ISAS - INTERNATIONAL SCHOOL FOR ADVANCED STUDIES

Thesis submitted for the degree of
Magister Philosophiae

QUATERNIONIC ANALYSIS

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Academic year 1988/1989

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INTRODUCTION

It was 1843 when Sir W.R. Hamilton perceived that it would have been possible to define a non-trivial four dimensional real associative division algebra simply by doing away with commutativity: it was the birth of the quaternions.
We will study the algebraic structure of the quaternions in the first chapter of this thesis, while a detailed descriptions of the features of non-commutative linear algebra will be given in the Appendix. The resulting approach to the problem is adopted in Chapter 1 when studing linear spaces, Banach spaces and Hilbert spaces over the quaternionic field.

The richness of the complex analysis makes it natural to look for a similar theory in the quaternionic case. Nevertheless, the lack of commutativity of the quaternions makes the two fundamental definitions of holomorphy useless when adapted to the quaternionic valued functions of a quaternionic variable. The existence of quaternionic derivative, in the obvious sense, is a to much restrictive requirement: it selects only constants and linear functions (and not all of them). Furthermore, polynomials in the quaternionic variable are just polynomials in the four real variables, and the same holds for power series: this condition it is not restrictive enough.

It was only in 1935 that R. Fueter put down a good definition of "regular function" of a quaternionic variable. His definition is obtained by an analogue of the Cauchy Riemann equation in the complex case, which is called the Cauchy Riemann Fueter equation.

Regular functions in quaternionic case behaves rather like holomorphic functions. In the second chapter of this thesis, we will try to translate the most important theorems of the complex analysis into the quaternionic analysis (see [S] and [PE] for more details)

In the third chapter, the existence of a reproducing kernel for the class of all regular and square integrable functions is demostrated; the fourth chapter is devoted to the study of the series expansions of regular functions, paying particular attention to the existence of a propety similar to the classical Abel's lemma.

Finally, in the last chapter we will generalize the regularity to functions defined between (possible infinite dimensional) quaternionic Banach spaces. This context exalts the algebraic character of the regularity requirement: we will prove that we can have regular functions only between differently sided quaternionic spaces.
It follows that all the holomorphic maps are regular maps too. Furthermore, we will be able to uniquely decompose the real derivative of a map in a sum of interesting differential operators, as in the complex case.

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CHAPTER 1

QUATERNIONS: ALGEBRAIC STRUCTURE

1. QUATERNIONS

1.1 Definition

It is well known that, up to algebra isomorphism, there exists only one four dimensional real division algebra, associative and with unity: it is called the algebra of quaternions

Let now ι_λ $(\lambda = 0,1,2,3)$ be the canonical base of $\mathbb{R}^4$, i.e.

$$(\iota_\lambda)^\mu = \delta_{\lambda\mu} \tag{1.1.1}$$

Then the real bilinear form on $\mathbb{R}^4$ defined by:

$$\iota_0 \iota_\lambda = \iota_\lambda = \iota_\lambda \iota_0 \qquad (\lambda = 0,1,2,3)$$

$$\iota_\lambda^2 = -\iota_0 \qquad (\lambda = 1,2,3)$$

$$\iota_\lambda \iota_\mu + \iota_\mu \iota_\lambda = 0 \qquad (\lambda, \mu = 1,2,3 \quad \lambda \neq \mu)$$

(1.1.2)

$$\iota_1 \iota_2 = \iota_3$$

$$\iota_2 \iota_3 = \iota_1$$

$$\iota_3 \iota_1 = \iota_2$$

endows $\mathbb{R}^4$ with an algebra structure.

More explicitly, the product is defined by

$$\begin{pmatrix} x^0 \\ \underline{x} \end{pmatrix} \cdot \begin{pmatrix} y^0 \\ \underline{y} \end{pmatrix} = \begin{pmatrix} x^0 y^0 - \underline{x} \cdot \underline{y} \\ x^0 \underline{y} + y^0 \underline{x} + \underline{x} \times \underline{y} \end{pmatrix} \tag{1.1.3}$$

where $\underline{x} = x^1 \iota_1 + x^2 \iota_2 + x^3 \iota_3$, $\underline{y} = y^1 \iota_1 + y^2 \iota_2 + y^3 \iota_3 \in \mathbb{R}^3$ and $\underline{x} \cdot \underline{y}$, $\underline{x} \times \underline{y}$ are the scalar and vector product in $\mathbb{R}^3$ respectively.

This is the standard representation of the algebra of quaternions, usually denoted by $\mathbb{H}$.
In the following we will use this representation to describe the main properties of quaternions

Formula (1.1.2) implies that the quaternionic product is not commutative : as we will see, this is the principal source of difficulties in studing quaternionic analysis.

Furthermore note that

$$\mathbb{R} \ni t \longmapsto t\iota_0 \in \mathbb{H}$$

is a real algebra monomorphism (actually, it is the only real al= gebra monomorphism from $\mathbb{R}$ to $\mathbb{H}$, as we shall see later).
So, in the following we talk about real numbers as a sub-algebra of $\mathbb{H}$, identifying ι_0 with 1.

Note that real numbers lie in the center of $\mathbb{H}$.

Moreover:

1.1.1 Lemma The real numbers $\mathbb{R}\iota_0 \subset \mathbb{H}$ are the center of $\mathbb{H}$

Proof If $x = x^0 + x^1\iota_1 + x^2\iota_2 + x^3\iota_3$ commutes with ι_1 then
$$x^2\iota_3 - x^3\iota_2 = -x^2\iota_3 + x^3\iota_2$$
and so $x^2 = x^3 = 0$. Similarly, since x commute with ι_2, we have $x^1 = 0$ ▲

We end this paragraph giving another useful representation of the field of quaternions, i.e. $\mathbb{C}^2$ with the product
$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \cdot \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} z_1 w_1 - z_2 \overline{w_2} \\ z_1 w_2 + z_2 \overline{w_1} \end{pmatrix} \qquad (1.1.4)$$
where "$-$" denotes the usual conjugation in $\mathbb{C}$
Note that
$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \cdot \left[\alpha \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \right] = \overline{\alpha} \left[\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \cdot \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \right] \qquad (1.1.5)$$
and that $\mathbb{C}^2$ with the product (1.1.4) is not a complex algebra.

Finally, this last representation, which we denote with $\tilde{\mathbb{H}}$, is isomorphic to $\mathbb{H}$, as real algebra. The map $\chi : \mathbb{H} \to \tilde{\mathbb{H}}$ defined as
$$\chi \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} a + ib \\ c + id \end{pmatrix} \qquad (1.1.6)$$
is a canonical isomorphism

1.2 Canonical conjugation

Let us define $(\bar{\ }) : \mathbb{H} \to \mathbb{H}$ by

$$\bar{x} = x^0 - x^1 \iota_1 - x^2 \iota_2 - x^3 \iota_3 \qquad (1.1.7)$$

for $x = x^0 + x^1\iota_1 + x^2\iota_2 + x^3\iota_3 \in \mathbb{H}$.

This map is an involutive $\mathbb{R}$-linear map:

$$\bar{\bar{x}} = x \qquad x \in \mathbb{H}$$

and an algebra anti-isomorphism:

$$\overline{xy} = \bar{y}\,\bar{x} \qquad x, y \in \mathbb{H}.$$

Note that:

1. $\bar{x} = x$ if and only if $x \in \mathbb{R}$

2. $\bar{x} = -x$ if and only if x belongs to the $\mathbb{R}$-linear space spanned by $\iota_1, \iota_2, \iota_3$, which we identify with $\mathbb{R}^3$

For $x = x^0 + x^1\iota_1 + x^2\iota_2 + x^3\iota_3 \in \mathbb{H}$, we define the real part of x by

$$\operatorname{Re} x = \frac{x + \bar{x}}{2} = x^0 \qquad (1.1.8)$$

and the pure part x by:

$$\operatorname{Pu} x = \frac{x - \bar{x}}{2} = x^1\iota_1 + x^2\iota_2 + x^3\iota_3 \qquad (1.1.9)$$

Remark that

1.1.2 Lemma For $x \in \mathbb{H}$, we have $x \in \mathbb{R}^3$ if and only if

$$x^2 \leqslant 0$$

Proof Since:

$$x^2 = (\mathrm{Re}\, x)^2 - \mathrm{Pu}\, x \cdot \mathrm{Pu}\, x + 2\, \mathrm{Re}\, x\, \mathrm{Pu}\, x$$

one has $x^2 \in \mathbb{R}$ if and only if $\mathrm{Re}\, x = 0$ or $\mathrm{Pu}\, x = 0$. If furthermore $\mathrm{Re}\, x \neq 0$, then $x^2 > 0$ ◢

Furthermore remark that, for $x, y \in \mathbb{R}^3 \subset \mathbb{H}$

$$xy = -x \cdot y + x \times y \qquad (1.1.10)$$

so that

$$x \cdot y = 0 \iff xy + yx = 0 \qquad (1.1.11)$$

i.e., two pure quaternions anticommute if and only if they are orthogonal elements of $\mathbb{R}^3$.

1.1.3 Lemma If $x \in \mathbb{H}$ then there exists $y \in \mathbb{R}^3$ such that

$$xy \in \mathbb{R}^3$$

Proof Let $y \in \mathbb{R}^3$ such that $\mathrm{Pu}\, x \cdot y = 0$. Then

$$xy = (\mathrm{Re}\, x + \mathrm{Pu}\, x)y = \mathrm{Re}\, x\, y - \mathrm{Pu}\, x \cdot y + \mathrm{Pu}\, x \times y =$$
$$= (\mathrm{Re}\, x\, y) + (\mathrm{Pu}\, x) \times y \in \mathbb{R}^3.$$

The involution "$-$" allows us to construct a "sesquilinear form" on $\mathbb{H}$, and precisely

$$\mathbb{H}\times\mathbb{H} \ni (x,y) \longmapsto x\bar{y} \in \mathbb{H} \qquad (1.1.12)$$

This form is biadditive and for $x, y \in \mathbb{H}$ the following relations hold:

$$x\bar{y} = \overline{y\bar{x}} \qquad (1.1.13)$$

$$(\alpha x)\bar{y} = \alpha(x\bar{y}) \qquad (1.1.14)$$

$$x\overline{(\beta y)} = (x\bar{y})\bar{\beta} \qquad (1.1.15)$$

Moreover

$$x\bar{x} = (\mathrm{Re}\,x)^2 + \mathrm{Pu}\,x \cdot \mathrm{Pu}\,x \geqslant 0 \qquad (1.1.16)$$

is the square of the euclidean norm of x in $\mathbb{R}^4$.
As usual, we define the norm of $x \in \mathbb{H}$ by

$$|x| = (x\bar{x})^{1/2} \qquad (1.1.17)$$

<u>1.1.4 Lemma</u> For $x, y \in \mathbb{H}$ we have

$$|xy| = |x|\,|y| \qquad (1.1.18)$$

Proof

$$|xy|^2 = xy\,\overline{xy} = x\,y\bar{y}\,\bar{x} = x\bar{x}\,y\bar{y} = |x|^2|y|^2$$

since $y\bar{y} \in \mathbb{R}$ ◢

With the aid of the norm, we can calculate the inverse of a non zero element of $\mathbb{H}$. Precisely, if $x \neq 0$ then its inverse is

$$x^{-1} = \bar{x}/|x|^2 \qquad (1.1.19)$$

1.3 Real subalgebras of $\mathbb{H}$

We want to classify the real subalgebras of $\mathbb{H}$ which are isomorphic to $\mathbb{R}$ or $\mathbb{C}$

Let us begin with $\mathbb{R}$.

Suppose that $\varphi : \mathbb{R} \to \mathbb{H}$ is a algebra monomorphism.

In particular

$$\varphi(1) = 1$$

and so necessarily

$$\varphi(t) = t \qquad \forall t \in \mathbb{R}.$$

Then the canonical immersion $\mathbb{R} \hookrightarrow \mathbb{H}$ is the unique real algebra monomorphism from $\mathbb{R}$ to $\mathbb{H}$.

The situation is more complicated for the case of $\mathbb{C}$.
If $\psi : \mathbb{C} \longrightarrow \mathbb{H}$ is an algebra morphism, then again

$$\psi(1) = 1$$

but the only condition on $\psi(i)$ is that:

$$\psi(i)^2 = -1$$

On the contrary, if:

$$a \in \mathbb{H} \text{ and } a^2 = -1 \qquad (1.1.20)$$

i.e. a is a pure quaternion with unitary norm, then the map

$$\psi(t + is) = t + sa$$

is a real algebra-monomorphism.
More explicitly the real subalgebras of $\mathbb{H}$ isomorphic to $\mathbb{C}$ are

$$\mathbb{R} + \mathbb{R}a$$

with $a \in \mathbb{H}$ satisfying (1.1.20): in particular, there exist many of such subalgebras, in contrast with the unicity of subalgebras isomorphic to $\mathbb{R}$.

We can now define different kinds of linear structures on $\mathbb{H}$ by restricting of quaternionic product to convient subalgebras. More explicitely, if $\mathbb{K}$ is a real algebra and

$$\varphi : \mathbb{K} \longrightarrow \mathbb{H} \tag{1.1.21}$$

is a real-algebra monomorphism, we can define $\mathbb{H}$ to be a module over $\mathbb{K}$ in the following alternative ways :

$$k * x = \begin{cases} \varphi(k)\, x & (1.1.22) \\ x\, \varphi(k) & (1.1.23) \end{cases}$$

Obviously, if $\varphi(\mathbb{K}) \subset C(\mathbb{H})$ - center of $\mathbb{H}$ - then (1.1.22) and (1.1.23) define the same operation.

This is the case when $\mathbb{K} = \mathbb{R}$. Then φ is the canonical immersion and so there is a canonical way to interpret $\mathbb{H}$ as a real linear space : precisely

$$t * x = tx = xt \tag{1.1.24}$$

i.e. the canonical four dimensional real linear space $\mathbb{R}^4$.

This is no longer true for $\mathbb{K} = \mathbb{C}$. In fact, for each $a \in \mathbb{H}$ such that

$$a^2 = -1$$

we can define a embedding of $\mathbb{C}$ in $\mathbb{H}$ by

$$t + si \longmapsto t + sa$$

and so we can define a complex linear structure on $\mathbb{H}$ in the following ways :

$$(t+si)*x = \begin{cases} (t+sa)x & (1.1.25) \\ x(t+sa) & (1.1.26) \end{cases}$$

Since $\mathbb{R}+\mathbb{R}a$ cannot be contained in $\mathbb{R}=C(\mathbb{H})$, (1.1.25) and (1.1.26) give different results.

When $a=i_1$ and (1.1.25) holds, we have the standard two dimensional complex structure; in this case, in fact, if $\alpha, z_1, z_2 \in \mathbb{C}$ then:

$$\alpha * \chi^{-1}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \chi^{-1}\begin{pmatrix} \alpha z_1 \\ \alpha z_2 \end{pmatrix} \qquad (1.1.27)$$

where χ is defined in (1.1.6).

Finally note that $\mathbb{H}$ has many different complex linear structures, but it cannot be endowed with the structure of a complex algebra, associative with unity, then

$$E := \{\alpha * 1 : \alpha \in \mathbb{C}\} \subset C(\mathbb{H})$$

which contradicts Lemma (AI.2.1)

2. AUTOMORPHISMS OF $\mathbb{H}$ AND ROTATIONS

2.1 Ring and algebra automorphisms

For definitions and properties of ring and algebra automorphisms we refer to paragraph 2 of the Appendix.

It is a general fact that a ring morphism between two algebras is not necessarily an algebra morphism. For example $\mathbb{C}$ has many non trivial ring automorphisms but only two real algebra automorphisms.

This is no longer the case for the real algebra $\mathbb{H}$.

1.2.1 Prop. Every ring (anti-) automorphism of $\mathbb{H}$ is a real algebra (anti-) automorphism.

Proof Let $\varphi: \mathbb{H} \to \mathbb{H}$ be a ring automorphism. Since φ is surjective and $C(\mathbb{H}) = \mathbb{R}$, then

$$\varphi(\mathbb{R}) \subset \mathbb{R}$$

So the restriction $\varphi_{\restriction}: \mathbb{R} \to \mathbb{R}$ is a ring morphism, not trivial since $\varphi(1) = 1$. Then $\varphi_{\restriction} = \mathrm{Id}_{\mathbb{R}}$, and the thesis follows from Lemma (A.2.2). ◢

From now on the word automorphism (or antiautomorphism) (of $\mathbb{H}$) will indicate a ring (or algebra) automorphism (or antiautomorphism)

The following Proposition shows that automorphisms of $\mathbb{H}$ are closely related to the orthogonal automorphisms of $\mathbb{R}^3$, wich we denote by $\mathrm{Ort}(\mathbb{R}^3)$.

