-1} < X, X_1 >_z) + \alpha\beta_1 + \alpha_1\beta \tag{2.5}$$

This form is *ad*-invariant in the sense that $< [\tilde{X}, \tilde{Y}]_\sim, \tilde{Z} >_\sim = < \tilde{X}, [\tilde{Y}, \tilde{Z}]_\sim >_\sim$ for any $\tilde{X}, \tilde{Y}, \tilde{Z} \in \overline{L\mathfrak{g}}$.

The connection with the abstract theory is found noticing that if $\mathfrak{h}$ is the Cartan subalgebra of the finite dimensional Lie algebra $\mathfrak{g}$ then the subspace

$$\tilde{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}\hat{c} \oplus \mathbb{C}\hat{d}$$

is a maximal abelian subalgebra, hence a Cartan subalgebra of $\overline{L\mathfrak{g}}$ [36]. The theory of roots can be carried on in this setting, see [36,35,45], and produces an extended Cartan matrix of (untwisted) affine type. Detailed proofs can be found in [45].

However, by the general theory outlined above, we have that

$$\begin{aligned} \widehat{L\mathfrak{g}} &= [\overline{L\mathfrak{g}}, \overline{L\mathfrak{g}}] \\ \overline{L\mathfrak{g}} &= \widehat{L\mathfrak{g}} \oplus \mathbb{C}\hat{d} \end{aligned}$$

and we stipulate the convention of calling "Kac-Moody algebra" the central extension of a loop algebra, while we will use the name "affine Kac-Moody algebra" for the full algebra $\overline{L\mathfrak{g}}$, unless otherwise explicitly stated. This obviously means that we will always restrict ourselves to the subclass of affine untwisted Kac-Moody algebras.

2.2 Loop Groups and their central extensions

2.2.1 Loop Groups as manifolds of maps

In this section we start to embed the algebraic theory outlined above into an analytic setup where it is possible to "exponentiate" the Lie algebra described in the previous rather abstract way to obtain a group, at least in the affine case [57] (but see [64,65] for the state of art in the general case).

First of all note that with the change of variable $z = e^{i\theta}$ [35] we can think of the Lie algebra $\mathbb{C}[z, z^{-1}] \otimes_{\mathbb{C}} \mathfrak{g}$ as an algebra $L_{pol}\mathfrak{g}$ of polynomial maps from S^1 to $\mathfrak{g}$, where we consider S^1 either as $\{z \in \mathbb{C} \mid |z| = 1\}$ or as $\mathbb{R}/2\pi\mathbb{Z}$. The connection of course is given by

$$\frac{d}{d\theta} = iz\frac{d}{dz} \quad \text{and} \quad \int_{S^1} = \int_0^{2\pi} \frac{d\theta}{2\pi} = \frac{1}{2\pi i}\oint \frac{dz}{z}$$

This observation quite naturally leads to consider the space $L\mathfrak{g} = Map(S^1, \mathfrak{g})$ of smooth (i.e. C^∞) maps from S^1 to $\mathfrak{g}$ or various completions of it, and one would be tempted to say that $L\mathfrak{g}$ is the Lie algebra of the group LG of maps from S^1 to G, where G is the finite dimensional Lie group whose Lie algebra is $\mathfrak{g}$. This is indeed the case, even for the general situation of groups $Map(M, G)$ of maps from a smooth compact manifold M in G, either endowed with the C^∞–topology [42,53,57] or with some Sobolev completion [32] – see also [54,26,25] for generalities on manifolds of maps. The local model for $Map(M, G)$ is $Map(M, \mathfrak{g})$, where an appropriate topology has been given: this furnishes a local chart near the identity (the map $M \ni x \mapsto e \in G$) and by left translations with $f \in Map(M, G)$ we can produce a local coordinate system.

The C^∞–topology is defined requiring "uniform convergence with all the derivatives" in the local model $Map(M, \mathfrak{g})$: this is achieved either by fixing an auxiliary connection on M to globalize the notion of derivative, or by specifying a basis of neighbourhoods of zero in $Map(M, \mathfrak{g})$ taking finite intersection of the sets [53]:

$$\mathcal{N}(U, \{x_i\}, K; m, \epsilon) = \{f : M \to \mathfrak{g} \mid \|D^\alpha f\| \le \epsilon; \quad |\alpha| \le m\}$$

for $U \subset M$ open with local coordinates $\{x_i\}$, $K \subset U$ compact, $\|\cdot\|$ = absolute value of the Killing form and $D^\alpha = \partial^{\alpha_1 + \cdots + \alpha_n} / \partial^{\alpha_1} x_1 \cdots \partial^{\alpha_n} x_n$.

For any real number s, Sobolev completions are defined giving to $Map(M, \mathfrak{g})$ the metric [32]

$$(X, Y)_{H_s} = \int_M < (1 + \triangle^s) X(x), Y(s) >_{\mathfrak{g}} dx \qquad X, Y \in Map(M, \mathfrak{g}) \tag{2.6}$$

for $\triangle = d^* d$ the Laplace operator on M and dx the Riemannian volume form. The Hilbert space completion is denoted $H_s(M, \mathfrak{g})$ and for $s > \frac{n}{2}$, $n = dim\, M$, it consists of continuous maps (Sobolev embedding). In this range the corresponding completion $H_s(M,G)$ exists and is modelled on $H_s(M,\mathfrak{g})$. Pointwise multiplication is defined by the Sobolev multiplication lemma and we have [32,34]

Proposition 2.2.1 *The group $H_s(M,G)$ is a Hilbert-Lie group for $s > \frac{n}{2}$.* □

The H_s metric on $Map(M,\mathfrak{g})$ defining the completion also gives an H_s metric on each tangent space, where the tangent space at a map $f \in Map(M,G)$ consists of H_s sections of the pulled back tangent bundle $f^*(TG)$. $Map(M,G)$ also inherits a natural family of weak metrics, namely the H_t–metrics defined in the same way as (2.6) but with $t < s$, the most important one being the L^2 (or H_0) metric given on $Map(M,\mathfrak{g})$ by

$$(X,Y) = \int_M < X(x), Y(x) >_{\mathfrak{g}} dx$$

which defines the Killing form on $Map(M,\mathfrak{g})$. See [32,33] for a detailed analysis of the interplay between this various metrics as far as curvature and characteristic classes are concerned.

All what we have said applies to the case $LG = Map(S^1,G)$: we choose to consider LG as a manifold of smooth maps in agreement with [57], even though Hilbert-Sobolev completions are oftenly considered [32,7]. Actually, for a certain range of purposes it doesn't matter which kind is used and, even more, only subsets of smooth or algebraic maps turn out to be really important [7,57]

From now on for a compact real (semi)simple Lie group G with Lie algebra $\mathfrak{g}$, we will deal with the smooth loop group $LG = Map(S^1,G)$. It has $L\mathfrak{g} = Map(S^1,\mathfrak{g})$ as Lie algebra and $\exp : L\mathfrak{g} \to LG$ defined pointwise by $(\exp X)(z) = \exp(X(z))$ is a local homeomorphism. Even though LG very often behaves like a compact group, we begin pointing out a difference: the exponential map is not surjective on the connected component of the identity, namely there are elements which do not lie in any one-parameter subgroup, as shown in explicit examples in [35,53,57]. Nevertheless, for G compact the image of exp is dense in the identity component of LG. Another obvious remark is that if G has a complexification $G_{\mathbb{C}}$ then $LG_{\mathbb{C}}$ is the complexification of LG and is a complex Lie group. This is true also for the more general group $Map(M,G)$.

Given LG, one of the most important subgroup is perhaps the group $\Omega G = Map_0(S^1,G)$ of *based maps*, i.e. those maps $f : S^1 \to G$ such that $f(1) = e$. It is obviously a normal subgroup and we can think of it in a variety of ways. First of all ΩG can be thought also as the homogeneous space $\Omega G \cong LG/G$ the correspondence being the one that associates to the coset $G \cdot f$, $f \in LG$, the element $\tilde{f}(\cdot) = f(1)^{-1}f(\cdot) \in \Omega G$. Using this correspondence, we can consider the tangent space at the identity of ΩG ($\cong$ coset G) either as the space $Map_0(S^1,\mathfrak{g})$ of maps

$X : S^1 \to \mathfrak{g}$ such that $X(1) = 0$, or as the quotient $L\mathfrak{g}/\mathfrak{g}$. ΩG can also be regarded as a homogeneous space of a larger group. $U(1)$ acts on LG by rigid rotations, i.e. we have

$$(e^{it} \cdot f)(z) = f(e^{-it}z) = f(e^{i(\theta - t)}) \tag{2.7}$$

for $f \in LG$. Form the semidirect product $U(1) \ltimes LG$ with the rules

$$\begin{aligned} (e^{it}, f)(e^{is}, g) &= (e^{i(t+s)}, (e^{-is} \cdot f)g) \\ (e^{it}, f)^{-1} &= (e^{-it}, e^{it} \cdot f^{-1}) \end{aligned}$$

it follows that [32,33,7]

$$\Omega G \cong (U(1) \ltimes LG)/(U(1) \times G)$$

via the map

$$(e^{it}, f) \longrightarrow f(1)^{-1} f(\cdot).$$

This result, in spite of its apparent triviality, leads to make the following remark. $U(1)$ is trivially a torus (in the sense of being a compact abelian subgroup) in $U(1) \ltimes LG$; its centralizer $C(U(1))$ is the set of elements in $U(1) \ltimes LG$ that commute with every $e^{it} \in U(1)$ and it's easy to show that

$$C(U(1)) = U(1) \times G$$

so that ΩG is the quotient of a Lie group by the centralizer of a torus, that is an infinite dimensional (intermediate) flag manifold [32,33,27,14,69]. This is very important, since at least in the finite dimensional case flag manifolds are essentially coadjoint orbits and indeed ΩG can be also realized in this way, as we will directly prove in the next section.

Since ΩG is a homogeneous space, the full group $U(1) \ltimes LG$ acts on it shifting the cosets in the following way. Represent a coset in ΩG by the element $(1, f)$ for $f : S^1 \to G$, $f(1) = e$. Then for $(e^{it}, g) \in U(1) \ltimes LG$ we have

$$\begin{aligned} (1, f)(e^{it}, g) &= (e^{it}, (e^{-it} \cdot f)g) \\ &= \left(e^{it}, f(e^{it})g(1)\right)\left(1, g(1)^{-1} f(e^{it})^{-1} f(e^{it} \cdot) g(\cdot)\right) \end{aligned}$$

and this proves the

Proposition 2.2.2 *There are two actions*

$$\begin{aligned} LG \times \Omega G \longrightarrow \Omega G &\qquad (g, f) \mapsto g(1)^{-1} f(\cdot) g(\cdot) \\ U(1) \times \Omega G \longrightarrow \Omega G &\qquad (e^{it}, f) \mapsto f(e^{it})^{-1} f(e^{it} \cdot) \end{aligned}$$

of the groups LG and $U(1)$ on ΩG. □

Note that the fibration

$$G \longrightarrow LG \longrightarrow \Omega G$$

is trivial, since being LG and ΩG groups we have that $LG \cong \Omega G \times G$ as a topological space.

2.2.2 Central extension

We now show how our previous algebraic discussion on Kac-Moody algebras fits into this framework. Consider the set $L_{pol}\mathfrak{g}$ of polynomial maps from S^1 to $\mathfrak{g}$, which of course can be identified with the restriction of $\mathbb{C}[z, z^{-1}] \otimes_{\mathbb{C}} \mathfrak{g}_{\mathbb{C}}$, our previous formally defined loop algebra. At the same time consider also the set $L_{pol}G$ of polynomial maps from S^1 into G: it can be defined directly by embedding G in some $SU(n)$ and considering matrix– valued functions of the form

$$\sum_{k=-N}^{N} A_k z^k$$

see [57], or more formally as the group of all maps $f : S^1 \rightarrow G$ which are the restriction of morphisms of complex varieties from $\mathbb{C}^{\times}$ to $G_{\mathbb{C}}$, the complexification of G, [7]. $\Omega_{pol}G$ is the subgroup of $L_{pol}G$ consisting of the basepoint–preserving maps. A result due to Segal is the following [57,7]

Proposition 2.2.3 *$L_{pol}G$ is dense in LG if G is semisimple. The same is true also for $L_{pol}\mathfrak{g}$.* □

Actually a more striking result can be proved, namely there exists a filtration

$$\Omega_0 \subset \Omega_1 \subset \Omega_2 \subset \cdots$$

of $\Omega_{pol}G$ by closed subvarieties of a finite dimensional complex Grassmanian such that $\Omega_{pol}G = \bigcup_{i=0}^{\infty} \Omega_i$ and the inclusions are algebraic embeddings; with the use of this filtration in [7] it is proved that $\Omega_{pol}G$ is dense in $\Omega_1 G$, its Sobolev completion with $s = 1$. Of course the same holds true for $L_{pol}G$ and $H_1(S^1, G)$.

