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Nonregular representations of CCR algebras and the problem of fermions bosonization in 1+1 dimensions.

Thesis submitted for the degree of

"Magister Philosophiæ"

Mathematical Physics sector

CANDIDATE

Fabio Acerbi

SUPERVISOR

Prof. Franco Strocchi

Academic year 1989/90

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CONTENTS

O. MOTIVATIONS AND RESULTS.

In the study of many models of Quantum Field Theory and of Statistical Mechanics the solution of the dynamics is usually obtained in terms of variables or fields that (formally) satisfy the Canonical Commutation Relations, but that it is not possible to represent as operators in a Hilbert space. A typical example is the massless scalar field in 1+1 dimensions, that does not exists as a field operator in a Hilbert space. Such a phenomenon has, in our opinion, very general roots (tied to infrared structures) and shows up, for instance, in the following models: in Statistical Mechanics, the infinite quantum harmonic lattice in thermal equilibrium and the free Bose gas, both in $d \leq 2$ space dimensions. For the former, the operators representing the deplacement from the equilibrium positions are not well defined. For the latter, the local particle number is not well defined as an operator in the Hilbert space. In QFT, in the Schwinger model (in its bosonized form) the fields which are charged under gauge transformations display the same problem as in the massless scalar field case, and the same happens with the Stückelberg-Kibble model in 1+1 and 2+1 dimensions.

Faced to this problem, the stategies followed in the literature are substantially two.

1. To give up representing the fields in a space with a positive metric. This strategy has been followed for the Schwinger model (see for instance [**LOW**], [**RAI**], [**PIE**]), in the standard treatement of the massless scalar field [**WIG**] and in the case of the Stückelberg-Kibble model.
2. To search for a formulation that does not introduce the variables that give rise to such a problems (for instance, to treat the massless scalar field in terms of the derivative of the field [**STR1**]).

The problems and the limitations of these approaches, especially if one looks at a systematics of such models, have already been displayed in the literature. The indefinite metric leads to unphysical degrees of freedom while, for instance, using only the derivative of the massless scalar field it is impossible to construct the Wick ordered exponentials, which play a crucial rôle in the solutions of twodimensional models.

The aim of this thesis is to reconsider the problem from its grounds, searching for a solution within the framework of canonical commutation relations and their representations. This in order to take as much as possible advantage of the results of the canonical formulation, in which the variables defining the model (and partially also the dynamics) have a mathematical status which is under control. Moreover, we want to clarify the relation between infrared structures and mathematical structures they give rise to *in the canonical formalism.*

As we will see, the solution of this problem is to use and to study the properties of the *nonregular representations* of algebras of variables satisfying the CCR. As we will show the so emerging mathematical structure is very compact; moreover, under quite general assumptions, nonregular representation of a CCR algebra are *univocally determined* by their (maximal) regularly represented subalgebra.

The usefulness of this strategy for the solution and the discussion of the properties of models of QFT and of Statistical Mechanics is outlined in **Section I.2**.

The study of nonregular representations of CCR algebras turns out to have interesting implications also in the discussion of a different but related problem, namely the structure of the representations of fields algebras $\mathcal{F}$ charged under a gauge (that is, with the property of leaving pointwise invariant the subalgebra $\mathcal{A}$ of the observables) group $\mathcal{G}$.

We study the case in which an observable algebra $\mathcal{A}$ is given in canonical form (that is, obeying the CCR). We will see that the charged fields can be introduced as *extension* of the algebra $\mathcal{A}$, so as to obtain a field algebra $\mathcal{F} \supset \mathcal{A}$ with a canonical structure. We will show how the charged fields identify (thanks to the canonical structure of $\mathcal{F}$) a gauge group $\mathcal{G}$.

Nonregular representations allow for an analysis of the representations of $\mathcal{F}$ in terms of those of $\mathcal{A}$. In particular, the relevant results are

A. If the gauge group is *unbroken*, then the (vacuum) representation of $\mathcal{F}$ cannot be regular and the representation space is a direct sum of *charged sectors.*

B. If the gauge group is completely broken, then the representation of $\mathcal{F}$ is regular and coincides with the vacuum representation of the observables.

C. If the gauge group is unbroken and β is a symmetry (not of the gauge type) of $\mathcal{F}$ which is unbroken in the vacuum sector, then this symmetry is unbroken in the representation of $\mathcal{F}$, too.

Connections between the strategy developped in the thesis and the approach of Doplicher, Haag and Roberts [**DHR2**] are briefly discussed.

Finally, nonregular representations and "extended CCR algebras" play a crucial rôle in the bosonization problem in 1+1 dimensions. The treatement given in the literature solved the problem in terms of correlation functions [**COL**] and the attempts of exhibiting an operatorial formulation which expresses the fermionic field in terms of bosonic fields are limited to an heuristic level [**MAN**] or do not completely solve the question [**RUIJ**].

In the framework of nonregular representations of CCR algebras we are able to

1. Construct fermionic degrees of freedom in terms of CCR algebras, independently of the dynamics.

2. Construct local fermionic fields, that is the ACR algebra, as *strong limits* of bosonic operators (solving in this way a problem which is completely open in the literature (*)).

(*) compare with [**STR2**] and [**RUIJ**] where the possibility of constructing fermion field operators as strong limits is discussed in rather pessimistic terms and almost ruled out.

I. NONREGULAR REPRESENTATIONS OF CCR ALGEBRAS.

I.0. Introduction.

In this thesis we study a class of nonregular representations of CCR algebras. In this analysis, we are strongly motivated by a phenomenon which occurs in models of Quantum Field Theory and Quantum Statistical Mechanics. Indeed, it happens that the solution of certain models, that is of the dynamics defining them, displays variables complying formally with the Canonical Commutations Relations, but that do not admit a representation as operators in a Hilbert space.

A very clear example of this phenomenon is given by the massless scalar field in 1+1 dimensions (QFT). It is well known that it can be quantized in a consistent way [**WIG**], in a positive metric, only if the testfunctions space with which we smear the canonical field ϕ is made of elements with zero integral: for instance

$$\partial\mathcal{S} := \{f \in \mathcal{S} : \int f(x)dx = 0\}.$$

This is equivalent to treat the model in terms of the derivative of the canonical field $\phi(x)$: in this way one gives up introducing the variable that gives rise to problems.

The same behaviour shows up if we try to quantize, again in a positive metric, the Stückelberg-Kibble model in 1+1 dimensions: the canonical field $\phi(f)$ does not exist as a field *operator* unless we restrict f to be in $\partial\mathcal{S}$.

In the case of QSM, it is interesting the example of the infinite harmonic crystal lattice in thermal equilibrium, in $d = 1, 2$ dimensions. The variables describing the deplacement from the single site equilibrium positions are not well defined operators, and so the equilibrium positions are not defined: the lattice is destroyed by thermal fluctuations.

As a final example we notice that also the field operators of the free Bose gas in $d = 1, 2$ dimensions are well defined only if we restrict the test function space to be $\partial\mathcal{S}$.

Such a behaviour has, as we shall see, a number of physically relevant implications, and we want to study it from its grounds, searching for a description of the underlying mathematical structures in the framework of canonical commutation relations algebras and their representations. In this way we can benefit of the advantages of the canonical formulation, in which the variables describing the models have a mathematical status which is under control.

These phenomena are indeed well described, in the canonical formlism, by the introduction of a particular type of *nonregular* representations of the CCR algebra we use to study the model. They come, by GNS costruction, from states with the following property:
* For some element F belonging to the symplectic space that labels the Weyl operators,

$$\omega(W(\lambda F)) = 0 \qquad \forall\lambda \in \mathbb{R}, \lambda \neq 0 \tag{0.1}$$

where ω is the state.

The map $\lambda \longmapsto \omega(W(\lambda F))$ is not continuous in $\lambda = 0$ since, by the normalization condition of the state, $\omega(W(0)) = 1$; the representation is thus nonregular, and the Stone's generator of the nonregularly represented Weyl operator does not exists as an operator in the Hilbert space. In the standard interpretation of the model (for instance by extrapolation from higher dimensions) these generators would be exactly the above mentioned ill-defined variables.

For instance, treating the massless scalar field, we show that the CCR algebra generated by the exponential of the canonical field $\phi(f)$ is represented nonregularly by the space and time translation invariant state. A detailed discussion of the above mentioned models from this point of view is given in **Section I.2**.

In general, we shall substantially confine our analysis to "generalized quasifree states", i.e. nonregular states which generalize in a natural way the notion of quasifree state, which has been extensively studied in the literature.

The analysis of the regularity properties of the representations of CCR algebras arising from generalized quasifree states (**g.q.s.**) will lead us to the notion of (maximal) regularly represented canonical subalgebra and of CCR extension of CCR algebras and to the problem of the extent to which a g.q.s. is determined by its restriction to the regularly represented subalgebra. It turns out that such a restriction uniquely determines the state if it is "maximally regular", i.e. if it has no regular extensions. On the basis of these notions, we establish a direct approach to the discussion of the structure of the representations of field canonical algebras charged under a gauge group.

This approach is established in **Sections I.3.1** and **I.3.2** where we study the case in which an observable algebra is given as a of CCR algebra. We will see that the charged fields can be introduced as an *extension* of this algebra, so as to obtain a field algebra obeying canonical commutation relations. We then show how the fields algebra identifies a gauge group under the action of which our algebra of observables is pointwise invariant. The structure of the set of the charged states is then analyzed.

At this level, several lines of development appear as workable. Firstly, we are interested in applications of our methods to QFT, beyond the problem we treat in **part II**. They can be relevant in studying models that present screening or confinement phenomena, or the Higgs mechanism. Second, we have to investigate the properties of the g.q.states. In **Appendix A** this is made as far as an explicit characterization of pure and primary states is concerned, the latter being not complete (see [**MAN2**] for the regular case). It can be interesting to try to fill this gap, and, for instance, to establish the type of the factors induced by primary g.q.s.. Lasty, one may wonder how to control nonregular states satisfying (0.1) but not quasifree in their regular part and how to introduce noncommutative gauge structures.

We give now a sketch of the content of **part I**: more details can be found in the introductions to the single sections.

In **section I.1.1** the essential properties af Weyl algebras are pointed out, togheter with the definition of quasifree state on it. We are in particular interested in the necessary steps towards the construction of a C^* structure for the generic CCR *-algebra. This problem had not been well focused in the first works on the argument ([**MAN1**] and Manuceau's contribute to [**CAR**]) In particular, in order to solve the existence problem it is necessary to exhibit a nondegenerate Hilbert representation. The only one which has been explicitely constructed for *every* CCR *-algebra originates from a nonregular state ([**SLA**] or th. 5.2.8 in [**BRA**]). The uniqueness of the C^* structure turns out then to be equivalent to the nondegeneracy of the symplectic form. We will recover these results in a way which is completely elementary, straightforward and *independent* of previous literature.

In **Section I.1.2** we introduce the notion of g.q.s.. The extension from the class of quasifree ones, characterized ([**MAN2**]) by a Hilbert quadratic form $q(\cdot)$ over the real symplectic space of interest (V, σ), occurs in a fully natural way with the introduction of the notion of generalized quadratic form (g.q.f.); the latter can assume the values zero or infinity over certain elements in V. **Prop 2.2** shows that every g.q.f. satisfing the condition

$$|\sigma(F,G)|^2 \leq q(F)q(G) \qquad \forall F, G \in V : q(F) < +\infty, q(G) < +\infty \tag{0.2}$$

(generalization of the standard positivity condition) gives rise to a state on the CCR *-algebra $\mathcal{A}(V, \sigma)$. Such states are said then generalized quasifree states (g.q.s.). The section ends with the explicit construction of the GNS representation induced by the state ω_∞: it will be shown to coincide with the one mentioned above and introduced in [**SLA**] to show the existence of at least a C^* structure.

In **Section I.1.3** we introduce the notion of *maximally regular* g.q.s., and we state some of its properties. The exact definition is the following. Let $\mathcal{A}(V_0, \sigma_0)$ be a *-subalgebra of $\mathcal{A}(V, \sigma)$. A quasifree state ω_q on the former is said maximally regular in $\mathcal{A}(V, \sigma)$ if there is no linear space V_1, $V_0 \not\subseteq V_1 \subseteq V$, such that ω_q admits regular extensions to $\mathcal{A}(V_1, \sigma)$. In the key **Prop. 3.3** it is shown that the maximal regularity of ω_q is equivalent to everyone of these statements.

a. ω_q has no regular *and quasifree* extensions.

b. Fixed anyhow $G \in V$ but not in V_0, $\sigma(\cdot, G)$ is an *unbounded* linear functional on V_0, equipped with the inner product induced by q.

c. ω_q has a *unique* extension to $\mathcal{A}(V, \sigma)$, namely that associated to the g.q.f. q_E obtained from q by extending it this way: we impose that $q_E(F) = +\infty$ for every F in V but not in V_0.

The striking property of uniqueness of extension of maximally regular states has several noticeable consequences. In particular the structure of the representation of $\mathcal{A}(V_0, \sigma_0)$ induced by the unique extended state Ω is completely characterized as follows:

consider the quotient space V/V_0. We show that as a representation of $\mathcal{A}(V_0, \sigma_0)$,

$$\pi_\Omega = \bigoplus_{F \in V/V_0} \pi_F$$

where π_F is the representation of $\mathcal{A}(V_0, \sigma_0)$ defined by

$$\pi_F(\cdot) = \pi_{\omega_q}(\delta(-F) \cdot \delta(F)), \qquad F \in V.$$

In **Section I.2** we treat the examples we have already mentioned. We adjoin to them a little investigation about the onedimensional harmonic oscillator; we obtain g.q.s. for limiting values of the physical parameters (mass, frequency and temperature). Most of the abstractly discussed features will be recovered already in this model.

In **Section I.3.1** we introduce the notion of extension in a CCR ambit of a *-algebra $\mathcal{A}(V,\sigma)$. It takes into account the structural properties of such algebras: one of such extensions is defined as the *-algebra labelled by a symplectic space (V_1,σ_1) such that $V_1 \supset V$ and $\sigma_{1|V\times V} = \sigma$.

It is then shown that those extended algebras admits an interpretation in terms of charged fields. Indeed, we show that to every algebra $\mathcal{A}(V_1,\sigma_1)$, viewed as an extension in a CCR ambit of $\mathcal{A}(V,\sigma)$, one can associate a group $G_{V_1/V}$ of *-automorphisms of the former such that the latter is the gauge invariant part. Moreover, we introduce the group of *-automorphisms of $\mathcal{A}(V,\sigma)$ implemented by the generators $\delta(\cdot)$ of $\mathcal{A}(V_1,\sigma_1)$.

In **Section I.3.2** we continue this analysis, supposing that a g.q.s. Ω_q is given on $\mathcal{A}(V_1,\sigma_1)$, which is regular on $\mathcal{A}(V,\sigma)$. We then characterize the unbroken part of $G_{V_1/V}$ and the decomposition of the Hilbert space in terms of representations of the regularly represented algebra $\mathcal{A}(V,\sigma)$. The resulting structure is compared to that arising in the Doplicher, Haag and Roberts [**DHR2**] construction of charged representations of the observable algebra.

In **Appendix A** we give an explicit characterization of pure and primary g.q.s.. The one for factor states is really, up to now, a conjecture: we were able to set up only the necessity proof. The whole job is developed as a suitable generalization of the standard case (see for instance [**MAN2**]).

I.1 GENERALIZED QUASIFREE STATES OVER CCR ALGEBRAS.

I.1.1. Generalities about Weyl systems and CCR algebras.

A *Weyl system* is a mapping W from a real linear space V, equipped with a nondegenerate symplectic form σ (nondegenerate symplectic space (*) (V, σ)) into the group of unitary operators on a Hilbert space.

It is composed by elements $W(F), F \in V$, such that, $\forall F, G \in V$

$$\begin{aligned} W(F)^* &= W(-F) \\ W(F)W(G) &= W(F+G)\exp\left(-\frac{i}{2}\sigma(F,G)\right). \end{aligned} \tag{1.1}$$

Finite linear combinations of such operators define an involutive algebra. If the mapping $\lambda \longmapsto W(\lambda F)$ is weakly continuous in $\lambda \in \mathbb{R}$ for every $F \in V$ then our Weyl system is said to be *regular*. This condition is equivalent, by classical Stone's theorem, to the existence of the generator of $W(F)$, that is selfadjoint operators $\Phi(F)$ such that

$$W(F) = \exp\left(i\Phi(F)\right). \tag{1.2}$$

If V is finite dimensional, by Von Neumann's theorem all regular Weyl systems over it are direct sums of Weyl systems unitarily equivalent to those defined by armonic oscillators. The ground state ψ_{osc} of a harmonic oscillator defines a functional over the single operators

$$(\psi_{osc}, W(F)\psi_{osc}) = \exp\left(-\frac{1}{4}q(F)\right) \tag{1.3}$$

where $q(\cdot)$ is a Hilbert (that is, arising from an inner product) quadratic form, nondegenerate on V.

In general, given a Weyl system on a Hilbert space $\mathcal{H}$ with a ciclic vector ψ that satisfies (1.3), from the positivity of the inner product in $\mathcal{H}$ it follows at once that $\forall F, G \in V$

$$|\sigma(F,G)|^2 \leq q(F)q(G). \tag{1.4}$$

Weyl systems of this type are called *quasifree.*

Conversely, given a nondegenerate symplectic space (V, σ) and a quadratic form $q(\cdot)$ on V which is nondegenerate *and that satisfies* (1.4), one introduces the vector space $\mathcal{H}_o$, given by the linear span of elements of form $\psi_F, F \in V$; one then defines operators $W(F), F \in V$, on $\mathcal{H}_o$ by

$$W(F)\psi_G := \exp\left(-\frac{i}{2}\sigma(F,G)\right)\psi_{F+G} \qquad \forall F, G \in V.$$

The so defined operators satisfy (1.1). By identifying $\psi_o \equiv W(0)$ it is immediate that

$$\mathcal{H}_o = Span\{W(F)\psi_o : F \in V\}.$$

(*) We will *always* use this explicit denomination if σ is nondegenerate.

A sesquilinear form $(W(F)\psi_0, W(G)\psi_0)$ is uniquely determined on $\mathcal{H}$ by the defining equation

$$(\psi_o, W(F)\psi_o)_q := \exp(-\frac{1}{4}q(F)) \qquad \forall F \in V$$

and by equation (1.1). The form $(\cdot,\cdot)_q$ is a true inner product: by explicit calculation, inequality (1.4) is equivalent to the positivity of the two point function

$$(\psi_o, (\Phi(F) + \lambda\Phi(G))^*(\Phi(F) + \lambda\Phi(G))\psi_o)_q \geq 0, \qquad \forall\lambda \in \mathbb{C}, \forall F, G \in V$$

a necessary condition and sufficient condition for the positivity of $(\cdot,\cdot)_q$. The proof of this claim is really elementary but it needs introducing a Fock space structure: see theorem 2 in [**MAN2**] for this. Then $\mathcal{H}_o$, equipped with this inner product, is a prehilbert space. In the standard way, by quotienting and completing, we obtain a Hilbert space $\mathcal{H}$: it contains ψ_o as ciclic vector. The $W(F)$ operators, $F \in V$, define then in an obvious way Weyl operators on $\mathcal{H}$: it follows from (1.1) and $q(0) = 0$ that they are isometric on the dense set $\mathcal{H}_o$ and extension to unitary operators on $\mathcal{H}$ is straightforward.

This construction, in particular positivity *as a consequence of* (1.4), is independent of nondegeneracy of σ, *if* $q(\cdot)$ is nondegenerate. Indeed, we can deduce that positivity is a property which one can study on the dense $\mathcal{H}_o$, and which is essentially related to Weyl systems in a finite number of canonical coordinates. The latter are always extendible to quasifree systems with σ nondegenerate: to do this it is enough to add a suitable number of degrees of freedom (*), extending then conveniently $q(\cdot)$.

Clearly, we cannot proceed in this way, if (1.4) holds and $q(\cdot)$ is degenerate. In his generality, this question will be treated in **Prop. 2.2**, where it will become clear that $q(\cdot)$ nondegeneracy is irrelevant, *if* property (1.4) is true.

We want to reformulate (and to generalize) these concepts in terms of C^* algebras and their representations. We collect our results in several steps.

1. Definition of the *-algebra $\mathcal{A}(V,\sigma)$.

Let then (V,σ) be a symplectic space, and let's denote with $\mathcal{A}(V,\sigma)$ the involutive algebra generated, as a linear space, by elements $\delta(\cdot)$, labelled by vectors in V, with the following properties, holding for every $F, G \in V$:

a. $\delta(F)\delta(G) = \delta(F+G)\exp(-\frac{i}{2}\sigma(F,G))$ and it follows that $\delta(0) = \mathbb{1}$.

b. The involution is defined by $\delta(F)^* = \delta(-F)$ and it follows that

$$\delta(F)^{-1} = \delta(F)^*. \tag{1.5}$$

Remarks. 1. Notice that we have not assumed that σ is nondegenerate. This property implies that $\mathcal{A}(V,\sigma)$ is *simple*, by the following argument (**). One needs only to prove that every nondegenerate representation π of $\mathcal{A}(V,\sigma)$ is faithful. To this end, following [**MAN1**], we will show inductively that $\{\pi(\delta(F)) : F \in V\}$ is system of linearly independent generators. Indeed, we have firstly that, $\forall F \in V, \pi(\delta(F)) \neq 0$ by (1.5). We argue

(*) In another context, see 2.1.1 in [**MAN1**].

(**) See 2.2.3 in [**MAN1**].

now by contradiction. Suppose that there exist a family $\{F_i\}_{i=1,\cdots,n}$ of vectors in V such that, for suitable $a_i \in \mathbb{C}$,

$$\sum_{i=1}^{n} a_i \pi(\delta(F_i)) = 0.$$

This implies that, for appropriate $b_i \in \mathbb{C}$,

$$\pi(\delta(0)) = \sum_{i=1}^{n-1} b_i \pi(\delta(G_i))$$

with $G_i = F_i - F_n$. From the fact that $\pi(\delta(F))\pi(\delta(0))\pi(\delta(-F)) = \pi(\delta(0))\ \forall F \in V$ it follows at once that

$$\sum_{i=1}^{n-1} b_i \pi(\delta(G_i)) = \sum_{i=1}^{n-1} b_i \exp\left(-i\sigma(F, G_i)\right)\pi(\delta(G_i)).$$

The inductive hypothesis implies then that $\exp\left(-i\sigma(F, G_i)\right) = 1\ \forall F \in V$, that is $F_1 = \cdots = F_n$ by σ nondegeneracy: we should have (for example) $\pi(\delta(F_n)) = 0$ and this contradicts our first observation. This proves that $\{\pi(\delta(F)) : F \in V\}$ is a system of independent generators. We will call explicitly $\mathcal{A}(V, \sigma)$ CCR^*-algebra if σ is nondegenerate.

2. If V_o is a linear subspace of V, on which σ is eventually degenerate, **a.** and **b.** imply that $\mathcal{A}(V_o, \sigma)$ (*) is a *-subalgebra of $\mathcal{A}(V, \sigma)$.

2. Existence and uniqueness of C^*-structures on $\mathcal{A}(V, \sigma)$.

Our aim is to build up a C^*-algebra starting from $\mathcal{A}(V, \sigma)$. To this end we need to exhibit at least one nondegenerate Hilbert representation of $\mathcal{A}(V, \sigma)$: the operator norm associated to this representation will induce over the algebra an abstract C^* structure. As a second step one will have the problem of classifing all possible C^* structures; for instance by taking as a reference one of them which is somehow "minimal" [**MAN3**]. The importance of the first step was not so clear in the early works on the subject (see for instance [**MAN1**]). On the other hand, where this aspect is taken into account (see [**MAN3**] or [**SLA**]), it is somewhat hidden into refined analysis of wider problems, from which it is not so easy to extract a syntetic and autonomous logical line referred to it. The following arguments, *independently* from [**MAN3**] or [**SLA**], solve the existence and uniqueness problem of the C^* structure, in a straightforward way.

Let's denote thus with $I(V, \sigma)$ the set of, even degenerate, norms over $\mathcal{A}(V, \sigma)$ which enjoy the C^* property (that is, norms such that $\|A^*A\| = \|A\|^2\ \forall A \in \mathcal{A}(V, \sigma)$); let $\|\ \|_*$ the generic element in $I(V, \sigma)$. Then the closure of $\mathcal{A}(V, \sigma)$ in the norm $\|\ \|_*$ is a C^*-algebra. For every $\|\ \|_*$

$$\mathcal{A}^*(V, \sigma) := \{A \in \mathcal{A}(V, \sigma) : \|A\|_* = 0\}$$

is an ideal of $\mathcal{A}(V, \sigma)$. If σ is nondegenerate on V, so that $\mathcal{A}(V, \sigma)$ is simple, then $\mathcal{A}^*(V, \sigma) = \emptyset$ for every $\|\ \|_* \in I(V, \sigma)$, which consequently contains only nondegenerate norms. Let's concentrate now on the structure of $I(V, \sigma)$.

(*) Here and in the following we will often use: $\sigma_{|V_o \times V_o} \equiv \sigma$.

i. We will show in the next section that $I(V,\sigma)$ is not empty, by explicitely constructing a nondegenerate, actually faithful, representation of $\mathcal{A}(V,\sigma)$.

ii. We prove now that every *nondegenerate* (and therefore faithful) representation defines on $\mathcal{A}(V,\sigma)$ the *same* C^* norm.

Lemma 1.1(*) *If $\|\ \|_1$ and $\|\ \|_2$ are two nondegenerate norms over $\mathcal{A}(V,\sigma)$ which enjoy the C^* property, then they are equal.*

Proof. If $\|\ \|$ is a nondegenerate norm in $I(V,\sigma)$, then the closure of $\mathcal{A}(V,\sigma)$ in the topology defined by it is a C^*-algebra, and as such isomorfic to a selfadjoint, norm closed algebra of operators on a Hilbert space (**). Let then $\|\ \|_1$ and $\|\ \|_2$ be two nondegenerate norms in $I(V,\sigma)$: we can thus construct them as operator norms over suitable Hilbert spaces $\mathcal{H}_{(1)}$ and $\mathcal{H}_{(2)}$. Let π_1 and π_2 the representations of $\mathcal{A}(V,\sigma)$ so defined.