<u>1.2.2 Prop.</u> Every automorphism or antiautomorphism of $\mathbb{H}$ has the form

$$x \longmapsto \mathrm{Re}\,x + \Lambda(\mathrm{Pu}\,x) \qquad (1.2.1)$$

where $\Lambda \in \mathrm{Ort}(\mathbb{R}^3)$

Conversely, (1.2.1) is

1. an automorphism when $\det \Lambda = +1$
2. an anti-automorphism when $\det \Lambda = -1$

<u>Proof</u> Let $\varphi : \mathbb{H} \to \mathbb{H}$ be an automorphism or anti-automorphism of $\mathbb{H}$. Then we know that

$$\varphi(t) = t \qquad \forall t \in \mathbb{R}$$

If now $x \in \mathbb{R}^3$, then

$$(\varphi(x))^2 = \varphi(x^2) = x^2 \leqslant 0$$

and so $\varphi(x) \in \mathbb{R}^3$

This prove that φ can be decomposed as follows

$$\varphi = \mathrm{id}_{\mathbb{R}} \oplus \Lambda$$

where $\Lambda : \mathbb{R}^3 \to \mathbb{R}^3$ is an $\mathbb{R}$-linear automorphism.
To prove that Λ is an orthogonal map, note that for each $x \in \mathbb{R}^3$ we have

$$|\Lambda(x)|^2 = |\Lambda(x)^2| = |\varphi(x^2)| = |x^2| = |x|^2.$$

Since furthermore, for $x, y \in \mathbb{R}^3$ one has:

$$\Lambda(x \times y) = \begin{cases} \Lambda(x) \times \Lambda(y) & \text{when } \det\Lambda = +1 \\ \Lambda(y) \times \Lambda(x) & \text{when } \det\Lambda = -1 \end{cases}$$

the converse follows immediately. ▲

As a consequence, the general involution of $\mathbb{H}$ has the form

$$x \longmapsto \mathrm{Re}\, x + \Gamma(\mathrm{Pu}\, x)$$

where Γ is a rotation of $\mathbb{R}^3$ such that $\Gamma^2 = id_{\mathbb{R}^3}$ (equivalentely, a simmetric rotation of $\mathbb{R}^3$), i.e. a rotation of $\mathbb{R}^3$ of an angle π around some axis.
Anti-involutions can be obtained by composition of involutions with conjugation.

2.2 Rotations of $\mathbb{R}^3$

We can describe the orthogonal group of $\mathbb{R}^3$ with the aid of the quaternionic multiplication.

1.2.3 Prop For any non zero pure quaternion q, the map
$$\mathbb{R}^3 \ni x \longmapsto -qxq^{-1} \in \mathbb{R}^3 \qquad (1.2.2)$$
is reflection around in the 3-plane $(\mathbb{R}q)^\perp$

Proof The fact that $qxq^{-1} \in \mathbb{R}^3$ follows from
$$(qxq^{-1})^2 = qxq^{-1}qxq^{-1} = qx^2q^{-1} = x^2 \leqslant 0$$
The map (2.1.2) is obviously linear. Furthermore it sends q into its opposite:
$$-q(q)q^{-1} = -q$$
and it fixes all vectors $x \in (\mathbb{R}q)^\perp$. In fact
$$-qxq^{-1} = xqq^{-1} = x$$
since orthogonal elements of $\mathbb{R}^3$ anticommute in $\mathbb{H}$ (see (1.1.11)) ◢

Now, it is well known that each orthogonal map can be decomposed in a certain number of hyperplane reflections. (see [P] Th. 9.41). This number is

1. even when the map is a rotation (det = +1)
2. odd when the map is an anti-rotation (det = -1)

Furthermore, for $p \in \mathbb{H}$ it is
$$p(qxq^{-1})p^{-1} = (pq)x(pq)^{-1}$$
and, as a consequence of Lemma (1.1.3), each quaternion

can be decomposed in the product of two pure quaternions. It follows then:

<u>1.2.4 Cor.</u> Every rotation of $\mathbb{R}^3$ has the form

$$x \longmapsto qxq^{-1} \qquad (1.2.3)$$

for a non zero quaternion q.

Conversely, for every non zero quaternion q, (1.2.3) is a rotation

Anti-rotations can be obtained by composition of rotations with $-id_{\mathbb{R}^3}$.

What we have proved allows us to construct the map

$$\phi : \mathbb{H}\setminus\{0\} \longrightarrow SOrt(\mathbb{R}^3)$$
$$q \longmapsto \phi q \qquad (1.2.4)$$

where $\qquad (\phi q)(x) = qxq^{-1}$

We have denoted by $SOrt(\mathbb{R}^3)$ the special orthogonal group of $\mathbb{R}^3$, i.e. the group of all orthogonal maps of $\mathbb{R}^3$ with unitary determinant.

This map is a group morphism and, by the next corollary, it is surjective.

<u>1.2.5 Prop</u> $\quad \operatorname{Ker}\phi = \mathbb{R}\setminus\{0\}$

<u>Proof</u> Let $q \in \mathbb{H}\setminus\{0\}$ such that

$$qxq^{-1} = x \qquad \forall x \in \mathbb{R}^3$$

i.e.

$$qx = xq \qquad \forall x \in \mathbb{R}^3$$

Then $q \in \mathbb{R} = C(\mathbb{H})$ ◢

It follows that each rotation of $\mathbb{R}^3$ can be represented by a unit quaternion, unique up to sign.

Finally, the following Proposition describes the idempotent rotations of $\mathbb{H}$.

<u>1.2.6 Prop.</u> For $q \in \mathbb{H} \setminus \{0\}$, ϕ_q is idempotent if and only if

$$\mathrm{Re}\, q = 0 \quad \text{or} \quad \mathrm{Pu}\, q = 0$$

<u>Proof</u> Suppose that

$$q(qxq^{-1})q^{-1} = x \qquad \forall x \in \mathbb{R}^3$$

i.e.

$$(q^2)x = x(q^2) \qquad \forall x \in \mathbb{R}^3$$

Then $q^2 \in \mathbb{R}$. But it is

$$q^2 = (\mathrm{Re}\, q)^2 - \mathrm{Pu}\, q \cdot \mathrm{Pu}\, q + 2\, \mathrm{Re}\, q\, \mathrm{Pu}\, q$$ ◢

2.3 Rotations of $\mathbb{R}^4$

Quaternions may also be used to represent rotation of $\mathbb{R}^4$.

For $q \in \mathbb{H}$, let us define the maps $\mathbb{R}^4 \longrightarrow \mathbb{R}^4$

$$\ell_q(x) = qx \qquad (1.2.5)$$

$$r_q(x) = xq \qquad (1.2.6)$$

where $\mathbb{R}^4$ is identified with $\mathbb{H}$.
Of course, these maps are $\mathbb{R}$-linear.
Furthermore

$$|\ell_q(x)| = |q||x| = |r_q(x)| \qquad (1.2.7)$$

and, as for determinants, we have:

$$\det \ell_q = |q|^4 = \det r_q \qquad (1.2.8)$$

Note that these determinants are always non negative numbers.
Therefore:

<u>1.2.7 Prop</u> For any unitary quaternion q it is

$$\ell_q, r_q \in SOrt(\mathbb{R}^4) \qquad (1.2.9)$$

Can every rotation of $\mathbb{R}^4$ be represented in this way?
The following Proposition give an answer to this question.
Here S^3 denotes the unit sphere or $\mathbb{R}^4$, i.e.
$\{q \in \mathbb{H} : |q| = 1\}$.

1.2.8 Prop. The map $\Psi : S^3 \times S^3 \longrightarrow SOrt(\mathbb{R}^4)$ defined by:

$$\Psi(p,q) = \ell_p \circ r_{\overline{q}} \qquad (1.2.10)$$

is a group surjection and

$$\operatorname{Ker}\Psi = \{(1,1), (-1,-1)\} \qquad (1.2.11)$$

Proof For any $p, p', q, q' \in S^3$ and any $x \in \mathbb{H}$ we have

$$\Psi(pp', qq')(x) = \ell_{pp'}\left(r_{\overline{qq'}}(x)\right) = pp'x\overline{q'}\,\overline{q} =$$

$$= \left(\Psi(p,q) \circ \Psi(p',q')\right)(x)$$

that is, Ψ is a group morphism.
Now, $(p,q) \in \operatorname{Ker}\Psi$ when

$$px\overline{q} = x \qquad \forall x \in \mathbb{H}$$

When $x = 1$ this implies $p\overline{q} = 1$, and so the above condition can be restated as follows:

$$px = xp \quad \forall x \in \mathbb{H}$$

This implies $p \in C(\mathbb{H}) = \mathbb{R}$, and the assertion (2.1.11) follow
To prove that Ψ is surjective, let w be any rotation of $\mathbb{R}^4$ and let $a = w(1)$.
Then $|a| = |w(1)| = 1$ and the map

$$\mathbb{R}^4 \ni x \longmapsto \overline{a}\, w(x) \in \mathbb{R}^4$$

is a rotation leaving 1 (and therefore each point of $\mathbb{R}$) fixed. It must have the form

$$id_{\mathbb{R}} \oplus \lambda$$

where $\lambda \in \mathrm{Sort}(\mathbb{R}^3)$.

So, by Cor (2.1.4) and Prop (2.1.5), there exists a unit quaternion q such that, for all $x \in \mathbb{R}^4$,

$$\bar{a}\, w(x) = q x q^{-1}$$

i.e.

$$w(x) = p x \bar{q}$$

where $p = aq$ ◢

Antirotations of $\mathbb{R}^4$ can be obtained by composition of a rotation with the conjugation, which is a particular antirotation.

3. LINEAR MAPS BETWEEN LINEAR SPACES OVER $\mathbb{H}$

3.1 A comparison among linearities

Each chain

$$\mathbb{R} \longrightarrow \mathbb{C} \longrightarrow \mathbb{H} \qquad (1.3.1)$$

of real algebra monomorphisms allows us to look at the $\mathbb{H}$ space as a $\mathbb{C}$-space or an $\mathbb{R}$-space.
Really, as we know, the left morphism and the composition of the two morphism are canonical inclusions. The only "free" morphism is the second one, which depends on the choice of a quaternion $a \in \mathbb{H}$ such that

$$a^2 = -1$$

We restrict ourself to the standart choice

$$a = i_1 \qquad (1.3.2)$$

So, the second morphism is defined by

$$(t + si) \longmapsto t + si_1 \qquad (1.3.3)$$

which gives the standard way to consider $\mathbb{C}$ as a real subalgebra of $\mathbb{H}$ (see §(1.3))

Let us now consider X, Y left linear space over

$\mathbb{H}$ and a map

$$\Lambda : X \to Y$$

We can talk about $\mathbb{R}$ or $\mathbb{C}$ or $\mathbb{H}$ linearity of Λ in correspondence to the fact that X and Y are linear spaces over different fields: our aim now is to compare this different linearities

Of course

$$\Lambda \ \mathbb{H}\text{-linear} \Rightarrow \Lambda \ \mathbb{C}\text{-linear} \Rightarrow \Lambda \ \mathbb{R}\text{-linear} \qquad (1.3.4)$$

Now we will study conditions to invert this implications.

1.3.1 Prop Let Λ be $\mathbb{R}$-linear. Then:

1. it is $\mathbb{C}$-linear if and only if for each $x \in X$
$$\Lambda(\imath_1 x) = \imath_1 \Lambda(x) \qquad (1.3.5)$$
2. It is $\mathbb{H}$-linear if and only if for each $x \in X$
$$\Lambda(\imath_\mu x) = \imath_\mu \Lambda(x) \quad \mu = 1,2,3 \qquad (1.3.6)$$

1.3.2 Prop Let Λ be $\mathbb{C}$-linear. Then it is $\mathbb{H}$-linear if and only if for each $x \in X$

$$\Lambda(\imath_2 x) = \imath_2 \Lambda(x) \qquad (1.3.7)$$

The proofs of the above Propositions are trivial, the second one depending on the equality

$$q^0+q^1\iota_1+q^2\iota_2+q^3\iota_3=(q^0+q^1\iota_1)+(q^2+q^3\iota_1)\iota_2$$

for $q^\mu \in \mathbb{R}$ $(\mu=0,1,2,3)$

Remark that, in general, the set of all $x\in X$ satisfying condition (1.3.6) or (1.3.7) is an $\mathbb{H}$-linear subspace of X.
Therefore conditions (1.3.6) or (1.3.7) have to be required only on a set whose cardinality is $\dim_{\mathbb{H}} X$.

3.2 Matrices associated to linear maps

In this paragraph we will explicit the conditions (1.3.5) – (1.3.7) when

$$X = Y = \mathbb{H} \tag{1.3.9}$$

Let $\Lambda : \mathbb{H} \to \mathbb{H}$ be an $\mathbb{R}$-linear map. In the standard bases of $\mathbb{R}^4$, identified with $\mathbb{H}$, the map Λ is represented by a matrix 4×4 with real entries:

$$\Lambda = (\alpha \mid \beta \mid \gamma \mid \delta) \tag{1.3.10}$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{R}^4$ are the columns of the matrix itself.

Condititions (1.3.5) – (1.3.7) obviously express some constraints on the entries of Λ: we will write explicitely these constraints.

Let us begin with the requiremtes for $\mathbb{C}$-linearity. To this aim, we choose $\{1, \iota_2\}$ as $\mathbb{C}$-basis of $\mathbb{H}$. Then our conditions become:

$$\beta = \iota_1 \alpha \quad \wedge \quad \delta = \iota_1 \gamma \tag{1.3.11}$$

and so the general $\mathbb{C}$-linear map from $\mathbb{H}$ to $\mathbb{H}$ is represented by a matrix having the form

$$\left(\alpha \mid i_1\alpha \mid \gamma \mid i_1\gamma \right) \tag{1.3.12}$$

Furthermore, if we want Λ to be $\mathbb{H}$-linear, we have to require that

$$\gamma = i_2\alpha \tag{1.3.13}$$

So, the general form of a matrix representing an $\mathbb{H}$-linear map is

$$\left(\alpha \mid i_1\alpha \mid i_2\alpha \mid i_3\alpha \right) \tag{1.3.14}$$

This matrix represents also the multiplication for the quaternion α on the right, i.e. the map

$$\Lambda(q) = q\alpha \qquad \forall q \in \mathbb{H} \tag{1.3.15}$$

Recall that, in this case:

$$\det \Lambda = |\alpha|^4 \geq 0$$

as we stated in (1.2.8).

Let us now consider the representation $\tilde{\mathbb{H}}$ of $\mathbb{H}$ (see § 1.1) and the chain (see (1.3.1))

$$\mathbb{R} \to \mathbb{C} \to \mathbb{H} \to \tilde{\mathbb{H}} \qquad (1.3.16)$$

when the right morphism is the algebra isomorphism defined in (1.1.6).
Remark that the resulting complex structure of $\tilde{\mathbb{H}}$ is the standard complex structure of $\mathbb{C}^2$.

Let now

$$\Gamma : \tilde{\mathbb{H}} \to \tilde{\mathbb{H}} \qquad (1.3.17)$$

be a $\mathbb{C}$-linear map. In the standard bases of $\mathbb{C}^2$, identified with $\tilde{\mathbb{H}}$, the map Γ is represented by a 2x2 matrix with complex entries

$$\Gamma = \begin{pmatrix} u & w \\ v & z \end{pmatrix} \qquad (1.3.18)$$

By choosing $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$ as the quaternionic base of $\tilde{\mathbb{H}}$ on which to test condition (1.3.7) we obtain

$$\begin{pmatrix} w \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -\bar{v} \\ \bar{u} \end{pmatrix} \qquad (1.3.19)$$

So, the generic matrix representing a quaternionic linear map have the following form:

$$\begin{pmatrix} u & -\overline{v} \\ v & \overline{u} \end{pmatrix} \tag{1.3.20}$$

for $u, v \in \mathbb{C}$

Of course its determinant is non negative:

$$|u|^2 + |v|^2 \geq 0.$$

4. HILBERT SPACES OVER $\mathbb{H}$

4.1 Definition and elementary properties

A left quaternionic vector space X is called an inner product space if it is endowed whith a function

$$(\, , \,) : X \times X \longrightarrow \mathbb{H} \qquad (1.4.1)$$

(called scalar product) whith the following properties

$$(y, x) = \overline{(x, y)} \qquad (1.4.2)$$

$$(x+y, z) = (x, z) + (y, z) \qquad (1.4.3)$$

$$(qx, y) = q(x, y) \qquad (1.4.4)$$

$$(x, x) \geqslant 0 \qquad (1.4.5)$$

$$(x, x) = 0 \text{ only if } x = 0 \qquad (1.4.6)$$

where $x, y, z \in X$ and $q \in \mathbb{H}$

These axioms imply that, for every $y \in X$, the mapping

$$X \ni x \longmapsto (x, y) \in \mathbb{H} \qquad (1.4.7)$$

is an $\mathbb{H}$-linear functional on X

Furthermore (1.4.2) and (1.4.3) imply that

$$(x, qy) = (x, y)\bar{q} \tag{1.4.8}$$

where $x, y \in X$ and $q \in \mathbb{H}$. So, for $x \in X$, the mapping

$$X \ni y \mapsto (x, y) \in \mathbb{H} \tag{1.4.9}$$

is $\mathbb{H}$-antilinear whith respect tothe canonical conjugation in $\mathbb{H}$ (see § I.1.5)

An alternative (simmetric to (1.4.4)) axiom for the homogeneity of the inner product would be

$$(qx, y) = (x, y)\bar{q}$$

REMARK: If X is a left space and if

$$(qx, y) = (x, y)q$$

or if

$$(qx, y) = \bar{q}(x, y)$$

for all $x, y \in X$ and $q \in \mathbb{H}$, then

$$(pq)(x, y) = (\overline{pq}\, x, y) = (\bar{q}\bar{p}\, x, y) =$$
$$= q(\bar{p}x, y) = (qp)(x, y)$$

which contradicts the non commutativity of $\mathbb{H}$ (unless $(,)$ is not trivial)

Similar reasoning holds for right quaternionic linear spaces.