Thus we have that the formal loop algebra described in the previous section can be considered as a dense subset of our "true" loop algebra. Actually all the untwisted affine Kac-Moody algebras we were referring to before arise in this way. The twisted case corresponds to algebras coming from sets of maps $f : S^1 \rightarrow \mathfrak{g}$ with the identification $f(1) = \alpha(f(0))$, for α an outer automorphism of G of finite order. Since in our discussion about formal loop algebras we have considered mainly central extensions, we do the same here. It will turn out that the algebras previously examined embed as dense subsets into the algebras we are going to construct. We can still use formula (2.2) to define a 2–cocycle φ on $L\mathfrak{g}$ by

$$\begin{aligned} \varphi(X,Y) &= \frac{1}{2\pi}\int_0^{2\pi} < X', Y >_{\mathfrak{g}} d\theta \\ &= \int_{S^1} < X', Y >_{\mathfrak{g}} \quad X, Y \in L\mathfrak{g}. \end{aligned} \tag{2.8}$$

By direct calculation on $L_{pol}\mathfrak{g}$ and using proposition 2.2.3 it can be shown that [57]

Proposition 2.2.4 *If $\mathfrak{g}$ is semisimple the only continuous G–invariant cocycles on $L\mathfrak{g}$ are those given by 2.8.* □

Therefore given φ we can form the central extension

$$0 \longrightarrow \mathbb{R} \longrightarrow \widehat{L\mathfrak{g}} \longrightarrow L\mathfrak{g} \longrightarrow 0 \tag{2.9}$$

with commutator

$$[X \oplus \alpha\hat{c}\,, Y \oplus \beta\hat{c}\,]_\wedge = [X,Y] \oplus \varphi(X,Y)\hat{c} \tag{2.10}$$

Since now we have a genuine group LG whose Lie algebra is $L\mathfrak{g}$, we can ask ourselves what is the Lie group (if it exists) whose Lie algebra is $\widehat{L\mathfrak{g}}$, or stated in other words, given the exact sequence (2.9) does a corresponding group extension

$$1 \longrightarrow U(1) \longrightarrow \widehat{LG} \longrightarrow LG \longrightarrow 1 \tag{2.11}$$

exists? This is not always the case, since certain integrability conditions must be satisfied.

First of all, use φ on $L\mathfrak{g}$ to produce a left–invariant 2–form on the whole LG, still denoted by φ; this form will be closed due to the cocycle condition. We start with the following

Theorem 2.2.5 *If G is simply connected the Lie algebra extension (2.9) corresponds to a group extension (2.11) if and only if the differential form $\varphi/2\pi$ represents an integral cohomology class. In this case $\widehat{LG}$ is a $U(1)$–principal bundle over the the base LG uniquely determined by the class $[\varphi/2\pi]$.*

Proof. LG is topologically a product and by definition of homotopy groups [15] we have

$$\begin{aligned} \pi_q(LG) &= \pi_q(\Omega G \times G) \\ &= \pi_q(\Omega G) \times \pi_q(G) \\ &= \pi_{q+1}(G) \times \pi_q(G); \end{aligned}$$

since G is a compact Lie group it has $\pi_2(G) = 0$ [43,39], and being also simply connected it follows that $\pi_0(LG) = \pi_1(LG) = 0$, that is the loop group itself is connected and simply connected. The theorem will follow from the following two important propositions.

Proposition 2.2.6 *Let X a connected and simply connected manifold (it may be also infinite dimensional).*

i) If φ is a closed two form on X such that $\varphi/2\pi$ represents an integral cohomology class then there is a circle bundle over X with a connection whose curvature is φ.

ii) If Y and Y' are circle bundles over X with connections α and α' which have the same curvature φ then there is a bundle isomorphism $\lambda : Y' \rightarrow Y$ such that $\lambda^*\alpha = \alpha'$. Furthermore ψ is unique up to the composition with the action of an element in $U(1)$.

Proof of the proposition. $\pi_1(X) = 0 \Rightarrow H_1(X, \mathbb{Z}) = 0$ and by the universal coefficient theorem this implies that the torsion part of $H^2(X, \mathbb{Z})$ is zero [15], so that the map $H^2(X, \mathbb{Z}) \rightarrow H^2(X, \mathbb{R})$ is injective. Since $H^2(X, \mathbb{Z})$ is the group of isomorphism classes of complex line bundles (as follows from the well known exponential sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{A} \xrightarrow{\exp} \mathcal{A}^* \longrightarrow 0$$

[70,16] where $\mathcal{A}$ is the sheaf of C^∞–functions [3]), $\frac{\varphi}{2\pi}$ determines uniquely a class of line bundles. Moreover the principal bundle of a line bundle is a $\mathbb{C}^\times$–bundle, and, being $\frac{\varphi}{2\pi}$ real, it determines a reduction to a $U(1)$–bundle [49,46]. To show that the bundle $U(1) \rightarrow Y \rightarrow X$ in question has a connection α whose curvature is $\frac{\varphi}{2\pi}$, proceed as follows. Given a covering [4] $\mathfrak{U} = \{U_i\}$ of the manifold let $\varphi_i = \varphi \mid_{U_i}$; then $d\varphi_i = 0 \Rightarrow \exists \alpha_i \in \Omega^1(U_i) \mid \varphi_i = d\alpha_i$, so that on $U_{ij} = U_i \cap U_j$ we have $d(\alpha_i - \alpha_j) = 0$, which implies that $\exists f_{ij} \in \Omega^0(U_{ij}) \mid \alpha_i - \alpha_j = df_{ij}$. Obviously $d(f_{ij} + f_{jk} - f_{ik}) = 0$, so that $\exists\ 2\pi a_{ijk} \in \mathbb{R} \mid f_{ij} + f_{jk} - f_{ik} = 2\pi a_{ijk}$. It follows that $\{a_{ijk}\}$ is the two-cocycle relative to φ ([16] and the "collating formula" in [15]). Since $\frac{\varphi}{2\pi}$ is integer, $\{a_{ijk}\} \in Z^2(\mathfrak{U}, \mathbb{Z})$. Taking $c_{ij} = \exp i f_{ij}$, we have that $\{c_{ij}\}$ satisfies the cocycle condition, i.e. $\{c_{ij}\} \in Z^1(\mathfrak{U}, \mathcal{A}^*)$ and therefore it can be taken as a set of transition functions. Moreover the α_i are the components of the connection, since

$$\begin{aligned} \alpha_i &= \alpha_j + df_{ij} \\ &= \alpha_j + \frac{1}{i}\frac{dc_{ij}}{c_{ij}} \end{aligned}$$

[3] In the sequel we shall often put $\mathcal{A}(U) = \Omega^0(U)$

[4] Čech cohomology applies to the manifold we are considering here [57]

To prove part *ii)*, let $\{\alpha_i'\} \in C^0(\mathfrak{U}, \Omega^1)$ such that $d\alpha_i' = \varphi_i$: there is a corresponding construction and $c_{ij}' = \exp i f_{ij}'$ are the transition function for a bundle $Y' \to X$. Now we have that $d\alpha_i' = d\alpha_i$ implies that $\exists \beta_i \in \Omega^0(U_i) \mid \alpha_i' = \alpha_i + d\beta_i$. It follows that $d(f_{ij}' - f_{ij} - \beta_i + \beta_j) = 0$, which in turn means that $\exists\ \{\mu_{ij}\} \in C^1(\mathfrak{U}, \mathbb{R}) \mid f_{ij}' = f_{ij} + \beta_i - \beta_j + \mu_{ij}$, and the condition $2\pi a_{ijk}' = 2\pi a_{ijk}$ forces $\{\mu_{ij}\}$ to be a cocycle. Since X is simply connected, $H_1(X, \mathbb{Z}) = 0$ and $\{\mu_{ij}\}$ is a coboundary, i.e. $\exists\ \{\nu_i\} \in C^0(\mathfrak{U}, \mathbb{R}) \mid \mu_{ij} = \nu_i - \nu_j$. From this we have that

$$c_{ij}' = c_{ij} \lambda_i \lambda_j^{-1} \quad \text{for} \quad \lambda_i = \exp i(\beta_i + \nu_i)$$

that is the bundle Y' is isomorphic with Y, and from

$$\alpha_i' = \alpha_i + d\beta_i = \alpha_i + \frac{1}{i}\frac{d\lambda_i}{\lambda_i}$$

we see that $\lambda^* \alpha = \alpha'$. Uniqueness follows from the fact that if $\mu_{ij} = \nu_i' - \nu_j' = \nu_i - \nu_j$ then $\nu_i' - \nu_i = \nu_j' - \nu_j$ and therefore for the transition functions we have

$$\lambda_i' = \exp i(\beta_i + \nu_i') = \lambda_i \, \exp i(\nu_i' - \nu_i)$$

namely they differ for a globally defined element in $U(1)$. $\bigtriangledown$

Proposition 2.2.7 *Suppose that a Lie group* Γ *acts on a connected and simply connected manifold* X*, leaving invariant an integral closed 2-form* $\frac{\varphi}{2\pi}$ *on* X *(both* Γ *and* X *may be infinite dimensional). Then there is an extension* $\tilde{\Gamma}$ *of* Γ *by* $U(1)$ *canonically associated to* (φ, X)*, and for any point* $x \in X$ *the associated extension of Lie algebras can be represented by the cocycle*

$$(\xi, \eta) \longmapsto \varphi(\xi_x, \eta_x)$$

where ξ_x *denotes the tangent vector at* $x \in X$ *corresponding to the action of the element* $\xi \in Lie(\Gamma)$.

Proof of the proposition. First construct a $U(1)$–bundle Y over X with connection α with curvature φ. For each $\gamma \in \Gamma$ take the pulled back bundle $\gamma^* Y$ which has a connection α_γ whose curvature is $\gamma^* \varphi$. We define $\tilde{\Gamma}$ as the group of all pairs (γ, λ) with $\gamma \in \Gamma$ and λ an isomorphism such that $\alpha_\gamma = \lambda^* \alpha$. By the previous proposition there is a circle of possible choices of λ for each γ. From this it should be apparent that $\tilde{\Gamma}$ is a $U(1)$–bundle over Γ. More accurately, let's note that with the pull back bundle $\gamma^* Y$ we can form the diagram

$$\begin{array}{ccccc} Y & \longrightarrow & \gamma^* Y & \xrightarrow{\tilde{\gamma}} & Y \\ & \searrow & \downarrow & & \downarrow \\ & & X & \xrightarrow{\gamma} & X \end{array}$$

where the upper map $\tilde{\gamma}$ is canonically determined by definition of pull back; any other isomorphism λ will differ from $\tilde{\gamma}$ by the action of an element in $U(1)$. Moreover there is a canonical way of producing the map $Y \rightarrow \gamma^*Y$ that covers the identity on X – as clearly explained, for instance, in [49] – simply by specifying how an arbitrary $y_0 \in Y$ goes into an arbitrarily chosen element of γ^*Y such that both project down on the same $x_0 \in X$. Due to our particular geometry this is exactly the same as specifying an element of the fiber of Y over $\gamma(x_0)$, so that the $U(1)$–freedom in the choice of $\lambda : \gamma^*Y \rightarrow Y$ can be transferred to the map from Y to γ^*Y, thereby indicating that the group $\tilde{\Gamma}$ can be equally described as the set of pairs (γ, λ) where λ is an isomorphism $\lambda : Y \rightarrow Y$ covering γ and such that $\lambda^*\alpha = \alpha$. Such an isomorphism λ is determined by γ and the choice of $\lambda(y_0)$ such that it projects on $\gamma(x_0)$. Thus as a manifold $\tilde{\Gamma}$ is the fibered product $\Gamma \times_X Y$, that is the pull back of Y by the map $\Gamma \rightarrow X$ which takes γ to $\gamma(x_0)$. ∇

Now the proof of the theorem is almost finished, since it is sufficient to apply the proposition just proved to the case $X = \Gamma = LG$. The other half of the statement is quite trivial, for if there is a group extension

$$1 \longrightarrow U(1) \longrightarrow \widehat{LG} \longrightarrow LG \longrightarrow 1$$

a choice of a splitting

$$Lie(\tilde{\Gamma}) \cong Lie(\Gamma) \oplus \mathbb{R}$$

determines a decomposition of the tangent space at each point of $\tilde{\Gamma}$, i.e. a connection. The splitting map $Lie(\Gamma) \rightarrow Lie(\tilde{\Gamma})$ can be identified with the horizontal lifting $\xi \rightarrow \tilde{\xi}$ of left-invariant vector fields. Since the curvature in principal bundle can be described by the lacking of the horizontal lift to be a Lie algebra homomorphism, wee see that the expression

$$\varphi(\xi, \eta) = [\tilde{\xi}, \tilde{\eta}] - \widetilde{[\xi, \eta]}$$

describes both the curvature and a Lie algebra cocycle which therefore has to be integer. □

When G is not simply connected the situation is much more intricated [57]. In this case the loop group will be neither connected nor simply connected, so the theorem above does not apply. We can write $G = \tilde{G}/Z$ where $\tilde{G}$ is the simply connected covering group of G and $Z \cong \pi_1(G)$ is a finite subgroup of the centre of $\tilde{G}$. Since the conditions under which a 2-cocycle on a loop algebra is integer can be ultimately deduced entirely from the bilinear form on the finite dimensional Lie algebra $\mathfrak{g}$ [57], by the preceeding theorem such a form gives rise to a unique central extension $\widehat{L\tilde{G}}$ of $L\tilde{G}$. The group $\tilde{G}$ can be thought as a subgroup of $\widehat{L\tilde{G}}$ (because the fibration over it as a subset of $L\tilde{G}$ is trivial) and therefore we can regard Z as a subgroup of $\widehat{L\tilde{G}}$. In fact Z belongs to the center of $\widehat{L\tilde{G}}$, because its adjoint action on

$\widehat{L\mathfrak{g}}$ is trivial (see proposition 2.3.1 in the next section). Thus we have an extension

$$1 \longrightarrow U(1) \longrightarrow \widehat{L\tilde{G}}/Z \longrightarrow (LG)^0 \longrightarrow 1$$

where $(LG)^0 \cong (L\tilde{G})/Z$ is the identity component of LG. One should be warned about the fact that the that extension is generally *not* the restriction of an extension of the whole group LG unless certain additional conditions are satisfied, but we do not proceed further into details.