Let $A \in \mathcal{A}(V,\sigma)$ and let $\|A\|_2$ be its norm as an element of the abstract C^*-algebra obtained by closing $\mathcal{A}(V,\sigma)$ in the $\|\ \|_2$ norm. Then, for every vector $\psi \in \mathcal{H}_{(1)}$

$$\frac{(\psi, \pi_1(A^*)\pi_1(A)\psi)}{(\psi,\psi)} \leq \|A^*A\|_2 = \|A\|_2^2. \qquad \forall A \in \mathcal{A}(V,\sigma)$$

As a consequence

$$\|A\|_1^2 := \sup_{\psi\in\mathcal{H}_1} \frac{(\psi, \pi_1(A^*)\pi_1(A)\psi)}{(\psi,\psi)} \leq \|A\|_2^2 \qquad \forall A \in \mathcal{A}(V,\sigma).$$

By inverting the argument we obtain $\|A\|_1 = \|A\|_2 \ \forall A \in \mathcal{A}(V,\sigma)$.

q.e.d.

It follows that, if σ is nondegenerate, there exists (by *i.*) a unique (by *ii.*) C^*-algebra, denoted by $\overline{\mathcal{A}(V,\sigma)}$, obtained from $\mathcal{A}(V,\sigma)$ by closure in the (unique) C^* norm. This uniqueness is obviously defined up to isomorphisms; it follows directly from *ii.* that also $\overline{\mathcal{A}(V,\sigma)}$ is simple (***).

3. Nonuniqueness of the C^* structure.

The nondegeneracy of σ is not only sufficient but also necessary in order to have a unique C^* structure over $\mathcal{A}(V,\sigma)$. Indeed, let π be a faithful representation of $\mathcal{A}(V,\sigma)$ (it exists by *i.*); if σ is degenerate, then

$$\{\pi(\delta(F)) : F \in \ker\sigma \subset V\} \subset \pi(\mathcal{A}(V,\sigma))'' \cap \pi(\mathcal{A}(V,\sigma))',$$

which is thus non trivial. It will contain in particular a projector $P \neq \mathbb{1}$. It is immediate to verify that setting

$$\|A\|_P := \|\pi(A)P\|_{\mathcal{H}_\pi} \qquad \forall A \in \mathcal{A}(V,\sigma)$$

(*) Compare with th. 3.7 in [**SLA**] and corollary 4.23 in [**MAN**].

(**) Th. 2.1.10 in [**BRA**].

(***) Compare with corollary (4.24) in [**MAN3**] and th. 3.7(iv) in [**SLA**].

we define a nondegenerate C^* norm $\| \;\|_P$, different from that induced by the operator norm on $\mathcal{H}_\pi$, which is then not unique.

Notice that if V_o is a subspace of V, $\mathcal{A}(V_o,\sigma)$ admits a unique C^* structure exactly when $\sigma_{|V_o\times V_o}$ is non degenerate.

4. Representations of $\mathcal{A}(V,\sigma)$.

a.By setting

$$W(F)=\pi(\delta(F)) \qquad \forall F\in V \tag{1.6}$$

we establish an isomorphism between Weyl systems and representations of the corresponding CCR C^*-algebra.

b. For every representation π of $\mathcal{A}(V,\sigma)$ it is true that

$$\|\pi(A)\|\leq \|A\|_* \qquad \forall A\in\mathcal{A}(V,\sigma)$$

where in the right hand side we have used any C^* norm in $I(V,\sigma)$. Thus π can be extended by continuity to a unique representation of the C^*-algebra obtained from $\mathcal{A}(V,\sigma)$ by closure in the $\|\;\|_*$ norm $(:=\overline{\mathcal{A}(V,\sigma)}^*)$.

c. Every state ω (that is, positive(*) and normalized linear functional) over $\mathcal{A}(V,\sigma)$ is continuous in every C^* norm in $I(V,\sigma)$. In fact, let's consider the generic C^*-algebra $\overline{\mathcal{A}(V,\sigma)}^*$. If $A\in\mathcal{A}(V,\sigma)$, then A^*A is a positive element in $\overline{\mathcal{A}(V,\sigma)}^*$ and as a consequence, by the spectral radius formula, $A^*A\leq \|A^*A\|_*\mathbb{1}$. By using this inequality, togheter with the Schwarz's inequality applied to ω as a state over $\mathcal{A}(V,\sigma)$, we obtain

$$|\omega(A)|^2\leq\omega(A^*A)\omega(\mathbb{1})\leq\|A^*A\|_*=\|A\|_*^2 \qquad \forall A\in\mathcal{A}(V,\sigma).$$

Then ω admits then a unique continuous extension to $\overline{\mathcal{A}(V,\sigma)}^*$. Such an extension is also positive. This follows immediately from the fact that $\omega(A^*A)$ is real and that $\|\mathbb{1}-\frac{A^*A}{\|A\|_*^2}\|_*\leq \mathbb{1}\ \forall A\in\overline{\mathcal{A}(V,\sigma)}^*$ (see prop. 2.3.11 in [**BRA**]).

d. An analysis of how to proceed to the GNS construction (see 2.3.3 in [**BRA**] or th. I.2.14 in [**EMC**]) shows that it goes trough *also* for *involutive algebras*, since the Banach (or C^*) structure is necessary *only* to guarantee that all the elements of the algebra are represented as *bounded* operators. In this sense, $\mathcal{A}(V,\sigma)$ is the best one would desire since the unitarity property (1.5) holds. It implies at once that, given a state ω over $\mathcal{A}(V,\sigma)$ and built up the GNS representation space $(\mathcal{H}_\omega,\Psi_\omega)$, the $\pi_\omega(\delta(F))$ are actually isometric on the *dense* set in $\mathcal{H}_\omega$ obtained by appling $\mathcal{A}(V,\sigma)$ to the cyclic vector Ψ_ω. Hence they are *unitary* on $\mathcal{H}_\omega$. As a consequence $(\pi_\omega,\mathcal{H}_\omega,\Psi_\omega)$ represents $\mathcal{A}(V,\sigma)$ with *bounded* operators. The continuity of ω established in the preceding remark implies that this property holds also for the generic C^*-algebra $\overline{\mathcal{A}(V,\sigma)}^*$.

Our interest will concentrate now on studying *nonregular* representations of the *-algebra $\mathcal{A}(V,\sigma)$ arising from states generalizing the quasifree ones.

(*) This means that $\omega(A^*A)\geq 0\ \forall A\in\mathcal{A}(V,\sigma)$.

I.1.2. Generalized quasifree states over CCR *-algebras.

We start by studying nonregular states (and relative representations) which generalize of the quasifree functionals mentioned in the previous section. We have seen that the latter are completely caracterized by Hilbert quadratic forms satisfing the positivity condition (1.4). Since we are looking to quasifree states, we are naturally led to the notion of *generalized quadratic form* q (see **Def. 2.1** below). It is a Hilbert quadratic form on V majorizing σ with two kinds of singularities: it can be degenerate (that is, there are in V vectors of zero lenght in the norm defined by it); it can take an infinite value over certain elements in V. As we will see, the nonregularity of the state we are going to associate to it is determined by the latter singularity.

Indeed, the main result in this section is the proof that *every* generalized quadratic form over the generic symplectic space (V, σ) gives rise in a natural way to a state over the *-algebra $\mathcal{A}(V, \sigma)$. To this end, the crucial point will be to get the *positivity* of the functional (linearity and normalization will be immediate): this is guaranteed by a condition analogous to the standard one. We suppose indeed that (1.4) is required only for every vector on which q is finite (vectors of finite length) As a second result we will show that nonregular states of this type are weak limits of regular quasifree states. Finally, our analysis will lead in a natural and elementary way to the proof that any *-algebra $\mathcal{A}(V, \sigma)$ admits at least one C^*-norm; this will be obtained by explicitely exhibiting a (nonregular) state over $\mathcal{A}(V, \sigma)$. In particular, we verify it generates the type II_1 factor introduced in the literature ([**BRA**],[**MAN3**],[**SLA**]).

Definition 2.1 We call *generalized quadratic form* **(g.q.f.)** over a symplectic space (V, σ) a map $q : V \longrightarrow \mathbb{R}^+ \cup \{+\infty\}$ such that the following properties hold, $\forall \lambda \in \mathbb{R} \backslash \{0\}, \forall F, G \in V$

A.

$$q(\lambda F) = \lambda^2 q(F) \qquad q(F) \geq 0 \qquad q(0) = 0$$
$$q(F+G)^{1/2} \leq q(F)^{1/2} + q(G)^{1/2}$$
$$q(F+G) + q(F-G) = 2q(F) + 2q(G)$$

(with the obvious convention when $q(F)$ or $q(G)$ equals infinity). It can be that

$$\{F \in V : q(F) = +\infty\} \neq \emptyset$$
$$V_q^o := \{F \in V : q(F) = 0\} \neq \{0\}.$$

B.

$$|\sigma(F, G)|^2 \leq q(F) q(G) \qquad \forall F, G \in V : q(F) < +\infty, q(G) < +\infty.$$

Remarks. 1. Let q be a g.q.f. and let

$$V_q = \{F \in V : q(F) < +\infty\}. \tag{2.1}$$

Then **A.** implies that V_q is a linear subspace of V and that $q_{|V_q}$ comes from an inner product, even degenerate, denoted with $[\cdot, \cdot]_q$. The form q defines then over the *-subalgebra $\mathcal{A}(V_q, \sigma)$ a *regular* quasifree state, as it is clear from the definition.

2. Property **B.** abstracts from the positivity of quasifree states, and it reduces to it if $V_q = V$ with q nondegenerate.

3. Obviously, given any symplectic space (V, σ), it always exists a g.q.f. over it: namely that defined by

$$q_\infty(0) = 0, \qquad q_\infty(F) = +\infty \qquad \forall F \in V, F \neq 0 \tag{2.2}$$

The following proposition shows that every g.q.f. over any symplectic space (V, σ) defines a state over $\mathcal{A}(V, \sigma)$.

Proposition 2.2 *Let (V, σ) be a symplectic space, $q(\cdot)$ a g.q.f. over it; $\mathcal{A}(V, \sigma)$ the *-algebra associated to (V, σ). Then the linear functional ω_q defined over the generators of $\mathcal{A}(V, \sigma)$ by*

$$\omega_q(\delta(F)) = \begin{cases} \exp\left(-\frac{1}{4}q(F)\right), & \text{if } q(F) < +\infty; \\ 0, & \text{if } q(F) = +\infty. \end{cases} \tag{2.3}$$

and then extended by linearity is positive and thus defines a state over $\mathcal{A}(V, \sigma)$. It will be called **generalized quasifree state (g.q.s.)** *associated to $q(\cdot)$.*

Proof. In order to study the positivity of ω_q it is useful "to approximate q by finite forms". To this end, we introduce over V nondegenerate inner products $[\cdot, \cdot]_{n,m}$, with $n, m \in \mathbb{N}$, defined by the following steps.

a. Let $V_q^0 := \{F \in V : q(F) = 0\}$. We decompose then $V = V_q + V'$ and $V_q = V_q^o + V_q'$. To do this one needs only to specify a basis in V_q^o. It is then always possible to extend it to V_q and finally to the whole V. As a consequence every $F \in V$ admits a decomposition (which is unique, once specified the basis): $F = F_q + F'; F_q = F_q^o + F_q'$ with $F_q^o \in V_q^o, F_q \in V_q, F' \in V'$

b. We choose arbitrary nondegenerate inner products: $[\cdot, \cdot]^o$ on V_q^o; $[\cdot, \cdot]'$ on V'. Then, $\forall F, G \in V$

$$[F, G]_{n,m} := [F_q, G_q]_q + n[F', G']' + \frac{1}{m}[F_q^o, G_q^o]^o.$$

We have then a two label sequence of Hilbert quadratic forms, nondegenerate over the whole V, defined by

$$q_{n,m}(F) := [F, F]_{n,m} \qquad \forall F \in V.$$

This sequence has the following properties:

i. Fixed anyhow $\overline{m} \in \mathbb{N}$ and a pair $< F, G > \in V \times V$, the positivity condition (1.4) applied to it is verified from a certain $n_{F,G}$ on. From a suitable n_E on, this is true also for any pair of vectors in any finite dimensional subspace E in V. The "counterterm" $\frac{1}{\overline{m}}[\cdot, \cdot]^o$ makes it possible that (1.4) holds, if n is large enough, also over pairs of vectors $< F, G > \in V \times V$ such that $q(F) = +\infty$ and $q(G) = 0$

ii. It converges pointwise to q (provided one takes the limits in the order specified above) the limits:

$$\lim_{m \to +\infty} \lim_{n \to +\infty} q_{n,m}(F) = q(F) \qquad \forall F \in V.$$

We show now that the functional ω_q in (2.3) is positive. This, togheter with

$$\omega_q(\mathbb{1}) = \omega_q(\delta(0)) = \exp(-\frac{1}{4}q(0)) = 1$$

implies that it is a state.
The generic $A \in \mathcal{A}(V,\sigma)$ is by definition of the form

$$A = \sum_{i=1}^{k} \lambda_i \delta(F_i) \qquad \lambda_i \in \mathbb{C}, F_i \in V; k < +\infty.$$

Thus

$$\omega_q(A^* A) = \omega_q((\sum_{i=1}^{k} \overline{\lambda_i}\delta(F_i)^*)(\sum_{j=1}^{k} \lambda_j \delta(F_j))).$$

The r.h.s. can be written in the form

$$\sum_{i=1}^{N} \alpha_i \omega_q(\delta(G_i)) \qquad \alpha_i \in \mathbb{C}, G_i \in V; N < +\infty$$

and it is equal to

$$\sum_{i=1}^{N} \alpha_i \exp(-\frac{1}{4}q(G_i)) = \lim_{m\to+\infty} \lim_{n\to+\infty} \sum_{i=1}^{N} \alpha_i \exp(-\frac{1}{4}q_{n,m}(G_i)). \tag{2.4}$$

The G_i vectors generate a finite dimensional subspace E, over which the $q_{n,m}$ forms satisfy the condition (1.4), for every fixed m, from a suitable n_E on. As a consequence (2.4) is *not negative* and ω_q is a *positive* functional.

q.e.d.

If ω is a g.q.s. we can use the GNS construction to obtain a representation $(\mathcal{H}_\omega, \pi_\omega, \Psi_\omega)$ of $\mathcal{A}(V,\sigma)$. As we have discussed in the previous section, property (1.5) guarantees that the $\delta(\cdot)$ are represented by bounded operators. It follows that

Corollary 2.3 *The set $I(V,\sigma)$ of C^*-norms which can be associated to $\mathcal{A}(V,\sigma)$ is not empty, for every symplectic space (V,σ).*
Proof. It is an immediate consequence of the preceding reasoning and of the fact that the functional ω_∞ associated by (2.3) to the g.q.f. q_∞, defined over every (V,σ) by (2.2), is a state, by **prop.2.2**.

q.e.d.

Thus the *-algebra $\mathcal{A}(V,\sigma)$ admits at least one C^* structure, for every (V,σ). By the discussion of the previous section, the nondegeneracy of σ is equivalent to the uniqueness of such a structure .

Remarks. **a.** The proof of **prop.2.2** gives a basis to the claim of the previous section that (in particular) q *nondegeneracy is irrelevant* for the positivity of quasifree states, *if* condition (1.4) holds.

b. If q is not everywhere finite on V, then ω_q is not regular. Indeed, if $q(F) = +\infty$ for some $F \in V$ one has ($t \in \mathbb{R}$)

$$\omega_q(\delta(tF)) = \begin{cases} 1, & \text{if } t = 0; \\ 0, & \text{if } t \neq 0 \end{cases} \tag{2.5}$$

and so $\pi_{\omega_q}(\delta(tF))$ is not weakly continuous in $t \in \mathbb{R}$. As we have seen in the previous section, ω_q and π_{ω_q} extend by continuity to every C^*-algebra $\overline{\mathcal{A}(V,\sigma)}^*$, with $\| \ \|_*$ in $I(V,\sigma)$, and such extensions are unique.

c. Let then V_o a subspace of V. Every quasifree state ω_q over the *-subalgebra $\mathcal{A}(V_o,\sigma)$ can be extended to a g.q.s. over $\mathcal{A}(V,\sigma)$; namely that associated to the g.q.f. q_E over (V,σ) defined by

$$\begin{aligned} q_E(F) &= q(F) \qquad \forall F \in V_o \\ q_E(F) &= +\infty \qquad \forall F \in V \setminus V_o. \end{aligned} \tag{2.6}$$

d. The g.q.s. ω_∞ associated to the g.q.f. (2.2) gives rise, by GNS construction, to a representation π_∞ of any *-algebra $\mathcal{A}(V,\sigma)$, as we have already seen. This state has a simple physical interpretation, namely as weak limit, for $\beta \to 0^+$, of equilibrium (KMS) states associated to suitable classes of dynamics on $\mathcal{A}(V,\sigma)$ (see par. 5 in [**ROC**] or, for explicit examples, **section I.2** in the following). A representation π_s, which we'll see is unitarily equivalent to the type II_1 factor π_∞, was introduced by Slawny (prop. 3.4 in [**SLA**]) as a decisive step in the proof of the existence of at least one representation of the CCRs over the generic *nondegenerate* space (V,σ) (see also th. 5.2.8 in [**BRA**] or lemma 3.1 in [**MAN3**]). We now show that π_s and π_∞, introduced on very different grounds, are unitarily equivalent.

Indeed, $(\pi_s, \mathcal{H}_s)$ is defined by:

α. The Hilbert space $\mathcal{H}_s$.

$$\mathcal{H}_s := l^2(V) = \{L : V \longrightarrow \mathbb{C} : \sum_{f \in V : L(f) \neq 0} |L(f)|^2 < +\infty\}.$$

Thus every element L in $l^2(V)$ has a unique representation as

$$L = \sum_{f_i \in V} \alpha_i L_{f_i}$$

where the $\alpha_i \in \mathbb{C}$ are such that $\sum_i |\alpha_i|^2 < +\infty$ and the L_{f_i} are defined for every $f \in V$ by

$$\begin{aligned} L_f(f) &= 1 \\ L_f(g) &= 0 \qquad \forall g \in V, g \neq f. \end{aligned}$$

A dense set in $\mathcal{H}_s$ is defined by

$$D_s := \{L \in \mathcal{H}_s : \exists N < +\infty, \{f_i\}_{i=1\cdots N} \in V : L = \sum_{i=1}^{N} \alpha_i L_{f_i}, \alpha_i \in \mathbb{C}\}.$$

The Hilbert structure of $\mathcal{H}_s$ is defined by the inner product

$$(\sum_{i\in I} \alpha_i L_{f_i}, \sum_{j\in J} \beta_j L_{g_j})_s := \sum_{i\in I:\exists j\in J: f_i=g_j} \overline{\alpha_i}\beta_i$$

where I and J are index sets (finite for vectors in D_s, otherwise countable).

β. The representation π_s.

$$\pi_s(\delta(f)) \sum_{i\in I} \alpha_i L_{f_i} := \sum_{i\in I} \alpha_i \exp(-\frac{i}{2}\sigma(f_i, f)) L_{f_i - f}.$$

On the other hand, $(\pi_\infty, \mathcal{H}_\infty)$ is defined by

α'. The Hilbert space is that associated to ω_∞ by GNS construction over $\mathcal{A}(V, \sigma)$. A dense set in it is by definition given by

$$D_\infty := \{\psi \in \mathcal{H}_\infty : \exists N < +\infty, \{f_i\}_{i=1\cdots N} \in V : \psi = \sum_{i=1}^{N} \alpha_i \pi_\infty(\delta(f_i))\psi_\infty, \alpha_i \in \mathbb{C}\}$$

where ψ_∞ is the cyclic vector state in $\mathcal{H}_\infty$ associated to ω_∞.
The Hilbert space structure is defined, by (2.2) and (2.3), by extending to $\mathcal{H}_\infty$ the following inner product on D_∞:

$$(\sum_{i=1}^{N} \alpha_i \pi_\infty(\delta(f_i))\psi_\infty, \sum_{j=1}^{M} \beta_j \pi_\infty(\delta(g_i))\psi_\infty)_\infty := \sum_{i:\exists j: f_i=g_j} \overline{\alpha_i}\beta_i.$$

β'. The representation π_∞ is obviously originated by

$$\pi_\infty(\delta(f)) \sum_{i=1}^{N} \alpha_i \pi_\infty(\delta(f_i))\psi_\infty := \sum_{i=1}^{N} \alpha_i \exp(-\frac{i}{2}\sigma(f, f_i))\pi(\delta(f + f_i))\psi_\infty$$

over D_∞ and then extended by continuity.
So it is clear that the operator $U : \mathcal{H}_s \longrightarrow \mathcal{H}_\infty$ with domain D_s and defined by

$$U(\sum_{i=1}^{N} \alpha_i L_{f_i}) := \sum_{i=1}^{N} \alpha_i \pi_\infty(\delta(-f_i))\psi_\infty$$

is isometric from the dense set D_s onto D_∞ and hence unitary from $\mathcal{H}_s$ onto $\mathcal{H}_\infty$ and implements the unitary equivalence of π_s and π_∞.

I.1.3. Extensions of generalized quasifree states. Maximal regularity and uniqueness of the extension.

In this section we study some very important (and carachteristic) properties of the g.q.s. defined over the generic *-algebra $\mathcal{A}(V,\sigma)$. The problem is the following.

Let a g.q.s. ω_q be given over a *-algebra $\mathcal{A}(V,\sigma)$; let π_{ω_q} the associated GNS representation. It is then univocally determined the *-subalgebra of $\mathcal{A}(V,\sigma)$ regularly represented in π_{ω_q}, and it coincides with $\mathcal{A}(V_q,\sigma_{|V_q\times V_q})$ (see formula (2.1)). One may wonder if and in which sense π_{ω_q} can be recovered from its restriction to $\mathcal{A}(V_q,\sigma_{|V_q\times V_q})$. A simple argument shows at once that this will never be the case for g.q.s. whose nonregularity has been "unnaturally" induced: we can have a tipical example of it by considering a g.q.f. q_0 which is finite over the whole space V and a second g.q.f. q which coincides with q_0 over a certain linear subspace V_q of V and it is infinite otherwise. Clearly the arbitrariness of this definition is such that the knoweledge of q over V_q does not uniquely determine q. We'll not consider this possibility in this section.

The need of studying g.q.s. is strongly motivated by explicit examples as clearly displayed in **Section I.2**.

With these motivations, we select in this section a class of g.q.s. univocally determined by their restrictions to the subalgebra they represent regularly: to this purpose it is convenient to introduce the concept of *maximally regular* g.q.s..

Definition: Given a *-algebra $\mathcal{A}(V,\sigma)$ and a *-subalgebra $\mathcal{A}(V_0,\sigma_0)$. A *regular* state over this latter is said *maximally regular* in $\mathcal{A}(V,\sigma)$ if there exists no linear space V_1, $V_0 \not\subseteq V_1 \subset V$, such that ω admits a regular extension to $\mathcal{A}(V_1,\sigma)$.

In a very analogous way the definition is given of maximally regular g.q.f.. **Proposition 3.3** plays a crucial rôle for the following sections. The following implications are worthwhile to be stressed.

i. If ω is quasifree, that is $\omega = \omega_q$ for some *finite* g.q.f. $q(\cdot)$ over (V_0,σ_0), then the absence of regular extensions is equivalent to the absence of regular *and* quasifree extensions.

ii. If ω is quasifree, its maximal regularity is equivalent to the following property: chosen anyhow $G \in V\backslash V_0$, $\sigma(\cdot,G)$ is an *unbounded* linear functional on V_0, the latter being equipped with the inner product $[\cdot,\cdot]_q$ induced by q. As we will see in a moment, this fact makes possible a precise analysis of the structure of the representation of $\mathcal{A}(V_0,\sigma_0)$ induced by the (unique!) extension of ω_q to $\mathcal{A}(V,\sigma)$.

iii. If ω is quasifree, its maximal regularity is equivalent to the existence of a *unique* extension of it to $\mathcal{A}(V,\sigma)$, exactly the one defined by the g.q.s. associated to the g.q.f. obtained by extending q this way:

$$q(F) = +\infty \qquad \forall F \in V\backslash V_0.$$

Thus in this case $V_0 = V_q$. It is clear that for these states a lot of properties can be decided, thanks to the uniqueness of extension, by studying their *regular* restriction, and hence with standard methods.

Thus, indipendently of the **Appendix A** in wich a general treatement is given, we can establish the purity (factoriality) of a g.q.s. as a consequence of the maximal regularity and of the purity (factoriality) of its regular restriction.

Interesting applications arise in studying problems of spontaneous symmetry breaking and in the search of states invariant under groups of automorphisms (like the dynamics, for example). We will study these problems in **Section I.3**

The second part of this section studies the structure of the representation π_{ω_q} of $\mathcal{A}(V_q,\sigma)$ in the case in which the restriction of ω_q to this algebra is maximally regular. If this restriction gives rise to the representation π_0 and we define

$$\pi_F := \pi_0(\delta(-F)\cdot\delta(F)) \qquad \forall F \in V$$

then we obtain (**prop. 3.6**)

$$\pi_{\omega_q} = \bigoplus_{F\in V/V_q} \pi_F .$$

The equivalence relation defining V/V_q is given by $F \sim F'$ iff $q(F-F') < +\infty$, with $F, F' \in V$. The essential input in order to obtain this result is the characterization of the maximal regularity anticipated in *ii.*. This last result will be very useful in **Section I.3.2.**

Definition 3.1 Given a *-algebra $\mathcal{A}(V,\sigma)$ and a *-subalgebra $\mathcal{A}(V_0,\sigma_0)$ of it, a regular state ω on this latter is said *maximally regular* in $\mathcal{A}(V,\sigma)$ if there is no vector space V_1, with $V_0 \not\subseteq V_1 \subset V$, such that ω admits a regular extension to $\mathcal{A}(V_1, \sigma_{|V_1\times V_1})$.

Definition 3.2 Given a symplectic (V,σ) space and a subspace V_0 of it, a g.q.f. $q(\cdot)$ over this letter is said *maximally regular* in (V,σ) if there is no linear space V_1, with $V_0 \not\subseteq V_1 \subset V$, such that q admits an extension to a g.q.f. finite over $(V_1, \sigma_{|V_1\times V_1})$.

In the case in which ω is a quasifree state we have the following result.