The schwarz inequality holds in $\mathbb{H}$ spaces

<u>1.4.1 Lemma</u> Properties (1.4.2)-(1.4.5) imply that

$$|(x,y)| \leq (x,x)^{1/2} (y,y)^{1/2} \qquad (1.4.8)$$

for all $x, y \in X$

<u>Proof</u> Put

$$\alpha = (x,x) \qquad \beta = |(x,y)| \quad \gamma = (y,y)$$

and let $\delta \in \mathbb{H}$ be such that $|\delta| = 1$ and

$$\delta\,(y,x) = \beta \in \mathbb{R}$$

Then, for every $t \in \mathbb{R}$ we have:

$$0 \leq (x - t\delta y, x - t\delta y) = (x,x) - t\delta\,(y,x) +$$
$$- t\,(x,y)\overline{\delta} + t^2 \delta (y,y) \overline{\delta} =$$
$$= \alpha - 2\beta t + \gamma t^2$$

If $\gamma = 0$ we have $\beta = 0$, and so (1.4.8) is verified. Otherwise, i.e. if $\gamma > 0$, we have $\beta^2 \leq \alpha\gamma$ ◢

Another foundamental property of the inner product is the "Triangle Inequality".

<u>1.4.2 Lemma</u> For $x, y \in X$ we have

$$(x+y, x+y)^{1/2} \leqslant (x,x)^{1/2} + (y,y)^{1/2} \qquad (1.4.9)$$

<u>Proof</u> By the Schwarz inequality

$$(x+y, x+y) = (x,x) + (y,x) + (x,y) + (y,y) \leqslant$$
$$\leqslant (x,x) + 2|(x,y)| + (y,y) \leqslant (x,x) +$$
$$+ 2(x,x)^{1/2}(y,y)^{1/2} + (y,y) = \left((x,x)^{1/2} + (y,y)^{1/2}\right)^2 \blacktriangle$$

Let us define now for $x \in X$

$$\|x\| = (x,x)^{1/2} \qquad (1.4.10)$$

If follows that, for $x, y \in X$ and $q \in \mathbb{H}$

$$\|x\| \geqslant 0 \qquad (1.4.11)$$

$$\|x\| = 0 \text{ implies } x = 0 \qquad (1.4.12)$$

$$\|x+y\| \leqslant \|x\| + \|y\| \qquad (1.4.13)$$

$$\|qx\| = |q| \, \|x\| \qquad (1.4.14)$$

The first three of the above relations tell us that the non negative function $d : X \times X \to \mathbb{R}$.

$$d(x,y) = \|x-y\|$$

is a metric for X. If X is complete with respect to d, then X is called a Hilbert space over $\mathbb{H}$.

Notice that ; for any fixed $y \in X$, the mappings

$$X \ni x \mapsto (x,y) \in \mathbb{H}$$

$$X \ni x \mapsto (y,x) \in \mathbb{H}$$

$$X \ni x \longmapsto \|x\| \in \mathbb{R}$$

are continuous functions on the metric space X.

A first example of a Hilbert space over $\mathbb{H}$ is the set $\mathbb{H}^n$ of all n-tuples

$$x = (\xi_1, - \xi_n)$$

where $\xi_1, - \xi_n$ are quaternionic numbers, with its obvious left (or right) linear structure and the following inner product

$$(x,y) = \sum_{j=1}^{n} \xi_j \overline{\eta_j}$$

where $y = (\eta_1, \dots, \eta_n)$

A more interesting example is constructed, for any positive measure μ on a set S, from the set $\mathcal{F}(\mu)$ of all functions $f : S \to \mathbb{H}$

1. f is μ-measurable

2. $\int_S |f|^2 d\mu < +\infty$.

The set $\mathcal{F}(\mu)$, with its obvious left or right $\mathbb{H}$-linear structure, quotiented with respect to the equivalence relation:

$$f \sim g \iff f = g \quad \mu \text{ a.e.}$$

and endowed with the scalar product

$$(f, g) = \int_S f \bar{g} \, d\mu$$

is denoted by $L^2(\mu)$, as it is usual.

4.2 Orthogonality and Riesz Theorem

If $(x, y) = 0$ for some $x, y \in \mathbb{H}$, we say that x is orhogoual to y, and write $x \perp y$. Of course, the relation $\perp$ is symmetric.

Let $x^{\perp}$ denote the set of all $y \in X$ which are orthogonal to x and, for M subset of X

$$M^{\perp} = \bigcap_{x \in M} x^{\perp}$$

i.e. the set of all $y \in X$ which are ortogonal to every $x \in M$. Note that, for $x \in X$, $x^{\perp}$ is a closed subspace of X. The same holds for $M^{\perp}$.

The properties of a quaternionic Hilbert space easily imply that

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

i.e. the parallelogram property. This identity implies a classical theorem:

1.4.3 Theorem Every nonempty, closed, convex set in a Hilbert space over $\mathbb{H}$ contains a unique element of smallest norm.

The proof is exactly the same as in the complex case.

The above theorem allows us to esthablish a decomposition of X in orthogonal closed subspaces, in a general way.

<u>1.4.3 Theorem</u> If M is a closed subspace of X then X admits the following decomposition in direct sum

$$X = M \oplus M^{\perp} \qquad (1.4.15)$$

<u>Proof</u>

For $x \in X$, $x \notin M$, $x+M$ is a nonempty, closed, convex subset of X. So it contains a unique element of smallest norm, say x_0. We have $(x-x_0) \in M$.

To prove that $x_0 \in M^{\perp}$ we show that $(x_0, y) = 0$ for all $y \in M$. Assume $\|y\| = 1$, without loss of generality.

The minimizing property of x_0 shows that

$$(x_0, x_0) = \|x_0\|^2 \leq \|x_0 - \alpha y\|^2 = (x_0 - \alpha y, x_0 - \alpha y)$$

for every $\alpha \in \mathbb{H}$. This simplifies to

$$0 \leq -(x_0, y)\bar{\alpha} - \alpha(y, x_0) + \alpha\bar{\alpha}$$

For $\alpha = (x_0, y)$ this gives

$$0 \leq -|(x_0, y)|^2$$

which implies $(x_0, y) = 0$.

To prove that the decomposition is unique, suppose

$$x' + x'' = y' + y''$$

with $x', y' \in M$ and $x'', y'' \in M^{\perp}$.

Then

$$M \ni x' - y' = y'' - x'' \in M^{\perp}$$

But, as a consequence of (1.4.6), $M \cap M^{\perp} = \{0\}$, and so we have $x' = y'$ and $x'' = y''$ ◢

We have already noticed that, for each $y \in X$, the map

$$X \ni x \longmapsto (x, y) \in \mathbb{H} \qquad (1.4.16)$$

is a continuous linear functional on X.
Let us denote with X' the space of all continuous elements of X^*. This right quaternionic space can be endowed with a complete norm by

$$\|\lambda\| = \sup_{\|x\| \leq 1} \|\Lambda x\| \qquad (1.4.17)$$

Then the map

$$\mathcal{J} : X \rightarrow X' \qquad (1.4.18)$$

with $\mathcal{J}(y)$ defined by (1.4.16) is well-defined, and the classical Riesz representation theorem holds:

<u>1.4.4 Theorem</u> The map $\mathcal{J}$ is an anti-isometry, with respect to conjugation in $\mathbb{H}$.

<u>Proof</u> The space $\mathcal{J}$ is additive and, for $q, x, y \in \mathbb{H}$ we have:

$$\mathcal{J}(qy)\cdot x = (x, qy) = (x, y)\bar{q} = (\mathcal{J}y \cdot x)\bar{q} = ((\mathcal{J}y)\bar{q})\cdot x$$

i.e. $\mathcal{J}(qy) = (\mathcal{J}y)\bar{q}$ and $\mathcal{J}$ is an anti-morphism with respect to quaternionic conjugation.

Furthermore

$$|(\mathcal{J}y)\cdot x| = |(x, y)| \leq \|x\| \|y\|$$

and

$$|(\mathcal{J}y)\cdot y| = \|y\|^2$$

which implies:

$$\|\mathcal{J}y\| = \|y\|$$

We have now to prove that $\mathcal{J}$ is surjective.

Let $\Lambda \in X'$ and define

$$M = \ker \Lambda$$

Then M is a closed linear subspace of X.

If $M = X$ then $\Lambda = \mathcal{J}(0)$. Otherwise $M^{\perp} \neq \{0\}$

After choosing $z \in M^{\perp}$ with $\|z\| = 1$, we can decompose the generic $x \in X$ in the following way

$$x = u + v$$

where

$$u = (\Lambda x)(\Lambda z)^{-1} z$$

$$v = x - u$$

Then $u \in M^{\perp}$ (since it belongs to the linear space spanned by z) and $v \in M$ since

$$\Lambda v = \Lambda x - (\Lambda x)(\Lambda z)^{-1}(\Lambda z) = 0$$

It follows that

$$(z, v) = 0$$

and so

$$(\Lambda x)(\Lambda z)^{-1} = (\Lambda x)(\Lambda z)^{-1}(z, z) = ((\Lambda x)(\Lambda z)^{-1} z, z) =$$
$$= ((\Lambda x)(\Lambda z)^{-1} z + v, z) = (u + v, z) = (x, z)$$

which implies

$$\Lambda x = (x, z)\Lambda z = (x, \overline{\Lambda z}\, z)$$

i.e.

$$\Lambda = \mathcal{T}(\overline{\Lambda z}\, z).$$

which proves the surjectivity of $\mathcal{T}$. ◢

CHAPTER 2

QUATERNIONS: DIFFERENTIAL STRUCTURE

1. ONE DIMENSIONAL REGULARITY

Can we define a good differential calculus for quaternionic maps?
Fréchet differentiability and quaternionic analiticity are not of interest.
Following the idea of Fueter (see [F]) we can give a definition of regularity that has properties similar to holomorphy. All we will state is proved in [S] and [PE].

1.1 Fréchet differentiability

Let $U \subset \mathbb{H}$ be an open set, $x_0 \in U$ and

$$f : U \rightarrow \mathbb{H}$$

a map. We will say that f is left quaternionic Fréchet differentiable (or $\mathbb{H}_\ell$-differentiable) at x_0 if there exists a map:

$$\Lambda \in \mathrm{Lin}_{\ell}(\mathbb{H};\mathbb{H}) \qquad (2.1.1)$$

(i.e. a left quaternionic linear map into $\mathbb{H}$) such that the usual

$$f(x_0+h)-f(x_0)-\Lambda(h)=o(\|h\|) \qquad (2.1.2)$$

Of course, Λ is also complex linear and real linear (we endow $\mathbb{H}$ with its standard structure). Thus f is $\mathbb{R}$-differentiable and $\mathbb{C}$-differentiable at x_0.

Now $\mathbb{R}$-differentiability does not imply the existence of the second derivative, while $\mathbb{C}$-differentiability implies analiticity. The $\mathbb{H}_{\ell}$-differentiabity is an even stronger requirement, and selects a very small class of maps.

<u>2.1.1 Theorem</u> Let U be a connected open set. If f is $\mathbb{H}_{\ell}$-differentiable at every point of U then:

$$f(x)=\alpha+x\beta \qquad (x\in U)$$

for some $\alpha,\beta\in\mathbb{H}$.

Proof

The map f is holomorphic on U, as a map from $U \subset \mathbb{C}^2$ into $\mathbb{C}^2$.

Thus, in order to prove the assertion, by the identity principle we only have to verify that

$$d^2 f \equiv 0 \quad \text{on } U$$

Suppose now that

$$f = u + v\iota_2 \qquad x = z + w\iota_2$$

where $u, v \in Hol(U, \mathbb{C})$ and $z, w \in \mathbb{C}$, from (1.3.20) we obtains that in U:

$$\frac{\partial v}{\partial w} = \overline{\frac{\partial u}{\partial z}}$$

$$\frac{\partial u}{\partial w} = -\overline{\frac{\partial v}{\partial z}}$$

Thus, for example

$$\frac{\partial^2 u}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z}\right) = \frac{\partial}{\partial z}\left(\overline{\frac{\partial v}{\partial w}}\right) = \overline{\frac{\partial}{\partial \bar{z}}\left(\frac{\partial v}{\partial w}\right)} = 0$$

since $\partial v/\partial w \in Hol(U, \mathbb{C})$. Analogous computations for the remaining derivatives imply the assertion ◭

1.2 Analiticity

If we require that a function f of one complex variable $z = x + iy$ is a complex polinomial, i.e. that

$$f(z) = \sum_{k=0}^{n} a_k z^k ,$$

with $a_k \in \mathbb{C}$ $(1 \leq k \leq n)$, we select a proper subset of the set of all polynomials in the real variables x, y.

In fact the real variables x, y cannot be written as complex polynomials in z. For example:

$$2x = z + \bar{z}$$

and the condition

$$z + \bar{z} = \alpha z \qquad \forall z \in \mathbb{C}$$

cannot be satisfied with $\alpha \in \mathbb{C}$.

On the other hand, in the quaternionic case if $q = x_0 + x_1 i_1 + x_2 i_2 + x_3 i_3$ then we have:

$$x_0 = \frac{1}{4}\left(q - i_1 q i_1 - i_2 q i_2 - i_3 q i_3\right)$$

$$x_1 = \frac{1}{4 i_1}\left(q - i_1 q i_1 + i_2 q i_2 + i_3 q i_3\right)$$

$$x_2 = \frac{1}{4i_2}\left(q + i_1 q i_1 - i_2 q i_2 + i_3 q i_3\right)$$

$$x_3 = \frac{1}{4i_3}\left(q + i_1 q i_1 + i_2 q i_2 - i_3 q i_3\right)$$

It follows that every real polynomial in the variables x_μ ($0 \leq \mu \leq 3$) is a quaternionic polynomial, i.e. a sum whose terms are :

$$a_0 q a_1 \cdots q a_{n-1} q a_n$$

with $a_k \in \mathbb{H}$.

Thus, the theory of quaternionic power series coincides with the theory of real analytic maps in four variables

1.3 Functions of one complex variable

In the previous paragraphs we have tryed to make a straightforward translation of the classical definitions of regularity for complex maps into the case of quaternionic maps. In order to find an interesting definition of regularity for the quaternionic case, we will revisit some aspects of the meaning of the holomorphicity of a complex map.

Let $U \subset \mathbb{C}$ be an open set and

$$f : U \to \mathbb{C}$$

an $\mathbb{R}$-differentiable map on U.
Then we have on U

$$df = \partial f\, dz + \bar{\partial} f\, d\bar{z} \tag{2.1.3}$$

where:

$$\partial = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \quad \bar{\partial} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \tag{2.1.4}$$

Now, holomorphy can be expressed in two equivalent ways:

1. $\bar{\partial} f \equiv 0$ on U (2.1.5)

2. $d(f\,dz) \equiv 0$ on U (2.1.6)

The first equation is known as the Cauchy Riemann equation.
Since for the laplacian Δ one has

$$\Delta = 4\partial\bar{\partial} = 4\bar{\partial}\partial \tag{2.1.7}$$

every holomorphic map is harmonic, as well as its real and imaginary component.

It follows that a holomorphic map f has the mean value property, i.e. for every $z \in U$ and every $r>0$ such that $B(z,r)$ is contained in U, we have

$$f(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(z + re^{i\vartheta})\, d\vartheta \qquad (2.1.9)$$

and

$$f(z) = \frac{1}{\pi r^2} \int_{B(z,r)} f(w) \left(-\frac{1}{2} d\bar{w} \wedge dw\right) \qquad (2.1.10)$$

The second equation has a more geometrical meaning.

The Stokes theorem (2.1.6) leads to the Cauchy integral formula in the following manner: given the Cauchy kernel

$$G(z) = 1/z \quad \forall z \in \mathbb{C} \setminus \{0\} \qquad (2.1.11)$$

then, for every $z_0 \neq 0$,

$$d\big(G(z-z_0)\,dz\big) = 0 \quad \forall z \neq z_0 \qquad (2.1.12)$$

and

$$d\big(G(z-z_0) f(z)\,dz\big) = 0 \quad \forall z \in U \setminus \{z_0\} \qquad (2.1.13)$$

Now, if $z_0 \in U$ and $r>0$ is such that $B(z_0, r) \subset U$, one obtains the classical Cauchy representation formula

$$\int_{\partial B(z_0,r)} G(z-z_0)f(z)\,dz = \lim_{n\to+\infty} \int_{\partial B(z_0,r/n)} G(z-z_0)f(z)\,dz =$$

$$= f(z_0) \lim_{n\to+\infty} \int_{\partial B(z_0,r/n)} G(z-z_0)\,dz =$$

$$= f(z_0) \int_{\partial B(z_0,r)} G(z-z_0)\,dz = 2\pi i\, f(z_0),$$

i.e., more explicitly:

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial B(z_0,r)} \frac{f(z_0)}{z-z_0}\,dz \qquad (2.1.14)$$

This integral representation formula is probably the most basic fact in the theory of holomorphic functions, and immediately implies, for example, the Liouville theorem and the Weierstrass Theorem.

In the quaternionic case we look for a definition of regularity which produces similar properties.

1.4 Real differential forms with quaternionic values

In the last paragraph we have used real differential forms with complex values (as dz, $d\bar{z}$ and df): since the complex field is a commutative field, they have exactly the same algebraic properties as the real differential forms with real values. For real differential forms with quaternionic values, which from now we call quaternionic differential forms, the situation is much more complicated, as $\mathbb{H}$ is not commutative.