2.3 Coadjoint orbits in Kac-Moody Algebras

In this section we specialize the general framework outlined in 1.1 to the case of the dual of a Kac–Moody algebra. Since such an algebra is infinite dimensional, we must specify the meaning of "dual". As we saw in 2.1 and 2.2 a Kac-Moody algebra is of the form

$$\widehat{L\mathfrak{g}} = L\mathfrak{g} \oplus \mathbb{R}$$

and is the tangent space at the identity of the "Kac-Moody group" $\widehat{LG}$ so that taking the topological dual is dangerous for two reasons:

1. the topological dual of a space of C^∞ maps such as $L\mathfrak{g}$ or $\widehat{L\mathfrak{g}}$ is made essentially of distributions, even though the C^∞ maps themselves form a dense subset; this is somewhat unpleasant from our point of view of studying coadjoint orbits, since we would like to avoid distributional objects;

2. the topological dual of a Fréchet vector space is *never* a Fréchet space itself, unless the space itself is finite dimensional [42], so that when dealing with infinite dimensional manifolds modelled on a Fréchet space one should not take dual spaces or, more generally, spaces of linear maps in order to remain into the chosen manifold model.

Of course we have the inclusion $L\mathfrak{g} \subset L\mathfrak{g}^*$, where $L\mathfrak{g}$ acts on itself by means of the Killing form as

$$L(X) =< L, X >= \int_{S^1} < L, X >_{\mathfrak{g}} \tag{2.12}$$

for $L, X \in L\mathfrak{g}$. Since this pairing is nondegenerate, we limit ourselves to consider only the smooth part of the dual, and when speaking of the dual $L\mathfrak{g}^*$ we will intend the space $L\mathfrak{g}$ itself as a set with the above pairing. It follows that for the centrally extended algebra we have

$$\widehat{L\mathfrak{g}}^* = L\mathfrak{g}^* \oplus \mathbb{R} \cong L\mathfrak{g} \oplus \mathbb{R} \tag{2.13}$$

with the duality

$$(L \oplus \lambda)(X \oplus \alpha\hat{c}\ ) = < L \oplus \lambda, X \oplus \alpha\hat{c}\ > = < L, X > + \lambda\alpha. \tag{2.14}$$

With this preliminaries we pass to discuss the action of the loop group. The ad-action of a Lie algebra on itself is defined by the commutator, so for $\widehat{L\mathfrak{g}}$ we have

$$\begin{aligned} ad(X \oplus \alpha) \cdot (Y \oplus \beta) &= [X \oplus \alpha, Y \oplus \beta]_\wedge \\ &= [X, Y] \oplus \varphi(X, Y) \\ &= ad(X \oplus 0) \cdot (Y \oplus \beta) \end{aligned} \tag{2.15}$$

and we see that the action of the centre is trivial. The finite version of the action (2.15) is the adjoint action of the loop group. Let Ad denote the adjoint action of LG on $L\mathfrak{g}$ obtained evaluating pointwise the one of G on $\mathfrak{g}$, and for $g \in LG$ let $g^{-1}\partial g$ be the element in $L\mathfrak{g}$ obtained first by differentiating the map g with respect to $d/d\theta$ and then left translating the vector so obtained by g [5]. We have the

Proposition 2.3.1 *The Adjoint action of the loop group LG on $\widehat{L\mathfrak{g}}$ is given by*

$$Ad_\wedge g \cdot (X \oplus \alpha) = Ad\, g \cdot X \oplus (\alpha + < g^{-1}\partial g, X >)$$

Proof. The formula above correctly reproduces the ad-action, in fact take $g_t = \exp tY$, $Y \in L\mathfrak{g}$, and differentiate with respect to t:

$$\begin{aligned} \left.\frac{d}{dt}\right|_{t=0} (Ad\, g \cdot X)(z) &= \left.\frac{d}{dt}\right|_{t=0} Ad_{g(z)} X(z) \\ &= [Y(z), X(z)] = [Y, X](z) \end{aligned}$$

and

$$\left.\frac{d}{dt}\right|_{t=0} < g^{-1}\partial g, X > = < \left.\frac{d}{dt}\right|_{t=0} (g^{-1}\partial g), X >$$

but

$$\left.\frac{d}{dt}\right|_{t=0} (g^{-1}\partial g) = \partial Y$$

and therefore

$$\left.\frac{d}{dt}\right|_{t=0} < g^{-1}\partial g, X > = < \partial Y, X > = \varphi(Y, X)$$

[5] We use notations as if G were a group of matrices

It must be checked that the formula in the proposition defines a representation of LG, but this follows from the relation

$$(hg)^{-1}\partial(hg) = g^{-1}\partial g + Ad\,g^{-1}\cdot(h^{-1}\partial h)$$

for $g, h \in LG$, and the obvious Ad-invariance of the form (2.12). □

Proposition 2.3.2 *Under the identification (2.13) the coadjoint action of LG on the Kac-Moody algebra $\widehat{L\mathfrak{g}}$ is given by*

$$Ad^*_\wedge\, g\cdot(L\oplus\lambda) = \left(Ad\,g^{-1}\cdot L + \lambda g^{-1}\partial g\right)\oplus\lambda$$

Proof. Trivial using the previous proposition and the duality relation (2.14). □

Remark 2.3.3 The coadjoint action of LG on $\widehat{L\mathfrak{g}}$ preserves the hyperplanes $\lambda = const$ in $\widehat{L\mathfrak{g}}$: once that any such hyperplane has been fixed by the choice of a particular value of λ, we can regard it as $L\mathfrak{g}$ equipped with an *affine* action of LG.

Remark 2.3.4 Note that taking $g_t = \exp tY$ and differentiating with respect to t the formula for the coadjoint action in the above proposition we get

$$\left.\frac{d}{dt}\right|_{t=0} Ad^*_\wedge\, g_t\cdot(L\oplus\lambda) = (-[Y,L]+\lambda\partial Y)\oplus 0 \tag{2.16}$$

which in turn is equal to $ad^*_\wedge\, Y\cdot(L\oplus\lambda)$, in fact we have

$$\begin{aligned} < ad^*_\wedge\, Y\cdot(L\oplus\lambda), X\oplus\alpha > &= < L\oplus\lambda, [Y, X\oplus\alpha]_\wedge > \\ &= < L, [Y,X] > + \lambda\varphi(Y,X) \\ &= < [L,Y], X > + \lambda < \partial Y, X > \\ &= < \lambda\partial Y + [L,Y], X > \end{aligned} \tag{2.17}$$

and the pairing $< \cdot,\cdot >$ is nondegenerate on $L\mathfrak{g}$. The previous remark, combined with the above formulas, says that tangent vectors to the orbits are entirely contained into the affine hyperplane where the orbit takes place, hence their component along the center is identically zero.

It follows from (2.16) and (2.17) that the stabilizer subgroup of the point $L\oplus\lambda$ can be constructed, modulo elements that cannot be reached by the exponential map, solving the differential equation on S^1

$$\lambda\,\partial Y + [L,Y] = 0 \tag{2.18}$$

for $Y \in L\mathfrak{g}$. More generally, calling $\mathcal{O}_{L\oplus\lambda}$ the coadjoint orbit passing through $L \oplus \lambda$, a tangent vector at this point can be thought as lying into $L\mathfrak{g}^* \oplus 0$ and therefore identified with an element $Z \in L\mathfrak{g}$. Again from (2.16) and (2.17) we have that an element $X \in L\mathfrak{g}$ such that $ad^*_\wedge X \cdot (L \oplus \lambda) = Z \oplus 0$ is determined solving

$$\lambda\, \partial X + [L, X] = Z \tag{2.19}$$

on S^1. Of course X determines the tangent vector in question up to the addition of a solution of (2.18), so that we could say that the tangent space to the orbit at the point coincides as a set with $L\mathfrak{g}$ modulo solutions of (2.18).

Now in the same spirit of definition 1.1.2 we can state the

Definition 2.3.5 *Given a coadjoint orbit $\mathcal{O} \subset \widehat{L\mathfrak{g}}^*$ and two tangent vectors $\tilde{X}, \tilde{Y}$ at the point $L \oplus \lambda$, the Kostant-Kirillov form ω is given by*

$$\begin{aligned} \omega(\tilde{X}, \tilde{Y}) &= < L \oplus \lambda, [X, Y] > \\ &= < L, [X, Y] > + \lambda < \partial X, Y > \end{aligned}$$

where $X, Y \in L\mathfrak{g}$ determine $\tilde{X}, \tilde{Y}$ in the sense of equation (2.19) above.

2.4 ΩG as a coadjoint orbit

Consider now the special coadjoint orbit $\mathcal{O}_\kappa$ in $\widehat{L\mathfrak{g}}^*$ passing through the point $0 \oplus \kappa$. From the general form of the coadjoint action stated in proposition 2.3 we see that the generic element in this orbit is of the form $L_g = \kappa g^{-1}\partial g$ for $g \in LG$. Therefore it follows that if g is a constant loop we have $L_g = 0$, that is the point $0 \oplus \kappa$ is left fixed by the elements in $G \subset LG$; furthermore, also the converse is true, since the equation $g^{-1}\partial g = 0$ implies that g is a constant loop. It follows that the stability subgroup of $0 \oplus \kappa$ is G and we can make the identification $\mathcal{O}_\kappa \cong LG/G \cong \Omega G$; this can also be inferred using the relation

$$L_{gh} = \kappa h^{-1}\partial h + \kappa Ad\, h^{-1} \cdot (g^{-1}\partial g)$$

which for g a constant loop reduces to

$$L_{gh} = \kappa h^{-1}\partial h = L_h.$$

As we anticipated in section 2.2 *the based loop group is therefore a coadjoint orbit.* We will make a complete identification between $\mathcal{O}_\kappa$ and ΩG speaking from now on of ΩG directly as a coadjoint orbit, since this will be our main object of interest. Sometimes, to be more definite, we will mention explicitly an identification map

$$j : \Omega G \longrightarrow \mathcal{O}_\kappa$$

such that

$$j(f) = \kappa f^{-1}\partial f \oplus \kappa \qquad f \in \Omega G$$

On ΩG many calculations simplify considerably, for instance it is easy to calculate the element in $L\mathfrak{g}$ that produces a specified tangent vector on ΩG via coadjoint action. The calculation is as follows. Suppose that δf is a tangent vector to ΩG at the loop f, that is a field $z \mapsto \delta f(z)$ on S^1 of tangent vectors at G such that $\delta f(1) = 0$. Choose any path $t \longmapsto f_t$ in ΩG such that $f_0 = f$ and $\delta f = \frac{d}{dt}\big|_{t=0} f_t$, therefore under the map j the corresponding tangent vector to the orbit at the point $\kappa f^{-1}\partial f \oplus \kappa$ is given by

$$\frac{d}{dt}\bigg|_{t=0} (\kappa f_t^{-1}\partial f_t \oplus \kappa) = \left(\kappa \frac{d}{dt}\bigg|_{t=0} (f_t^{-1}\partial f_t) \oplus 0\right)$$

and therefore we have [6]

$$\begin{aligned}
\frac{d}{dt}\bigg|_{t=0} f_t^{-1}\partial f_t &\equiv \delta(f^{-1}\partial f) \\
&= (\delta f^{-1})\partial f + f^{-1}\delta(\partial f) \\
&= -f^{-1}\delta f\, f^{-1}\partial f + f^{-1}\delta(\partial f) \\
&= -f^{-1}\delta f\, f^{-1}\partial f + f^{-1}\partial(\delta f) \\
&= \partial(f^{-1}\delta f) + f^{-1}\partial f\, f^{-1}\delta f - f^{-1}\delta f\, f^{-1}\partial f \\
&= \partial(f^{-1}\delta f) + [f^{-1}\partial f, f^{-1}\delta f]
\end{aligned}$$

Now note that $f^{-1}\delta f$ is an element in $L\mathfrak{g}$ and since also $\frac{d}{dt}\big|_{t=0} f_t^{-1}\partial f_t$ is in $L\mathfrak{g}$, we have obtained exactly the structure of equation (2.19), and $f^{-1}\delta f$ is the vector in $L\mathfrak{g}$ that determines the tangent vector δf to f through infinitesimal coadjoint action.

Now using the calculation above and the explicit identification between ΩG and $\mathcal{O}_\kappa$ furnished by the map j we can state how the Kostant-Kirillov form on the orbit translates to ΩG. Let $f \in \Omega G$ and let $\delta_1 f, \delta_2 f \in T_f(\Omega G)$; then by definition we have

$$\begin{aligned}
(j^*\omega)_f(\delta_1 f, \delta_2 f) &\equiv \omega_{j(f)}(T_f(\delta_1 f), T_f(\delta_2 f)) \\
&= \kappa \int_{S^1} < f^{-1}\partial f, [f^{-1}\delta_1 f, f^{-1}\delta_2 f] >_{\mathfrak{g}} \\
&+ \kappa \int_{S^1} < \partial(f^{-1}\delta_1 f), f^{-1}\delta_2 f >_{\mathfrak{g}} \qquad (2.20)
\end{aligned}$$

Thus our object of investigation will be the space ΩG equipped with the symplectic form (2.20). We point out that in [7,32] a symplectic form which coincides with the second half of the ours, and therefore coincides with the cocycle giving the central extension of $L\mathfrak{g}$, is used. We plan to put some comment on this fact in the next chapter.