Proposition 3.3 *Let $\mathcal{A}(V,\sigma)$ be a *-algebra, $\mathcal{A}(V_0,\sigma)$ a *-subalgebra of it, ω_q a regular quasifree state defined on $\mathcal{A}(V_0,\sigma)$, characterized by a finite g.q.f. $q(\cdot)$ on (V_0,σ).*
The following statements are equivalent.

1. *ω_q is maximally regular.*
2. *ω_q has no regular and quasifree extensions.*
3. *For every $G \in V\backslash V_0$, a sequence $\{F_n\}_{n\in\mathbb{N}}$ in V_0 exists such that*

$$\lim_{n\to+\infty} \sigma(F_n, G) = \alpha \neq 0 \tag{3.1}$$

$$\lim_{n\to+\infty} q(F_n) = 0. \tag{3.2}$$

4. *ω_q admits as* unique *extension Ω to $\mathcal{A}(V,\sigma)$ that defined by the g.q.f. on (V,σ):*

$$Q(F) = q(F) \qquad \forall F \in V_0$$
$$Q(F) = +\infty \qquad \forall F \in V\backslash V_0 .$$

(hence $V_0 = V_q$)

In particular if ω_q is pure then also Ω is.

Proof. *1.*$\Rightarrow$ *2.* In fact, if ω_q has no regular extensions, then in particular has no regular and quasifree ones.
2.$\Rightarrow$ *3.* If *3.* does not hold, $G \in V \backslash V_0$ exists such that

$$\sup_{F \in V_0} \frac{|\sigma(F,G)|^2}{q(F)} < +\infty.$$

We can then extend q to $V_0 + Span[G]$ defining

$$[G,F]_q = 0 \qquad \forall F \in V_0$$
$$q(G) = \sup_{F \in V_0} \frac{|\sigma(F,G)|^2}{q(F)}.$$

Such a form gives rise to a quasifree state which extends ω_q to $\mathcal{A}(V_0 + Span[G], \sigma)$ and this contradicts *2.*.
3.$\Rightarrow$ *4.* In fact, let Ω be an extension of ω_q to $\mathcal{A}(V,\sigma)$. Fixed anyhow $G \in V \backslash V_0$, take $\{F_n\}$ as in *3.*. Then

$$\lim_{n \to +\infty} \Omega(\delta(F_n)\delta(G)\delta(-F_n)) =$$
$$= \lim_{n \to +\infty} \Omega(\delta(G)) \exp(i\sigma(G,F_n)) = \Omega(\delta(G)) \exp(i\alpha) \tag{$*$}$$

by (3.1). But (3.2) implies that

$$\lim_{n \to +\infty} \Omega(\delta(F_n)) = \lim_{n \to +\infty} \omega_q(\delta(F_n)) =$$
$$= \lim_{n \to +\infty} \exp(-\frac{1}{4} q(F_n)) = 1.$$

It follows that $\|\pi_\Omega(\delta(F_n) - \mathbb{1})\psi_\Omega\|_{\mathcal{H}_{\pi_\Omega}} = 2 - 2\Omega(\delta(F_n)) \longrightarrow 0$ from which

$$s - \lim_{n \to +\infty} \pi_\Omega(\delta(F_n))\psi_\Omega = \psi_\Omega.$$

Hence that

$$\lim_{n \to +\infty} \Omega(\delta(F_n)\delta(G)\delta(-F_n)) = \Omega(\delta(G)). \tag{$**$}$$

Comparing $(*)$ and $(**)$, $\alpha \neq 0$ implies that

$$\Omega(\delta(G)) = 0 \qquad \forall G \in V \backslash V_0.$$

Then Ω coincides with the g.q.s. associated to the g.q.f. in *4.*. Clearly, if ω_q is pure so is Ω by uniqueness.

4.⇒ *1.* If ω_q is not maximally regular then there exists at least one extension of it different from the one in *4.*, since this latter does not extend the regularity space. This contradicts the uniqueness of the extension asserted in *4.*.

q.e.d.

Remarks a. It is very important to notice that condition *3.* is equivalent to the fact that, chosen anyhow $G \in V\backslash V_0$, $\sigma(\cdot, G)$ is an *unbounded* linear functional on V_0, when V_0 is equipped with the inner product induced by $q(\cdot)$. This is easily seen by noticing that the negative of both these assertions is equivalent to the existence of a $G \in V\backslash V_0$ such that

$$\sup_{F\in V_0} \frac{|\sigma(F,G)|^2}{q(F)} \leq c < +\infty. \tag{$*$}$$

The only nonobvious point in this claim is to verify that

$$\sup_{F\in V_0} \frac{|\sigma(F,G)|^2}{q(F)} = +\infty \qquad \forall G \in V \tag{$**$}$$

implies condition *3.* Now, choose anyhow $G \in V$. If $(**)$ is true, there exists a sequence $\{F_n\} \in V_0$ such that

$$\lim_{n\to+\infty} \frac{|\sigma(F_n,G)|^2}{q(F_n)} = +\infty.$$

One notices then that the ratio in $(**)$ is an homogeneous function in the variable F_n. Hence condition *3.* follows by rescaling.

b. It is clear that, if ω_q is a quasifree state regular over $\mathcal{A}(V_0, \sigma_0)$, its maximal regularity is equivalent to that of the form q that defines it. By **Prop. 3.3** condition *3.* is equivalent to the maximal regularity of the form q. To verify maximal regularity for states reduces then, in this case, to verify condition *3.* (or the equivalent unboundedness condition in the preceding remark) that is maximal regularity of forms.

We show now how the structure of the representation space of $\mathcal{A}(V,\sigma)$ associated by the GNS construction to a maximally regular g.q.s. is rather special and in particular contains only once the GNS representation of the regular *-subalgebra $\mathcal{A}(V_q,\sigma)$.

We have seen how, given a *-algebra $\mathcal{A}(V,\sigma)$, a g.q.s. ω_q on it defines, by the GNS construction, a representation $\pi_{\omega_q}(\mathcal{A}(V,\sigma))$ as a *-algebra of bounded operators on a Hilbert space $\mathcal{H}_{\omega_q}$. Our aim is now to analyze the structure of $\mathcal{H}_{\omega_q}$.

Let $\mathcal{A}(V_q,\sigma)$ be the *-subalgebra of $\mathcal{A}(V,\sigma)$ regularly represented by ω_q and let's denote with $\mathcal{H}_0$ the Hilbert space of the GNS representation π_0 of $\mathcal{A}(V_q,\sigma)$ induced by the restriction of ω_q to it (we call it ω_0). For every fixed $F \in V$, let ρ_F be the automorphism of $\mathcal{A}(V_q,\sigma)$ defined by

$$\rho_F : \delta(G) \longmapsto \delta(G)\exp\left(i\sigma(F,G)\right) \qquad \forall G \in V_q. \tag{3.3}$$

Lemma 3.4 *Let ω_q be a g.q.s. over the *-algebra $\mathcal{A}(V,\sigma)$; let $F \in V$. Let the regular-restriction ω_0 of ω_q be maximally regular in $\mathcal{A}(V,\sigma)$. Then ρ_F is unitarily implemented on $\mathcal{H}_0$ exactly when $q(F) < +\infty$. In this case ρ_F is inner in $(\mathcal{A}(V_q,\sigma)$.*

Proof. Since ω_0 is a regular quasifree state, it is well known(*) that a necessary and sufficient condition for the implementability of ρ_F is that $\sigma(F,\cdot)$ be a bounded linear functional on V_q, equipped with the real Hilbert structure induced by q.
Let us suppose now that $q(F) < +\infty$: then $F \in V_q$ and the positivity condition **B.** in **Def 2.1** immediately implies the boundedness, in the sense already defined, of the linear functional $\sigma(F,\cdot)$ on V_q. It follows that ρ_F is unitarily implemented on $\mathcal{H}_0$: the unitary operator on it $\pi_0(\delta(F)) \in \pi_0(\mathcal{A}(V_q,\sigma))$ implements ρ_F which is thus inner. Clearly, $\pi_0(\delta(F))$ is also the only, up to factors in $\pi_0(\mathcal{A}(V_q,\sigma))'$, operator in $\mathcal{B}(\mathcal{H}_0)$ that does this job.
Conversely, let $q(F) = +\infty$. Then $F \in V \backslash V_q$. The supposed maximal regularity of ω_q implies, by condition *3.* in **Prop 3.3**:

- $\sigma(F,\cdot)$ is an *unbounded* linear functional on V_q, equipped with the inner product $[\cdot,\cdot]_q$, as we have observed in **Remark a.** just above. Hence ρ_F cannot be implemented on $\mathcal{H}_0$.

q.e.d.

Let's denote with ω_F the states obtained from ω_0 by composition with the ρ_F:

$$\omega_F := \omega_0 \circ \rho_F.$$

They are well defined as states on $\mathcal{A}(V_q,\sigma)$ and let $(\pi_F, \mathcal{H}_F, \psi_F)$ be the GNS representation of this algebra induced by them. We observe that, since the ρ_F are automorphisms, they are such that

π_0 is irreducible iff π_F is.
π_0 is a factor iff π_F is.

Let us define now in V an equivalence relation: given $F, F' \in V$, we say that

$$F \sim^q F' \; if \; F - F' \in V_q. \tag{3.4}$$

We denote $V/^q V_q$ the quotient space of equivalence classes. As a simple consequence of the last lemma we have the following

Proposition 3.5 *In the hypothesis of* **lemma 3.4,** *π_F is equivalent to $\pi_{F'}$ iff $F \sim^q F'$.*

Proof. Since $\pi_F = \pi_0 \circ \rho_F$, the generators of $\mathcal{A}(V_q,\sigma)$ are represented, in the different π_F's, by operators of the form

$$\pi_F(\delta(G)) = (\pi_0 \circ \rho_F)(\delta(G)) = \exp\left(i\sigma(F,G)\right)\pi_0(\delta(G)).$$

Thus it is clear that π_F is equivalent to $\pi_{F'}$ iff $\rho_F \circ \rho_{F'}^{-1}$ is unitarily implemented in π_0.

(*) Theorem pag. 155 in [SEG]

By **Lemma 3.4** and by the fact that

$$\rho_F \circ \rho_{F'} : \delta(G) \longmapsto \delta(G) \exp\left(i\sigma(F - F', G)\right) \qquad \forall G \in V_q$$

it follows that this happens exactly when $q(F - F') < +\infty$, that is $F \sim^q F'$.

q.e.d.

Clearly, the class of automorphisms of $\mathcal{A}(V_q, \sigma)$

$$\{\rho_F \text{ as in } (3.3) \text{ with } F \in V\}$$

coincides by definition with the class

$$\{Ad \circ (\delta(F))(\cdot) \text{ with } F \in V\}.$$

Hence we have

$$\omega_F(\cdot) := (\omega_0 \circ \rho_F)(\cdot) = \omega_0(\delta(-F) \cdot \delta(F)) \tag{3.5}$$

as states on $\mathcal{A}(V_q, \sigma)$, where $\delta(F) \in \mathcal{A}(V, \sigma)$.
Now, it follows from the definition of GNS construction that, for every $F \in V$, π_F is a representation of $\mathcal{A}(V_q, \sigma)$ contained in the representation π_{ω_q} of $\mathcal{A}(V, \sigma)$. As a consequence, $\mathcal{H}_F$ is a subspace of $\mathcal{H}_{\omega_q}$. Again from the definition of GNS construction it follows at once that:

$$\pi_{\omega_q} \subset \bigcup_{F \in V} \pi_F \qquad \mathcal{H}_{\omega_q} \subset \bigcup_{F \in V} \mathcal{H}_F.$$

There is again a large redundancy in this description: in fact, for every $F \in V_q$, it is true that

$$\psi_F := \pi_0(\delta(F))\psi_0 \in \mathcal{H}_0.$$

But then (*) π_F is a subrepresentation of π_0 and $\mathcal{H}_F$ is a subspace of $\mathcal{H}_0$. By applying the same argument to ρ_F^{-1} (which exists), we obtain $\mathcal{H}_F = \mathcal{H}_0 \quad \forall F \in V_q$. These considerations and the previous proposition imply the following result.

Proposition 3.6 *In the hypothesis on ω_q of* **Lemma 3.4** *we have*

$$\mathcal{H}_{\omega_q} = \bigoplus_{F \in V/^q V_q} \mathcal{H}_F \qquad \pi_{\omega_q} = \bigoplus_{F \in V/^q V_q} \pi_F.$$

q.e.d.

(*) Lemma just before th. II.1.2 in [EMC]

I.2. EXAMPLES OF GENERALIZED QUASIFREE STATES IN MODELS OF QUANTUM STATISTICAL MECHANICS AND QUANTUM FIELD THEORY.

We give in this section some examples of g.q.s. in quantum systems. Very surprisingly, the first one is the one-dimensional harmonic oscillator in thermal equilibrium. It is identified by a quasifree state (canonical Gibbs factor) on the Weyl algebra labelled by $\mathbb{R}^2$. For limiting values of the physically relevant parameters (mass, frequency and temperature) one obtains g.q.s.. We can, already at this level, find out some of the phenomena we abstractly studied: in this case maximal regularity and existence of a nontrivial null space for the g.q.f. associated to the state. Thus, the main interest of this example comes from the possibility of having a first control on what happens with nonregular states.

In order to escape from limiting cases (we have already noticed that the state ω_∞ can be interpreted as the infinite temperature limit of the canonical ensemble on the generic Weyl algebra in a finite number of degrees of freedom, for certain classes of dynamics) one can study system with an infinite number of degrees of freedom.

Indeed, as a second example we study quantum harmonic cristals in thermal equilibrium. It is well known they do not exist, in dimension $d \leq 2$: the argument, due to Peierls, consists in showing that the dispersion of the deplacement from the equilibrium positions vector diverges, *in the thermodynamic limit* (*). Thermal fluctuations are responsible of this divergence, but we'll observe that , unlike the $d = 2$ case, the onedimensional lattice is destroyed even at T=0 by quantum fluctuactions. In order to treat this example, we introduce the Weyl algebra associated to the finite lattice and we explicitely calculate the action of the state determined by the canonical density matrix (it will come out obviously a quasifree state). By going to the thermodynamical limit it gives rise to a g.q.s. which represent nonregularly the Weyl operators associated to the single deplacements from the equilibrium positions. This is indeed not surprising, given the form (1.3) of quasifree expectations, since $q(\cdot)$ is nothing but the mean value of the square of the Weyl operators' generators.

The third model is the free Bose gas. We'll observe that, while in $d \geq 3$ the well known condensation phenomenon occur, in $d \leq 2$ it is substituted by the appearing of nonregular representations of the Weyl algebra describing the kinematics of the system (that is, the one labelled by $\mathcal{S}(\mathbb{R}^d)$). This fact has been observed (see for instance sect. 5.2.5 in [BRA]) but not so much attention has been paied to it.

One may wonder what is happening if we remove the ultraviolet cutoff represented (in our second example) by the crystal lattice. From our point of view, and hence for $d \leq 2$ and since we are studing quasifree theories, this removal involves no problem and we obtain quantum field theories without ultraviolet complications. The quantum lattice is characterized by the dispersion functions $\omega_s(\underline{k})$, whose behaviour is linear in $|\underline{k}|$ in the range of small frequencies, in full generality. Our fourth model will be the analogous, in QFT, of the harmonic lattice, namely the free massless scalar field in $d + 1$ dimensions,

(*) see 137/138 in [LAN] or problem 3 in chap 24 in [ASH] with $d = 1, 2$. We construct the ground and equilibrium state on a suitable CCR C^*-algebra. We read in our language the statement that " the massless scalar field in 1+1 does not exists" (*) by showing that the standard algebra, namely the one labelled by $\mathcal{S} \times \mathcal{S}$, is represented nonregularly by the vacuum. Even more, the well known cure for this phenomenon (preserving the positivity of the state space) gives rise to a Weyl subalgebra of it on which the vacuum state is maximally regular.

The last example is the Stückelberg-Kibble model in 1+1 dimensions. It is the twodimensional version of a model that, in 3+1 dimensions, is usually regarded as the prototype of gauge theories exhibiting the Higgs phenomenon. Since we choose the Coulomb gauge, which gives rise to a long range interaction, we give a short account on the procedures which are necessary to properly define the dynamics (it needs to take a suitable limit of infrared cutoffed dynamics). The standard CCR algebra is not going into itself under the dynamics. We select one of its subalgebras, which is stable and pointwise invariant under the gauge automorphism. On it we determine the space and time translations invariant state, and we discuss the (maximal) regularity properties.

Not considering the first, there are substantial affinities between these models. The basic idea is that the divergences of the form q that defines the state identify a certain class of collective effects. Having in mind for instance the form (4.1) of the thermal state for the harmonic oscillator, one tries to couple an infinite number of them so that the frequency $\omega(k)$ develops a continuous spectrum and, this is the essential point, without a mass gap. We have indeed in our last three examples that $\omega(\underline{k}) \sim |\underline{k}|^p$, with $p = 1$ or 2 (in the free Bose gas), for $|\underline{k}| \to 0$.

Thus, it is clear that the rising of nonregular representations is strictly tied to infrared effects, which are magnified in lower dimensions. This receives a support from the fact that it is necessary to go to the thermodynamical limit: for instance even the massless scalar field, if infrared cutoffed, has no singular behaviour (at least for a class of boundary conditions).

0. The onedimensional harmonic oscillator.

We treat the general case of finite temperature. In order to fix constants, let

$$H = \frac{1}{2m}(\hat{p}^2 + m^2\omega^2\hat{x}^2)$$

be the quantum hamiltonian. The Weyl algebra describing the kinematics of the model is generated by the operator

$$W(\underline{u}) := \exp\left(i(u_1\hat{x} + u_2\hat{p})\right)$$

with $\underline{u} := < u_1, u_2 > \in \mathbb{R}^2$. The symplectic form is

$$\sigma(\underline{u}, \underline{v}) := u_1v_2 - v_1u_2 \tag{4.1}$$

which is nondegenerate on $\mathbb{R}^2 \times \mathbb{R}^2$.

(*) see the classical treatement in [WIG]

It is an easy calculation to find the canonical density matrix, at inverse temperature β, associated to the above hamiltonian operator; it comes out the quasifree state defined by

$$\omega_\beta(W(\underline{u})) = \exp\left(-\frac{1}{4}\{\frac{u_1^2}{m\omega} + m\omega u_2^2\}\coth\frac{\omega\beta}{2}\right). \tag{4.2}$$

There are several interesting limitig cases: we discuss the $\omega \to 0$ one, in the whole range of temperatures, since it summarizes the relevant behaviours. We analyze first the ground state, by taking the limit $\beta \to +\infty$.

$\omega \to 0$. It is clear from (4.2) that in this case we obtain the g.q.s. ω_0 on $\mathcal{A}(\mathbb{R}^2, \sigma)$ defined by

$$\omega_0(W(<0,u>)) = 1 \qquad \forall u \in \mathbb{R}$$
$$\omega_0(W(<v,0>)) = 0 \qquad \forall v \in \mathbb{R}\ v \neq 0.$$

It is indeed easy to verify it is complying with **Def. 2.1**, with an associated g.q.f. q_0 such that (the notations are those of **Section I.1.2**)

$$V_{q_0} = V_{q_0}^0 = \{\underline{t} \in \mathbb{R}^2 : \underline{u} = <0, u_2>\}.$$

The restriction of ω_0 to $\mathcal{A}(V_{q_0}, \sigma)$ is maximally regular in $\mathcal{A}(\mathbb{R}^2, \sigma)$. This is easily seen if one consider that if there is a regular quasifree extension, it would be associated to a finite 2×2 matrix q. Since ω_q extends ω_0, we would have $q(<0,u>) = 0 \quad \forall u \in \mathbb{R}$. Given the positivity condition

$$|\sigma(\underline{u}, \underline{v})|^2 \leq q(\underline{u})q(\underline{v}) \qquad \forall \underline{v}, \underline{u} \in \mathbb{R}^2$$

this contradicts the nondegeneracy of σ. In the nonzero temperature case, the $\omega \to 0^+$ limit produces a state with a different behaviour. Indeed we obtain the g.q.s. ω_β^0 defined by

$$\begin{aligned} \omega_\beta^0(W(<v,0>)) &= 0 \qquad v \neq 0 \\ \omega_\beta^0(W(<0,u>)) &= \exp\left(-\frac{m}{2\beta}u^2\right). \end{aligned} \tag{4.3}$$

Hence only the variables in the abelian C^*-algebra generated by $\{\exp(is\hat{x}), s \in \mathbb{R}\}$ are regularly represented, while thermal fluctuactions allow nonzero momenta for the equilibrium state, unlike the $T = 0$ case.

In general, as we will see later, thermal states define representations of the Weyl algebra with regularity properties not better than those of the representations associated to the corresponding ground states. Finally, it is clear that the $\beta \to 0^+$ limit (for $\omega \neq 0$) gives the central state ω_∞.

1. The quantum harmonic lattice.

We treat in this example quantum crystal lattices in harmonic approximation. We introduce first the necessary notation.

We consider a Bravais lattice in a finite volume in d dimensions: let $\mathbf{R}$ be the vector identifying the single lattice site, $\mathbf{K}$ the one identifying the sites of the reciprocal lattice, v the volume of the primitive cell. We suppose to have a quantum harmonic oscillator in every site. Its kinematics is described by the canonical coordinates $q(\mathbf{R})$ and $p(\mathbf{R})$: we fix here the origin in the centre of mass of the lattice, hence the variable $q(\mathbf{R})$ represents the deplacement from the mean value of the position as to referred the origin. One has, forgetting about domain problems, the CCR

$$[q_\mu(\mathbf{R}), p_\nu(\mathbf{R}')] = i\delta_{\mu\nu}\delta_{\mathbf{R}\mathbf{R}'} \qquad \mu, \nu = 1, \cdots, d.$$

The dynamics is given, supposing one has only a nearest-neighbor interaction, by the quantum hamiltonian

$$H = \frac{1}{2m}\sum_{\mathbf{R}} p(\mathbf{R})^2 + \frac{1}{2}\sum_{<\mathbf{R}\mathbf{R}'>,\mu,\nu} q_\mu(\mathbf{R}) D_{\mu\nu}(\mathbf{R}-\mathbf{R}') q_\nu(\mathbf{R}). \tag{4.4}$$

We have thus a system of coupled harmonic oscillators.

Since our aim is to take the thermodynamical limit, we construct now a CCR algebra that describes kinematically the infinite lattice, identifying the subalgebras associated to finite subsystems. To this end, we denote with V the set of lattice functions with values in $\mathbb{R}^d \times \mathbb{R}^d$ such that, if $\underline{\alpha}(\mathbf{R}) := <\alpha_1(\mathbf{R}), \alpha_2(\mathbf{R})> \in V$, it holds

$$\sum_{\mathbf{R}} \alpha_1(\mathbf{R})^2 < +\infty \qquad \sum_{\mathbf{R}} \alpha_2(\mathbf{R})^2 < +\infty.$$

Let then σ be the nondegenerate symplectic form on V defined by

$$\sigma(\underline{\alpha}, \underline{\beta}) = \sum_{\mathbf{R}} \alpha_1(\mathbf{R}) \cdot \beta_2(\mathbf{R}) - \sum_{\mathbf{R}} \alpha_2(\mathbf{R}) \cdot \beta_1(\mathbf{R}).$$

V is a real linear space and hence our infinite lattice is described by the CCR *-algebra $\mathcal{A}(V,\sigma)$. We use instead one of its subalgebras, enough to describe every finite subsystem. Indeed, we think of the lattice as imbedded in $\mathbb{R}^d$ and let $\Lambda_N \subset \mathbb{R}_d$ be open, bounded and containing in its interior N sites of the lattice. We denote with V_N the real linear space of maps from Λ_N with values in $\mathbb{R}^d \times \mathbb{R}^d$. The restriction of σ to $V_N \times V_N$ is still nondegenerate: hence $\mathcal{A}(V_N,\sigma)$ is a *-subalgebra of $\mathcal{A}(V,\sigma)$. Lasty, $\bigcup_{\Lambda_N} V_N$ is a subspace of V and the * subalgebra associated to it, we call it $\mathcal{A}$, contains the whole information relative to finite subsystems. If $\delta(\underline{\alpha})$ is a generator of $\mathcal{A}(V,\sigma)$ contained in $\mathcal{A}$, then it exists a Λ_N such that $\underline{\alpha} \in V_N$. Such a generator admits an obvious representation as Weyl operator

$$W(\underline{\alpha}) = \exp\,(i \sum_{\mathbf{R}\in\Lambda_N} \{q(\mathbf{R}) \cdot \alpha_1(\mathbf{R}) + p(\mathbf{R}) \cdot \alpha_2(\mathbf{R})\}).$$

We identify $\mathcal{A}(V_N,\sigma)$ with this representation.

We now proceed to the calculation, chosen $\Lambda_N \in \mathbb{R}^d$, of the Gibbs canonical equilibrium state

$$\omega^\beta_{\Lambda'_N}(A) := \frac{Tr\, A \exp\,(-\beta H_{\Lambda'_N})}{Tr \exp\,(-\beta H_{\Lambda'_N})} \qquad \beta \in \mathbb{R}^+$$

with $A \in \mathcal{A}(V_N, \sigma)$. $H_{\Lambda'_N}$ is the above hamiltonian with sums restricted to the sites in Λ'_N. For simplicity, we always choose Λ_N to be a parallalepiped centered at the origin and with suitable (referring to the structure of the lattice) sides. In order to calculate $\omega^{\beta}_{\Lambda_N}$ one needs to diagonalize the hamiltonian: the procedure is standard. One introduces wave vectors $\underline{k}$ and imposes Born-Karman periodic boundary conditions. This restrict the allowed wave vectors to the first Brillouin zone: their number is thus N. Polarization vectors $\underline{\epsilon}_s(\underline{k})$ and normal frequencies $\omega_s(\underline{k})$ are obtained as solutions of the eigenvalue problem

$$D_{\mu\nu}(\underline{k})\epsilon_\nu(\underline{k}) = m\omega(\underline{k})^2\epsilon^\mu(\underline{k})$$

where

$$D_{\mu\nu}(\underline{k}) := \sum_{\mathbf{R}} D_{\mu\nu}(\mathbf{R}) \exp(i\underline{k}\cdot\mathbf{R})$$

with the sum restricted to nearest neighbors of the origin.

