ISAS - INTERNATIONAL SCHOOL FOR ADVANCED STUDIES

LOGARITHMIC SOBOLEV INEQUALITIES

Thesis submitted for the degree of

"Magister Philosophiae"

CANDIDATE

Fabio Cipriani

SUPERVISOR

Prof. G.F. Dell'Antonio

October 1990

TRIESTE

SISSA - SCUOLA INTERNAZIONALE SUPERIORE STUDI AVANZATI

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Strada Costiera 11

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CONTENTS

INTRODUCTION

The Logarithmic Sobolev Inequality were introduced and studied as a tool in the analysis of operators in infinitely many dimensions. A Classical Sobolev Inequality in $\mathbb{R}^d$ states that if a function f and its first distributional derivatives are in $L^p(\mathbb{R}^d)$ then the function is also in L^q with $q=(p^{-1}-d^{-1})^{-1}$ Now this expression shows that when the dimension d increase to infinite, q tends to p, and the implication of the theorem becames less meaningful. Moreover in infinite many dimensions (for example on an infinite dimensional Banach space), there is no Lebesgue mesure (i.e. translation invariant mesure). Now, a Logarithmic Sobolev Inequality is an expression that appears as the following one

$$(L-S) \qquad \int_{\mathbb{R}^d} d\mu(x)\,|f(x)|^p \cdot \log|f(x)| \le \frac{p}{2(p-1)} \cdot \int_{\mathbb{R}^d} d\mu(x)\,|\nabla f(x)|^p + \|f\|^p_{L^p(\mu)} \cdot \log\|f\|_{L^p(\mu)}$$

for the case of the Gaussian mesure on $\mathbb{R}^d$, $p>1$

$$d\mu(x) = (2\pi)^{-d/2} \cdot dx \cdot e^{-\|x\|^2/2} \qquad x \in \mathbb{R}^d$$

The appeal of the inequality above is that replacing the Lebesgue mesure with the Gaussian mesure we have an expression whose coefficient are indipendent on the dimension d of the space and at the same time if f and its gradient are in $L^p(\mu)$ $p>1$ then f is also in the so called Orlicz space of function for which the l.h.s. of (L-S) is bounded.

At the beginning the theory started showing the equivalence between certain Logarithmic Sobolev Inequalities and the Hypercontractivity of the semigroup generated by the Dirchlet form $\int d\mu \cdot |\nabla f|^2$. In this way was possible to recover the result of Nelson in the sixties, about the hypercontractivity of the semigroups of the Free Euchidean Quantum Field Theory (see Ref. [N2]). In recent years with the work of R. Holley, D. Stroock, E. Carlen, J. Deuschel and B. Zegarlinski, appeared more and more clear the role that the Logarithmic Sobolev Inequalities play in the area of the Infinite Lattice Systems. A motivation for this, was the idea, followed by Holley and Stroock in a series of papers, that Gibbs mesures of a Statistical Mechanical System can be seen as the invariant mesure of a Stochastic Dynamics and the fact that certain Logarithmic Sobolev Inequalities eniply exponential convergence to equilibrium of the semigroup dynamics on the lattice. This permits to have a bridge between the ergodic properties (especially the mixing one) of the Gibbs states and the corrisponding analytical properties of the semigroup that determines the Stochastic Dynamics. Closely related to this approach there is the use of the Logarthmic Sobolev Inequalities in Large Deviation theory, where certain characteristic inequalities of the Large Deviation theory for Ergodic Systems can be seen as Logarithmic Sobolev Inequalities.

Beside to this infinite dimensional setting for the Log-

Sobolev Inequalities, in the last four or five years another one appeared in the field of elliptic operators on locally compact spaces. There, B. Davies and Simon B. used Log-Sobolev Inequalities to obtain sharp upper and lower bounds for the integral kernels of certain semigroups. Moreover that theory permits to obtain the bounds in a gaussian form.
We stop here with this excursus and start to sketch the organization of the work.

DESCRIPTION OF THE CHAPTERS

We divided the work in two Parts. In the first one we treated the general aspects and the methods for finite dimensional situations. The second part is devoted to the problems arising in the infinite dimensional setting.

Chapter 1 contains the theory of the equivalence between Log-Sobolev Inequalities and certain L^p-smoothing properties.

In Chapter 2 is shown the relation between the Log-Sobolev Inequalities, the Paincaré Inequality and the Spectral Gap.

In Chapter 3 we review two methods originated by the Bakry-Emery's one, and develop some explicit computation of the Sobolev's constants.

In Chapter 4 we devolop the method of Rosen and in Chapter 5 we apply it to the study of the intrinsic hypercontractivity of Schrödinger operators.

In Chapter 1 of Part 2 we give a short introduction to the theory of Gibbs mesures and to the Dobrushin uniqueness theorem.

In Chapter 2 we quote general results about Dirichlet forms for an infinite lattice system.

In Chapter 3 we develop the anlysis of the Log-Sobolev Inequalities on infinite lattice systems.

PART 1

LOG-SOBOLEV INEQUALITIES, METHODS AND APPLICATIONS TO SCHRÖDINGER OPERATORS

CHAPTER 1

GENERAL SETTING FOR LOG-SOBOLEV-INEQUALITY

1.1 Logarithmic Sobolev Inequality and -smoothness

In this section we'll define the Logarithmic Sobolev Inequalities (shortly (L-S)) satisfyied by special operators H in L^p spaces, and we'll prove the relation between these inequalities and the L^p -smoothing properties of the semigroup e^{-tH} generated by H.

Since now on we'll denote with $(\Omega, \mathcal{M}, \mu)$ a mesurable space with σ -algebra M and positive mesure μ.

For any $p \in (1;+\infty)$ and any complex valued function f we fix $f_p \equiv (\operatorname{sgn} f) \cdot |f|^{p-1}$ where $\operatorname{sgn} z = z/|z|$ if $z \neq 0$ and $\operatorname{sgn} 0 \equiv 0$.

In the next lemma we give (without proof) useful properties of the map $f \longmapsto f_p$ on L^p - spaces:

Lemma 1

i) $f \in L^p(\Omega;\mu) \Longrightarrow f_p \in L^q(\Omega;\mu)$, $p \in (1;+\infty)$ $p^{-1} + q^{-1} = 1$

ii) the map $f \in L^p \longmapsto f_p \in L^q$ is continuous, injective and surjective with inverse $g \in L^q \longmapsto g_q \in L^p$.

Definition 1.

The operator H in $L^p(\Omega\mu)$ $p \in (1;+\infty)$ is said to be a "Sobolev generator of index p" if it is the generator of a strongly continuous semigroup e^{-tH} and for some $c>0$ and $\gamma\in\mathbb{R}$ the following inequality is satisfied

$$\text{(L-S)}\qquad \int_\Omega d\mu\, |f|^p \cdot \ln|f| - \|f\|^p_{L^p}\cdot \ln\|f\|_{L^p} \leq c\langle Hf; f_p\rangle + \gamma\cdot\|f\|^p_{L^p} \quad f\in D(H)$$

This inequality is said to be a "Logarithmic Sobolev Inequality" for H, and the constants c and γ are called respectively "Sobolev coefficient (or constant)" and "local norm".

The bilinear form $\langle ;\rangle$ being the duality between L^p and L^q, i.e: $\langle f;g\rangle = \int_\Omega d\mu\, \bar{f}\cdot g$.

Our main interest is not in the individual properties of the operator H_p on L^p, but in those of the "coherent family" of operators on any L^p $p\in(a;b)\subset[1;+\infty]$, coming, maybe, from H_p.

Definition 2

Let $(a;b)\subset[1;+\infty]$. A family $(e^{-tH_p})_{p\in(a;b)}$ of strongly continuous semigroups on $(L^p)_{p\in(a;b)}$ is said to be "coherent" if $\forall r\in(a;b)$ we have:

i) $e^{-tH_p}(L^r\cap L^p)\subset L^r$

ii) $e^{-tH_p}\restriction L^r\cap L^p: L^r\cap L^p \longrightarrow L^r$ is continous $\forall t>0$ in the $\|\cdot\|_r$-topologies of this spaces.

iii) $e^{-tH_p}\restriction L^r\cap L^p = e^{-tH_r}\restriction L^r\cap L^p$

the family $(H_p)_{p\in(a;b)}$ is called a "coherent family of operators an $(a;b)$.

Remark 1.

A coherent family can be re-constructed by any element e^{-tH_p}. This because $L^r \cap L^p$ is $\|\cdot\|_r$-dense in L^r $\forall r \in (a;b)$. In this sense we can say that H_p (or e^{-tH_p}) generates the whole family.

An operator H_p on $L^p(\Omega;\mu)$ is said to be a "Sobolev generator on $(a;b)$" if:

i) H_p generates a coherent family on $(a;b)$

ii) $\forall p \in (a;b)$ H_p is a Sobolev generator with constant $c(p) > 0$ and $\gamma(p) \in \mathbb{R}$

iii) the functions $c : (a;b) \longrightarrow \mathbb{R}^+$ and $\gamma : (a;b) \longrightarrow \mathbb{R}$ are continuous.

Some properties of the Sobolev generators are summarized in the following proposition:

Proposition 1.

1) (Homogeneity) If H is an operator on L^p, then (L-S) hold for some $f \in D(H)$ if and only if they hold for the element $f/\|f\|_p \in D(H)$

2) (L-S) hold on $D(H_p)$ if and only if they hold on any other core D of Hp

3) if (L-S) hold for $p=2$ on $D(H_2)$ then they hold on all the form domain $D(H^{1/2})$

4) if $\mu(\Omega)=1$ and Hp is a Sobolev generator of index p, with constants c and γ then $\| e^{-tH_p}\|_{p,p} \le e^{-t\gamma/c}$ in particular if in (L-S) $\gamma=0$, e^{-tH_p} is a contraction

semigroup on L^p .

Proof:

1) follows because the function $\int_\Omega d\mu |f|^p \cdot \ln|f| - \|f\|_p^p \cdot \ln\|f\|_p$ is homogeneous of degree p in f

2) If $D \subset D(H_p)$ is a core for Hp, then $\forall f \in D(H_p)$ exist sequence $(f_n)_{n\in\mathbb{N}} \subset D$ such that $f_n \to f$, $H_p f_n \to H_p f$ in L^p and $f_n \to f$ μ-a.e. The sentence follows applying Fatou's lemma to $|f_n|^p \cdot \ln|f_n|$ and observing that, by lemma 1, the function $c \cdot \mathrm{Re}\langle Hf; f_p\rangle + \gamma\|f\|_p^p + \|f\|_p^p \ln\|f\|_p$ continous function of $f \in D$ in the graphic norm.

3) if $g \in D(H_2^{1/2})$ them $g_\delta \equiv e^{-\delta H} \cdot g \in D(H_2)$ we have $g_\delta \to g$ $\delta \to 0$ in L^2 , by the strong continuity of e^{-tH_2} .

Moreover $(H_2 g_\delta ; g_\delta)_{L^2} \to (H_2^{1/2} g ; H_2^{1/2} g)_{L^2}$ $\delta \to 0$

We can now assume g_δ converging pointwise μ-a.e. to g: using Fatou's lemma we obtain

$$\int d\mu \cdot |g|^2 \cdot \ln|g| - \|g\|_2^2 \cdot \ln\|g\| \leq c\,\mathcal{E}(g;g) + \gamma\|g\|_2^2$$

where $\mathcal{E}$ is the form associated to H_2 .

The following lemma is the key to prove the equivalence between (L-S) and L^p smoothing properties It is based on a theorem of Mazur about the differentiability of the L^p norms.

Lemma 2.

Let $(\Omega; \mathcal{M}; \mu)$ be a mesurable space with positive mesure μ

$p\in(1;+\infty)$ and $\varepsilon>0$. Let $s:[0\,\varepsilon)\to\mathbb{R}$ be a continous function differentiable in $t=0$ and such that $s(0)=p$.

Moreover for $\delta>0$

$$\bar{D}(p;\delta)=\{r\in(1;+\infty):|p-r|\le\delta\}$$

let $f:[0\,\varepsilon)\longrightarrow\bigcap_{r\in\bar{D}(p;\delta)}L^r(\Omega;\mathcal{M};\mu)$ be a continous function in all L^r- norms $r\in D(p;\delta)$, differentiable in $t=0$ in the L^p- norm and such that $f(0)\equiv v\neq0$.

Then the function $g:[0\,\varepsilon)\to\mathbb{R}$ defined by $g(t)\equiv\|f(t)\|_{s(t)}\;\forall t\in[0\,\varepsilon)$ is continous and differentiable in $t=0$ with derivate

$$\left.\frac{dg}{dt}\right|_{t=0}=\|v\|_p^{1-p}\cdot\left[p^{-1}\cdot s'(0)\cdot\left\{\int d\mu\cdot|v|^p\cdot\ln|v|-\|v\|_p^p\cdot\ln\|v\|_p\right\}+Re\langle f'(0);v_p\rangle\right]$$

Remark 2.

With $f'(0)$ we denote the derivative with respect to the L^p norm, and with $v_p\equiv\operatorname{sgn}v\cdot|v|^{p-1}$

Proof.

First of all we have

$$|f(t)(x)|^{s(t)}\le|f(t)(x)|^{p-\delta}+|f(t)(x)|^{p+\delta}$$

for some $t>0$, because s is continous and $s(0)=p$.

Since f is L^r- norm continous $r\in\bar{D}(p;\delta)$ we have that

$f^{p-\delta} + f^{p+\delta}$

is continous in L^1 and then the functions $f(t)^{p+\delta} + f(t)^{p-\delta}$ are uniformly integrable for sufficiently small $t \in [0\ \varepsilon)$.

By the dominate convergence theorem we obtain that

$$t \longmapsto \|f(t)\|_{s(t)}^{s(t)} = \int_\Omega d\mu(x)\, |f(t)(x)|^{s(t)}$$

is continous in a neighbourhood of $t=0$. But since $s(\cdot)$ is continous, also g is continous. We may assume that g is continous on all $[0\ \varepsilon)$.

About the differentiability of g in $t=0$, we start writing

$$\lim_{t\to 0} \frac{1}{t}\left(\|f(t)\|_{s(t)} - \|f(0)\|_{s(0)}\right) = \lim_{t\to 0} \frac{1}{t}\left(\|f(t)\|_{s(t)} - \|f(t)\|_{s(0)}\right) + \lim_{t\to 0} \frac{1}{t}\left(\|f(t)\|_{s(0)} - \|f(0)\|_{s(0)}\right)$$

By Mazur's theorem, our hypothesis about the differentiability of f in $t=0$ in the L^p-norm and by the chain rule for derivatives, we have

$$\lim_{t\to 0} \frac{1}{t}\left(\|f(t)\|_{s(0)} - \|f(0)\|_{s(0)}\right) = \|v\|_p^{1-p} \cdot Re\langle f'(0); v_p\rangle \qquad (s(0)=p \quad f(0)=v\neq 0)$$

We have, now, just to calculate

$$\lim_{t\to 0} \frac{1}{t}\left(\|f(t)\|_{s(t)} - \|f(t)\|_{s(0)}\right).$$

Let $u \in \bigcap_{r\in \overline{D}(p;\delta)} L^r$ and let's calculate the derivatives of $t \longmapsto \|u\|_{s(t)}$:

$$\|u\|_{s(t)}^{-s(t)} \cdot \frac{d}{dt}\left(\|u\|_{s(t)}^{s(t)}\right) = \frac{d}{dt}\left(\ln \|u\|_{s(t)}^{s(t)}\right) = \frac{d}{dt}\left(s(t)\cdot \ln \|u\|_{s(t)}\right) = s'(t) \cdot \ln \|u\|_{s(t)} + \frac{s(t)}{\|u\|_{s(t)}} \cdot \frac{d}{dt} \|u\|_{s(t)}$$

and then

$$\frac{d}{dt}\|u\|_{s(t)} = s(t)^{-1}\cdot\|u\|_{s(t)}^{1-s(t)}\cdot\frac{d}{dt}\left(\|u\|_{s(t)}^{s(t)}\right) - s(t)^{-1}\cdot s'(t)\cdot\|u\|_{s(t)}\cdot \ln\|u\|_{s(t)}$$

$$\frac{d}{dt}\|u\|_{s(t)}\Big|_{t=0} = p^{-1}\cdot\|u\|_p^{1-p}\cdot\left[\frac{d}{dt}\left(\|u\|_{s(t)}^{s(t)}\right)\Big|_{t=0} - p^{-1}\cdot s'(0)\cdot\|u\|_p^{p}\cdot \ln\|u\|_p\right]$$

Now we have just to show that

$$\frac{d}{dt}\left(\|u\|_{s(t)}^{s(t)}\right)\Big|_{t=0} = s'(0)\cdot\int_\Omega d\mu(x)\cdot|u(x)|^p\cdot\ln|u(x)|$$

Fixing $h_s(x) = \dfrac{|u(x)|^s - |u(x)|^p}{s-p}$ $\quad s\in\bar{D}(p;S)$ we can write

$$\frac{d}{ds}\|u\|_s^s\Big|_{s=p} = \lim_{s\to p}\int_\Omega d\mu(x)\,\frac{|u(x)|^s-|u(x)|^p}{s-p} = \lim_{s\to p}\int_\Omega d\mu(x)\,h_s(x)$$

Since $u\in\bigcap_{r\in\bar{D}(p;S)} L^r$ we have $(h_s)_{s\in\bar{D}(p;S)}\subset L^1(\mu)$.
This family is uniformely bounded by the function $L^1(\mu)$
$K(x)\equiv|u(x)|^{p-S}+|u(x)|^{p+S}$ Then by dominate convergence theorem

$$\frac{d}{ds}\|u\|_s^s\Big|_{s=p} = \lim_{s\to p}\int d\mu\, h_s = \int d\mu\lim_{s\to p} h_s = \int d\mu\cdot|u(x)|^p\cdot\ln|u(x)|$$

Using the chain rule for derivatives we have

$$\frac{d}{dt}\cdot\left(\|u\|_{s(t)}^{s(t)}\right)\Big|_{t=0} = s'(0)\cdot\int_\Omega d\mu(x)\cdot|u(x)|^p\cdot\ln|u(x)| .$$

Applying the theorem of medium value to the function $t \mapsto \|u\|_{s(t)}$ with $u = f(t)$, there exist $t_1 \in [0\, t)$ such that

$$\frac{1}{t}\left(\|f\|_{s(t)} - \|f(t)\|_{s(0)} \right) = s'(t_1) \cdot s(t_1)^{-1} \cdot \|f(t)\|_{s(t_1)}^{1-s(t_1)} \cdot \left\{ \int d\mu \, |f(t)|^{s(t_1)} \cdot \ln |f(t)| - \|f(t)\|_{s(t_1)}^{s(t_1)} \cdot \ln \|f(t)\|_{s(t_1)} \right\}$$

Since $t \mapsto f(t)$ is continous for any L^r- norm with $r \in \bar{D}(p;S)$ and since $s(t_1) \in \bar{D}(p;S)$ $t \in [0;\varepsilon)$, by the cointinuity of s, last term under integral sign is uniformely dominated in $t \in [0;\varepsilon)$ by $K(x)$. Since this function is in L^1 , we obtain the thesis by the dominated convergence theorem. ///

Next proposition prooved by Gross in [G 1] , shows that $(L-S)$ implies smoothing properties of the semigroup of a Sobolev generator.

Proposition 2.

Let H be a Sobolev generator on $(a;b) \subset [1;+\infty]$ with coefficient function $c : (a;b) \to \mathbb{R}^+$ and local norm $\gamma : (a;b) \to \mathbb{R}$ Fixing $q \in (a;b)$ let $p(t;q)$ the solution of the problem

$$\begin{cases} c(p) \cdot \dfrac{dp}{dt} = p & t > 0 \\ p(0;q) = q & \end{cases}$$

defined for t such that $p(t;q) < b$. Moreover we define

$$M(t;q) \equiv \int_0^t ds \, \gamma(p(t;s)) \cdot c(p(t;q))^{-1}$$

Then:

$$\| e^{-tH} \|_{q,\, p(t;q)} \leq e^{M(t;q)} .$$

Proof.

Let D be the linear manifold generated by the vectors

$$\int_0^\infty dt\, g(t)\cdot e^{-tH} \cdot v \qquad \text{with } g \in C_c^\infty(\mathbb{R}^+) \quad \text{and } v \in L^1 \cap L^\infty .$$

First of all we want to show that $D \subset D(H_p)\ \forall p \in (a;b)$, $e^{-tH} D \subset D\ \forall t>0\ \forall p \in (a;b)$ and that D is L^p dense with $p \in (a;b)$

Since $L^1 \cap L^\infty \subset L^p\ \forall p$, we have that $D \subset L^p\ \forall p \in (a;b)$.

Let

$$h = \int_0^\infty dt\, g(t)\, e^{-tH_p} \cdot v \in D \subset L^p .$$

We have

$$\lim_{s \to 0} \frac{1}{s} \left(e^{-sH_p} \cdot h - h \right) = \int_0^\infty dt\, e^{-tH_p} \cdot v \cdot g'(t) \in D .$$

Then $D \subset D(H_p)\ \forall p \in (a;b)$. Obviously D is invariant for $e^{-tH_p}\ \forall t \geq 0\ \forall p \in (a;b)$ Now, let $w \in L^q = (L^p)'$ $q^{-1} + p^{-1} = 1$ such that $w(D) = 0$. Then $0 = \langle w; \int_0^\infty dt\, g(t) \cdot e^{-tH} \cdot v \rangle = \int_0^\infty dt\, g(t) \langle w, e^{-tH} \cdot v \rangle$ $\forall g \in C_c^\infty(\mathbb{R}^+)$ and so $\langle w; e^{-tH} \cdot v \rangle = 0\ \forall t > 0\ \forall v \in L^1 \cap L^\infty$.

By the strong continuity of the semigroup we have $\langle w; v \rangle = 0$ $\forall v \in L^1 \cap L^\infty$, and by the density in any L^p of $L^1 \cap L^\infty$ we have $w = 0$. Hence D is dense in any L^p $p \in (a;b)$.

Let $h \in D$, $h \neq 0$ and let's put

$$f : \mathbb{R}^+ \longrightarrow \bigcap_{r \in (a;b)} L^r \qquad f(t) \equiv e^{-tH} \cdot h \qquad t \in \mathbb{R}^+$$

The function f is continous in any norm L^r norm with $r \in (a;b)$ and since $D \subset D(H_p)\ \forall p \in (a;b)$, it is also differentiable in $t = 0$ with respect to any L^r- norm $r \in (a;b)$. But the continuity of the Sobolev coefficient $c(\cdot)$, the function $t \mapsto p(t;q)$ is C^1 where it is defined, in particular this is true in $t = 0$.