[6] we prefer to give a more transparent formal derivation treating the loop as a matrix-valued function

Finally we make a few remarks on the invariance of the Kostant-Kirillov form. Even though invariance should be apparent from the construction itself of the symplectic form, if one had the need to see this explicitly the procedure is as follows. First note in general that if $L \oplus \kappa$ is a point in a coadjoint orbit $\mathcal{O} \in \widehat{L\mathfrak{g}}^*$ and for $X \in L\mathfrak{g}$

$$\delta L \equiv ad^*_\wedge X \cdot (L \oplus \kappa) = \kappa\,\partial X + [L, X]$$

is a tangent vector to it, the coadjoint action of the loop group LG

$$\Phi_g : \mathcal{O} \longrightarrow \mathcal{O}$$

$$\Phi_g(L \oplus \kappa) = Ad^*_\wedge g \cdot (L \oplus \kappa) = \left(\kappa\, g^{-1}\partial g + Ad\, g^{-1} \cdot L\right) \oplus \kappa$$

sends δL into

$$\begin{aligned} T\Phi(\delta L) &= \kappa\,\partial\left(Ad\, g^{-1} \cdot X\right) + \left[Ad^*_\wedge g \cdot (L \oplus \kappa), Ad\, g^{-1} \cdot X\right] \\ &= ad^*_\wedge(Ad\, g^{-1} \cdot X) \cdot Ad^*_\wedge g \cdot (L \oplus \kappa) \\ &= ad^*_\wedge(Ad_\wedge g^{-1} \cdot X) \cdot Ad^*_\wedge g \cdot (L \oplus \kappa) \end{aligned}$$

as is easily checked by making only derivatives. In the last equality we have used the fact that using the affine action is the same since the central term is unessential. Now invariance can be proved in the same way as we did in the finite dimensional case in section 1.1.

In the case of ΩG, we had the explicit identification j with the orbit $\mathcal{O}_\kappa$. For $f \in \Omega G$ and $g \in LG$ the relation

$$\kappa\, g^{-1}\partial g + \kappa Ad\, g^{-1} \cdot (f^{-1}\partial f) = \kappa\,(f\,g)^{-1}\partial(f\,g) = j(g(1)^{-1} f\,g)$$

shows that the map j is equivariant with respect to both the coadjoint actions of LG on $\mathcal{O}_\kappa$ and ΩG and therefore it follows that $j^*\omega$ is invariant.

Chapter 3

WZW model from coadjoint orbits

3.1 The symplectic form on ΩG

We ended the previous chapter presenting the based loop group ΩG as a coadjoint orbit. For convenience we transcript here the symplectic form

$$\begin{aligned} \omega_f(\delta_1 f, \delta_2 f) &= \kappa \int_{S^1} < f^{-1}\partial f, [f^{-1}\delta_1 f, f^{-1}\delta_2 f] >_{\mathfrak{g}} \\ &+ \kappa \int_{S^1} < \partial(f^{-1}\delta_1 f), f^{-1}\delta_2 f >_{\mathfrak{g}} \end{aligned} \tag{3.1}$$

where $f \in \Omega G$ and $\delta_1 f, \delta_2 f \in T_f(\Omega G)$. We have also suppressed any reference to the identification between ΩG and the orbit passing through the point $0 \oplus \kappa \in \widehat{L\mathfrak{g}}^*$. The second half of the symplectic form is nothing more than the 2-cocycle defining the central extension of $L\mathfrak{g}$ extended to the whole group LG by left translations and restricted to ΩG. This observation suggests that this symplectic form cannot represent a trivial cohomological class on ΩG and in fact we now recast the expression in (3.1) into a form suited for our purposes in which its cohomological meaning is more transparent.

For a compact, connected, not necessarily simply connected (semi)-simple Lie group G fix an invariant positive defined bilinear form $< \cdot, \cdot >_{\mathfrak{g}}$ on its Lie algebra $\mathfrak{g}$, such that the 2-cocycle determining the central extension of the loop algebra be integral. (typically there is a smallest one which is minus the Killing form divided two times the dual Coxeter number [45,19]). Consider the left invariant differential 3-form on G given by

$$TQ(\theta) = \frac{1}{4\pi} < \theta, [\theta, \theta] >_{\mathfrak{g}} \tag{3.2}$$

where θ is the Maureer-Cartan form with values in $\mathfrak{g}$. The value of the form $TQ(\theta)$

at the identity of G on $\xi, \eta, \zeta \in \mathfrak{g}$ is

$$TQ(\theta)_e(\xi, \eta, \zeta) = \frac{1}{4\pi} < \xi, [\eta, \zeta] >_{\mathfrak{g}}.$$

This form is closed and actually represents a generator in $H^3(G, \mathbb{Z})$. Now with the aid of the *evaluation map*

$$ev : S^1 \times \Omega G \longrightarrow G \quad (z, f) \longmapsto f(z)$$

where $z \in S^1$ and $f \in \Omega G$, it is possible to obtain a closed 2-form on ΩG from this generator in the following way. Denoting with ∂ the tangent vector [1] to S^1 given by $\partial/\partial\theta = iz\, d/dz$, we state

Definition 3.1.1 *The* transgression mapping $\tau : H^q(G) \to H^{q-1}(\Omega G)$ *is given by*

$$\tau(\psi) = \int_{S^1} i_\partial \left(ev^* \psi\right) \tag{3.3}$$

for ψ an element in $H^q(G)$ and ev the evaluation map above.

In the definition the role of the internal product with ∂ is to isolate the $(1, q-1)$ part which is integrated over S^1.

Remark 3.1.2 Some word must be spent on the rather non-standard use of the term "transgression". Strictly speaking, the Transgression refers to fibre bundles. Suppose to have a fibration $\pi : E \to M$ with fibre F in the differentiable category: then an element β in $H^q(F)$ is said to be transgressive if it is the restriction to the fibre of a global form ψ on E such that $d\psi = \pi^*\alpha$ for some form α on the base M. Since in the case of a fibration π^* is injective, we have $\pi^* d\alpha = d\pi^*\alpha = d\,d\psi = 0$ and therefore $d\alpha = 0$, that is the form α represents a cohomology class in $H^{q+1}(M)$ on the base manifold M. Thus the Transgression mapping is the assignment $[\beta] \to [\alpha]$ [15,12]. Of course replacing forms with singular cochains, the definition translates word by word to the singular setting with arbitrary coefficients [12]. A typical situation is when one has a principal G-bundle $\pi : P \to M$ with connection A. Then the differential form (3.2) is transgressive since it is the restriction to the fibre of the *Chern-Simons form* [17] on P

$$TQ(A) = \frac{1}{4\pi}\left(< A, dA > + \frac{1}{3} < A, [A, A] >\right) \tag{3.4}$$

which in turn satisfies

$$dTQ(A) = \frac{1}{4\pi} < F, F > \tag{3.5}$$

[1] we oscillate continuosly between the coordinates θ and z

where F is the curvature form of the connection A. Therefore the name $TQ(\theta)$ for the differential form (3.2) is appropriate with this definition of transgression. Nevertheless we follow here Pressley and Segal [57] in calling transgression the map (3.3) simply because it takes a generator in the cohomology of G into a generator for the cohomology of ΩG in one dimension less.

To come back to (3.2), it is very easy to check by direct calculation that the form $\tau(\psi)$ is closed if so is ψ. Instead of doing this calculation in general we prefer to concentrate ourselves on the case we are interested in, namely the 3-form (3.1). First note that the tangent vector

$$(\partial, \delta f) \in T_{(z,f)}(S^1 \times \Omega G)$$

is sent by the derivative of the evaluation map into

$$T_{(z,f)}\, ev\, (\partial, \delta f) = \partial\, f(z) + \delta f(z)$$

so that finally calling W_Q the transgression of $TQ(\theta)$ we have

$$(W_Q)_f(\delta_1 f, \delta_2 f) = \frac{1}{4\pi}\int_{S^1} < f^{-1}\partial\, f, [f^{-1}\delta_1 f, f^{-1}\delta_2 f] >_{\mathfrak{g}} \tag{3.6}$$

(remember that the Maureer-Cartan θ is formally $f^{-1}df$). This expression clearly reproduces, modulo a factor, the first half of (3.1). To take into account also the second half in a coherent way consider the 1-form β over ΩG:

$$\beta_f(\delta f) = \frac{1}{2}\int_{S^1} < f^{-1}\partial\, f, f^{-1}\delta f >_{\mathfrak{g}} \qquad \delta f \in T_f(\Omega G) \tag{3.7}$$

whose differential turns out to be given by the following expression

$$\begin{aligned} d\beta_f(\delta_1 f, \delta_2 f) &= \frac{1}{2}\int_{S^1} < f^{-1}\partial\, f, [f^{-1}\delta_1 f, f^{-1}\delta_2 f] >_{\mathfrak{g}} \\ &= \int_{S^1} < \partial\,(f^{-1}\delta_1 f), f^{-1}\delta_2 f >_{\mathfrak{g}} \end{aligned} \tag{3.8}$$

Proof. Pull back β by a two-parameter map γ to, say, $\mathbb{R}^2$ and use the relation $\gamma^* d\beta = d\gamma^*\beta$. This can ultimately be considered as a device to coordinatize the space. Now $\gamma^*\beta$ is a 2-form over $\mathbb{R}^2$ and by the usual calculus of differential forms we have [2]:

$$d(\gamma^*\beta)_{(s,t)}\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial t}\right) = \frac{\partial}{\partial s}\left((\gamma^*\beta)\left(\frac{\partial}{\partial t}\right)\right) - \frac{\partial}{\partial t}\left((\gamma^*\beta)\left(\frac{\partial}{\partial s}\right)\right) - (\gamma^*\beta)\left(\left[\frac{\partial}{\partial s}, \frac{\partial}{\partial t}\right]\right)$$

but the term with the commutator obviously vanishes. Now by definition of pull back we have

$$\begin{aligned} \frac{\partial}{\partial s}\left((\gamma^*\beta)(\frac{\partial}{\partial t})\right) &= \frac{\partial}{\partial s}\left(\beta_{\gamma(s,t)}\left(T\gamma\left(\frac{\partial}{\partial t}\right)\right)\right) \\ &= \frac{\partial}{\partial s}\left(\frac{1}{2}\int_{S^1} < \gamma^{-1}\partial\,\gamma, \gamma^{-1}\left(\frac{\partial}{\partial t}\gamma\right) >_{\mathfrak{g}}\right) \end{aligned}$$

and using the relations

$$\frac{\partial}{\partial s}\left(\gamma^{-1}\partial\gamma\right) = \partial\left(\gamma^{-1}\frac{\partial}{\partial s}(\gamma)\right) + \left[\gamma^{-1}\partial\gamma, \gamma^{-1}\frac{\partial}{\partial s}\gamma\right] \quad (3.9)$$

$$\frac{\partial}{\partial s}\left(\gamma^{-1}\frac{\partial}{\partial t}(\gamma)\right) - \frac{\partial}{\partial t}\left(\gamma^{-1}\frac{\partial}{\partial s}(\gamma)\right) = \left[\gamma^{-1}\frac{\partial}{\partial t}(\gamma), \gamma^{-1}\frac{\partial}{\partial s}(\gamma)\right] \quad (3.10)$$

the claim is easily proved. □

Comparing expressions (3.1, 3.6, 3.8) proves the following

Proposition 3.1.3 *The Kostant-Kirillov form (3.1) on ΩG can be put into the form*

$$\omega = 2\pi\kappa\, W_Q + \kappa\, d\beta \quad (3.11)$$

where W_Q is the transgression via the evaluation map of the generator (3.2) in $H^3(G.)$
□

Remark 3.1.4 We could have also put things differently to recast the Kostant-Kirillov form into this other expression

$$\omega = -\kappa\,\varphi + 2\kappa\, d\beta$$

where φ is the cocycle generating the central extension of $L\mathfrak{g}$ thought as a left-invariant 2-form over ΩG. This is essentially the reason why in [7,32] the form φ is directly taken as a symplectic form.

Remark 3.1.5 Using the (3.11) and the expression for ω in the previous remark we see that ω can be eliminated to obtain the following relation between W_Q and φ:

$$W_Q = -\frac{1}{2\pi}\varphi + \frac{1}{2\pi}d\beta$$

showing that W_Q and ω are in fact cohomologous and represent the same element in $H^2(\Omega G)$. Note that If G is simply connected the transgression mapping $\tau : H^3(G,\mathbb{Z}) \to H^2(\Omega G,\mathbb{Z})$ is actually an isomorphism: it is in fact the transpose of the isomorphism in homology $H_3(G,\mathbb{Z}) \cong H_2(\Omega G,\mathbb{Z})$ obtained applying Hurewicz theorem: $H_3(G,\mathbb{Z}) \cong \pi_3(G) = \pi_2(\Omega G) \cong H_2(\Omega G,\mathbb{Z})$; recall that ΩG is *always* simply connected, since $\pi_1(\Omega G) = \pi_2(G) = 0$. Simply connectedness, forcing torsion effects to vanish, implies that the integer classes in $H^2(\Omega G,\mathbb{Z})$ are representable by means of classes in $H^2(\Omega G,\mathbb{R})$. Transgressing the generators in $H^*(G)$, which are given by left invariant forms, produce classes on ΩG which are not invariant anymore, but it is a fairly easy result to show that each one of this generators is in fact cohomologous to left-invariant form [57] just in the way the form W_Q above is cohomologous to the form φ.

Remark 3.1.6 The decomposition (3.11) of the Kostant-Kirillov form into a cohomologically non trivial part plus an exact part is in fact the global version of the procedure adopted by Alekseev and Shatashvili [4,5] in their investigation of WZW–model and $2D$–Quantum Gravity in terms of geometric quantization of coadjoint orbits: they write essentially a version of the form which is valid only locally.

3.2 The action on symplectic manifolds

Let M be a symplectic manifold with symplectic form ω, and let $h : M \to \mathbb{R}$ a Hamiltonian function on M. The Hamiltonian will play no role in the sequel, but to include it into the discussion requires no extra effort. If ω were exact, i.e.

$$\omega = d\alpha$$

for a 1-form α on M, then we could easily form the "action" functional

$$S(\gamma) = \int_\gamma \alpha(\dot{\gamma}) - \int h(\gamma(t))\, dt$$

where γ is any path in M. Variations of γ produce the equation of motion in the hamiltonian form

$$i(\dot{\gamma})\,\omega = d\,h$$

Of course when ω is not exact the above mechanism does not apply so that we must modify the procedure. One method is to lift all to a space where in some sense it is possible to "exactify" the (lifted) symplectic form and therefore to construct an action functional. Obviously the final result at the classical level, namely the equation of motion, should not depend on the lifting in order to have a meaningful construction. However at the quantum level things change and other global features of the action must be taken into account.

