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Some

Qualitative and Quantitative Results

on a Differential Inclusion

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1. Introduction

Any differential inclusion may be interpreted as a system whose evolution is not uniquely determined, i.e. as a **non–deterministic dynamical system.** From this point of view, it is natural to look for some way to distinguish among all solutions those which are **more probable, why** and **how much.** A possible way to answer these questions might consist in a multivalued analog of stochastic calculus, which would require to accept nowhere differentiable functions as "solutions" in some generalized sense and to choose some measure in the (enlarged) set of solutions.

We preferred an **intrinsic** approach. The questions above may be answered relying exclusively on the structures **naturally** related to differential inclusions. In fact, measure spaces may be substituted by **metric** spaces and **probability** by **likelihood.** The latter, introduced in [B] according to an idea of A. Cellina, is based on the **measure of non–compactness.** This choice, although arbitrary, may be justified observing that in any complete metric space compact sets are **negligible** in the sense that they can be approximated (according to the Hausdorff distance) to any degree of accuracy by sets with a finite number of elements. Therefore, the use of measures of non–compactness may be interpreted as a generalization of the idea of relative frequencies.

In the present thesis the ideas above are applied to the differential inclusion

$$(\mathrm{P}) \qquad \begin{cases} \dot{x}(t) \in v(t) + r(t)\partial U \\ x(0) = 0 \end{cases} \qquad t \in [0,T]$$

where ∂U stands for the relative boundary of a compact convex subset of $\mathbf{R}^n$.

A natural way to disinguish among the different solutions of (P) is to consider the point reached at time T. This leads to introduce the map assigning to any point in the attainable set of (P) the set of solutions reaching it. Even in this simple and apparently very regular case, this map may fail to be continuous. More precisely, it is the lower semicontinuity that may be lost, as will be shown by an example, while the upper semicontinuity holds in general. Notice, however, that the results obtained hold also in the case ∂U is replaced by U.

Using these results, a formula allowing the computation of the Kuratowski index of the set of solutions reaching a given point is estabilished. Next we estimate the same index for the family of the solutions passing through a finite number of given points and of the solutions reaching a closed subset of the attainable set. All this allows also to obtain an integral formula for the metric likelihood based on the Kuratowski index α and evaluated in any $\mathbf{L^p}$. The reason for the choice of α instead of Hausdorff's β (as considered in [B] only in the case of $\mathbf{L^2}$) relies mainly in its being more **intrinsic**. That is to say, if A is a bounded subset of a metric space (M, d), then $\alpha_{(M,d)}(A) = \alpha_{(A,d_{|A})}(A)$. In other words, the quantity $\alpha(A)$ may be defined independently from the metric space M, using only the set A and the restriction to A of the distance.

After Chapter 2 that contains the notations and some preliminary results, Chapter 3 is concerned with the qualitative theory. The quantitative part is deferred to Chapter 4 while the most technical lemmas are collected in the last Chapter 5.

2. Notations and Preliminary Results

In any vector space V normed with $\|\cdot\|_V$, the following quantities will be of use:

$$
\begin{aligned}
d_M(x,y) &= \|y-x\|_V && \text{(point-to-point distance)}\\
d_M(x,Y) &= \inf\{d_v(x,y): y \in Y\} && \text{(point-to-set distance)}\\
\mathrm{diam}_V(X) &= \sup\{d_v(x_1,x_2): x_1,x_2 \in X\} && \text{(diameter)}\\
\mathrm{d^*}_V(X,Y) &= \sup\{d_V(x,Y): x \in X\} && \text{(excess)}\\
\mathrm{d}_V(X,Y) &= \max\{\mathrm{d^*}_V(X,Y), \mathrm{d^*}_V(Y,X)\} && \text{(Haudorff distance)}
\end{aligned}
$$

where $x, y \in V$ and $X, Y \subseteq V$. $B_V(x,r)$ stands for the closed ball about x with radius r.

Let A be any bounded subset of V; its Kuratowski measure of non–compactness is defined as

$$\alpha_V(A) = \inf\left\{\varepsilon > 0 : \begin{array}{c} A \text{ can be covered by a finite} \\ \text{number of sets of diameter} < \varepsilon \end{array}\right\}.$$

The main properties of α_V are:

$$
\begin{aligned}
\alpha_V(A) = 0 &\iff A \text{ is relatively compact}\\
B \subseteq A &\iff \alpha_V(B) \le \alpha_V(A)\\
\alpha_V(A \cup B) &= \max\{\alpha_V(A), \alpha_V(B)\}\\
\alpha_V(A \cap B) &\le \min\{\alpha_V(A), \alpha_V(B)\}\\
\alpha_V(A) &= \alpha_V(\overline{\mathrm{co}}(A))\\
\alpha_V(A) &\le \mathrm{diam}_V(A)
\end{aligned}
$$

where $B \subset V$ is bounded and $\overline{\mathrm{co}}(A)$ is the closed convex hull of A. The Hausdorff measure of non–compactness, referred to in the introduction, is defined as

$$\beta_V(A) = \inf\left\{\varepsilon > 0 : \begin{array}{c} A \text{ can be covered by a finite} \\ \text{number of balls of radius} < \varepsilon \end{array}\right\}$$

and its relation with the Kuratowski measure is $\beta_V \leq \alpha_V \leq 2\beta_V$ (both the inequalities being optimal, see [FM]). As already pointed out in the introduction, $\alpha_V(A)$ depends exlusively on A and on the metric structure, while $\beta_V(A)$ depends also on how A is embedded in V. In fact, to evaluate $\beta_V(A)$ it is essential to specify the set where the centres of the balls have to be chosen (see [D], pag. 42). General references about measures of non–compactness are [Sa] and [BG].

Remark that when $V = \mathbf{R}^n$ the subscript V will always be omitted, $\|\cdot\|$ denoting any norm in $\mathbf{R}^n$. In some proofs the choice of a particular norm, the Euclidean one for instance, may lead to some simplifications without implying any loss of generality. In those cases, such choice will be explicitely underlined. In particular, when the Euclidean norm will be of use, the Euclidean scalar product shall be denoted by a dot.

For any real interval $[a,b]$ equipped with the Lebesgue measure m, $\mathbf{L^p}([a,b])$ (with $p \in [1,\infty[$) is the Banach space of the measurable functions $f\colon [a,b] \to \mathbf{R}^n$ such that the usual norm

$$\|f\|_{\mathbf{L^p}([a,b])} = \left(\int_a^b \|f(t)\|^p dt \right)^{1/p}$$

is finite. $\mathbf{AC^p}([a,b])$ denotes the Banach space of the absolutely continuous functions $x\colon [a,b] \to \mathbf{R}^n$ with derivative in $\mathbf{L^p}([a,b])$, normed with

$$\|x\|_{\mathbf{AC^p}([a,b])} = \|x(a)\| + \|\dot{x}\|_{\mathbf{L^p}([a,b])} \ .$$

Whenever S is a subset of $\mathbf{AC^p}([a,b])$, $\dot{S}$ is the set of the derivatives of the elements of S; that is to say

$$\dot{S} = \{f \in \mathbf{L^p}([a,b]) \colon \exists\, x \in S,\ f = \dot{x}\} \ .$$

All along the thesis, I stands for $[0,T]$ with $T > 0$ and, for brevity, we put $\mathbf{AC^p} = \mathbf{AC^p}(I)$ and $\mathbf{L^p} = \mathbf{L^p}(I)$.

A decomposable combination of two functions f, g in $\mathbf{L^p}([a,b])$ is any function of the form $f\chi_{A} + g\chi_{A^C}$, where χ_A is the characteristic function of a measurable subset A of $[a,b]$

and $A^C = [a,b]\backslash A$ is its complement in $[a,b]$. A subset D of $\mathbf{L^p}$ is decomposable if for any two functions f, g in D, any of their decomposable combinations is in D. In the spirit of an analogous definition given in [HU], a decomposable set D is said $\mathbf{L^p}$–integrably bounded if there exists a function φ in $\mathbf{L^p}$ such that $\|f\| \leq \varphi$ for any f in D. Given a subset A of $\mathbf{L^p}$, $\mathrm{de}(A)$ is its decomposable hull, that is to say it is the smallest decomposable subset of $\mathbf{L^p}$ containing A.

Concerning the theory of set–valued maps, general reference will be made to [AC] or to [HU], the former for what concerns continuity properties, the latter for measurability properties. Given a set–valued map $F: I \to \mathcal{P}(\mathbf{R}^n)$, the set of its $\mathbf{L^p}$–selections will be denoted by $\mathrm{Sel}_p(F)$. Such a map is called $\mathbf{L^p}$–integrably bounded if there exists a function $k \in \mathbf{L^p}$ such that $\|F\| \leq k$ a.e. in I, where $\|F\|(t) = \|F(t)\| = \sup\{\|x\| : x \in F(t)\}$.