Then the function

$$g : [0\ \varepsilon) \longrightarrow \mathbb{R} \qquad g(t) = \| f(t) \|_{p(t;q)}$$

satisfies the hipotheses of lemma 2 and we have:

$$\frac{d}{dt} g = \| f(t) \|_p^{1-p} \cdot \left[c(p)^{-1} \cdot \left\{ \int_\Omega d\mu\, |f(t)|^p \cdot \ln |f(t)| - \| f(t) \|_p^p \cdot \ln \| f(t) \|_p \right\} - Re \langle H f(t) ; f(t)_p \rangle \leq \right.$$

$$\leq c(p)^{-1} \cdot \gamma(p) \cdot \| f(t) \|_p \qquad p \equiv p(t;q)$$

and then $\frac{d}{dt} \left(\ln g(t) \right) \leq c(p)^{-1} \cdot \gamma(p)$.

This implies $\ln g(t) \leq \ln g(0) + M(t;q)$ and then

$$\| e^{-tH} h \|_{p(t;q)} \leq e^{+M(t;q)} \cdot \| h \|_q$$

Since D is dense in any L^p $p \in (a;b)$, we have

$$\| e^{-tH} \|_{q;\, p(t;q)} \leq e^{M(t;q)} .$$

///

Remark 3.

Since $c(p) > 0 \ \forall p \in (a, b)$, $\frac{d}{dt} p(t; q) > 0 \ \forall t \in [0\ \varepsilon)$. So $p(t;q) \geq p(0;q) = q$.

Remark 4.

Before to pass to proove the inverse of Proposition 2, we want to show another property of the set D, that we needed in the last proof, and that we'll use again.

Lemma 3.

The linear mainfold D, generated by vectors of the form

$$\int_0^\infty dt\, g(t) \cdot e^{-tH_p} \cdot v \qquad g \in C_c^\infty(\mathbb{R}^+)\ ,\ v \in L^1 \cap L^\infty$$

is a core for Hp.

Proof.

By arguments above we have just to prove that any $h \in D(H_p)$ can be approximated by a sequence $(h_n)_n$ in D, in the graphic norm topology of Hp.

Let $(g_n)_n \subset C_c^\infty(\mathbb{R}^+)$ such that, $g_n \to \delta_0, g_n' \to \delta_0'$ in the distributional sense. Let $h_n \equiv \int_0^\infty dt\, g_n(t) \cdot e^{-tH} \cdot h \in D \ \forall n \geq 1$.

Obwiously $h_n \to h$ in L^p. Moreover

$$-H h_n = \lim_{s \to 0} s^{-1}(e^{-sH} h_n - h_n) = \lim_{s\to 0} s^{-1} \left(\int_0^\infty dt\, g_n(t) e^{-(s+t)H} \cdot h - \int_0^\infty dt\, g_n(t) \cdot e^{-tH} \cdot h \right) =$$

$$= \lim_{s \to 0} s^{-1} \int_0^\infty dt\, (g_n(t-s) - g_n(t)) \cdot e^{-tH} h = \int_0^\infty dt\, g_n'(t) \cdot e^{-tH} \cdot h \to \int_0^\infty dt \cdot \delta_0'(t) e^{-tH} h \equiv -Hf$$

In the L^p norm. ///

In the next proposition we'll prove that L^p-smoothing properties imply Logarithmic Sobolev Inequalities.

Proposition 3.

Let $(e^{-tH_p})_{p\in(a;b)}$ $(a;b)\subset[1;+\infty]$ a coherent family of semigroups strongly continuous on $(L^p(\Omega,\mu))_{p\in(a;b)}$ (see Definition 2).

Suppose that $\forall q\in(a;b)$ we have two continous functions

$$p(\cdot;q),\ m(\cdot;q):[0;\varepsilon(q))\longrightarrow\mathbb{R}$$

such that $p(0;q)=q$ and $m(0;q)=1$, and that

$$\|e^{-tH}\|_{q,p(t;q)}\leq m(t;q)\quad\forall t\in[0;\varepsilon(q))\quad q\in(a;b).$$

Moreover suppose that $p(\cdot;q)$ and $m(\cdot;q)$ be differentiable in $t=0$ and that the functions $c,\bar{\gamma}:(a;b)\to\mathbb{R}$ defined by

$$c(q)\equiv q^{-1}\left.\frac{dp(t;q)}{dt}\right|_{t=0}\qquad\qquad\bar{\gamma}(q)\equiv\left.\frac{dm(t;q)}{dt}\right|_{t=0}$$

are continous with c strictly positive. Then H is a Sobolev generator an $(a;b)$ with coefficient function c and local norm $\gamma=\bar{\gamma}\cdot c$.

Proof.

By lemma 3, the linear manifold D generated by vectors of the form

$$\int_0^\infty dt\, g(t)\cdot e^{-tH_p}\cdot v \qquad g\in C_0^\infty(\mathbb{R}^+)\ ,\quad v\in L^1\cap L^\infty$$

is a core for Hp, it's sufficient to verify $(L\text{-}S)$ on D (by Proposition 1). Let $v\in D-\{0\}$ and $f:[0\,\infty)\to D$ defined by $f(t)\equiv e^{-tH}\cdot v$ Since $D\subset\bigcap_{q\in(a;b)} L^q$ and since the function $p(\cdot\,;q)$ is continous and differentiable in $t=0$, applying lemma 2 with $s(t)\equiv p(t;q)$ we obtain from the inequality

$$\frac{1}{t}\cdot\left(\|f(t)\|_{p(t;q)}-\|f(0)\|_{p(0;q)=q}\right)\le\|v\|_q\cdot\frac{(m(t;q)-1)}{t}$$

the following one

$$\|v\|_q^{1-q}\cdot\left\{q^{-1}\cdot q\cdot c(q)^{-1}\left[\int_\Omega d\mu\,|v|^q\ln|v|-\|v\|_q^q\cdot\ln\|v\|_q\right]-\mathrm{Re}\langle Hv;v_q\rangle\right\}\le\gamma(q)\cdot\|v\|_q$$

This last inequality is precisely $(L\text{-}S)$ with local norm $\tilde{\gamma}=\gamma\cdot c$.

1.2 Markovian Semigroups

In Proposition 2 and 3 we saw the equivalence between $(L\text{-}S)$ and certain smoothing properties, in the context of a generic mesure space $(\Omega;\mathcal{M};\mu)$. In practice we meet often, more definite situations:

i) $(\Omega;\mathcal{M};\mu)$ is a probability space

ii) e^{-tH} is positive preserving

iii) e^{-tH} is a Markovian semigroup on $L^2(\Omega;\mathcal{M};\mu)$.

The case i) is the context in which L. Gross pro ved, for the first time, Proposition 2 and 3. (see Ref. [G1]). The importance of the case i) derive from applications in Euclidean Quantum Field Theory. In those cases the probability mesure is on a space of distributions as $\mathcal{S}'(\mathbb{R}^d)$. Moreover the property to preserve the particular ordering given by pointwise positivity of functions and the hypercontractivity of a semigroup have strict relation with the problem of the existence and uniqueness of the ground state of selfadjoint operator. L. Gross applied these result to hamiltomain operators in Quantum Field Theory (see Ref. [G 2] .).

The case iii) (that cover ii) is that one we meet frequently in the applications to Euclidean Quantum Field Theory, Statistical Mechanics (see Ref. [CA-S] , [Z 1] , [Z 2] , [Z 3]) and in the study of elliptic and generalized Schrödinger operators (see Ref. [DA 1]).

The common point of all these directions is that the interesting operators are usually given by quadratic forms that are also Dirichlet forms.

This representation permits to study operators with coefficient functions with a weaker regularity, and also gives us a bridge between analytical aspects (for example spectral properties) and probabiliste aspects (see Ref. [A-H-K]) close to the theory of Markov Processes and Potential Theory.

In this section we'll show that in the cases ii) and iii) verify [L-S] is much simpler that in the general case. With respect the point ii) we have the following.

Proposition 4.

Suppose that e^{-tH} is a positive preserving operator. Then H is a Sobolev generator on $(a;b) \subset [1;+\infty]$ if and only if (L-S) is verified only on the cone of positive functions in $D(H_p)$ $\forall p \in (a;b)$ (or in any other core of D).

Proof.

By lemma 3 the linear manifold D generated by the vector of the form $h \equiv \int_0^\infty dt\, g(t) \cdot e^{-tH} \cdot v$ $g \in C_0^\infty(\mathbb{R}^+)$, $v \in L^1(\Omega\mu) \cap L^\infty(\Omega\mu)$ is a core for H_p $p \in (a;b)$. As in Proposition 2, where we show that D is dense in L^p $p \in (a;b)$, it's possible to show that the subset of vector generated by non-negative elements g and v, is dense in the subset of non-negative functions in L^p.
In particular the positivity of e^{-tH} implies that $g, v \geq 0 \Rightarrow h \geq 0$.
At the same time the positivity of e^{-tH} implies that $f(t) = e^{-tH} \cdot h$ takes its values in the set of non-negative elements of D. We can now apply lemma 2, and obtain as in the proof of lemma 2, the following inequality:

$$\| e^{-tH} h \|_{p(t;q)} \leq e^{M(t;q)} \cdot \| h \|_q \qquad \forall\, h \in D_+ \equiv \{ f \in D : f \geq 0 \} .$$

Since e^{-tH} is positive preserving, we have that $| e^{-tH} h | \leq e^{-tH} |h|$ for any element of **D**. ///

Next Proposition show that in the case of Markovian semigroups, the (L-S) inequalities for $p=2$ imply those for

$p>2$. So a Sobolev generator for index 2 is automatically a Sobolev generator of on $[2;\infty)$.

The proposition was proved by L. Gross in [G1] in the case where $\Omega=\mathbb{R}^d$ and the Markovian operator comes from a classical Dirichlet form. Our proof follows that of [DA 1], but uses the following lemma (see Ref.[V]) in order to deal with more general mesure spaces, non-necessarely locally compact.

Lemma 4.

Let $(\Omega\mathcal{M}\mu)$ be a Polish mesure space with positive mesure μ and let $(\mathcal{E};D(\mathcal{E}))$ be a Dirichlet form on $L^2(\Omega\mathcal{M}\mu)$. Let also P a Markovian operator on $L^2(\Omega\mathcal{M}\mu)$

Then:

i) $\forall f\in L^2$ $\alpha,\beta\geq 0$ $\alpha+\beta=2$ we have $\alpha\cdot\beta\cdot((I-P)f;f)_{L^2}\leq((I-P)f^\alpha;f^\beta)_{L^2}$

ii) $\forall\, 0\leq f\in D(\mathcal{E})$ $\alpha,\beta\geq 0$ $\alpha+\beta=2$ we have $\alpha\cdot\beta\cdot\mathcal{E}(f;f)\leq\mathcal{E}(f^\alpha;f^\beta)$

Proposition 5.

Let $(\Omega\mathcal{M}\mu)$ be a Polish mesure space with positive mesure μ , and let be a Dirichlet form whose corrisponding operator $(H_2;D(H_2))$ on L^2 is a Sobolev generator of index 2 with constants c and γ . Then the coherent family $(e^{-tH_p})_{p\in[1+\infty)}$ generated by $(\mathcal{E};D(\mathcal{E}))$ is a Sobolev generator on $[2;+\infty)$ with coefficient functions

$$c(p)=\frac{c\cdot p}{2(p-1)} \qquad \gamma(p)=\frac{2\gamma}{p}$$

Proof.

Let $p > 2$ and D be the common care of all operators described in lemma 3:

By Proposition 1 point 2) it's sufficient to show that $(L-S)$ holds on $D \subset D(H_p)$ and by Proposition 4 it's also sufficient to consider only the positive elements of D : D_+ .

Let $g \in D_+$ be a generic element and suppose we know that $g^p \in D(\mathcal{E})$. Taking $(L-S)$ with $p = 2$ and $f = g^{p/2}$ we have

$$\int d\mu\, g^p \ln g - \|g\|_p^p \cdot \ln \|g\|_p \leq \frac{2c}{p} \mathcal{E}(g^{p/2}; g^{p/2}) + \frac{2\gamma}{p} \cdot \|g\|_p^p$$

Applying lemma 4 ii) with $f = g^{p/2} \in D_+ \subset D(H_2) \subset D(\mathcal{E}) \equiv D(H^{1/2})$
$\alpha = 2/p \quad \beta = 2 - \alpha = 2\frac{(p-1)}{p}$ we obtain

$$(*) \qquad \int d\mu\, g^p \ln g - \|g\|_p^p \cdot \ln \|g\|_p \leq \frac{c \cdot p}{2(p-1)} \cdot \mathcal{E}(g; g^{p-1}) + \frac{2\gamma}{p} \|g\|_p^p$$

Since $g \in D \subset D(H_2)$ $\mathcal{E}(g; g^{p-1}) = \langle H_2 g; g^{p-1} \rangle$ and by coherence of the family $(H_p)_p$ we have $H_2 \restriction D = H_p \restriction D$ because $D \subset D(H_2) \subset$ $\subset D(H_p)$ Then $\langle H_2 g; g^{p-1} \rangle = \langle H_p g; g^{p-1} \rangle$

Taking (*) we have the thesis with $c(p) = \frac{c \cdot p}{2(p-1)}$ $\gamma(p) = \frac{2\gamma}{p}$.

Now we have to show that $g^{p/2} \in D(\mathcal{E})$. Applying lemma 4 i) to $P = e^{-tH_2}$

We have

$$\left\langle \frac{(1 - e^{-tH_2})}{t} \cdot g^{p/2}; g^{p/2} \right\rangle \leq \frac{p^2}{4(p-1)} \cdot \left\langle \frac{(1 - e^{-tH_2})}{t} g; g_p \right\rangle$$

Again by coherence of $(e^{-tH_p})_p$, since $g \in D \subset D(H_2) \cap D(H_p)$ we can change $e^{-tH_2} \cdot g$ with $e^{-tH_p} \cdot g$:

$$\left\langle \frac{(1-e^{-tH_2})}{t} g^{p/2}; g^{p/2} \right\rangle \leq \frac{p^2}{4(p-1)} \cdot \left\langle \frac{(1-e^{-tH_p})}{t} g; g_p \right\rangle$$

Now by definition of D(Hp) $\lim_{t\to} (r.h.s.)$ is equal to $\frac{p^2}{4(p-1)} \cdot \langle H_p g ; g_p \rangle$ Since $(l.h.s)$ is, as function of t, monotone decreasing, because is bounded, admits its limit as $t \to 0$. By definition of $D(\mathcal{E}) = D(H_2^{1/2})$: $g^{p/2} \in D(\mathcal{E})$. ///

Remark 5.

A Markovian semigroup e^{-tH} on $L^2(\mu)$ is said to be hypercontractive if $\exists T_0 > 0$ such that $e^{-T_0 H}$ is bounded from L^2 to L^4 . Using Stein's interpolation theorems is not difficult to show that an hypercontractive semigroup is a Sobolev generator on $(1;+\infty)$ with constants,for equal to:

$$c = 2T_0 \qquad \gamma = \frac{1}{T_0} \ln \| e^{-T_0 H} \|_{2,4}$$

Moreover $\gamma = 0$ if and only if $\| e^{-T_0 H} \|_{2,4} \leq 1$.

1.3 Log-Sobolev Inequalities for Dirichlet forms on $\Omega = \{+1;-1\}$.

We will see in Chapter 3 methods and example of Sobolev generator on many general spaces. Here we want to show that any probability mesure on $\Omega = \{+1;-1\}$ gives rise to a

Dirichlet form that is a "Sobolev generator" on $[2;+\infty)$

Definition 3.

Let Ω be a differentiable manifold or the space $\{+1;-1\}$. and ∇ the gradient operator or the "finite difference operator" $Bf \equiv (f_{|\omega=+1} - f_{|\omega=-1})\omega \quad \omega \in \{+1;-1\}$
If the form

$$\mathcal{E}(f;f) \equiv \int_{\Omega} d\mu \cdot |\nabla f|^2$$

$$D(\mathcal{E}) = \{ f \in L^2(\mu) : \mathcal{E}(f;f) < \infty \}$$

is a Dirichlet form satisfying (L-S) then we say that μ satisfy (L-S) or that μ is a Log-Sobolev mesure.

Proposition 6.

Any probability mesure μ on $\Omega = \{+1;-1\}$ is a Log-Sobolv mesure with zero local norm. Moreover if a family $(\mu_\alpha)_{\alpha \in I}$ of probability satisfy $0 < \inf_{\alpha \in I} \mu_\alpha(\{+1\})$

$\sup_{\alpha \in I} \mu_\alpha(\{+1\}) < 1$ then the Sobolev constant can be choosen indipendent on $\alpha \in I$.

Proof.

It's sufficient to prove (L-S) for the set of functions $f_s(\sigma) = 1 + s\sigma \quad s \in [-1;+1]$. Let $p \equiv \mu\{+1\}$ $q \equiv 1-p$ If $p \in \{0;1\}$ there is nothing to prove, so let $p \neq 0,1$. . We have

$$\mu f_s^2 = p(1+s)^2 + q(1-s)^2 \quad , \quad \mu f_s^2 \ln f_s = p(1+s)^2 \ln(1+s) + q(1-s)^2 \ln(1-s)$$

Defining for $s \in [-1;+1]$ $h(s) = \mu f_s^2 \ln f_s - (\mu f_s^2) \ln (\mu f_s^2)^{1/2}$
and observing that $\nabla f_s(\sigma) = s\sigma$, we have to show that exist a costant $C > 0$ such that $h(s) \le C s^2 \quad \forall s \in [-1;+1]$.

Now $h(0) = 0$

$$\frac{d}{ds} h(s) = 2p(1+s)\log(1+s) - 2q(1-s)\log(1-s) - [(p-q)+s]\cdot\log(1+2(p-q)s+s^2) \,,$$

for same $a > 0$ we have

$$|(1+s)\log(1+s)| \le (2\log 2)\cdot|s| \qquad |\log(1+2(p-q)s+s^2)| \le a|s|$$

and so $(L\text{-}S)$ holds. From last inequality we can see that $a > 0$ can be choose indipendent on $p_\alpha = \mu_\alpha\{+1\}$ if our hypoteses hold.

///

Remark 6.

In Ref.[G1] L. Gross proved $(L\text{-}S)$ for Gaussian mesure in R, starting from the result in Proposition 6 for the uniform mesure μ: $\mu\{\pm 1\} = 1/2$ and using an approximation processes.
We'll prove $(L\text{-}S)$ for Gaussian mesure in Chapter 3, as an application of the method of Bakry-Emery.
The fact that on $\Omega = \{+1;-1\}$ the Sobolev constant can be choosen indipendently with respect a certain set of mesures $(\mu_\alpha)_{\alpha \in I}$ if they weigh the set $\{+1\}$ in "uniform way" $0 < \inf_{\alpha \in I} \mu_\alpha\{+1\} \le$

$\leq \sup\limits_{\alpha \in I} \mu_\alpha\{+1\} < 1$, will be crucial for us in Part 2, when we'll investigate $(L-S)$ for mesure on $\{+1;-1\}^{\mathbb{Z}^d}$ $d \geq 1$.

CAPTER 2

LOGARITHMIC SOBOLEV INEQUALITIES, POINCARÉ INEQUALITIES AND SPECTRAL PROPERTIES

In this chapter we will be intersted in logarithmic Sobolev inequalities with zero local norm. Therefore from now on we will denote with $(L\text{-}S)$ an inequality of the form:

$$(L\text{-}S) \qquad \int d\mu\, f^2 \log \frac{f^2}{\|f\|^2} \leq \frac{2}{\alpha}\mathcal{E}(f;f) \quad f \in D(\mathcal{E}) \quad \alpha > 0 .$$

2.1 Log-Sobolev Inequalities toward Poincaré Inequalities

In Chapter 1 we have seen that $(L\text{-}S)$ is equivalent to a certain type of smoothness of the semigroup. In the next proposition we'll see that $(L\text{-}S)$ implies the Poincaré Inequality (or Spectral Gap Inequality) and also how all that is closely related to the property of the generator to have a gap above the infimum of his spectrum.

Shortly, we choose a probability space $(X\mathcal{B}\mu)$ and we'll consider a Markov semigroup on $L^2(X\mu)$ $P_t = e^{-tL}$ with generator L and Dirichlet form $\mathcal{E}(ff) = (f;Lf)$

We remark that $P_t 1 = 1$ implies $\sigma(L) \ni 0 = \inf \sigma(L) \quad (\sigma(L) \subset [0\ +\infty))$.

Proposition 1.

The following properties holds:

1) the semigroup is hypercontractive, i.e. exist $T_0 > 0$ such that $\| P_{T_0} \|_{2,4} < +\infty$, iff (L-S) holds and and the Sobolev costants are relate by $2T_0 = 1/\alpha$

2) L has a gap above zero in his spectrum, $\sigma(L) \subset \{0\} \cup [c\,;+\infty)$ for some , iff the Poincaré inequality is satisfied

(S-G) $\quad \| f - \langle f \rangle_\mu \|^2 \leq \frac{1}{c} \mathcal{E}(f;f) \quad f \in D(\mathcal{E})$

3) (S-G) is satisfied with costant C iff the following "exponental convergence to equilibrium (ECE) holds:

(ECE) $\quad \| P_t f - \langle f \rangle_\mu \| \leq e^{-ct} \cdot \| f - \langle f \rangle_\mu \| \quad f \in L^2(\mu)$

4) (L-S) with Sobolev costant $1/\alpha$ implies a gap in the spectrum of L: $\sigma(L) \subset \{0\} \cup [\alpha\,;+\infty)$

5) (L-S) implies (S-G) with $\alpha \leq c$

Proof.

1) See Remark after Proposition Prop 5 of Chapter 1

2) with the spectral mesure $dE_{f,f}(\lambda)$ $f \in D(L)$

we can write $\mathcal{E}(f;f) - c\| f - \langle f \rangle \|^2 = \int_{\sigma(L)} dE_{f,f}(\lambda)\left[\lambda - c(1 - \chi_{\{0\}}(\lambda))\right]$

The function $g(\lambda) = \lambda - c(1 - \chi_{\{0\}}(\lambda)) = \begin{cases} 0 & \lambda = 0 \\ \lambda - c & \lambda \neq 0 \end{cases}$

$E_{f,f}$-a.e. non negative $E_{f,f}\{(0;c)\} = 0$ and this implies (S-G) if there is a gap.