Anyway the procedure is set up [20,9] considering the *path space* $\mathcal{P}_{x_0}M$ of M, that is the space of all paths $p : I \to M$ such that $p(0) = x_0$, where $x_0 \in M$ is an arbitrary base-point and $I = [0,1]$. It is well known [15,62,63] that the path space is contractible, and explicitly the contracting homotopy is

$$H : I \times \mathcal{P}_{x_0}M \longrightarrow \mathcal{P}_{x_0}M \qquad (s,p) \longmapsto p^s(\cdot) = p(s\cdot) \tag{3.12}$$

It also well known that the path space is a fibration

$$\begin{array}{ccc} \Omega_{x_0}M & \longrightarrow & \mathcal{P}_{x_0}M \\ & & \big\downarrow \pi \\ & & M \end{array}$$

where $\Omega_{x_0}M$ is the space of based loops at x_0 and π the obvious projection $\pi(p) = p(1)$. We now lift the symplectic form ω to a closed form $\tilde{\omega} = \pi^*\omega$ on the path space $\mathcal{P}_{x_0}M$: this form is exact, as follows from the following general

Proposition 3.2.1 ([15,2]) *Every closed form defined on a contractible manifold is exact.*

Proof. We need the following

Lemma 3.2.2 (Deformation lemma) *Given a manifold M consider the mapping $i_t : M \to I\times M$ given by $i_t(x) = (t, x)$. Define the operator $\mathcal{K} : \Omega^{p+1}(I\times M) \to \Omega^p(M)$ by*

$$\mathcal{K}\alpha = \int_0^1 i_s^*(i_{\partial_s}\alpha)\, ds$$

for $\alpha \in \Omega^{p+1}(I \times M)$. Then we have

$$d\mathcal{K} + \mathcal{K}d = i_1^* - i_0^*$$

Proof of the lemma. The flow of the vector field ∂_s over $I \times M$ is given by $F_\lambda(s, x) = (\lambda + s, x)$ that is $i_{\lambda+s} = F_\lambda \circ i_s$. Then for any form $\beta \in \Omega^{p+1}(I \times M)$ we have

$$\begin{aligned} i_s^*\mathcal{L}_{\partial_s}\beta &= i_s^* \left.\frac{d}{d\lambda}\right|_{\lambda=0} F_\lambda^*\beta \\ &= \left.\frac{d}{d\lambda}\right|_{\lambda=0} i_s^* F_\lambda^*\beta = \left.\frac{d}{d\lambda}\right|_{\lambda=0} (F_\lambda \circ i_s)^*\beta \\ &= \left.\frac{d}{d\lambda}\right|_{\lambda=0} i_{\lambda+s}^*\beta = \frac{d}{ds} i_s^*\beta \end{aligned}$$

Therefore we have

$$\begin{aligned} d\mathcal{K}\alpha + \mathcal{K}d\alpha &= d\int_0^1 i_s^*(i_{\partial_s}\alpha)\, ds + \int_0^1 i_s^*(i_{\partial_s} d\,\alpha)\, ds \\ &= \int_0^1 i_s^*(d\, i_{\partial_s}\alpha + i_{\partial_s} d\,\alpha)\, ds \\ &= \int_0^1 i_s^*\mathcal{L}_{\partial_s}\alpha\, ds \\ &= \int_0^1 i_s^* \frac{d}{ds} i_s^*\alpha\, ds = i_1^*\alpha - i_0^*\alpha \end{aligned}$$

as claimed. ▽

Consider two homotopic maps $f, g : M \to N$: it exists a homotopy map $F : I \times M \to N$ such that $F(0, \cdot) = f(\cdot)$ and $F(1, \cdot) = g(\cdot)$. and therefore $F^* : \Omega^*(I\times M) \leftarrow \Omega^*(N)$

so that we can define an operator $\mathcal{H} := \mathcal{K} \circ F^* : \Omega^{*-1}(M) \leftarrow \Omega^*(N)$ which by the deformation lemma verifies

$$d\mathcal{H} + \mathcal{H}d = (F \circ i_1)^* - (F \circ i_0)^*$$

On the other hand we have that $F \circ i_1 = g$ and $F \circ i_0 = f$ so that the result is

$$d\mathcal{H}\alpha + \mathcal{H}d\,\alpha = g^*\alpha - f^*\alpha.$$

Now the proposition follows at once from the above reasoning, for if M is contractible we can take $g = id_M$ and f the map that takes M to the base-point x_0. Then f^* is the zero mapping and

$$\alpha = d(\mathcal{H}\alpha)$$

that is the primitive of α is $d(\mathcal{H}\alpha)$ □

Remark 3.2.3 Note that the sequence of applications $i_s^* \circ i_{\partial_s}$ extracts the $(1, p-1)$–part of the form $F^*\alpha$ and localizes it on the manifold M.

In the case of the form $\tilde{\omega}$ on the path space $\mathcal{P}_{x_0}M$ the primitive form is given by

$$\begin{aligned} \tilde{\alpha} &= \mathcal{H}\tilde{\omega} \\ &= \int_0^1 i_s^*(i_{\partial_s} H^*\tilde{\omega}) \\ &= \int_0^1 i_s^*(i_{\partial_s}(\pi \circ H)^*\omega) \end{aligned}$$

where H is the contracting homotopy (3.12). But note that the composition of the contracting homotopy with the projection is actually an evaluation map, since

$$(\pi \circ H)(s, p) = \pi(p^s(\cdot)) = p^s(1) = p(s),$$

therefore we can rewrite the form $\tilde{\alpha}$ as

$$\tilde{\alpha} = \int_0^1 i_s^*(i_{\partial_s} \overline{ev}^*\omega)$$

where we have defined

$$\overline{ev} := \pi \circ H : I \times \mathcal{P}_{x_0}M \longrightarrow M$$

Now for a path $I \ni t \to \gamma(t) \in \mathcal{P}_{x_0}M$ form the action functional

$$S(\gamma) = \int_\gamma \tilde{\alpha}(\dot{\gamma}) - \int_I h(\pi(\gamma(t))\,dt \tag{3.13}$$

Due to the expression which defines the differential 1-form $\tilde{\alpha}$ the first part of the action functional actually involves two integrations, in fact we have the

Lemma 3.2.4 *The functional (3.13) can be rewritten as*

$$S = \int_{I\times I} \gamma_1^* \omega - \int_{\partial(I\times I)} \gamma_1^* h \qquad (3.14)$$

where the map γ_1 is the composite

$$I \times I \xrightarrow{id\times\gamma} I \times \mathcal{P}_{x_0}M \xrightarrow{\overline{ev}} M$$

$$(s,t) \longmapsto (s,\gamma(t)) \longmapsto \gamma(t)(s) =: \gamma_1(s,t)$$

and the second half of (3.14) briefly indicates that the integration takes place only on (part of) the boundary of the 2-dimensional region in M described by γ_1.

Proof. It is a direct calculation starting from the observation that that the tangent vector

$$(\partial_s, \delta p) \in T_{(s,p)}(I \times \mathcal{P}_{x_0}M)$$

is sent into

$$T\overline{ev}_{(s,p)}(\partial_s, \delta p) = \delta p(s) + \partial_s p(s)$$

On the other hand i_{∂_s} selects the $(1,1)$–part of $\overline{ev}^*\omega$ as

$$i_{\partial_s}(\overline{ev}^*\omega)_{(s,p)}(\partial_s, \delta p) = \omega_{p(s)}(\partial_s p(s), \delta p(s) + \partial_s p(s))$$

while i_s^* acts non trivially only on forms of degree higher than two, so that we have

$$\tilde{\alpha}_p(\delta p) = \int_0^1 \omega_{p(s)}(\partial_s p(s), \delta p(s))\, ds\ .$$

Now integrating $\tilde{\alpha}$ over the path γ means exactly that

$$\int_I \gamma^* \tilde{\alpha} = \int_0^1 dt \int_0^1 ds\, \omega_{\gamma(t)(s)}(\partial_s\gamma(t)(s), \partial_t\gamma(t)(s))$$

proving the assertion. Obviously the part containing h is trivial □

For future reference, note that the previous lemma can be generalized to the following one

Lemma 3.2.5 *Consider a p–form ψ on a manifold N and the sequence of applications*

$$M \times K \xrightarrow{id\times g} M \times Map(M,N) \xrightarrow{ev} N$$

with $dimM + dimK = p$, $dimM = m$. Then we have that

$$\int_K g^* \left(\int_M (ev^*\psi)_{(m,p-m)} \right) = \int_{M\times K} g_1^* \psi$$

where $g_1 = ev \circ (id \times g)$.

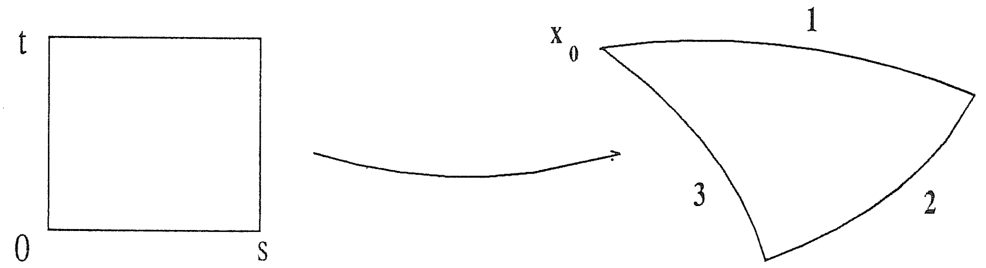

Figure 3.1: The map γ_1

Proof. A multidimensional analogue of the proof of the preceeding lemma. □

Due to the base-point condition on the path space $\mathcal{P}_{x_0}M$ the map γ_1 satisfies the condition $\gamma_1(t)(0) = x_0 \; \forall t \in I$ and the situation can be schematically represented as in figure 3.1: the $[0,t]$–edge of $I \times I$ is in fact collapsed to the base-point of $\mathcal{P}_{x_0}M$. Actually various other choices are possible, depending on the map $\gamma : I \longrightarrow \mathcal{P}_{x_0}M$ and we can list other interesting cases.

i) The map γ is such that $\gamma(0) = \gamma(1)$: we can think of it as a map

$$\gamma : S^1 \longrightarrow \mathcal{P}_{x_0}M$$

and therefore we obtain a further condition: $\gamma_1(s,0) = \gamma_1(s,1) \; \forall s \in I$, that is the boundaries **1** and **3** in figure 3.1 are identified, as depicted in figure 3.2. The topology is that of a disc based with its centre at the base-point of $\mathcal{P}_{x_0}M$.

ii) The map γ is itself based, that is $\gamma(0) = p_0$, where p_0 is the base-point of $\mathcal{P}_{x_0}M$, namely the constant path $s \mapsto p_0(s) \equiv x_0$, and 0 is chosen as the base-point of I. In this case, compared with the initial one of an unbased map, we have the further condition $\gamma_1(s,0) = x_0 \; \forall s \in I$ and this produces the effect of collapsing to the point x_0 also the $[0,s]$–edge, as shown in figure 3.3 where we see that the boundary **3** in figure 3.1 has been collapsed to the base-point.

iii) Consider the combination of the two previous possibilities, that is a based periodic map

$$\gamma : (S^1, 1) \longrightarrow (\mathcal{P}_{x_0}M, p_0)$$

In this case referring to figures 3.1,3.2,3.3 we see that we have to satisfy all the listed conditions $\gamma_1(0,t) = \gamma_1(s,0) = \gamma_1(s,1) = x_0$, so that, still referring to the figures, only the boundary **2** survives. The situation is depicted in figure 3.4.

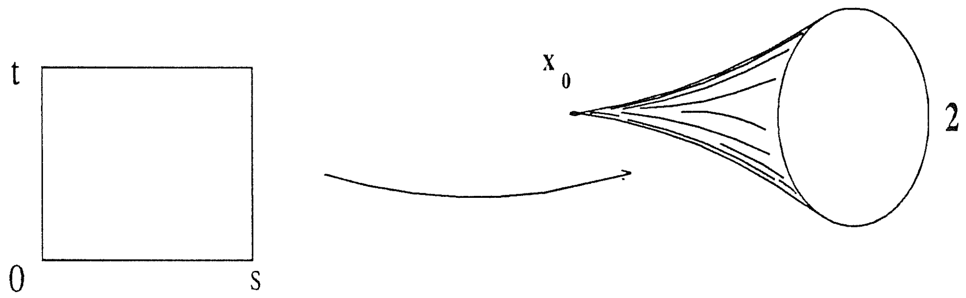

Figure 3.2: The map $S^1 \to \mathcal{P}_{x_0}M$

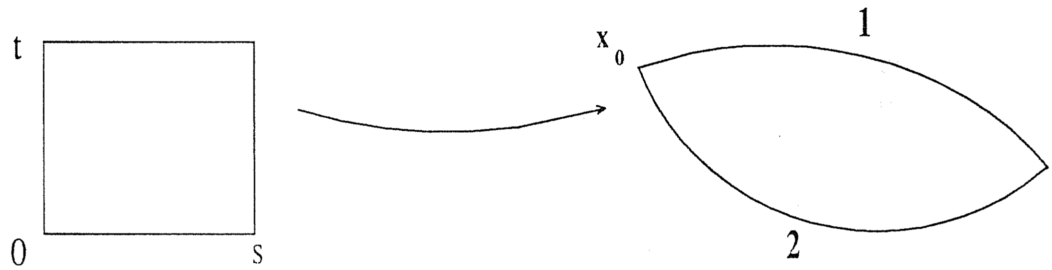

Figure 3.3: The based map $I \to \mathcal{P}_{x_0}M$

Actually what we have stated so far is the toy version of a general fact from algebraic topology [52,62]. For a brief account we refer to Appendix A.