It is absolutely essential to notice that

$$\omega(\underline{k}) \approx |\underline{k}|$$

in the range of small $|\underline{k}|$ (see for instance chap. 23 in **[ASH]**). Furthermore, since we use coordinates which are referred to to centre of mass, there are no zero modes. This done, a very long but standard calculation gives for the Gibbs state in a finite volume

$$\omega^{\beta}_{\Lambda_N}(\delta(\underline{\alpha})) = \exp\Big(- \frac{1}{4N}\sum_{k,s}\Big\{\frac{|\tilde{\alpha}_1(\underline{k})\cdot\underline{\epsilon}_s(\underline{k})|^2}{m\omega_s(\underline{k})} + m\omega_s(\underline{k})|\tilde{\alpha}_2(\underline{k})\cdot\underline{\epsilon}_s(\underline{k})|^2\Big\}\coth\frac{\beta\omega_s(\underline{k})}{2}\Big) \tag{4.5}$$

where $\tilde{\underline{\alpha}}(\underline{k}) := \sum_{\mathbf{R}\in\Lambda_N}\exp(-i\underline{k}\cdot\underline{\mathbf{R}})\underline{\alpha}(\mathbf{R})$ and $\delta(\underline{\alpha}) \in \mathcal{A}(V_N,\sigma)$. It is then easy to go to the thermodynamical limit. Let indeed $\delta(\underline{\alpha}) \in \mathcal{A}$ and let $\Lambda'_N \nearrow \mathbb{R}^d$ in the sense that it eventually contains every bounded $\Lambda \subset \mathbb{R}^d$. It is then immediate that

$$\lim_{\Lambda'_N\nearrow\mathbb{R}^d}\omega^{\beta}_{\Lambda'_N}(\delta(\underline{\alpha})) = \omega^{\beta}(\delta(\underline{\alpha}))$$

where ω^β is the g.q.s. on $\mathcal{A}$ defined by

$$\omega^{\beta}(\delta(\underline{\alpha})) = \exp\Big(- \frac{v}{4}\int\frac{d^dk}{(2\pi)^d}\sum_{s=1}^{d}\Big\{\frac{|\tilde{\alpha}_1(\underline{k})\cdot\underline{\epsilon}_s(\underline{k})|^2}{m\omega_s(\underline{k})} + m\omega_s(\underline{k})|\tilde{\alpha}_2(\underline{k})\cdot\underline{\epsilon}_s(\underline{k})|^2\Big\}\coth\frac{\beta\omega_s(\underline{k})}{2}\Big). \tag{4.6}$$

Remembering the small wave vector behaviour of the frequencies $\omega_s(\underline{k})$ we have the following results, since the sums in (4.5) approximate the integral in (4.6)

if $d \leq 2$ and if $\sum_{\mathbf{R}}\alpha_1(\mathbf{R}) = \tilde{\alpha}_1(0) \neq \underline{0}$ then $\omega^\beta(\delta(\underline{\alpha})) = 0$ for every $\beta \in \mathbb{R}^+$.

if $d = 1$ and if $\tilde{\underline{\alpha}}_1(0) \neq \underline{0}$ then also $\lim_{\beta\to+\infty} \omega^\beta(\delta(\underline{\alpha})) = 0$.

Moreover, since $\underline{\alpha}(\mathbf{R}) \in \bigcup_{\Lambda_N} V_N$, it follows that $\tilde{\underline{\alpha}}(\underline{k})$ is analytic in $\underline{k}$. Hence the above conditions are also necessary for the expectation value to be zero.

The standard argument consists in showing that

$$\lim_{\Lambda_N \nearrow \mathbb{R}^d} \omega^\beta_{\Lambda_N}(q(\mathbf{R}^*)^2) = +\infty$$

for every site $\mathbf{R}^*$ in the lattice. It is easy to recover it. Indeed, for expectation different from zero, and hence also for the all states $\omega^\beta_{\Lambda_N}$, it is true that

$$\omega^\beta_{\Lambda_N}(e^{iq(\mathbf{R}^*)}) = \exp\left(-\frac{1}{2}\omega^\beta_{\Lambda_N}(q(\mathbf{R}^*))\right).$$

But

$$\exp iq(\mathbf{R}^*) = \delta(\underline{\alpha}_{\delta_{\mathbf{R}^*}}) \in \mathcal{A}$$

with $\underline{\alpha}_{\delta_{\mathbf{R}^*}}(\mathbf{R}) = < \delta_{\mathbf{R}\mathbf{R}^*}\underline{1}, \underline{0} >$ and hence

$$\tilde{\underline{\alpha}}_{\delta_{\mathbf{R}^*}}(0) = \underline{1}.$$

As a consequence of

$$\lim_{\Lambda_N \nearrow \mathbb{R}^d} \omega^\beta_{\Lambda_N}(e^{iq(\mathbf{R}^*)}) = 0$$

we obtain

$$\lim_{\Lambda_N \nearrow \mathbb{R}^d} \omega^\beta_{\Lambda_N}(q(\mathbf{R}^*)^2) = +\infty.$$

More abstractly, we can say that

$$\omega^\beta(\delta(\underline{\alpha}_{\delta_{\mathbf{R}^*}})) = 0$$

for every site in the lattice implies that $\pi_{\omega^\beta}(\delta(\lambda\underline{\alpha}_{\delta_{\mathbf{R}^*}})$ is not strongly continuous in λ. This implies that in the representation π_{ω^β} of $\mathcal{A}$ the Stone's generators of the group $\pi_{\omega^\beta}(\delta(\lambda\underline{\alpha}_{\delta_{\mathbf{R}^*}})$ do not exist, if $d \leq 2$.

These last would have an interpretation as the operators that, in the quantum case, describe the deplacements from the equilibrium positions of the single lattice sites. The fact that they do not exist implies that, for an infinite crystal lattice in thermal equilibrium in $d \leq 2$, the absolute (that is, referred to the centre of mass) site positions are not well defined; are instead well defined the *relative* positions, and more generally all correlation functions of vectors of the form $\sum_{\mathbf{R}} \alpha(\mathbf{R}) \cdot q(\mathbf{R})$ with $\sum_{\mathbf{R}} \alpha(\mathbf{R}) = 0$.

2. The free Bose gas.

The free Bose gas gives a good example of nonregular representation of a CCR algebra in quantum statistical mechanics. We use the treatement given in chap 5.2.5 in [**BRA**].

We suppose that the gas is confined in a volume $\Lambda \in \mathbb{R}^d$. It is described kinematically by the CCR C^*-algebra $\mathcal{A}(L^2(\Lambda), \sigma) \equiv \mathcal{A}_\Lambda$, where

$$\sigma(f,g) = Im(f,g) \qquad \forall f,g \in L^2(\Lambda). \tag{4.7}$$

The dynamics acts at the one particle level and defines on $\mathcal{A}_\Lambda$ the one-parameter group of *-automorphisms α^t :

$$\alpha^t(\delta(f)) = \delta(e^{itH_\Lambda} f) \qquad \forall f \in L^2(\Lambda)$$

where H_Λ is the selfadjoint extension of the Laplacian $-\triangle$ on $L^2(\Lambda)$ corresponding to *Dirichlet* boundary conditions on $\partial\Lambda$.

It is immediate (prop. 5.2.28 in [**BRA**]) that it is well defined on $\mathcal{A}_\Lambda$ the Gibbs grancanonical factor with chemical potential $\mu \in \mathbb{R}$ and inverse temperature $\beta \in \mathbb{R}^+$, we call it $\omega_\Lambda^{\mu\beta}$, associated to H_Λ. It is, by explicit calculation, the gauge invariant quasifree state defined by

$$\omega_\Lambda^{\mu\beta}(\delta(f)) := \exp\{-\frac{(f, (\mathbb{1} + ze^{-\beta H_\Lambda})(\mathbb{1} - ze^{-\beta H_\Lambda})^{-1} f)}{4}\} \tag{4.8}$$

for every $f \in L^2(\Lambda)$, with $z = e^{\mu\beta}$. This state is regular.

For $d \geq 3$, in the thermodynamical limit there are two distinct regimes. In the first one there is a single phase, characterized by high temperature and low density (corresponding to $z < 1$ for Dirichlet boundary conditions). In the second one (low temperature and high density, corresponding to $z = 1$) a finite fraction of particles occupies the lowest energy level (Bose-Einstein condensation). There is a multiplicity of phases, overyone of them characterized by its own value of the particle density, all values in the interval $[\rho_c(\beta), +\infty[$ being possible for it. $\rho_c(\beta)$ is the critical value of the particle density (local particle number per unit volume: it is independent of the shape of Λ, for these boundary conditions (see again 5.2.5 in [**BRA**]))

$$\rho_c(\beta) = (2\pi)^{-d} \int d^d p e^{-\beta p^2}(1 - e^{-\beta p^2})^{-1}. \tag{4.9}$$

This is the value, in $z = 1$, of the density as a function of β and of the activity z given by

$$\rho(\beta, z) = (2\pi)^{-d} \int d^d p e^{-\beta p^2}(1 - ze^{-\beta p^2})^{-1}. \tag{4.10}$$

The integral defining $\rho_c(\beta)$ is divergent in $d \leq 2$: it is obtained as the (divergent) limit of well defined Riemann sums giving the value of the local particle number per unit volume when the system is confined in Λ. The critical density goes to infinity but nonregular representations appear, as it shows the following

Proposition (5.2.31 in [**BRA**]) *With the above notations,*

$$\omega^{\mu\beta}(A) = \lim_{\Lambda' \nearrow \mathbb{R}^d} \omega_{\Lambda'}^{\mu\beta}(A)$$

exists for $z = 1$ *and for every* $\beta \in I\!R^+$, $A \in \overline{\bigcup_\Lambda \mathcal{A}_\Lambda}$ *when* $\Lambda' \nearrow I\!R^d$ *in the sense that eventually contains every* $\Lambda \subseteq I\!R^d$. *The limit state is the gauge invariant quasifree state defined by*

$$\omega^{\mu\beta}(\delta(f)) := e^{-\frac{\|f\|}{4}} \exp\{-\frac{1}{(2\pi)^d}\int d^d p |\tilde{f}(p)|^2 e^{-\beta p^2}(1-e^{-\beta p^2})^{-1}\} \tag{4.11}$$

for every $f \in \bigcup_\Lambda L^2(\Lambda)$. *In particular,* $\omega^{\mu\beta}(\delta(f)) = 0$ *if* $d = 1, 2$ *and* $\int d^d x f(x) \neq 0$.

q.e.d.

We obtain hence, in $d \leq 2$, a g.q.s. with

$$V_q = \{f \in \bigcup_\Lambda L^2(\Lambda) : \tilde{f}(0) = 0\} \tag{4.12}$$

and $\tilde{f}(0)$ is well defined since $\tilde{f}(p)$ is analytic.

Remark. It is very important to notice that $\omega^{\mu\beta}$ is not locally normal, thanks to the discontinuity of $\omega^{\mu\beta}(\delta(\lambda f))$ in $\lambda = 0$ for every $f \in L^2(\mathcal{O})$ with $\tilde{f}(0) \neq 0$, *chosen anyhow* an open set $\mathcal{O} \in \mathrm{I\!R}^d$. In this model, this property is confirmed by the nonexistence of the local particle number discussed above.

3. The massless scalar field in 1+1 dimensions.

We start with a brief presentation of the model and of our notations. Since we are working in canonical formalism, it needs first to identify the CCR algebra describing the kinematical structure. On general grounds, it is generated by the canonical "time zero" fields $\phi(f)$ and $\pi(g)$, where f and g belongs to a suitable testfunctions space, usually identified with $\mathcal{S}_{\mathrm{I\!R}}(\mathrm{I\!R})$ (we'll always write $\mathcal{S}$, for simplicity). It is useful to introduce the operator

$$\Phi(F) := \phi(f_1) + \pi(f_2) \tag{4.13}$$

with $F := < f_1, f_2 > \in \mathcal{S} \times \mathcal{S}$. As operators acting on some Hilbert space, $\phi(f_1)$ and $\pi(f_2)$ give a representation of the CCR

$$[\Phi(F), \Phi(G)] = -i\{(f_1, g_2) - (f_2, g_1)\} := -i\sigma(F, G). \tag{4.14}$$

It is immediate a formulation in terms of abstract CCR *-algebras. Identifying in this case the objects introduced in **Section I.1.1**, one has that $V = \mathcal{S} \times \mathcal{S}$ and σ is the nondegenerate symplectic form on V defined in (4.14). This last property implies, as we have seen, that to the CCR algebra $\mathcal{A}(V, \sigma)$ is associated a unique CCR C^*-algebra, we denote with $\mathcal{A}$.

$\mathcal{A}(V, \sigma)$ it is generated by elements $\delta(F)$, with $F \in \mathcal{S} \times \mathcal{S}$, of simple interpretation: it is clear indeed that, chosen anyhow a *regular* representation π of $\mathcal{A}(V, \sigma)$, we can identify

$$\pi(\delta(F)) = \exp(i\Phi(F)) = \exp(i\{\phi(f_1) + \pi(f_2)\}) \tag{4.15}$$

by representing $\mathcal{A}(V,\sigma)$ as the concrete algebra generated by exponentials of the fields. Particular properties of the fields depend of the chosen representation; everyone of them corresponds to a definite model of QFT. We study here a particular representation of $\mathcal{A}(V,\sigma)$, denoted with π_0, whose construction proceeds as follows.

The massless scalar field is defined by the quasifree dynamics acting on $\mathcal{A}(V,\sigma)$ as a one-parameter group α_0^t of Bogolubov *-automorphisms by

$$\alpha_0^t(\delta(F)) = \delta(T_0^t F) \qquad \forall F \in V$$

where $(\tilde{T_0^t}F)(p) = \tilde{T}_0^t(p)\tilde{F}(p)$ and

$$\tilde{T}_0^t(p) = \begin{matrix} \cos\omega(p)t & -\omega(p)\sin\omega(p)t \\ \omega(p)^{-1}\sin\omega(p)t & \cos\omega(p)t \end{matrix} \tag{4.16}$$

with $\omega(p) = |p|$.

It is important to notice that $\tilde{T}_0^t : \mathcal{S}\times\mathcal{S} \to \mathcal{S}\times\mathcal{S}$ for every fixed t. We will recover the standard treatement of the model if we can exibit a Hilbert space, containing a space and time translations invariant cyclic vector, and (unbounded) operators such that their expectation values over this vector give the correlation functions of the massless scalar field. This program is implemented by searching for a state Ω_0 on $\mathcal{A}(V,\sigma)$ such that

$$\alpha_0^{t*}\Omega_0 = \Omega_0 \qquad \alpha_x^*\Omega_0 = \Omega_0 \tag{4.17}$$

where α_x is the space translations automorphism. The GNS construction gives us a representation $(\pi_0, \mathcal{H}_0, \psi_0)$ in which to carry on the analysis of the model. In particular, if Ω_0 is regular, we can calculate by differentiation the correlation functions of the fields, using (4.15).

It is well known [WIG], instead, that, if one wants to preserve the positivity of the Hilbert space metric, it needs to *restrict* the testfunctions space labelling the field ϕ, by allowing exactly the subspace of $\mathcal{S}$ identified by the condition $\tilde{f}(0) = 0$. We set thus

$$\partial\mathcal{S} := \{f \in \mathcal{S} : f = \partial g \ for\ some\ g \in \mathcal{S}\}. \tag{4.18}$$

It seems then to be natural to introduce the subspace $V_0 := \partial\mathcal{S}\times\mathcal{S}$ of V, and the non-degenerate symplectic space (V_0,σ_0), where σ_0 is the restriction of σ to $V_0\times V_0$. One costructs then the associated CCR *-algebra $\mathcal{A}(V_0,\sigma_0)$; we denote with $\mathcal{A}_0$ its unique C^* closure.

We are wondering which form the just discussed phenomenon assumes in our formalism. As we have seen, the first step consists in searching for states on $\mathcal{A}(V,\sigma)$ such that (4.17) holds. We'll not completely characterize this set.

We concentrate instead ourselves on a particular (ground) state, containing already many relevant informations. We require as a first thing that Ω_0 is pure. Since the dynamics is a quasifree one, we search for a state in the class of quasifree functionals, in particular of the form

$$\Omega_0(\delta(F)) = \exp\left(-\frac{1}{4}q_0(F)\right)$$

with q_0 a quadratic form on V. The space translations invariance implies that q_0 is of the form

$$q_0(F) = \int dp \bar{\tilde{f}}_i(p) \tilde{f}_j(p) M^0_{ij}(p) \tag{4.19}$$

with $i, j = 1, 2$ and where $M^0_{ij}(p)$ is an hermitian matrix. The purity of Ω_0 implies easily that $Det M^0_{ij}(p) = 1$ for every $p \in \mathbb{R}$. By imposing the time translation invariance condition one obtains, after *formal* calculations

$$M^0(p) = \begin{matrix} \omega(p)^{-1} & 0 \\ 0 & \omega(p) \end{matrix} \tag{4.20}$$

The meaning of the word "formal" lies in the fact that, by inserting this matrix in $q_0(\cdot)$, one obtains ill-defined expressions for every $F \in V$ such that $\tilde{f}_1(0) \neq 0$. We notice also that we can follow the very same procedure restricted to $\mathcal{A}(V_0, \sigma_0)$, since

$$T_0^t : V_0 \to V_0,$$

this time being all well defined. We recognize then an examplification of the abstract structure discussed in the previous sections. Indeed it is immediate to verify that the restriction ω_0 of Ω_0 to $\mathcal{A}(V_0, \sigma_0)$ is a regular quasifree state. We want to show that it is maximally regular in $\mathcal{A}(V, \sigma)$. Hence our state Ω_0 will be the unique extension of ω_0 to this algebra, whose regularity space V_q, as defined in (2.1), coincides with $\partial\mathcal{S} \times \mathcal{S}$. To this end one needs only, by **Prop 3.3** and following remarks, to show

Lemma 4.1 *Fixed anyhow $G \in V \backslash V_0$, $\sigma(\cdot, G)$ is an unbounded linear functional on V_0, when this is equipped with the inner product induced by q_0.*
Proof. We argue by contradiction. Indeed, if, fixed anyhow $G \in V \backslash V_0$, $\sigma(\cdot, G)$ is bounded, we can extend it by continuity to $\overline{V_0}^{q_0}$. This space contains elements of the form $F_n = <0, f_n>$, with

$$\tilde{f}_n(p) = \begin{cases} |p|^{-1+\frac{1}{n}} & |p| < \delta \\ 0 & |p| \geq \delta. \end{cases}$$

It is easy to see, for instance using mollifiers, that the continuous extensions of σ and q_0 to $\overline{V_0}^{q_0}$ are still given by the standard integral expressions. It is then immediate that

$$q_0(F_n) = \int \omega(p) |\tilde{f}_n(p)|^2 dp = \int_{-\delta}^{\delta} |p|^{-1+\frac{2}{n}} dp = n\delta^{\frac{2}{n}}.$$

If $G \in V \backslash V_0$, then $\epsilon > 0$, $b > 0$ exist such that

$$|Re\tilde{g}_1(p)| > b \qquad if \ |p| < \epsilon.$$

With $\epsilon = \delta$ we obtain

$$|\sigma(F_n, G)| = |\int_{-\delta}^{\delta} |p|^{-1+\frac{1}{n}} \tilde{g}_1(p) dp| = \int_{-\delta}^{\delta} |p|^{-1+\frac{1}{n}} |Re\tilde{g}_1(p)| dp > 2bn\delta^{\frac{1}{n}}.$$

It follows that, for every $G \in V \backslash V_0$,

$$\frac{|\sigma(F_n, G)|^2}{q(F_n)} > 4b^2 n$$

and $\sigma(\cdot, G)$ is not bounded.

q.e.d.

By **Proposition 3.3**, ω_0 admits as unique extension to $\mathcal{A}(V,\sigma)$ the pure state defined by

$$\Omega_0(\delta(F)) = \begin{cases} \omega_0(\delta(f)) & \forall F \in V_0 \\ 0 & \text{otherwise.} \end{cases} \tag{4.21}$$

Hence Ω_0 is determined by ω_0 and it is invariant under α_0^{t*}. Such a state is thus identified, as unique extension, by the restriction to $\partial\mathcal{S} \times \mathcal{S}$ of equations (4.16) and (4.17), that is by the solution of the same problem on a subalgebra, solution that gives a regular state.

As a further step, we search now for equilibrium states associated to the dynamics α_0^t, as acting on $\mathcal{A}(V_0,\sigma_0)$. We do that by determining a solution of the (α_0^t,β)-KMS boundary condition on $\mathcal{A}(V_0,\sigma_0)$ (we are not interested to the most general solution). To this end we introduce the following sesquilinear form on V_0:

$$\mathcal{S}_\beta(F,G) := (S_\beta F, G) \qquad \forall F,G \in V_0$$

where $(\cdot,\cdot)$ is the usual L^2 inner product and S_β is the matrix

$$S_\beta = \begin{matrix} \omega(p)^{-1}\coth\frac{\omega(p)\beta}{2} & 0 \\ 0 & \omega(p)\coth\frac{\omega(p)\beta}{2} \end{matrix} \tag{4.23}$$

with $\beta \in \mathbb{R}^+$. It is then easy to verify that the linear functional ω_β on $\mathcal{A}(V_0,\sigma_0)$ defined by

$$\omega_\beta(\delta(F)) := \exp\left(-\frac{1}{4}\mathcal{S}_\beta(F,F)\right) \qquad \forall F \in V_0$$

is a regular quasifree state: the positivity condition (1.4) is satisfied since $|\coth\frac{\omega(p)\beta}{2}| > 1 \quad \forall p \in \mathbb{R}$. Moreover, this state is complying with the (α_0^t,β)-KMS boundary condition on $\mathcal{A}(V_0,\sigma_0)$.

Indeed, both maps $t \mapsto \sigma_0(F, T_0^t G)$ and $t \mapsto \mathcal{S}_\beta(F, T_0^t G)$ are analytic in the strip $0 < Imt < \beta$ and continuous on the boundary, chosen anyhow $F,G \in V_0$, as one can see by inspection. The explicit form of S_β is then determined by the fulfillement of the KMS boundary condition. It is important to stress that all integral expressions we are using are meaningful, since we are dealing with $\partial\mathcal{S} \times \mathcal{S}$.

In order include in our treatement the algebra $\mathcal{A}(V,\sigma)$ we state the following well known result (see th. 4.11 in [**ROC**])

Lemma 4.2 *Let ω_β be the state on $\mathcal{A}(V_0,\sigma_0)$ defined by (4.22), (4.23). Then it admits an unique extension to $\mathcal{A}(V,\sigma)$ in the class of (α_0^t,β)-KMS states, namely the state Ω_β defined by*

$$\Omega_\beta(\delta(F)) = \begin{cases} \omega_\beta(\delta(F)) & \forall F \in V_0 \\ 0 & \textit{otherwise.} \end{cases} \tag{4.24}$$

q.e.d.

Indeed, Ω_β is a (α_0^t,β)-KMS state on the C^*-algebra $\mathcal{A}$. It clarly suffices to check the validity of this condition for the generators of $\mathcal{A}$ not contained in $\mathcal{A}_0$. Choose $\delta(F)$ and $\delta(G)$, say, as the generic generators. Then, if $F - G \notin V_0$, we have

$$\Omega_\beta(\delta(F)\alpha_0^t(\delta(G))) = 0 = \Omega_\beta(\delta(G)\delta(F)) \qquad \forall t \in \mathbb{R}$$

since $(T_0^{\tilde{t}}F)_1(0) = \tilde{f}_1(0)$, as it is easily verified. We now extend $\Omega_\beta(\delta(F)\alpha_0^t(\delta(G)))$, as a function of t, to the whole complex plane by setting it identically equal to the zero function. Hence the analiticity properties and the boundary condition required by the KMS condition are trivially satisfied.

If, instead, $F - G \in V_0$, then also $F - T_0^t g \in V_0 \quad \forall t \in \mathbb{R}$. In this case we can use ω_β instead of Ω_β in all calculations, and so the KMS condition is satisfied by construction.

As a consequence, we notice that Ω_β is primary. Indeed, it is well known that ω_β is primary (one can check for instance that the continuous extension of σ to $\overline{V_0}^{q_\beta}$ is nondegenerate), and hence it is an extremal (α_0^t,β)-KMS state on $\mathcal{A}_0$. But then also Ω_β, as unique extension of it in the class of (α_0^t,β)-KMS states, is an extremal (α_0^t,β)-KMS state on $\mathcal{A}$. By prop 5.3.30 in [**BRA**] the extended state is also primary.

4. The Stückelberg-Kibble model in 1+1 dimensions.

This model is an extrapolation to 1+1 dimensions of the original one, defined in 3+1 dimensions, in canonical formalism, by the formal Hamiltonian

$$H_K = \frac{1}{2}\int d^3x[(\nabla\phi)^2 + \pi^2] + \frac{1}{2}\int d^3x d^3y \pi(x)\pi(y)V(x-y)$$

where $V(x-y)$ is the Coulomb potential. The model is also characterized by a gauge automorphism τ_K^μ, namely the one acting as shift on the field $\phi(x)$: $\phi \to \phi + \mu$.

The kinematics of the model is thus described by the standard CCR C^*-algebra $\mathcal{A}$ labelled by testfunctions in $\mathcal{S}_{real}(\mathbb{R}^3) \times \mathcal{S}_{real}(\mathbb{R}^3)$.

To give a precise meaning to this Hamiltonian, one introduces (see [**MOR2**]) an infrared cutoff. A careful handling of his removal, in representations defined by phisically relevant states, gives rise to a well defined limiting dynamics of $\overline{\mathcal{A}}$, where the bar denotes the closure in the topology induced on $\mathcal{A}$ by the class of relevant states.

A similar procedure can be followed also in 1+1 dimensions (we use the notations of the preceding example). The standard algebra $\mathcal{A}(V,\sigma)$ is not stable under the limiting dynamics α_K^t: it goes into the CCR *-algebra labelled by the space $\mathcal{S} \times \partial^{-2}\mathcal{S}$, where

$$\partial^{-2}\mathcal{S} := \{f \in C^\infty : \partial^2 f \in \mathcal{S}\}. \tag{4.25}$$

This algebra is not, unlike the 3+1 dimensional case, contained in $\overline{\mathcal{A}(V,\sigma)}$, with the bar taken in the above sense, and we treat it in the following section as an example of extended CCR algebra. Instead, we select in this section a *-subalgebra of $\mathcal{A}(V,\sigma)$, using the criterion that it must be the largest among those which are gauge (in the sense precised above) invariant and stable under α_K^t.