Viceversa if (S-G) holds, taking the infimum in each side over $\{ f \in D(L) : \|f\| = 1\ ,\ f \in 1^\perp \}$ we have $\sigma(L) \subset \{0\} \cup [c;+\infty)$.

3) if (S-G) is satisfied, then by 2) we have for the spectral mesure $E_{f,f}(\lambda)$ $f \in D(L)$ that $E_{f,f}\{(0;c)\}=0$. Representing $\|P_t f - \langle f\rangle\|^2$ as $\int dE_{f,f}(\lambda)\cdot(e^{-2t\lambda} - \chi_{\{0\}}(\lambda))$ and $\|f-\langle f\rangle\|^2$ as $\int dE_{f,f}(\lambda)(1-\chi_{\{0\}}(\lambda))$ we have that $e^{-2t\lambda} - \chi_{\{0\}}(\lambda) - e^{-2tc}\cdot(1-\chi_{\{0\}}(\lambda))$ is $E_{f,f}(\lambda)$a.e. non-negative and then $\|P_t f - \langle f\rangle\|^2 \leq e^{-2tc}\cdot \|f-\langle f\rangle\|^2$.

In order to prove the opposite is sufficient to derive (ECE) with respect to t.

4) with 1) we know that (L-S) holds iff $\|P_{T_0}\|_{2,4} \leq 1$ with $T_0 = 1/2\alpha$. For $f \in L^2(\mu)$ we can split as: $f = \gamma + g$ $g \in \{1\}^{\perp}, \gamma \in \mathbb{R}$. Then $\|P_{T_0}\|_4^4 = \gamma^4 + 4\gamma^3\cdot\langle 1; P_{T_0} g\rangle + 6\cdot\gamma^2\cdot\|P_{T_0} g\|_2^2 + O(\gamma) = \gamma^4 + 6\gamma^2\cdot\|P_{T_0} g\|_2^2 \leq$
$\leq \|f\|_2^4 = \gamma^4 + 2\cdot\gamma^2\cdot\|g\|_2^2 + \|g\|_2^4$

Taking γ large we have $\|P_{T_0} g\|_2 \leq 3^{-1/2}\cdot\|g\|_2$ and so $\sigma(L) \subset \{0\} \cup \left[\frac{\log 3}{2T_0}; +\infty\right)$

5) If (L-S) holds with Sobolev constant $1/\alpha$, then by 1) and 2) we have that (S-G) holds with $\alpha \leq c$.

Remark 1.

The name of "exponential convergence to equilibrium" (ECE) to the formula that appears in 3), refers to the applications to Statistical Mechanics (as we'll see in the next chapter) in which the equilibrium states of the statistical system (Gibbs states) can be seen as the states to whom a stochastic dynamics leads the system$(P_t f \to \langle f\rangle_\mu$ in $L^2(\mu))$. Here the problem is about the equivalence of this dynamics (see Ref. [H-S1]) . If (S-G) holds the convergence is exponentially fast.

2.2 The Laguerre Semigroup

In Ref. [K-S] the authors investigated the hyper-contractive property of the Laguerre semigroup. This is defined on the space $\Omega = (0;+\infty)$ with the probrability mesure $d\mu(x) \equiv e^{-x} \cdot dx$

The generator of the Markovian semigroup is taken to be:

$$L = -x \cdot \frac{d^2}{dx^2} - (1-x)\frac{d}{dx} \qquad \text{on} \quad D(L) \equiv C_c^\infty(\Omega)$$

The analysis is devoted to the calculation of the "best" constant for which (L-S) and (S-G) hold. In this example they are different and are equal to 2 and 1 respectively:

$$\text{(L-S)} \quad \int_0^\infty d\mu\, f^2 \cdot \log f \leq 2 \cdot \mathcal{E}(f;f) + (\mu f^2) \cdot \log (\mu f^2)^{1/2} \quad f \geq 0$$

$$\text{(S-G)} \quad \| f - \langle f \rangle_\mu \|^2_{L^2(\mu)} \leq \mathcal{E}(f f) \qquad f \in D(\mathcal{E}).$$

CHAPTER 3

THE METHODS OF BARKY-EMERY

In this chapter we start to show methods to prove Logarithmic Sobolev Inequality. Between the others, the method of Bakry and Emery had an increasing popularity. The reasons for this are at least two: first of all is simple in applicatins, and furthermore when works it furnish the best Sobolev constant. As we shall see is based on "convexity" arguments.

3.1 The general criterion of Bakry-Emery

The original proof (see Ref.[B-E1]) of Bakry-Emery, took in account the general form of (L-S), i.e. with local norm not necessarly zero. Because of the use of the method in Part 2, we give a recent proof of Deuschel that deals with the case where the local norm is zero.

Proposition 1.

Let Ω a Polish space and μ a probability mesure an Ω. We denote with $\mathbb{B}(\Omega)$ the space of bounded mesurable functions on Ω. Suppose L is the generator of a Markovian diffusion semigroup P_t on $L^2(\mu)$ and suppose that exist an algebra

$\cdot \mathcal{A} \subset \mathbb{B}(\Omega) \cap D(L)$ invariant under L, P_t and composition with functions, dense in L^2. Suppose also that $\forall f \in \mathcal{A}$

$$\lim_{t \to +\infty} P_t f = \int d\mu f .$$

Define the two bilinear forms an $\mathcal{A} \times \mathcal{A}$:

$$\Gamma(f;g) \equiv \frac{1}{2} \cdot \left[L(fg) - f \cdot Lg - g \cdot Lf \right]$$

$$\Gamma_2(f;g) \equiv \frac{1}{2} \left[L\Gamma(f;g) - \Gamma(Lf;g) - \Gamma(f;Lg) \right] \qquad f, g \in \mathcal{A}$$

If exist $\rho \in \mathbb{B}(\Omega)$ such that

$$\Gamma_2(f;f)^{(x)} \geq \frac{\rho(x)}{2} \cdot \Gamma(f;f)(x) \qquad \forall f \in \mathcal{A} \quad \forall x \in \Omega$$

and the Green fucntion G^ρ of the semigroup P_t^ρ with generatior $L-\rho$ such that $G^\rho 1$ is in $\mathbb{B}(\Omega)$, then (L-S) holds with Sobolev constant $c \leq \| G^\rho 1 \|_{\mathbb{B}(\Omega)} / 2$.

Proof.

Because P_t is a diffusion, L is a local operator and we have the following rules:

A) $L(\phi \circ f) = \phi' \circ f + \phi'' \circ f \cdot \Gamma(f;f)$

B) $\Gamma(\phi \circ f; g) = \phi' \circ f \cdot \Gamma(f;g)$ See Ref. [B-E1] .

By partial integration in A) we have $-\langle f; Lg \rangle_\mu = \langle \Gamma(f;g) \rangle_\mu$.

By theorems of Chapter 1, since P_t is Markovian, we have to prove (L-S) just on positive function $f \in \mathcal{A}$ with $\|f\|_2 = 1$.

For any such a function set $g_t \equiv P_t f^2$ and $\mathcal{M}_t \equiv \Gamma(g_t; \log g_t)$.

We then have by the mixing assumption

$$\lim_{t\to\infty} \langle g_t \cdot \log g_t \rangle_\mu = 0$$

Moreover since $-\frac{d}{dt}\langle g_t \log g_t\rangle_\mu = \langle (1+g_t)\, L g_t \rangle =$

$$= \langle \Gamma(g_t; \log g_t)\rangle_\mu = \langle u_t \rangle_\mu .$$

We have

$$\int_0^\infty dt \, \langle u_t\rangle_\mu = -\int_0^\infty dt \, \frac{d}{dt}\langle g_t \cdot \log g_t\rangle_\mu = \langle f^2 \cdot \log f^2 \rangle_\mu$$

Using B) we have $u_0 = \Gamma(f^2; \log f^2) = 4\Gamma(f;f)$. With this position we can re-write (L-S) in the following form

$$\int_0^\infty dt \, \langle u_t\rangle_\mu \leq 2c \, \langle u_0\rangle_\mu$$

From A) and B) it's easy to obtain the following rules:

$$\Gamma_2(g_t; \log g_t) = g_t \, \Gamma_2(\log g_t; \log g_t) + \Gamma(g_t; \Gamma(\log g_t; \log g_t))$$

$$\frac{d}{dt}\Gamma(g_t; \log g_t) = L\Gamma(g_t; \log g_t) - 2 g_t \, \Gamma(\log g_t; \log g_t)$$

$$\frac{d}{dt} u_t \leq L u_t - \rho u_t \; .$$

Hence $u_t \leq P_t^\rho u_0$. Finally:

$$\int_0^\infty dt \langle u_t\rangle_\mu \leq \int_0^\infty dt \langle P_t^\rho u_0\rangle_\mu = \int_0^\infty dt \, (P_t^\rho u_0; 1)_{L^2(\mu)} = \int_0^\infty dt \, (u_0; P_t^\rho 1)_{L^2(\mu)} =$$

$$= \int_0^\infty dt \int d\mu \; P_t^\rho 1 \cdot u_0 = \int d\mu \, u_0 \cdot \int_0^\infty dt \, P_t^\rho 1 = \int d\mu \, u_0 \cdot G^\rho 1 \leq \| G^\rho 1 \|_{\mathbb{B}(\Omega)} \cdot \langle u_0\rangle_\mu .$$

///

Remark.

The original "Γ_2-criterion" of Bakry-Emery is just a little different from the one above. Their condition was stronger, in the sense that they require $\rho \geq \varepsilon > 0$

for some constant ε .

It's easy to recover the Bakry-Emery condition using the Feynman-Kac formula for $P_t^\rho \mu_0(x)$:

$$\mu_t(x) \leq P_t^\rho \mu_0(x) = \int_{\mathcal{C}(\mathbb{R}^+;\Omega)} P_x(d\omega) \cdot e^{+\int_0^t ds\,[-\rho(X_s(\omega))]} \cdot \mu_0(X_t(\omega)) .$$

where $X_t : \mathcal{C}(\mathbb{R}^+;\Omega) \longrightarrow \Omega$ $X_t(\omega) = \omega(t)$ is the processes associated to P_t and P_x is the path-space mesure of the process conditionated to start from $x \in \Omega$. Now if $\rho \geq \varepsilon$, $-\rho \leq -\varepsilon$ $P_t^\rho 1(x) \leq e^{-\varepsilon t}$ $x \in \Omega$ and so $\| P_t^\rho 1 \|_{\mathbb{B}(\Omega)} \leq e^{-\varepsilon t}$ and $\| G^\rho 1 \|_{\mathbb{B}(\Omega)} \leq \frac{1}{\varepsilon} < +\infty$.

However is not clear have use the greater generality of proof of Deuschel (see Ref. [D 1]).

3.2 The criterion of Lichnerowicz and that one of Bakry-Emery on a Riemannian manifold

Since in application one deals often with operators on Riemannian manifold, is useful to investigated deeper how the Bakry and Emery appear in this situation. As we shall see the condition is more clear and its "convexity" character is evident.

Moreover in the proof one find out that the method is very close to that of Lichnerowitz for the estimate of the constant in the Poincaré inequality (see Ref. [DA1] for example).

Let Ω be a compact Riemannian manifold with natural normalized mesure Let $U \in C^\infty(\Omega)$ and consider the mesure

$$\mu^U \equiv \frac{\mu \cdot e^{-U}}{Z(U)} \qquad Z(U) = \int_\Omega d\mu \, e^{-U} .$$

With this mesure we contruct the Dirichlet form

$$\mathcal{E}^U(f;f) \equiv \int_\Omega d\mu \, |\nabla f|^2 \qquad f \in C^\infty(\Omega)$$

and the corrisponding operator

$$L^U f = \Delta f - \nabla U \cdot \nabla f \qquad f \in C^\infty(\Omega)$$

$$\mathcal{E}^U(f;f) = -(f; Lf)_{L^2(\mu)}$$

(For the closability problem see Ref. [A-Rö]). These are the operators on which we concentrate our attention.

The method of Lichnèrowicz for the Paincaré inequality

$$(S\text{-}G) \qquad \| f - \langle f \rangle \|^2 \leq \alpha \cdot \mathcal{E}(f;f)$$

starts with the observation that (S-G) is equivalent to the following formula (by Spectral Theorem):

$$\left\langle \| \nabla f \|^2 \right\rangle_{\mu^U} \leq \alpha \cdot \| L^U f \|^2_{L^2(\mu)}$$

and with the Bochner-Weitzenböck formula

$$\Gamma^U(f;f) \equiv \frac{1}{2}\left[L^U |\nabla f|^2 - 2 \nabla f \cdot \nabla L^U f \right] = \| \mathrm{Hess} f \|^2 + (\mathrm{Ric} + \mathrm{Hess}\, U)(\nabla f; \nabla f)$$

where Hess is the Hessian of f and Ric is the Ricci's curvature tensor. An integration by parts gives us (recall $L^U 1 = 0 \Rightarrow \langle L^U f\rangle_\mu = 0$):

$$\left\langle |L^U f|^2 \right\rangle_{\mu^U} = \left\langle \Gamma^U(f;f)\right\rangle_{\mu^U} = \left\langle \|\mathrm{Hess} f\|^2 + (\mathrm{Ric} + \mathrm{Hess}\, U)(\nabla f;\nabla f)\right\rangle_{\mu^U}$$

So we can caracterize in (S-G) as the constant for which

$$\left\langle \|\nabla f\|^2\right\rangle_{\mu^U} \leq \alpha \left\langle \|\mathrm{Hess} f\|^2 + (\mathrm{Ric} + \mathrm{Hess}\, U)(\nabla f;\nabla f)\right\rangle_{\mu^U}$$

This gives us the following estimate for α:

$$\alpha \leq \frac{1}{\rho(U)} \qquad \rho(U) \equiv \sup\left\{\rho\in\mathbb{R} : (\mathrm{Ric}+\mathrm{Hess}\,U)(X;X) \geq \rho\|X\|^2 \quad X\in\mathcal{X}(\Omega)\right\} .$$

We shall see in next proposition that $\rho(U)$ gives us also an estimate of c in (L-S): $c \leq \frac{1}{\rho(U)}$.

Proposition 2

We the above notation, if $\rho(U) > 0$ then (L-S) holds with $c \leq \frac{1}{\rho(U)}$

(LS) $$\int d\mu f^2 \log f \leq \frac{1}{\rho(U)} \cdot \mathcal{E}(ff) + \|f\|_2^2 \cdot \log \|f\|_2$$

Proof.

First of all we note that (S-G) is equivalent to

$$\frac{d^2}{dt^2} \cdot \left\langle (P_t^U f)^2\right\rangle_{\mu^U} \leq -2 \cdot \frac{1}{\alpha}\frac{d}{dt}\left\langle (P_t^U f)^2\right\rangle_{\mu^U}$$

Now we see that a corrisponding expression holds for (L-S) changing $\left\langle (P_t^U f)^2\right\rangle_{\mu^U}$ with $H(t) \equiv \langle f_t \log f_t\rangle_{\mu^U}$ where $f_t \equiv P_t^U f^2$ with $\|f\|_2 = 1$. In fact $\dot H(t) = -4\,\mathcal{E}(f;f) = -\left\langle \frac{\|\nabla f_t\|^2}{f_t}\right\rangle_{\mu^U}$

and so (L-S) becames

(L-S)' $$H(0) \leq -\frac{c}{2}\dot{H}(0).$$

Since $H(t) \to 0$ $t \to 0$ because $P_t^U f^2 \to \langle f^2 \rangle_{\mu^U} = 1$, (L-S) comes from $-\ddot{H}(t) \leq \frac{2}{c}\dot{H}(t)$. Now expressing $H(t)$ as a function of Γ^U and using the Bochner-Weitzenböck formula we have that

$$-\dot{H}(t) = \langle \nabla \log f_t \cdot \nabla f_t \rangle_{\mu^U}$$

$$+\ddot{H}(t) = +2 \left\langle f_t \cdot \Gamma^U (\log f_t ; \log f_t) \right\rangle_{\mu^U}$$

Finally (L-S)' is implied by

$$\left\langle \frac{\|\nabla f\|^2}{f^2} \right\rangle_{\mu^U} \leq c \left\langle f \cdot \left[\mathrm{Hess} f + (\mathrm{Ric} + \mathrm{Hess}\, U)(\nabla f ; \nabla f) \right] \right\rangle_{\mu^U}$$

From the definition of $\rho(U)$ we extract that $c \leq \frac{1}{\rho(U)}$.

///

Remark 1.

From the proof it's clear that the compactness hypoteses of the manifold can be replaced by the following weaker one:

$$e^{-U} \in L^1(\mu).$$

Remark 2.

In a recent paper [D-S1], Deuschel and Strook proved long the lines of the proof of Proposition 2, a better estimate for the Sobolev constant, i.e. for the quantity

$$\rho(U): \quad \frac{1}{c} \geq \left[\frac{e^{-\delta(U)} \cdot c(0)}{N} + \rho(U) \right] \vee \left[\frac{3c(0) \cdot e^{-\delta(U)} + N\rho\left(\frac{N+2}{N} U\right) + 2\left(1 - e^{-\delta(U)}\right)(\rho(0) \wedge 0)}{N+2} \right]$$

where $\delta(U) \equiv \sup U - \inf U \qquad U \in C^{\infty}(\Omega)$ bounded.

3.3 Logaritmic Sobolev Inequalities for the Gaussian Mesure on and Nelson's hypercontractive estimates

The first calculation of the Sobolev constant for the Gaussian mesure $(2\pi)^{-1} dx\, e^{-x^2/2}$ on $\mathbb{R}$ was given by Gross in [G1]. The author used the central limit theorem and the calculation of the Sobolev constant for the uniform mesure on $\{+1; -1\}$ to prove that

$$(2\pi)^{-1} \cdot \int dx\, e^{-x^2/2} f^2(x) \log f(x) - \|f\|_2^2 \log \|f\|_2 \leq 1 \cdot (2\pi)^{-1} \int dx\, e^{-x^2/2} \cdot |f'|^2$$

this result and the use of Proposition 5 of Chapter 1, permits to deduce the result of Nelson (see [N2]) about the hypercontractivity of the Orstein-Uhlenbeck semigroup, whose generator is:

$$L = -\frac{d^2}{dx^2} + x \frac{d}{dx} \qquad \text{on} \quad C_c^{\infty}(\mathbb{R}) \subset L^2\left((2\pi)^{-1} e^{-x^2/2} \cdot dx\right).$$

In particular the (L-S) above and Proposition 5 of chapter 1 gives the Sobolev constant on L^p $p \geq 1$: $c(p) = \frac{p}{2(p-1)}$. Now by Proposition 2 of Chapter 1 we calculate:

$$\begin{cases} c(p)\dfrac{dp}{dt} = p \\ p(0;q) = q \end{cases} \quad \begin{cases} \dfrac{p}{2(p-1)}\cdot\dfrac{dp}{dt} = p \\ p(0;q) = q \end{cases} \quad \begin{cases} \dfrac{d}{dt}\left(\log(p-1)\right) = 2 \\ p(0;q) = q \end{cases} \quad \begin{cases} p-1 = \mathrm{Cost.}\ e^{2t} \\ p(0;q) = q \end{cases}$$

$$p(t;q) = (q-1)e^{+2t} + 1$$

And finally we obtain the result of Nelson

$$\|e^{-tL}f\|_p \leq \|f\|_q$$

$$e^{-t} \leqslant \left(\frac{q-1}{p-1}\right)^{1/2} \qquad q, p \geqslant 1 .$$

For $q=2$ and $p=4$ we have $e^{-t} \leq 3^{-1/2} \iff t \geqslant \log 3/2$

So if $t \geqslant T_0 \equiv \log 3/2$ e^{-tL} is a contraction from L^2 to L^4, and by Proposition 1 of chapter 2, we have that the gap in the spectrum of L is at least 1. B. Simon proved in [S3] that the gap is exatly 1 (this is the reason why people refers to the result as "Nelson's best hypercontractive estimates").

3.4 Log-Sobolev Inequalities for Gaussian Mesures

Proposition 2 of this chapter easely applies to the case of Gaussian mesure on $\mathbb{R}^d$ $d \geqslant 1$.

Proposition 3.

Let's consider the Gaussian Mesures on given by the positive definite matrix $G \in M^{d\times d}(\mathbb{R})$

$$d\mu^G(x) = \frac{1}{Z}\, dx\, e^{-\frac{1}{2}(x;Gx)}$$

dx being the Lebesgue mesure on $\mathbb{R}^d$.

Then μ^G is a Log-Sobolev mesure with local norm zero and Sobolev constant given by:

$$c \leq (\inf \text{Spectrum } G)^{-1}.$$

Proof.

Here $Ric = 0$ and $U = \frac{1}{2}(\cdot;G\cdot)$. So applying Proposition 2 we calculate $\rho(U) = \sup\{\rho \in \mathbb{R} : (x;Gx) \geq \rho \|x\|^2 \ x \in \mathbb{R}^d\} = \inf \text{Spectrum } G$. ///

3.5 Log-Sobolev inequalities for the Riemannian Mesures on spheres S^d $d \geq 2$.

This was the first application of the theorem of Bakry-Emery, showing, between other things, the simplicity of the methods. We consider the sphere S^d $d \geq 2$ and the associated normalized Riemannian mesure

Proposition 4.

The mesure μ^d satisfy (L-S) with zero local norm and Sobolev constant given by $c^d = (d-1)^{-1}$

Proof

We apply the arguments of Section 3.2 to the case $V = 0$. Moreover on S^d $d \geq 2$ $Ric = (d-1)g$ where g is the Riemannian metric. This gives us: $\rho(0) = (d-1)$.

///

Remark 3.

The method of Bakry-Emery does not applies to circle This case has been treated Weissler in Ref. [W].

CHAPTER 4

THE "ROSEN'S LEMMA"

The Rosen's lemma furnish a criterion for the "intrisic hypercontractivity" of strongly continous semigroups. To be clear we start with the following:

Def. (Intrinsic hypercontractivity and "Ground State Representation").