Given three resonable spaces X, Y, Z there is an "association map"

$$\alpha : Map_0(X \wedge Y, Z) \longrightarrow Map_0(X, Map(Y, Z))$$

where all spaces and maps are supposed to be based, and $X \wedge Y$ is the *smash product.* Cases *ii)* and *iii),* which are obtained considering $Map_0(I, \mathcal{P}M)$ and $Map_0(S^1, \mathcal{P}M)$ respectively, therefore both correspond to $Map_0(D^2, M)$, since we have that $S^1 \wedge I \cong D^2$ and $I \wedge I \cong D^2$ as can be easily verified by hand.

Therefore the main case and cases *ii)* and *iii)* suggest the possibility of setting the action functional directly on the space $Map_0(D^2, M)$ of (based) maps $(D^2, 1) \to$

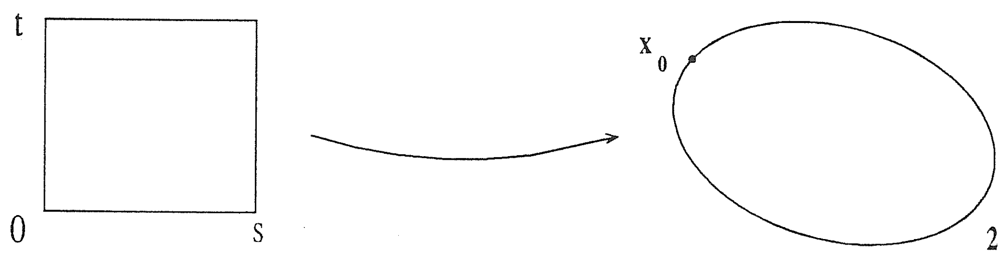

Figure 3.4: The based map $S^1 \to \mathcal{P}_{x_0} M$

(M, x_0), with the constraint that the functional should depend only on the restriction of the map to $S^1 = \partial D^2$, since in all the above cases what is relevant is the restriction of the integration to the boundary of the domain determined in M by the map γ_1. However it will turn out that at least the dependence on the topological character of the maps in $Map_0(D^2, M)$ cannot be eliminated, as we will show later on. First we reformulate (3.13, 3.14) in a form suitable to set the variational principle on $Map_0(D^2, M)$ and we will prove our claim that indeed the equation of motions in the Hamiltonian form are produced.

Given the symplectic manifold (M, ω) with base-point x_0,consider those loops $\ell \in (\Omega_{x_0} M)_0$, the connected component in $\Omega_{x_0} M$ of the constant loop x_0. Since $\pi_0(\Omega_{x_0} M) = \pi_1(M, x_0)$, the loop $\ell \in (\Omega_{x_0} M)_0$ is homotopically trivial, that is it represents the zero element in $\pi_1(M, x_0)$, and therefore it can be extended to mapping [62]

$$u : (D^2, 1) \longrightarrow (M, x_0).$$

Now form the functional [2]

$$S'(\ell) = \int_{D^2} u^* \omega - \int_{S^1} \ell^* h \tag{3.15}$$

simply adapting expressions (3.13, 3.14) to the present case. Note that this type of functional has been widely used in the mathematical literature concerning the Arnol'd conjecture, the theory of Lagrangian intersections and Floer's Homology, [59,18,29,30,31]: in [27] Feigin and Frenkel call it *the Conley-Zehnder function* and claim to be able to use its critical points and the associated homology to extract informations about the topology of the flag manifolds associated to Kac-Moody groups. Sometimes we will retain this name to mean exactly (3.15). We prove the

[2]later on we shall discuss its behaviour under non trivial changes of the map u with fixed boundary

Lemma 3.2.6 *The differential of the Conley-Zehnder function on the space $(\Omega_{x_0}M)_0$ is given by*

$$dS'(\ell)(\delta\ell) = \int_{S^1} \left(-\omega_{\ell(t)}(\dot{\ell}(t), \delta\ell(t)) + dh_{\ell(t)}(\delta\ell(t))\right) dt$$

where $\delta\ell$ is a vector tangent to $(\Omega_{x_0}M)$ at the point ℓ (and therefore satisfying the condition $\delta\ell(1) = 0$).

Proof. Consider a path $\epsilon \to \ell_\epsilon$ in $(\Omega_{x_0}M)_0$ such that $\delta\ell = \frac{d}{d\epsilon}\Big|_{\epsilon=0} \ell_\epsilon$ and the corresponding variation $\frac{d}{d\epsilon}\Big|_{\epsilon=0} S'(\ell_\epsilon)$. As usual the only nontrivial part is the one containing the symplectic form and we will concentrate ourselves only on that. Now $\ell_\epsilon \in (\Omega_{x_0}M)_0$ determines a map $u_\epsilon : D^2 \to M$ and

$$\frac{d}{d\epsilon}\bigg|_{\epsilon=0} \int_{D^2} u_\epsilon^* \omega = \int_{D^2} \frac{\partial}{\partial\epsilon} u_\epsilon^*\omega\bigg|_{\epsilon=0}$$

where $u_\epsilon^*\omega$ depends on three coordinates, the two integration variables on D^2, say x and y, and ϵ. Since ω is closed, $u_\epsilon^*\omega$ is also closed and using the relation

$$d\omega(X,Y,Z) = X \cdot \omega(Y,Z) - \omega([X,Y],Z) + \textit{ cyclic perm.}$$

with the coordinate tangent vectors $\partial_x, \partial_y, \partial_\epsilon$ we can find with an easy calculation that

$$\frac{\partial}{\partial\epsilon} u_\epsilon^*\omega(\partial_x, \partial_y) = d(i_{\partial_\epsilon} u_\epsilon^*\omega)(\partial_x, \partial_y)$$

from which we can infer that

$$\begin{aligned} \int_{D^2} \frac{\partial}{\partial\epsilon} u_\epsilon^*\omega\bigg|_{\epsilon=0} &= \int_{S^1} (i_{\partial_\epsilon} \ell_\epsilon^*\omega)|_{\epsilon=0} \\ &= -\int_{S^1} \omega_{\ell(t)}(\dot{\ell}(t), \delta\ell(t))\, dt \end{aligned}$$

and this finishes the proof. □

Remark 3.2.7 Sometimes it is also convenient to remove the base-point condition and to set the variational principle for the Conley-Zehnder function on $(LM)_0$, the connected component of the loop space LM of M made of contractible loops: these loops are always extendable to maps from D^2 to M. The proof of the lemma 3.2.6 applies without modifications.

The functionals (3.13,3.14,3.15) are actually defined only modulo periods of ω, namely they can depend in a non trivial way on the surface itself described in M. Let us recall that (3.13,3.14) are set up from a path in the path space $\mathcal{P}_{x_0}M$: what is really interesting is the projection of this path down on M, or stated in a more formal way, given a path in M we can lift it to $\mathcal{P}_{x_0}M$ via the Homotopy Lifting

Property of the fibration $\Omega_{x_0}M \to \mathcal{P}_{x_0}M \to M$ [63] and all should be independent of the lifting chosen. However two different liftings with the same initial and final conditions describe a closed two-dimensional surface in M as well as two different extensions to D^2 of a map $\ell : S^1 \to M$, so that as far as the subsequent discussion is concerned it does not matter whether we choose (3.14) or (3.15). We choose to do it with the last one, so let u, u' be two different extensions of the (based or unbased) loop ℓ. Obviously one has that the difference between the two resulting values of S' is

$$\int_{D^2} u^*\omega - \int_{D^2} u'^*\omega$$

which can be thought as

$$\int_{S^2} \overline{u}^*\omega \tag{3.16}$$

where $\overline{u}$ is the "glue" of the mappings u, u'. The last integral will be zero or not depending on whether the map $\overline{u} : S^2 \to M$ represents the zero in $\pi_2(M)$: if it is so it can be extended to a map $\tilde{u} : D^3 \to M$ and

$$\int_{S^2} \overline{u}^*\omega = \int_{D^3} \tilde{u}^* d\omega = 0$$

since ω is closed. Thus we have that (3.15) is certainly single-valued if $\pi_2(M) = 0$. Otherwise, requiring the integral (3.16) to be zero is equivalent to demand special conditions on the symplectic form ω, the most obvious one being exactness. By the way, the request of the integral of ω to vanish on "holomorphic 2-spheres" in Floer's proof of the Arnol'd conjecture is nothing else than the requirement on the integral (3.16) to vanish.

3.3 Wess-Zumino-Witten action

Now we specialize the discussion in the previous section to the case of $M = \Omega G$ with symplectic form (3.11). Therefore for a loop $f_0 \in \Omega G$ we consider the path space $\mathcal{P}_{f_0}(\Omega G)$ and for a path $\gamma : I \to \mathcal{P}_{f_0}(\Omega G)$ we form the action

$$\int_{I\times I} \gamma_1^*(2\pi\kappa W_Q + \kappa\, d\beta) \tag{3.17}$$

as in (3.14). Recall that the forms W_Q and β are respectively given by

$$(W_Q)_f(\delta_1 f, \delta_2 f) = \frac{1}{4\pi}\int_{S^1} < f^{-1}\partial f, [f^{-1}\delta_1 f, f^{-1}\delta_2 f] >_{\mathfrak{g}} \tag{3.18}$$

for $f \in \Omega G$ and $\delta_1 f, \delta_2 f \in T_f(\Omega G)$, and

$$\beta_f(\delta f) = \frac{1}{2}\int_{S^1} < f^{-1}\partial f, f^{-1}\delta f >_{\mathfrak{g}} \qquad \delta f \in T_f(\Omega G). \tag{3.19}$$

As preannounced in the previous section, the hamiltonian has been put to zero [20,66], and the action functional is purely topological in the sense that it depends only on the symplectic form and the manifold ΩG itself.

Now we can state our main result, namely that the action functional (3.17) *is nothing else than the Wess-Zumino-Witten action.* First we can see it in fairly simple case, namely the case *i)* of the periodic mapping $\gamma : S^1 \to \mathcal{P}_{f_0}(\Omega G)$ (see figure 3.2); the parameter of the map γ becomes an angular variable [20], call it τ, and it is easy to see writing all in coordinates that (3.17) becomes

$$\begin{aligned} S &= \frac{\kappa}{4\pi}\int_0^{2\pi} d\tau \int_0^{2\pi} d\theta < \gamma_0^{-1}\partial\gamma_0, \gamma_0^{-1}\partial_\tau\gamma_0 >_{\mathfrak{g}} \\ &+ \frac{\kappa}{4\pi}\int_0^1 ds \int_0^{2\pi} d\tau \int_0^{2\pi} d\theta < \gamma_1^{-1}\partial\gamma_1, [\gamma_1^{-1}\partial_s\gamma_1, \gamma_1^{-1}\partial_\tau\gamma_1] >_{\mathfrak{g}} \end{aligned} \tag{3.20}$$

where we have denoted with γ_0 the restriction of the mapping γ_1 to the boundary of the disc, that is the region with $s = 1$. From this simple example we can start to see that the kinetic term in the Wess-Zumino-Witten action arises exactly from the exact part of the symplectic form on ΩG, while the transgression of the form (3.2) on the group gives rise to the topological term in the action. This remains true in more complicated cases, since this phenomenon in some sense is entirely encoded in the symplectic form constructed via coadjoint orbit method.

To proceed further in this direction, we need to improve lemma 3.2.5 in order to manipulate intrinsically both terms in (3.17). First note the

Lemma 3.3.1 *If $ev : M \times Map(M,N) \to N$ and ψ is a q-form on N, then we have*

$$(ev^*\psi)_{(q,0)}(x,f) = (f^*\psi)(x)$$

for $x \in M$ and $f \in Map(M,N)$.

Proof. This follows noticing that for $X \oplus \delta f \in T_{(x,f)}(M \times Map(M,N))$ it happens that

$$T_{(x,f)}ev(X \oplus \delta f) = T_x f(X) + \delta f(x)$$

and by definition of pull-back of differential forms we have

$$\begin{aligned} ev^*\psi_{(x,f)}(X_1 \oplus \delta_1 f, \ldots, X_q \oplus \delta_q f) &= \psi_{f(x)}(T_x f(X_1) + \delta_1 f(x), \ldots, T_x f(X_q) + \delta_q f(x)) \\ &= \psi_{f(x)}(T_x f(X_1), \ldots, T_x f(X_q)) + \cdots \\ &= (ev^*\psi)_{(q,0)}(x,f)(X_1 \oplus \delta_1 f, \ldots, X_q \oplus \delta_q f) \\ &+ (ev^*\psi)_{(q-1,1)}(x,f)(\cdots) + \cdots \end{aligned}$$

□

Remark 3.3.2 A consequence of this lemma is that if we had considered $\psi = d\phi$ for ϕ a $(q-1)$-form on N, we would have obtained

$$(ev^* d\phi)_{(q,0)}(x, f) = d_M(f^*\phi)(x)$$

where d_M is the exterior differential relative to M.