If K is a convex set, ∂K and $\mathrm{ri}\, K$ denote its relative boundary and its relative interior, while $\mathrm{extr}\, K$ is the set of its extreme points. For the definitions and properties concerning convex sets, we refer to [Ro]. Moreover, a set K is said strictly convex whenever $\partial K = \mathrm{extr}\, K$.

The following assumptions on (P) will always be assumed: p is fixed in $[1, \infty[$, $v \in \mathbf{L^p}$, r is measurable and $r(t) \in [r_m, r_M]$ a.e. in I, for some $r_M \geq r_m > 0$. U is a non empty convex and compact subset of $\mathbf{R}^n$. Finally, $\Sigma(v, r, I)$ denotes the subset of $\mathbf{AC^p}$ consisting of the solutions to (P) and $\mathcal{A}_T$ the attainable set at time T.

These regularity conditions allow to pass to a simpler but equivalent problem.

Proposition 2.1: *There exists a $\Lambda > 0$ such that (P) is equivalent to the normalized problem*

$$(\mathrm{P_n}) \qquad \begin{cases} y'(\lambda) \in \partial U \\ y(0) = 0 \end{cases} \qquad \lambda \in [0, \Lambda] \quad .$$

In other words, there exists a bijective and continuous operator $A: \mathbf{AC^p}([0,\Lambda]) \to \mathbf{AC^p}(I)$ transforming the set of solutions to $(\mathrm{P_n})$ into the set of solutions to (P) bijectively. Further-

more, for any pair of solutions y_1, y_2 *to* $(\mathrm{P_n})$*, it holds that*

$$(2.1) \qquad r_m^{1-\frac{1}{p}} \|y_2 - y_1\|_{\mathbf{AC^p}([0,\Lambda])} \leq \|Ay_2 - Ay_1\|_{\mathbf{AC^p}(I)} \leq r_M^{1-\frac{1}{p}} \|y_2 - y_1\|_{\mathbf{AC^p}([0,\Lambda])}$$

$$(2.2) \qquad (Ay)(T) = y(\Lambda) + w$$

Proof. Let $\rho(t) = \int_0^t r(\tau)d\tau$, $\omega(t) = \int_0^t v(\tau)d\tau$ and $\Lambda = \rho(T)$. Define

$$(Ay)(t) = y\big(\rho(t)\big) + \omega(t) \quad .$$

Remark that $\frac{d}{dt}(Ay)(t) = r(t)y\big(\rho(t)\big) + v(t)$ which shows that if $y \in \mathbf{AC^p}([0,\Lambda])$, then $Ay \in \mathbf{AC^p}(I)$. The invertibilty of A follows directly from that of ρ, which is due to $\dot{\rho} > 0$. Continuity follows from (2.1) and to prove it, simply compute

$$\|Ay_2 - Ay_1\|_{\mathbf{L^p}(I)}{}^p = \int_0^T \left\| \frac{dy_2\big(\rho(t)\big)}{dt} - \frac{dy_1\big(\rho(t)\big)}{dt} \right\|^p dt$$
$$= \int_0^\Lambda \|y_2'(\lambda) - y_1'(\lambda)\|^p \left[r\big(\rho^{-1}(\lambda)\big)\right]^{p-1} d\lambda$$

and (2.1) follows via the bounds on r. (2.2) is trivial, with $w = \omega(T)$. *Q.E.D.*

3. Qualitative Results

Aiming at a distinction among the different solutions of (P), it is natural to introduce the set–valued map

$$\begin{aligned} S\colon \mathcal{A}_T &\to \mathcal{P}(\mathbf{AC^p}) \\ \xi &\to \{x \in \Sigma(v,r,I)\colon x(T) = \xi\} \end{aligned}$$

and, according to the convention above, also the map

$$\begin{aligned} \dot{S}\colon \mathcal{A}_T &\to \mathcal{P}(\mathbf{L^p}) \\ \xi &\to \{f \in \dot{\Sigma}(v,r,I)\colon \int_I f = \xi\} \,. \end{aligned}$$

Purpose of this chapter is to study some qualitative properties of S. In particular, we show that it is upper semicontinuous in the ε sense (see [AC] p. 45) from $\mathcal{A}_T$ to $\mathbf{AC^p}$. An example shows that stronger continuity properties require further assumptions on U.

The chosen norm in $\mathbf{AC^p}$ makes derivation (i.e. the map assigning to a function in $\mathbf{AC^p}$ its derivative in $\mathbf{L^p}$) an isometry. Therefore, it is equivalent to consider $\dot{S}$ and S. Moreover, in view of Proposition 2.1, it is enough to study the "normalized" version of (P) with $v = 0$ and $r = 1$.

What follows is the first part of Lemma 1 in [O2]. Isolating it is useful in view of the next generalization of the same lemma to any $\mathbf{L^p}$.

Lemma 3.1: *Let K be a non empty decomposable subset of $\mathbf{L^p}$ and $Q = \overline{\text{co}}\left\{\int_I f\colon f \in K\right\}$. Call e an extreme point of Q. Then, given any $\varepsilon > 0$ there exists a $\delta > 0$ such that for any pair of functions f, g in K, the inequalities $\left\|e - \int_I f\right\| \leq \delta$ and $\left\|e - \int_I g\right\| \leq \delta$ imply that for any measurable subset A of I, $\left\|\int_A (f-g)\right\| \leq \varepsilon$.*

Proof. Without any loss of generality, this lemma may be proved in the case where $\|\cdot\|$ is the Euclidean norm.

If Q is a singleton, using the decomposability of K it is immediate to prove that also K contains only one element. In this simple case the conclusion of the proof is trivial.

Assume Q is not a singleton, hence there exists a $\vartheta > 0$ such that $\vartheta \leq \varepsilon/2$ and $Q \backslash B(e, \vartheta)$ is not empty. Observe first that clearly $e \notin Q \backslash B(e, \vartheta)$ and since $e \in \text{extr}(Q)$ it follows that $e \notin \overline{\text{co}}\left(Q \backslash B(e, \vartheta)\right)$.

Therefore, e can be (strongly) separated from $\overline{\text{co}}\left(Q \backslash B(e, \vartheta)\right)$ by a hyperplane of the form $\{\eta \in \mathbf{R}^n : p \cdot \eta = c\}$, where $p \in \mathbf{R}^n$, $\|p\| = 1$ and $c = \sup\left\{p \cdot \eta : \eta \in Q \backslash B(e, \vartheta)\right\}$.

Introduce $\delta = p \cdot e - c$ and the half-space $H = \{\eta \in \mathbf{R}^n : p \cdot \eta \geq c\}$. The separation stated above implies that

$$Q \cap B(e, \delta) \subseteq Q \cap H \subseteq Q \cap B(e, \vartheta) . \tag{3.1}$$

Call χ the characteristic function of A. Both the functions $w_+ = f + (f - g)\chi$ and $w_- = f - (f - g)\chi$ are in K, hence their integrals are in Q. More precisely, due to (3.1), to $\left\|e - \int_I f\right\| \leq \delta$ and to $\left\|e - \int_I g\right\| \leq \delta$, both $\int_I w_+$ and $\int_I w_-$ belong to $Q \cap H$. If $\int_I [(f - g)\chi] \geq 0$, then $\int_I w_+$ is in H, otherwise $\int_I [(f - g)\chi] \leq 0$ and $\int_I w_-$ is in H. In any case, $\left\|\int_I (f - g)\chi\right\| < 2\vartheta \leq \varepsilon$, thanks to (3.1). *Q.E.D.*

Remark that both the previous lemma and the next proposition may be proved without any change also for the case of functions defined on an arbitrary measure space. For the purposes of this thesis, this wider generality is useless.

Proposition 3.2: *Let $K \subset \mathbf{L}^\mathbf{p}$ be non empty, decomposable and $\mathbf{L}^\mathbf{p}$–integrably bounded. Define $Q = \overline{\text{co}}\left\{\int_I f : f \in K\right\}$. Call e an extreme point of Q. Then, given any $\varepsilon > 0$ there exists a $\delta > 0$ such that for any pair of functions f, g in K the inequalities $\left\|e - \int_I f\right\| \leq \delta$ and $\left\|e - \int_I g\right\| \leq \delta$ imply that $\|f - g\|_{\mathbf{L}^\mathbf{p}} \leq \varepsilon$.*

Proof. As before, the use of a particular norm is useful, without leading to any loss of generality. Here, the choice $\|\xi\| = \max\{|\xi_1|, \ldots, |\xi_n|\}$ shows that it is enough to consider the one dimensional case, i.e. $n = 1$.

Fix $\varepsilon > 0$, the previous lemma ensures that for any $\bar{\varepsilon} > 0$, with $\bar{\varepsilon} \le \varepsilon/2$, there exists a $\delta > 0$ such that if $\left\|e - \int_I f\right\| \le \delta$ and $\left\|e - \int_I g\right\| \le \delta$ then $\int_A |f - g| \le \bar{\varepsilon}$ for any measurable subset A of I.