Let $(\Omega, \mathcal{M}, \mu)$ be a mesure space with positive mesure μ, and e^{-tH} a strongly continous semigroup on $L^2(\Omega, \mathcal{M}, \mu)$ with selfadjoint generator H. If $\phi : \Omega \to \mathbb{R}$ is a mesurable function on Ω and $\phi \in L^1_{loc}(\mu)$, we'll consider the space $L^2(\Omega, \mathcal{M}, \phi^2 d\mu)$ and the unitary transformation

$$U_\phi : L^2(\mu) \longrightarrow L^2(\phi^2 \mu) \qquad U_\phi(f) \equiv \phi^{-1} f \qquad f \in L^2(\mu).$$

We'll say that e^{-tH} (or H) in "intrisically hypercontractive" if the operator $H_\phi \equiv U_\phi \circ H \circ U_\phi^*$ defined on $L^2(\phi^2 \mu)$ is a Dirichlet operator and a Sobolev Generator of index 2.

The operator U_ϕ will be called "ground state transformation" and the operator H_ϕ "ground state representation of H".

Remark 1.

The language of Def. 1 is obviously lended from the fact that in main application H will be a Schrödinger on $\mathbb{R}^d$ and ϕ

will be its "ground state" (if exist), i.e. the eigenvector corrisponding to the eigenvalue $E = \inf \sigma(H)$.

Remark 2.

To discuss the Rosen's lemma we are forced to assume (as we did in Def. 1) H_ϕ to be a Dirichlet operator. However the problem of intrinsic hypercontractivity can be studied in general.

Proposition 1 - (Rosen's lemma).

We consider on the mesure space $(\Omega \mathcal{M} \mu)$ with positive mesure μ, two selfadjoint operators, defined respectively on $L^2(\mu)$ and on $L^2(\phi^2\mu)$, and related by the ground state representation specified by the function $\phi : \Omega \to \mathbb{R}$ in $L^1_{loc}(\mu)$ a.e. positive. Suppose that:

i) H_ϕ is a Dirichlet operator

ii) $\exists\, \alpha, \delta, \mu \in \mathbb{R}$ with $\mu > 2$ such that $\forall g \in L^{\mu/2}(\mu)$ $\|g\|_{\mu/2} = 1$ as quadratic forms, the following inequality holds $g \le \alpha(H+\delta)$

iii) $\exists\, c' > 0$, and $\gamma' \in \mathbb{R}$ such that the following inequality holds

$$-\log \phi \le c' H + \gamma'$$

Then H_ϕ is a Sobolev generator of index 2 with constants $c = c' + \alpha \cdot \beta^{4/\mu} \cdot b^{2/\mu}$ $\quad \gamma = \gamma' - \log \beta + \alpha \cdot \beta^{4/\mu} \cdot b^{2/\mu}$ $\quad \forall \beta > 0$.

Proof.

The idea of the proof is very simple and consist to note that if, as quadratic forms, on $L^2(\phi^2\mu)$, $\log \leq c\, H\phi + \gamma$ is true $\forall f \in L^2(\phi^2\mu)$ positive with norm 1, then (L-S) for $H\phi$ sudden follows, just taking the expetation on $f \in D_+(\mathcal{E}\phi)$ ($\mathcal{E}_\phi$ being the Dirichlet form associated to $H\phi$).
On the other hand, using the "ground state transformation" U_ϕ the inequality $\log f \leq c\, H\phi + \gamma$ $f \in L^2(\phi^2\mu)$ is equivalent to an inequality on $L^2(\mu)$ of the same type

$$\log f \leq cH + \gamma \qquad f \in L^2(\phi^2\mu) \quad f \geq 0 \quad \| f \|_{L^2(\phi^2\mu)} = 1 .$$

Hypoteses i) and ii) are used just to verify this last inequality. Let's note that $\forall s \geq 1$ (exist $b > 0$ (it depends on μ) such that

$$(\log s)^{\mu/2} \leq b\, s^2 \quad \forall s \geq 1$$

Let $\beta > 0$ $f \in L^2_+(\phi^2\mu)$ $\|f\| = 1$ and define

$$\chi = \begin{cases} 1 & \text{if } \beta f \phi \geq 1 \\ 0 & \text{if } \beta f \phi < 1 \end{cases}$$

Where χ is zero $\beta f \phi < 1$ and then $\log(\beta f \phi) < 0$. Hence we have that $\log(\beta f \phi) \leq \chi . \log(\beta f \phi)$. Moreover $\int d\mu |\chi \log(\beta f \phi)|^{\mu/2} =$

$$= \int d\mu\, \chi \cdot (\log(\beta f \phi))^{\mu/2} \leq \int d\mu\, \chi . b . (\beta f \phi)^2 = b\beta^2 . \int d\mu\, \chi\, f^2 \phi^2 \leq b\beta^2 \int d\mu\, f^2 \phi^2 = b\beta^2 . \|f\|^2 = b . \beta^2 .$$

(Note that χ depends on β !).

So $\chi \log(\beta f \phi) \in L^{\mu/2}(\mu)$, $\|\chi \log(\beta f\phi)\|_{\mu/2} \leq (b\beta^2)^{2/\mu}$.

By hypotesis ii)

$$\log(\beta f \phi) \leq \chi \cdot \log(\beta f\phi) \leq \alpha \|\chi \log(\beta f\phi)\|_{\mu/2} \cdot (H+\delta) \leq \alpha \cdot (b\beta^2)^{\frac{2}{\mu}} \cdot (H+\delta).$$

Hence by hypotesis iii)

$$\log f \leq \alpha \beta^{\frac{4}{\mu}} \cdot b^{2/\mu} \cdot (H+\delta) - \log\beta + c' \cdot H + \gamma' \equiv cH + \gamma \qquad \forall f \in L^2_+(\phi^2\mu) \ \|f\|=1$$

$$c \equiv c' + \alpha \cdot \beta^{4/\mu} \cdot b^{2/\mu} \qquad \gamma = \gamma' - \log\beta + \alpha \cdot \beta^{4/\mu} \cdot b^{2/\mu}.$$

///

Remark 1.

In principle it's possibile, acting on the free parameter $\beta > 0$ to reduce γ (increasing). If the parameters α, b and μ are such that $\exists\, \beta > 0$ for which $\gamma = 0$, we could say that for H_ϕ, may-be increasing the Sobolev coefficient c, the local norm can be taken to be zero. This property of certain Sobolev generators is strictly related, as we have seen in previous chapters, with a gap in the spectrum of H_ϕ (and H).

Remark 2.

The criterion of Rosen makes sure the hypercontractivity of e^{-tH_ϕ} in terms of the operator H (hypotesis iii).

CHAPTER 5

APPLICATIONS TO SCHRODINGER OPERATORS

5.1 Introduction

From now on Ω represents an open set in $\mathbb{R}^d$, M its Lebesgue σ-algebra, and $d\mu = dx$ the Lebesgue mesure.
In order to apply the Rosen's criterion to the Schrödinger operator $H = H_0 + V$ $\left(H_0 = -\Delta = -\sum_{i=1}^{d} \partial_i^2\right)$ we must, first of all, make us sure that the potential V is such that $E \equiv \inf \sigma(H)$ as eigenvalue and in a way that the "ground state representation" carry us to a Dirichlet operator H_ϕ (hypotesis i) in Proposition 1 of Chapter 4). One we made this, we'll show criteria that enable us to verify hypotesis ii) and iii) of the Rosen's lemma.
Now we want to note that, in what we said about the Schrödinger operators we are dealing with, are implicit the classical methods of construction of "good operators". For example when the potential V is in the Kato class (i.e. $V_- \in K_d$, $V_+ \in K_d^{loc}$ $V = V_+ - V_-$) it's possible to define $H_0 + V$ as the operator corrisponding to the closed form constructed by the sum of the forms of H_0 and V. (see Ref. [S 1]). Alternatively it's possible to construct decent operators (closable and lower bounded) using the Feyuman-Kac formula (see Ref. [S1] page 459 and the article of McKean quoted therein). By the way we note that these methods are equivalent for potential in the Kato's

class.

Recently the "Dirichlet approach" to the Quantum Mechanics, appeared. This approach (originated from the functional point of view in Quantum Fiel Theory: see Ref. [A-H-K].) concentrates the attention on the Dirichlet form $\mathcal{E}_\phi$ corrisponding to H_ϕ, and then meets the Schödinger operator H, or its form, by an invers ground state representation. The simplicity of this method resides on the fact that one can use "workable" closability criteria for quadratic forms as $\mathcal{E}_\phi(f;f) = \int_\Omega dx\, \phi^2 |\nabla f|^2$ (see Ref. [A-Rö]). This permits to focus all the attention on the "ground state mesure" $d\mu = dx \cdot \phi^2$.

The success of this approach resides on the fact that it enable us to construct perturbations of the free hamiltonian H_0, that are concentrated on zero mesure sets, and so a whole series of Schrödinger operators with "singular potential". With respect to this approach, the method of Rosen should marry in a nice way, at least expressing the condition iii) $-\log\phi \leq c' \cdot H + \gamma'$ in terms of ϕ, H_ϕ or $\mathcal{E}_\phi$.

Coming back to the problem of justification of the "ground state representation "we can say that the Kato's class Kd, represents for the potentials V a sufficient setting in which the "grand state representation" has the required properties.

Obviously in such a general setting we cannot assume the existence of the "ground state" . This problem will be solved in smaller class of potentials.

5.2 Criteria for the Rosen's lemma

In this section we summarize, without proofs, some properties of the Schrödinger operators with Kato's potentials. (see Ref. [51]).

Proposition 1.

Suppone $H = H_0 + V$ $H_0 = -\Delta$ on $\mathbb{R}^d$ $V \equiv V_+ - V_-$ $V_\pm \geq 0$ such that $V_+ \in K_d^{loc}$ $V_- \in K_d$. Then:

1) the form $H_0 + V$ is closed on $D(\mathcal{E}(H_0)) \cap D(\mathcal{E}(V_+))$ and also lower bounded

2) for e^{-tH} the Feyuman-Kac formula holds:

$$\left(e^{-tH} f\right)(x) = E_x\left(\exp\left(-\int_0^t ds\, V(\omega(s))\right) \cdot f(\omega(t))\right)$$

3) $C_0^\infty(\mathbb{R}^d)$ is a form-core for H

4) Sobolev estimates hold: $\alpha > 0, p \leq q, \frac{1}{p} - \frac{1}{q} < \frac{2\alpha}{d}, \operatorname{Re} z < \inf \sigma(H) \Rightarrow \|(H-z)^{-\alpha}\|_{p,q} < +\infty$

5) if $V \in L^2_{loc}$ $C_0^\infty(\mathbb{R}^d)$ is an operator care for H

6) ("L^p -smoothing"): $t > 0, p \leq q \Rightarrow$ e^{-tH} is bounded from L^p to L^q

7) $p \in [1, +\infty], f \in L^p \Rightarrow$ $e^{-tH} f$ is a continous function

8) $p \in [1, +\infty], f \in L^p \Rightarrow$ $\nabla e^{-tH} f \in L^2_{loc}$ (in distributional sense)

9) ("Harnack's inequalities") $\forall\, Hu = Eu$ $E \in \mathbb{R}$ with u continous and non-negative, given $\Omega \subset \mathbb{R}^d$ open, $x, y \in \Omega$

$\exists\ c$ such that $u(x) \leq c\, u(y)$

10) if $Hu = Eu$ $u \in L^2$ $E \in \mathbb{R}$ then $u(x) \to 0$ $x \to \infty$.

///

From these properties we can deduce that if the ground state ϕ exist : $H\phi = E\phi$ $E = \inf \sigma(H)$ $\phi \in L^2$ then ϕ is continous, positive, bounded, locally bounded away from zero with derivatives in L^2_{loc}. This properties are enough to prove that if the ground state exist then $\mathcal{E}_\phi$ (and H_ϕ) has all good properties required above.

Proposition 2.

Let $H = H_0 + V$ a Schrödinger operator on $\mathbb{R}^d$, with potential $V = V_+ - V$ such that $V_+ \in K_d^{loc}$ $V_- \in K_d$ Suppose that $E = \inf \sigma(H)$ is an eigenvalue with eigenfunction $\phi \in L^2$ and consider the "ground state transformation"

$$U_\phi : L^2(dx) \longrightarrow L^2(\phi^2 dx) \qquad U_\phi(f) \equiv \phi^{-1} f \quad f \in L^p(dx)$$

and the form $\mathcal{E}_\phi$ associated to an operator H_ϕ, equivalent, by U_ϕ, to $\mathcal{E}$ and H respectively. Then

i) $D(\mathcal{E}_\phi) = \{ f \in L^2(\phi^2 dx) : \nabla f \in L^2(\phi^2 dx)$

ii) $\mathcal{E}_\phi(f, f) = (H_\phi f, f)_{L^2(\phi^2 dx)} = \int dx\, \phi^2 |\nabla f|^2 \qquad f \in D(\mathcal{E}_\phi)$

iii) $C_0^\infty(\mathbb{R}^d)$ is a form core for $\mathcal{E}_\phi$.

Proof.

See Ref. [D-S], [C]. The closability of $\mathcal{E}_\phi$ is given by the criterion of [Rö-W] pag. 129, applied to $\rho=\phi$ since $\phi>0$ is continous.

Remark.

In the article [C] of Carmona a different class of potentials is considered, that is almost all included in the class of potentials we considered. Carmona choose $V=V_+-V_-$ with $V_-\in L^p$ for some $p>\max(1;d/2)$ and $V_+\in L^{d/2}_{loc}$. But with the notation of [S] pag. 456, $L^p\subset L^p_u\subset K_d$ if $p>\max(1;d/2)$ Moreover $L^q_{loc}\subset K^{loc}_d$ if $q>\max(1;d/2)$. Next step deals with condition ii) of the Rosen's lemma. In the next proposition we'll see that for Kato's potentials, that condition is always verified.

Proposition 3.

Let $H=H_0+V$ with $V=V_+-V_-$, $V_+\in K_d^{loc}$, $V_-\in K_d$ Then there exist constants $\alpha, \delta, \mu>2$ such that

$$g\in L^{\mu/2}(\phi^2 dx) \quad \|g\|_{\mu/2}=1 \implies g\leq \alpha(H+\delta)$$ (as forms)

Proof.

The proof is based on the Sobolev's estimates for $H_0=-\Delta$

and on the fact that if V is as in hypotesis, then $H_0 \leq 2H + K$ (see Ref.[D-S] page 356, Ref.[S 1] pag. 458).

To satisfy the condition iii) of the Rosen lemma $-\log\phi \leq c' H + \gamma'$ we have to restrict the class of potentials. Next proposition exhibit a general criterion "well shaped" for a class of potentials lower polynomially bounded.
The proof is based on the following lemma and on the "sub-harmonic comparison inequalities".

Lemma 1.

With previous notations, suppose that the ground state ϕ exist and that is C^2. We can suppose $E = \inf \sigma(H) = 0$. Suppose exist a function $W: \mathbb{R}^d \to \mathbb{R}$ of class C^2 such that:

i) $W \leq c' H + \gamma'$ with $c', \gamma' \in \mathbb{R}$
ii) $W(x) \to \infty \quad x \to \infty$
iii) $|\nabla W|^2 - \Delta W \geq V_+$ outside a compact subset $K \subset \mathbb{R}^d$
iv) $e^{-W} \leq \phi$ on K

Then:

$$-\log\phi \leq c' H + \gamma'$$

Proof.

First of all consider a region $\Omega \subset \mathbb{R}^d$ and functions C^2 $W', \psi, \phi' : \Omega \to \mathbb{R}$ such that $\psi \leq \phi'$ on $\partial\Omega \cup \{\infty\}$, $-\Delta\phi' + V\phi' \geq 0$ on Ω, $-\Delta\psi + W'\psi \leq 0$.

an Ω and $V_+ \leq W'$ $W' \geq 0$ on Ω. We prove than that $\psi \leq \phi'$ on the whole Ω.

Fix $h = \psi - \phi \in C^2(\Omega)$ and consider $x \in \Omega^+ = \{y \in \Omega : h(y) > 0\}$, by our hypoteses we have $\Delta h(x) \equiv \Delta\psi - \Delta\phi' \geq$ (since $-\Delta\psi + W\psi \leq 0$ on Ω) $\geq$

$\geq$ (since $-V_+ \geq -W$ on Ω) $\geq W\psi - W\phi' + V - \phi' \geq$ $(V_- \geq 0)$ $\geq W(\psi - \phi) =$

$= Wh$.

Hence if $x \in \Omega^+$ then $\Delta h(x) \geq Wh(x) > 0$. So h is subharmonic in Ω^+ and obviously on $\partial\Omega^+$ is zero. From this follows that $h \leq 0$ on the whole Ω^+. By the definition of Ω^+ we have $\Omega^+ = \phi$.

Now we can start to prove the lemma. Fix $\Omega = K^c$, that will be an open subset since is compact. Let's define $\psi \equiv e^{-W}$ $X \equiv |\nabla W|^2 - \Delta W$. By our hypothesis, $X \geq V_+$ on Ω, $\psi \leq \phi$ on $\partial\Omega$ and $-\Delta\phi + V\phi = 0$. By construction we have $-\Delta\psi + X\psi = 0$ on Ω. Since ψ and ϕ are C^2, we can deduce, applying the above reasoning with $\phi' = \phi$ and $W' = X$, that $\psi \equiv e^{-W} \leq \phi$ on Ω and hence on the whole $\mathbb{R}^d$

Hence

$$-\log\phi \leq W \leq c' \cdot H + \gamma' \qquad ///$$

We want now recall that if the potential V is continous and diverge to the infinite, then ther exist the ground stte and in C^2. (see Ref. [D1] page 120 and Ref. [S2] page 56).

Proposition 4 - (Intrinsic hypercontractivity for Schödinger operators with potential with polynomial grow).

Let $V:\mathbb{R}^d \to \mathbb{R}$ continous, subject to the following condition:

$$c_1 \cdot |x|^{a_1} - c_2 \leq V(x) - E \leq c_3 \cdot |x|^{a_2} + c_4 \qquad \forall x \in \mathbb{R}^d \quad d > 2$$

$$a_i > 2 \ , \quad c_i > 0 \quad / \quad a_2 < 2a_1 - 2 \ .$$

where $E = \inf \sigma(H_0 + V)$ is the lowest eigenvalue with ergenvector $\phi \in L^2$.

Then $H = H_0 + V$ is intrinsically hypercontractive.

Proof.

We have already seen that for continous divergent potentials the ground state exist. The potentials in hypoteses are clearly of the type $V = V_+ - V_-$ with $V_+ \in K_d^{loc}$, $V_- \in K_d$. Hence in order to verify the intrinsic hypercontractivity we may deduce, first of all (using Proposition 3) that the problem is well posed, in the sense that $(\mathcal{E}; D(\mathcal{E}))$ is a Dirichlet form on $L^2(\phi^2 dx)$. Since our potentials are in the Kato's class, we can say (using Proposition 4) that $\exists \mu > 2, \alpha, \delta$ such that

$$g \leq \alpha (H + \delta) \qquad \forall g \in L^{\mu/2}(dx) \qquad \|g\|_{\mu/2} = 1$$

To verify the condition $-\log \phi \leq c' H + \gamma'$, we apply lemma 1 with the function $W: \mathbb{R}^d \to \mathbb{R}$ defined as

$$W(x) \equiv |x|^a \qquad a_2 < 2a - 2 < 2a_1 - 2$$

It's easy to verify that $s^{a} \leq c'' \cdot s^{a_1} + c_6 \cdot c''^{-a(a_1-a)}$, and then fixing $c'' = c' \cdot c_1$ $c' > 0$ we have

$$W(x) = |x|^{a} \leq c'(c_1|x|^{a_1} - c_2) + c_2 \cdot c' + c_6 \cdot (c_1 \cdot c'')^{-a(a_1-a)} \leq (Hp\ on\ V\ c_5 = c_6 \cdot c_1) \leq$$

$$\leq c'(V-E) + c_2 \cdot c' + c_5 \cdot c'^{-a(a_1-a)} \leq (-\Delta \geq 0) \leq$$

$$\leq c' \cdot (H-E) + c_2 \cdot c' + c_5 \cdot c'^{-a(a_1-a)} = \left(\gamma' \equiv c_2 \cdot c' + c_5 \cdot c'^{-a(a_1-a)}\right) =$$

$$= c'(H-E) + \gamma'$$

From the above calculation we see that the hypotesis i) of lemma 1 is verified.

It's not difficult to show that $|\nabla W|^2 - \Delta W = a^2 \cdot |x|^{2a-2} - a(a+d-2) \cdot |x|^{a-2}$ $\forall x \in \mathbb{R}^d$ Since $2a-2 > a_2$ and $V-E \leq c_3 \cdot |x|^{a_2} + c_4$ we have that $\exists R > 0$ such that

$$\forall x \ / \ |x| > R \quad ; \quad |\nabla W|^2 - \Delta W \geq (V-E)_+$$

With $K \equiv \bar{D}(0;R)$ hypotesis iii) of lemma 1 is verified. To verify hypotesis iv) of the same lemma, we can take advantage from Harnack's inequality in [S1] (page 493) and admit that there exist a constant $k > 0$ such that $e^{-W} \leq e^{k}\phi$ on $K \equiv \bar{D}(0;R)$.

///

Remark 1

The possibility to act on the constants c_2 and C_4, in the hypotesis of Proposition tell us that is the behavior at infinity of the potential V that implies intrisic hupercontractivity.

Remark 2.

The Proposition 4 does not cover the case $V(x) \cong X^2$ since a_1 must be greater than 2. However is well known that in this case the ground state exist and is proportional to $e^{-X^2/2}$
We can recognize the case treated in the previous chapter.

Remark 3.

Proposition 4 with its condition on the potential V, makes more clear which characteristics of the potential V give intrinsic hypercontractivity. In particular the condition $a_2 < 2a_1 - 2$ tell us that the grow at infinity must be regular in some sense.
We finish the discussion about the intrinsic hypercontractivity with the following proposition that shows that the case $V(x) = X^2$ is on the borderline for intrinsic hypercontracitivity.

Proposition 5.

Let $H = -\Delta + V$ on $L^2(R^d)$ where $V \in L^1_{loc}$ is bounded below. Then if the ground state exist, is in $L^2(R^d)$ and is positive and

e^{-tH} is intrisically hypercontractive the there exist constants $\gamma > 0$, $\beta \in \mathbb{R}$ such that

$$H \geq \gamma \cdot X^2 - \beta \qquad \text{(as forms)}$$

For the proof see [DA1] page 125. ///

5.3 A class of potentials with local singularities

Now we know that is the grow at infinite of the potential that implies intrisic hypercontractivity. However it's interesting to show situations in which potentials with local singularities are intrinsically hypercontractive.
The method is based on comparison between the (L-S) we have to prove and that one of the Gaussian case.
The class of potential is the following one:

Definition 1.