Indeed, it results that the limiting dynamics acts on $\mathcal{A}(V,\sigma)$ in the same way as in formula (4.16), with the matrix

$$\tilde{T}_K^t(p) = \begin{matrix} \cos\omega_m(p)t & -\frac{p^2}{\omega_m(p)}\sin\omega_m(p)t \\ \frac{\omega_m(p)}{p^2}\sin\omega_m(p)t & \cos\omega_m(p)t \end{matrix} \tag{4.26}$$

with $\omega_m(p) = (p^2+m^2)^{\frac{1}{2}}$. The subalgebra we use is then $\mathcal{A}(\partial^2\mathcal{S}\times\mathcal{S},\sigma_0)$ with

$$\partial^2\mathcal{S} := \{f \in \mathcal{S} : f = \partial^2 g \ for\ some\ g \in \mathcal{S}\}. \tag{4.27}$$

One can easily verify that it satisfies the above requirements.

Since α_K^t is a linear dynamics, it is easy to find a (regular) pure quasifree state on $\mathcal{A}(\partial^2\mathcal{S}\times\mathcal{S},\sigma_0)$ invariant under it. Again in the notations of the preceding example, it is given by the state ω_K associated, as in (4.19) and (4.20), to

$$M_K(p) = \begin{matrix} \frac{\omega_m(p)}{p^2} & 0 \\ 0 & \frac{p^2}{\omega_m(p)}. \end{matrix} \tag{4.28}$$

We call q_K the finite g.q.f. on $\partial^2\mathcal{S}\times\mathcal{S}$ associated, by (4.19), to $M_K(p)$.

It is interesting to notice that ω_K gives rise to a representation in which the limit of the infrared cutoffed dynamics (restricted to $\mathcal{A}(\partial^2\mathcal{S}\times\mathcal{S},\sigma_0)$) exists. The state ω_K admits an extension to $\mathcal{A}(V_0,\sigma_0)$, call it Ω_K, defined by the natural extension of q_K to (V_0,σ_0) (recall that $V_0 = \partial\mathcal{S}\times\mathcal{S}$. We want to show that Ω_K is maximally regular in $\mathcal{A}(V,\sigma)$. This is proved, using **Prop. 3.3** in the next

Lemma 4.3 *Chosen anyhow $G \in V\backslash V_0$, a sequence $\{F_n\} \in V_0$ exists such that*

$$\lim_{n\to+\infty}\sigma(F_n,G) = \tilde{g}_1(0)$$

$$\lim_{n\to+\infty}q_K(F_n) = 0.$$

Proof. Let $G =< g,0 >\in V\backslash V_0$. Let $\delta_n(p)$ a sequence of functions in $\mathcal{S}$, obtained by scaling of a positive and symmetric functions $f(p)\in\mathcal{S}$, approximating in $\mathcal{S}'$ Dirac's delta function. We set as our sequence

$$\tilde{F}_n(p) =< 0, -\delta_n(p) > .$$

It is then easy to see that

$$\lim_{n\to+\infty}\sigma(F_n,G) = \lim_{n\to+\infty}\int\delta_n(p)\tilde{g}(p)dp = \tilde{g}(0)$$

and that, using the dominated convergence theorem,

$$\lim_{n\to+\infty}q_K(F_n) = \lim_{n\to+\infty}\frac{1}{n}\int p^2\omega_K(\frac{p}{n})^{-1}|f(p)|^2dp = 0.$$

q.e.d.

We continue our discussion on this model in **Section 1.3.2**, where we treat the dynamics as acting on $\mathcal{A}(V,\sigma)$ and we show that, unlike the 3+1 case, the gauge symmetry is *unbroken.*

I.3 EXTENSIONS OF CCR ALGEBRAS.

I.3.1. Charged fields as extended CCR algebras.

The structures we introduce in this section are motivated by looking at the generic situation in Quantum (Field) Theory, where the variables are described by an algebra of observables $\mathcal{A}$ and an algebra of gauge charged fields $\mathcal{F}$, the former being contained in the latter. The observable algebra coincides with the algebra whose elements are pointwise invariant under the gauge group. We want here to study the situation in which the observable algebra is given in canonical form, call it for instance $\mathcal{A}(V_0, \sigma_0)$. We want to show that, in this case, a charged field algebra can be exhibited within the class of CCR algebras, by constructing a suitable extension of $\mathcal{A}(V_0, \sigma_0)$.

Fixed an algebra $\mathcal{A}(V_0, \sigma_0)$, its extensions in a CCR ambit admit an immediate interpretation in terms of *charged fields*. We shall discuss a number of examples of this situation in models; the general structure is characterized in terms of a "gauge group" of *-automorphisms of the extension $\mathcal{A}(V, \sigma)$. From this point of view, it is relevant to observe that it is easy to construct a gauge group of *-automorphisms of $\mathcal{A}(V, \sigma)$, we call it G_{V/V_0}, characterized by the fact that $\mathcal{A}(V_0, \sigma_0)$ is left pointwise invariant by the action of this group. In this way, we can interpretate an element in $\mathcal{A}(V, \sigma)$ as charged under the action of the group G_{V/V_0}.

The peculiar structure of CCR *-algebras is supplying to us a very natural way in which to extend them. Clearly this means, generally speaking, to find an algebra of which the starting one is a proper subalgebra; the problem has a sense if the former has some structural affinities with the latter. In our case, CCR algebras are characterized by the symplectic space (V_0, σ_0) (we assume here σ_0 to be nondegenerate) and to search for extensions of $\mathcal{A}(V_0, \sigma_0)$ *in the class of CCR algebras* reduces to the problem of finding symplectic spaces (V, σ) such that $V_0 \subset V$ and $\sigma_{|V_0 \times V_0} = \sigma_0$.

In concrete terms, we have to establish whether it is possible to extend σ_0 to linear spaces $V \supset V_0$ which are in someway relevant. If it is so, **Section I.1.1** tells us how to proceed in order to build up (at least) one CCR C^*-algebra associated to (V, σ). It can be that σ admits several extensions, or that there is uniqueness but with degeneracy: these events will lead to a plurality of possible C^* structures. We'll not treat this aspect.

We establish now

Definition 5.1 Let $\mathcal{A}(V_0, \sigma_0)$ be a CCR *-algebra. Let (V, σ) be a symplectic space such that $V \supset V_0$ and $\sigma_{|V_0 \times V_0} = \sigma_0$. Then the *-algebra $\mathcal{A}(V, \sigma)$ is said to be an *extension in a CCR ambit* of $\mathcal{A}(V_0, \sigma_0)$.

The construction of such extensions is sometimes immediate. The question is more interesting if a regular g.q.s. state ω_q is given on $\mathcal{A}(V_0, \sigma_0)$. One may wonder how to characterize extensions of ω_q to $\mathcal{A}(V, \sigma)$, for instance if there are regular ones, and how the canonical representations of $\mathcal{A}(V, \sigma)$ defined by everyone of them represent $\mathcal{A}(V_0, \sigma_0)$. From this point of view, it is relevant the case in which ω_q is maximally regular in $\mathcal{A}(V,\sigma)$. We'll develop this point in the next section.

We now show that, given a *-algebra $\mathcal{A}(V_0,\sigma_0)$ and an extension $\mathcal{A}(V,\sigma)$ of it in a CCR ambit, this latter admits an interpretation as a *charged fields algebra*. To this end we proceed to the construction of a gauge group, namely of a group of *-automorphisms *univocally* associated to $\mathcal{A}(V,\sigma)$ as an extension in a CCR ambit of a $\mathcal{A}(V_0,\sigma_0)$, and acting as the identity on this last. For this reason, the single elements belonging to the extended algebra but not to $\mathcal{A}(V_0,\sigma_0)$ are called *charged* under the gauge group they determine.

Let then $\mathcal{A}(V_0,\sigma_0)$ be a CCR *-algebra and let $\mathcal{A}(V,\sigma)$ be an extension of it in a CCR ambit. The equivalence relation

$$F \sim^0 F' \qquad \text{iff} \qquad F - F' \in V_0,$$

with $F, F' \in V$, defines the quotient space V/V_0. It is a real linear space with the following, elementary, property (whose routine proof we omit)

Lemma 5.2 *The algebraic dual of V/V_0 is isomorphic to the set $V'_{(0)}$ of real linear functionals on V that are zero on V_0:*

$$(V/V_0)' \sim V'_{(0)} := \{\phi \in V'_{I\!R} : \phi(F) = 0 \quad \forall F \in V_0\}. \tag{5.1}$$

q.e.d.

In many cases, $V/V_0 \sim \mathrm{I\!R}^n$ for some n and so $(V/V_0)' \sim \mathrm{I\!R}^n$.

We can thus associate to every extension in a CCR ambit of the algebra $\mathcal{A}(V_0,\sigma_0)$ the real linear space $V'_{(0)}$. Even more, we assign to every $\phi \in V'_{(0)}$ the *-automorphism of $\mathcal{A}(V,\sigma)$ defined by

$$\alpha_\phi(\delta(F)) = \exp\left(i\phi(F)\right)\delta(F) \qquad \forall F \in V. \tag{5.2}$$

This way we establish an isomorphism between the set $V'_{(0)}$ (additive structure) and the set

$$G_{V/V_0} := \{\alpha_\phi \in Aut(\mathcal{A}(V,\sigma)) \quad \text{as in} \quad (5.2) : \phi \in V'_{(0)}\} \tag{5.3}$$

(equipped with the abelian group structure arising from morphisms composition). This is our (abelian) gauge group. It is clar that, for every $\phi \in V'_{(0)}$ and for every $F \in V_0$, $\alpha_\phi(\delta(F)) = \delta(F)$ holds. Hence $\mathcal{A}(V_0,\sigma_0)$ is pointwise invariant under the action of G_{V/V_0}.

As a further point, there is class of automorphisms of $\mathcal{A}(V_0,\sigma_0)$ which is "naturally" attached to $\mathcal{A}(V,\sigma)$, viewed as extension of $\mathcal{A}(V_0,\sigma_0)$ in a CCR ambit. We introduce indeed the class $\tau_{(V,\sigma)}$. To this end, let $G \in V$. Then

$$\tau^\sigma_G(\delta(F)) = \exp\left(i\sigma(G,F)\right)\delta(F) \qquad \forall F \in V_0 \tag{5.4}$$

defines an automorphism of $\mathcal{A}(V_0,\sigma_0)$.

As a consequence we let

$$\tau_{(V,\sigma)} := \{\tau^\sigma_G \ \textit{in}\ (5.4)\ \textit{with}\ G \in V\}. \tag{5.5}$$

In the same way one introduces $\tau_{(V_0,\sigma_0)}$. This class of *-automorphisms of $\mathcal{A}(V_0,\sigma_0)$ is contaned in $\tau_{(V,\sigma)}$. It is clear that every automorphism in $\tau_{(V_0,\sigma_0)}$ is inner since it is implemented by the corresponding generator:

$$\tau_G^{\sigma_0}(\delta(F)) = \delta(-G)\delta(F)\delta(G) := Ad(\delta(G))(\delta(F)) \qquad \forall F \in V_0.$$

We will use these automorphisms in the following section. We'll study the state space of the field algebra $\mathcal{A}(V,\sigma)$ in terms of the state space of the observable algebra $\mathcal{A}(V_0,\sigma_0)$. More precisely, we put a g.q.s. on $\mathcal{A}(V,\sigma)$ and we construct the GNS representation of $\mathcal{A}(V,\sigma)$ associated to it. The question is - how this representation is described in terms of the representation the restriction to $\mathcal{A}(V_0,\sigma_0)$ of the state gives rise to? The relevance of the class $\tau_{(V,\sigma)}$ is thus clear if one looks at **Prop. 3.6.**

Example. We discuss an application to QFT of the above ideas. As an example we use the CCR algebras suited to describe the kinematics of canonical models in 1+1 dimensions. We described these algebras when dealing with the massless scalar field, the fourth example in the previous section: we refer to it for the notations. We had as algebras of interest the ones labelled by $V = \mathcal{S} \times \mathcal{S}$ and by $V_0 = \partial\mathcal{S} \times \mathcal{S}$; the corresponding symplectic forms σ and σ_0 are defined in formula (4.14) and in the discussion just after (4.18). In order to construct a sensible extension of $\mathcal{A}(V,\sigma)$ in a CCR ambit, let

$$\partial^{-1}\mathcal{S} := \{f \in C^\infty(\mathbb{R}) : \partial f \in \mathcal{S}(\mathbb{R}), \lim_{x\to-\infty} f(x) = 0\}. \tag{5.6}$$

We defer to the following section and to **part II** for a motivation of this choice.

Let then $V_1 := \mathcal{S} \times \partial^{-1}\mathcal{S}$. Clearly, V is a linear subspace of the vector space V_1. In order to naturally extend the symplectic form we define on $V_1 \times V_1$

$$\sigma_1(F,G) := \int g_2(x)f_1(x) - f_2(x)g_1(x))dx \tag{5.7}$$

It follows immediately that σ_1 is nondegenerate on $V_1 \times V_1$ and that

$$\sigma_{1|V\times V} = \sigma.$$

Thus $\mathcal{A}(V_1,\sigma_1)$ is an extension in a CCR ambit of $\mathcal{A}(V,\sigma)$ (and hence of $\mathcal{A}(V_0,\sigma_0)$).

We come to the gauge group G_{V_1/V_0}. We notice that

$$V_1/V_0 \simeq \mathbb{R} \times \mathbb{R}$$

and every equivalence class of functions $F \in V_1$ is labelled by

$$< q_1, q_2 >_F := < \tilde{f}_1(0), (\tilde{\partial f_2})(0) > .$$

This is easily seen by fixing a vector $\hat{F}$ in V_1 with $< q_1, q_2 >_{\hat{F}} = < 1, 1 >$ and by noticing that $F - < q_1, q_2 >_F \cdot \hat{F} \in V_0$ for every $F \in V_1$ (with obvious notation). Hence, one can always write $F = Q_F \hat{F} + F_0$ with $F_0 \in V_0$. The algebraic dual of V_1/V_0 is spanned by the real linear functionals

$$\begin{aligned} \phi_1(F) &= \tilde{f}_1(0) \\ \phi_2(F) &= (\partial \tilde{f}_2)(0) \qquad \forall F \in V_1/V_0 . \end{aligned}$$

To them one can associate the two one-parameter groups of automorphisms of $\mathcal{A}(V_1, \sigma_1)$

$$\begin{aligned} \alpha_1^\lambda(\delta(F)) &= \exp\left(i\lambda \tilde{f}_1(0)\right)\delta(F) \\ \alpha_2^\mu(\delta(F)) &= \exp\left(i\mu(\partial \tilde{f}_2)(0)\right)\delta(F) \end{aligned} \tag{5.8}$$

for every $F \in V_1$; $\lambda, \mu \in \mathbb{R}$. As a consequence, G_{V_1/V_0} is isomorphic to $\mathbb{R} \times \mathbb{R}$.

In different models, the two one-parameter groups in the above formula admit different interpretations. Thus, in the massless scalar field, viewed as the bosonized form (we will dedicate the whole **Part II** to this problem) of the massless fermionic field, α_1^λ corresponds to chiral transformations and α_2^μ to gauge transformations (fermionic charge). In the free massive scalar field, which can be interpreted as the bosonized version of the Schwinger model, the first group implements the (broken) chiral symmetry, while the second represents the (unbroken) gauge transformations associated to the electric charge. In the Stückelberg-Kibble model the gauge transformations are instead given by the first group. We'll better study these correspondences in the next section.

I.3.2. Nonregular representations of extended CCR algebras, spontaneous symmetry breaking and charged states.

In this section the notions introduceed in the previous section are combined with the introduction of states and representations. The resulting structure is discussed from two points of view:

1. its interpretation in terms of observables and field algebras:

we are indeed thinking at a structure tipical in QFT, namely a field algebra $\mathcal{F}$ and an observable algebra $\mathcal{A}$, the latter being left pointwise invariant by the gauge group. We suppose we are given a representation of $\mathcal{F}$, for instance arising from a state ω on it (call π_ω the representation). One may wonder whether the representation of $\mathcal{A}$ associated to the restriction to it of ω someway determines the structure of π_ω. In particular, one looks at a decomposition of π_ω in terms of representations of $\mathcal{A}$ and at a discussion of the properties of the gauge group, for instance its (partial or complete) breaking. We study this problem in the setting of the previous section, where, given a CCR algebra $\mathcal{A}(V_0, \sigma_0)$ and an extension in a CCR ambit $\mathcal{A}(V, \sigma)$ of it, we reconstructed the gauge group G_{V/V_0}. In this section we let a g.q.s. Ω_q be defined on $\mathcal{A}(V, \sigma)$. Then we show how to characterize the unbroken part of G_{V/V_0}. Moreover, we exhibit the decomposition of π_{Ω_q} in terms of *charged sectors*, namely of representations of $\mathcal{A}(V_0, \sigma_0)$ containing states which are charged with respect to the gauge group G_{V/V_0}.

2. The appearence of spontaneous symmetry breaking:

we explore the consequences of the observation that if a state has a unique extension to a larger algebra then the extension is invariant under those automorphisms whose restriction to the smaller algebra leaves invariant the original state.

In particular, in the search for g.q.s. invariant under physically relevant automorphisms groups, like α_t or α_x, it is enough to verify the invariance properties on the restriction to the regularly represented algebra, provided this restriction is maximally regular.

Furthermore, it is sufficient to study the latter in order to establish whether a symmetry defined on the whole algebra is spontaneously broken or not. A typical case will be that of automorphisms reducing to the identity over the regular algebra.

An example are the gauge automorphisms naturally associated to every extension in a CCR ambit of a *-algebra $\mathcal{A}(V_0, \sigma_0)$, which we constructed in the previous section.

We start the above sketched analysis by introducing a CCR algebra $\mathcal{A}(V_0, \sigma_0)$ and an extension in a CCR ambit $\mathcal{A}(V, \sigma)$ of it. Moreover, let Ω_q be a g.q.s. on $\mathcal{A}(V, \sigma)$ which represents regularly $\mathcal{A}(V_0, \sigma_0)$. Let ω_q be the restriction of Ω_q to $\mathcal{A}(V_0, \sigma_0)$.

Besides V_0 and V_q (this latter defined in formula (2.1) as the maximal subspace of V over which the g.q.f. q is finite) we introduce

$$V_c := \{F \in V : \sigma(F, \cdot) \textit{ is a continuous linear functional on } V_0, \\ \textit{equipped with the inner product induced by } q\}. \tag{6.1}$$

It is very important to stress that in general

$$V_0 \subseteq V_q \subseteq V_c \subseteq V.$$

Indeed, if $F \in V_q$, then it belongs also to V_c since σ is majorized by q, in the sense of **Def. 2.1,B**. One notices also that if the restriction ω_q is maximally regular in $\mathcal{A}(V,\sigma)$, then $V_0 = V_q = V_c$. We then define, in strict analogy with the definition of $V'_{(0)}$ (see equation (5.1))

$$V'_{(q)} := \{\phi \in V'_{\mathbb{R}} : \phi(F) = 0 \quad \forall F \in V_q\} \tag{6.2}$$

$$V'_{(c)} := \{\phi \in V'_{\mathbb{R}} : \phi(F) = 0 \quad \forall F \in V_c\}. \tag{6.3}$$

It is then clear that

$$V'_{(c)} \subseteq V'_{(q)} \subseteq V'_{(0)}.$$

In the same way in which we defined G_{V/V_0} in (5.2) and (5.3), we define the two abelian groups G_{V/V_q} and G_{V/V_c}. We have then the natural immersions

$$G_{V/V_c} \subseteq G_{V/V_q} \subseteq G_{V/V_0}.$$

The group G_{V/V_0} is the gauge group associated to $\mathcal{A}(V,\sigma)$ as extension of $\mathcal{A}(V_0,\sigma_0)$ in a CCR ambit. We want to show that

1. G_{V/V_q} is the subgroup of G_{V/V_0} which leaves the vector representing Ω_q invariant.

2. V/V_c describes the decomposition of π_{Ω_q} into charged sectors (i.e. inequivalent representations of $\mathcal{A}(V_0,\sigma_0)$).

1.

Proposition 6.1 *Let Ω_q a g.q.s. on $\mathcal{A}(V,\sigma)$, regular on $\mathcal{A}(V_0,\sigma_0)$. Then $\alpha_\phi \in G_{V/V_0}$ leaves the state Ω_q invariant iff $\phi \in V'_{(q)}$.*

Proof. We prove first the necessity part. To this end we show that $\alpha^*_\phi \Omega_q = \Omega_q$.

Notice first of all that, since $\phi \in V'_{(q)}$, α_ϕ is the identity automorphism on $\mathcal{A}(V_q,\sigma)$. Furthermore, for every $F \in V$ such that $\phi(F) \neq 0$ one must have $F \in V \backslash V_q$ and hence $\omega_q(\delta(F)) = 0$. It follows that, for every F of this type,

$$(\alpha^*_\phi \Omega_q)(\delta(F)) = \Omega_q(\delta(F)) \exp(i\phi(F)) = 0 = \Omega_q(\delta(F)).$$

Combining these two statements we are done, since $\mathcal{A}(V,\sigma)$ is generated by the $\delta(\cdot)$. Conversely, if $\phi \notin V'_{(q)}$, then there is $F \in V_q$ such that $\phi(F) \neq 0$. Since $F \in V_q$ it follows that $\Omega_q(\delta(F)) \neq 0$. hence

$$(\alpha^*_\phi \Omega_q)(\delta(F)) = \Omega_q(\delta(F)) \exp(i\phi(F)) \neq \Omega_q(\delta(F)).$$

q.e.d.

We have then the following, consequence, which is connected with the basic motivations of our work:

Corollary 6.2 *If G_{V/V_0} is nontrivial and it leaves Ω_q invariant, then Ω_q cannot be a regular state on $\mathcal{A}(V,\sigma)$.*

Proof. If G_{V/V_0} leaves Ω_q invariant, then $V'_{(q)} = V'_{(0)}$ and $V_0 = V_q$. Hence, since $V_0 \not\subseteq V$ by assumption, $V_q \not\subseteq V$ and Ω_q cannot be a regular state on $\mathcal{A}(V, \sigma)$.

q.e.d.

By the very proof of the above proposition we get even more

Corollary 6.3 *If $\phi \in V'_{(q)}$, then α_ϕ is implemented in $(\pi_{\Omega_q}, \mathcal{H}_{\Omega_q}, \psi_{\Omega_q})$ by a unitary operator leaving ψ_{Ω_q} invariant: $U_\phi \psi_{\Omega_q} = \psi_{\Omega_q}$.*

Proof. It is a standard result, having proved that $\alpha^*_\phi \Omega_q = \Omega_q$.

q.e.d.

We concentrate now our attention on particular subgroups of G_{V/V_q}. Fixed $\phi \in V'_{(q)}$, consider the one-parameter subgroups of *-automorphisms α^λ_ϕ defined by

$$\alpha^\lambda_\phi(\delta(F)) = \exp(i\lambda\phi(F))\delta(F) \qquad \forall F \in V, \forall \lambda \in \mathbb{R}.$$

The first information about them is

Lemma 6.4 *Let $\phi \in V'_{(q)}$. Then the one-parameter group of unitary operators $U_\phi(\lambda)$ in $\mathcal{H}_{\Omega_q}$ implementing α^λ_ϕ in π_{Ω_q} is strongly continuous in $\lambda \in \mathbb{R}$.*

Proof. $\mathcal{A}(V, \sigma)$ is generated by the $\delta(\cdot)$'s. Hence one needs only to calculate, fixed $F \in V$,

$$\begin{aligned}&\|(U_\phi(\lambda) - \mathbb{1})\pi_{\Omega_q}(\delta(F))\psi_{\Omega_q}\| = \|(\exp(i\lambda\phi(F)) - 1)\pi_{\Omega_q}(\delta(F))\psi_{\Omega_q}\| = \\ &= |\exp(i\lambda\phi(F)) - 1| \longrightarrow 0 \qquad \text{for} \quad \lambda \to 0.\end{aligned}$$

q.e.d.

We are so equipped with a set of "charges": the Stone's generators Q_ϕ (*) of the single one-parameter groups $U_\phi(\lambda)$:

$$U_\phi(\lambda) := \exp(i\lambda Q_\phi). \tag{6.4}$$

It is immediate from this whole construction that the set of charges

$$Q_{\Omega_q} := \{Q_\phi \text{ in } (6.4) : \phi \in V'_{(q)}\}$$

has the structure of a linear space, isomorphic to $V'_{(q)}$. It coincides with the set of unbroken charges. We also have

Lemma 6.5 $U_\phi(\lambda) \in \pi_{\Omega_q}(\mathcal{A}(V_0, \sigma_0))' \qquad \forall \phi \in V'_{(q)}$.

Proof. It follows at once from the fact that α^λ_ϕ is the identity automorphism on $\mathcal{A}(V_0, \sigma_0)$, if $\phi \in V'_{(q)}$.

q.e.d.

(*) see th. viii.8 in [**REE**]

2.

Our aim is now to analyze the charged sectors coming out of our setting. To this end it is relevant the space V/V_c introduced above. We first look at the automorphisms τ_G^σ, $G \in V$, of $\mathcal{A}(V_0, \sigma_0)$ defined in (5.4) of the previous section. We then notice that the set of representations of $\mathcal{A}(V_0, \sigma_0)$

$$\pi_{\omega_q}^{(V,\sigma)} := \{\pi_{\omega_q,G} : \pi_{\omega_q,G} = \pi_{\omega_q} \circ \tau_G^{\sigma *}, \quad G \in V\}$$

has the structure of an abelian group, isomorphic to V/V_0.

By the definition of GNS representation, $\mathcal{H}_{\Omega_q}$ is contained in the (disjoint) union of the Hilbert spaces associated to the single elements in $\pi_{\omega_q}^{(V,\sigma)}$:

$$\mathcal{H}_{\Omega_q} \subset \bigcup_{G \in V} \mathcal{H}_G.$$

By the very same arguments used in the analysis given in the second part of **Sect. I.1.3** it follows that

$$\mathcal{H}_{\Omega_q} = \bigoplus_{G \in V/V_c} \mathcal{H}_G.$$

Indeed, looking at the proof of **Lemma 3.4** it is clear that G_1 and G_2 define disjoint representations iff $G_1 - G_2 \notin V_c$. Hence V/V_c describes the decomposition of π_{Ω_q} into charged sectors.