Now if we take K, M, N, ψ as in lemma 3.2.5, we are led to consider the whole space $Map(K, Map(M, N))$ to analyze $\int_{M\times K} g_1^*\psi$ as a function of $g : K \longrightarrow Map(M, N)$. Introduce the "double" evaluation map

$$Ev : M \times K \times Map(K, Map(M, N)) \longrightarrow N$$

simply by identifying g with $g_1 = ev \circ (id_M \times g)$ and putting

$$Ev(x, y, g) = g(y)(x) = g_1(x, y)$$

To be more pedantic, we should have introduced another evaluation map

$$ev_1 : K \times Map(K, Map(M, N)) \longrightarrow Map(M, N)$$

and defined Ev as the composite mapping $Ev = ev \circ (id_M \times ev_1)$. By lemma 3.3.1 we have that $(Ev^*\psi)_{(p,0)}(x, y, g) = (g_1^*\psi)(x, y)$ with $(x, y) \in M \times K$. This proves the

Lemma 3.3.3 *If we take M, N, K, ψ as in lemma 3.2.5, then*

$$\int_K g^* \left(\int_M (ev^*\psi)_{(m,p-m)} \right) = \int_{M\times K} (Ev^*\psi)_{(p,0)}(\cdot, g)$$

and therefore, as a function of on $Map(K, Map(M, N))$, the left hand side is equal to

$$\int_{M\times K} (Ev^*\psi)_{(p,0)}$$

□

Lemmas 3.2.5 and 3.3.3 will be iteratively used in the sequel.

To come back to our problem, we briefly recall the situation. The basic object is ΩG with the associated evaluation map

$$ev : S^1 \times \Omega G \longrightarrow G$$

which we used to produce the 2-form W_Q by means of transgression. Then we have the path space $\mathcal{P}_{f_0}(\Omega G) \equiv Map_0(I, \Omega G)$ with evaluation map

$$\overline{ev} : I \times \mathcal{P}_{f_0}(\Omega G) \longrightarrow \Omega G$$

used to obtain the 1-form

$$\tilde{\alpha} = \int_I (\overline{ev}^*\omega)_{(1,1)}$$

Furthermore, for $\gamma : I \to \mathcal{P}_{f_0}(\Omega G)$ or $\gamma : S^1 \to \mathcal{P}_{f_0}(\Omega G)$ we considered the action functional

$$S(\gamma) = \int \gamma^* \tilde{\alpha}$$

If p_0 is the base-point in $\mathcal{P}_{f_0}(\Omega G)$, that is the constant path $p(s)(\cdot) = f_0(\cdot)$, the two cases above correspond to $\gamma \in \mathcal{P}_{p_0}(\mathcal{P}_{f_0}(\Omega G))$ and $\gamma \in \Omega_{p_0}(\mathcal{P}_{f_0}(\Omega G))$ respectively. As previously noted in the last section, both amount to consider γ as lying in $Map_0(D^2, \Omega G)$, where the base-points in D^2 and ΩG are taken to be the point 1 and the loop f_0 respectively. To avoid complicated notations let us fix the ideas on the case $\gamma \in \mathcal{P}_{p_0}(\mathcal{P}_{f_0}(\Omega G))$, the other one being completely analogous.

Applying lemmas 3.2.5 and 3.3.3 we get

$$\begin{aligned} S(\gamma) &= \int_I \gamma^* \left(\int_I (\overline{ev}^*\omega)_{(1,1)} \right) \\ &= \int_{D^2} (\widetilde{ev}^*\omega)_{(2,0)}(\cdot, \gamma) \end{aligned}$$

(having suppressed the distinction between γ and γ_1) where $\widetilde{ev}$ is the evaluation map relative to $Map_0(D^2, \Omega G)$, that is

$$\widetilde{ev} : D^2 \times Map_0(D^2, \Omega G) \longrightarrow \Omega G$$

In the above statement we have directly exploited the equivalence

$$Map_0(I, Map_0(I, \Omega G)) \cong Map_0(D^2, \Omega G)$$

Let us note that it is somewhat unpleasant to have as base-point on ΩG a generic loop f_0 instead of the constant loop e itself. Nevertheless, being G a group, we can "shift" the base-point to e by a rigid multiplication by $f_0(\cdot)^{-1}$. Namely, defining

$$\tilde{\gamma}(x)(z) = f_0(z)^{-1}\gamma(x)(z) \qquad x \in D^2, z \in S^1$$

we obtain a map $\tilde{\gamma} \in Map_0((D^2, 1), (\Omega G, e))$, restoring a more natural choice of the base-points. Therefore, by suspension [52,62], we have

$$Map_0(D^2, \Omega G) \cong Map_0(S(D^2), G) \cong Map_0(D^3, G)$$

and from now on we shall consider γ as lying in anyone of these spaces.

Since $\omega = 2\pi\kappa W_Q + \kappa d\beta$, the action S splits into two parts and it is convenient to analyze them separately. The so called "kinetic term" is

$$S_{kin} = \kappa \int_{D^2} (\widetilde{ev}^* d\beta)_{(2,0)}$$

and by the remark after lemma 3.3.1 we have that for $x \in D^2$

$$(\widetilde{ev}^* d\beta)_{(2,0)}(x,\gamma) = d_2(\gamma^*\beta)(x)$$

where d_2 is the exterior differential on D^2. Thus we obtain

$$\begin{aligned} S_{kin}(\gamma) &= \kappa \int_{S^1} \gamma^*\beta \\ &= \kappa \int_{S^1} \gamma_0^*\beta \end{aligned}$$

where we have denoted with γ_0 the restriction of γ to $S^1 = \partial D^2$. Therefore $\gamma_0 \in Map_0(S^1, \Omega G) \equiv \Omega_{f_0}(\Omega G)$. Recalling the explicit form of β given in (3.19), we have that

$$S_{kin}(\gamma) = \frac{\kappa}{4\pi} \int_{S^1 \times S^1} < \gamma_0^{-1}\partial_\theta \gamma_0, \gamma_0^{-1}\partial_t \gamma_0 >_{\mathfrak{g}} d\theta dt \tag{3.21}$$

where $z = e^{i\theta}$ is the loop parameter[3] and t is taken as the coordinate on $S^1 = \partial D^2$ by means of e^{it}. Applying the above reasoning and considering all the base-point conditions, we obviously obtain $\gamma_0 \in Map_0(S^2, G)$, since $S^1 \wedge S^1 \cong S^2$ [52,62]. Therefore the integration in (3.21) actually takes place on S^2.

Let us now consider the so called "topological term", that is the one involving the 2-form W_Q on ΩG. We have

$$S_{top}(\gamma) = 2\pi\kappa \int_{D^2} (\widetilde{ev}^* W_Q)_{(2,0)}(\cdot,\gamma) = 2\pi\kappa \int_{D^2} \gamma^* W_Q$$

but W_Q itself is obtained through the evaluation map ev as

$$W_Q = \int_{S^1} (ev^* TQ(\theta)_{(1,2)}$$

so that we have

$$S_{top}(\gamma) = 2\pi\kappa \int_{D^2} \gamma^* \left(\int_{S^1} (ev^* TQ(\theta)_{(1,2)} \right).$$

Combining ev and $\widetilde{ev}$ together to form

$$Ev : S^1 \times D^2 \times Map_0(D^2, \Omega G) \longrightarrow G$$

just as done in general in lemma 3.3.3 and, applying it another time, we get

$$S_{top} = 2\pi\kappa \int_{D^3} (Ev^* TQ(\theta))_{(3,0)}$$

that is, by lemma 3.3.1,

$$S_{top}(\gamma) = 2\pi\kappa \int_{D^3} \gamma^* TQ(\theta) \tag{3.22}$$

[3] remember that a factor $1/2\pi$ has been always included when integrating over S^1 with respect to the loop parameter

where, in these last two formulas, we have considered directly γ as a based map $D^3 \to G$.

Collecting all terms together, we see that the abstract action functional on the coadjoint orbit ΩG produces the action for the WZW model on the sphere S^2, *including* the kinetic term

$$\begin{aligned} S(\gamma) &= \frac{\kappa}{4\pi}\int_{S^2} < \gamma_0^{-1}\partial_\mu\gamma_0, \gamma_0^{-1}\partial^\mu\gamma_0 >_{\mathfrak{g}} dvol \\ &+ \frac{\kappa}{4\pi}\int_{D^3} < \gamma^{-1}d\gamma, [\gamma^{-1}d\gamma, \gamma^{-1}d\gamma] >_{\mathfrak{g}} \end{aligned}$$

We also see that the topological term happens to be integrated over D^3, which is exactly a 3-manifold whose boundary is S^2, the source-space for the kinetic term.

3.4 The integrality condition

We now address to the question of the multivaluedness of the action functional constructed so far. By pursuing the analysis done at the end of section 3.2, we shall show that we cannot demand the functional

$$S = \int_{D^2} (\widetilde{ev}^*\omega)_{(2.0)} \tag{3.23}$$

to descend on the space $Map_0(S^1, \Omega G)$[4].

To be more definite, in order the functional (3.23) to descend on $Map_0(S^1, \Omega G)$, it should depend only on $\gamma_0 \equiv \gamma\mid_{S^1}$ for any $\gamma \in Map_0(D^2, \Omega G)$. Equivalently, for any two maps $\gamma, \gamma' \in Map_0(D^2, \Omega G)$ such that $\gamma\mid_{S^1} = \gamma'\mid_{S^1} = \gamma_0$, it should happen that

$$S(\gamma) = \int_{D^2} \gamma^*\omega = \int_{D^2} \gamma'^*\omega = S(\gamma')$$

that is, "glueing" γ and γ' along S^1 to form a map $\bar{\gamma} : S^2 \to \Omega G$[5]

$$\int_{S^2} \bar{\gamma}^*\omega = 0 \tag{3.24}$$

as already said in section 3.3. But this cannot always be true, since first ω is not exact, and second ΩG is not 2- connected, being $\pi_2(\Omega G) = \pi_3(G) = \mathbb{Z}$, at least. Therefore, if the glued mapping $\bar{\gamma}$ does not represent the zero element in $\pi_2(\Omega G)$, the integral in (3.24) cannot be zero and it will happen that $S(\gamma) \neq S(\gamma')$. Thus we see that looking at the functional (3.23) only, the best we can expect is that it descend on the space $\overline{Map_0(S^1, \Omega G)}$ of classes of maps in $Map_0(D^2, \Omega G)$ defined as follows.

[4] As previously noted, we can take the base-point of ΩG to be either the generic loop f_0 or the more natural constant loop e

[5] See Appendix B for further details

Definition 3.4.1 *Two maps $\gamma, \gamma' \in Map_0(D^2, \Omega G)$ are defined to be in the same class if*

1. $\gamma|_{S^1} = \gamma'|_{S^1}$
2. *γ and γ' are homotopic, that is the glued mapping $\bar{\gamma} : S^2 \to \Omega G$ represents the zero element in $\pi_2(\Omega G)$*

There is an obvious projection from $\overline{Map_0(S^1, \Omega G)}$ to $Map_0(S^1, \Omega G)$ sending the class $[\gamma]$ to $\gamma|_{S^1}$. Actually $\overline{Map_0(S^1, \Omega G)}$ is the covering space of $Map_0(S^1, \Omega G) \cong Map_0(S^2, G)$, see Appendix B.

Therefore the functional (3.23) itself is well-defined on the covering space of the space of dynamical variables of the theory. A more careful look at the shape of ω and S immediately suggests that only the topological term S_{top} is responsible for this phenomenon. Nevertheless, this is not a true problem, since full variations of (3.23) produce the right equations of motion, as we have already shown in section 3.2 for a generic symplectic manifold M. Therefore, from a classical point of view, (3.23) causes no troubles, since we are ultimately interested in the equations of motion.

Of course, at the quantum level we are not interested in the action in itself, as the right object to look at is rather the functional $\exp iS$. Since non trivial phenomena come only from the topological term, we shall focus only on that one.

We saw that

$$\begin{aligned} S_{top} &= 2\pi\kappa \int_{D^2} (\widetilde{ev}^* W_Q)_{(2,0)} \\ &= 2\pi\kappa \int_{D^3} (Ev^* TQ(\theta))_{(3,0)} \end{aligned}$$

where $Ev : D^3 \times Map_0(D^3, G) \to G$ and $Map_0(D^3, G) \cong Map_0(D^2, \Omega G)$. Since looking back at what we have done before this last space comes from $Map_0(I, \mathcal{P}_{f_0}(\Omega G))$, we can also consider Ev as a map

$$Ev : S^1 \times I \times I \times Map_0(I, \mathcal{P}_{f_0}(\Omega G)) \longrightarrow G.$$

Exploiting the equivalence

$$\mathcal{P}_{f_0}(\Omega G)) \xrightarrow{\cong} \Omega(\mathcal{P}_e G)$$

obtained sending the path p into the loop $\bar{p}$ defined by

$$\bar{p}(z)(s) := f_0(z)^{-1} p(s)(z) \qquad z \in S^1, s \in I$$

by suspension we obtain $Map_0(I, \mathcal{P}_{f_0}(\Omega G)) \cong Map_0(S^1 \wedge I, \mathcal{P}_e G)$. Therefore we can consider Ev as a map

$$Ev : S^1 \wedge I \times I \times Map_0(S^1 \wedge I, \mathcal{P}_e G) \longrightarrow G$$

Recall moreover that $S^1 \wedge I \cong D^2$. We have now placed ourselves in the same general framework as in [10], and it is possible to use the general analysis carried out there for any dimension to infer that $\exp\left(2\pi i\kappa \int_{D^3}(Ev^*TQ(\theta))_{(3,0)}\right)$ is well defined only when κ is an integer. Since here the situation is considerably simpler than in [10], we can derive the same result directly, without appealing to highly general results.