Let $\mathcal{V}^{(d)}$ the set of functions V: $\mathbb{R}^d - \{0\} \longrightarrow \mathbb{R}$ such that:

i) $V(x) \equiv V_R(|x|)$ $x \in \mathbb{R}^d - \{0\}$ $V_R : (0; +\infty) \longrightarrow \mathbb{R}$

ii) $V_R \in C^\infty(0; +\infty)$

iii) a) $\exists p > d/2$, $p > 1$ such that $V_- \in L^p(\mathbb{R}^d; dx)$

b) $\overline{\lim}_{r \to 0} V_R(r) < +\infty$

c) V_R is monotone in a neighborhood of zero

iv) a) V_R is positive at infinity

b) at infinity $V_R' / V_R^{1/2}$ is positive and definitively different from zero

c) V_R'/V_R is uniformely bounded at infinity

v) admits a finite number of zero.

We note that $V_- \in L^p \subset K_d$ since $p > \max(1; d/2)$ and that, by the continuity of V_R, $V_+ \in K_d^{loc}$ Then we can apply the previous general results. Obviously, also in this case, is basic to know that

Lemma 2.

If $V \in \mathcal{V}^{(d)} d \geq 1$ then the Schrödinger operator $H = H_0 + V$ has a ground state with eigenvalue E.

For the proof see Ref. [E-P]. ///

Lemma 3.

Let $V \in \mathcal{V}^{(d)}$ and ϕ the ground state of $H = H_0 + V$. Since ϕ is strictly positive we fix $\phi = e^{-h/4}$. Then there exist constants $\alpha > 4, r_0 > 0, \beta \in \mathbb{R}$ such that

$$|x| > r_0 \Rightarrow -\log \phi = h(x) \leq \alpha \cdot V(x) - \beta .$$ ///

Proposition 6.

If $V \in \mathcal{V}^{(d)}$ and $\phi = e^{-h/4}$ is the ground state, then $\exists\, \gamma > 1, \delta \in \mathbb{R}$ such that

(*) $$-\log \phi = \frac{h}{4} \leq (\gamma - 1) \cdot \left(-\Delta + \frac{\gamma}{\gamma - 1} \cdot V\right) - \delta .$$

Proof.

By lemma 3 $\exists r_0 > 0 \ / \ |x| \geq r_0 \quad \frac{\alpha}{4} . V(x) - \beta/4 \geq h/4$, $\alpha > 4$.

Put $\gamma \equiv \alpha/4 > 1$. By Harnack's inequality h is bounded on the disk $|x| \leq r_0$ (since is compact!). Then the function W = $\gamma . V - h/4 - \beta/4$ has its negative part in L^p (by hypoteses on V) and a positive part that is continous by hypoteses on V_R and 7) in Proposition 1. Then the operator $-(\gamma-1)\Delta + W$ is lower bounded since W is in the Kato's class. Now we can choose $\delta \in \mathbb{R}$ to obtain the thesis. ///

The formula (*) is nothing else that the condition of the Rosen lemma. Anyway we'll follow the "comparison method" of Eckmann and Pearson.

Proposition 7 - (Intrinsic hypercontractivity for potential in the class of Eckmann and Pearson).

Let $V \in \mathcal{V}^{(d)}$ and $H = H_0 + V$, $H_0 = -\Delta$ its Schrödinger operator. Let ϕ the ground state, U_ϕ its ground state representation, $H_\phi = U_\phi \cdot (H - E) \cdot U_\phi^*$ $E = \inf \sigma(H)$ the operator corrisponding to the Dirichlet form $(\mathcal{E}_\phi ; D(\mathcal{E}_\phi))$:

$$D(\mathcal{E}_\phi) = \{ f \in L^2(\phi^2 dx) : \nabla f \in L^2(\phi^2 dx) \}$$

$$\mathcal{E}_\phi(f;f) = \int_{\mathbb{R}^d} dx \cdot \phi^2 \cdot |\nabla f|^2$$

Then H_ϕ is a Sobolev generator of index 2.

Proof.

Since "$V \in Kato$" by Proposition 1, $C_0^\infty(\mathbb{R}^d)$ is a form case for $\mathcal{E}_\phi$.

Now consider (L-S) for the Gauss mesure

[1] $f \in C_0^\infty(\mathbb{R}^d)$ $\quad (2\pi)^{-d/2} \cdot \int dx \cdot e^{-x^2/2} \cdot |f|^2 \cdot \log|f| - \frac{1}{2}(2\pi)^{-d/2} \cdot \left(\int dx\, e^{-x^2/2} |f|^2\right) \cdot \log\left((2\pi)^{-d/2} \int dx\, e^{-x^2/2} |f|^2\right) \leq$

$$\leq (2\pi)^{-d/2} \cdot \int dx\, e^{-x^2/2} \cdot |\nabla f|^2$$

Put $f = g \cdot e^{x^2/2}$ $\; g \in C_0^\infty(\mathbb{R}^d)$ and substitute in the previous formula, taking advantage from the following

[2] $\int dx\, e^{-x^2/2} |\nabla(g e^{x^2/2})|^2 = \int dx\, |\nabla g|^2 + \frac{1}{2} \int dx\; x \cdot \nabla |g|^2 + \frac{1}{4} \int dx\, |x|^2 \cdot |g|^2 =$

$$= \int dx\, |\nabla g|^2 - \frac{d}{2} \int dx\, |g|^2 + \frac{1}{4} \int dx\, |x|^2\, |g|^2.$$

Then [1] is equivalent to

[3] $\int dx\, |g|^2 \log|g| - \frac{1}{2}\left(\int dx\, |g|^2\right) \log\left(\int dx\, |g|^2\right) + \frac{d}{2}\left(1 + \frac{1}{2}\log 2\pi\right) \cdot \int dx\, |g|^2 \leq \int dx\, |\nabla g|^2$

$g \in C_0^\infty(\mathbb{R}^d)$.

Now we use a similar substitution to transform (L-S) for $\mathcal{E}_\phi$:

$f = g \cdot \phi^{-1} = g \cdot e^{h/4}$ $\qquad$ where $\phi = e^{-h/4}$.

Let's transform

[4] (L-S) $\quad \int dx \cdot e^{-h/2} \cdot |f|^2 \lg|f| - \frac{1}{2} \cdot \left(\int dx\, |f|^2 \cdot e^{-h/2}\right) \lg\left(\int dx\, e^{-h/2} \cdot |f|^2\right) \leq$

$$\leq c \cdot \int dx \cdot e^{-h/2} \cdot |\nabla f|^2 + \gamma \cdot \int dx \cdot e^{-h/2} \cdot |f|^2 \qquad f \in C_0^\infty(\mathbb{R}^d),$$

using the identity

$$[5] \quad \int dx \cdot e^{-h/2} \cdot |\nabla(g e^{h/4})|^2 = \int dx\, |\nabla g|^2 - \frac{1}{4} \cdot \int dx\, |g|^2 (\Delta h) + \frac{1}{16} \cdot \int dx\, |g|^2 \cdot |\nabla h|^2$$

in the previous formula. We obtain

$$[6] \quad \int dx\, |g|^2 \log|g| - \frac{1}{2}\left(\int dx\, |g|^2\right) \log\left(\int dx\, |g|^2\right) + \frac{1}{4}\int dx \cdot h \cdot |g|^2 - \gamma \cdot \int dx \cdot |g|^2 \leq$$

$$\leq c \cdot \int dx\, |\nabla g|^2 - \frac{c}{4} \cdot \int dx \cdot |g|^2 (\Delta h) + \frac{c}{16} \cdot \int dx\, |g|^2 \cdot |\nabla h|^2 \qquad g \in C_0^\infty(\mathbb{R}^d)$$

To verify [6] is than sufficient, by [3] check the following:

[7]

$$-(c-1)\int dx\, \bar{g} \cdot \Delta g - c\int dx\, |g|^2 (\Delta h/4) + c\int dx\, |g|^2 \left(\frac{\nabla h}{4}\right)^2 + \gamma \cdot \int dx\, |g|^2 - \frac{1}{4} \cdot \int dx\, |g|^2 \cdot h \geq 0$$

But this is equivalent to

$$[8] \quad -(c-1)\Delta - c\,\Delta(h/4) + c\left(\frac{\nabla h}{4}\right)^2 + \gamma - \frac{h}{4} \geq 0$$

Since $V = \Delta\phi/\phi = -(\nabla h/4)^2 - \Delta(h/4)$, we have just to verify the following

$$(c-1)\left(-\Delta + \frac{c}{c-1} V\right) + \gamma \geq \frac{h}{4} = -\log \phi \ .$$

This was done in Proposition 6. ///

Remark 1.

In application of the Rosen's lemma that we have seen, we always verified the inequality in a "strong sense", i.e. we verified $-\log \phi \leq c' \cdot V + \gamma'$ and then we used the boundedness of V- with respect to $H_0 = -\Delta$, V- being in K_d.

5.4 The probabilistic method

We want now show another criterion due to R. Carmona (see Ref. [C]). It's originality resides in its "probabilistic approach" to the proof of the inequality of the Rosen's lemma.

In the article of Carmona is prevailing the probabilistic approach. He use the "Kac average" and with respect to a class of potentials $V = V_+ - V_-$ with V_+ mesurable and with $V_- \in L^\infty + L^p$ $p > \max(1; \frac{d}{2})$, he define strongly continous semigroups by the Feynman-Kac formula.

Since the Feynman-Kac formula works for potentials in the "Kato class" $V = V_+ - V_-$ $V_+ \in K_d^{loc}$ $V_- \in K_d$ and since, as we'll see, the proof of Carmona depends only on the properties of the Wiener mesure, we'll suppose V as above.

Moreover we'll suppose that $H = H_0 + V$ has its ground state with regularity that we can deduce from Proposition 1.

Let $(\Omega; X_t; W_x; x \in \mathbb{R}^d)$ the Brownian Motion in $\mathbb{R}^d$, in the sense that

$\Omega = C(\mathbb{R}_+; \mathbb{R}^d)$, $X_t : \Omega \to \mathbb{R}^d$, $X_t(\omega) = \omega(t)$ $\omega \in \Omega$,

W_x being the Wiener mesure conditionated at $x \in \mathbb{R}^d$.

Let $p_t(x;y)$ the transition function of the process, Kernel of the Markovian semigroup P_t on $L^2(\mathbb{R}^d)$:

$$p_t(x;y) = (2\pi t)^{-d/2} \cdot e^{-|x-y|^2/2t} \qquad t>0 \quad x,y \in \mathbb{R}^d.$$

Proposition 8.

With the above assumptions on V, we suppose also that

i) $\exists\, \alpha>0, t, a', b'$ such that $\forall x \in \mathbb{R}^d$

$$W_x\{|X_t|_\infty \le \alpha\}^{-1} \cdot \int_0^t ds \int_{|y|_\infty \le \alpha} dy \cdot \int_{\mathbb{R}^d} dz \cdot V_+(z) \cdot p_s(x;z) \cdot p_{t-s}(z;y) \le a' \cdot V_+(x) + b'$$

ii) $\exists K_1, K_2 \in \mathbb{R}$ such that $1 + |X|^2 + V_+(X) \le K_1 + K_2 \cdot V_+(x)$.

Then $\exists\, d_1, d_2 \in \mathbb{R}$ such that

$$-\log \phi \le d_1 \cdot V_+ + d_2$$

and H_ϕ is a Sobolev generator of index 2 on $L^2(\phi^2 dx)$.

Proof.

Put $|y|_\infty \equiv \max_{i=1\dots d} |y_i|$.

Since V is in the Kato's class, the ground state is locally bounded away from zero. So, since $\{y \in \mathbb{R}^d : |y|_\infty \le \alpha\}$

is compact, $\mathcal{E}(\alpha) \equiv \inf\{\phi(y) : |y|_\infty \leq \alpha\}$ is strictly greater than zero: $\mathcal{E}(\alpha) > 0$

Now we have using Feynmann-Kac: $\phi(x) = e^{tE} \cdot (e^{-tH}\phi)(x) =$

$$= e^{+tE} \cdot \mathbb{E}_{W_x}\left\{\phi(X_t(\cdot)) \cdot e^{-\int_0^t ds\, V_+(X_s(\cdot)) + \int_0^t ds\, V_-(X_s(\cdot))}\right\} \geq \left(V_- \geq 0 \Rightarrow e^{+\int_0^t ds\, V_-(X_s(\cdot))} \geq 1\right) \geq$$

$$\geq e^{+tE} \cdot \mathbb{E}_{W_x}\left\{\phi(X_t(\cdot)) \cdot e^{-\int_0^t ds\, V_+(X_s(\cdot))}\right\} \geq \left(\{\omega \in \Omega : |X_t(\omega)|_\infty \leq \alpha\} \subset \Omega\right) \geq$$

$$\geq e^{+tE} \cdot \mathbb{E}_{W_x} \cdot \left\{\phi(X_t(\cdot)) \cdot e^{-\int_0^t ds\, V_+(X_s(\cdot))} \;\middle|\; |X_t(\cdot)|_\infty \leq \alpha\right\} \geq \left(|X_t(\omega)| \leq \alpha \Rightarrow \phi(X_t(\omega)) \geq \mathcal{E}(\alpha)\right) \geq$$

$$\geq e^{+tE} \cdot \mathcal{E}(\alpha) \cdot \mathbb{E}_{W_x} \cdot \left\{e^{-\int_0^t ds\, V_+(X_s(\cdot))} \;\middle|\; |X_t(\cdot)|_\infty \leq \alpha\right\} \geq$$

and using Jensen's inequality

$$\geq e^{+tE} \cdot \mathcal{E}(\alpha)\, \exp\left\{-\int_0^t ds\, \mathbb{E}_{W_x}\left(V_+(X_s(\cdot)) \;\middle|\; |X_s(\cdot)| \leq \alpha\right)\right\} \geq$$

$$\geq \left(K \geq 0 \,, \bar{K} \in (0;1] \Rightarrow e^{-K} \geq \bar{K} \cdot e^{-K/\bar{K}} \right) \geq$$

$$\geq e^{+tE} \cdot \mathcal{E}(\alpha) \cdot W_x\left\{|X_t(\cdot)|_\infty \leq \alpha\right\} \cdot \exp\left\{- W_x\left[|X_t(\cdot)|_\infty \leq \alpha\right]^{-1} \cdot \right.$$

$$\left. \cdot \int_0^\infty ds\, \mathbb{E}_{W_x}\left\{V_+(X_s(\cdot)) \;\middle|\; |X_s(\cdot)|_\infty \leq \alpha\right\}\right.$$

Hence we obtain

$$-\log \phi(x) \leq -tE - \log \mathcal{E}(\alpha) - \log W_x\left\{|X_t(\cdot)|_\infty \leq \alpha\right\} + W_x\left\{|X_t(\cdot)|_\infty \leq \alpha\right\}^{-1} \cdot$$

$$\cdot \int_0^t ds \cdot \int_{|y|_\infty \leq \alpha} dy \cdot \int_{\mathbb{R}^d} dz \cdot V_+(z) \cdot p_s(x;z) \cdot p_{t-s}(z;y) \leq$$

$$\leq -tE - \log \mathcal{E}(\alpha) + a' V_+(x) + b' - \log W_x\left\{|X_t(\cdot)|_\infty \leq \alpha\right\} .$$

But $\quad -\log W_x\{|X_t(\cdot)|_\infty \leq \alpha\} \leq a'' \cdot (1+|x|^2)$

for some a'' , and then

$$-\log \phi \leq -tE - \log \mathcal{E}(\alpha) + b' + a' \cdot V_+(x) + a'' \cdot (1+|x|^2) \leq$$

$$\leq -tE - \log \mathcal{E}(\alpha) + b' + (a' \vee a'') \cdot \left(1+|x|^2 + V_+(x)\right) \leq$$

$$\leq d_1 \cdot V_+(x) + d_2$$

$$\begin{cases} d_1 \equiv K_2 \cdot (a' \vee a'') \\ d_2 \equiv -tE - \log \mathcal{E}(\alpha) + b' + K_1 \cdot (a' \vee a'') \end{cases} .$$

///

To verify i) we can take advantage from the Knowledge that V is a continous function such that:

there exist a polynomial $P: \mathbb{R}^d \to \mathbb{R}$ and costants $a_i > 0$ $b_i \in \mathbb{R}$ such that

$$a_1 \cdot P + b_1 \leq V_+ \leq a_2 \cdot P + b_2$$

with $$P(x_1; \cdots; x_d) = \sum_{j_1 \cdots j_d \geq 0} a_{j_1 \cdots j_d} \cdot x_1^{2 \cdot j_1} \cdots\cdots x_d^{2 j_d}$$

$$a_{j_1 \cdots j_d} \geq 0 \quad .$$

For this potential the ground state exist (see Ref. [S2] page 56). The control of i) by previous estimates riduce to the calculation of gaussian integrals of polynomials.

5.5 Convex combinations of Log-Sobolev mesures

Here we want to prove the following lemma, in which we show that the convex combination of Log-Sobolev mesure is again a Log-Sobolev mesure. However the local norm increases.

Lemma.

Suppose μ_1, μ_2 are two Log-Sobolev mesure with Sobolev constants c_1, c_2 and local norms γ_1, γ_2 (obviously we are in the situation of Definition 3).

Fixing $\alpha \in (0;1)$, we have that $\mu \equiv \alpha\mu_1 + (1-\alpha)\mu_2$ is a Log-Sobolev mesure with constants

$$c = c_1 \vee c_2$$

$$\gamma = \gamma_1 \vee \gamma_2 + \frac{K_\alpha}{2} \qquad K_\alpha = \max(-\log\alpha; -\log(1-\alpha)).$$

Proof.

$$(L-S:\mu_1) \qquad \int \mu_1 f^2 \log f^2 \leq 2c_1 \cdot \int \mu_1 |\nabla f|^2 + \left(\int \mu_1 f^2\right)\log\left(\int \mu_1 f^2\right) + 2\gamma_1\left(\int \mu_1 f^2\right)$$

$$(L-S:\mu_2) \qquad \int \mu_2 f^2 \log f^2 \leq 2c_2 \cdot \int \mu_2 |\nabla f|^2 + \left(\int \mu_2 f^2\right)\log\left(\int \mu_2 f^2\right) + 2\gamma_2\left(\int \mu_2 f^2\right)$$

$$((L-S):\mu_1)\times\alpha \qquad \alpha\int \mu_1 f^2 \log f^2 \leq 2\alpha c_1 \int \mu_1 |\nabla f|^2 + \left(\alpha\int \mu_1 f^2\right)\log\left(\int \mu_1 f^2\right) + 2\alpha\gamma_1\left(\int \mu_1 f^2\right)$$

$$((L-S):\mu_2)\times(1-\alpha) \qquad (1-\alpha)\int \mu_2 f^2 \log f^2 \leq 2(1-\alpha)c_2 \int \mu_2 |\nabla f|^2 + (1-\alpha)\int \mu_2 f^2 \log\left(\int \mu_2 f^2\right) + 2(1-\alpha)\gamma_2\left(\int \mu_2 f^2\right)$$

$$\alpha\int \mu_1 f^2 \log f^2 + (1-\alpha)\int \mu_2 f^2 \log f^2 = \int \mu f^2 \log f^2 \leq 2(c_1 \vee c_2)\cdot\int \mu |\nabla f| + 2(\gamma_1 \vee \gamma_2)\cdot\left(\int \mu f^2\right) +$$

$$+ \alpha\left(\int \mu_1 f^2\right)\log\left(\int \mu_1 f^2\right) + (1-\alpha)\left(\int \mu_2 f^2\right)\log\left(\int \mu_2 f^2\right) \quad . \text{ But :}$$

$$\alpha\left(\int \mu_1 f^2\right)\log\left(\int \mu_1 f^2\right) + (1-\alpha)\left(\int \mu_2 f^2\right)\log\left(\int \mu_2 f^2\right) =$$

$$= \alpha\left(\int \mu_1 f^2\right)\log\left(\alpha\int \mu_1 f^2\right) - \alpha\left(\int \mu_1 f^2\right)\log\alpha + (1-\alpha)\left(\int \mu_2 f^2\right)\log\left((1-\alpha)\int \mu_2 f^2\right) - (1-\alpha)\left(\int \mu_2 f^2\right)\log(1-\alpha) =$$

$$\leq \alpha\left(\int \mu_1 f^2\right)\log\left(\alpha\int \mu_1 f^2 + (1-\alpha)\int \mu_2 f^2\right) + (1-\alpha)\left(\int \mu_2 f^2\right)\log\left(\alpha\int \mu_1 f^2 + (1-\alpha)\int \mu_2 f^2\right) + K_\alpha \cdot \int \mu f^2 =$$

$$= \left(\int \mu f^2\right)\log\left(\int \mu f^2\right) + K_\alpha \int \mu f^2 \qquad K_\alpha \equiv \max(-\log\alpha; -\log(1-\alpha)).$$

Then :

$$\int \mu f^2 \log f^2 \leq 2c \cdot \int \mu |\nabla f|^2 + \left(\int \mu f^2\right) \log \left(\int \mu f^2\right) + 2\left(\gamma_1 \vee \gamma_2 + \frac{K_\alpha}{2}\right) \cdot \int \mu f^2 =$$

$$\leq 2c \int \mu |\nabla f|^2 + \left(\int \mu f^2\right) \log \left(\int \mu f^2\right) + 2\gamma_\alpha \cdot \int \mu f^2$$

$c = c_1 \vee c_2$

$\gamma_\alpha = \gamma_1 \vee \gamma_2 + K_\alpha / 2$ $\qquad$ $K_\alpha / 2 = \frac{1}{2} \max(-\log \alpha ; -\log(1-\alpha))$.

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PART 2

LOGARITHMIC SOBOLEV INEQUALITIES FOR INFINITE LATTICE SYSTEMS

INTRODUCTION

In this second part we'll study some recent criteria due to B. Zegarlinski to decide when a Gibbs mesure satisfies Log-Sobolev Inequalities.

We want to remark that the interest about hypercontractive semigroups originated just in studies of infinite dimensional problem, as the problems of Quantum Field Theory.

E. Nelson showed (see Ref.[N 1] and [N 2].) that operators built by second quantization of contraction operators are bounded from L^2 to L^p for some $p > 2$ (i.e. are hyper-contractive). Using these result Segal, and then Simon and Hoegh-Krohn (see Ref.[S-H-K]), showed the selfadjointness and the lower boundedness of operators of the form $H_0+V(g)$ where H_0 is the free hamiltonian of a bidimensional field theory and V(g) is a cut-off potential of type $P(\phi)_2$. However just recently, with the work of E. Carlen, D. Stroock and B. Zegarlinski, we have at our disposal methods for (L-S) for Gauss-Dirichlet from on infinite dimensional space.