Call φ a function in $\mathbf{L^p}$ such that $|h| \le \varphi$ for any h in K. Let

$$A_{\ge} = \{t \in I \colon |f(t) - g(t)| \ge 1\}$$
$$A_{<} = \{t \in I \colon |f(t) - g(t)| < 1\}$$

Clearly,

$$\begin{aligned} \|f - g\|_{\mathbf{L^p}}{}^p &= \int_{A_{\ge}} |f - g|^p + \int_{A_<} |f - g|^p \\ &\le 2^p \int_{A_{\ge}} |\varphi|^p + \int_{A_<} |f - g| \,. \end{aligned}$$

For the former term in the last line, observe that

$$m(A_{\ge}) = \int_{A_{\ge}} 1 \le \int_{A_{\ge}} |f - g| \le \bar{\varepsilon}$$

hence, due to the absolute continuity of the integral, this term may be made smaller than, say, $\varepsilon/2$ if $\bar{\varepsilon}$ is small enough. The latter term may be made smaller than $\bar{\varepsilon} \le \varepsilon/2$ if δ is chosen properly, due to the preceding lemma. *Q.E.D.*

A simple consequence of this result is that an extreme point of Q is the integral of only one function in K.

Notice that the requirement of $\mathbf{L^p}$–integrable boundedness may be avoided in the case $p = 1$ (as shown in [O2]), but is strictly necessary when $p > 1$ as shown by the next example.

Example 3.3: The set $K = \{f \in \mathbf{L}^p([0,1],\mathbf{R}): f \geq 0\}$ is decomposable but not $\mathbf{L}^p$–integrably bounded. 0 is an extreme point of its Aumann integral $Q = [0,+\infty[$. Define $f_n = n\chi_{[0,n^{-p}]}$, for $n \in \mathbf{N}\setminus\{0\}$. It is straightforward to check that $\int_0^1 f_n = n^{1-p}$ and this quantity tends to 0 for n that goes to infinity. Nevertheless, $\|f_n\|_{\mathbf{L}^p} = 1$ for any n. This shows that the functions f_n are far from the null function, although $\int_0^1 f_n$ is arbitrarily near to 0.

Proposition 3.4: *S and $\dot{S}$ are Hausdorff continuous in any extreme point of $\mathcal{A}_T$.*

This follows directly from Proposition 3.2. Passing from extreme points to extremal faces, we have the following

Lemma 3.5: *Let L be a face of $\mathcal{A}_T$ and call H the smallest affine space containing L. For any η in the relative interior of L, set $\delta_\eta = d(\eta, H\setminus L)$. Then for any $\varepsilon > 0$ it holds that*

$$\left.\begin{array}{r} \xi \in L \\ \|\xi-\eta\| \leq \min\left\{\dfrac{1}{2}\delta_\eta\,,\ \dfrac{\varepsilon^p}{T[\mathrm{diam}(U)]^p}\delta_\eta\right\} \end{array}\right\} \implies \mathrm{d}^*{}_{\mathbf{L}^p}\left(\dot{S}^p(\eta), \dot{S}^p(\xi)\right) < \varepsilon\,.$$

(For the definition of face, see [Ro] p. 162)

Proof. The idea of the following procedure consists in modifying the function f_η in order to obtain a function f_ξ in $S(\xi)$ whose $\mathbf{L}^p$–distance from f_η is "small" and independent from ξ. The chosen modification is "large" but on a set of "small" measure.

a) Consider the following continuous parametrization of the half–line leaving from ξ in the direction of the vector $\xi - \eta$:

$$\begin{aligned} \varphi : [0,1[&\to H \\ \lambda &\to \xi + \frac{\lambda}{1-\lambda}(\xi-\eta)\,. \end{aligned}$$

Due to the convexity of L, there exists a unique λ_* such that $\varphi(\lambda_*) \in \partial L$. From the definition of δ_η, $\delta_\eta > 0$ and $\delta_\eta \leq d(\eta, \varphi(\lambda_*)) = \frac{1}{1-\lambda_*} \|\xi - \eta\|$, i.e.

$$1 - \lambda_* \leq \frac{1}{\delta_\eta} \|\xi - \eta\| \ .$$

b) Let f_η be in $\dot{S}(\eta)$. Applying Liapunov's Theorem (see, for example, [Ru], p. 113) we infer the existence of a measurable M_* such that

$$\int_{M_*} f_\eta = \lambda_* \eta \quad \text{and} \quad m(M_*) = \lambda_* T \ .$$

Set

$$f_\xi = f_\eta \chi_{M_*} + \frac{1}{T} \varphi(\lambda_*) \chi_{I \setminus M_*} \ .$$

For a.e. t in I, $f_\xi(t) \in \partial U$ since the fact that L is a face of $\mathcal{A}_T$ implies that $\frac{1}{T} L$ is a face of U. Hence, f_ξ is the derivative of a solution to (P). Moreover,

$$\int_I f_\xi = \int_{M_*} f_\eta + \int_{I \setminus M_*} \frac{1}{T} \varphi(\lambda_*) = \lambda_* \eta + \frac{1}{T} \frac{1}{1 - \lambda_*} (\xi - \lambda_* \eta)(T - \lambda_* T) = \xi$$

so that $f_\xi \in \dot{S}(\xi)$. Furthermore,

$$d_{\mathbf{L^p}}(f_\eta, \dot{S}(\xi)) \leq \|f_\eta - f_\xi\|_{\mathbf{L^p}} = \left(\int_{I \setminus M_*} \|f_\eta - f_\xi\|^p \right)^{\frac{1}{p}} \leq \operatorname{diam}(U) \big(T(1 - \lambda_*)\big)^{\frac{1}{p}} \leq \varepsilon$$

concluding the proof. *Q.E.D.*

Theorem 3.6: *Let L be a face of $\mathcal{A}_T$ and $\operatorname{ri} L$ be its relative interior. Then $S_{|\operatorname{ri} L}$ and $\dot{S}_{|\operatorname{ri} L}$ are Hausdorff continuous.*

Proof. The lower semicontinuity of $\dot{S}_{|\operatorname{ri} L}$ is a consequence of the preceding lemma. The upper semicontinuity follows from noting that δ_η may be bounded from below by a (strictly) positive constant whenever η varies in a compact subset of $\operatorname{ri} L$. *Q.E.D.*

The situation is as follows: the set–valued map S, defined on the compact convex set $\mathcal{A}_T$, is continuous on $\operatorname{ri}\mathcal{A}_T$ and on the extreme points of $\mathcal{A}_T$. On the other extremal faces of positive dimension, only its restriction to the relative interior of the face is continuous. The next step consists in the attempt of "glueing" all these continuities together. In some cases it is possible.

Proposition 3.4 together with Theorem 3.6 yield

Corollary 3.7: *If U is strictly convex, that is to say $\partial U = \operatorname{extr} U$, then S and $\dot{S}$ are Hausdorff continuous in all $\mathcal{A}_T$.*

The next result allows to widen the class of those compact convex sets for which S and $\dot{S}$ are continuous.

Proposition 3.8: *For $i = 1, \ldots, m$ let U_i be a nonempty compact and convex subset of $\mathbf{R}^{n_i}$. Denote by $S_i(\xi_i)$ the subset of $\mathbf{AC^P}$ consisting of the solutions to*

$$\begin{cases} \dot{x}_i(t) \in U_i & (x_i \in \mathbf{R}^{n_i}) \\ x_i(0) = 0 & \\ x_i(T) = \xi_i & \end{cases}$$

Assume that the maps S are continuous. If $U = U_1 \times \ldots \times U_m$, then the map S is (uniformly) continuous.

The proof is straightforward.

For the case of the particular decomposable sets here considered the last two results are a generalization of Lemma 1 in [O2]. From the single–valued continuity in the extreme points of $\mathcal{A}_T$ we passed to the Hausdorff continuity on all $\mathcal{A}_T$. For more general decomposable sets, the continuity of the map S (e.g. U a generic compact convex subset of $\mathbf{R}^n$) need not be true, as shown by

Example 3.9: Let $n = 3$. $D = \{(x,y,z) \in \mathbf{R}^3 : x^2 + y^2 = 1,\ z = 0\}$ is the unit circumference on the xy–plane and define $U = \overline{\text{co}}\left(D \cup \{(1,0,-1),(1,0,1)\}\right)$. U is the usual example of a compact convex subset of $\mathbf{R}^3$ such that $\text{extr}\, U$ is not closed. Put $T = 1$, so that $\mathcal{A}_T = U$. In this case, both the maps $\dot{S}$ and S are not lower semicontinuous in $\eta = (1,0,0)$. In fact, the function

$$f_\eta = (1,0,-1)\chi_{[0\,,1/2]} + (1,0,1)\chi_{]1/2\,,1]}$$

is in $\dot{S}(\eta)$ but is far from any function f_ξ belonging to $\dot{S}(\xi)$ with $\xi \neq \eta$ and ξ in the circumference D. For any such f_ξ, $\|f_\xi - f_\eta\|_{\mathbf{L}^p} \geq 2$ (in the Euclidean norm), although the distance between η and ξ may become arbitrarily small.