We proceed as follows. We saw that $S_{top}(\gamma)$ actually depends only on $[\gamma]$ in the sense of the previous definition, so that we should ask ourselves what is the difference $S_{top}(\gamma) - S_{top}(\gamma')$ if $[\gamma] \neq [\gamma']$. In this case the class $[\bar{\gamma}]$ will be a non trivial element in $\pi_2(\Omega G) = \pi_3(G)$. Note that, again by suspension, a map $\bar{\gamma} : S^2 \to \Omega G$ can be also considered as a map $\bar{\gamma} : S^3 \to G$. Therefore this map will represent a non trivial element in $\pi_3(G)$. By the Hurewicz homomorphism [52], see also Appendix A, this map corresponds to an element in $H_3(G, \mathbb{Z})$. Since

$$S_{top}(\gamma) - S_{top}(\gamma') = 2\pi\kappa \int_{S^3} \bar{\gamma}^* TQ(\theta)$$

the difference on the left-hand-side is given evaluating $TQ(\theta)$, which represents an integer class in $H^3(G, \mathbb{Z})$, on an element of $H_3(G, \mathbb{Z})$. Thus we have that with the correct normalization of $TQ(\theta)$ the difference in question is an integer, and if the coupling constant κ is an integer, the functional $\exp\left(2\pi i\kappa \int_{D^3} Ev^*TQ(\theta)\right)$ is single-valued, that is it descends on the space $Map_0(S^1, \Omega G) \cong Map_0(S^2, G)$, unlike the classical action. This is in agreement with the standard treatments of the WZW model [71,28,38,48]. From the Kac-Moody point of view, we have that at the quantum level, only those coadjoint orbits with integer central charge are allowed.

Conclusions

In this thesis partial results have been obtained in globalizing the approach of [4,5] to the construction of WZW models. In particular the geometrical action for the WZW model on a sphere has been constructed from a specific coadjoint orbit in a Kac-Moody algebra.

At this point a number of unsolved questions arise. First, it would be interesting to analyze also other coadjoint orbits, besides the fundamental one ΩG, and to see whether the related geometrical action has anything to do with known models in $2D$-field theory. Closely related to this is the interesting mathematical problem of the study of flag manifolds relative to Kac-Moody groups [55,51,50]: if some of the results stated in [27] proved to be really founded, an unexpected deep connection with Floer's theory [29] may appear. Also the generalization of the approach outlined in this thesis to non trivial topologies would be very interesting. It is suspected that in order to obtain the WZW action for higher genus Riemann surfaces, coadjoint orbits in more general infinite dimensional Lie algebras, such as the Krichever-Novikov generalization of Kac-Moody algebras described for instance in [11], must be studied.

Even more interesting, is the Virasoro counterpart of what we have treated here. One hopes to be able to explain the appearance of the $SL(2,\mathbb{R})$-symmetry in $2D$-gravity from general principles. Besides the fact that, in the usual Hamiltonian approach, the Virasoro geometrical action can be obtained from $SL(2,\mathbb{R})$-Kac-Moody one via a Drinfel'd-Sokolov type reduction [8,21], the presence of this symmetry still remains in great part unexplained. Therefore it would be a progress to explain the presence of the $SL(2,\mathbb{R})$ current algebra in $2D$-gravity exploiting the method of orbits.

Just to end this partial list of unsolved problems, let us mention the suggested connection [67] between WZW models and $2D$-gravity on one side and Chern-Simons gauge theory on the other. It would be very interesting to be able to clarify whether this connection really exists, and in the case of an affirmative answer to put it on more rigorous fundaments.

Appendix A

Some facts from Algebraic Topology

We recall here some simple facts from Algebraic Topology; for a complete account on the subject see [52,62]. We start defining the notion of *smash product* of two spaces. Take two based spaces (X, x_0) and (Y, y_0), that is two spaces X, Y together with a choice of base points $x_0 \in X$ and $y_0 \in Y$ respectively. Then the *wedge product*[1] (or one-point union or "bouquet") is the space

$$X \vee Y = (X \sqcup Y)/\{x_0, y_0\}.$$

In other words, $X \vee Y$ is the space obtained from the disjoint union of X, Y identifying together the base-points x_0, y_0. The base-point of $X \vee Y$ is of course the point corresponding to $\{x_0, y_0\}$. Now the smash product (or reduced product) is defined to be the quotient space

$$X \wedge Y = (X \times Y)/(X \vee Y)$$

whose base-point is of course the point corresponding to $(X \vee Y)$. It is convenient to denote by $x \wedge y$ the image of $(x, y) \in X \times Y$ in $X \wedge Y$.

The point in working with the smash product, rather than the ordinary product, is that its properties are more often convenient when dealing with based spaces. In particular, the smash product is appropriate when dealing with spaces of maps. Given two based spaces X, Y, we can consider the space $Map_0(X, Y)$ of mappings $f : X \to Y$ such that $f(x_0) = y_0$ [2] . Then $Map_0(X, Y)$ is a based space itself with base-point the constant map $f_0(x) = y_0$ for every $x \in X$.

[1]The name wedge is rather unfortunate, both for the name itself and for the symbol used to denote it, but in spite of this it is widely used in standard texts on Algebraic Topology

[2]If we are in the category of topological (based) spaces the we can endow $Map_0(X, Y)$ with the compact-open topology

Given three based spaces X, Y, Z, form the mapping space $Map_0(X, Map_0(Y, Z))$. For f in this mapping space one is tempted to say that it corresponds to an $\bar{f} \in Map_0(X \times Y, Z)$ by defining $\bar{f}(x, y) := f(x)(y)$. The requirement on f to be a based map forces $\bar{f}$ to send the whole $X \vee Y$to the base-point z_0 of Z, in fact

1. $\bar{f}(x, y_0) = f(x)(y_0) = z_0$
2. $\bar{f}(x_0, y) = f(x_0)(y) = g_0(y) = z_0$ where g_0 is the base-point of $Map_0(Y, Z)$.

Therefore $\bar{f}$ can be considered as a map in $Map_0(X \wedge Y, Z)$. Moreover, if $\bar{f}'$ is another map in $Map_0(X \wedge Y, Z)$ which corresponds to f, it must be $\bar{f} = \bar{f}'$, since in this case we must have $\bar{f}(x \wedge y) = \bar{f}'(x \wedge y)$ for all $x \in X$ and $y \in Y$. This observation suggests that the stated correspondence between $Map_0(X, Map_0(Y, Z))$ and $Map_0(X \wedge Y, Z)$ should be one-to-one, and in fact it exists an *association map* [52]

$$\alpha : Map_0(X \wedge Y, Z) \longrightarrow Map_0(X, Map_0(Y, Z))$$

defined by $[\alpha(f)(x)](y) = f(x \wedge y)$, which under rather general conditions turns out to be an isomorphism. In fact if we remain in the category of topological spaces, if X is Hausdorff then α is continuous and it is onto if Y is locally compact and Hausdorff.

Particular examples of mapping spaces are the path space $\mathcal{P}X = Map_0(I, X)$ and the loop space $\Omega X = Map_0(S^1, X)$ where we can take as the base points of I and S^1 the points 0 and 1 (when viewing S^1 as the set of complex numbers of modulus 1) respectively. Therefore, if we consider the mapping spaces $Map_0(X, \mathcal{P}Y)$ and $Map_0(X, \Omega Y)$, we obtain the equivalences

$$\begin{aligned} Map_0(X, \mathcal{P}Y) &\cong Map_0(X \wedge I, Y) \\ Map_0(X, \Omega Y) &\cong Map_0(X \wedge S^1, Y) \end{aligned}$$

The objects $X \wedge I$ and $X \wedge S^1$ are respectively known as the "cone" over X, denoted with $C(X)$, and the "suspension" of X, denoted with $S(X)$ (Sometimes they are also called reduced cone and reduced suspension respectively).

The suspension $S(X)$ of X can be also realized as

$$(X \times I)/(X \times \{0\} \cup \{x_0\} \times I \cup X \times \{1\}) \tag{A.1}$$

see figure A.1, where the thick line is supposed to be identified to a point. The suspension (or more generally the smash product) is particularly well suited to deal with spheres. A theorem proved virtually in any text-book on Algebraic Topology says that

$$S^n \wedge S^m \cong S^{n+m}$$

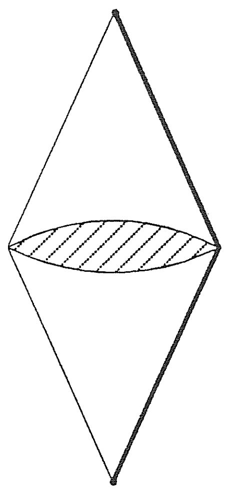

Figure A.1: The Suspension $S(X)$

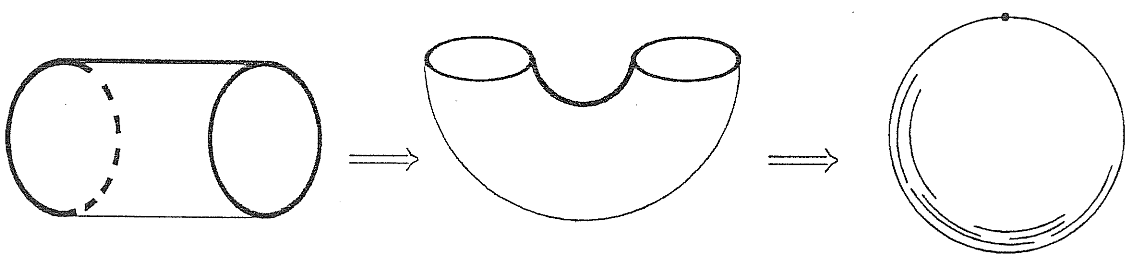

Figure A.2: The Suspension of S^1

In particular we have that $S(S^n) \cong S^{n+1}$. The realization (A.1) of the suspension $S(X)$ is very useful to visualize the suspension process $S(S^1) \cong S^2$ as shown in figure A.2. This also shows that $S(D^2) \cong D^3$.

On the other hand, for any space the cone $C(X)$ over it is always a contractible space [52], and the relationship between $S(X)$ and $C(X)$ is that $S(X) \cong C(X)/X$.

Appendix B

A covering space

In this appendix we show that the space $\overline{Map_0(S^1, \Omega G)} \equiv \overline{\Omega_{f_0}(\Omega G)}$ introduced in section 3.3 is in fact the covering space of $\Omega_{f_0}(\Omega G)$.

Since G is a group, we can shift the base-point of ΩG to e, as already done in the main text in section 3.3. Moreover, we can be slightly more general and consider directly the unbased loop space $L(\Omega G) = Map(S^1, \Omega G)$. Since $\pi_1(\Omega G) = \pi_2(G) = 0$, each loop is contractible and it extends to a map $u : D^2 \to \Omega G$. Given two maps $u, u' \in Map(D^2, \Omega G)$, we define their "glue" $\bar{u} : S^2 \to \Omega G$, shortly denoted with $u - u'$, as follows. We consider S^2 as the union of two emispheres, the northern and the southern ones, $S^2 \cong D^2_+ \cup D^2_-$, along the equatorial line $S^1 = D^2_+ \cap D^2_-$. Obviously, the two emispheres must be glued together in such a way to have an overall correct orientation. This happens to be true if they are connected by an orientation reversing diffeomorphism φ of S^1, so that the orientations of D^2_+ and D^2_- will agree one with the other to build up the orientation of the whole connected sum $D^2_+ \cup D^2_-$. Then we define $\bar{u} = u - u'$ to be the map equal to u on D^2_+ and to u' on D^2_- with reversed orientation. For this machinery to work, we need the restrictions of u and u' to S^1 to be connected by on orientation *preserving* diffeomorphism of S^1, since composing this last one with the exchange in the orientation of D^2_-, will furnish the diffeomorphism φ that glues together the two emispheres. In our specific case the glueing diffeomorphism will be $\varphi(e^{i\theta}) = e^{-i\theta}$, since we shall consider maps u, u' such that $u\,|_{S^1} = u'\,|_{S^1}$. In fact we define two maps $u, u' \in Map(D^2, \Omega G)$ to be in the relation $\sim$ iff

1. $u\,|_{S^1} = u'\,|_{S^1}$

2. $u - u' = 0$ in homotopy, that is $[u - u'] = 0$ in $\pi_2(\Omega G) = \pi_3(G)$.

This is an equivalence relation on $Map(D^2, \Omega G)$; define $\overline{L(\Omega G)}$ to be the space of classes

$$\overline{L(\Omega G)} := Map(D^2, \Omega G)/\sim$$

with the obvious projection

$$\begin{array}{rl} \overline{L(\Omega G)} & \xrightarrow{\pi} L(\Omega G) \\ [u] & \mapsto u\,|_{S^1} \end{array}$$

Now, since G is a group, ΩG and $L(\Omega G)$ are themselves groups, so that we have the decomposition $L(\Omega G) \cong \Omega G \times \Omega\Omega G$, just as in the case $LG \cong G \times \Omega G$. Of course, $L(\Omega G)$ decomposes in this way only as a topological space, without considering its group structure. This allow us to say that

$$\begin{aligned} \pi_1(L(\Omega G)) &= \pi_1(\Omega G) \times \pi_1(\Omega\Omega G) \\ &= \pi_2(G) \times \pi_3(G) \\ &= \pi_3(G) \end{aligned}$$

On the other hand, we have that

$$L(\Omega G) \cong \overline{L(\Omega G)}/\pi_3(G)$$

since for each $\ell : S^1 \to \Omega G$ classes $[u] \in \overline{L(\Omega G)}$ such that $\pi([u]) = \ell$ are in correspondence with homotopy classes in $\pi_2(\Omega G)$. To see this it is sufficient to fix $u_0 \in Map(D^2, \Omega G)$, $u_0\,|_{S^1} = \ell$, and consider those u which agree with u_0 on S^1 but do not satisfy $[u - u_0] = 0$. Therefore we obtain that $\overline{L(\Omega G)}$ is the universal cover of $L(\Omega G)$, which is what we intended to show.

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