In this perspective many wor in the seventies about intrinsic hypercontractivity of Schrödinger operators are not of own interest, but they seen to be steps toward the solution of infinite dimensional problems. Only on the middle of eighties, with the works of B. Davies and B. Simon (see Ref. [DA 2] and [DA-S]), was possible to see the whole fecundity of an approach with Logarithmic Sobolev Inequalities to problems as upper and lower bound of kernels and eigenfunctions of elliptic operators on domain of $\mathbb{R}^d$, or estimates of eigenvalues of Schrödinger operators.

CHAPTER 1

GIBBS MESURE AND DOBRUSHIN CONTRACTION TECHNIQUE

1.1 Local Specifications and Gibbs Mesures

The set of the Gibbs mesure is specified when a Local Specification is given. To be precise let S be a Polish space and L (lattice) a countable set of sites. The elements of the product space $\Omega = S^L$, i.e. by definition the functions $\omega : L \to S$ are called fields or configurations. Since S is a Polish space (separable complete metric space) so is Ω with the product topologes, and is compact if S is compact.

In order to define on Ω a family of σ-algebras that we need we consider for each $\Lambda \subset L$ a projection $p_\Lambda : \Omega \to S^\Lambda$ as $p_\Lambda(\omega) \equiv \omega \upharpoonright \Lambda$. This function is continous, since the topology of Ω is the product topology, and can be used to define the σ-algebra Σ_Λ on Ω as the smallest one for which p_Λ is mesurable (on S^Λ we consider the Borel σ-algebra). We'll denote Σ_L as Σ. Biside the family (Σ_Λ) ($\Lambda \subset L$) great importance has the σ-algebra $\Sigma_\infty \equiv \bigcap_{\substack{\Lambda \subset L \\ \text{FINITE}}} \Sigma_{\Lambda^c}$ of "events at infinity", called "tail field". We can say that its elements are mesurable outside each finite region Λ.

Now if μ a mesure an $(\Omega ; \Sigma)$ then we can condition with respect each σ-algebra of the family (Σ_Λ) ($\Lambda \subset L$ finite) and obtain a family of "stochastic kernel" E_Λ from $(\Omega ; \Sigma_{\Lambda^c})$ to $(\Omega ; \Sigma)$.

With this is mind we define a "Local Specification" $\mathbb{E}=(E_\Lambda)$ ($\Lambda\subset L$ finite) as a family with the following properties:

a) for each $\omega\in\Omega$ $E_\Lambda(\omega;\cdot):(\Omega;\Sigma)\to\mathbb{R}$ is a probability mesure

b) for each $A\in\Sigma$ $E_\Lambda(\cdot;A):\Omega\to\mathbb{R}$ is Σ_{Λ^c} -mesurable

c) if $f\in\Sigma_{\Lambda^c}$ (i.e. f is Σ_{Λ^c} -mesurable): $E_\Lambda(\cdot;f)=f$
$\left(E_\Lambda(\omega;f)\equiv\int_\Omega E_\Lambda(\omega;d\omega')\,f(\omega')\right)$

d) $\tilde\Lambda\supset\Lambda \Rightarrow E_{\tilde\Lambda}E_\Lambda=E_{\tilde\Lambda}$.

When we are dealing with a kernel $E_\Lambda:\Omega\times\Sigma\to\mathbb{R}$ we'll write also E_Λ for the function that takes a Σ -mesurable function f , and give us an Σ_{Λ^c}-mesurable function $E_\Lambda f$ defined by
$(E_\Lambda f)(\omega)\equiv E^\omega_\Lambda f\equiv\int_\Omega E_\Lambda(\omega;d\omega')\cdot f(\omega')$.
So by $E_{\tilde\Lambda}E_\Lambda$ we denote the composition of this functions.
If $\mathbb{E}=(E_\Lambda)$ ($\Lambda\subset L$ FINITE) is a local specification we define the set $G(\mathbb{E})$ of the Gibbs mesures as the set of mesure μ on $(\Omega;\Sigma)$ such that the D.L.R. equation is satisfied:

$$\mu E_\Lambda=\mu \qquad \forall\Lambda\subset L \quad \text{finite}.$$

In an other may we can say that $G(\mathbb{E})$ is the set of mesures μ such that its conditional probabilities with respect to the sub σ-algebra Σ_{Λ^c} is E_Λ .
The crucial fact is that $G(\mathbb{E})$ doesn't need to be a singleton, a situation in which, we say, a phase transition occours.
When $E_\Lambda f$ is a continous function any time that f is, we say that $\mathbb{E}$ has the "Feller property" and this implies that the function $M(\Omega)\ni\mu\longmapsto\mu E_\Lambda\in M(\Omega)$ (probability mesure) is

continous in the weak topology. Since $M(\Omega)$ is a compact convex subset in the space of all mesure and $G(\mathbb{E}) = \bigcap_{\substack{\Lambda\subset L \\ \text{FINITE}}} \{\mu : \mu E_\Lambda = \mu\}$ the Schouder-Tychonow fixed point theorem, tells us that $G(\mathbb{E}) \neq \phi$.

If S is not compact some additional requirement on specification is needed. From the definition it's easy to see that. $G(\mathbb{E})$ is a convex compact topological space and the theory of the integral representation of Choquet, of the elements of $G(\mathbb{E})$ works. In particular each Gibbs mesure $\mu \in \mathcal{G}(\mathbb{E})$ can be represented in terms of the extremal elements of $G(\mathbb{E})$: the "extremal Gibbs mesure $\partial G(\mathbb{E})$".

A fondamental property of the extremal Gibbs mesure $\mu \in \partial\mathcal{G}(\mathbb{E})$ is that they can be caracterized as the elements of $G(\mathbb{E})$ for which the "tail field" Σ_∞ is trivial: $A \in \Sigma_\infty \Rightarrow \mu(A) \in \{0,1\}$. Hence any function "mesurable at infinity", $f \in \Sigma_\infty$, is μ-a.e. constant with respect to any extrem Gibbs mesure $\mu \in \partial\mathcal{G}(\mathbb{E})$ and $\dim L^p(\mu) = 1 \quad p \geq 1$.

In Statistical Mechanics one build Local Specification starting from an "interaction" ϕ, i.e. a family (ϕ_X) ($X \subset L$ finite) such that $\phi_X \in \Sigma_X$. ϕ is called differentiable if ϕ_X is $C^1\ \forall X$. The "interaction energy in the region $\Lambda \subset L$ is defined by:

$$U_\Lambda \equiv \sum_{X\cap\Lambda\neq\phi} \phi_X \quad .$$

Then one choose a "single spin space mesure" ρ on the space S and, putting $\rho_\Lambda = \bigotimes_{i\in\Lambda} \rho$ for $\Lambda \subset L$, on defines:

$$dE_\Lambda^\omega(\bar\omega_\Lambda \otimes \bar\omega_{\Lambda^c}) \equiv \frac{d\rho_\Lambda(\bar\omega_\Lambda)\, e^{-U_\Lambda(\bar\omega_\Lambda\otimes\omega_{\Lambda^c})}}{\int_{S^\Lambda} d\rho_\Lambda(\bar\omega_\Lambda)\cdot e^{-U_\Lambda(\bar\omega_\Lambda\otimes\omega_{\Lambda^c})}} \otimes \delta_{\omega_{\Lambda^c}}(\bar\omega_{\Lambda^c})$$

where $\delta_{\omega_{\Lambda^c}}$ is the Dirac mesure on S^{Λ^c} in the point $\omega_{\Lambda^c} \in S^{\Lambda^c}$ (we took $\Omega = S^L = S^\Lambda \otimes S^{\Lambda^c}$). Often one speaks about the mesure E_Λ^ω as a mesure on S^Λ referring to

$$\frac{d\rho_\Lambda(\bar\omega_\Lambda)\cdot e^{-U_\Lambda(\bar\omega_\Lambda\otimes\omega_{\Lambda^c})}}{Z_\Lambda(\omega_{\Lambda^c})} \qquad Z_\Lambda(\omega_{\Lambda^c}) \equiv \int_{S^\Lambda} d\rho_\Lambda(\bar\omega_\Lambda)\, e^{-U_\Lambda(\bar\omega_\Lambda\otimes\omega_{\Lambda^c})}$$

In what follows we shall indicate $E_{\{i\}}^\omega$ as E_i^ω, $i \in L$.

1.2 Dobrushin Uniqueness theorem

The following criterion of Dobrushin has the advantage to be very general about the nature of the space S and provides also the decay of the correlation function for the unique Gibbs mesure.

Proposition 1.

Let S be a Polish space and $\mathbb{E}$ a Local Specification. Let's define "the Dobrushin interaction matrix" $(C_{ij})_{i,j\in L}$ as follows:

(D) $\quad c_{ij} \equiv \sup\{ \| E_j(\omega;\cdot) - E_j(\omega';\cdot) \| : \omega = \omega' \text{ off } i \in L \} \quad i,j \in L.$

Then if $\lim_{n\to\infty} \sum_i (C^n)_{ij}$ $\forall j \in L$ or at least $\sum_i c_{ij} \leq \alpha < 1$ $\forall j \in L$

then $\# \mathcal{G}(\mathbb{E}) = 1$.

In proof of the Dobrushin theorem (see for example [Gorss], [Lanford] the central fact is that the condition (D) implies the strong mixing property of the semigroup $(T^n)_{n \in \mathbb{N}}$ on $\mathcal{C}(\Omega)$ generated by the operator $T = \lim_p E_p \cdots E_1$. This property implies not only the uniqueness of the Gibbs mesure but also its representation in terms of the strong mixing semigroup $(T^n)_{n \in \mathbb{N}}$

$$\mu(f) = \lim_{n \to \infty} T^n f \quad \text{in } \|\cdot\|_\infty \text{ on } \mathcal{C}(\Omega).$$

As we'll see in the next Chapter 3, this mixing property will allows us to carry the Log-Sobolev property for a mesure from finite volume to infinite volume.

CHAPTER 2

DIRICHLET FORMS FOR INFINITE LATTICE SYSTEMS

2.1 Introduction

In recent years there was on increasing attention to the theory of Dirichlet forms on infinite dimensional spaces. Mainly for application to the Euclidean Quantum Field Theory where the infinite dimensional space is sometimes the space of tempered distribution $\mathcal{S}'(\mathbb{R}^d)$ (see Ref. [A-Rö]).

A parallel problem, but technically simpler, arise when we want to consider semigroups (operators, forms) on the space of all configurations of a statistical systems as $\Omega = S^L$ where L is an infinite lattice and S is a "good space".

At our knowledge the problem was studied firstly by the needs of the theory of the Stochastic Ising Model.

Recently, in the paper [D-S 1] , Deuschel and StroocK proved a more general result.

In this chapter we want to show these results without proving them, in order to have a precise background for what we shall describe in Chapter 3.

We limit us to the case where S is Remannian manifold because the case $S=\{+1;-1\}$ is completely similar.

2.2 The theorem of J. Deuschel and D. Stroock.

Let's choose a family $(X^i)_{i=1}^{r}$ of vector fields on S such that $X^1(x), \ldots\ldots, X^r(x)$ span $T_x S$ $\forall x \in S$. For $j \in L$ $\underline{m} = (m_1; \cdots ; m_r) \in \mathbb{N}^r$ SET $X_j^{\underline{m}} = (X_j^1)^{m_1} \circ \cdots\cdots \circ (X_j^r)^{m_r}$ and $|\underline{m}| \equiv \sum_{i=1}^{r} m_i$ where X_j^K is a field on j-th spin space.

Let $C(\Omega)$ the space of continous functions on $\Omega = S^L$ and $C^\infty(\Omega)$ the space of functions whose restrictions to finite subset Λ of L are C^∞ on S^Λ.

Define

$$D(L) \equiv \left\{ f \in C^\infty(\Omega) : \|f\|_m \equiv \|f\|_{C(\Omega)} + \sum_{j \in L} \sum_{1 \le |\underline{m}| \le m} \| X_j^{\underline{m}} \|_{C(\Omega)} < +\infty \quad \forall m \in \mathbb{Z}_1^+ \right\}$$

and the operator

$$Lf \equiv -\sum_{j \in L} e^{+U_j} \cdot (\nabla_j (e^{-U_j} f)). \qquad f \in D(L).$$

We can now state the following:

Proposition 1.

1) $\mu \in \mathcal{G}(E)$ iff $\int_\Omega d\mu\, f \cdot Lg = \sum_{j \in L} \int d\mu\, (\nabla_j f | \nabla g)$ $\quad f, g \in D(L)$

2) there exist a unique Markovian semigroup P_t on $C(\Omega)$ (the space of continous functions on Ω) with the property that

$$P_t f = f + \int_0^t ds\, P_s L f \qquad t > 0 \quad f \in D(L)$$

3) D(L) is P_t invariant and $\forall n \in \mathbb{N}$ there is a $K_n \in [0; +\infty)$ such that

$$\| P_t f \|_n \leq K_n \cdot e^{K_n \cdot t} \cdot \| f \|_n \quad t > 0 \quad f \in D(L)$$

4) for each $\mu \in \mathcal{G}(\mathbb{E})$ there is a unique strongly continous semigroup $\overline{P}_t$ of selfadjoint contractions on $L^2(\mu)$ such that $\overline{P}_t f = P_t f \quad \forall t > 0 \quad \forall f \in C(\Omega)$

5) $\mu \in \partial \mathcal{G}(\mathbb{E})$ if and only if $P_t f \to \langle f \rangle_\mu$ in $L^2(\mu)$ and if this is the case then $\overline{P}_t f \longrightarrow \langle f \rangle_\mu$ in $L^2(\mu)$.

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CHAPTER 3

THE THEORY OF ZEGARLINSKI FOR LOG-SOBOLEV INEQUALITIES ON INFINITE LATTICE SYSTEMS

In Part 1 we saw that the (L-S) property behaves well with respect to the product of mesures: if the mesures μ and ν on $\mathbb{R}^n$ and $\mathbb{R}^m$ satisfly (L-S) with costant c_μ and c_ν then so does $\mu\otimes\nu$ on $\mathbb{R}^n\times\mathbb{R}^m$ with costant c = $\max(c_\mu; c_\nu)$. Since Gibbs mesures of Local Specification coming from on interaction can be considered as local perturbation of the "free" mesure, there is a chance that the inductive property of (L-S) with respect to product of mesures can be generalized to Gibbs mesures. Anyway the method we are going to describe deals with general Local Specifications.

3.1 Criteria for Log-Sobolev Inequality: continous single spin space

We want to consider a C^1-Local Specification $\mathbb{E}=(E_\Lambda)$ ($\Lambda \subset L$ finite) on the space $\Omega = S^L$ where S is a complete smooth Riemannian manifold and L is the lattice (for example $\mathbb{Z}^d$ $d\geq 1$). The two main qualitative hypotheses we require on the Local Specification are following:

A) $\forall i\in L$, $\forall \omega\in\Omega$ E_i^ω satisfy (L-S) with a constant indipendent on i and ω

B) there exist matrix $(C_{ij})\ (i,j\in L)$ whose elements are non negative such that

$$\left|\nabla_j (E_i \rho^2)^{1/2}\right| \leq \left(E_i |\nabla_j \rho|^2\right)^{1/2} + C_{ij}\cdot\left(E_i |\nabla_i \rho|^2\right)^{1/2} \quad i,j\in L$$

for ρ differentiable. (∇_j is the gradient operator with respect to the site $j\in L$).

Because the function $E_i \rho^2$ doesn't depend on the site $i\in L$ we can choose $C_{ii}=0 \ \forall i\in L$.

We remark that conditions A) and B) deal just with the kernels E_i^{ω} and nothing is required on the kernels E_{Λ}^{ω} for finite regions $\Lambda\subset L$.

The following lemma is the first step towards the proof of (L-S) and it enable us to understand why conditions A) and B) are reasonable.

Lemma 1.

Assume that A) holds and let μ be a Gibbs mesure $\mu\in\mathcal{G}(\mathbb{E})$. For any sequence $i:\{1\cdots n\}\longrightarrow L$ the following inequality holds:

(L1) $$\mu\rho^2\log\rho \leq c_0\left(\mu|\nabla_{i_1}\rho|^2 + \sum_{k=1}^{n-1}\mu\left|\nabla_{i_{k+1}}(E_{i_k}\cdots E_{i_1}\rho^2)^{1/2}\right|^2\right) + \mu\left[(E_{i_n}\cdots E_{i_1}\rho^2)\log(E_{i_n}\cdots E_{i_1}\rho^2)^{1/2}\right]$$

for ρ positive differentiable with $\sum_{i\in L}\mu|\nabla_i\rho|^2 < +\infty$.

Proof. $$\mu\rho^2\log\rho = \mu E_{i_1}\rho^2\log\rho = c_0\,\mu E_{i_1}|\nabla_{i_1}\rho|^2 + \mu\left[(E_{i_1}\rho^2)\log(E_{i_1}\rho^2)^{1/2}\right] = c_0\,\mu|\nabla_{i_1}\rho|^2 +$$
$$+\,\mu\left[(E_{i_1}\rho^2)\log(E_{i_1}\rho^2)^{1/2}\right]$$

so (L1) is true for $n=1$. By induction:

$$\mu f^2 \log f \leq c_0\Big(\mu|\nabla_{i_1} f|^2+\sum_{k=1}^{n-2}\mu|\nabla_{i_{k+1}}(E_{i_k}\cdots E_{i_1}f^2)^{1/2}|^2\Big)+\mu\Big[(E_{i_{n-1}}\cdots E_{i_1}f^2)\log(E_{i_{n-1}}\cdots E_{i_1}f^2)^{1/2}\Big]\leq$$

$$\leq c_0\Big(\mu|\nabla_{i_1} f|^2+\sum_{k=1}^{n-2}\mu|\nabla_{i_{k+1}}(E_{i_k}\cdots E_{i_1}f^2)^{1/2}|^2\Big)+\mu E_{i_n}\Big[(E_{i_{n-1}}\cdots E_{i_1}f^2)\log(E_{i_{n-1}}\cdots E_{i_1}f^2)^{1/2}\Big]\leq$$

$$\leq c_0\Big(\mu|\nabla_{i_1} f|^2+\sum_{k=1}^{n-1}\mu|\nabla_{i_{k+1}}(E_{i_k}\cdots E_{i_1}f^2)^{1/2}|^2+\mu\Big[(E_{i_n}\cdots E_{i_1}f^2)\log(E_{i_n}\cdots E_{i_1}f^2)^{1/2}\Big].$$

Before to pass to the proof at the following two lemmas we want to discuss briefly condition B).

To go from formula (L1) to (L-S) for $\mu\in\mathcal{G}(\mathbb{E})$ we have to establish two estimates in the right hand of (L1):

$$\text{(L'1)}\qquad \mu|\nabla_{i_1} f|^2+\sum_{k=1}^{n-1}\mu|\nabla_{i_{k+1}}(E_{i_k}\cdots E_{i_1}f^2)^{1/2}|^2\leq \mathbb{C}\cdot\sum_{i\in L}\mu|\nabla_i f|^2\equiv\mathbb{C}\,\mathcal{E}(f,f)$$

for same $0<\mathbb{C}<+\infty$.

$$\text{(L"1)}\qquad \mu\Big[(E_{i_n}\cdots E_{i_1}f^2)\log(E_{i_n}\cdots E_{i_1}f^2)^{1/2}\Big]\leq(\mu f^2)\log(\mu f^2)^{1/2}$$

(L"1) will be verified if the sequence of operators $E_{i_n}\cdots E_{i_1}$ possess a mixing property (see [Schaefer]) with respect to the uniform topology: $E_{i_n}\cdots E_{i_1}f\longmapsto\mu f$ $n\to\infty$ uniformly on Ω. Such a mixing property is used in the proof of Dabrushin Uniqueness theorem (see for example [Lanford], [Gross] and is derived from the following estimates:

$$\text{(B')}\qquad \delta_j(E_i f)\leq\delta_j(f)+c_{ij}\cdot\delta_i(f)\qquad\qquad \delta_j(f)\equiv\sup\{|f(\omega)-f(\omega')|:\omega=\omega'\text{ off } j\}$$

and $(c_{ij})_{i,j\in L}$ is the Dobrushin interaction matrix.

The condition (B) has only same "minimal" change with respect to (B'), but we'll obtain from it the same mixing property. On the other hand in order to verify (L'1) we need a control on the gradients of the projections $E_i f^2$. The same condition (B) contains these estimates and (L'1) follows by the its remarkable interation property.

Besides these qualitative considerations we have to add some quantitative one, otherwise the mixing property will not always holds. As in the Dobrushin theorem the condition is about the "smallness" of the matrix $(c_{ij})_{ij \in L}$. We require the following:

(C) $$\gamma \equiv \sup_{i \in L} \max \left(\sum_{j \in L} c_{ij} ; \sum_{j \in L} c_{ji} \right) < 1 .$$

Before to prove the main statements of this section we have to fix the sequence $i : \mathbb{N} \to L$ on which we will iterate condition (B).

In order to have the mixing property we choose a sequence that visit any site of L infinitely many times. It's precisely this averaging property of the sequence that will imply, besides all above considerations, the mixing property.

Let's choose on order $\preceq$ on L and an increasing sequence $\{\Lambda_n\}_{n \geq 1}$ of regions such that $\bigcup_n \Lambda_n = L$ and $i \in \Lambda_n , j \in \Lambda_m , n \leq m \Rightarrow i \preceq j$

Fixing $K_n = \sum_{\ell=1}^{n} |\Lambda_\ell|$ ($|\Lambda_\ell|$=cardinality of Λ_ℓ) we define the sequence $i : \mathbb{N} \to L$ as follows:

$k \in [1; K_1]$ $i_k \in \Lambda_1$ and $i_k \preceq i_{k'} \iff k \leq k'$, $k, k' \in [1; K_1]$

$k \in [K_n; K_{n+1}]$ $i_k \in \Lambda_{n+1}$ and $i_k \preceq i_{k'} \iff k \leq k'$, $k, k' \in [K_n; K_{n+1}]$

So when K goes from K_n to K_{n+1}, since $K_{n+1} - K_n = |\Lambda_{n+1}|$, i_k visits all sites of Λ_{n+1} following the order of L. In the short sequence $\{i_1, \dots, i_{K_n}\}$ the sites of Λ_1 appear in times, the sites of Λ_1 (n-times, the sites of Λ_2 $(n-1)$-times and so on. When $n \to \infty$ the number of times that the sequence visit any site tends to infinite.