Furthermore, the representation that π_{Ω_q} gives of $\mathcal{A}(V, \sigma)$ has the property that automorphisms in $\tau_{(V,\sigma)}$ are implemented in it by unitary operators in $\pi_{\Omega_q}(\mathcal{A}(V, \sigma))$, inner ones (those in $\tau_{(V_0,\sigma_0)}$) being implemented in $\pi_{\omega_q}(\mathcal{A}(V_0, \sigma_0))$. Hence the fields algebra $\mathcal{A}(V, \sigma)$ is entirely represented in it. Even more, π_{Ω_q} contains every sector exactly once (compare with **Prop. 3.6**).

In the following examples we are mainly interested in the situation in which $V_0 = V_q = V_c$.

Example 1.

The massless scalar field in 1+1 contains structures that go beyond the previous, simple, analysis. It is indeed possible to give a precise identification of the unitary implementers and of the associated charges constructed above. We have seen at the end of the previous section that

$$V_1/V_0 \simeq \mathbb{R} \times \mathbb{R}.$$

It follows that

$$V'_{(0)} \simeq \mathbb{R} \times \mathbb{R}.$$

We had also a basis for $V'_{(0)}$: it is formed by the following two real linear functionals

$$\begin{aligned} \phi_1(F) &:= \tilde{f}_1(0) \\ \phi_2(F) &:= (\partial \tilde{f}_2)(0) \qquad \forall F \in V_1/V_0 \end{aligned} \tag{6.5}$$

to which correspond the two one-parameter groups of automorphisms of $\mathcal{A}(V_1, \sigma_1)$

$$\begin{aligned} \alpha_1^{\lambda}(\delta(F)) &= \exp{(i\lambda \tilde{f}_1(0))}\delta(F) \\ \alpha_2^{\mu}(\delta(F)) &= \exp{(i\mu(\partial \tilde{f}_2)(0))}\delta(F) \end{aligned} \tag{6.6}$$

for every $F \in V_1$; $\lambda, \mu \in \mathbb{R}$. As a consequence, G_{V_1/V_0} is isomorphic to $\mathbb{R} \times \mathbb{R}$.

We now consider the state ω_0 on $\mathcal{A}(V_0, \sigma_0)$ defined by (4.19) and (4.20). It was showed in **Lemma 4.1** that it is maximally regular in $\mathcal{A}(V, \sigma)$. One can use the very same proof of it, by using this time the sequence $\{F_n^2\} := \{< f_n^2, 0 >\} \in \overline{V_0}^{q_0}$ given by

$$f_n^2 = \begin{cases} \frac{1}{2n}|p|^{\frac{1}{n}} & |p| < \frac{1}{n} \\ 0 & |p| > \frac{1}{n}, \end{cases}$$

in order to show

Lemma 6.6 *The regular g.q.s. ω_0 on $\mathcal{A}(V_0, \sigma_0)$ is maximally regular in $\mathcal{A}(V_1, \sigma_1)$.*

q.e.d.

We conclude that ω_0 admits a unique extension to this larger algebra, we call it Ω. We obtain also that

$$V_0 = V_q = V_c.$$

Using **Prop. 3.6** we have then that

$$\mathcal{H}_\Omega = \bigoplus_{F \in V_1/V_0} \mathcal{H}_F$$

where $\mathcal{H}_F$ carries an irreducible representation of $\mathcal{A}(V_0, \sigma_0)$.

Furthermore, as a consequence of **Prop. 6.1**, both automorphisms in (6.6) are unbroken in π_Ω, and by **Corollary 6.3** and **Lemma 6.4** they are implemented by strongly continuous one-parameter groups of unitary operators in $\mathcal{H}_{\omega_0}$.

We call $U_1(\lambda)$ and $U_2(\mu)$ the respective unitary implementers and Q_1, Q_2 the associated generators. We can now give a precise identification of them following **Lemma 4.1.** Indeed let $\{F_n^1\} :=< 0, f_n^1 >$ with

$$\tilde{f}_n^1(p) = \begin{cases} \frac{1}{2n}|p|^{-1+\frac{1}{n}} & |p| < \frac{1}{n} \\ 0 & |p| > \frac{1}{n}. \end{cases} \tag{6.7}$$

For n fixed, $F_n^1 \in \overline{V_0}^{q_0}$ and so $\pi_{\omega_0}(\delta(F_n^1))$ is an unitary operator in $\pi_{\omega_0}(\mathcal{A}(\overline{V_0}^{q_0}, \sigma_0)) \subset \pi_{\omega_0}(\mathcal{A}(V_0, \sigma_0))''$. Indeed, if $F_n^1 \in \overline{V_0}^{q_0}$, there is a sequence $\{F_{n,m}^1\} \in V_0$ approximating it in the q norm and hence σ-weakly with respect to V_0. It follows that $\pi_{\omega_0}(\delta(F_n^1))$ is unitary as a strong limit of a sequence of unitary operators with strongly convergent adjoints.

By the same argument, $s - \lim_{n \to +\infty} \pi_{\omega_0}(\delta(F_n^1))$ exists and it is an unitary operator in $\pi_{\omega_0}(\mathcal{A}(\overline{V_0}^{q_0}, \sigma_0))'' = \pi_{\omega_0}(\mathcal{A}(V_0, \sigma_0))''$.

Moreover, let's notice that the sequence $\{\tilde{f}_n^1(p)\}$ tends, as n goes to infinity, in $\mathcal{S}'$ to $\delta(p)$, and that $\lim_{n\to+\infty} q(F_n^1) = 0$.

Hence

$$U_1(\lambda) = s - \lim_{n\to+\infty} \pi_{\omega_0}(\delta(\lambda F_n^1)). \tag{6.8}$$

It follows from this and from **Lemma 6.5** that

$$U_1(\lambda) \in \pi_{\omega_0}(\mathcal{A}(V_0,\sigma_0))' \cap \pi_{\omega_0}(\mathcal{A}(V_0,\sigma_0))''. \tag{6.9}$$

If we fix $n < +\infty$, there exists the Stone's generator $\Phi_{\omega_0}(F_n^1)$ of $\pi_{\omega_0}(\delta(\lambda F_n^1))$, $\lambda \in \mathbb{R}$.

$\{\Phi_{\omega_0}(F_n^1)\}$ is a sequence of selfadjoint operators on $\mathcal{H}_{\omega_0}$, in the same way as it is Q_1. It follows then from (6.8) that

$$\Phi_{\omega_0}(F_n^1) \longrightarrow Q_1$$

in the strong resolvent sense (*).

The same holds for $U_2(\mu)$, with a sequence $\{F_n^2\} := \{< f_n^2, 0 >\} \in \overline{V_0}^{q_0}$ given by

$$\tilde{f}_n^2 = \begin{cases} \frac{1}{2n}|p|^{\frac{1}{n}} & |p| < \frac{1}{n} \\ 0 & |p| > \frac{1}{n}. \end{cases} \tag{6.10}$$

Let finally be i=1,2. We can differentiate with respect to the parameter and obtain

$$\begin{aligned} \phi_i(G)\pi_\Omega(\delta(G)) &= \frac{d}{d\lambda}\alpha_i^\lambda(\pi_\Omega(\delta(G))_{|\lambda=0} = \\ &= [Q_i, \pi_\Omega(\delta(G))] \end{aligned} \tag{6.11}$$

the last equality being true at least in a weak sense on a dense domain.

Example 2.

We analyze here, with the same notations of the previous example, the canonical free scalar field, with mass m, in 1+1 dimensions. It is well known that this model describes also the vacuum sector of the bosonized version of the Schwinger model, in the Coulomb gauge ([**MOR**]). The kinematics is described by the canonical CCR algebra $\mathcal{A}(V,\sigma)$, already introduced in the previous example. The time translation automorphism α_m^t is defined as in (4.16), with the only difference that this time we have

$$\omega(p) = (p^2 + m^2)^{\frac{1}{2}} \equiv \omega_m(p). \tag{6.12}$$

Calculations absolutely analogous to those of the previous section lead to a space and time translation invariant state ω_m, the vacuum state of the model. It is a regular quasifree state on $\mathcal{A}(V,\sigma)$ defined as in equations (4.18), (4.19). We call q_m the g.q.f. associated to it. Explicitely, the analogous of (4.19) is

$$M^m(p) = \begin{matrix} \omega_m(p)^{-1} & 0 \\ 0 & \omega_m(p). \end{matrix} \tag{6.13}$$

(*) th. viii.21 in [**REE**].

We introduce also in this case the algebra $\mathcal{A}(V_1, \sigma_1)$. In this interpretation, $\mathcal{A}(V_1, \sigma_1)$ corresponds to the algebra of the fields charged under the gauge symmetry, so that $\mathcal{A}(V, \sigma)$ is the neutral algebra (and hence describes the vacuum sector).

Indeed, the gauge symmetry is represented by the one-parameter group α_2^μ of *-automorphisms of $\mathcal{A}(V_1, \sigma_1)$.

We want to show that ω_m is maximally regular in this extended algebra, and hence it admits a unique extension, call it Ω_m, to $\mathcal{A}(V_1, \sigma_1)$. It follows that the gauge symmetry is unbroken, thanks to **Prop. 6.1**, in π_{Ω_m}

By **Prop. 3.3** one needs only to show

Lemma 6.7 *Chosen anyhow $G \in V_1 \backslash V$, a sequence $\{F_n\} \in V$ exists such that*

$$\lim_{n \to +\infty} \sigma_1(F_n, G) = (\partial \tilde{g_2})(0)$$

$$\lim_{n \to +\infty} q_m(F_n) = 0.$$

Proof. Let $G =< 0, g_2 >\in V_1 \backslash V$. Let $\delta_n(p)$ be a sequence of functions in $\mathcal{S}$, obtained by scaling of a positive function $f(p)$ in $\mathcal{S}$, approximating in $\mathcal{S}'$ Dirac's delta distribution. We set as our sequence

$$\tilde{F}_n(p) =< -ip\delta_n(p), 0 > .$$

It is then easy to see that

$$\lim_{n \to +\infty} \sigma_1(F_n, G) = \lim_{n \to +\infty} i \int p\tilde{g_2}(p)\delta_n(p)dp = (\partial \tilde{g_2})(0)$$

and that, using the dominated convergence theorem,

$$\lim_{n \to +\infty} q_m(F_n) = \lim_{n \to +\infty} \frac{1}{n} \int p^2 \omega_m(\frac{p}{n})^{-1} |f(p)|^2 dp = 0.$$

q.e.d.

Since the gauge symmetry is unbroken in π_{Ω_m}, we can speak of gauge charged fields as really distinct from the vacuum sector.

The status of the would-be chiral symmetry is different. It is well known that it is spontaneously broken in this model. Hence the fields which are charged under it are degrees of freedom contained in the vacuum sector. In the bosonic setting, the chiral symmetry corresponds to the one-parameter group of *-automorphisms of $\mathcal{A}(V, \sigma)$ α_1^λ (shift of the field $\phi \to \phi + \lambda_1$: we give in **Section II.1** a detailed discussion of the reasons of these correspondences between automorphisms). The vacuum state ω_m is regular on $\mathcal{A}(V, \sigma)$ and hence it is not invariant under $\alpha_1^{\lambda *}$:

$$\alpha_1^{\lambda *} \omega_m(\delta(F)) = e^{i\lambda \tilde{f}_1(0)} \omega_m(\delta(F))$$

Hence α_1^λ is *broken* in π_{ω_m}.

Example 3.

We study now the Stuckelberg-Kibble model, the last example we treated in **Section I.2.** We already noticed that, if we insist that the dynamics α_K^t acts on the algebra $\mathcal{A}(V,\sigma)$, then, given the form (4.26) of the dynamical matrix, we are forced to consider the CCR *-algebra labelled by the symplectic space (V_2, σ_2) where $V_2 = \mathcal{S} \times \partial^{-2}\mathcal{S}$ as defined in (4.25) and σ_2 is defined by the same expression as in (4.14). Clearly, V is a subspace of V_2 and $\sigma_{2|V\times V} = \sigma$. Hence $\mathcal{A}(V_2,\sigma_2)$ is an extension in a CCR ambit of $\mathcal{A}(V,\sigma)$. One of its *-subalgebras, namely the one labelled by (V', σ_2) where $V' = \partial\mathcal{S} \times \partial^{-1}\mathcal{S}$ (for the suitable definitions, see (4.18) and (5.6)), has the property of being maximal with respect to the criterion that select gauge invariant and α_K^t stable algebras, as it is easily verified. The state ω_K admits a natural pure regular and quasifree extension to $\mathcal{A}(V',\sigma_2)$, call it Ω'_K, namely the one associated to the g.q.f. q_K, viewed as acting on V'. We want to show that Ω'_K has a unique extension to $\mathcal{A}(V_2,\sigma_2)$. We use **Prop. 3.3** and so we are reduced to prove, taking into account that half of our work has already been given in **Lemma 4.2** and thanks to the product structure of our symplectic spaces

Lemma 6.8 *Chosen anyhow $G \in V_2 \backslash V'$, a sequence $\{F_n\} \in V'$ exists such that*

$$\lim_{n\to+\infty} \sigma_2(F_n, G) = (\partial^2 \tilde{g}_2)(0)$$
$$\lim_{n\to+\infty} q_m(F_n) = 0.$$

Proof. Let $G =< 0, g_2 >\in V_2\backslash V'$. Let $\delta_n(p)$ be a sequence of functions in $\mathcal{S}$, obtained by scaling of a positive function $f(p)$ in $\mathcal{S}$, approximating in $\mathcal{S}'$ Dirac's delta distribution. We set as our sequence

$$\tilde{F}_n(p) =< p^2\delta_n(p), 0 > .$$

It is then easy to see that

$$\lim_{n\to+\infty} \sigma_2(F_n, G) = \lim_{n\to+\infty} \int p^2 \tilde{g}_2(p)\delta_n(p)dp = (\partial^2 \tilde{g}_2)(0)$$

and that, using the dominated convergence theorem,

$$\lim_{n\to+\infty} q_m(F_n) = \lim_{n\to+\infty} \frac{1}{n} \int p^2 \omega_m(\frac{p}{n})^{-1} |f(p)|^2 dp = 0.$$

q.e.d.

We call Ω''_K the unique extension of Ω'_K to $\mathcal{A}(V_2,\sigma_2)$.

In the definition of the Stückelberg-Kibble model it is also contained the gauge symmetry τ_K^λ, which acts as a shift on the field ϕ. In the exponentiated form, it corresponds to the *-automorphism of $\mathcal{A}(V,\sigma)$ defined by

$$\alpha_1^\lambda(\delta(F)) = e^{i\lambda \tilde{f}_1(0)}\delta(F).$$

It is then easy to show that this symmetry is, *unlike the 3+1 case*, unbroken in the representation of $\mathcal{A}(V,\sigma)$ defined by the natural extension of the time translation invariant state we discussed in the previous section.

Indeed $\mathcal{A}(V_0,\sigma_0)$ is, as we have seen, a maximal gauge invariant (even if not time invariant) algebra contained in $\mathcal{A}(V,\sigma)$. Furthermore, the state Ω_K is, by the previous lemma, maximally regular in $\mathcal{A}(V,\sigma)$, and so it admits an unique extension, call it Ω_K^0, to $\mathcal{A}(V,\sigma)$. Hence **Prop. 6.1** implies that α_1^λ is unbroken in $\pi_{\Omega_K^0}$.

It is thus interesting to observe that infrared effects (in this case due to the lowering of the dimension) can change the status of the relevant symmetries of the model.

We now want to study more general symmetries of the field algebra $\mathcal{A}(V,\sigma)$. We assume now a slightly different point of view, namely we start with a state on the observable algebra $\mathcal{A}(V_0,\sigma_0)$ and we ask whether its extension preserves its symmetries. In general, a symmetry of a physical system is translated to an automorphism of the algebras used to describe it. By duality, the automorphism is extended to the states. An automorphism β is said to be broken in the representation π if $\pi\circ\beta$ is not unitarily equivalent to π. Called β^* the dual automorphism, the breaking of β from the state ω is equivalent to

$$\beta^*\omega \not\simeq \omega$$

where we means inequivalence of the respective GNS representations.

We are interested in investigating the relations between symmetries of states and symmetries of their own extensions. In this direction goes the following

Proposition 6.1 *Let be given a *-algebra $\mathcal{A}(V,\sigma)$, a *-subalgebra $\mathcal{A}(V_0,\sigma_0)$ of it and a g.q.s. ω_q on $\mathcal{A}(V_0,\sigma_0)$ maximally regular in $\mathcal{A}(V,\sigma)$. Let β be an automorphism of $\mathcal{A}(V,\sigma)$ such that*

$$\beta(\mathcal{A}(V_0,\sigma_0)) \subset \mathcal{A}(V_0,\sigma_0) \qquad (*)$$

and let β be a symmetry for ω_q:

$$\beta^*\omega_q = \omega_q.$$

Then, ω_q has a unique extension Ω to $\mathcal{A}(V,\sigma)$, satisfying

$$\beta^*\Omega = \Omega.$$

Proof. Since $(*)$ holds, $\beta^*\omega_q$ is a state on $\mathcal{A}(V_0,\sigma_0)$. By **Prop. 3.3**, ω_q has a unique extension to $\mathcal{A}(V,\sigma)$ and so our statement immediately follows from the fact that $\beta^*\Omega$ is an extension of $\beta^*\omega_q$.

q.e.d.

This proposition holds obviously even if β is substituted by an one or more parameter automorphism group $\beta(g)$ It has two fundamental consequences.

1. In the search of invariant states. In the assumed hypothesis, one needs only to verify the invariance of the regular part: maximal regularity provides us the invariance of the whole state. Relevant applications arise in the case of the space and time translation automorphisms, and hence in the search for ground states in field thories.

2. In the analysis of phenomena of spontaneous symmetry breaking. Proposition says that to this end, it is enough to limit ourselves, in the assumed hypothesis, in studying the action of the symmetries on the "regular part" of the state.

Thus, with this proposition, we can treat, in the maximal regularity hypothesis, symmetries which are not of the gauge type, like for instance those considered in the first part of this section. Indeed, we tacitly used the above proposition in **Section I.2.3**, when we said that the state Ω_0 of the free massless field is, thanks to its maximal regularity, space and time translations invariant.

Appendix A. Pure and primary generalized quasifree states.

We collect in this appendix two problems in the study of the properties of the generic g.q.s.. Indeed these questions, while being of some interest, have a technical character; in the same way the treatement is slightly less expository and more bristly of theorems. In fact, we study how to characterize factoriality and purity for g.q.s..

The idea is to find suitable generalizations of the well known characterizations of the same properties in the case of regular quasifree states: our basic reference is thus [**MAN2**], dedicated to this very problem. From this point of view, the method of presentation of our results is the same in both analysis: we formulate first, introducing the appropriate notation where it needs, the standard result. We circumscribe then the aspects we want to generalize, with suitable new definitions (verifing they reduce to the standard ones in the regular case). Looking at them we'll conjecture about a possible extension to the g.q.s. of the characterization. Finally, we prove such a conjecture, in the case by subsequent generalizations.

This program is not yet complete: it is not so clear how to set up the sufficiency proof for the condition we think to be equivalent to the factoriality, while the necessity one is easily carried through.

We briefly explain the content of this appendix; as an aside, we notice that the following proofs elaborate and enlarge ideas and logical line used in [**MAN2**] in the regular case.

Factoriality. A regular quasifree state ω_q over the algebra $\mathcal{A}(V,\sigma)$ is primary exactly when the continuous extension of σ to $\overline{V}^q$ is nondegenerate. This is the standard case. Notice that σ is supposed to be nondegenerate on V. The new aspect consists in introducing on V a topology, appropriate to the case in which q is a g.q.f. and σ is degenerate, reducing, when we have regularity and nondegeneracy, to the one induced by q on V. It is called the σq-topology. Our conjecture consists simply in sobstituting $\overline{V}^{\sigma q}$ to $\overline{V}^q$ in the above. The necessity of this condition is then verified. We notice also the very relevant fact that we are able to tackle the case, not treated in [**MAN2**], in which ω_q is regular but σ and q are degenerate. We end with a proposition classifying all extension a primary g.q.s. on $\mathcal{A}(V,\sigma)$ admits to $\mathcal{A}(\overline{V}^{\sigma q},\sigma)$.

Purity. In the regular case, the positivity condition (1.4) implies that there is $D_q \in \mathcal{B}(\overline{V}^q)$ such that $\sigma(\cdot,\cdot) = [\cdot, D_q \cdot]_q$. One then shows that $A_q = -D_q^{-1}$ exists. The standard equivalence says that a regular quasifree state ω_q on $\mathcal{A}(V,\sigma)$ is pure exactly when $A_q^\dagger A_q = \mathbb{1}$. Our generalization goes in the following direction: we introduce the notion of *minimal* g.q.f. over a symplectic space (V,σ). This means that there is no g.q.f. q' different from q such that $q'(F) \leq q(F)$ for every $F \in V$. We show that, in the regular and nondegenerate case, this property reduces to the previous one. On these grounds we hypotize that minimality of a g.q.f. is equivalent to the state associated to it being pure. The proof is by no means immediate and requires a series of intermediate propositions, in the direction of a gradually increasing generality. Also in this case we are able to treat regularity with degeneracy, as an improvement to [**MAN2**]. Finally, we explicitely control that our construction is consistent, that is purity implies factoriality (both in our sense).

Primary states

We first of all recall the standard characterization of primary quasifree states (prop. 11 and th. 3 in [**MAN2**]).

Let $\mathcal{A}(V,\sigma)$ be a CCR *-algebra (σ is assumed to be nondegenerate). Let ω_q be the quasifree state associated to the finite and nondegenerate g.q.f. $q : V \to \mathbb{R}$. Then ω_q is primary iff the continuous extension of σ to $\overline{V}^q$ is nondgenerate.

We start our analysis by noticing that, if ω_q is primary, then σ degenerate on V implies q degenerate on V.

Indeed, if $F \in V$ exists such that $\sigma(F,G) = 0 \quad \forall G \in V$, then, in *every* representation π of $\mathcal{A}(V,\sigma)$, $\pi(\delta(F))$ belongs to the centre of it. If π is factorial, then unitarity implies $\pi(\delta(F)) = \lambda \mathbb{1}$ with $|\lambda| = 1$. In particular, if π arises from a quasifree state, $\pi = \pi_{\omega_q}$, it follows that

$$1 = |(\psi_{\omega_q}, \pi_{\omega_q}(\delta(F))\psi_{\omega_q})| = \exp{(-\frac{1}{4}q(F))}$$

that is $q(F) = 0$ and q is degenerate on V. The next step is the introduction on V of a topology generalizing to the case of g.q.f. that induced, in the regular one, by q.

Definition A.1 Let (V,σ) be a symplectic space and q a g.q.f. over it. We call σq-topology the locally convex topology defined by the neighborhood basis at the origin $I_{\epsilon,\epsilon_i,G_i}$ where, for every finite set $\{G_i\} \in V$ and with $\epsilon, \epsilon_i \in \mathbb{R}$

$$I_{\epsilon,\epsilon_i,G_i} = \{F \in V : q(F) < \epsilon \quad , \quad |\sigma(F,G_i)| < \epsilon_i\}. \qquad (A.1)$$

In other words, let $G \in V$. Then $|\sigma(G,\cdot)| : V \to \mathbb{R}$ is a seminorm on V. The σq-topology is generated by the set of these seminorm and by the $q(\cdot)$ norm. We notice that we can take the G_i to be in $V \backslash V_q$. Indeed, if $G_i \in V_q$ for some i, then the second condition is implied by the first, with $\epsilon_i = \epsilon q(G_i)$, by the positivity condition (1.4). Hence, if $V = V_q$, this topology reduces to the strong topology on V induced by the q norm.
The quotient topology induced on V/V_q by the σq-topology is the discrete one.

In a generic situation, that is if both σ and q are degenerate, there can be elements $F \in V$ such that $q(F) = 0$ and $\sigma(F,\cdot) \equiv 0$ on V. Hence the σq-topology is not Haussdorff. This problem is removed if we notice that we are not really interested to V as a space equipped with the σq-topology. Therefore we introduce

Definition A.2 We call $\overline{V}^{\sigma q}$ the space obtained from V adjoining to it all limit points of σq-Cauchy nets and then going to the quotient with respect to the nets that admits the origin as a limit point.

$\overline{V}^{\sigma q}$ is the space we are interested in. If $V = V_q$, $\overline{V}^{\sigma q}$ coincides with $\overline{V}^q$, the Hilbert space canonically obtained from $(V, [\cdot,\cdot]_q)$ by completion and quotient over the zero sequences. As a first result we obtain at once

Lemma A.3 *Given a decomposition* $V = V_q + V'$, *then* $\overline{V}^{\sigma q} = \overline{V_q}^{\sigma q} + V'$. *Furthermore,* $q(\cdot)$ *and* $\sigma(\cdot,\cdot)$ *are continuous (*σ *is jointly continuous) in the* σq*-topology and they have then a unique continuous extension to* $\overline{V}^{\sigma q}$.

Proof. The first statement is an immediate consequence of the definition of $\overline{V}^{\sigma q}$. It follows that, if the net $\{F_\alpha\} \in V$ is σq-Cauchy, one can write $F_\alpha = G_\alpha + H$, $\{G_\alpha\} \in V$ a σq-convergent net and $H \in V'$, fixed. The continuity of $q(\cdot)$ follows at once. The joint continuity of $\sigma(\cdot,\cdot)$ follows from the preceding decomposition, from the σ-weak and the q-strong convergence of $\{G_\alpha\}$ combined with the standard positivity condition, true on pairs of vectors in V_q.

q.e.d.

q and σ can well be degenerate on $\overline{V}^{\sigma q}$, but if there is $F_0 \in \overline{V}^{\sigma q}$ such that $q(F_0) = 0$ and $\sigma(F_0, G) = 0 \quad \forall G \in \overline{V}^{\sigma q}$, then $F_0 = 0$ in $\overline{V}^{\sigma q}$ by definition. We have also the important

Lemma A.4 *Let $\{F_\alpha\}$ be a net in V σq-convergent to $F \in \overline{V}^{\sigma q}$. Then $\{F - F_\alpha\}$ is convergent to zero q-strongly and σ-weakly. In particular* $\lim_\alpha \pi_{\omega_q}(\delta(F_\alpha)) \in \pi_{\omega_q}(\mathcal{A}(V,\sigma))''$.