Remark that in the example above, lower semicontinuity is lost in spite of the extreme semplicity of the decomposable set considered: it is the set of the $\mathbf{L}^p$–selections of a costant set–valued map with **compact** and **convex** values.

In the example above, only lower semicontinuity is lost. This is a general feature, as shows the following

Theorem 3.10: *The map S is upper semicontinuous on $\mathcal{A}_T$.*

Proof. Without any loss of generality, we may restrict $\|\cdot\|$ to be the Euclidean norm in $\mathbf{R}^n$. This is useful since the proof makes use of some ortogonal projections and scalar products.

Fix η in $\mathcal{A}_T$. By Theorem 3.6, we can assume $\eta \in \partial\mathcal{A}_T$. Moreover, we can as well consider $\eta \in \text{ri}\, L_T$, L_T being a closed extremal face of $\mathcal{A}_T$. L_T may be supposed of positive dimension, since otherwise Proposition 3.4 would complete the proof.

Fix $\varepsilon > 0$. Let $\bar{\delta}$ be the modulus of continuity of $\dot{S}_{|\text{ri}\, L_T}$ in η relative to $\varepsilon/2$. Set $\varepsilon_1 = \frac{1}{2}\min\{\varepsilon, \bar{\delta}T^{1-1/p}\}$. Let $\xi \in \mathcal{A}_T$ and $f_\xi \in \dot{S}(\xi)$. Consider Π, the projection of minimal

(Euclidean) distance on $L = \frac{1}{T}L_T$ and set $g = \Pi \circ f_\xi$. Then:

i) g is in $\mathbf{L^p}$;

ii) for a.e. $t \in I$, $g(t) \in L \subset \partial U$;

iii) $\int_I g \in L_T$.

Moreover, when $\|f_\xi - g\|_{\mathbf{L^p}} < \varepsilon_1$ we have

$$\left\|\xi - \int_I g\right\| \leq \int_I \|f_\xi - g\| \leq T^{\frac{1}{p}-1} \|f_\xi - g\|_{\mathbf{L^p}} < \bar{\delta} \tag{3.2}$$

and

$$\|g - f_\xi\|_{\mathbf{L^p}} \leq \frac{1}{2}\varepsilon \quad . \tag{3.3}$$

It is now necessary to find a $\delta > 0$ such that:

$$\|\xi - \eta\| < \delta \ \text{ implies } \ \|f_\xi - g\|_{\mathbf{L^p}} < \varepsilon_1 \quad .$$

Let H be a hyperplane intersecting U only along L, (for the existence of H see [Ro], p. 100).

Claim: For every $\nu > 0$ there exists a $\gamma(\nu) > 0$ such that $u \in U$ and $d_E(u, H) < \gamma(\nu)$ imply $d_E(u, L) < \nu$.

Proof: Proceed by contradiction. If it is false, then there exists a sequence $\{u_j : j = 1, 2, \ldots\}$ of elements of U such that $d_E(u_j, H) < 1/j$ and $d_E(u_j, L) > \nu_* > 0$. By the compactness of U, there is a subsequence of the $\{u_j\}$ converging to a $u_* \in U$. Furthermore, $u_* \in H$ by the closure of H but $d_E(u_*, L) > \nu_* > 0$. This contradicts the choice of H.

Let $h(t) = d_E(f_\xi(t), H)$. Clearly, h is measurable and $h(t) \in [0, \mathrm{diam}_E(U)]$. Furthermore, denoting by k the normal to H external to U with $\|k\| = 1$, it follows that $h(t) = k \cdot$

$\left(\frac{1}{T}\eta - f_\xi(t)\right)$ (the dot represents the usual scalar product). Let $J_\Gamma = \{t \in I : h(t) > \Gamma\}$; by Proposition 5.1 it follows that

$$\begin{aligned} m(J_\Gamma) &\leq \frac{1}{\Gamma}\int_I h = \frac{1}{\Gamma}\int_I k \cdot \left(\frac{1}{T}\eta - f_\xi(t)\right) dt \\ &= \frac{1}{\Gamma} k \cdot (\eta - \xi) = \frac{1}{\Gamma} d_E(\xi, T\,H) \\ &\leq \frac{1}{\Gamma}\,\|\xi - \eta\| \quad . \end{aligned} \tag{3.4}$$

Now, choose:

$$\nu = \left(\frac{\varepsilon_1}{2T}\right)^{1/p} \; ; \quad \Gamma = \gamma(\nu) \; ; \quad \delta = \min\left\{\frac{1}{2}\Gamma \, , \, \frac{\Gamma \varepsilon_1}{2[\mathrm{diam}_E(U)]^p}\right\} .$$

Whenever $\|\xi - \eta\|_E < \delta$, we have

$$\begin{aligned} \|f_\xi - g\|_{\mathbf{L}^p}{}^p &= \int_{I \setminus J_\Gamma} d_E(f_\xi, L)^p + \int_{J_\Gamma} d_E(f_\xi, L)^p \\ &\leq \nu^p T + \frac{1}{\Gamma}[\mathrm{diam}_E(U)]^p \frac{\Gamma \varepsilon_1}{2[\mathrm{diam}_E(U)]^p} = \varepsilon_1 \end{aligned}$$

where we used the claim to estimate the first term and (3.4) for the latter.

To conclude the proof, it is enough to observe that

$$d_{\mathbf{L}^p}\left(f_\xi, \dot{S}(\eta)\right) \leq d_{\mathbf{L}^p}(f_\xi, g) + d_{\mathbf{L}^p}\left(g, \dot{S}(\eta)\right) < \varepsilon$$

since both the terms on the right hand side are estimated from above by $\varepsilon/2$, due to (3.3) and to (3.2), respectively. *Q.E.D.*

4. Quantitative Results

The principal aim of the present chapter is to derive some quantitative informations about (P) from the previous qualitative results. Preliminarly, it is useful to generalize some of the results in [CM]. The computation of the Kuratowski index of both the set of solutions reaching a compact subset of $\mathcal{A}_T$ and the set of solutions arbitrarily near to a given function y follow. Finally, a fully explicit example is given.

4.1. *The Kuratowski Index and the Diameter of Some Subsets of* $\mathbf{L}^p$.

The results of this section hold in the case of set–valued maps defined on an arbitrary σ–finite measure space with values in a separable Banach space. This generality is useless for the scope of the present thesis. Nevertheless, the proofs are carried out so that the simple "tipographical" substitutions of I with an arbitrary σ–finite measure space and of $\mathbf{R}^n$ with a separable Banach space lead to the general case.

It is straightforward to prove the following necessary and sufficient conditions for $\mathbf{L}^p$–boundedness:

Proposition 4.1: *Let* $F : I \to \mathcal{P}(\mathbf{R}^n)$ *be measurable with non empty closed values. Then the following statements are equivalent:*

(i) F *is* $\mathbf{L}^p$*–bounded;*

(ii) $\|F\|$ *is in* $\mathbf{L}^p$

(iii) $\mathrm{Sel}_p(F)$ *is not empty and bounded.*

The following proposition gives an explicit formula for the computation of the diameter of the set of the selections of a set–valued map.

Proposition 4.2: *Let $F: I \to \mathcal{P}(\mathbf{R}^n)$ be measurable, with non empty closed values and $\mathbf{L}^p$–bounded. Then*

$$\|\operatorname{diam} F\|_{\mathbf{L}^p} = \operatorname{diam}_{\mathbf{L}^p}(\mathrm{Sel}_p(F)) \ .$$

(where $\|\operatorname{diam} F\|_{\mathbf{L}^p} = \left\{\int_I [\operatorname{diam}_{\mathbf{R}^n}(F(t))]^p dt\right\}^{1/p}$*.)*

Proof. Some easy calculations give a first inequality.

$$\begin{aligned}
\operatorname{diam}_{\mathbf{L}^p}(\mathrm{Sel}_p(F)) &= \left(\sup\left\{\int_I \|f-g\|^p : f,g \in \mathrm{Sel}_p(F)\right\}\right)^{1/p} \\
&\le \left(\int_I \sup\{\|f-g\|^p : f,g \in \mathrm{Sel}_p(F)\}\right)^{1/p} \\
&\le \left[\int_I (\sup\{\|f-g\| : f,g \in \mathrm{Sel}_p(F)\})^p\right]^{1/p} \\
&\le \left[\int_I (\sup\{\|x-y\| : x,y \in F(t)\})^p \, dt\right]^{1/p} \\
&= \|F\|_{\mathbf{L}^p} \ .
\end{aligned}$$

To complete the proof, it is sufficient to find for any $\varepsilon > 0$ two functions $f_\varepsilon, g_\varepsilon$ in $\mathrm{Sel}_p(F)$ such that $\|f_\varepsilon - g_\varepsilon\|_{\mathbf{L}^p} \ge \|\operatorname{diam} F\|_{\mathbf{L}^p} - \varepsilon$. Let $\{f_n(t): n \in \mathbf{N}\}$ be a countable subset of $\mathrm{Sel}_p(F)$ such that $F(t)$ is the closure of the set $\{f_n(t): n \in \mathbf{N}\}$. Define

$$\begin{aligned}
\varphi: I \times \mathbf{R}^n &\to \mathbf{R} \\
(t,x) &\to \sup_{n \in \mathbf{N}} \|f(t) - x\|^p \ .
\end{aligned}$$