With these positions we may investigate the mixing property:

Lemma 2.

Let μ be an extremal Gibbs mesure $\mu \in \partial \mathcal{G}(E)$ of C^1- Local Specification satisfying condition A) B) and C) above. Than for the sequence $\{i_K\}_{K\in\mathbb{N}}$ we have:

$$\lim_{n\to\infty} E_{i_n}\cdots E_{i_1} f = \mu f \qquad \text{uniformly on } \Omega$$

for $\qquad f \in D(L)$ (see Def. 2.2 in Chapter 2)

Proof.

In order to manipulate expression (B) let's fix the matrices

$$A^{(i)}_{jK} = \begin{cases} \delta_{jK} & K \neq i \\ c_{Kj} & K = i \end{cases} \qquad B^{(K)} \equiv A^{(i_K)} \cdot A^{(i_{K+1})} \cdots\cdots A^{(i_1)} \qquad i \in L$$

then (B) will appear $\left|\partial_j (E_i f^2)^{1/2}\right| \leq \sum_{K\in L} A^{(i)}_{jK} \cdot \left(E_i |\partial_K f|^2\right)^{1/2}$

Now let's define $\tau_i f \equiv (E_i f^2)^{1/2}$ in such a way we have

$$(E_{i_K}\cdots E_{i_1} f^2)^{1/2} = \tau_{i_K}(E_{i_{K-1}}\cdots E_{i_1} f^2)^{1/2} = \cdots\cdots = \tau_{i_K}\cdots\tau_{i_1} f.$$

From (B) by interation we have

$$\left|\partial_i (E_{i_K}\cdots E_{i_1} f^2)^{1/2}\right| = \left|\partial_i \tau_{i_K}\cdots\tau_{i_1} f\right| \leq \sum_{j_K\in L} A^{(i_K)}_{i j_K} \cdot \left(E_{i_K} |\partial_{j_K} \tau_{i_{K-1}}\cdots\tau_{i_1} f|^2\right)^{1/2} =$$

$$= \sum_{j_K\in L} A^{(i_K)}_{i j_K} \cdot \|\partial_{j_K} \tau_{i_{K-1}}\cdots\tau_{i_1} f\|_{L^2(E_{i_K})} \leq \sum_{j_K\in L} A^{(i_K)}_{i j_K} \cdot \left\| \sum_{j_{K-1}\in L} A^{(i_{K-1})}_{j_K j_{K-1}} \left(E_{i_{K-1}} |\partial_{j_{K-1}} \tau_{i_{K-2}}\cdots\tau_{i_1} f|^2\right)^{1/2} \right\|_{L^2(E_{i_{K-1}})} \leq$$

$$\leq \sum_{j_K j_{K-1}} A^{(i_K)}_{i j_K} \cdot A^{(i_{K-1})}_{j_K j_{K-1}} \left\| \left(E_{i_{K-1}} |\partial_{j_{K-1}} \tau_{i_{K-2}} \cdots \tau_{i_1} f|^2 \right)^{1/2} \right\| = \sum_{j_K j_{K-1}} A^{(i_K)}_{i j_K} \cdot A^{(i_{K-1})}_{j_K j_{K-1}} \left(E_{i_K} E_{i_{K-1}} |\partial_{j_{K-1}} \tau_{i_{K-2}} \cdots \tau_{i_1} f|^2 \right)^{1/2} \leq$$

$$\leq \cdots \leq \sum_{j_K \cdots j_1} A^{(i_K)}_{i j_K} A^{(i_{K-1})}_{j_K j_{K-1}} \cdots A^{(i_1)}_{j_2 j_1} \cdot \left(E_{i_K} E_{i_{K-1}} \cdots E_{i_1} |\partial_{j_1} f|^2 \right)^{1/2} = \sum_{j_1} B^{(K)}_{i j_1} \tau_{i_K} \cdots \tau_{i_1} |\partial_{j_1} f| = \sum_j B^{(K)}_{ij} \left(\tau_{i_K} \cdots \tau_{i_1} |\partial_j f| \right)$$

From the definition of the matrices $A^{(i)}$ and $B^{(K)}$ it's not difficult to prove that:

$$B^{(K)}_{ij} = \begin{cases} i = i_K & 0 \\ i \neq i_K & C_{i_K i} \cdot B^{(K-1)}_{i_K j} + B^{(K-1)}_{ij} \end{cases}$$

So we can show that if $i \neq i_K$

$$B^{(K)}_{ij} = B^{(1)}_{ij} + \sum_{m=1}^{K} \sum_{\substack{j:\{1\cdots m\} \to \{1\cdots K\} \\ j_\ell \leq j_{\ell'} \text{ if } \ell < \ell'}} C_{i_K i_{j_m}} \cdot C_{i_{j_m} i_{j_{m-1}}} \cdot \cdots \cdot C_{i_{j_1} i}$$

Actually the second sum is over all sub-chain $\{i_{j_1} \cdots i_{j_m}\}$ of the chain $\{i_1 \cdots i_K\}$.

Now if $K = K_m$ for $m \geq 1$ and $i \in \Lambda_K$ the number of times $\{i_1 \cdots i_{K_m}\}$ visit i is $(m-K+1)$ and this produce a factor in the last formula proportional to $\gamma^{(m-K+1)}$, that go to zero when $m \to \infty$.

If f depends only on a finite subset of L we can estimate $\tau_{i_K} \cdots \tau_{i_1} |\partial_j f|$ with $\| \partial_j f \|_\infty$ and obtain that sup $\sup | \partial_i (E_{i_K} \cdots E_{i_1} f^2)^{1/2} | \to 0$ as $m \to \infty$. We obtain the same result for general f approximating them with "cylindric" functions.

So all-derivatives of the functions of the sequence $\{f_K \equiv \tau_{i_K} \cdots \tau_{i_1} f\}$ $(K \in \mathbb{N})$ tend uniformily to zero and then the possible accumulation point (in the uniform topology) are functions that don't depend on finite subset $\Lambda \subset$ L. They are mesurable at the infinity, i.e. with respect to the "tail field" $\Sigma_\infty = \bigcap_{\substack{\Lambda \subset L \\ \text{FINITE}}} \Sigma_{\Lambda^c}$

From the definition of the Local Specification we can see that the Kernels E_i are Markovian operators in the sense that they are positive preserving and $E_i 1 = 1$. So if f is a bounded function the sequence $\{f_K\}$ is uniformly bounded : $\|f_K\|_\infty \leq \|f\|_\infty$. This with the above property implies existence of an accumulation point in the uniform topology for the sequence $\{f_K\}$.

Now if $c_1(f)$ and $c_2(f)$ are two accumulation point for the two subsequences $\{f_{K_i}\}, \{f_{n_j}\}$ we have: $\mu(c_1(f)) = \mu\left(\lim_i f_{K_i}\right) =$

$$= \lim_i \mu(f_{K_i}) = \lim_i \mu E_{i_{K_i}} \cdots E_{i_1} f = \lim_i \mu f = \mu f \quad ,$$

$$\mu(c_2(f)) = \cdots\cdots\cdots = \mu f \, .$$

So $\mu(c_1(f)) = \mu f = \mu(c_2(f))$. But since μ is extreme, the tail field is trivial, so functions mesurable with respect to it (as $c_i(f)$ $i=1,2$ are) are constant.

This means that $c_1(f) = c_2(f) = \mu(f)$ μ-a.e. thus we have proved that the only accumulation paint in the set of all classes of functions μ-a.e. equal is the constant μf (or its class) this finish the proof.

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Lemma 3.

In the hypotheses of lemma 2 we have the following estimate:

$$\mu |\nabla_{i_1} f|^2 + \sum_{K=1}^{n-1} \mu \left| \nabla_{i_{K+1}} \left(E_{i_K} \cdots E_{i_1} f^2\right)^{1/2} \right|^2 \leq (1-\gamma)^2 \mu |\nabla f|^2$$

$f \in D(L)$ (see Def. in 2.2 Chapter 2).

Proof.

From the following estimate (proved in Lemma 2)

$$\left|\partial_{i_{K+1}}\left(E_{i_K}\cdots E_{i_1}\rho^2\right)^{1/2}\right| \leq \sum_j B^{(K)}_{i_{K+1}j}\cdot\left(E_{i_K}\cdots E_{i_1}|\partial_j \rho|^2\right)^{1/2}$$

we obtain, squaring both sides: $\left|\partial_{i_{K+1}}\left(E_{i_K}\cdots E_{i_1}\rho^2\right)^{1/2}\right|^2 \leq$

$$\leq \sum_{jj'} B^{(K)}_{i_{K+1}j}\cdot B^{(K)}_{i_{K+1}j'}\cdot\left(E_{i_K}\cdots E_{i_1}|\partial_j\rho|^2\right)^{1/2}\cdot\left(E_{i_K}\cdots E_{i_1}|\partial_{j'}\rho|^2\right)^{1/2} \leq$$

$$\leq \frac{1}{2}\sum_{jj'} B^{(K)}_{i_{K+1}j}\cdot B^{(K)}_{i_{K+1}j'}\cdot\left(E_{i_K}\cdots E_{i_1}|\partial_j\rho|^2 + E_{i_K}\cdots E_{i_1}|\partial_{j'}\rho|^2\right) =$$

$$= \left(\sum_j B^{(K)}_{i_{K+1}j}\right)\cdot\left(\sum_j B^{(K)}_{i_{K+1}j}\, E_{i_K}\cdots E_{i_1}|\partial_j\rho|^2\right)$$

Taking the expdation of both sides with respect to $\mu \in \partial\mathcal{G}(\mathbb{E})$ we have

$$\mu\left|\partial_{i_{K+1}}\left(E_{i_K}\cdots E_{i_1}\rho^2\right)^{1/2}\right|^2 \leq \left(\sum_j B^{(K)}_{i_{K+1}j}\right)\cdot\left(\sum_j B^{(K)}_{i_{K+1}j}\cdot \mu|\partial_j\rho|^2\right)$$

The sum of the elements of any row of $B^{(K)}$ does not exceed $\sum_{j=0}^{K}\gamma^j$ and so for $K\geq 0$ $\sum_j B^{(K)}_{i_{K+1}j} \leq (1-\gamma)^{-1}$. Developping $\sum_{K\geq 0} B^{(K)}_{i_{K+1}j}$ in terms of the matrix (c_{ij}) we obtain $\sum_j\sum_{K\geq 0}\sum_{K/\, i_{K+1}=i} B^{(K)}_{i_{K+1}j} \leq$ $\leq (1-\gamma)^{-1}$ and the Lemma.

///

Combining Lemmas 2 and 3 we obtain the Log-Sobolev inequalities for an extremal Gibbs mesure whose Local Specification satisfy condition A) B) and C).

Proposition 1.

Let $\mathbb{E}$ be a Local Specification satisfying conditions A) B), C) and let μ be an extremal Gibbs mesure $\mu\in\partial\mathcal{G}(\mathbb{E})$. Then we have:

(L-S) $\mu f^2 \log f \leq c_0 (1-\gamma)^{-2} \mathcal{E}(f;f) + (\mu f^2) \log (\mu f^2)^{1/2} \quad f \in D(\mathcal{E})$.

Proof.

Combine Lemmas 1) 2) 3). ///

3.2 Applications to models arising in Statistical Mechames

General criteria for the inspection of hypothese (A) are often too strong because they require for example the boundedness of the potentials U_i uniformly with respect to $i \in L$ (see Part. 1).
Because of this we think it's better to verify hypotheses (A) in concrete example when the criterion of Bakry-Emery, for example, holds.
In the next proposition we give a general criterion fro hypoteses (B) restricting the "game" to the case of compact single spin space.

Proposition 2 - Compact single spin space

Let's consider a C^1-Local Specification arising from an interaction ϕ as shown in Chapter 1, and suppose that the single spin space S is compact. If condition (A) holds with the Sobolev constant c_0 then the following estimate for (B) is true:

$$0 \leq e_{ij} \leq \frac{1}{2} e_0^{1/2} \cdot \sup_{\omega,\omega'} \cdot \left| \nabla_i \cdot U_j(\omega) - \nabla_i U_j(\omega') \right| .$$

Proof.

Let $f \in D(\mathcal{E})$ (the domain of the Dirichlet form: see Chapter 2).

Then $(E_i f^2)^{1/2} = \tau_i f$ differentiable and $\left|\nabla_j (E_i f^2)^{1/2}\right| = \dfrac{|\nabla_j E_i f^2|}{2(E_i f^2)^{1/2}}$.

Moreover $\nabla_j E_i f^2 = E_i \nabla_j f^2 + E_i (f^2; \nabla_j U_i)$, where $E_i(f^2; \nabla_j U_i) =$

$= E_i(f^2 \nabla_j U_i) - (E_i f^2)(E_i \nabla_j U_i)$ is the truncated correlation function of f^2 and $\nabla_j U_i$. Hence we have : $|\nabla_j E_i f^2| \leq |E_i \nabla_j f^2| + |E_i(f^2; \nabla_j U_i)|$. Now we evaluate the last two terms separately:

$$|E_i \nabla_j f^2| = |E_i 2 f \cdot \nabla_j f| \leq 2 \cdot \|f\|_{L^2(E_i)} \cdot \|\nabla_j f\|_{L^2(E_i)} = 2 \cdot (E_i f^2)^{1/2} (E_i |\nabla_j f|^2)^{1/2} .$$

To evaluate the second term we use the indentity:

$$E_i(f^2; \nabla_j U_i) = \frac{1}{2} E_i \otimes \tilde{E}_i \left(f^2(\omega) - f^2(\tilde{\omega}) \right) \cdot \left(\nabla_j U_i(\omega) - \nabla_j U_i(\tilde{\omega}) \right)$$

where $E_i \otimes \tilde{E}_i$ is the product of the mesure E_i and E_i with integration variable equal to w and w respectively. From this we have:

$$\left| E_i(f^2; \nabla_j U_i) \right| \leq \frac{1}{2} E_i \otimes \tilde{E}_i \left| f^2(\omega) - f^2(\tilde{\omega}) \right| \cdot \left| \nabla_j U_i(\omega) - \nabla_j U_i(\tilde{\omega}) \right| \leq$$

$$\leq \frac{1}{2} \left(\sup_{\omega,\omega'} \left| \nabla_j U_i(\omega) - \nabla_j U_i(\omega') \right| \right) E_i \otimes \tilde{E}_i \left| f(\omega) + f(\tilde{\omega}) \right| \cdot \left| f(\omega) - f(\tilde{\omega}) \right| \leq$$

$$\leq \frac{1}{2} \left(\sup_{\omega,\omega'} \left| \nabla_j U_i(\omega) - \nabla_j U_i(\omega') \right| \right) \cdot (E_i f^2)^{1/2} \cdot \left(E_i \otimes \tilde{E}_i \left(f(\omega) - f(\tilde{\omega}) \right)^2 \right)^{1/2} .$$

Since E_i satisfy (L-S) with constant c_0 so does $E_i \otimes E_i$ with the same constant. Hence the Poincaré inequality is true for the form arising from $E_i \otimes \tilde{E}_i$ with mass gap equal to $\frac{1}{c_0}$.

Since $\langle f(\omega) - f(\tilde{\omega}) \rangle_{E_i(\omega) \otimes \tilde{E}_i(\tilde{\omega})} = E_i \otimes \tilde{E}_i \cdot (f(\omega) - f(\tilde{\omega})) = 0$

we have:

$$| E_i(f^2; \nabla_j U_i) | \leq \frac{1}{2} \left(\sup_{\omega,\omega'} | \nabla_j U_i(\omega) - \nabla_j U_i(\omega') | \right) \cdot (E_i f^2)^{1/2} \cdot c_0^{1/2} \cdot \left[E_i \otimes E_i \left(|\nabla_i f(\omega)|^2 + |\nabla_i f(\tilde{\omega})|^2 \right) \right]^{1/2} \leq$$

$$\leq c_0^{1/2} \cdot \left(\sup_{\omega,\omega'} | \nabla_j U_i(\omega) - \nabla_j U_i(\omega') | \right) \cdot (E_i f^2)^{1/2} \cdot \left(E_i |\nabla_i f|^2 \right)^{1/2}$$

Finally we obtain: $|\nabla_j (E_i f^2)^{1/2}| \leq \frac{1}{2(E_i f^2)^{1/2}} \cdot |\nabla_j E_i f^2| \leq$

$$\leq \frac{1}{\cancel{2(E_i f^2)^{1/2}}} \left[2 (E_i f^2)^{1/2} \cdot \cancel{(E_i f^2)^{1/2}} \cdot (E_i |\nabla_j f|^2)^{1/2} + c_0^{1/2} \cdot \left(\sup_{\omega, \omega'} |\nabla_j U_i(\omega) - \nabla_j U_i(\tilde{\omega})| \right) \cdot \cancel{(E_i f^2)^{1/2}} \cdot (E_i |\nabla_i f|^2)^{1/2} \right]$$

and this conclude the proof.

///

Example 1: Stochastic Heisenberg Model

This model was investigated by Halley and Stroock in [H-S 6], where they proved with other methods that Log-Sobolev inequality holds at sufficiently high "temperature". Here the single spin space is the sphere $S \equiv S^n$ in $\mathbb{R}^{n+1}$ for $n \geq 2$ with its natural Riemannian probability mesure ρ. When we discussed the method of Bakry-Emery we proved that ρ satisfy (L-S) with Sobolev constant $c_0 = (n-1)^{-1}$

$$\int_{S^n} d\rho\, f^2 \log f \leq c_0 \cdot \int_{S^n} d\rho\, \|\nabla f\|^2 + \left(\int_{S^n} d\rho\, f^2 \right) \log \left(\int_{S^n} d\rho\, f^2 \right)^{1/2}$$

The statistical model is defined by the following nearest-neighbrhood interaction ϕ, where we have inserted the "inverse-temperature" parameter β :

$$\phi_X = \begin{cases} \beta\cdot(\omega_i ; \omega_j) & \text{if } X=\{i,j\} \quad |i-j|=1 \quad i,j\in L \equiv \mathbb{Z}^d \quad d\geq 1 \\ 0 & \text{otherwise} \end{cases}$$

($(\,;\,)_{\mathbb{R}^{n+1}}$ indicates the scalar product in $\mathbb{R}^{n+1}$).

The potential at the site $i\in\mathbb{Z}^d$, U_i is:

$$U_i = \sum_{X\ni i} \phi_X = \beta\cdot\Big(\omega_i ; \sum_{\substack{|j-i|=1 \\ j\in\mathbb{Z}^d}} \omega_j\Big) .$$

Applying the Bakry-Emery criterion to the measure $dE_i \equiv \dfrac{dp\, e^{-U_i}}{\int dp\, e^{-U_i}}$ we see that, because the Hessian of U_i is zero, the mesure E_i satisfy (L-S) uniformly in $i\in\mathbb{Z}^d$ and $\omega\in(S^n)^{\mathbb{Z}^d\setminus\{i\}}$ with constant $c_0=(n-1)^{-1}$ (U_i is linear in the variable $\omega_i\in S^n$).

To calculate c_{ij} we see that $\nabla_j U_i = \beta\omega_i$ vector in the tangent space $T_{\omega_i}S^n$. And so: $c_{ij} = \frac{1}{2} c_0^{1/2} \sup_{\omega\omega'} \| \nabla_j U_i(\omega) - \nabla_j U_i(\omega') \|_{T_{\omega_i}S^n} = \frac{1}{2} c_0^{1/2}\cdot\beta\|\omega_i-\omega_i'\| = c_0^{1/2}\cdot\beta$ if $|i-j|=1$

and $c_{ij}=0$ otherwise.

Than we obtain for $\gamma = \max\left(\sup_i \sum_j c_{ij} ; \sup_j \sum_i c_{ij}\right)$ in condition (C):

$$\gamma = \sup_i \sum_{\substack{j\in\mathbb{Z}^d \\ |j-i|=1}} \beta c_0^{1/2} = 2d\beta\, c_0^{1/2} = 2d\cdot\beta\,(n-1)^{-1/2} .$$

From this we see that the Stochastic Heisenberg Model satisfy (L-S) at sufficiently high temperature (or sufficiently small inverse temperature):

$$\beta < \frac{1}{2d}\cdot(n-1)^{1/2} \quad ,$$

To see how we case satisfy hypotheses (B) in the non-compact case we restrict attention to a specific class of models. This mainly in order to verify the condition (A) without too restrictive general assumption on the potential U_i.

We choose as single spin space the linear space $\mathbb{R}$: $S=\mathbb{R}$ (but the method apply word by word to $\mathbb{R}^n$ $n>1$). Let's consider a real matrix $(G_{ij})_{i,j\in\mathbb{Z}^d}$ with constant terms on the diagonal: $G_{ii}=G_{oo}>0$ $i\in\mathbb{Z}^d$. The umperturbed mesure at each site is the gaussian mesure:

$$d\rho(x)=\left(\frac{G_{oo}}{2\pi}\right)^{1/2}\cdot dx\cdot e^{-\frac{G_{oo}}{2}x^2}\qquad x\in\mathbb{R}\equiv S.$$

This mesure we have shown satisfy (L-S) with constant $c_o=G_{oo}$.

The interaction ϕ is a two body interaction.

$$\phi_X=\begin{cases}\frac{1}{2}G_{ij}\cdot\omega_i\cdot\omega_j & \text{if } X=\{i;j\}\\ 0 & \text{if } |X|>2 \\ V & \text{if } X=\{i\}\end{cases}\qquad |X|=\textit{cardinality of } X$$

where $V:\mathbb{R}\to\mathbb{R}$ is a C^1-function such that $e^{-V}\in L^1(\rho)$. In order to define the interaction potentials U_i we have to restrict the space of configurations from $\Omega\equiv\mathbb{R}^{\mathbb{Z}^d}$ to:

$$\Omega_G=\left\{\omega\in\Omega:\left|\sum_{j\in\mathbb{Z}^d}G_{ij}\omega_j\right|<+\infty\ \forall i\in\mathbb{Z}^d\right\}.$$

On this space we consider a Local Specification E_G defined by the interaction potentials:

$$U_i=\sum_{j\neq i}G_{ij}\omega_i\omega_j+V(\omega_i)$$

Now we want to prove the following proposition.

Proposition 3.