Proof. The first statement follows from **Def. A.2** and **Lemma A.3**. The second one from the fact that q-strong and σ-weak convergence imply in an absolutely standard way that there exists

$$s - \lim_\alpha \pi_{\omega_q}(\delta(F_\alpha)) \in \pi_{\omega_q}(\mathcal{A}(V,\sigma))''.$$

q.e.d.

We can now state our

Conjecture A.5 *Let (V,σ) be a symplectic space, q a g.q.f. on it and ω_q the associated g.q.s.. Then ω_q is primary iff the continuous extension of σ to $\overline{V}^{\sigma q}$ is nondegenerate.*

Lemma A.4 allows to control the elements of the centre and this let us to set up the necessity proof of the conjecture. This proof is a simple generalization of th. 3 proof in [**MAN2**]. Notice that we cover also the case, not treated in it, in which q is finite but degenerate and σ is degenerate.

Proposition A.6 *Let (V,σ) be a symplectic space and q a g.q.f. on it. If the continuous extension of σ to $\overline{V}^{\sigma q}$ is degenerate, then ω_q is not primary.*

Proof. If σ is degenerate on $\overline{V}^{\sigma q}$, let $F_0 \neq 0$ in $\overline{V}^{\sigma q}$ such that $\sigma(F_0,\cdot) \equiv 0$ on $\overline{V}^{\sigma q}$ and let $\{F_\alpha\}$ be a net in V σq-convergent to F_0. By **Lemma A.4**, $\lim_\alpha \pi_{\omega_q}(\delta(F_\alpha))$ exists and defines an element U in $\pi_{\omega_q}(\mathcal{A}(V,\sigma))''$. Since σ is degenerate on F_0 and $\{F_\alpha\}$ is σq convergent to F_0, U is in the centre $\pi_{\omega_q}(\mathcal{A}(V,\sigma))' \cap \pi_{\omega_q}(\mathcal{A}(V,\sigma))''$. Notice then that U is unitary: it is a strong limit of unitary operators with, by the definition of adjoint in Weyl algebras, strongly convergent adjoints. If ω_q is primary, and so the above centre is made of multiple of the identity, we must have

$$U = \lambda \mathbb{1} \qquad |\lambda| = 1.$$

But $F_0 \neq 0$ in $\overline{V}^{\sigma q}$ and since σ is already degenerate on F_0, necessarily $q(F_0) \neq 0$ and so

$$1 = |\lambda| = |(\psi_{\omega_q}, U\psi_{\omega_q})| = \lim_\alpha \exp\left(-\frac{1}{4}q(F_\alpha)\right) = \exp\left(-\frac{1}{4}q(F_0)\right) < 1$$

where the last equality follows again from **Lemma A.4**.

Hence ω_q is not primary.

q.e.d.

We end the discussion about factoriality by studing whether it is possible, given a primary g.q.s. ω_q on $\mathcal{A}(V,\sigma)$, to characterize its primary extension to $\mathcal{A}(\overline{V}^{\sigma q},\sigma)$. It turns out that such an extension is unique up to phases.

Proposition A.7 *Let $\mathcal{A}(V,\sigma)$ be a CCR *-algebra, ω_q a* primary *g.q.s. on it. Then primary states exist extending it to $\mathcal{A}(\overline{V}^{\sigma q},\sigma)$ and they all are of the form*

$$\Omega_\phi(\delta(F)) := \exp(i\phi(F))\exp(-\frac{1}{4}q_e(F)) \qquad \forall F \in \overline{V}^{\sigma q} \tag{A.2}$$

with $q_e(\cdot)$ the unique continuous extension of $q(\cdot)$ to $\overline{V}^{\sigma q}$ and $\phi(\cdot)$ a real additive functional on $\overline{V}^{\sigma q}$. Furthermore, $\mathcal{A}(V,\sigma)$ is strongly dense in $\mathcal{A}(\overline{V}^{\sigma q},\sigma)$ in any representations defined by the Ω_ϕ.

Proof. We organize the proof in three steps.
1. Let $\Omega_0(\delta(F)) := \exp(-\frac{1}{4}q_e(F)) \quad \forall F \in \overline{V}^{\sigma q}$. Then the g.q.s. Ω_0 on $\mathcal{A}(\overline{V}^{\sigma q},\sigma)$ is primary iff ω_q is primary on $\mathcal{A}(V,\sigma)$. Indeed **Lemma A.4** and the definition of $\overline{V}^{\sigma q}$ immediately imply that $\pi_{\omega_q}(\mathcal{A}(V,\sigma))$ is strongly dense in $\pi_{\Omega_0}(\mathcal{A}(\overline{V}^{\sigma q},\sigma)$. In particular the centres of the associated Von Neumann algebras coincide and this proves our statement.
2. Ω_0 primary implies Ω_ϕ primary, for every real additive functional $\phi(\cdot)$. Indeed, if $\phi(\cdot)$ is such (linearity is not needed!!), then $\exp(i\phi(\cdot))$ defines an *automorphism* of $\mathcal{A}(\overline{V}^{\sigma q},\sigma)$. Hence Ω_ϕ is obtained from Ω_0 by composition with an automorphism and this proves our statement.
This way we have shown that all states Ω_ϕ are primary.
3. Let, conversely, Ω be a primary state extending ω_q to $\mathcal{A}(\overline{V}^{\sigma q},\sigma)$ (it exists by **1.**). We want to show that $\Omega = \Omega_\phi$ for some real additive functional ϕ.. Let then $F \in \overline{V}^{\sigma q}$. If $\{F_\alpha\}$ is a net σq-convergent to F, it is clear that, for every $G \in \overline{V}^{\sigma q}$,

$$\lim_\alpha [\delta(F_\alpha - F), \delta(G)] = 0 \tag{$*$}$$

in the C^*-norm sense (the unique C^*-norm on $\mathcal{A}(\overline{V}^{\sigma q},\sigma)$: ω_q is primary and hence the continuous extension of σ to $\overline{V}^{\sigma q}$ is nondegenerate!); indeed $\sigma(F_\alpha - F, G) \to 0 \quad \forall G \in \overline{V}^{\sigma q}$ by definition.

By the same reason and using **Lemma A.4** we obtain

$$s - \lim_\alpha \pi_\Omega(\delta(F_\alpha - F)) = s - \lim_\alpha \pi_\Omega(\delta(F_\alpha))\pi_\Omega(\delta(F)).$$

Since π_Ω is a factor it follows from $(*)$ that

$$s - \lim_\alpha \pi(\delta(F_\alpha)) = \exp(-i\phi(F))\pi_\Omega(\delta(F))$$

where $\phi(F) \in \mathbb{R}$ by the already observed unitarity of this strong limit.

As a consequence, fixed anyhow $F \in \overline{V}^{\sigma q}$,

$$\Omega(\delta(F)) = \exp(i\phi(F))\Omega(s - \lim_{\alpha} \pi_\Omega(\delta(F_\alpha)) =$$
$$= \exp(i\phi(F)) \lim_{\alpha} \omega_q(\delta(F_\alpha)) = \exp(i\phi(F)) \lim_{\alpha} \exp(-\frac{1}{4} q(F_\alpha)) =$$
$$= \exp(i\phi(F)) \exp(-\frac{1}{4} q_e(F)).$$

This shows also that $\phi(F)$ is independent of the net used to approximate F. The additivity of $\phi(\cdot)$ comes from the fact that $\exp(i\phi(\cdot))$ is obtained as a strong limit and that the strong operators topology is jointly continuous on the unit ball. By the above construction it is also clear that $\mathcal{A}(V,\sigma)$ is strongly dense in $\mathcal{A}(\overline{V}^{\sigma q}, \sigma)$ in all representations defined by the $\Omega'_\phi s$.

q.e.d.

In the very same way one can prove the following, useful in applications

Corollary A.8 *Let (V,σ), (W,σ') be symplectic spaces such that $V \subset W$, $\sigma'_{|V\times V} = \sigma$. Let ω_q be a primary g.q.s. on $\mathcal{A}(V,\sigma)$ and let V be σq-dense in W. Then primary states exist extending ω_q to $\mathcal{A}(W,\sigma')$ and they all are as in formula (A.2) where q_e is the unique continuous extension of q to W and ϕ a real additive functional on W.*

q.e.d.

Pure states

Our aim is to characterize the set of pure g.q.s.. We remember to this end the one given in the regular case, introducing also the relative notation (we always refer to [**MAN2**]).

Let $\mathcal{Q}$ be the set of quasifree states on the CCR *-algebra $\mathcal{A}(V,\sigma)$ associated to g.q.f. q satisfing the following condition

q is a finite g.q.f. over the nondegenerate symplectic space (V,σ) and the continuous extension of σ to $\overline{V}^q$ is nondegenerate.

Then (*) $D_q \in \mathcal{B}(\overline{V}^q)$ exists such that $D_q^\dagger = -D_q$, $D_q^\dagger D_q \leq \mathbb{1}$ and $\sigma(\cdot,\cdot) = [\cdot, D_q \cdot]_q$ on $\overline{V}^q$. The polar decomposition $D_q = J|D_q|$ gives $J^2 = -\mathbb{1}$ and $J^\dagger = -J$.

One shows that $A_q = -D_q$ exists with a dense domain and that it is a normal operator on it. Furthermore, one obtains that $A_q = J|A_q|$ with $|A_q| = |D_q|^{-1} \geq \mathbb{1}$; $A_q^\dagger A_q \geq \mathbb{1}$.

Remember that if $D_q^\dagger D_q = \mathbb{1} = A_q^\dagger A_q$ then $|D_q| = |A_q| = \mathbb{1}$ by the uniqueness of the positive square root. By the above, this is equivalent to $D_q^2 = -\mathbb{1} = A_q^2$.

We arrive finally at the standard characterization:

$\omega_q \in \mathcal{Q}$ is pure iff $A_q^\dagger A_q = \mathbb{1}$.

Our generalization is based on the following

(*) prop. 3 in [**MAN2**] and following ones.

Definition A.9 Let q be a g.q.f. on a symplectic space (V, σ). q is said minimal on (V, σ) if there is no g.q.f. q' on (V, σ) different of q such that

$$q'(F) \leq q(F) \qquad \forall F \in V \tag{A.3}$$

The existence of minimal forms in the space of g.q.f. over the generic space (V, σ) is a direct consequence of Zorn's lemma. Our goal is to prove the following

Proposition A.10 *Let q be a g.q.f. over the symplectic space (V, σ). Then the g.q.s ω_q on $\mathcal{A}(V, \sigma)$ is pure exactly when q is minimal.*

The proof is given by a series of lemmas, the first following immediately. Let us denote with $Q(V, \sigma)$ the set of g.q.f. over the symplectic space (V, σ).

Lemma A.11 *Let $q \in Q(V, \sigma)$. If q is minimal then $q_{|V_q}$ it is also minimal on (V_q, σ) and maximally regular in (V, σ).*
Conversely, every $q \in Q(V, \sigma)$ which is not minimal is contained in one of the following disjoint sets:
$Q_1(V, \sigma) = \{q \in Q(V, \sigma) : q_{|V_q}$ is maximally regular in (V, σ) but there is a g.q.f dominated by it, in the sense of (A.3), on $(V_q, \sigma)\}$.
$Q_2(V, \sigma) = \{q \in Q(V, \sigma) : q_{|V_q}$ is not maximally regular in $(V, \sigma)\}$.

q.e.d.

The first step consists in the proof that we are generalizing the standard treatement. We need a preliminary result.

Lemma A.12 *Let q be a finite g.q.f. over the nondegenerate symplectic space (V, σ). If q is minimal then the continuous extension of σ to $\overline{V}^q$ is nondegenerate.*

Proof. We notice that nondegeneracy of σ on V, the positivity condition (1.4) and finiteness of q immediately imply that also q is nondegenerate on V. This is true also for the inner product $[\cdot, \cdot]_q$.
Suppose now that σ is degenerate on $\overline{V}^q$: in this case zero is an eigenvalue of the operator D_q and so $D_q^\dagger D_q < \mathbb{1}$ (strictly); hence $|D_q| < \mathbb{1}$. Let's introduce then the sesquilinear form on V

$$[\cdot, \cdot]_{q_{min}} := [\cdot, |D_q| \cdot]_q.$$

It is easily verified that it is a finite g.q.f. on (V, σ): one needs only to notice that $\sigma(\cdot, \cdot) = [\cdot, J \cdot]_{q_{min}}$ on V with J normal and $||J|| \leq 1$.
On the other hand, it follows at once from $|||D_q||| \leq 1$ that, for every $F \in V$,

$$q_{min}(F) := [F, F]_{q_{min}} \leq q(F).$$

By the minimality hypothesis we have then that $q_{min}(\cdot) = q(\cdot)$ on V and by density this equality is true on $\overline{V}^q$, too. But this means $|D_q| = \mathbb{1}$ and we are arrived to an absurd. This proves our statement.

q.e.d.

Lemma A.13 *Let q be a finite g.q.f. over the nondegenerate symplectic space (V, σ). Then $\omega_q \in \mathcal{Q}$ is pure iff q is minimal.*

Proof. We argue by contradiction; first we prove the necessity part.
Let then $\omega_q \in \mathcal{Q}$ be nonpure. Hence $D_q^\dagger D_q < \mathbb{1}$ strictly and the proof is the same as in the previous lemma.

Conversely, let $q \in Q(V, \sigma)$ be finite and nonminimal, and let q_m be a finite and minimal g.q.f. in $Q(V, \sigma)$ dominated by it. We have then on $V \times V$

$$\sigma(\cdot, \cdot) = [\cdot, D_q \cdot]_q \qquad \sigma(\cdot, \cdot) = [\cdot, D_{q_m} \cdot]_{q_m}$$

with $D_q^\dagger D_q \leq \mathbb{1}$ and $D_{q_m}^\dagger D_{q_m} = \mathbb{1}$ (this last equality comes from the sufficiency part just proved).
But if q_m is dominated by q, then $I_q \in \mathcal{B}(\overline{V}^q)$ exists such that, on $\overline{V}^q \times \overline{V}^q$,

$$[\cdot, \cdot]_{q_m} = [\cdot, I_q \cdot]_q$$

with $I_q^+ = I_q$ and $0 \leq I_q \leq \mathbb{1}$.
So we have that $D_q = I_q D_{q_m}$ and

$$D_q^\dagger D_q = D_{q_m}^\dagger I_q^\dagger I_q D_{q_m} < \mathbb{1}$$

strictly. Hence ω_q (which is in $\mathcal{Q}$ by the preceding lemma) is not pure.

q.e.d.

We go ahead in generalizing by firstly removing the nondegeneracy assumption in the finite case. As before, we state a preliminary lemma.

Lemma A.14 *Let q be a minimal and finite g.q.f. over (V, σ). Then $\ker \sigma = \ker q$.*

Proof. Finiteness of q and positivity condition (1.4) imply that $\ker q \subset \ker \sigma$. Let then $F_0 \in \ker \sigma$, that is such that $\sigma(F_0, G) = 0 \quad \forall G \in V$. Since $\sigma(\cdot, \cdot) = [\cdot, D_q \cdot]_q$ we have that $F_0 \in \ker D_q = \ker |D_q|$. By the definition of q_{min} in **Lemma A.12**, this implies $q_{min}(F_0) = 0$. But also q is minimal and so $q(F_0) = q_{min}(F_0) = 0$. Hence $F_0 \in \ker q$ and $\ker \sigma \subset \ker q$.
We conclude that $\ker \sigma = \ker q$.

q.e.d.

Lemma A.15 *Let q be a finite minimal g.q.f. on (V, σ). Then the state ω_q is pure on $\mathcal{A}(V, \sigma)$.*

Proof. **Lemma A.14** shows that $\ker \sigma = \ker q$. It easily follows from it that

$$\pi_{\omega_q}(\delta(F_0)) = \mathbb{1} \qquad \forall F_0 \in \ker q. \tag{$*$}$$

Consider now the quotiet space $V/\ker q$. Schwartz's inequality and $\ker \sigma = \ker q$ imply respectively that q and σ are well defined on it. $(V/\ker q, \sigma)$ is thus a nondegenerate symplectic space and q a finite nondegenerate g.q.f. on it; so $\mathcal{A}(V/\ker q, \sigma)$ is a well defined CCR *-algebra and q induces a g.q.s. ω'_q on it. But then from $(*)$ it follows that

$$\pi_{\omega_q}(\mathcal{A}(V,\sigma)) \simeq \pi_{\omega_q}(\mathcal{A}(V/\ker q, \sigma)). \qquad (**)$$

One can verify that q is still minimal on $(V/\ker q, \sigma)$ and so, by **Lemma A.12**, the continuous extension of σ to $\overline{V/\ker q}^q$ is nondegenerate. By **Lemma A.13** ω'_q is pure and $\pi_{\omega_q}(\mathcal{A}(V/\ker q, \sigma)$ is irreducible. Using $(**)$ one concludes that so it is also $\pi_{\omega_q}(\mathcal{A}(V,\sigma))$, that is ω_q is pure.

q.e.d.

We can finally establish

Proposition A.16 *Let (V,σ) be a symplectic space. Then minimal g.q.f. on (V,σ) define pure states on $\mathcal{A}(V,\sigma)$.*

Proof. If q is minimal on (V,σ) then, by **Lemma A.11**, so it is its restriction to (V_q, σ). By **Lemma A.15** the restriction of ω_q to $\mathcal{A}(V_q,\sigma)$ is pure. But the above restriction of q is also maximally regular, by **Lemma A.11**, and so is also the restriction of ω_q. By **Prop. 3.3** we conclude that ω_q is pure as a state on $\mathcal{A}(V,\sigma)$.

q.e.d.

In order to prove the second half of **Prop. A.10**, we take advantage of **Lemma A.11** and we start with

Lemma A.17 *If $q \in Q_1(V,\sigma)$ then ω_q is not pure on $\mathcal{A}(V,\sigma)$.*

Proof. We know that $q_{|V_q}$ comes from an inner product, say $[\cdot,\cdot]_q$. Since $q \in Q_1(V,\sigma)$, let $q_< \in Q(V_q,\sigma)$ dominated by the restriction of q, but different of it. Clearly, $\ker q \subset \ker q_<$. Being a closed subspace of $\overline{V_q}^q$, $\ker q^\perp$ is a Hilbert space, over which $[\cdot,\cdot]_q$ is nondegenerate.

Also $q_<$ comes from an inner product on V_q, and since is dominated by q on it we have

$$[\cdot,\cdot]_{q_<} = [\cdot, A\cdot]_q$$

with $A^\dagger = A$ and $0 \le A < \mathbb{1}$.
Since $\mathbb{1} - A \neq 0$, $\epsilon > 0$ exists such that there exists a onedimensional projection P of $\mathbb{1}-A$ relative to a spectral interval contained in $[\epsilon, 1]$, projecting on a vector $H_P \in V_q$. As a consequence

$$\epsilon P \le \mathbb{1} - A. \qquad (*)$$

We define then on $V_q \times V_q$

$$[\cdot,\cdot]_* := [\cdot,\cdot]_q - \epsilon[\cdot, P\cdot]_q.$$

Clearly, $q_*(F) \le q(F) \quad \forall F \in V_q$, where $q_*(\cdot)$ is defined in an obvious way. Furthermore, $(*)$ implies easily that

$$q_<(F) \le q_*(F) \qquad \forall F \in V_q.$$

We extend q_* to V by $q_*(F) = +\infty \quad \forall F \in V \backslash V_q$ and similarly for $q_<$. It is easily verified they are g.q.f. on (V, σ) and

$$q_<(F) \leq q_*(F) \leq q(F) \qquad \forall F \in V.$$

So also q_* is dominated by q on (V, σ).

On the grounds of **Prop. 2.2**, let ω_α^* the state on$\mathcal{A}(V, \sigma)$ defined by

$$\omega_\alpha^*(\delta(F)) = \begin{cases} \exp\left(i\alpha[H_P, F]_q\right) \exp\left(-\frac{1}{4} q_*(F)\right) & \forall F \in V_q \\ 0 & \text{otherwise} \end{cases}$$

with $\alpha \in \mathbb{R}$. It is easy to verify that

$$\omega_q = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} d\alpha \exp\left(-\alpha^2\right) \omega_\alpha^*$$

so that ω_q is not pure.

q.e.d.

Lemma A.18 *If $q \in Q_2(V, \sigma)$ then the state ω_q is not pure on $\mathcal{A}(V, \sigma)$.*

Proof. If $q \in Q_2(V, \sigma)$ then $G_0 \in V$ and a g.q.f. q' on V exist such that $q'^{(F)} = q(F) \forall F \in V$ except that $q'(G_0) < +\infty$ while $q(G_0) = +\infty$. Following the notations of **Prop.2.2**, fix $V'' \subset V$ such that $V' = V'' + Span(G_0)$. For every $F \in V$, we decompose then with obvious notation

$$F = F_q + \lambda(F) G_0 + F''.$$

Let $\omega_{q'}$ the state on $\mathcal{A}(V, \sigma)$ associated to q'. Since $\lambda(F)$ is a real linear functional on V, the state on $\mathcal{A}(V, \sigma)$ given by

$$\omega_{q'}^\alpha(\delta(F)) = \exp\left(i\alpha\lambda(F)\right) \omega_{q'}(\delta(F)) \qquad \forall F \in V, \ \alpha \in \mathbb{R}$$

is well defined. One easily verifies that

$$\omega_q = w^* - \lim_{n \to +\infty} \frac{1}{2n\pi} \int_{-n\pi}^{n\pi} \omega_{q'}^\alpha.$$

We now exhibit a decomposition of ω_q.

To this end, take $I_{a,0} = [-a, a]$ with $0 < a < \pi$ and $I_{a,n}$ its translate by $2n\pi$, n a relative number. Let then $I_a := \bigcup_n \{I_{a,n}\}$. Let's now define

$$\omega_q^{(1)} := w^* - \lim_{n \to +\infty} \frac{1}{2na} \int_{I_a \cap [-n\pi, n\pi]} \omega_{q'}^\alpha \, d\alpha$$

$$\omega_q^{(2)} := w^* - \lim_{n \to +\infty} \frac{1}{2n(\pi - a)} \int_{\{\mathbb{R} \backslash I_a\} \cap [-n\pi, n\pi]} \omega_{q'}^\alpha \, d\alpha.$$

They are states on $\mathcal{A}(V,\sigma)$ as w^*-limits of sequences of states; furthermore the decomposition

$$\omega_q = \frac{a}{\pi}\omega_q^{(1)} + \frac{(\pi - a)}{\pi}\omega_q^{(2)}$$

holds so that ω_q is not pure.

q.e.d.

Collecting the last two lemmas and **Lemma A.11** we obtain

Proposition A.19 *Let (V,σ) be a symplectic space. Then nonminimal g.q.f. on it define nonpure states on $\mathcal{A}(V,\sigma)$.*

q.e.d.

Proof of **Proposition A.10.** It needs only to do the logical union of **Prop. A.16** and **Prop. A.19.**

q.e.d.

We show that our characterizations of pure and factor g.q.s. are consistent.

Lemma a.20 *Let q be a g.q.f. on (V,σ). If q is minimal on (V,σ) then the continuous extension of σ to $\overline{V}^{\sigma q}$ is nondegenerate.*

Proof. Suppose indeed that the second statement is not true and let $F_0 \in \overline{V}^{\sigma q}$ be an element, different from zero, giving rise to σ degeneracy. Looking at **Lemma A.3**, we decompose $F_0 = F_1 + F_2$, with $F_1 \in \overline{V_q}^{\sigma q}$ and $F_2 \in V'$. Now, F_2 is zero: indeed, the very definition of F_0 implies that

$$\sigma(F_2, G) = \sigma(-F_1, G) \qquad \forall G \in \overline{V}^{\sigma q}.$$

Thus $\sigma(F_2,\cdot)$ would define a bounded linear functional on V_q, equipped with the inner product induced by q. But this implies q being not maximally regular (see **Remark a.** to **Prop. 3.3**) and this contradicts, by **Lemma A.11**, the minimality hypothesis. Now let's consider $V_q/\ker\sigma = V_q/\ker q$ (the equality is true by the minimality hypothesis). If $F_1 \neq 0$, then $\{F_{0,\alpha}\} \in V_q$ exists such that $q(F_\alpha - F_1) \to 0$ but $\lim_\alpha q(F_\alpha) \neq 0$ since the degeneracy condition already holds. Thus $\{F_\alpha\}$ defines an element different from zero in $\overline{V_q/\ker q}^q$ and this contradicts **Lemma A.12**. So $F_0 \equiv 0$ and σ is nondegenerate on $\overline{V}^{\sigma q}$.

q.e.d.

II. FERMIONS BOSONIZATION IN 1+1 DIMENSIONS.

II.0. Introduction.

In general, by "bosonization" it is meant the possibility of constructing, in theories with an infinite number of degrees of freedom, fermionic kinematical variables out of bosonic ones, and conversely.

In one of the two directions, namely to sort bosons out of fermions, the question is simpler and more transparent. One of the basic aspects was pointed out by Schwinger in 1959 [**SCH1**] with the observation of the "paradoxical consequencies" on the algebra of fermionic currents which follows from the most general principles of QFT, in particular from the positivity of the hamiltonian: anomalous commutators appeared, nonzero precisely because of the so-called Schwinger term. These last never come out in first quantization; and that this state of affairs is general enough was realized later.

Simply stated, relativistic hamiltonians like Dirac's one are not lower-bounded; in order to remedy this, one introduces a ground state with the procedure of "filling the Fermi sea" (the above way of second quantizing the system). This entails a redefinition of creation and destruction operators (a canonical transformation): products of operators, and hence commutators, must be rearranged in order to be normal ordered in the new fields operators. This induces subtraction of vacuum expectation values that originates the above anomalous commutators.

This behaviour appears to be common to a large class of relativistic models with Dirac hamiltonians; in any case, being necessary the particular Fock structure of the CAR's, it is not intrinsic, that is algebraic: it needs to make reference to well defined models. Even more, it is not typical of QFT, but more generally of canonical models: in particular, in many-fermions systems it is illuminating the way in which Lieb and Mattis [**MAT**] treat the Luttinger model (onedimensional!!), whose exact solubility is due to this very phenomenon.

Now, the whole mechanism goes when one analyzes the right kinematical variables, in the just cited case the operator density of momentum $\rho(\vec{p})$ or, in QFT, the currents. These are always composite operators in the fundamental fermionic fields (it is just their product nature that allows anomalous terms). One is thus faced the problem of their construction.