All problems refer to the definition of wedge product. If α, β are quaternionic differential forms on an open set $U \subset \mathbb{H}$ and $x_0 \in U$, then we define

$$(\alpha \wedge \beta)(x_0; h_1, \dots, h_{\sigma(m+n)}) = \sum_{\sigma}{}' \alpha(x_0; h_1, \dots, h_{\sigma(m)})\, \beta(x_0; h_{\sigma(m+1)}, \dots, h_{\sigma(m+n)}) \tag{2.1.15}$$

where $h_k \in \mathbb{H}$ $(1 \leq k \leq m+n)$ and the sum is taken over all permutations of $\{1, \dots, m+n\}$ satisfying

$$\sigma(1) < \dots < \sigma(m) \quad \text{and} \quad \sigma(m+1) < \dots < \sigma(m+n) \tag{2.1.16}$$

It follows that in general

$$\alpha \wedge \beta \neq (-1)^{mn} \beta \wedge \alpha$$

For example if dx denotes, as usual, the exterior derivative of the identity map of $\mathbb{H}$, then

$$(dx \wedge dx)(x_0; h_1, h_2) = h_1 h_2 - h_2 h_1 \neq 0$$

if h_1 does not commute with h_2.
Of course, the exterior derivative is defined as usual and has the usual properties. Precisely

$$d\alpha(x_0; h_0, \ldots, h_m) = \tag{2.1.17}$$
$$= \sum_{k=0}^{m} (-1)^k (\alpha'(x_0) \cdot h_k) \cdot (h_0, \ldots, \hat{h}_k, \ldots, h_m)$$

where $h_k \in \mathbb{H}$ $(0 \leq k \leq m)$ and where $\hat{h}_k$ means that h_k is missing. Furthermore α' denotes the usual real Fréchet derivative of the real differential form α (see [C] for exact definition of all terms)

Note that the exterior derivative does not depend on the definition of the wedge product.

For example Stokes Theorem holds in its classical form:

$$\int_A d\alpha = \int_{\partial A} \alpha \qquad (2.1.18)$$

for A open set relatively compact in U, and with regular boundary. Moreover (see [PE])

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta \qquad (2.1.19)$$

Now, if v denotes the usual volume form of $\mathbb{R}^4$, i.e. if

$$v = dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \qquad (2.1.20)$$

then we can define a 3-form Dx on $\mathbb{H}$ by requiring the following condition

$$\left(h_0, Dx(x_0; h_1, h_2, h_3)\right)_{\mathbb{R}} = v(h_0, h_1, h_2, h_3) \qquad (2.1.21)$$

for all h_0, h_1, h_2, h_3 in $\mathbb{H}$, where $(\ ,\ )_{\mathbb{R}}$ denotes the usual scalar product on $\mathbb{R}^4$, i.e.

$$(x, y)_{\mathbb{R}} = \mathrm{Re}(x\bar{y}) \qquad x, y \in \mathbb{H}$$

Geometrically, $Dx(h_1,h_2,h_3)$ is orthogonal to h_1, h_2, h_3 and its norm is equal to the volume of the 3-dimensional parallelepiped whose edges are h_1, h_2 and h_3. The following algebraic expression holds for Dx:

$$Dx(h_1,h_2,h_3) = \frac{1}{2}\left(h_3\bar{h}_1 h_2 - h_2\bar{h}_1 h_3\right) \qquad (2.1.22)$$

The form Dx will play a key-role in the definition of quaternionic regularity.

1.5 Regular functions of one quaternionic variable

Let $U \subset \mathbb{H}$ an open set, and

$$f : U \to \mathbb{H}$$

a real differentiable function on U.
Remark that, as opposed to what happens in § (2.2.3) for complex maps,

$$df \neq dx\, \partial_\ell f + d\bar{x}\, \bar{\partial}_\ell f \qquad (2.1.23)$$

$$df \neq \partial_r f\, dx + \bar{\partial}_r f\, d\bar{x} \qquad (2.1.24)$$

where, we have defined

$$\partial_\ell f = \frac{1}{2}\sum_{\mu=0}^{3} i_\mu \frac{\partial f}{\partial x^\mu} \qquad (2.1.25)$$

$$\bar{\partial}_\ell f = \frac{1}{2}\sum_{\mu=0}^{3} \bar{i}_\mu \frac{\partial f}{\partial x^\mu} \qquad (2.1.26)$$

and

$$\partial_r f = \frac{1}{2}\sum_{\mu=0}^{3} \frac{\partial f}{\partial x^\mu} i_\mu \qquad (2.1.27)$$

$$\bar{\partial}_r f = \frac{1}{2}\sum_{\mu=0}^{3} \frac{\partial f}{\partial x^\mu} \bar{i}_\mu \qquad (2.1.28)$$

Instead, the following algebraic identity holds

$$d(g\, Dx\, f) = \{(\bar{\partial}_r g) f + g(\bar{\partial}_\ell f)\}\, v \qquad (2.1.29)$$

on U (see [S]). It follows, by setting $g \equiv 1$ and $f \equiv 1$, respectively

$$\bar{\partial}_\ell f \equiv 0 \Longleftrightarrow d(Dx\, f) \equiv 0 \qquad (2.1.30)$$

$$\bar{\partial}_r g \equiv 0 \Longleftrightarrow d(g\, Dx) \equiv 0 \qquad (2.1.31)$$

Furthermore the four differential operators ∂_ℓ, $\bar{\partial}_\ell$, ∂_r and $\bar{\partial}_r$ commute pairwise, and the laplacian operator Δ can be expressed in the usual way:

$$\Delta = 4\,\partial_\ell \bar{\partial}_\ell = 4\,\partial_r \bar{\partial}_r \qquad (2.1.32)$$

All this suggests the following

2.1.2 Definition

The real differentiable function $f: U \to \mathbb{H}$ is called

1. left regular at x_0

$$\bar{\partial}_\ell f(x_0) = 0 \qquad (2.1.33)$$

2. left antiregular at x_0 when

$$\partial_\ell f(x_0) = 0 \qquad (2.1.34)$$

Of course, a similar Definition is given for right regularity and antiregularity.
From now on, we will deal mainly with left regularity. All the results obtained can be translated for the right-regularity case.

Notice that every twice differentiable, regular map is harmonic. Actually, we can prove that:

<u>2.1.3 Theorem</u> If f is regular on U then it is continuously differentiable on U.

The proof is non-trivial, and can be found in [PE]

Furthermore, the Cauchy Fueter kernel G, defined by

$$G(x) = x^{-1}/|x|^2 = \bar{x}/|x|^4 \quad x \neq 0, \qquad (2.1.35)$$

(which is a left and right regular map on $\mathbb{H} \setminus \{0\}$) appears in the Cauchy Fueter integral formula:

<u>2.1.4 Theorem</u> If f is left regular on U and if $B(x_0, r) \subset U$ $(r>0)$ then

$$f(x_0) = \frac{1}{2\pi^2} \int_{\partial B(x_0,r)} G(x-x_0)\, Dx\, f(x) \qquad (2.1.36)$$

<u>Dim</u> To prove the assertion it is enough to note that

$$\int_{\partial B(x_0,r)} G(x-x_0)\,Dx = \int_{\partial B(x_0,r)} \frac{(x-x_0)^{-1}}{|x-x_0|^2}\,\frac{x-x_0}{|x-x_0|}\,d\sigma =$$

$$= \frac{1}{r^3}\int_{\partial B(x_0,r)} d\sigma = 2\pi^2$$

Since the Cauchy Fueter kernel is real analytic except at the origin, the integrand of (2.1.36) is a continuous function of (x, x_0) in $\partial B(x_0,r) \times S(x_0,r)$. Moreover, for each fixed $x \in \partial B(x_0,r)$, the integrand is a real analytic function of x_0 in $S(x_0,r)$

Thus, by the previous Theorem

2.1.5 Theorem A function which is left regular in an open set $U \subset \mathbb{H}$ is real analytic in U.

In particular, a left regular function is harmonic.

Real analyticity of regular maps implies the following identity principle:

2.1.6 Theorem If $U \subset \mathbb{H}$ is a connected open set and f is a left regular function in U which is zero in an open subset of U, then

$$f \equiv 0 \quad \text{in } U.$$

As usual harmoniticity yields the mean value theorem:

<u>2.1.7 Theorem</u> If $f : U \subset \mathbb{H} \to \mathbb{H}$ is left regular and if for $x_0 \in U$ and $\varepsilon > 0$ $\{x \in \mathbb{H} : |x - x_0| \leq \varepsilon\}$ is contained in U, then

$$f(x_0) = \int_{|x-x_0|=\varepsilon} f(x) \Big/ \mathrm{vol}\,\{|x-x_0|=\varepsilon\} \tag{2.1.37}$$

The Liouville theorem holds:

<u>2.1.8 Theorem</u> If $f : \mathbb{H} \to \mathbb{H}$ is a bounded left regular function then f is constant.

as well as the maximum modulus theorem

<u>2.1.9 Theorem</u> Let $U \subset \mathbb{H}$ be an open connected set, and $f : U \to \mathbb{H}$ a non-constant left regular map in U. Then $|f|$ cannot have a local maximum in U.

The Weierstrass theorem follows from the Cauchy-Fueter integral representation formula:

<u>2.1.10 Theorem</u> Let $\{f_n : U \to \mathbb{H}\}$ $(n \in \mathbb{N})$ a sequence of left regular functions which is uniformly convergent on compact subset of U to a function $f : U \to \mathbb{H}$.
Then f is left regular.

The following two results have much harder proofs than the analogous results in the complex case:

<u>2.1.11 Theorem</u> Let u be a real valued function defined on a star-shaped open set $U \subset \mathbb{H}$.
If u is harmonic and has continuous second derivatives, there is a left regular function f defined on U such that $\operatorname{Re} f = u$.

and Morera's theorem:

<u>2.1.12 Theorem</u> If $f : U \subset \mathbb{H} \to \mathbb{H}$ is a continuous function such that

$$\int_{\partial R} Dq\, f = 0 \qquad (2.1.38)$$

for every region R with regular boundary and relatively compact in U, then f is left regular.

For complete proofs of the above Theorems and for others interesting feautures of regular maps, see [S] and [PE]. We will try to investigate, in the quaternionic case, some classical subjects of complex analysis.

We are expecially interested in the following argu= ments :

1. series expansions for regular functions
2. existence of Bergman reproducing kernel
3. generalisation of regularity to the infinite dimensional case.

This will be done in the next chapters, while now we are going to show the algebraic nature of regu= larity, looking for special classes of linear regular functions.

2. REGULARITY AND LINEARITY

2.1 Algebraic nature of regularity

Let

$$\Lambda : \mathbb{H} \to \mathbb{H}$$

be an $\mathbb{R}$-linear map. Of course, Λ is $\mathbb{R}$-differentiable at every point x in $\mathbb{H}$. and

$$\frac{\partial \Lambda}{\partial x^{\mu}}(x) = \Lambda(i_{\mu}) \qquad \mu = 0, \ldots, 3$$

Therefore, left regularity of Λ is equivalent to

$$\sum_{\mu=0}^{3} i_{\mu} \Lambda(i_{\mu}) = 0 \qquad (2.2.1)$$

More precisely, note that a left regular Λ is such that:

$$\Lambda(x) = \sum_{k=1}^{3} \pi_k(x)\, \alpha_k \qquad (2.2.2)$$

where

$$\alpha_k = -\iota_k \wedge (\iota_k) \quad k = 1, 2, 3$$

and

$$\pi_k : \mathbb{H} \longrightarrow \operatorname{span}\{1, \iota_k\} \subset \mathbb{H}$$

is the usual orthogonal projection.

Conversely, the right member of (2.2.2) always defines a left regular $\mathbb{R}$-linear function.

Denote the class of left regular $\mathbb{R}$-linear maps by

$$R\ell(\mathbb{H})$$

(In general, R, AR stay for Regular and Antiregular while ℓ, r for left, right respectively).

If $U \subset \mathbb{H}$ is an open set, $x_0 \in U$ and if

$$f : U \longrightarrow \mathbb{H}$$

then

<u>2.2.1 Lemma</u> The function f is left regular at x_0

if and only if there is a map Λ such that:

1. $\Lambda \in \mathcal{RL}(\mathbb{H})$ (2.2.4)

2. $\|f(x_0+h)-f(x_0)-\Lambda(h)\| = o(\|h\|)$ (2.2.5)

<u>Proof</u> If f is left regular at x_0 then 2. holds with $\Lambda = df(x_0) \in \mathcal{RL}(\mathbb{H})$.
Conversely suppose that 2. is true for a $\Lambda \in \mathcal{RL}(\mathbb{H})$. Since $\mathcal{RL}(\mathbb{H})$ is a subset of the class of $\mathbb{R}$-linear map, then f is $\mathbb{R}$-differentiable at x_0, and $\Lambda = df(x_0)$ ▲

As pointed out in the proof, the map Λ satisfying tangency condition (2.2.5) is unique:

$$\Lambda = df(x_0) \tag{2.2.6}$$

So, left regularity differs from $\mathbb{R}$-Fréchet differentia= bility only for the choice of the approximation class for the tangency condition (2.2.5): in the second case we choose all $\mathbb{R}$-linear maps, while in the first case we restrict ourselves to the proper subset $\mathcal{RL}(\mathbb{H})$.

The class $\mathcal{R}(\mathbb{H})$ is closed with respect to pointwise addition. As for $f, g : \mathbb{H} \to \mathbb{H}$ real differentiable maps, we have

$$d(f+g) = df + dg,$$

the sum of regular functions is regular.

If α is a quaternion and if its product with a function is defined pointwise, then

$$\Lambda \in \mathcal{R}(\mathbb{H}) \Rightarrow \Lambda\alpha \in \mathcal{R}(\mathbb{H}) \qquad (2.2.7)$$

since

$$\sum_{\mu=0}^{3} \iota_\mu (\Lambda\alpha)(\iota_\mu) = \left(\sum_{\mu=0}^{3} \iota_\mu \Lambda(\iota_\mu)\right)\alpha$$

The space $\mathcal{R}(\mathbb{H})$ is a <u>right</u> quaternionic space only; in fact $\alpha\Lambda \notin \mathcal{R}(\mathbb{H})$ in general.

Note that

$$d(fg)(x;h) = df(x;h)g(x) + f(x)\,dg(x;h);$$

hence fg is not necessarily regular if f and g are.

Moreover

$$id_{\mathbb{H}} \notin \mathcal{RE}(\mathbb{H}) \tag{2.2.8}$$

since

$$\sum_{\mu=0}^{3} i_\mu^2 = -2 \neq 0$$

Furthermore $\mathcal{RE}(\mathbb{H})$ is not closed with respect to composition. For example, the $\mathbb{R}$-linear map φ such that

$$\varphi(i_\mu) = \begin{cases} i_\mu & \mu = 0,1,2 \\ -i_3 & \mu = 3 \end{cases}$$

belongs to $\mathcal{RE}(\mathbb{H})$, but

$$\varphi \circ \varphi = id_{\mathbb{H}} \notin \mathcal{RE}(\mathbb{H})$$

Analogously, the $\mathbb{R}$-linear map defined by

$$\psi(i_\mu) = \begin{cases} i_0 & \mu = 0 \\ i_\mu/3 & \mu = 1,2,3 \end{cases}$$

belongs to $\mathcal{RE}(\mathbb{H})$ but ψ^{-1} does not.

Notwithstanding these phenomena, regular functions remain an interesting class of maps.

We will generalize the Definition of regularity to the case of infinite quaternionic variables

A left regular function f will be an $\mathbb{R}$-differentiable function such that (with the notations used above)

$$df \in \mathcal{R}\ell\,(X;Y)$$

for a suitable choice of the class $\mathcal{R}\ell(X;Y)$ (see Chapter 5)

2.2 Regular algebra automorphisms

In §(2.2.1) we have characterized real linear maps which are left regular.
Doing the same for complex quaternionic left linear maps, we obtain:

2.2.1 Lemma Suppose that $\Lambda : \mathbb{H} \to \mathbb{H}$ is $\mathbb{C}$-linear. Then it is left regular if and only if

$$\Lambda(\iota_2) = 0 \qquad (2.2.9)$$

If furthermore Λ is $\mathbb{H}_e$-linear then

$$\Lambda \equiv 0 \qquad (2.2.10)$$

Proof Since Λ is $\mathbb{C}$-linear then

$$\sum_{\mu=0}^{3} \iota_\mu \Lambda(\iota_\mu) = \Lambda(1) + \iota_1^2 \Lambda(1) + $$
$$+ \iota_2 \Lambda(\iota_2) + \iota_3 \iota_1 \Lambda(\iota_2) = 2\iota_2 \Lambda(\iota_2)$$

▲

Much more interesting are the decomposable maps, i.e. the $\mathbb{R}$-linear maps which acts separately on $\mathbb{R}$ and $\mathbb{R}^3$ as subsets of $\mathbb{H}$

$$\Lambda = \varphi \oplus \Phi \qquad (2.2.11)$$

where

$$\varphi : \mathbb{R} \to \mathbb{R}$$

and

$$\Phi : \mathbb{R}^3 \to \mathbb{R}^3$$

are $\mathbb{R}$-linear maps

The following Lemma holds:

2.2.2 Lemma Λ is left regular if and only if

1. Φ is symmetric (2.2.12)
2. $tr\varphi = tr\,\Phi$ (2.2.13)

Proof Let

$$\left(\begin{array}{c|c} u & 0 \\ \hline 0 & A \end{array}\right)$$

with $u \in \mathbb{R}$ and $A = (\alpha_{ij})$ $(i,j = 1,2,3)$, be the associated matrix to Λ in the real orthonormal standard basis i_μ $(\mu = 0, ,3)$.