Suppose that $\inf_{x \in \mathbb{R}} V''(x) = m^2 > G_{oo}$ and $\sup_{x \in \mathbb{R}} |V'(x)| < \infty$. Then $\mathbb{E}_G$ is a C^1- Local Specification on Ω_G , the mesure E_i^{ω} satisfy (L-S) uniformly in $i \in \mathbb{Z}^d$ and $\omega \in \Omega_G$ with a constant $c_o \leq G_{oo} - m^2$ and condition (B) hold with the following estimate:

$$0 \leq c_{ij} \leq |G_{ij}/G_{oo}| \cdot \left(1 + \frac{1}{2} c_o^{1/2} \cdot \sup_{x,y \in \mathbb{R}} |V'(x) - V'(y)|\right) .$$

Proof.

Since V is C^1, $\mathbb{E}_G$ is a C^1-Local Specification. To check condition (A) we have just to apply Bakry-Emery criterion and note that the Ricci's tensor is zero.

To verify the last point we take the proof of Proposition 2 with the following changes: $E_i(f^2; \nabla_j U_i) = E_i(f^2; G_{ij}\omega_i) = G_{ij} \cdot E_i(f^2; \omega_i) =$

$$= G_{ij} \cdot G_{oo}^{-1} \cdot \left[E_i \nabla_i f^2 - E_i(f^2; V')\right] = G_{ij} \cdot G_{oo}^{-1} \cdot \left[2\, E_i f \nabla_i f - E_i(f^2; V')\right]$$

$$|E_i(f^2; \nabla_j U_i)| \leq |G_{ij}/G_{oo}| \cdot \left\{2\,(E_i f^2)^{1/2} \cdot (E_i |\nabla_i f|^2)^{1/2} + c_o^{1/2} \left(\sup_{x,y} |V'(x) - V'(y)|\right) (E_i f^2)^{1/2} (E_i |\nabla_i f|^2)^{1/2}\right\}$$

$$|\nabla_j (E_i f^2)^{1/2}| \leq \frac{|\nabla_j E_i f^2|}{2(E_i f^2)^{1/2}} \leq \frac{1}{2(E_i f^2)^{1/2}} \left(2 (E_i f^2)^{1/2} (E_i |\nabla_j f|^2)^{1/2} + |E_i(f^2; \nabla_j U_i)|\right) \leq$$

$$\leq (E_i |\nabla_j f|^2)^{1/2} + |G_{ij}/G_{oo}| \cdot \left(1 + \frac{c_o^{1/2}}{2} \sup_{x,y} |V'(x) - V'(y)|\right) \cdot (E_i |\nabla_i f|^2)^{1/2} .$$

The essential point in the proof is $E_i(f^2; \omega_i) = G_{oo}^{-1} \cdot \left[E_i \nabla_i f^2 - E_i(f^2; V)\right]$ where we used integration by part in the ω_i variable.

///

Remark 1.

Unfortunatly this proposition does not apply to models of statistical mechanics coming from discrete approximation of models of Enchidean Quantum Field Theory with polynomial interaction (see Ref.[G-R-S] page 191). In these case the approximation of the operator $-\Delta + m^2$ by "finite-difference operator" provide an approximation of the "Free Gaussian mesure" by a Gibbs mesure of the Local Specification constructed with a matrix (G_{ij}) as above. But the polynomial interaction in the theory gives rise to a local term V of a polynomial type.

3.3 Criteria for Log-Sobolev Inequalities for descreete single spin space and applications to the Stochastic Ising Models

In this section we prove that the criterion given in Section 3.1 for the case of "continous" single spin space S, can be extended to include cases where S is descreete. For definiteness we consider

$$S = \mathbb{Z}_2 = \{+1; -1\} .$$

We saw in Part 1 that any probability mesure on S satisfy (L-S) and that the Sobolev constant can be choosen uniformly for a set of mesure $(\mu_\alpha)_{\alpha \in I}$ if exist constants $0 < m \leq M < 1$ such that $0 < m \leq \mu_\alpha\{+1\} \leq M < 1 \quad \forall \alpha \in I$. This property as we shall see simplify partly the work.

We have already fixed in Chapter 2 what we intend for the Dirchlet form in this setting where instead of the gradient operator ∇ on a Riemannian manifold we have now the "finite difference operator"

$$B_j f = \frac{1}{2}\cdot\left(f_{|\sigma_j=+1} - f_{|\sigma_j=-1}\right)\sigma_j \qquad f : \{+1;-1\}^{\mathbb{Z}^d} \longrightarrow \mathbb{R} \qquad j \in \mathbb{Z}^d.$$

The method we are going to prove works on the class of Local Specification coming from a **"Gibbsian interaction"** ϕ :

$$\|\phi\| \equiv \sup_{i \in L} \sum_{\substack{X \subset L \text{ FINITE} \\ X \ni i}} \|\phi_X\|_\infty < +\infty$$

and from the uniform mesure μ_0 on $\{+1;-1\}$: $\mu\{+1\} = \mu\{-1\} = 1/2$.

This mesure satisfy (L-S) with constant 1.

The arguments we use to prove (L-S) for this "discrete" case are completely similar to those used in Section 3.1. Instead of condition (B) we can use this weaker one:

$$\text{(B')} \qquad |B_j(E_i f^2)^{1/2}| \leq \alpha\cdot\left(E_i |B_j f|^2\right)^{1/2} + c_{ij}\cdot\left(E_i |B_i f|^2\right)^{1/2}$$

for a constant $1 \leq \alpha + \infty$

Conditions (A') and (B') (changing ∇_j with B_j) permit us to state the following analogue of Proposition 1 in 3.1.

Proposition 4.

Let $\mathbb{E}$ be a Local Specification satisfying condition (A) (B') (C). Let μ be an extremal Gibbs mesure $\mu \in \partial\mathcal{G}(\mathbb{E})$

Then we have:

$$\mu f^2 \log f \leq \alpha^2 \cdot c_0 (1-\gamma)^{-2} \cdot \mathcal{E}(f;f) + (\mu f^2) \log (\mu f^2)^{1/2} \qquad 0 \leq f \in D(\mathcal{E}) .$$

The following simple result tell us that for Gibbsian interaction the condition (A) is redundant.

Proposition 5.

Let E be a Local Specification coming from a Gibbsian interaction ϕ. Then the Kernels E_i^ω satisfy (L-S) with a constant $c_0 < \infty$ indipendent of $i \in L$ and $\omega \in \Omega$.

Proof.

In Part 1 we saw that any probability mesure on $S = \{+1;-1\}$ satisfy (L-S). Moreover we have for Gibbsian interaction that:

$$0 < \left(1 + e^{2\|\phi\|}\right)^{-1} \leq E_i^\omega(\{+1\}) \leq \left(1 + e^{-2\|\phi\|}\right)^{-1} < 1 \qquad \forall i \in L .$$

Since the bounds $\left(1+e^{2\|\phi\|}\right)^{-1}$, $\left(1+e^{-2\|\phi\|}\right)^{-1}$ are indipendent of $i \in L$ we have that the Sobolev constant can be choosen uniform. ///

Now we can restrict our attention on methods to check condition (B). We shall prove the following analogue of Proposition 2.

Proposition 6.

Let $\mathbb{E}$ be a Local Specification builded by a Gibbsian potential ϕ. Then condition (B') holds:

$$|B_j (E_i f^2)^{1/2}| \leq \alpha \cdot (E_i |B_j f|^2)^{1/2} + c_{ij} \cdot (E_i |B_i f|^2)^{1/2}$$

for any function $f : \Omega \to \mathbb{R}$ and with constants

$$1 \leq \alpha \leq 2^{1/2} \cdot e^{12\|\phi\|}$$

$$0 < c_{ij} \leq e^{8 \cdot \|B_j U_i\|_\infty} \cdot c_0^{1/2} \cdot \|B_j U_i\|_\infty .$$

Proof. of Proposition 6.

In order to use the fundamental theorem of calculus we introduce the following functions and mesures: $s_j \in [-1;+1]$

$$f_{s_j} \equiv A_j f + B_j f \cdot \frac{s_j}{\sigma_j} \qquad A_j = I - B_j \implies \partial_{s_j} f_s = \frac{B_j f}{\sigma_j}$$

$$U_{i;s_j} \equiv A_j U_i + B_j U_i \frac{s_j}{\sigma_j}$$

$$E_{i;s_j} \equiv \frac{\mu_0 \cdot e^{-U_{i,s_j}}}{\int \mu_0 \, e^{-U_{i,s_j}}} \implies \left| \frac{d E_{i,s_j}}{d E_i} \right| \leq e^{4\|B_j U_i\|_\infty} . \quad (*)$$

Note that $f_{\sigma_j} = f$. We start calculating $|B_j E_i f^2|$:

$$|B_j E_i f^2| = \frac{1}{2} \left| (E_i f^2)_{|\sigma_j=+1} - (E_i f^2)_{|\sigma_j=-1} \right| = \frac{1}{2} \left| \int_{-1}^{+1} ds_j \frac{d}{ds_j} (E_{i,s_j} f^2_{i,s_j}) \right| =$$

$$= \frac{1}{2} \left| \int_{-1}^{+1} ds_j \left[2 \, E_{i;s_j}(f_{s_j} \partial_{s_j} f_{s_j}) + E_{i;s_j}(f^2_{s_j} ; \partial_{s_j} U_{i;s_j}) \right] \right| \leq$$

$$\leq \int_{-1}^{+1} ds_j \, | E_{i;s_j}(f_{s_j} \partial_{s_j} f_{s_j}) | + \frac{1}{2} \int_{-1}^{+1} ds_j | E_{i;s_j}(f^2_{s_j} ; \partial_{s_j} U_{i,s_j}) |$$

The first term in the last inequality gives:

$$\int_{-1}^{+1} ds_j |E_{i;s_j}(f_{s_j} \partial_{s_j} f_{s_j})| \leq \int_{-1}^{+1} ds_j (E_{i;s_j} f^2_{s_j})^{1/2} \cdot (E_{i;s_j} |\partial_{s_j} f_{s_j}|^2)^{1/2} \leq (\text{by } (*)) \leq$$

$$\leq e^{4\|B_j U_i\|_\infty} \cdot (E_i |B_j f|^2)^{1/2} \cdot \int_{-1}^{+1} ds_j \cdot (E_{i;s_j} f^2_{s_j})^{1/2} \leq$$

$$\leq 2\cdot e^{4\|B_jU_i\|_\infty}\cdot(E_i|B_jf|^2)^{1/2}\cdot(E_iA_jf^2)^{1/2}\leq \text{(by (*) and Def. of } A_j) \leq$$
$$\leq 2\cdot 2^{1/2}\cdot e^{6\|B_jU_i\|_\infty}\cdot A_j(E_if^2)^{1/2}\cdot(E_i|B_jf|^2)^{1/2}.$$

To evaluate the term $\frac{1}{2}\int_{-1}^{+1}ds_j\,|E_{i,s_j}(f^2_{s_j};\partial_{s_j}U_{s_j})|$ we use same method used in Proposition 2 and the "tricks" above. At the end we obtain: $\frac{1}{2}\int_{-1}^{+1}ds_j\,|E_{i,s_j}(f^2_{s_j};\partial_{s_j}U_{s_j})|\leq 2^{1/2}\cdot A_j(E_if^2)^{1/2}\cdot e^{8\|B_jU_i\|_\infty}\cdot\left(\sup_{\sigma\sigma'}\left|\frac{B_jU_i(\sigma)}{\sigma_j}-\frac{B_jU_i(\sigma')}{\sigma_j'}\right|\right)\cdot$

$$\cdot\left[\left(\frac{c_0}{2}\right)^{1/2}\cdot(E_i|B_if|^2)^{1/2}+2(E_i|B_jf|^2)^{1/2}\right].$$

Combining the two result above we have: $|B_j E_if^2|\leq$

$$\leq 2A_j(E_if^2)^{1/2}\cdot\Big\{2^{1/2}\cdot e^{6\|B_jU_i\|_\infty}\cdot(E_i|B_jf|^2)^{1/2}+2^{-1/2}\cdot e^{8\|B_jU_i\|_\infty}\cdot\left(\sup_{\sigma,\sigma'}\left|\frac{B_jU_i(\sigma)}{\sigma_j}-\frac{B_jU_i(\sigma')}{\sigma_j'}\right|\right)\cdot$$
$$\cdot\left[\left(\frac{c_0}{2}\right)^{1/2}\cdot(E_i|B_if|^2)^{1/2}+2(E_i|B_jf|^2)^{1/2}\right]\Big\}$$

Using the property $B_jf^2=2A_jf\cdot B_jf$ we obtain the final result with
$$\alpha\leq 2^{1/2}\cdot e^{6\|B_jU_i\|_\infty}\cdot\left(1+2^{1/2}\cdot e^{2\|B_jU_i\|_\infty}\cdot\sup_{\sigma\sigma'}\left|\frac{B_jU_i(\sigma)}{\sigma_j}-\frac{B_jU_i(\sigma')}{\sigma_j'}\right|\right).$$

To obtain the estimate that appear in the statement of the proposition we have to use $\|B_jU_i\|_\infty\leq\|\phi\|$ and $\sup_{\sigma\sigma'}\left|\frac{B_jU_i(\sigma)}{\sigma_j}-\frac{B_jU_i(\sigma')}{\sigma_j'}\right|\leq 2\|B_jU_i\|_\infty$.

///

Applications: Stochastic Ising Models

In the series of paper [H-S;i] $i=1\cdots 6$ Holley and Stroock studied a class of stochastic process on the space $\Omega=\{+1;-1\}^{\mathbb{Z}^d}$ with the property that every Gibbs mesure of a finite range interaction is a stationary mesure every such a processes. So the equilibrium states in Statistical Mechamics (The Gibbs mesures) can be seen as the equilibrium states of a (stochastic) evolution on Ω. They firstly caracterized this processes as solutions of martingale problems whose data are given in terms of a finite range (hence Gibbsian) Local Specification. In order to investigate the link between the ergodic properties of the processes and these of the Gibbs mesures they computed the generators of the semigroups of the processes. If $\mu \in \mathcal{G}(\mathbb{E})$ the related Dirichlet forms are of the following form: $\mathcal{E}'(f,f) = \sum_{j \in \mathbb{Z}^d} c_j \cdot |B_j f|^2$

where $c_j : \Omega \rightarrow \mathbb{R}$ are definite coefficient functions (solutions of the "detailed balance equation" see Ref. [H-S 1]).

For example the c_j's can be choosen to be:

$$c_j = \frac{1}{2}\cdot\left(1 - \frac{B_j e^{-U_j}}{A_j e^{-U_j}}\right) \qquad A_j = I - B_j \qquad j \in \mathbb{Z}^d .$$

We want to prove that the Dirichlet form $\mathcal{E}'$ satisfy a Log-Sobolev inequality.

Proposition 7.

Suppose μ is an extremal Gibbs mesure $\mu \in \partial\mathcal{G}(\mathbb{E})$ of a Gibbsian Local Specification . Then if

$$\gamma \equiv \max\left(\sup_i \sum_j c_{ij} ; \sup_i \sum_j c_{ji}\right) < 1$$

where $c_{ij} = c_0^{1/2} \cdot e^{8\|B_j U_i\|_\infty} \cdot \|B_j U_i\|_\infty$ then $\mathcal{E}'$ satisfy (L-S) with a Sobolev constant $\alpha^2 \cdot c_0 \cdot (1-\gamma)^{-2}$.

Proof.

The proof follows from the fact that since ϕ is

Gibbsian exist a constant $a>0$ such that: $0<a\leqslant c_j\leqslant 1$ $\forall j\in\mathbb{Z}^d$.
Now since $\gamma<1$ Proposition 5 imply that the form $\mathcal{E}(f,f)=\sum_j |B_j f|^2$ satisfy (L-S) with constant $c_0\cdot(1-\gamma)^{-2}$ and then: $f\geqslant 0$

$$\mu f^2\log f-(\mu f^2)\log(\mu f^2)^{1/2}\leq c_0\cdot(1-\gamma)^{-2}\cdot\sum_{j\in\mathbb{Z}^d}|B_j f|^2=$$

$$=c_0(1-\gamma)^{-2}\cdot\sum_{j\in\mathbb{Z}^d}\frac{c_j}{c_j}|B_j f|^2\leq c_0(1-\gamma)^{-2}\cdot\sum_{j\in\mathbb{Z}^d}\frac{c_j}{a}|B_j f|^2=\frac{c_0}{a}\cdot(1-\gamma)^{-2}\cdot\mathcal{E}'(f,f)\,.\quad ///$$

3.4 Finite range translation invariation Local Specification and Log-Sobolev Inequalities

We want to close this section investigating condition (B') and (C) in the high temperature region.

Lemma.

Suppose $\mu\in\partial\mathcal{G}(\mathbb{E}_\beta)$ where $\mathbb{E}$ is a finite range translation invariant Local Specification and $\mathbb{E}_\beta$ is the related "Specification at temperature $\frac{1}{\beta}$".
In this situation if β is a sufficiently small μ satisfies (L-S).

Proof.

Since $\mathbb{E}_\beta$ is finite range and translation invariant it is Gibbsian for any $\beta>0$. Moreover since $\|B_j U_i\|\leq\|\phi\|$ we have by Proposition 6 that

$$c_{ij} \equiv c_o^{1/2} \cdot e^{\beta \cdot 8 \|B_j U_i\|_\infty} \cdot \beta \cdot \|B_j U_i\|_\infty \leq c_o^{1/2} \cdot e^{8 \cdot \beta \cdot \|\phi\|} \cdot \beta \|B_j U_i\|_\infty .$$

(c_0 being the Sobolev constant of any E_i^ω : see Proposition 5). But for $j \in L$ fixed $\|B_j U_i\|_\infty$ is different from zero only for a finite number of $i \in L$. Because of the translation invariance the same is true for j if we fix $i \in L$. So we can say that exist a constant $K > 0$ such that:

$$\gamma \leq c_o^{1/2} \cdot e^{8 \cdot \beta \cdot \|\phi\|} \cdot K \cdot \beta \cdot \|\phi\| .$$

. From this expression we can see that if $\beta > 0$ is sufficiently small $\gamma < 1$ and Proposition 4 apply. ///

3.5 The connection between the theory of Zegarlinski and the Dobrushin Uniqueness Theorem

As we saw in last sections the technique of Zegarlinski is very chose to that of Dobrushin. But we were able to prove Log-Sobolev inequality just for extreme Gibbs mesure. Many arguments suggest that if a phase transition appears ($|\mathcal{G}(\mathbb{E})| > 1$) then no exponential convergence to equilibrium should hold. In particular this would imply that (L-S) holds only if there is a unique phase (see Ref. [F2] for arguments based on Large Deviation theory, and Ref. [D2] for recent investigation). Whith this in mind,
in order to compare the criterion of Zegarlinski ad that one of Dobrushin, we formulate in the next proposition a condition that is a little bit stronger than that one of

Zegarlinski in Proposition 2, but implies also the uniqueness of the Gibbs mesure. We formulate the result in the "continous" case and use notation of Section 3.1.

Proposition 8.

Let's consider a C^1-local specification coming from a potential ϕ when the spin space S is compact.

Suppose the following estimate holds:

(B")
$$|\nabla_j E_i f| \le (E_i |\nabla_j f|) + c_{ij} \cdot (E_i |\nabla_i f|)$$

and condition (C) on γ holds too.

Then condition (B) holds and there is a unique Gibbs mesure.

Proof.

For the uniqueness we note that our conditions imply

$$|\nabla_j E_i f| \le E_i |\nabla_j f| + c_{ij} \cdot E_i |\nabla_i f| \le \|\nabla_j f\|_\infty + c_{ij} \cdot \|\nabla_i f\|_\infty \quad \Longrightarrow$$

$\|\nabla_j E_i f\|_\infty \le \|\nabla_j f\|_\infty + c_{ij} \cdot \|\nabla_i f\|_\infty$. We can limit us to prove the Proposition just for cylindric f. Since $c_{ii}=0$ we have:

$$\sum_j \|\nabla_j E_i f\|_\infty \le \sum_j \|\nabla_j f\|_\infty - (1 - \sum_j c_{ij}) \cdot \|\nabla_i f\|_\infty \le \sum_j \|\nabla_j f\|_\infty - (1-\gamma) \cdot \|\nabla_i f\|_\infty$$

This is precisely the starting point of the proof. of Lanford in [L], of the Dobrushin theorem where instead of $\|\nabla_j f\|_\infty$ there is $\delta_j(f) \equiv \sup\{|f(\omega) - f(\omega')| : \omega = \omega' \text{ off } j \in L\}$.

In the same way we deduce that there is a sequence $i_1 \cdots i_k \in L$ such that

$$\lim_{K\to\infty} \sum_{j\in L} \| \nabla_j E_{i_K} \cdots E_{i_1} f \|_\infty = 0$$

Since S is compact there exist $M > 0$ such that

$\delta_j(f) \leq M \cdot \|\nabla_j f\|_\infty$ So we have:

$$\lim_{K\to\infty} \sum_j \delta_j (E_{i_K} \cdots E_{i_1} f) = 0$$

Having in mind that $\sup f - \inf f \leq \sum_{j\in L} \delta_j(f)$ (see Ref.) we may apply word by word the first proof. of Lonford and deduce the uniqueness of the Gibbs mesure.

We now verify that our estimates imply condition (B):

$$|\nabla_j (E_i f^2)^{1/2}| = \frac{|\nabla_j E_i f^2|}{2\cdot(E_i f^2)^{1/2}} \leq \frac{1}{2(E_i f^2)^{1/2}} \cdot \left[2 E_i |f \partial_j f| + 2 C_{ij} E_i |f \partial_i f| \right] \leq$$

$$\leq (E_i |\partial_j f|^2)^{1/2} + C_{ij} \cdot (E_i |\partial_i f|^2)^{1/2} . \qquad ///$$

Corollary.

In the same hypotheses of the last Proposition, if $\gamma < 1$ then a unique Gibbs mesure satisfy (L-S) with Sobolev constant $c_0 \cdot (1-\gamma)^{-2}$. ///

Example

As esample we take the situation of Proposition 3 of this chapter with V=0. Then an integration by parts shows that $E_i(f ; \partial_j U_i) = G_{ij} \cdot E_i(f ; \omega_i) = G_{ij}/G_{00}\, E_i \partial_i f$,

$$\partial_j E_i f = E_i \cdot \partial_j f \quad + C_{ij} \cdot E_i \cdot \partial_i f$$

with $C_{ij} \equiv G_{ij} / G_{00}$.

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