By means of this function it is possible to express the $\mathbf{L}^p$–norm of $\operatorname{diam} F$:

$$\begin{aligned}\|\operatorname{diam} F\|_{\mathbf{L}^p}{}^p &= \int_I (\sup\{\|x-y\| : x,y \in F(t)\})^p\, dt \\ &= \int_I \left(\sup_{x\in F(t)} \sup_{n\in\mathbf{N}} \|f(t)-x\| \right)^p dt \\ &= \int_I \left(\sup_{g\in \mathrm{Sel}_p(F)} \sup_{n\in\mathbf{N}} \|f(t)-g(t)\| \right)^p dt \\ &= \int_I \sup_{g\in \mathrm{Sel}_p(F)} \left(\sup_{n\in\mathbf{N}} \|f(t)-g(t)\| \right)^p dt \\ &= \int_I \sup_{g\in \mathrm{Sel}_p(F)} \varphi\big(t, g(t)\big)\, dt \\ &= \sup_{g\in \mathrm{Sel}_p(F)} \int_I \varphi\big(t, g(t)\big)\, dt\end{aligned}$$

where to get the last line Theorem 2.2 of [HU] was used. Choose $g_\varepsilon \in \mathrm{Sel}_p(F)$ such that

$$\int_I \varphi\big(t, g_\varepsilon(t)\big)\, dt \geq \sup_{g\in \mathrm{Sel}_p(F)} \int_I \varphi\big(t, g(t)\big)\, dt - \frac{\varepsilon}{4} \, .$$

Let $\rho \in \mathbf{L}^1$ be positive a.e. and such that $\int_I \rho = \varepsilon/4$. Introduce the sets

$$\begin{aligned} A_n &= \left\{ t \in I : \varphi\big(t, g_\varepsilon(t)\big) - \|f_n(t) - g_\varepsilon(t)\|^p \leq \rho(t) \right\} \\ B_0 &= A_0 \\ B_n &= A_n \setminus \bigcup_{m=1}^{n} A_m \end{aligned}$$

The sets $\{E_n : n \in \mathbf{N}\}$ are a measurable partition of I. The $\mathbf{L}^p$–boundedness of F implies that there exists a function $k \in \mathbf{L}^p$ such that $\|F\| \leq k$ a.e.. Let $N \in \mathbf{N}$ be such that $\int_{\bigcup_{N+1}^{\infty} B_m} k < \varepsilon/4$ and, finally, introduce the function

$$f_\varepsilon = \chi_{\bigcup_{N+1}^{\infty} B_m} f_n + \sum_{m=0}^{N} \chi_{B_n} g_\varepsilon \, .$$

Clearly, $g_\varepsilon \in \mathrm{Sel}_p(F)$ and moreover

$$\begin{aligned}
\|f_\varepsilon - g_\varepsilon\|_{\mathbf{L^p}}{}^p &= \int_{\bigcup_0^N B_m} \|f_\varepsilon - g_\varepsilon\|^p \\
&= \sum_{m=0}^{N} \int_{B_m} \|f_\varepsilon - g_\varepsilon\|^p \\
&\geq \sum_{m=0}^{N} \int_{B_m} \left[\varphi\big(t, g_\varepsilon(t)\big) - \rho(t)\right] dt \\
&\geq \int_I \varphi\big(t, g_\varepsilon(t)\big) dt - 2 \int_{\bigcup_{N+1}^\infty B_m} k - \int_I \rho \\
&\geq \sup_{f \in \mathrm{Sel}_p(F)} \int_I \varphi\big(t, g_\varepsilon(t)\big) dt - \frac{\varepsilon}{4} - 2\frac{\varepsilon}{4} - \frac{\varepsilon}{4} \\
&= \|\mathrm{diam}\, F\|_{\mathbf{L^p}}{}^p - \varepsilon
\end{aligned}$$

which completes the proof. *Q.E.D.*

The following result generalizes Theorem 4 in [CM] in two different ways. On the one side, the same formula are proved in any $\mathbf{L^p}$, analogously to the preceding proposition. On the other side, the requirement of decomposability is relaxed: any (non empty) intersection of a decomposable set with some closed hyperplane satisfies the same equations.

Proposition 4.3: *Let S be a bounded decomposable subset of $\mathbf{L^p}(I)$. Let $H_1, \ldots, H_m$ be m closed hyperplanes in $\mathbf{L^p}(I)$. Then*

$$\alpha_{\mathbf{L^p}(I)}(S \cap H_1 \cap \ldots \cap H_m) = \mathrm{diam}_{\mathbf{L^p}(I)}(S \cap H_1 \cap \ldots \cap H_m) \quad .$$

Proof. Since the proof is a straightforward generalization of the theorem cited above, we follow the notations of [CM] supplying only the details needed to obtain the result.

Replace S^n with the set $S^n = \{x \in \mathbf{R}^{n+1} : \sum_{i=1}^{n+1} |x_i|^p = 1\}$. Let $k_1, \ldots, k_m$ be elements of $\mathbf{L}^q$ (q conjugate to p) such that $H_i = \{\psi \in \mathbf{L}^p : \int_I k_i \cdot \psi = h_i\}$ for some suitable $h_i \in \mathbf{R}$, $i = 1, \ldots, m$.

In **i)**, modify b) to

b) for every $\alpha \in [0,1]$

$$\begin{cases} \int_{A(\alpha)} \|f\|^p \, d\mu = \alpha^p \int_I \|f\|^p \, d\mu \\ \int_{A(\alpha)} \|g\|^p \, d\mu = \alpha^p \int_I \|g\|^p \, d\mu \\ \int_{A(\alpha)} k_i \cdot f d\mu = \alpha^p \int_I k_i \cdot f d\mu & i = 1, \ldots, n \\ \int_{A(\alpha)} k_i \cdot g d\mu = \alpha^p \int_I k_i \cdot g d\mu & i = 1, \ldots, n \end{cases}$$

Change the definition of the p_j to: $p_j(z) = |z_0|^p + \ldots + |z_j|^p$. As a consequence, the properties of the partition $N_i(x)$ become

$$\begin{aligned} \int_{N_i(x))} \|f\|^p \, d\mu &= |x_i|^p \int_I \|f\|^p \, d\mu \\ \int_{N_i(x))} \|g\|^p \, d\mu &= |x_i|^p \int_I \|g\|^p \, d\mu \\ \int_{N_i(x)} k_i \cdot f d\mu &= |x_i|^p \int_I k_i \cdot f d\mu \\ \int_{N_i(x)} k_i \cdot g d\mu &= |x_i|^p \int_I k_i \cdot g d\mu \end{aligned} \qquad i = 1, \ldots, n$$

The remainder of the proof is obtained by simply replacing in [CM] $\|\cdot\|_{\mathbf{L}^1}$ with $\|\cdot\|_{\mathbf{L}^p}{}^p$. Remark that the two new conditions imposed on the $A(\alpha)$ ensure that the values of φ_n are in $S \cap H_1 \cap \ldots \cap H_m$. *Q.E.D.*

The result above allows to compute the Kuratowski index of the set of all the selections of F. In connection with it, it may be of interest to compute the Kuratowski index of the minimal family of selections of F through which the whole F may be described. It is known that there exists a countable family $\mathcal{F}$ of selections of F such that $F(t)$ is the closure of $\{f(t) : f \in \mathcal{F}\}$. In [CC] it is proved that $\mathcal{F}$ may be chosen relatively compact, i.e. $\alpha_{L^p}(\mathcal{F}) = 0$.

4.2. *Likelihood Estimates*

Remark that due to (2.1) and (2.2), the computations for the case $v = 0$ and $r = 1$ lead to estimates for the general case. Therefore, it is enough to consider only $v = 0$ and $r = 1$, while the dependence on the interval will be of use. Hence, let $\Sigma^p(I) = \Sigma^p(0, 1, I)$.

As introduced in Chapter 1, the quantity $\alpha_{\mathbf{AC^P}}\big(S(\xi)\big)$ may serve as an estimator of the likelihood of a given point ξ to be reached at time T by a solution of (P). In the same way, if K is a compact subset of $\mathcal{A}_T$, $\alpha_{\mathbf{AC^P}}\big(S(K)\big)$ evaluates the likelihood that a point of K may be reached. In view of the results of the previous chapter, these two quantities are related.