Then we have:

$$\sum_{\mu=0}^{3} i_\mu \Lambda(i_\mu) = u + \sum_{h=1}^{3} i_h \sum_{k=1}^{3} \alpha_{kh} i_k =$$

$$= u - \alpha_{11} - \alpha_{22} - \alpha_{33} + (\alpha_{32} - \alpha_{23}) i_1 + (\alpha_{13} - \alpha_{31}) i_2 +$$

$$+ (\alpha_{21} - \alpha_{12}) i_3$$

◢

From the proof we can deduce that exactly the same charaterisation holds for right regular decompo= sable maps, while for antiregularity (left and right) condition 2. has to be replaced by:

$$\mathrm{tr}\varphi + \mathrm{tr}\Phi = 0 \tag{2.2.14}$$

As we already know, every algebra automorphism or anti-automorphism of $\mathbb{H}$ acts separately on $\mathbb{R}$ and $\mathbb{R}^3$ (see Proposition 1.2.2). Since an ortho= gonal and symmetric map is also idempotent, it follows that:

<u>2.2.3 Lemma</u> An algebra automorphism (or anti-auto= morphism) $\omega : \mathbb{H} \longrightarrow \mathbb{H}$ is left re= gular if and only if

$$1.\ \omega^2 = id_{\mathbb{H}} \tag{2.2.15}$$

$$2.\ \mathrm{tr}\,\omega = 2 \tag{2.2.16}$$

To each left regular algebra automorphism ω we can associate a new concept of left regularity in order to include the iden= tity map among the left regular functions.

Precisely, let us denote by:

$$Re_\omega(\mathbb{H}) \tag{2.2.17}$$

the class of all $\mathbb{R}$-linear maps $\Lambda : \mathbb{H} \to \mathbb{H}$ which satisfy the condition:

$$\sum_{\mu=0}^{3} w(i_\mu)\,\Lambda(i_\mu) = 0 \qquad (2.2.18)$$

When $w = id_{\mathbb{H}}$ we have the usual Definition of left regularity. Therefore now

$$id_{\mathbb{H}} \in Re_w(\mathbb{H}) \qquad (2.2.19)$$

since in general

$$\Lambda \in Re_w(\mathbb{H}) \iff w \circ \Lambda \in Re(\mathbb{H}) \qquad (2.2.20)$$

As we can see, the class of left regular functions now contains the identity map but does not change in a significatively deep way.

Let us end the section by defining special algebra anti-involutions, which will be useful in Chapter 5 while looking for an interpretation of the Cauchy Riemann Fueter equation for left regularity.

Precisely, for $\mu = 1,2,3$ we define the algebra anti-involutions $\vartheta_\mu : \mathbb{H} \to \mathbb{H}$ as the maps such that

$$\vartheta_\mu(i_k) = \begin{cases} -i_k & \text{if } k = \mu \\ i_k & \text{otherwise} \end{cases} \qquad (2.2.21)$$

and furthermore

$$\vartheta_0 = \vartheta_1 \circ \vartheta_2 \circ \vartheta_3 \qquad (2.2.22)$$

These maps are all left and right regular in $\mathbb{H}$. Moreover ϑ_0 is the usual conjugation of $\mathbb{H}$, usually denoted by "$-$".

Such anti-involutions commute pairwise (with respect to composition), i.e.

$$\vartheta_\mu \circ \vartheta_\lambda = \vartheta_\lambda \circ \vartheta_\mu \qquad (2.2.23)$$

for every $\mu, \lambda = 0,1,2,3$.

Furthermore we can easily prove that, for every $\lambda, \mu = 0,1,2,3$

$$\frac{1}{4}\sum_{k=0}^{3}\overline{\vartheta}_{\mu}(\iota_k)\vartheta_{\lambda}(\iota_k)=\begin{cases}1 & \text{if } \mu=\lambda\\ 0 & \text{otherwise}\end{cases}\qquad (2.2.24)$$

Finally, note that every $\overline{\vartheta}_{\mu}$ is an involution of the real algebra $\mathbb{H}$.

CHAPTER 3

QUATERNIONIC BERGMAN KERNEL

1. REPRODUCING KERNELS

1.1. Definition

Let X be a set, and $\mathcal{F}$ a left quaternionic Hilbert space of quaternionic valued functions on X. Let $(,)$ be the scalar product of $\mathcal{F}$ and $\| \| = (,)^{1/2}$ the associated norm

Following [V] (pg 42) we say that

3.1.1. Definition

A function

$$K : X \times X \longrightarrow \mathbb{H} \qquad (3.1.1)$$

is a reproducing kernel of $\mathcal{F}$ when

1. for every $y \in X$ the function

$$K_y : x \longmapsto K(x, y) \qquad (3.1.2)$$

belongs to $\mathcal{F}$

2. for every $x \in X$ and $f \in \mathcal{F}$

$$f(x) = (f, K_x) \tag{3.1.3}$$

Note that

$$K(x,y) = \overline{K(y,x)} \tag{3.1.4}$$

$$K(x,x) = \|K_x\|^2 \geq 0 \tag{3.1.5}$$

for every $x, y \in X$. In fact, since $K_y \in \mathcal{F}$

$$K(x,y) = K_y(x) = (K_y, K_x) = \overline{(K_x, K_y)} =$$

$$= \overline{K_x(y)} = \overline{K(y,x)}$$

It follows that

<u>3.1.2 Lemma</u> The space $\mathcal{F}$ can have at most one reproducing kernel

<u>Proof</u> If K' is an other reproducing kernel of $\mathcal{F}$, then necessarily

$$K'(x,y) = K'_y(x) = (K'_y, K_x) = \overline{(K_x, K'_y)} =$$

$$= \overline{K_x(y)} = \overline{K(y,x)} = K(x,y) \quad \blacktriangle$$

Furthermore note that

<u>3.1.3 Lemma</u> If $\mathfrak{F}$ admits a reproducing kernel, then the linear subspace of $\mathfrak{F}$ defined by

$$\{K_x : x \in X\} \qquad (3.1.6)$$

is dense in $\mathfrak{F}$.

<u>Proof</u> Let be $f \in \mathfrak{F}$. Then $f(x)=0$ if and only if f is orthogonal to K_x. The Riesz representation theorem, implies the assertion. ◭

1.2 Characterisation

Note that, for every $x \in X$, the "valutation" map

$$\mathfrak{F} \ni f \xmapsto{\phi_x} f(x) \in \mathbb{H} \qquad (3.1.7)$$

belongs to the algebraic dual $\mathfrak{F}^*$ of $\mathfrak{F}$.
Now (3.1.3) tells us that the above defined map ϕ_x is a continuous map, i.e. belongs to $\mathfrak{F}'$.
More precisely, we can prove that :

3.1.4 Theorem

The space $\mathcal{F}$ admits a reproducing kernel if, and only if, for every $x \in X$ there exists a constant $k_x > 0$ such that

$$|f(x)| \leq k_x \|f\| \qquad (3.1.8)$$

for every $f \in \mathcal{F}$

Proof If $\mathcal{F}$ admits a reproducing kernel K, then

$$|f(x)| \leq \|K_x\| \, \|f\|$$

Conversely, suppose that (3.1.8) is true. Then the functional

$$\mathcal{F} \ni f \xmapsto{\phi_x} f(x) \in \mathbb{H}$$

belong to $\mathcal{F}'$. From the Riesz representation theorem it follows that there exists one and only one $K_x \in \mathcal{F}$ that reproduces ϕ_x, i.e. such that:

$$f(x) = (f, K_x)$$

for every $f \in \mathcal{F}$. Then $K(x,y)$ defined by

$$K(x,y) := K_y(x)$$

is a reproducing kernel of $\mathcal{F}$ ▲

As for the norm of ϕ_x, we have

3.1.5 Lemma For every $x \in X$

$$\sup_{\|f\| \leq 1} \phi_x(f) = \sup_{\|f\| \leq 1} |f(x)| = K(x,x)^{1/2} \qquad (3.1.9)$$

the supremum being attained (when $K(x,x) > 0$) on:

$$f = \alpha K_x \qquad (3.1.10)$$

with

$$|\alpha| = K(x,x)^{-1/2} \qquad (3.1.11)$$

Proof If $x \in X$ then

$$|f(x)| \leq \|K_x\| \, \|f\| \qquad (3.1.12)$$

for every $f \in \mathfrak{F}$. So

$$\sup_{\|f\| \leq 1} |f(x)| \leq \|K_x\| = K(x,x)^{1/2}$$

To show the converse, note that equality holds in (3.1.12) only when

$$f = \alpha K_x$$

If furthermore $\|f\|=1$ then

$$1 = (f,f) = (\alpha k_x, \alpha k_x) = \alpha (k_x, k_x)\overline{\alpha} =$$
$$= |\alpha|^2 K(x,x)$$

$\blacktriangle$

1.3 Topological questions

Equation

$$|f(x)| \leqslant \|f\| \cdot K(x,x)^{1/2} \qquad (3.1.13)$$

yields that convergence in $\mathcal{F}$ implies pointwise convergence.

In the complex case, when X is an open subset of some $\mathbb{C}^n$ and $\mathcal{F}$ contains only holomorphic maps, the Hartogs Theorem implies that

$$X \times \overline{X} \ni (x,y) \longmapsto K(x,\overline{y}) \in \mathbb{H}$$

is an holomorphic map. So

$$X \ni x \longmapsto K(x,x) \in \mathbb{H}$$

is a continuous map, and convergence on $\mathcal{F}$ implies uniform convergence on all compact subsets of X.

In the quaternionic case, we are not able to prove the Hartogs Theorem in the usual form, and so cannot prove that the map

$$x \longmapsto k(x,x)$$

is continuous.

Suppose now that

$$X \text{ is a separable topological space} \tag{3.1.14}$$

and that

$$\mathcal{F} \subset \mathcal{C}(X;\mathbb{H}) \tag{3.1.15}$$

Then we have

<u>3.1.6 Lemma</u> If $\mathcal{F}$ admits a reproducing kernel, then it is separable

<u>Proof</u> Let $\{e_\jmath\}_{\jmath \in S}$ a complete orthonormal set in $\mathcal{F}$. Then $\mathcal{F}$ is separable if and only if S is at most countable.

Suppose that $\mathcal{F}$ admits a reproducing kernel k.

For every $y \in X$ there exists a sequence $\jmath_y : \mathbb{N} \to S'$ and a sequence α_ν $(\nu \in \mathbb{N})$ of quaternionic numbers such that

$$k_y = \sum_{\nu \in \mathbb{N}}' \alpha_\nu e_{\jmath_y(\nu)}$$

Moreover

$$\alpha_\nu = (k_y, e_{\jmath_y(\nu)}) = \overline{(e_{\jmath_y(\nu)}, k_y)} =$$
$$= \overline{e_{\jmath_y(\nu)}(y)}$$

and so

$$k_y = \sum_{\nu \in \mathbb{N}}' \overline{e_{\jmath_y(\nu)}(y)}\, e_{s_y(\nu)}$$

For every $\jmath \in S \setminus \jmath_y(\mathbb{N})$ we have

$$e_\jmath(y) = (e_\jmath, k_y) = 0$$

Let now y_n $(n \in \mathbb{N})$ be a dense subset of X, and

$$T = \bigcup_{n \in \mathbb{N}} \jmath_{y_n}(\mathbb{N})$$

We prove that

$$S = T$$

In fact, if $j \in S \setminus T \neq \emptyset$, then

$$e_j(y_n) = 0 \qquad \forall n \in \mathbb{N}$$

Now, the map e_j is continuous, and so $e_j = 0$, whic is a contradiction.

▲

If e_ν $(\nu \in \mathbb{N})$ is an orthonormal basis of $\mathcal{F}$ having a reproducing kernel K, then

$$K_y = \sum_{\nu \in \mathbb{N}}' \overline{e_\nu(y)}\, e_\nu \tag{3.1.16}$$

Convergence in $\mathcal{F}$ implies pointwise convergence. Therefore

3.1.7 Lemma

For every $x, y \in X$:

$$K(x,y) = \sum_{\nu \in \mathbb{N}}' \overline{e_\nu(y)}\, e_\nu(x) \tag{3.1.17}$$

2. EXISTENCE OF A BERGMAN KERNEL

Let

$$U \subset \mathbb{H}$$

be an open set, $L^2(U)$ the class of square integrable functions from U to $\mathbb{H}$ with respect to the Euclidean measure, $\mathcal{R}r(U;\mathbb{H})$ the space of all right regular functions in U and:

$$\mathcal{F}_U := L^2(U) \cap \mathcal{R}r(U;\mathbb{H}) \qquad (3.2.1)$$

The space $\mathcal{F}_U$ is a left quaternionic linear space, and a pre-Hilbertian space respect to the usual scalar product in $L^2(U)$.

3.2.1 Lemma If the subset S of U has strictly positive distance from U^c then there exists a constant $c>0$ such that:

$$\sup_{x\in S} |f(x)| \leq c\,\|f\|_2 \qquad (3.2.2)$$

for every $f \in \mathcal{F}_U$

Proof Let $r>0$ be such that $B(x,r) \subset U$ for every $x \in S$. Since f is an harmonic map in U, then

$$f(x) = \alpha^{-1} \int_{B(x,r)} f v$$

where v is the usual volume element of $\mathbb{R}^4$, and

$$\alpha = v(B(0,r))$$

Since $f \in L^2(U)$, and since the Hölder inequality implies that

$$\int_{B(x,r)} |f| v \leq \|f\|_2 \, \alpha^{1/2}$$

Then

$$|f(x)| \leq \alpha^{-1} \int_{B(x,r)} |f| v \leq \alpha^{-1/2} \|f\|_2$$

for every $x \in S$. ▲

An easy consequence of the above lemma is the followin

3.2.2 Theorem

The space $\mathcal{F}_U$ is a left quaternionic Hilbert space.
Furthermore, it admits a reproducing kernel

The reproducing kernel of $\mathcal{F}_U$ will be called Bergman kernel of $\mathcal{F}_U$.

Proof

It is left to prove that $\mathcal{F}_U$, with the norm of $L^2(U)$, is a complet space.

Let f_n $(n \in \mathbb{N})$ be a Cauchy sequence in $\mathcal{F}_U$. Since $\mathcal{F}_U \subset L^2(U)$, which is a complete space, then there exists $f \in L^2(U)$ such that

$$\|f_n - f\| \to 0 \quad \text{for } n \to +\infty$$

From Lemma (3.2.1) it follows that $f_n \to f$ uniformely on compact subset of U. Thus the Weierstrass Theorem (2.1.10) implies that f is right regular, and so that $f \in \mathcal{F}_U$ ▲

CHAPTER 4

SERIES EXPANSIONS FOR REGULAR FUNCTIONS

1. REGULAR POLYNOMIALS

1.1 Definition

A homogeneous polynomial of degree $k \in \mathbb{N}_+$

$$P : \mathbb{H} \to \mathbb{H}$$

is a function from $\mathbb{H}$ into $\mathbb{H}$ for which there exists a k-real multilinear map A from $\mathbb{H}^k$ to $\mathbb{H}$ such that

$$P(x) = A(x, x, \dots, x) = \hat{A}(x) \tag{4.1.1}$$

for every $x \in \mathbb{H}$.

We denote this class of functions by

$$\mathcal{P}^k(\mathbb{H})$$

Of course, $\mathcal{P}^k(\mathbb{H})$ is a real Banach space with respect to pointwise operations, and with norm $\| \ \|$ defined by

$$\|P\| = \sup_{|x| \leq 1} |P(x)| \tag{4.1.2}$$

Furthermore we have:

$$|P(x)| \leq \|P\| \, |x|^k \tag{4.1.3}$$

for every $x \in \mathbb{H}$.

Also set

$$\mathcal{P}^0(\mathbb{H}) = \mathbb{H} \tag{4.1.4}$$

Finally note that $\mathcal{P}^1(\mathbb{H})$ consists of all the real linear and continuous functions into $\mathbb{H}$.

Clearly every polynomial is a real analytic function. If S_k is the permutation group of $\{1, \dots, k\}$ and if

$$A_s(x_1, \dots, x_k) = \frac{1}{k!} \sum_{\sigma \in S_k} A(x_{\sigma(1)}, \dots, x_{\sigma(k)}) \tag{4.1.5}$$

is the symmetric part of $A \in \mathcal{L}^k(\mathbb{H})$, then the following lemma holds.

4.1.1 Lemma

For every $x, x_1, \ldots, x_k \in \mathbb{H}$ the equality

$$\frac{1}{k!}\, d^k \hat{A}(x; x_1, \ldots, x_k) = A_s(x_1, \ldots, x_k) \qquad (4.1.6)$$

holds.

Proof

Since
$$\hat{A} = \widehat{A_s}$$
we can suppose that A is symmetric.
If (for $h = 1, \ldots, k$)
$$\Delta_h : \mathbb{H} \longrightarrow \mathbb{H}^k$$
is the diagonal map
$$\Delta_h(x) = (\underbrace{x, \ldots, x}_{k \text{ times}})$$
then

$$d(A \circ \Delta_k)(x; x_1) = \big(dA(\Delta_k(x)) \circ d\Delta_k(x)\big)(x; x_1) =$$

$$= dA(\Delta_k(x))\cdot x_1^k = A(x_1, x, \dots, x) + \dots + A(x, \dots, x, x_1) =$$
$$= k\, A(x_1, x, \dots, x) = k\,(A_{x_1} \circ \Delta_{k-1})(x)$$

where

$$A_{x_1}(x_2, \dots, x_k) := A(x_1, \dots, x_k)$$

is a $(k-1)$ real multilinear symmetric map.

Then, with the same argument:

$$d^2(A\circ\Delta_k)\cdot(x; x_1, x_2) = k\, d(A_{x_1}\circ\Delta_{k-1})(x)\cdot x_2 =$$
$$= k(k-1)\,(A_{x_1, x_2} \circ \Delta_{k-2})(x)$$

By repeating the same argument k times, we prove the assertion. ▲

1.2 Regularity requirement

Not all polinomials are regular: we will look for regularity conditions.