Proposition 4.4: *Let K be a compact subset of $\mathcal{A}_T$. Then*

$$\alpha_{\mathbf{AC^P}}\big(S(K)\big) = \sup_{\xi \in K} \alpha_{\mathbf{AC^P}}\big(S(\xi)\big)$$

Proof. The inequality $\alpha_{\mathbf{AC^P}}\big(S(K)\big) \geq \sup_{\xi \in K} \alpha_{\mathbf{AC^P}}\big(S(\xi)\big)$ is straightforward. To prove the opposite one, by the upper semicontinuity of S, it follows that for any $\varepsilon > 0$ there are some $\xi_1, \ldots, \xi_N$ in $\mathcal{A}_T$ such that $S(K) \subseteq \bigcup_1^N B_{\mathbf{AC^P}}\big(S(\xi_i), \varepsilon\big)$, so that $\alpha_{\mathbf{AC^P}}\big(S(K)\big) \leq \max_i \alpha_{\mathbf{AC^P}}\big(S(\xi_i)\big) + 2\varepsilon$, the proof follows. *Q.E.D.*

Therefore, it is enough to study the function $\alpha_{\mathbf{AC^P}} \circ S : \xi \to \alpha_{\mathbf{AC^P}}\big(S(\xi)\big)$. The following proposition is of key importance.

Proposition 4.5: *For any ξ in $\mathcal{A}_T$, it holds:*

$$\alpha_{\mathbf{AC^p}}\left(S(\xi)\right) = \alpha_{\mathbf{L^p}}\left(\dot{S}(\xi)\right) = \mathrm{diam}_{\mathbf{L^p}}\left(\dot{S}(\xi)\right) = \mathrm{diam}_{\mathbf{AC^p}}\left(S(\xi)\right)$$

Proof. The first and the last equalities follow from the fact that derivation is an isometry, as already remarked. To prove the middle equality, notice that $\dot{\Sigma}(I)$ is decomposable and that $\dot{S}(\xi) = \dot{\Sigma}(I) \cap H_1 \cap \ldots \cap H_n$ where H_i is the closed hyperplane $H_i = \{f \in \mathbf{L^p}: \int_I f_i = \xi_i\}$, for $i = 1, \ldots, n$. The proof ends by applying Proposition 4.3.

Q.E.D.

The next theorem summarizes the main properties of $\alpha_{\mathbf{AC^p}}\left(S(\xi)\right)$.

Theorem 4.6: *There exists a function $\Delta: U \to \mathbf{R}$ such that:*

(a) For any ξ in $\mathcal{A}_T$, $\alpha_{\mathbf{AC^p}}\left(S(\xi)\right) = \Delta(\frac{1}{T}\xi)T^{1/p}$.

(b) S continuous at ξ implies Δ continuous at $\frac{1}{T}\xi$; the continuity of the restriction of S to the interior of an extremal face L implies the continuity of the restriction of Δ to $\frac{1}{T}L$.

(c) $0 \leq \Delta(u) \leq \mathrm{diam}(U)$.

(d) $u \in \mathrm{extr}\, U$ if and only if $\Delta(u) = 0$.

(e) If U is symmetric, then $\Delta(u) = \Delta(-u)$ and $\Delta(0) = \max_U \Delta = \mathrm{diam}(U)$.

Proof. For any real $k>0$, let $\Lambda = kT$ and consider the two boundary value problems

$$(\mathrm{BVP}) \quad \begin{cases} \dot{x}(t) \in \partial U \\ x(0) = 0 & t \in [0,T] \\ x(T) = \xi \end{cases} \qquad\qquad (\mathrm{BVP})_k \quad \begin{cases} y'(\lambda) \in \partial U \\ y(0) = 0 & \lambda \in [0,\Lambda] \\ y(\Lambda) = k\xi \end{cases}$$

The transformation $A\colon \mathbf{AC^p}([0,T]) \to \mathbf{AC^p}([0,\Lambda])$ defined by $(Ax)(\lambda) = kx(\frac{1}{k}\lambda)$ is linear. Furthermore, when restricted to the set of solutions to (BVP), A turns out to be a bijection onto the set of solutions to $(\mathrm{BVP})_k$. Besides, if x is a solution to (BVP)

$$\|Ax\|_{\mathbf{AC^p}([0,\Lambda])} = k^{1/p}\,\|x\|_{\mathbf{AC^p}([0,T])} \quad .$$

Set $\Phi(\xi,T) = \alpha_{\mathbf{AC^p}([0,T])} S(\xi)$. The previous remarks about A imply that $\Phi(k\xi, kT) = k^{1/p}\Phi(\xi,T)$. Choose in particular $k = 1/T$, define $\Delta(u) = \Phi(u,1)$ and (a) follows.

(b) follows from Proposition 3.4, from Theorem 3.6 and from the Hausdorff continuity of the Kuratowski index.

To prove (c), observe that for any ξ, $\alpha_{\mathbf{AC^p}}\big(S(\xi)\big) \le \alpha_{\mathbf{AC^p}}\big(\Sigma(I)\big) = \alpha_{\mathbf{L^p}}\big(\dot{\Sigma}(I)\big)$. Since $\dot{\Sigma}(I)$ is decomposable, its Kuratowski index may be computed via Proposition 4.2.

(d) is a straightforward consequence of Proposition 3.2.

Passing to (e), assume U is symmetric. The correspondence $x \to -x$ is a bijective isometry between $S(\xi)$ and $S(-\xi)$. Therefore, $\Delta(\frac{1}{T}\xi) = \Delta(-\frac{1}{T}\xi)$, which proves the first part of (e). Due to Proposition 4.5, $\alpha_{\mathbf{AC^p}}\big(S(0)\big) = \mathrm{diam}_{\mathbf{L^p}}\big(\dot{S}(0)\big)$. Define $A\colon \dot{\Sigma}([0,T/2]) \to \dot{S}(0)$ by

$$(Ag)(t) = g(t)\chi_{[0\,,\,T/2]}(t) - g(T-t)\chi_{]T/2\,,\,T]}(t) \quad .$$

Since A is bijective and $\|Ag\|_{\mathbf{L^p}(I)} = 2^{1/p}\,\|g\|_{\mathbf{L^p}([0,T/2])}$ it follows that

$$\mathrm{diam}_{\mathbf{L^p}(I)}\big(\dot{S}(0)\big) = 2^{1/p}\mathrm{diam}_{\mathbf{L^p}([0,T/2])}\big(\dot{\Sigma}([0,T/2])\big) = \mathrm{diam}(U)T^{1/p} \quad .$$

were we used the same argument as in (c) to compute $\operatorname{diam}_{\mathbf{L^p}([0,T/2])}(\dot\Sigma([0,T/2]))$.

Q.E.D.

The following result will be of fundamental importance in the computation of the likelihood. Nevertheless, it is interesting on its own since it gives an explicit formula for the $\alpha_{\mathbf{AC^p}}$ of those solutions passing through a finite number of given (admissible) points.

Proposition 4.7: *Let N be a positive integer and fix $t_0,\ldots,t_N$ be in $\mathbf{R}$ with $t_i > t_{i-1}$ for $i = 1,\ldots,N$. Define*

$$(4.1)\quad Q(\xi_0,\ldots,\xi_N;t_0,\ldots,t_N) = \{y \in \mathbf{AC^p}([t_0,t_N]): \dot y \in \partial U,\ y(t_i) = \xi_i,\ i = 0,\ldots,N\}$$

where $\xi_1,\ldots,\xi_N$ are points in $\mathbf{R}^n$. Then:

(a) $Q(\xi_0,\ldots,\xi_N;t_0,\ldots,t_N) \neq \emptyset \iff \dfrac{\xi_i - \xi_{i-1}}{t_i - t_{i-1}} \in U \quad \forall i = 1,\ldots,N.$

(b) If $Q(\xi_0,\ldots,\xi_N;t_0,\ldots,t_N)$ is non empty, then

$$\alpha_{\mathbf{AC^p}}\Big(Q(\xi_0,\ldots,\xi_N;t_0,\ldots,t_N)\Big) = \left\{\sum_{i=1}^{N}\left[\Delta\left(\frac{\xi_i - \xi_{i-1}}{t_i - t_{i-1}}\right)\right]^p (t_i - t_{i-1})\right\}^{1/p}.$$

(c) For fixed $t_0,\ldots,t_N$, the map $Q: (\xi_1,\ldots,\xi_N) \to Q(\xi_1,\ldots,\xi_N)$ is upper semicontinuous on its domain. If U is such that S is Hausdorff continuous on $\mathcal{A}_T$, then Q is Hausdorff continuous on its domain.