We will consider left regularity, the right regularity being completely analogous.

4.1.2 Lemma

If $A \in \mathcal{L}^k(\mathbb{H})$ $(k \geq 1)$ and $\overline{\partial}_\ell \hat{A} \equiv 0$ in $\mathbb{H}$ then

$$\hat{A}(x) = \sum_{\lambda_1, \dots, \lambda_k = 1}^{3} (x_{\lambda_1} - x_0 i_{\lambda_1}) \cdots (x_{\lambda_k} - x_0 i_{\lambda_k}) \, A_j(i_{\lambda_1}, \dots, i_{\lambda_k}) \qquad (4.1.7)$$

for every $x \in \mathbb{H}$.

Proof Since $\hat{A}$ is real differentiable at x and homogeneous of degree k, it follows that

$$\hat{A}(x) = \frac{1}{k}(d\hat{A})(x) \cdot x = \frac{1}{k} \sum_{\lambda=0}^{3} x_\lambda \frac{\partial \hat{A}}{\partial x_\lambda}(x)$$

But $\hat{A}$ is regular, and so

$$\frac{\partial \hat{A}}{\partial x_0}(x) = - \sum_{\lambda=1}^{3} i_\lambda \frac{\partial \hat{A}}{\partial x_\lambda}(x)$$

Then

$$\hat{A}(x) = \frac{1}{k} \sum_{\lambda=1}^{3} (x_\lambda - x_0 i_\lambda) \frac{\partial \hat{A}}{\partial x_\lambda}(x)$$

Now, $\partial\hat{A}/\partial x_\lambda$ is a regular polynomial of degree $k-1$. By iterating the same argument we obtain:

$$\hat{A}(x) = \frac{1}{k!} \sum_{\lambda_1,\dots,\lambda_k=1}^{3}{}' (x_{\lambda_1} - x_0 i_{\lambda_1})\cdots(x_{\lambda_k} - x_0 i_{\lambda_k}) \frac{\partial^k \hat{A}}{\partial x_{\lambda_1}\cdots\partial x_{\lambda_k}}(x)$$

The statement follows from Lemma (4.1.1). ▲

The following notations will be used:

$$\Gamma_0 = \{0\} \qquad \Gamma_k = \{1,2,3\}^{\{1,\dots,k\}} \quad (k \geq 1) \qquad (4.1.8)$$

$$\Gamma = \bigcup_{k\in\mathbb{N}} \Gamma_k \qquad (4.1.9)$$

For $\underline{\nu} \in \mathbb{N}^3$, let us define:

$$\Gamma_{\underline{\nu}} = \{\underline{\lambda} \in \Gamma : \# \underline{\lambda}^{-1}(h) = \nu_h \text{ per } h = 1,2,3\} \qquad (4.1.10)$$

$$\phi_{\underline{\nu}}(x) = \sum_{\underline{\lambda}\in\Gamma_{\underline{\nu}}} (x_{\lambda_1} - x_0 i_{\lambda_1})\cdots(x_{\lambda_k} - x_0 i_{\lambda_k}) \qquad (4.1.11)$$

We have

$$\# \Gamma_{\underline{\nu}} = |\underline{\nu}|! / \underline{\nu}! \qquad (4.1.12)$$

where

$$|\underline{\nu}| = \nu_1 + \nu_2 + \nu_3 \qquad \underline{\nu}! = \nu_1!\, \nu_2!\, \nu_3! \qquad (4.1.13)$$

and a direct computation shows that every $\phi_{\underline{\nu}}$ is left (and also right) regular.

These functions $\phi_{\underline{\nu}}$ are a basis of the space of regular polynomials.
Precisely, the following holds:

4.1.3 Lemma

Every left regular homogeneous polynomial of degree k can be uniquely written in the form

$$\sum_{|\underline{\nu}|=k}' \phi_{\underline{\nu}}\, a_{\underline{\nu}} \qquad (4.1.14)$$

where

$$a_{\underline{\nu}} \in \mathbb{H}$$

for every $\underline{\nu} \in \mathbb{N}^3$ with $|\underline{\nu}| = k$
Conversely, (4.1.14) always defines a left regular polynomial of degree k.

Proof Let A be a left regular polynomial of degree k.
Since:

$$\lambda, \mu \in \Gamma_{\underline{\nu}} \Rightarrow A_j(\iota_{\lambda_1}, \ldots, \iota_{\lambda_k}) = A_j(\iota_{\mu_1}, \ldots, \iota_{\mu_k})$$

from Lemma (4.1.2) we conclude that A can be written in the form (4.1.14).
Conversely, the left regularity of (4.1.14) follows from the left regularity of every $\phi_{\underline{\nu}}$.

As for the uniqueness of the representation, suppose that

$$\sum_{|\underline{\nu}|=k} \phi_{\underline{\nu}}(x)\, a_{\underline{\nu}} = 0 \qquad (4.1.15)$$

for every x belonging to a neighbourhood of zero in $\mathbb{H}$.
In particular, (4.1.15) is true for every x in a neighbourhood of zero in the purely imaginary subspace $\mathbb{R}^3 \subset \mathbb{H}$.
For a pure x we have

$$\phi_{\underline{\nu}}(x) = \frac{k!}{\underline{\nu}!}\, x^{\underline{\nu}} \qquad (4.1.16)$$

where

$$x^{\underline{\nu}} = x_1^{\nu_1}\, x_2^{\nu_2}\, x_3^{\nu_3} \qquad (4.1.17)$$

and $x = x_1 \iota_1 + x_2 \iota_2 + x_3 \iota_3$

Therefore (4.1.15) becomes

$$\sum_{|\underline{\nu}|=k} \left(\frac{k!}{\underline{\nu}!} a_{\nu}\right) x^{\underline{\nu}} = 0 \qquad (4.1.18)$$

The uniqueness of the representation in the theory of real analytic functions implies the assertion. ▲

We conclude this paragraph by remarking that a non homogeneous polynomial of degree k is a sum of homogeneous polynomials of degree h with

$$0 \leq h \leq k$$

in a unique way.

Moreover remark that

$$P \in \mathcal{P}^{k}(\mathbb{H}) \Rightarrow \bar{\partial} P \in \mathcal{P}^{k-1}(\mathbb{H}) \qquad (4.1.19)$$

2. SERIES EXPANSION FOR REGULAR FUNCTIONS

2.1 Regular analytic functions

Regular functions are real analytic, i.e. locally represented by a power series

$$\sum_{k\in\mathbb{N}} P_k \qquad (4.2.1)$$

where

$$P_k \in \mathcal{P}^k(\mathbb{H}) \quad k\in\mathbb{N} \qquad (4.2.2)$$

To represent an analytic function, (4.2.1) needs to have a non-zero radius of uniform convergence R, defined to be the supremum of $r \geqslant 0$ such that (4.2.1) is uniformly convergent on the closed ball $B(0, r)$

We have:

$$1/R = \limsup \|P_k\|^{1/k} \qquad (4.2.3)$$

i.e. R is actually the radius of convergence of the real power series:

$$\sum_{k\in\mathbb{N}} \|P_k\| \xi^k \tag{4.2.4}$$

A proof of this fact can be found in [M].

When a real analytic function is regular?

4.2.1 Lemma

Suppose $R>0$. Then (4.2.1) defines a left regular function if and only if

$$\bar{\partial}_\ell P_k \equiv 0 \qquad k\in\mathbb{N} \tag{4.2.5}$$

in $\mathbb{H}$.

Proof Since (4.2.1) is uniformely convergent in a neighbourhood of zero, in such a neighbourhood we have

$$\bar{\partial}\Big(\sum_{k\in\mathbb{N}} P_k\Big) = \sum_{k\in\mathbb{N}} \bar{\partial}_\ell P_{k+1}$$

Now, $\bar{\partial}_\ell P_{k+1}$ is an homogeneous polynomial of degree

k_1 and the assertion follows from the identity principle for analytic maps and from the Weierstrass Theorem for regular functions. ▲

Formula (4.1.7) yields that the left regular power series (4.2.1) can be written as:

$$\sum_{k \in \mathbb{N}} \left(\sum_{|\underline{\nu}|=k} \phi_{\underline{\nu}} a_{\underline{\nu}} \right) \tag{4.2.6}$$

with

$$a_{\underline{\nu}} \in \mathbb{H} \qquad \underline{\nu} \in \mathbb{N}^3$$

In a formal way, (4.2.6) can be rewritten as what we call a "multiple series"

$$\sum_{\underline{\nu} \in \mathbb{N}^3} \phi_{\underline{\nu}} a_{\underline{\nu}} \tag{4.2.7}$$

2.2 Convergence

A natural way to define the convergence of the multiple series (4.2.7), is to consider the summability of the family

$$\{\phi_{\underline{\nu}}\, a_{\underline{\nu}}\}_{\underline{\nu}\in\mathbb{N}^3} \qquad (4.2.8)$$

Let us be more precise

Suppose A is a set (no order structures) and X is a topological vector space.

Definition A family

$$x_\alpha \in X \quad (\alpha\in A)$$

is said to be summable with sum $x\in X$ when the net (see [D])

$$\sum_{\beta\in\Gamma} x_\beta \quad (\Gamma\in \mathcal{F}(A))$$

with $\mathcal{F}(A)$ class of all finite subset of A, ordered by inclusion, is convergent to x.

When X is a normed space, a similar definition holds for absolute summability.

For the study of summability we refer to [D]. Summability in the above sense is a very strong requirement.

When $A = \mathbb{N}$, it implies usual convergence, and when $X = \mathbb{R}$ or $\mathbb{C}$ it is equivalent to absolute convergence. Furthermore, the requirement of summability for $\{x_\alpha\}(\alpha \in A)$ implies the summability of every rearrangement of the same family.

In our context, summability of (4.2.7) at a point $x \in \mathbb{H}$ implies convergence of (4.2.6) at the same point.

The converse it is not true, but something interesting can be proved.

Let us define for $\lambda = 1, 2, 3$

$$\rho_\lambda : \mathbb{H} \to [0, +\infty) \qquad (4.2.9)$$

In such a way that:

$$\rho_\lambda(x) = \left(x_0^2 + x_\lambda^2\right)^{1/2} \qquad (4.2.10)$$

when: $x = x_0 \iota_0 + x_1 \iota_1 + x_2 \iota_2 + x_3 \iota_3 \in \mathbb{H}$.

Then the following Lemma holds:

4.2.2 Lemma

If the power series (4.2.6) has radius of uni= form convergence $R > 0$, then (4.2.7) is absolutely uniformly summable on all the compact subsets of

$$\mathcal{D}_R := \left\{ x \in \mathbb{H} : \sum_{\lambda=1}^{3} \rho_\lambda(x) < R/e \right\} \qquad (4.2.11)$$

Proof Suppose that:

$$\hat{A}_k = \sum_{|\underline{\nu}|=k} \phi_{\underline{\nu}} \, a_{\underline{\nu}}$$

for every $k \in \mathbb{N}$, with $A_k \in \mathcal{L}^k(\mathbb{H})$.

Then

$$\sum_{k \in \mathbb{N}} \sum_{|\underline{\nu}|=k} |\phi_{\underline{\nu}}(x)| \, |a_{\underline{\nu}}| \leq$$

$$\leq \sum_{k \in \mathbb{N}} \|A_k\| \sum_{|\underline{\nu}|=k} \frac{k!}{\underline{\nu}!} \rho(x)^{\underline{\nu}} =$$

$$= \sum_{k\in\mathbb{N}} \|A_k\| \left(\sum_{\lambda=0}^{3} \rho_\lambda(x)\right)^k$$

where as usual

$$\rho(x)^{\nu} = \rho_1(x)^{\nu_1}\, \rho_2(x)^{\nu_2}\, \rho_3(x)^{\nu_3}$$

Now, the radius of convergence of the power series

$$\sum_{k\in\mathbb{N}} \|A_k\|\, \xi^k$$

(also said reduced radius of convergence) is R/e (see [N], [M]), and this concludes the proof. ▲

As a result the power series (4.2.6) is (absolutely) uniform ly convergent in a neighbourhood of zero if and only if (4.2.7) is (absolutely) uniformely summable in a neighbourhood of zero (not necessarily equal to the first one).

As a remarkable consequence, either (4.2.6) or (4.2.7) can be used to define locally a left regular function.

3. ABEL'S LEMMA

In studying convergence properties of sums which are local expansions of regular functions, i.e. the power series:

$$\sum_{k \in \mathbb{N}} \left(\sum_{\substack{|\underline{\nu}|=k \\ \underline{\nu} \in \mathbb{N}^3}} \phi_{\underline{\nu}} \, a_{\underline{\nu}} \right) \qquad (4.3.1)$$

and the multiple series

$$\sum_{\underline{\nu} \in \mathbb{N}^3} \phi_{\underline{\nu}} \, a_{\underline{\nu}} \qquad (4.3.2)$$

with coefficients

$$a_{\underline{\nu}} \in \mathbb{H} \quad \underline{\nu} \in \mathbb{N}^3 \qquad (4.3.3)$$

we look for a lemma which is similar to the well-known Abel's lemma: let us first recall this lemma in a classical situation.

at the condition (4.3.10) implies that

$$|x_j| \neq 0 \quad j=1, \dots, n \qquad (4.3.13)$$

$D(|x_1|, \dots, |x_n|)$ is well-defined.

series

ultiple

ition selects an open and dense subset of
ich subset we denote by $\check{\mathbb{R}}^n$. (4.3.4)

r

Lemma (4.3.5)

every $x \in \mathbb{R}^n$ and $U \subset \mathbb{R}^n$ neighbourhood
x, we have that:

$$\{y \in U \cap \check{\mathbb{R}}^n : x \in D(|y^1|, \dots, |y^n|)\} \neq \emptyset \qquad (4.3.14)$$

(4.3.6)

erty is useful in studing summability domains. (4.3.7)
notice that

$$x \in \partial D(|x^1|, \dots, |x^n|)$$

is the
ph.

s give the following:

finition (4.3.8)

e call domain of summability of the

ii $r_1, \dots, r_n$ the

multiple power series

$$\sum_{\alpha \in \mathbb{N}^n} c_\alpha x^\alpha \qquad (4.3.15)$$

the class of all points $x \in \mathbb{R}^n$ such that it is summable in a neighborhood of x.

If it is not empty, the domain of summability of (4.3.15) is an open neighbourhood of zero.

Remark that, if the multiple series (4.3.15) is summable at a point $x \in \mathbb{H}$, then its generic term

$$c_\alpha x^\alpha \qquad \alpha \in \mathbb{N}^n \qquad (4.3.16)$$

is bounded, as function of $\alpha \in \mathbb{N}^n$.

Furthermore, if x belongs to the domain of summability of the multiple series (4.3.15), then the generic term (4.3.16) is bounded in a neighbourhood of x.

This fact implies that:

4.3.4 Lemma

The summability domain of the multiple power series (4.3.15) is an union of polydiscs. Furthermore, the series (4.3.15) is actually absolutely uniformly summable on all the compact subsets of its domain of summability.

As we can see, Abel's lemma makes possible an elementary geometrical description of the domain of summability of the multiple series (4.3.15).

Let us now consider the Taylor series

$$\sum_{k \in \mathbb{N}} \Bigg(\sum_{\substack{|\alpha| = k \\ \alpha \in \mathbb{N}^n}} c_\alpha x^\alpha \Bigg) \qquad (4.3.17)$$

whose convergence is the usual convergence.
We cannot prove nothing similar to the Abel's lemma. All we can do is to define a radius of uniform convergence (see [H]).

3.2 About a quaternionic Abel's lemma

We want to study results similar to the Abel's lemma for the quaternionic multiple series

$$\sum_{\underline{\nu} \in \mathbb{N}^3}' \phi_{\underline{\nu}}\, a_{\underline{\nu}} \qquad (4.3.18)$$

with coefficients $a_{\underline{\nu}} \in \mathbb{H}$ $(\underline{\nu} \in \mathbb{N}^3)$.

To this aim, recall that we have already defined, for $\lambda = 1,2,3$, the non negative functions:

$$-\ \rho_\lambda : \mathbb{H} \to [0, +\infty) \qquad (4.3.19)$$

such that

$$\rho_\lambda(x) = \left((x^0)^2 + (x^\lambda)^2\right)^{1/2} \qquad (4.3.20)$$

if $x = x^0 i_0 + x^1 i_1 + x^2 i_2 + x^3 i_3 \in \mathbb{H}$.

Furthermore, let us define for all ρ_1, ρ_2, ρ_3 such that

$$\rho_\lambda > 0 \quad \lambda = 1,2,3 \qquad (4.3.21)$$

the open neighbourhood of zero:

$$P(\rho_1, \rho_2, \rho_3) = \{ y \in \mathbb{H} : \rho_\lambda(y) < \rho_\lambda \quad \lambda = 1,2,3 \} \qquad (4.3.22)$$

to be the polycylinder having radii ρ_1, ρ_2, ρ_3.

Such a polycylinder seems a natural domain on which to state an "Abel's lemma" in the quaternionic case.

In fact, for the multiple series (4.3.18) we have that

4.3.5 Theorem

If, for a pure imaginary quaternion x such that

$$\rho_1(x)\, \rho_2(x)\, \rho_3(x) \neq 0$$

there exists a constant $M > 0$ such that:

$$|\phi_{\underline{\nu}}(x)\, a_{\nu}| \leq M \qquad \underline{\nu} \in \mathbb{N}^3$$

then the multiple series (4.3.18) is absolutely uniformly summable on all the compact subsets of

$$P(\rho_1(x), \rho_2(x), \rho_3(x))$$

Proof

Notice that for every $z \in \mathbb{H}$ one has that :

$$|\Phi_{\underline{\nu}}(z)| \leq \frac{|\underline{\nu}|!}{\underline{\nu}!} \rho(z)^{\underline{\nu}} \qquad \underline{\nu} \in \mathbb{N}^3$$

and that the equality holds when z is a real quaternion.
Thus, for every y such that

$$\rho_\lambda(y) / \rho_\lambda(x) < \vartheta_\lambda < 1 \quad \lambda = 1, 2, 3$$

we have

$$\sum_{\underline{\nu} \in \mathbb{N}^3}' |\Phi_{\underline{\nu}}(y)||a_{\underline{\nu}}| \leq M \sum_{\underline{\nu} \in \mathbb{N}^3}' |\Phi_{\underline{\nu}}(y)| / |\Phi_{\underline{\nu}}(x)| \leq$$

$$M \sum_{\underline{\nu} \in \mathbb{N}^3}' \rho(y)^{\underline{\nu}} / \rho(x)^{\underline{\nu}} \leq M \sum_{\vartheta \in \mathbb{N}^3}' \vartheta^{\underline{\nu}} < +\infty \qquad \blacktriangle$$

Unfortunely, nothing similar to the above Theorem holds for real quaternions.
In fact, let

$$x \in \mathbb{R}, \quad x \neq 0$$

be so that

$$\rho_1(x) = \rho_2(x) = \rho_3(x) = |x| \neq 0$$

and let $U \subset \mathbb{H}$ be an arbitrary neighbourhood of zero. We want to prove that there exist coefficients $a_{\underline{\nu}}$ ($\underline{\nu} \in \mathbb{N}^3$) for the multiple series (4.3.18), and

$$y \in U \cap \mathbb{R}^3$$

such that

1. $\phi_{\underline{\nu}}(x)\, a_{\underline{\nu}} = 0 \quad \forall \underline{\nu} \in \mathbb{N}^3$

2. the family $\phi_{\underline{\nu}}(y)\, a_{\underline{\nu}}$ ($\underline{\nu} \in \mathbb{N}^3$) is not bounded

In particular, it will turn out that our multiple series cannot be summable at the point y.

To show this, let us define for $k \in \mathbb{N}$

$$\underline{\nu}_k = (2k+1, 1, 0) \in \mathbb{N}^3$$

Note that (see definition of $\Gamma_{\underline{\nu}_k}$)

$$\# \Gamma_{\underline{\nu}_k} = 2(k+1)$$

is an even number, and therefore

$$\sum_{\underline{\lambda}\in\Gamma_{\underline{\nu}_k}}{}' \imath_{\underline{\lambda}} = \imath_1^{2k+1}\imath_2 + \imath_1^{2k}\imath_2\imath_1 + \cdots + \imath_1\imath_2\imath_1^{2k} + \imath_2\imath_1^{2k+1} = 0$$

Now, since x is a real quaternion we have that

$$\phi_{\underline{\nu}}(x) = (-x)^{|\underline{\nu}|} \sum_{\underline{\lambda}\in\Gamma_{\underline{\nu}}}{}' \imath_{\underline{\lambda}}$$

and that

$$\phi_{\underline{\nu}_k}(x) = 0 \qquad \forall k \in \mathbb{N}$$

Furthermore, if $y = y^1\imath_1 + y^2\imath_2 + y^3\imath_3$ is a pure imaginary quaternion, then

$$\phi_{\underline{\nu}}(y) = \frac{|\underline{\nu}|!}{\underline{\nu}!}\, y^{\underline{\nu}}$$

Now, in order to prove our assertions, we can choose, as we like, $y^1 \neq 0$, $y^2 \neq 0$ and y^3, and furthermore:

$$a_{\underline{\nu}} = \begin{cases} 1/y^{\underline{\nu}} & \text{if } \underline{\nu} = \underline{\nu}_k \text{ for some } k \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

so that

rthwhile to notice that, if (5.1.3) is
some $h, k \in X$, then it is true for all

$$\alpha * h + \beta * k \tag{5.1.4}$$

α and β are arbitrary complex numbers

equation (5.1.3) holds for every element
pace X if and only if it holds for all
ents of a complex basis of X.

case $X = Y = \mathbb{C}$, choosing $\{1\}$ as complex
$\mathbb{C}$, equation (5.1.3) becomes

$$\frac{\partial f}{\partial x}(x_0) + i * \frac{\partial f}{\partial y}(x_0) = 0 \tag{5.1.5}$$

usual Cauchy Riemann equation.

e can go further.
ow clear, the problem of checking the
of a real differentiable map is a purely
blem.

consider our Banach spaces X, Y as
ces, and let

$$\Lambda : X \to Y \tag{5.1.6}$$

be a real linear map

We can easily prove that

5.1.1 Lemma

The map Λ can be uniquely decomposed in a sum

$$\Lambda = L + A \qquad (5.1.7)$$

where L is a complex linear map, and A is a complex antilinear map

Furthermore we have:

$$L(x) = \frac{1}{2}\left\{\Lambda(x) - i * \Lambda(i * x)\right\} \qquad (5.1.8)$$

$$A(x) = \frac{1}{2}\left\{\Lambda(x) + i * \Lambda(i * x)\right\} \qquad (5.1.9)$$

Proof

The maps L, A are clearly complex linear and complex antilinear respectively, and (5.1.7) holds. Furthermore, suppose that

$$\Gamma : X \to Y$$

is both a complex linear and a complex antilinear

2. GENERALIZED REGULARITY AND PARITY OF SPACES

Let X, Y be quaternionic Banach spaces (possible infinite dimensional), $U \subset X$ an open set and

$$f : U \rightarrow Y \qquad (5.2.1)$$

a map which is real diffentiable at $x_0 \in U$.

As we stated in §(2.2), to define what does it mean regularity of f at x_0, we have to formulate a set of algebraic conditions to which the real derivative of f at x_0 have to satisfy. Such conditions, which we call generalized Cauchy Riemann Fueter conditions, are obtained using the analogy with the Cauchy Riemann conditions in the complex case: the aim of this paragraph is to show how we can do it.

It turns out that the problem of studing a set of Cauchy Riemann conditions is, at first, purely algebraic. Therefore we forget the metric structure on X and Y, and consider a real linear map:

$$\Lambda : X \to Y \qquad (5.2.2)$$

In the following a left (right) quaternionic linear space will be said to be a space of left (right) parity.

Now we want to extend the definition of <u>left regularity</u> for a real linear map into $\mathbb{H}$ (see § (2.2.1)) to the real linear map Λ.

To begin with we suppose that Y is a left space and X is bilateral.

Then the classical generalized Cauchy Riemann equation (5.1.10) seems to have two possible generalisations to the quaternionic case, in order to define left regularity of Λ, and precisely

$$\sum_{\mu=0}^{3} i_\mu \Lambda(i_\mu x) = 0 \quad x \in X \qquad (5.2.3)$$

$$\sum_{\mu=0}^{3} i_\mu \Lambda(x i_\mu) = 0 \quad x \in X \qquad (5.2.4)$$

We prove that only the last one is a good generalisation of the one dimensional Cauchy Riemann Fueter equation.

To this aim, let us define

$$S(x) = \sum_{\mu=0}^{3} \iota_\mu \wedge (\iota_\mu x) \tag{5.2.5}$$

and

$$R(x) = \sum_{\mu=0}^{3} \iota_\mu \wedge (x \iota_\mu) \tag{5.2.6}$$

The maps R and S are additive and:

5.2.1 Lemma

For every $q \in \mathbb{H}$ and $x \in X$

$$R(xq) = \bar{q} R(x) \tag{5.2.7}$$

Proof

In fact, if $q = q^0\iota_0 + q^1\iota_1 + q^2\iota_2 + q^3\iota_3$ then we have

$$R(xq) = \sum_{\mu=0}^{3} \iota_\mu \sum_{k=0}^{3} q^k \wedge (x(\iota_k \iota_\mu)) =$$

$$= \sum_{k=0}^{3} q^k \sum_{\mu=0}^{3} \iota_\mu \wedge (x(\iota_k \iota_\mu)) =$$

$$= \sum_{k=0}^{3} q^k \bar{\iota}_k \sum_{\mu=0}^{3} \iota_k \iota_\mu \Lambda(x(\iota_k \iota_\mu)) =$$

$$= \sum_{k=0}^{3} q^k \bar{\iota}_k \sum_{\mu=0}^{3} \iota_\mu \Lambda(x \iota_\mu) = \bar{q} R(x)$$

In particular, for a general real linear map Λ, it turns out that <u>ker R is a right quaternionic linear subspace of X</u>. Moreover, ker R is not a left quaternionic linear subspace of X, and ker S is neither a left nor a right quaternionic linear subspace of X.

For example:

$$S(qx) = \sum_{k=0}^{3} q^k \sum_{\mu=0}^{3} \iota_\mu \Lambda((\iota_\mu \iota_k) x) =$$

$$= \sum_{k=0}^{3} q^k \bar{\iota}_k \sum_{\mu=0}^{3} \iota_k \iota_\mu \Lambda((\iota_\mu \iota_k) x) =$$

$$= q^0 S(x) - q^1 \iota_1 \{ \iota_1 \Lambda(\iota_1 x) + \iota_0 \Lambda(\iota_0 x) +$$

$$- \iota_3 \Lambda(\iota_3 x) - \iota_2 \Lambda(\iota_2 x) \} - q^2 \iota_2 \{ \iota_2 \Lambda(\iota_2 x) +$$

$$- \iota_3 \Lambda(\iota_3 x) + \iota_0 \Lambda(\iota_0 x) - \iota_1 \Lambda(\iota_1 x) \} +$$

$$-q^3 \imath_3 \{\imath_3 \wedge (\imath_3 x) - \imath_2 \wedge (\imath_2 x) - \imath_1 \wedge (\imath_1 x) +$$
$$+ \imath_0 \wedge (\imath_0 x)\} = \sum_{k=0}^{3} q^k \bar{\imath}_k \sum_{\mu=0}^{3} \overline{\vartheta_k(\imath_\mu)} \wedge (\imath_\mu x)$$

where the anti-involutions ϑ_k $(k=0,1,2,3)$ are defined in (2.2.21) and (2.2.22):

$$\vartheta_k(\imath_\lambda) = \begin{cases} -\imath_\lambda & \text{if } \lambda = k \\ \imath_\lambda & \text{otherwise} \end{cases} \tag{5.2.8}$$

for $k = 1,2,3$ and

$$\vartheta_0 = \vartheta_1 \circ \vartheta_2 \circ \vartheta_3 \tag{5.2.9}$$

which is the usual conjugation of $\mathbb{H}$.
Thus we have, for generic $\wedge$ and $q \in \mathbb{H}$:

$$S(x) = 0 \not\Rightarrow S(qx) = 0$$

Lemma 5.2.1 yields that the fact that equation (5.2.4) holds for every element of the space X is equivalent to the fact that it holds for all the elements of a quaternionic base of X.
In the case $X = \mathbb{H}$, choosing $\{1\}$ as a quaternio=

nic base of $\mathbb{H}$, equation (5.2.4) becomes:

$$\sum_{\mu=0}^{3} i_\mu \Lambda(i_\mu) = 0$$

i.e. the usual Cauchy Riemann Fueter equation, when $\Lambda = df(x_0)$:

$$\sum_{\mu=0}^{3} i_\mu \frac{\partial f}{\partial x^\mu}(x_0) = 0$$

In conclusion, the definition of left regularity at some point $x_0 \in U$ for a function

$$f : X \supset U \longrightarrow Y \qquad (5.2.10)$$

requires that it is:

$$\begin{array}{l} X \text{ right quaternionic space} \\ Y \text{ left quaternionic space} \end{array} \qquad (5.2.11)$$

Analogus arguments for right regularity lead to the construction of the following table:

	X	Y
left regularity	right	left
right regularity	left	right
left antiregularity	left	left
right antiregularity	right	right

(5.2.12)

It is worthwhile to remark again that the algebraic character of the definition of regularity forces the parity of the involved spaces.

From now on, we will consider only left regularity and suppose therefore satisfied conditions (5.2.11).

3. REGULARITY AND HOLOMORPHY

To understand the relations between regularity and holomorphy we have to rewrite, at first, Cauchy Riemann Fueter equation in a complex form, i.e. having decomposed the real linear map Λ in its complex linear part L and its complex antilinear part A (see (5.1.7)).

The right space X and the left space Y will be endowed with their standard complex structures (see § (1.3.1)), for which:

$$\alpha * x = x\alpha \qquad \alpha \in \mathbb{C},\ x \in X \tag{5.3.1}$$

$$\alpha * y = \alpha y \qquad \alpha \in \mathbb{C},\ y \in Y \tag{5.3.2}$$

define the multiplication for complex scalars.

As stated in § (5.1), we can uniquely decompose the real linear map Λ in a sum:

$$\Lambda = L + A \tag{5.3.3}$$

where L is complex linear

$$L(x) = \frac{1}{2}\left\{\Lambda(x) - \imath_1 \Lambda(x\imath_1)\right\} \tag{5.3.4}$$

and A is complex antilinear

$$A(x) = \frac{1}{2}\left\{\Lambda(x) + \imath_1 \Lambda(x\imath_1)\right\} \tag{5.3.5}$$

Notice that, because of the different parity of the involved spaces, we have to pay attention in checking the complex linearity of a map.
For example, when $X = Y = \mathbb{H}$, the identity map is not a complex linear map (since $\mathbb{C} \not\subset C(\mathbb{H})$).

It turns out that the operator R associated to Λ (see (5.2.6)) depends only on the complex linear part A of Λ. Precisely:

<u>5.3.1 Lemma</u>

For every $x \in X$ we have

$$\frac{1}{2} R(x) = A(x) + \imath_2 A(x\imath_2) \tag{5.3.6}$$

Proof.

By definition of L, A and R (see (5.2.6)):

$$R(x) = \sum_{\mu=0}^{3} i_\mu (L+A)(x i_\mu) = L(x) + A(x) +$$

$$+ i_1 \{ i_1 L(x) - i_1 A(x) \} + i_2 L(x i_2) + i_2 A(x i_2) +$$

$$+ i_3 \{ -(L+A)(x i_2) i_1 \} = 2A(x) + i_2 L(x i_2) +$$

$$+ i_2 A(x i_2) + i_3 \{ -i_1 L(x i_2) + i_1 A(x i_2) \} =$$

$$= 2A(x) + 2 i_2 A(x i_2) \quad \blacktriangle$$

Thus the real linear map Λ is left regular if and only if:

$$A(x) + i_2 A(x i_2) = 0 \qquad x \in X \tag{5.3.7}$$

This clearly yields the following fact:

5.3.1 Corollary

Every holomorphic function is left regular.

Remark that Pertici in [PE] does not obtain the result stated in the Corollary. This is due to the fact that Pertici considers $X=Y=\mathbb{H}$ both viewed as left quaternionic spaces.

Suppose in fact that X is a bilateral space, and Λ a real linear map from X to Y (left space) as usual.
Then

$$\tilde{L}+\tilde{A}=\Lambda=L+A. \qquad (5.3.8)$$

where $\tilde{L}$, $\tilde{A}$ are the complex linear and the complex antilinear part of Λ respectively, when we choose on X the left (quaternionic and standard induced complex) structures, i.e.

$$\tilde{L}(x)=\frac{1}{2}\left\{\Lambda(x)+i_1\Lambda(i_1 x)\right\} \qquad (5.3.9)$$

and

$$\tilde{A}(x)=\frac{1}{2}\left\{\Lambda(x)-i_1\Lambda(i_1 x)\right\} \qquad (5.3.10)$$

and where L, A are defined in §(5.1) for the right structure on X.

If we consider the left structure on X, we obtain that

$$\frac{1}{2} R(x) = \tilde{A}(x) + \imath_2 \tilde{L}(\imath_2 x) \qquad (5.3.11)$$

i.e. that R cannot be expressed anymore in function of the complex antilinear part of Λ only, as it happens in equation (5.3.6).

4. CANONICAL DECOMPOSITION OF THE REAL DERIVATIVE

In the quaternionic case we prove a decom= position result for a real linear map similar to Lemma 5.1.1, stated in the complex case

5.4.1 Lemma

The real linear map Λ can be uniquely de= composed in a sum

$$\Lambda = \sum_{\mu=0}^{3} \Lambda_\mu \tag{5.4.1}$$

where, for every $\mu = 0, 1, 2, 3$ Λ_μ is ϑ_μ - quaternionic antilinear.