Proof. (a) is trivial. To prove (b) and (c), assume first that $N = 1$ and define

$$A: \dot Q(\xi_0,\xi_1;t_0,t_1) \to \dot S\left(\frac{\xi_1 - \xi_0}{t_1 - t_0}T\right) \quad \text{with} \quad (Ag)(t) = g\left(\frac{t_1 - t_0}{T}t + t_0\right) \quad .$$

A is bijective and $\|Ag\|_{\mathbf{L^p}(I)} = [T/(t_1 - t_0)]^{1/p} \|g\|_{\mathbf{L^p}([t_0,t_1])}$, therefore

$$\alpha_{\mathbf{L^p}}\left(\dot{Q}(\xi_0, \xi_1; t_0, t_1)\right) = \Delta\left(\frac{\xi_1 - \xi_0}{t_1 - t_0}\right)(t_1 - t_0)^{1/p}$$

and for any $\bar{\xi}_0, \bar{\xi}_1$ in the domain of Q

$$\mathbf{d}_{\mathbf{L^p}([t_0,t_1])}\left(\dot{Q}(\xi_0, \xi_1; t_0, t_1), \dot{Q}(\bar{\xi}_0, \bar{\xi}_1; t_0, t_1)\right) = \\ = \left(\frac{T}{t_1 - t_0}\right)^{1/p} \mathbf{d}_{\mathbf{L^p}}\left(\dot{S}\left(\frac{\xi_1 - \xi_0}{t_1 - t_0}T\right), \dot{S}\left(\frac{\bar{\xi}_1 - \bar{\xi}_0}{t_1 - t_0}T\right)\right)$$

and the case $N = 1$ is proved.

To prove the general case, put $J = [t_0, t_N[$ and $J_i = [t_{i-1}, t_i[$, for $i = 1, \ldots, N$. Define $\mathcal{L}$ and Υ as in Proposition 5.2 and notice that

$$\Upsilon\left(\prod_{i=1}^{N} \dot{Q}(\xi_{i-1}, \xi_i; t_{i-1}, t_i)\right) = \dot{Q}(\xi_0, \xi_1; t_0, t_1) \quad .$$

Applying Proposition 5.3, the diameter of the set in parentheses on the left hand side may be computed using the previous result for $N = 1$. Since Υ is an isometry, the proof of (b) follows.

To prove (c), simply observe that Υ induces an isometry between the metric spaces of the closed bounded subsets of $\mathbf{L^p}([t_0, t_N])$ and of $\prod_1^N \mathbf{L^p}([t_{i-1}, t_i])$. So the continuity properties of $Q(\xi_1, \ldots, \xi_N; t_1, \ldots, t_N)$ are equivalent to the analogous properties of the $Q(\xi_{i-1}, \xi_i; t_{i-1}, t_i)$, which follow from the case $N = 1$. In fact, it is easily seen that $Q(\xi_1, \xi_2; 0, T) = \xi_1 + S(\xi_2 - \xi_1)$. *Q.E.D.*

Let y be an absolutely continuous function on $[0, T]$ and consider the set $C(y, \varepsilon) = \{x \in \Sigma \colon \|y - x\|_{\mathbf{L^\infty}} \le \varepsilon\}$. In the spirit of [B], we wish to compute

$$\lim_{\varepsilon \to 0} \alpha_{\mathbf{AC^p}}\left(C(y, \varepsilon)\right)$$

as a measure of the likelihood of y of describing the evolution of (P). The set C is nonempty for small ε's only for those y satisfying $y(0) = 0$, and $\dot{y}(t) \in U$ for a.e. $t \in I$.

We have the following

Theorem 4.8: *If U is such that Δ is continuous, $\dot{y} \in U$ a.e. and $y(0) = 0$, then*

$$\lim_{\varepsilon \to 0} \alpha_{\mathbf{ACP}}\left(C(y,\varepsilon)\right) = \left(\int_I \left[\Delta(\dot{y}(t))\right]^p dt\right)^{1/p} = \|\Delta \circ \dot{y}\|_{\mathbf{L^p}}$$

Proof. For any positive integer N set t_i^N to be $\frac{i}{N}T$, $i = 0, \ldots, N$, and denote

$$R(\xi_1, \xi_2, \ldots, \xi_N) = Q(0, \xi_1, \xi_2, \ldots, \xi_N; t_0^N, t_1^N, t_2^N, \ldots, t_N^N)$$

for $\xi_1, \xi_2, \ldots, \xi_N$ in $\mathbf{R}^n$ and with Q given by (4.1).

For N large enough, $C(y,\varepsilon) \supseteq R\big(y(t_1^N), y(t_2^N), \ldots, y(t_N^N)\big) \neq \emptyset$, hence

$$\begin{aligned}\alpha_{\mathbf{ACP}}\left(C(y,\varepsilon)\right) &\geq \alpha_{\mathbf{ACP}}\left(R\big(y(t_1^N), y(t_2^N), \ldots, y(t_N^N)\big)\right) \\ &= \left\{\sum_{i=1}^{N} \left[\Delta\left(\frac{y(t_i^N) - y(t_{i-1}^N)}{t_i^N - t_{i-1}^N}\right)\right]^p (t_i^N - t_{i-1}^N)\right\}^{1/p}\end{aligned}$$

by Proposition 4.7. Passing to the limit over N and applying Proposition 5.4, we have

$$\lim_{\varepsilon \to 0} \alpha_{\mathbf{ACP}}\left(C(y,\varepsilon)\right) \geq \|\Delta \circ \dot{y}\|_{\mathbf{L^p}} \quad .$$

To prove the opposite inequality, set $B_i = B\big(y(t_i^N), \varepsilon\big)$ and observe that for any positive integer N

$$C(y,\varepsilon) \subseteq \bigcup_{\xi_1 \in B_1} \cdots \bigcup_{\xi_N \in B_N} R(\xi_1, \ldots, \xi_N) \quad .$$

By (c) of Proposition 4.7, R is upper semicontinuous. Hence, for any $\nu > 0$, there is a suitable net $\xi_i^1, \dots \xi_i^{m_i}$ in B_i, for $i = 1, \dots, N$, such that

$$\bigcup_{\xi_1 \in B_1} \cdots \bigcup_{\xi_N \in B_N} R(\xi_1, \dots, \xi_N) \subseteq \bigcup_{j_1=1}^{m_1} \cdots \bigcup_{j_N=1}^{m_N} B\big(R(\xi_1^{j_1}, \dots, \xi_N^{j_N}), \nu\big) \quad .$$

This amounts to

$$\alpha_{\mathbf{ACP}}\big(C(y,\varepsilon)\big) \leq 2\nu + \max\left\{\alpha_{\mathbf{ACP}}\big(R(\xi_1^{j_1}, \dots, \xi_N^{j_N})\big) : j_i = 1, \dots, m_i\,,\; i = 1, \dots, N\right\} \,.$$

Due to $d\big(\xi_i^{j_i}, y(t_i^N)\big) \leq \varepsilon$, one may pass to the limit first for $\varepsilon \to 0$, then for $N \to \infty$ and, finally, using arbitraryness of ν the proof is completed. *Q.E.D.*

4.3. An Explicit Example

All the previous estimates depend from the function Δ. In some cases, Δ may be written explicitely.

Theorem 4.9: *In* $\mathbb{R}^n$ *define the norm* $\|\xi\| = \left(\sum_{i=1}^n |\xi_i|^p\right)^{1/p}$ *and let* $U = \prod_1^n [-c_i, c_i]$, *with* $c_i \geq 0$, $i = 1, \dots, n$. *Assume* $v = 0$ *and* $r = 1$. *Then*

$$\Delta(u) = 2\left[\sum_{i=1}^n c_i^{p-1}\big(c_i - |u_i|\big)\right]^{1/p} \tag{4.2}$$

Proof. Consider first the case $n = 1$. If $c_1 = 0$, (4.2) trivially holds. Assume $c_1 > 0$, Proposition 2.1 allows to consider only the standard case $c_1 = 1$. Define

$$f_1 = -\chi_{[0, \frac{T-\xi}{2}]} + \chi_{[\frac{T-\xi}{2}, T]} \qquad \text{and} \qquad f_2 = \chi_{[0, \frac{T+\xi}{2}]} - \chi_{[\frac{T+\xi}{2}, T]}$$

for some $\xi \in \mathcal{A}_T = [-1,+1]$. Clearly, $f_1, f_2 \in \dot{S}(\xi)$ and some calculations lead to $\|f_2 - f_1\|_{\mathbf{L}^p} = 2(1-|\xi|/T)^{1/p}T^{1/p}$, providing a lower bound for $\operatorname{diam}_{\mathbf{L}^p}\left(\dot{S}(\xi)\right)$.

Take g_1, g_2 in $\dot{S}(\xi)$. Call $D = \{t \in I : g_1(t) \neq g_2(t)\}$. Then $\|g_2 - g_1\|_{\mathbf{L}^p} \leq 2m(D)^{1/p}$. Since $\int_I g_i = \xi$ and $g_i(t) \in \{-1,+1\}$, it follows that

$$(4.3) \qquad m\left(\{t \in I : g_i(t) = -1\}\right) = \frac{T-\xi}{2} \qquad m\left(\{t \in I : g_i(t) = +1\}\right) = \frac{T+\xi}{2}$$

for $i = 1,2$. D may be written as

$$(4.4) \qquad D = \{t \in I : g_1(t) = -1\,,\ g_2(t) = +1\} \cup \{t \in I : g_1(t) = +1\,,\ g_2(t) = -1\}$$

which implies

$$m(D) \leq m(\{t \in I : g_1(t) = -1\}) + m(\{t \in I : g_2(t) = -1\}) \leq T - \xi$$
$$m(D) \leq m(\{t \in I : g_1(t) = +1\}) + m(\{t \in I : g_2(t) = +1\}) \leq T + \xi$$

hence $m(D) \leq \min\{T-\xi, T+\xi\} = T - |\xi|$ and, therefore,

$$\operatorname{diam}_{\mathbf{L}^p}\left(\dot{S}(\xi)\right) \leq \|g_2 - g_1\|_{\mathbf{L}^p} \leq 2(T-|\xi|)^{1/p} = 2\left(1 - \frac{|\xi|}{T}\right)^{1/p} T^{1/p}$$

concluding the one-dimensional case.