Furthermore we have, for every μ

$$\Lambda_\mu(x) = \frac{1}{4} \sum_{k=0}^{3} \overline{\vartheta_\mu(1_k)} \Lambda(x 1_k) \tag{5.4.2}$$

Proof

If we suppose that (5.4.1) is true, then we have, by using (2.2.24):

$$\sum_{k=0}^{3} \overline{\vartheta_\mu}(\iota_k)\Lambda(x\iota_k) = \sum_{k=0}^{3} \overline{\vartheta_\mu}(\iota_k) \sum_{\nu=0}^{3} \Lambda_\nu(x\iota_k) =$$

$$= \sum_{\nu k=0}^{3} \overline{\vartheta_\mu}(\iota_k)\vartheta_\nu(\iota_k)\Lambda_\nu(x) =$$

$$= \sum_{\nu=0}^{3}\left(\sum_{k=0}^{3} \overline{\vartheta_\mu}(\iota_k)\vartheta_\nu(\iota_k)\right)\Lambda_\nu(x) = \Lambda_\mu(x).$$

On the converse, the maps defined by (5.4.2) are clearly ϑ_μ-quaternionic antilinear.
In fact, by using the fact that $\overline{\vartheta_\mu}$ is an automorphism of $\mathbb{H}$, we have for every μ, $\nu = 0,1,2,3$

$$\Lambda_\mu(x\iota_\nu) = \frac{1}{4}\sum_{k=0}^{3} \overline{\vartheta_\mu}(\iota_k)\Lambda(x(\iota_\nu\iota_k)) =$$

$$= \frac{1}{4}\vartheta_\mu(\iota_\nu)\sum_{k=0}^{3} \overline{\vartheta_\mu}(\iota_\nu)\overline{\vartheta_\mu}(\iota_k)\Lambda(x(\iota_\nu\iota_k)) =$$

$$= \frac{1}{4}\,\vartheta_\mu(\iota_\nu)\sum_{k=0}^{3}\overline{\vartheta}_\mu(\iota_\nu\iota_k)\,\Lambda(x(\iota_\nu\iota_k)) =$$

$$= \vartheta_\mu(\iota_\nu)\,\frac{1}{4}\sum_{k=0}^{3}\overline{\vartheta}_\mu(\iota_k)\Lambda(x\iota_k) = \vartheta_\mu(\iota_\nu)\,\Lambda(x)$$

▲

More explicitly

$$\Lambda_0(x) = \frac{1}{4}\left\{\Lambda(x) + \iota_1\Lambda(x\iota_1) + \iota_2\Lambda(x\iota_2) + \iota_3\Lambda(x\iota_3)\right\}$$

$$\Lambda_1(x) = \frac{1}{4}\left\{\Lambda(x) + \iota_1\Lambda(x\iota_1) - \iota_2\Lambda(x\iota_2) - \iota_3\Lambda(x\iota_3)\right\}$$

$$\Lambda_2(x) = \frac{1}{4}\left\{\Lambda(x) - \iota_1\Lambda(x\iota_1) + \iota_2\Lambda(x\iota_2) - \iota_3\Lambda(x\iota_3)\right\}$$

$$\Lambda_3(x) = \frac{1}{4}\left\{\Lambda(x) - \iota_1\Lambda(x\iota_1) - \iota_2\Lambda(x\iota_2) + \iota_3\Lambda(x\iota_3)\right\}$$

Therefore, Λ is left regular when its quaternionic antilinear part vanishes, i.e.

$$\Lambda_0 \equiv 0 \tag{5.4.3}$$

The meaning of the decomposition given above is clarified by the following table

$$\begin{array}{ccc} & \Lambda & \\ & \parallel & \\ A & + & L \\ \parallel & & \parallel \\ \Lambda_0 + \Lambda_1 & & \Lambda_2 + \Lambda_3 \end{array} \qquad (5.4.4)$$

where A, L are respecitvely the complex linear part and the complex anti-linear part of Λ.

When

$$\Lambda = df(x_0) \qquad (5.4.5)$$

the above results lead us to define four dif= ferential operators:

$$\bar{\partial}_0 f(x_0) = \Lambda_0 \qquad (5.4.6)$$

$$\bar{\partial}_1 f(x_0) = \Lambda_1 \qquad (5.4.7)$$

$$\bar{\partial}_2 f(x_0) = \lambda_2 \qquad (5.4.8)$$

$$\bar{\partial}_3 f(x_0) = \lambda_3 \qquad (5.4.9)$$

which decompose the real derivative of f at x_0:

$$df(x_0) = \sum_{\mu=0}^{3} \bar{\partial}_\mu f(x_0) \qquad (5.4.10)$$

The differential operator $\bar{\partial}_0$ it is usually called the Cauchy Riemann Fueter differential operator.

APPENDIX

ALGEBRAIC PRELIMINARIES

1. MODULES AND LINEAR MAPS

1.1 Introduction and notations

Our aim is to define linear structures on classes of linear maps: we will try to describe the typical situation.

Let X be a set and $(S, +, *)$ a (left or right) module over a ring R (not necessarily commutative).
Then S^X is a module over R with respect to the operations

$$(f+g)(x) = f(x) + g(x) \qquad (A.1.1)$$

$$(a*f)(x) = a*f(x) \qquad (A.1.2)$$

Furthermore, $(S^X, +, *)$ is a module of the same kind as $(S, +, *)$ since

$$(a*(b*f))(x) = a*(b*f(x)) \qquad (A.1.3)$$

Suppose now $\mathcal{F} \subset S^X$ (non empty) is a subset of "special maps". Then $\mathcal{F}$ is a sub-module of S^X if and only if

$$f, g \in \mathcal{F} \implies f+g \in \mathcal{F} \qquad (A.1.4)$$

$$a \in R \text{ and } f \in \mathcal{F} \implies a*f \in \mathcal{F}. \qquad (A.1.5)$$

In the following, for $a \in R$ and $y \in \mathcal{S}$, we will assume that

$$a * y = \begin{cases} ay & \text{if } \mathcal{S} \text{ is a left space} \\ ya & \text{if } \mathcal{S} \text{ is a right space} \end{cases}$$

This convention is particularly useful when $\mathcal{S}$ is a bimodule. Then we simply write

$$a(yb) = (ay)b \qquad (A.1.6)$$

for every $a, b \in R$ and $y \in \mathcal{S}$.

1.2 Linear maps

Let $(M, +, *)$ be another module over R.
Suppose $X \equiv M$ and $f \in \mathcal{F} \subset \mathcal{S}^M$ such that:

$$f(x+y) = f(x) + f(y) \quad \text{(additivity)} \quad (A.1.7)$$

$$f(a * x) = a * f(x) \quad \text{(homogeneity)} \quad (A.1.8)$$

for every $x, y \in M$ and $a \in R$.
First of all note that:

<u>A.1.1 Lemma</u> If R is a non commutative division ring and if M, S are not both right or left modules, then $\mathcal{F} = \{0\}$.

<u>Proof</u> Suppose M left and S right, and $f \in \mathcal{F}$ with $f(x) \neq 0$. Then, for $a, b \in R$ we have:

$$f(x)(ab) = (f(x)a)b = f(ax)b = f(b(ax)) = f((ba)x) = (ba)f(x)$$

and so $ab = ba$ ///

In other words, linearity makes sense between same sided modules
If M, S are <u>left</u> spaces, we denote with <u>$\mathrm{Lin}_\ell(M, S)$</u> the class of maps wich satisfy (A.1.7) & (A.1.8): analogously for right spaces.

Now, is $\mathrm{Lin}_\ell(M, S)$ a sub-module of M^S?
Relation (A.1.4) is satisfied, but

<u>A.1.2 Lemma</u> If R is a division ring, $a \in R$ and $f \in \mathrm{Lin}_\ell(M, S)$ is not trivial, then the following statements are equivalent

i) $af \in \mathrm{Lin}_\ell(M, S)$

ii) $ab = ba \quad \forall b \in R$

<u>Proof</u> Let $x \in M$ such that $f(x) \neq 0$ and $b \in R$.
If i) is true then

$$(ba)f(x) = b(af(x)) = b((af)(x)) = (af)(bx) =$$

$$= a f(bx) = f(a(bx)) = f((ab)x) = (ab) f(x)$$

and so $ab = ba$.

Suppose now that *ii)* is true. Then for every $x \in M$ we have:

$$(af)(bx) = af(bx) = a(bf(x)) = (ab)f(x) = (ba)f(x) =$$

$$= b(af(x)) = b((af)(x))$$

Furthermore af is additive, and so $af \in \mathrm{Lin}_\ell(M, S)$ ///

This allows us to define in $\mathrm{Lin}_\ell(M, S)$ a linear structure over the centre of R only. If we want a linear structure over R we have to require some additional structure on S. Precisely, S have to be a bimodule over R.

Then, as before, for $a \in R$ we have

$$af \notin \mathrm{Lin}_\ell(M, S)$$

but

$$fa \in \mathrm{Lin}_\ell(M, S)$$

In fact, the map fa is additive and furthermore

$$(fa)(bx) = f(bx)a = (bf(x))a = b(f(x)a) = b((fa)(x))$$

Finally, $\mathrm{Lin}_\ell(M, S)$ – the class of <u>left</u> linear maps – is a <u>right</u> module over R.

A special case occurs when $S = R$ (an obvious bimodule!), and we write:

$$\mathrm{Lin}_\ell(M, S) = M^*$$

for the algebraic dual of M.
Note that M is a left space but M^* is a right space.
Dual maps are defined as in the commutative case.

1.3 Bilinear maps

Suppose, for exemple, that M, N, S are left modules over R, and $\phi : M \times N \to S$ a map.
The usual meaning of bilinearity for ϕ does not make sense, since we can have

$$\phi(ax, by) = a\,\phi(x, by) = (ab)\,\phi(x, y)$$

or

$$\phi(ax, by) = b\,\phi(ax, y) = (ba)\,\phi(x, y)$$

according to the fact that the first scalar taken out is a or b.

When looking for a good definition of bilinearity, we have to consider that the prototype for a bilinear form is the canonical form of duality (or duality form), i.e. the form defined by $\langle \cdot , \cdot \rangle : M \times M^* \to \mathbb{R}$

$$\langle x, f \rangle = f(x)$$

Note that for $a, b \in R$ we have

$$\langle ax, fb \rangle = a \langle x, f \rangle b$$

The spaces involved here are of different sides (M is a left space and M^* a right space) and the duality form takes values in R itself.
Generalizing this, we obtain

A.1.3 Def. Let M, N, S be a left module, right module and bimodule respectively.
A map $\phi : M \times N \to S$ is said to be bilinear when
$$\phi(ax, yb) = a\,\phi(x, y)\,b$$
for every $x \in M$, $y \in N$ and $a, b \in R$.

Denote this class of maps by $\mathrm{Bil}\,(M, N; S)$
As in the commutative case, such a map ϕ is associated with two linear maps, i.e.:

i) a left linear map $\tilde{\phi}_\ell : M \to \mathrm{Lin}_r(N, S)$

ii) a right linear map $\tilde{\phi}_r : N \to \mathrm{Lin}_\ell(M, S)$

defined by (respectively)
$$(\tilde{\phi}_\ell\, x)\, y = \phi(x, y) = (\tilde{\phi}_r\, y)\, x$$
for every $x \in M$ and $y \in N$.

Note that we have no chances to define on $\mathrm{Bil}\,(M, N; S)$ a linear structure over a ring bigger than $C(R)$. In fact neither $a\phi$ nor ϕa belong to $\mathrm{Bil}(M, N; S)$ when $a \in R \setminus C(R)$.

1.4 Multilinear maps

If R is not commutative, there are no chances to define multilinearity in a reasonable way (intuitively, we dont know how to take out the scalars $a_j \in R$ from the expression:

$$\phi(a_1 x_1, \dots, a_n x_n)$$

ϕ being a "multilinear" map).

All we can do, is to define a multilinearity with respect to scalars lying in a commutative sub-ring of R, typically the centre of R. When doing this, we deal with modules over a commutative ring and so all can be defined in the usual way.

If M, S are (left or right) modules over R, we denote by $\mathrm{Mul}_k(M; S)$ the class of k-linear maps over $C(R)$ from M to S.

Then $\mathrm{Mul}_k(M, S)$ is an R-module of the same kind of S as sub-module of $S^{M \times \dots \times M}$

1.5 Generalisation of linearity: anti-linear maps

Let us go back to the argument pointed out in § (A.1.2). For maps between modules of different sides, linearity fails to be interesting because of Lemma (A.1.1): to beat this difficulty we can generalize the concept of linearity.

Suppose

$$\omega : R \to R$$

is a ring automorphism or anti-automorphism.

Define a map

$$f : M \to S$$

(where M, S are modules over R), to be an ω-linear map if:

1. f is additive
2. f is ω- homogeneus, i.e. for $\alpha \in R$ and $x \in M$
$$f(\alpha * x) = \omega(\alpha) * f(x)$$

Following the proof of Lemma (A.1.1), we can easily prove that:

i) if ω is an automorphism, ω-linearity is interesting when M, S are modules of the same kind.

ii) if ω is an anti-automorphism, ω-linearity is interesting when M, S are modules of different sides.

In fact note that, for $\alpha, \beta \in R$ and $x \in M$, if f is an ω-linear map then

$$f((\alpha\beta) * x) = \omega(\alpha\beta) * x$$

$$f(\alpha * (\beta * x)) = \omega(\alpha) * f(\beta * x) = \omega(\alpha) * (\omega(\beta) * f(x))$$

where $*$ denote the external operation in M, S.

We call anti-linearity the ω-linearity for ω an anti-automorphism. This concept is particulary useful in studing relations between a module and its dual, which are of different sides (see § (1.2)).

2. ALGEBRAS

2.1. Definition

Let $\mathbb{K}$ be a commutative field. A linear algebra over $\mathbb{K}$ (or $\mathbb{K}$-algebra) is, by definition, a linear space over $\mathbb{K}$ together with a $\mathbb{K}$-bilinear map

$$A \times A \longrightarrow A \qquad (x,y) \longmapsto xy$$

called the algebra product (or multiplication).
Note that the bi-additivity of the product corresponds to the distributivity of the product with respect to the addition in the linear space A.
As for the homogeneity of the product, note the following construction.
Suppose A is a ring (not necessarily commutative) and

$$\varphi : \mathbb{K} \longrightarrow A$$

a ring monomorphism.
Then

$$k * a := \varphi(k)\, a \qquad k \in \mathbb{K},\ a \in A$$

defines a $\mathbb{K}$-linear structure on A. Is A a linear algebra over $\mathbb{K}$?
Certainly the product in A is bi-additive, but not

necessarily homogeneous. In fact, for $k \in \mathbb{K}$ and $a, b \in A$ we have

$$(k * a)\, b = (\varphi(k)\, a)\, b = \varphi(k)\,(ab) = k * (ab)$$

but this is no longer true for:

$$a\,(k * b) = a\,(\varphi(k)\, b) = (a\,\varphi(k))\, b.$$

More precisely, our construction gives rise to a linear algebra over $\mathbb{K}$ if and only if

$$\operatorname{Im}\varphi \subset C(A)$$

Finally remark that, if A, B are rings and if

$$\psi : A \longrightarrow B$$

is a ring morphism, in general it is not true that

$$\psi(C(A)) \subset C(B) \qquad (A.2.1)$$

We only have that, for $a \in C(A)$ and $a' \in A$

$$\psi(a)\,\psi(a') = \psi(aa') = \psi(a'a) = \psi(a')\,\psi(a)$$

i.e. $\psi(a)$ commutes with every element belonging to $\operatorname{Im}\psi$. So, (A.1.9) is true only when ψ is surjective.

2.2. Some basic properties

For an algebra, the existence of the unity is not required and the algebra product is not required to be commutive or associative, though it is usual, (as in the case of rings), to mention explicitly any failure of associativity. Furthermore, an algebra is said to be with division if it is as ring.

An algebra morphism
$$\psi : A \to B$$
between two $\mathbb{K}$-algebras A, B is a $\mathbb{K}$-linear map such that
$$\psi(aa') = \psi(a)\psi(a')$$
and if A and B have unity
$$\psi(1) = 1$$

An algebra anti-morphism has similar obvious properties.

An algebra (anti-) involution is an algebra (anti-) auto=morphism whose square is the identity map.

When A is a $\mathbb{K}$-algebra with unity, $\mathbb{K}$ can be seen as a subalgebra of A in a canonical way.
Precisely, the following statement holds:

<u>A.2.1 Lemma</u> Let A be a $\mathbb{K}$-algebra with unity.
Then the map
$$\mathbb{K} \ni k \mapsto k * 1 \in A \qquad (A.2.2)$$
is the only algebra morphism from $\mathbb{K}$ to A,

it injective and such that:

$$\mathbb{K}*1 \subset C(A) \qquad (A.2.3)$$

We always identify $\mathbb{K}$ with its image via (A.2.2)

Now, let us examine the differences between a ring morphism and an algebra morphism.

I.2.2 Lemma Let A, B be $\mathbb{K}$-algebras with unity, and

$$\psi : A \to B$$

a ring morphism.
Then ψ is an algebra morphism if and only if

$$\psi(k) = k, \ \forall k \in \mathbb{K} \qquad (A.2.4)$$

Proof If ψ is an algebra morphism then, for $k \in \mathbb{K}$ it is:

$$\psi(k) = \psi(k*1) = k*\psi(1) = k*1 = k$$

via identification (A.2.2)
Conversely, if (A.2.4) holds and $k \in \mathbb{K}$, $a \in A$ then

$$\psi(k*a) = \psi((k*1)a) = \psi(k*1)\,\psi(a) =$$
$$= (k*1)\,\psi(a) = k*\psi(a) \ \blacktriangle$$

Of course, a similar result holds for anti-morphisms.

To conclude this Chapter let us classify the automorphisms and automorphisms

Iu the case of $\mathbb{R}$, the situation is very simple. There is one, and only one, ring automorphisu, and this is the identity map.

This is no longer true in the second case.
In fact, $\mathbb{C}$ has many not trivial ring automorphisms or antiautomorphisms: they are not necessarily continuous and may be different from the identity on real numbers [P].

The situation is simpler for real algebra automorphisms (or anti-automorphisms) of $\mathbb{C}$. In fact, if

$$\varphi : \mathbb{C} \to \mathbb{C}$$

is a real algebra automorphisms or (anti-automorphisms) then

$$\varphi(t) = t \qquad t \in \mathbb{R}$$

$$\varphi(i)^2 = \varphi(i^2) = -1$$

So we have only two possibilities:

1. $\varphi(i) = i$, and φ is the identity map, i.e. the only algebra automorphism

2. $\varphi(i) = -i$, and φ is the usual conjugation; i.e. the only algebra anti-automorphisms.

Moreover they both are involutions.

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