Let $n > 1$ and fix a ξ in $\mathcal{A}_T$. For $i = 1, \ldots, n$, define $S_i^c(\xi_i)$ and $S_i(\xi_i)$ to be the set of solutions (in $\mathbf{AC}^p(I, \mathbf{R})$) to the boundary value problems

$$\begin{cases} \dot{x}(t) \in [-c_i, c_i] \\ x(0) = 0 & t \in [0,T] \\ x(T) = \xi \end{cases} \qquad\qquad \begin{cases} \dot{x}(t) \in \{-c_i, c_i\} \\ x(0) = 0 & t \in [0,T] \\ x(T) = \xi \end{cases}$$

respectively. Observe that

$$(4.5) \qquad \operatorname{diam}_{\mathbf{L}^p}\left(\dot{S}_i^c(\xi_i)\right) = \operatorname{diam}_{\mathbf{L}^p}\left(\dot{S}_i(\xi_i)\right) = 2\left[c_i^{p-1}(c_i - |u_i|)T\right]^{1/p} \quad .$$

To prove the first of these equalities, simply repeat the previous argument replacing "=" with "≤" in (4.3) and "=" with "⊆" in (4.4). The second equality is (4.2) for $n = 1$.

Call $\mathcal{L} = \left(\mathbf{L}^p(I, \mathbf{R})\right)^n$ normed with $\|(f_1, \ldots, f_n)\|_{\mathcal{L}} = \left(\sum_{i=1}^n \|f_i\|_{\mathbf{L}^p}\right)^{1/p}$. Due to the choice of the norms, there is a canonical linear isometry A between $\mathbf{L}^p = \mathbf{L}^p(I, \mathbf{R}^n)$ and $\mathcal{L}$. Clearly,

$$\prod_{i=1}^n \dot{S}_i(\xi_i) \subseteq A\big(\dot{S}(\xi)\big) \subseteq \prod_{i=1}^n \dot{S}_i^c(\xi_i)$$

so, by Proposition 4.5, Proposition 5.3 and (4.5), it follows that

$$\alpha_{\mathbf{L}^p}(V_j) = \mathrm{diam}_{\mathbf{L}^p}(V_j) = 2\left[\sum_{i=1}^n c_i^{p-1}\big(c_i - |u_i|\big)\right]^{1/p} T^{1/p}$$

(where Proposition 5.3 has been applied with $\psi(\xi_1, \ldots, \xi_n) = \|\xi\|$). Passing to the union over j completes the proof. *Q.E.D.*

5. Technical Proofs

Proposition 5.1: *Let $f: I \to [a,b]$ be measurable and let $c \in\ ]a,b[$. Define $J_c = \{t \in I: f(t) \geq c\}$ and $\bar{f} = \frac{1}{m(I)} \int_I f$. Then*

$$m(J_c) \leq \frac{\bar{f} - a}{c - a} m(I)$$

Proof. Let $\delta \in \mathbf{R}$ be such that $m(J_c) = (\frac{\bar{f}-a}{c-a} + \delta) m(I)$. Then

$$\bar{f} = \frac{1}{m(I)} \int_{J_c} f + \frac{1}{m(I)} \int_{I \setminus J_c} f$$

$$\geq \left(1 - \frac{\bar{f} - a}{c - a} - \delta\right) a + \left(\frac{\bar{f} - a}{c - a} + \delta\right) c \tag{5.1}$$

On the other hand,

$$\bar{f} = \left(\frac{b - \bar{f}}{b - a} - \frac{b - c}{b - a} \frac{\bar{f} - a}{c - a}\right) a + \left(\frac{\bar{f} - a}{c - a}\right) c \tag{5.2}$$

Subtracting (5.1) from (5.2) we obtain $\delta \leq 0$. *Q.E.D.*

Proposition 5.2: *Let $J_1, \ldots, J_N$ be measurable and disjoint subsets of the bounded interval J such that $\bigcup_1^N J_i = J$. Define $\mathcal{L} = \prod_1^N \mathbf{L}^p(J_i)$ normed with*

$$\|(f_1, \ldots, f_N)\|_{\mathcal{L}} = \left(\sum_{i=1}^{N} \|f_i\|_{\mathbf{L}^p(J_i)}{}^p\right)^{1/p}.$$

Then, the operator $\Upsilon: \mathcal{L} \to \mathbf{L}^p(J)$ defined by

$$(\Upsilon(f_1, \ldots, f_N))(t) = f_i(t) \quad \text{for a.e. } t \text{ in } J_i$$

is a linear and bijective isometry.

Proof. Linearity and bijectivity are trivial. Due to the choice of the norm in $\mathcal{L}$, it follows that

$$\|(f_1,\ldots,f_N)\|_{\mathcal{L}} = \left(\sum_{i=1}^{N} \|f_i\|_{\mathbf{L}^p(J_i)}{}^p\right)^{1/p} = \left(\sum_{i=1}^{N} \int_{J_i} \|f_i(t)\|^p dt\right)^{1/p}$$
$$= \left(\int_I \|(\Upsilon(f_1,\ldots,f_N))(t)\|^p dt\right)^{1/p} = \|\Upsilon(f_1,\ldots,f_N)\|_{\mathbf{L}^p(J)}$$

which completes the proof. *Q.E.D.*

Proposition 5.3: *Let $(M_1,d_1),\ldots,(M_N,d_N)$ be N metric spaces. Define $M = \prod_1^N M_i$ and assume there exists a continuous function $\psi: \mathbf{R}^N \to \mathbf{R}$ non–decreasing in every argument and such that the function $d: M \times M \to \mathbf{R}$ given by*

$$d\big((x_1,\ldots,x_N),(y_1,\ldots,y_N)\big) = \psi\big(d_1(x_1,y_1),\ldots,d_N(x_N,y_N)\big)$$

is a metric on M. Then, for any choice of N bounded sets A_i with $A_i \subseteq M_i$

$$\operatorname{diam}_M\left(\prod_{i=1}^{N} A_i\right) = \psi\big(\operatorname{diam}_{M_1}(A_1),\ldots,\operatorname{diam}_{M_N}(A_N)\big) \quad .$$

Proof. Assume $N = 2$. Observe that if E_1, E_2 are bounded subsets of $\mathbf{R}$, then

$$\sup \psi(E_1 \times E_2) = \psi(\sup E_1, \sup E_2).$$

This in particular holds for $E_i = \{d_i(a,a'): a,a' \in A_i\}$.

The general case $N > 2$ follows directly by induction. *Q.E.D.*

Lemma 5.4: *Let $\varphi: U \to \mathbf{R}$ be a continuous function and x be in AC^{P} with $\dot{x}(t) \in U$ for a.e. t in I. For any positive integer N define $t_i^N = \frac{i}{N}T$, $i = 0, \ldots, N$. Then*

$$\lim_{N\to\infty} \sum_{i=1}^{N} \varphi\left(\frac{x(t_i^N) - x(t_{i-1}^N)}{t_i^N - t_{i-1}^N}\right)(t_i^N - t_{i-1}^N) = \int_I \varphi(\dot{x}(t))\,dt \quad .$$

Proof. It is clear that the arguments of φ are in U for any i and N. For any $t \in I$ let

$$s_N(t) = \sum_{i=1}^{N} \frac{1}{t_i^N - t_{i-1}^N} \int_{I_i^N} \dot{x}(\tau)d\tau \; \chi_{I_i^N}(t)$$

where $I_i^N = [t_{i-1}^N, t_i^N[$. A straightforward calculation leads to

$$\sum_{i=i}^{N} \varphi\left(\frac{x(t_i^N) - x(t_{i-1}^N)}{t_i^N - t_{i-1}^N}\right)(t_i^N - t_{i-1}^N) = \int_I \varphi(s_N(t))\,dt \quad .$$

Let $\mathcal{L}_x$ be the Lebesgue set of x. By the Lebesgue Differentiation Theorem, it follows that if $t \in \mathcal{L}_x \setminus \bigcup_{N,i} \{t_i^N\}$ then $\lim_{N\to\infty} s_N(t) = \dot{x}(t)$. Thanks to the continuity of φ, this leads to $\lim_{N\to\infty} \varphi(s_N(t)) = \varphi(\dot{x}(t))$ and, by the Dominated Convergence Theorem, the proof is completed. *Q.E.D.*

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