By referring to QFT, it is well known that in simple models, and hence in a definite representation of the field algebra, this is possible by following standard procedures: they include normal ordering (this is sufficient for the free fermionic field), limits of non-local expressions obtained with a point splitting (this is necessary in interacting models like Thirring's one, see [**SCH1**], [**JOH**], sect. 4.4 in [**WIG**], [**KLA**]). In gauge field theories models, like Schwinger's one [**SCH2**], it needs also to insert, before the limit, the "olonomy" operator

$$P \exp\,(i \int_x^y A_\mu(z)dz)$$

to make the whole expression gauge invariant.

In 1+1 dimensions, the currents so defined satisfy canonical commutation relations. On this basis explicit identifications between fermionic and bosonic models were proposed. In particular Coleman [COL] establish a correspondence between the Thirring model and the quantum sine-Gordon equation (in certain parameter regions). This happens by comparing the perturbative expansions of the correlation functions of relevant *observable* quantities. More precisely, by looking at the Green's functions for $m\bar{\psi}\psi$, on the one hand, and for $\frac{\alpha}{\beta^2}\cos\beta\phi$ on the other, one notices that they coincide if

$$\frac{4\pi}{\beta^2} = 1 + \frac{g}{\pi}$$
$$-m\bar{\psi}\psi = \frac{\alpha}{\beta^2}\cos\beta\phi$$

where g is the coupling constant of the Thirring model.

Further, the commutation relations of $j^\mu(x)$ with $\bar{\psi}\psi$ on the one side, and of $\partial_\mu\phi(x)$ with $\cos\beta\phi$ on the other suggest that

$$\frac{\beta}{2\pi}\epsilon^{\mu\nu}\phi\partial_\nu = j^\mu.$$

It has to be noticed that for $\beta^2 = 4\pi$ one obtains the vacuum sector of the free massive Dirac field. These "bosonization formulas" do not establish per se a bond between the two models: the weak sense in which they holds does not even allow, for instance, of establishing if, at the level of their field operators, these last live in the same Hilbert space. In this sense improvements have been proposed, for the very same pair of systems, by Mandelstam [MAN], but assuming the converse point of view.

Indeed, the other horn of the problem, namely the reconstruction of the foundamental fermionic fields from the current algebra (this side is typical of the QFT) is even more ancient: apart from the pioneering works by Jordan, it was Skyrme in 1961 [SKY] to introduce the argument in QFT. This has been then studied, on the basis of the Thirrig model, in [DELL] and precisely in [MAN]. The basic idea is the following.

Anticommuting variables are obtained from canonical bosonic fields by constructing suitable "vertex operators" (in [MAN], following the current fashion, they were dubbed "soliton operators"), that is, immaginary exponentials of the fields. The word 'suitable' indicates which condition one has to impose in order to obtain variables that satisfy the CAR's. It consists, apart of fixing certain constants, in "smearing the exponentiated fields with step functions". There are, so to speak, two ways of doing this. The first one, already due to Skyrme, suggests directly formulas like Mandelstam's one

$$\psi(x) \equiv: \exp\{-2\pi i\beta^{-1}\int_{-\infty}^{x} d\xi\pi(\xi) - \frac{i}{2}\beta\phi(x)\} :$$

and the field π is smeared with the step function $\theta(x-\xi)$ (ϕ and π are the canonical bosonic fields of the sine Gordon).

It is clear that one is able to control such an expression only in a restrict ambit: in this case comparison of commutators with the currents they give rise to and checking that the right equations of motion are satisfied. The whole job is done in terms of correlation functions for a class of models. Formulas like the above are so given a meaning only in definite representation, and it is not clear what is the status of the so constructed variables with respect to the bosonic field variables.

In our opinion, it is essential for the control of the above expression to separate the various aspects of the problem.

1. An infrared problem, originated by the fact that one is smearing bosonic field operators with test functions which do not vanish at infinity, and of an essentially algebraic nature, as we will see.

2. An ultraviolet problem, consisting in searching for a sense to give to the limit in which our test functions are going to a step function, where the challenge is trying to prove the limit operator exists in the strongest possible sense, in this case in the strong operator topology on the Hilbert space of the (bosonic) model.

The status of these problems can be understood on the basis of [**DELL**] where the Thirring model is treated, [**STR1**] interested in the (infrared side in the) massless scalar field (and [**STR2**] that gives an account of both) and more recently [**RUIJ**], with a bibliography on this subject (refs. 14-42). The best results obtained in these papers, with respect to the ultraviolet problem, arrive to a proof of the existence of the limit, starting from *special* test functions, in the sense of the strong operator topology *on a dense set* in the Hilbert space of the bosonic model. If this result is natural in [**DELL**], since the Thirring model is not canonical, it seems not to be the best possible in [**RUIJ**], in which the free massive Dirac field, which is satisfying CAR's and hence a *bounded* operator, is reconstructed. As we'll explain in a moment, our improvement will concern also this point.

We have indeed something to say even at an algebraic level. In [**STR1**], [**STR2**] it is proposed that the right environment in order to treat the infrared side is DHR's scheme we have already mentioned. In fact, the bosons-fermions correspondence in 1+1 dimensions can be seen as a face of a basic problem in QFT, namely the different status of charged fields (the fermions) and observables (the currents). About it two opinion lines are available, very differentiated both in the philosophical presuppositions and in the mathematical methods used.

The first of them, Wightman's one, assumes the point of view that *all* types of fields enter in the basic formalism, in particular the fermionic (that is *charged*) ones. The fields are then the fundamental objects (and hence use of operator and distribution-theoretic techniques).

Alternatively, Haag and Kastler *algebraic* approach takes the point of view in which the basic object is the algebra of local observables (with its associated state space). DHR construction is the proof that this program is a practicable one, at least for localizable charges. It is very important to observe that in DHR's "charged field" is a *derived* concept, not only in a suborder with respect to the observables, but even consequent, in its *construction*, to the structural properties of the charged sectors, which are exhibited first: thus charged fields are *always* constructed in a definite representation of the observable algebra.

We are thinking that in the case of bosonization something better is possible (nearer to Wightman's approach). We show in fact how, on the basis of **part I**, one can construct, from the current algebra in its abstract form, with *algebraic* methods, that is as elements of a *strong closure* of an *extension in a CCR ambit* of it, Wightman fermionic *fields*. This without passing from the states, as in DHR.

The strong closure is due to the necessity of operating the above mentioned ultraviolet limit; our qualifying point is that we are able of removing, in the case of canonical fermions, the limitation of the dense set. The use of extensions in a CCR ambit, that is of an infrared construction on the observable algebra, gives us the tools we need to formulate and to directly solve the problem of the construction of charged fermionic fields.

We now briefly sum up the contents of this **part II.**

In **Section II.1** we set up a treatement of the algebraic properties of the problem. This is the infrared side already cited. We rely here upon the structures we have developped in **part I.** In particular we give motivations for the identification of $\mathcal{A}(\partial\mathcal{S}\times\mathcal{S},\sigma_0)$ as the "current algebra" relevant to our purposes. We introduce the algebra $\mathcal{A}(\mathcal{S}\times\partial^{-1}\mathcal{S},\sigma_1)$ and give reasons, completing the analysis carried through the examplifications in **Sections I.2 I.3**, for its recognizing as the algebra of charged fields naturally related to our current algebra. In particular, we study the commutation properties of elements localized in disjoint intervals. This leads us to "ulraviolet cutoff fermions". We show in two steps that one can remove the ultraviolet cutoff in a large class of representations of the current algebra. Working first in a definite representation we state then the main proposition of this part, namely the one showing that the ultraviolet limit already discussed exists in the ultrastrong operator topology. Finally, we identify a class of representations in which one can control this very problem for free, on the basis of this result.

The whole **Section II.2** is devoted to the proof of the main proposition. The representation π of $\mathcal{A}(\partial\mathcal{S}\times\mathcal{S},\sigma_0)$ we use is that induced by the vacuum state of the massless scalar field. We construct canonical fermions whose correlation functions coincide with those of the free massless Dirac field.

In **Section II.3** we construct the canonical (chiral and fermionic) currents from these very fields. We show then that they coincide, at least in a weak sense on a dense set, with Stone's generators of the Weyl operators defining the algebra $\pi(\mathcal{A}(\partial\mathcal{S}\times\mathcal{S},\sigma_0))$.

II.1. Extended CCR algebras and fermionic degrees of freedom.

The extensions in a CCR ambit introduced in **Section I.3.1** induce representations of the starting algebra containing states which are *charged* with respect to a suitable group of *-automorphisms, constructed togheter with the extension (see **Section I.3.2**). We show in this section how to construct, given a CCR algebra which is interpreted as the algebra of fermionic currents in 1+1 dimensions, extensions of it in a CCR ambit giving rise to operators with *fermionic* quantum numbers. In the next section we'll show that building canonical *fermion fields* starting from a canonical pseudoscalar field in 1+1 dimensions reduces, in this framework, to the control of a particular ultraviolet limit, on the grounds of the extended algebra of this section.

The choice of the "current algebra" is based on the well known equal time commutation relations for the currents of free massless fermions [**WIG**]

$$[j^\mu(x), j_5^\nu(y)] = \frac{i}{\pi}\delta^{\mu\nu}\delta'(x-y). \tag{1.1}$$

Given the current algebra, we construct an extension that provides "fermionic fields with an ultraviolet cutoff". The identification of this extended algebra is made on the grounds of the explicit form of the automorphisms defined by the chiral charge and the fermionic charge. We'll state then the proposition which will be proved in the next section, namely that *in a class of representation* of the current algebra the ultraviolet cutoff can be removed and the limit exists in the ultrastrong operator topology. The class of representations is defined by the property of being locally quasi equivalent to that induced on the current algebra by the vacuum state of the (massless) bosonic field.

We start by introducing our "current algebra": on it we set up our treatement of the bosons-fermions correspondence. It is

$$\overline{\mathcal{A}(\partial\mathcal{S}\times\mathcal{S},\sigma_0)} \equiv \mathcal{A}_0 \tag{1.2}$$

where the symplectic form σ_0 is defined in (4.14). This choice follows from the *equal time* commutation relations of the canonical fermionic currents

$$[j^\mu(x), j_5^\nu(y)] = \frac{i}{\pi}\delta^{\mu\nu}\delta'(x-y)$$

togheter with the fact that, in 1+1 dimensions, it is true that $\gamma_5\gamma^\mu = \epsilon^{\mu\nu}\gamma_\nu$. It is indeed an easy consequence of these two relations that the algebra generated by the (exponentials of the) chiral current $j_5^\mu(f)$ (or of the fermionic current $j^\mu(g)$) coincides with $\mathcal{A}(\partial\mathcal{S}\times\mathcal{S},\sigma_0)$.

The correspondence is given by

$$j^0(x) = \frac{1}{\sqrt{\pi}}\partial^1\phi(x) \qquad j^1(x) = \frac{1}{\sqrt{\pi}}\pi(x)$$

It is very important to stress that this identification is dynamics-independent (and hence largely model-independent). The choice of a model only enters in the definition of a (vacuum) state, which has consequences on the properties of the *representation* of the currents, for instance determining the breaking of the charges associated to them, as we will see.

In order to identify the "field algebra", we start from the explicit form of the automorphisms defining the chirality and the fermionic number. We use to this end a procedure which substantially consists in costructing the *local charges* associated to our currents. Indeed, let's introduce the following elements of $\mathcal{A}(\partial\mathcal{S}\times\mathcal{S},\sigma_0)$: $\delta(<0,f_R>)$ and $\delta(<\partial f_R,0>)$, where $f_R(x)=f(\frac{x}{R})$ with $f\in\mathcal{D}$ and

$$f(x)=\begin{cases}\frac{1}{\sqrt{\pi}} & |x|<1\\ 0 & |x|>2.\end{cases}$$

Hence there exists the pointwise limit $\lim_{R\to+\infty}f_R(x)$ and it is equal to one for every $x\in\mathbb{R}$. We now search for our field algebra within the class of extensions in a CCR ambit of $\mathcal{A}(\partial\mathcal{S}\times\mathcal{S},\sigma_0)$. Call $\mathcal{A}(V_1,\sigma_1)$ the generic algebra in this class. The choice criterion is thus that the two one-parameter groups of *-automorphisms of $\mathcal{A}(V_1,\sigma_1)$

$$\begin{aligned}\alpha^\lambda_{1R}&:=\delta(<0,\lambda f_R>)^{-1}\delta(G)\delta(<0,\lambda f_R>)\\ \alpha^\mu_{2R}&:=\delta(<\mu\partial f_R,0>)^{-1}\delta(G)\delta(<\mu\partial f_R,0>)\qquad\forall G\in V_1\end{aligned}$$

are well defined in the limit $R\to+\infty$ and nontrivial. Now, the existence of the limit is guaranteed, in the norm sense (for every C^*-norm on $\mathcal{A}(V_1,\sigma_1)$), if there exists the limit of the phases these automorphisms give rise to. At this level, there are more than one possible choices for our field algebra. We are not interested here in classifying them; we make instead a "minimal" choice, looking at the example given in **Section I.3.1**.

We are thus led, in order to introduce fermionic degrees of freedom, to the algebra $\mathcal{A}(V_1,\sigma_1)$. $V_1=\mathcal{S}\times\partial^{-1}\mathcal{S}$ is defined in **Section I.3.1** and σ_1 as in formula (5.7,I). We call $\mathcal{A}$ the C^*-algebra obtained from it by closure in the unique C^* norm. We could still add the constants to the space $\partial^{-1}\mathcal{S}$; we do not consider these possibilities here, since we are mainly interested in discussing the *existence* of fermionic degrees of freedom in CCR algebras. On $\mathcal{A}(V_1,\sigma_1)$ the automorphisms α^λ_{1R} and α^μ_{2R} converge in norm to the automorphisms defined by

$$\begin{aligned}\alpha^\lambda_1(\delta(G))&:=\lim_{R+\infty}\alpha^\lambda_{1R}(\delta(G))=\exp\left(i\sqrt{2}\lambda\tilde{g}_1(0)\right)\delta(G)\\ \alpha^\mu_2(\delta(G))&:=\lim_{R+\infty}\alpha^\mu_{2R}(\delta(G))=\exp\left(i\sqrt{2}\mu(\partial\tilde{g}_2)(0)\right)\delta(G).\end{aligned}$$

We notice finally that the group of automorphisms generated by α^λ_1 and α^μ_2 coincides with the gauge group G_{V_1/V_0} (see **Section I.3.1**) associated to $\mathcal{A}(V_1,\sigma_1)$ as extension of $\mathcal{A}(\partial\mathcal{S}\times\mathcal{S},\sigma_0)$ in a CCR ambit.

From this analysis it follows that we can say that $\delta(G)\in\mathcal{A}(V_1,\sigma_1)$ has chiral and fermionic charges

$$<q_1,q_2>:=\sqrt{2}<\tilde{g}_1(0),(\partial\tilde{g}_2)(0)>\,. \tag{1.3}$$

The fact that we are on the right way is confirmed by the following observations:

The generators of $\mathcal{A}$ define a class of *localized* automorphisms on $\mathcal{A}_0$. Before seeing this, a little discussion of the localization properties of this last is in order. Indeed, it is important the following remark. Since $\mathcal{A}_0$ is labelled by a function space, the locality properties of the single generator is determined by the union of the supports of the pair $< f_1, f_2 >$ labelling it. But if $Suppf_1$ is not connected, the condition $\int f(x)dx = 0$ defining ∂S may fail to be true on every single component. Hence, given an open interval I, one must to request that the whole interval spanned by $Suppf_1$ is contained in I, in order to $\delta(< f_1, 0 >)$ being properly localized in it. For this reason, in the following, when dealing with localization properties, we'll always refer to intervals of the real line.

Thus, following the notations of **Section I.3.1**, we notice that $\delta(G)$, $G \in V_1$, implements in $\mathcal{A}$ the *-automorphism $\tau_G^{\sigma_1}$ acting on $\mathcal{A}_0$ by

$$\tau_G^{\sigma_1}(\delta(F)) = \exp\left(i\sigma_1(G,F)\right)\delta(F) \qquad \forall F \in V_0. \tag{1.4}$$

Fixed anyhow $G \in V_1$, $\tau_G^{\sigma_1}$ is a *local* automorphism of $\mathcal{A}_0$, localized in every interval I_G containing $Suppg_1 \cup Supp(\partial g_2)$.

Indeed, it is immediately verified, thanks to the explicit form of σ_1, to the factor ∂S in the product defining V_0 and to the fact that G is locally constant on $\mathbb{R} \backslash I_G$, that

$$\tau_G^{\sigma_1}(\delta(F)) = \delta(F) \tag{1.5}$$

whenever $\delta(F) \in \mathcal{A}_0$ is localized in an interval I_F disjoint from I_G. It is important to notice that, if $G_0 \in V_0$, $\tau_{G_0}^{\sigma_1}$ is localizable in every interval in which also $\delta(G_0)$, its implementer in $\mathcal{A}_0$ as inner automorphism, is. It is important to stress the above localization property, since our aim is to construct canonical fermions with the aid of $\mathcal{A}$: these fields must be local with respect to the current algebra they give rise to.

We continue our analysis of the properties of $\mathcal{A}$ which are relevant for our purposes by sketching the commutation properties of the generators of the abstract algebra $\mathcal{A}$. We are mainly interested in those between elements with the same charge. Given $F, G \in V_1$, this means that

$$\begin{aligned} \frac{1}{\sqrt{\pi}}\int f_1(x)dx = \frac{1}{\sqrt{\pi}}\int g_1(x)dx \qquad (= q_1) \\ \frac{1}{\sqrt{\pi}} f_2(+\infty) = \frac{1}{\sqrt{\pi}} g_2(+\infty) \qquad (= q_2) \end{aligned} \tag{1.6}$$

Let then $\delta(F)$ be localized in I_F and $\delta(G)$ in I_G, with $I_F \cap I_G = \emptyset$. It is then immediate that

$$\delta(F)\delta(G) = \exp\left(\mp i\pi q_1 q_2\right)\delta(G)\delta(F) \tag{1.7}$$

where the minus sign holds if I_G is on the left of I_F and the plus sign conversely. In particular

Proposition 1.1 *Let $F, G \in S \times \partial^{-1}S$, and let (1.6) hold, with $q_1 q_2 = (2n+1)$, n a relative integer. Then*

$$\{\delta(F), \delta(G)\} = 0 \tag{1.8}$$

if $I_F \cap I_G = \emptyset$.

q.e.d.

This way, $\mathcal{A}$ has good properties in order to exhibit fermionic degrees of freedom. The features discussed up to now are representation independent. The construction of the CAR algebra is tied to the introduction of a class of representations of $\mathcal{A}$. To this end it is indeed necessary to adopt a limiting procedure, of a local (ultraviolet) character. This limit does not exists in the norm sense, and we are led to introduce representations. Here is the procedure we'll use.

As a first step, we choose a reference representation of $\mathcal{A}(\partial\mathcal{S} \times \mathcal{S}, \sigma_0)$, namely the one defined by the vacuum state of the massless scalar field in 1+1 dimensions. We follow the notations of **Sections I.2.3, I.3.1** and **I.3.2**, so that this state is denoted by ω_0 and Ω the unique (by **Lemma 6.6**) extension of it to $\mathcal{A}(V_1, \sigma_1)$.

Chosen $f \in \mathcal{D}(\mathbb{R})$, $f(x) \geq 0$, with $\int f(x)dx = \sqrt{\pi}$, let $f_a(x) := \frac{1}{a} f(\frac{x}{a}) \quad \forall x \in \mathbb{R}, a \in \mathbb{R}^+$. By defining

$$F_y^a := < f_a(x-y), (\theta * f_a)(x-y) > \tag{1.9}$$

we obtain an "approximated (right-handed) fermion localized in y". An "approximated left-handed fermion" is defined by

$$G_y^a := < -f_a(x-y), (\theta * f_a)(x-y) > \tag{1.10}$$

Then the construction of the CAR algebra results from the following

Proposition 1.2 *Let F_y^a and G_y^a as defined in (1.9), (1.10). Then, with C a suitable constant and if $g \in \mathcal{S}$,*

$$\psi_a^r(g) := \frac{C}{a^{\frac{1}{2}}} \int \pi_\Omega(\delta(F_y^a)) g(y) dy \tag{1.11}$$

defines, $\forall g \in \mathcal{S}$, a family of operators which are uniformly bounded in norm. The limit for $a \to 0^+$ exists in the ultrastrong (equivalently, in the strong) operator topology on $\mathcal{H}_\Omega$. It defines a canonical (right handed) fermion $\psi^r(f)$:

$$\{\psi^r(f)^\dagger, \psi^r(g)\} = (f, g) \tag{1.12}$$

$$\{\psi^r(f), \psi^r(g)\} = 0. \tag{1.13}$$

The same is true for left-handed fermions $\psi_a^l(f)$ in terms of $\delta(G_y^a)$, with the same C. It also holds that

$$\{\psi^r(f), \psi^l(g)\} = 0 \tag{1.14}$$

Proof. It is the content of the next section.

q.e.d.

As a second step, we want now to show how to characterize a class of representations in which the same strong convergence result holds. We associate to every open subset $\mathcal{O} \in \mathbb{R}$ the local CCR C^*-algebra $\mathcal{A}_0(\mathcal{O})$ defined as norm closure of the *-subalgebra of $\mathcal{A}(\partial\mathcal{S} \times \mathcal{S}, \sigma_0)$ labelled by functions $F \in \partial\mathcal{S} \times \mathcal{S}$ such that $Supp f_1 \cup Supp f_2 \in \mathcal{O}$.

Proposition 1.3 *Let ω be a state on $\mathcal{A}_0$ which is locally normal with respect to ω_0 ($\mathcal{A}_0$ is equipped with the above local structure). Choose anyhow an extension of it to $\mathcal{A}$, call it ω'. Define $\psi_a(g)_{(\pi_{\omega'})}$ as above, using the representation $\pi_{\omega'}$ (we do not explicitely write the superscript l or r). Then $\psi_a(g)_{(\pi_{\omega'})}$ is convergent, as $a \to 0^+$, in the ultrastrong operator topology on $\mathcal{B}(\mathcal{H}_{\omega'})$. Moreover, equations (1.12), (1.13) and (1.14) hold for the limiting operators.*

Proof. We refer to the proof of the previous proposition given in the next section.

Lemmas 2.1 and **2.2** are independent of the representation one uses, as we already noticed. Concerning the other steps, let's introduces the operator $\pi_\Omega(\delta(\hat{\rho})) \in \pi_\Omega(\mathcal{A})$. We first show that the convergence of $\psi_a(g)$ in the ultrastrong operator topology on $\mathcal{B}(\mathcal{H}_\Omega)$ is equivalent to the convergence of $\pi_\Omega(\delta(-\hat{\rho}))\psi_a(g)$ in the ultrastrong operator topology on $\mathcal{B}(\mathcal{H}_{\omega_0})$. Notice first that, obviously, $\pi_\Omega(\delta(\hat{\rho}))$ has the same (fermion and chiral) charges as $\psi_a(g)$. It is clear that all the lemmas in the next section remain true when we use

$$\pi_\Omega(\delta(-\hat{\rho}))\psi_a(g) \in \pi_{\omega_0}(\mathcal{A}_0)$$

instead of $\psi_a(g)$ and we refer to the vacuum sector. The only nonobvious point is **Lemma 2.3**, where we explicitely use vectors $\psi \in \mathcal{H}_\Omega$. But it is clear, by the first observation in its proof, that only the vectors ψ such that $\pi_\Omega(\delta(-\hat{\rho}))\psi \in \mathcal{H}_{\omega_0}$ are relevant. Moreover, $\pi_\Omega(\delta(\hat{\rho}))D$ is dense in $\mathcal{H}_{\omega_0}$. This implies we can use the very same proof as above and hence $\pi_\Omega(\delta(-\hat{\rho}))\psi_a(g)$ converges in the ultrastrong operator topology on $\mathcal{B}(\mathcal{H}_{\omega_0})$.

But $\pi_\Omega(\delta(\hat{\rho}))$ is a *fixed* unitary operator and hence the convergence of $\psi_a(g)$ in the ultrastrong operator topology on $\mathcal{B}(\mathcal{H}_\Omega)$ is equivalent to the convergence of $\pi_\Omega(\delta(-\hat{\rho}))\psi_a(g)$ in the ultrastrong operator topology on $\mathcal{B}(\mathcal{H}_{\omega_0})$. This way, we have shown that our whole proof can be carried through with out leaving the vacuum sector, and estimating only the convergence of uncharged operators.

Choose now $f \in \mathcal{D}$. Then, since $\rho(x)$ has compact support, $\pi_\Omega(\delta(-\hat{\rho}))\psi_a(x)$ is in $\pi_{\omega_0}(\mathcal{A}_0(\mathcal{O}))$ for some $\mathcal{O} \in \mathbb{R}$ and hence there is a suitable $\mathcal{O}^1$, determined by $\mathcal{O}$ and $Suppf$, such that

$$\pi_\Omega(\delta(-\hat{\rho}))\psi_a(g) \in \pi_{\omega_0}(\mathcal{A}_0(\mathcal{O}^1))''$$

for every $a > 0$. Hence also the strong limit of this expression belongs to $\pi_{\omega_0}(\mathcal{A}_0(\mathcal{O}^1))''$. Take now a representation π of $\mathcal{A}$ and consider the operator $\pi(\delta(-\hat{\rho}))\psi_a(g)_{(\pi)}$. By the above, it is ultrastrongly convergent in all representations π of $\mathcal{A}_0$ which define on the local algebras $\mathcal{A}_0(\mathcal{O}^1)$ the *same* ultrastrong operator topology as π_{ω_0}. Among these representations it is well known that there are the ones arising, by GNS construction, from states ω on $\mathcal{A}_0$ which are *locally normal* with respect to ω_0. Hence, by the same argument as before, $\psi_a(g)_{(\pi_{\omega'})}$ is ultrastrongly convergent on $\mathcal{H}_{\omega'}$, with ω' an arbitrary extension of ω to $\mathcal{A}$. In the same way anticommutators are controlled. Indeed, they are obtained as strong limits in π_Ω of the corresponding approximated expressions. Moreover, using a suitable charged operator as $\pi_\Omega(\delta(-\hat{\rho}))$ above (the anticommutator $\{\psi_a(g)^\dagger, \psi_a(g)\}$ needs not it!) we can reduce the proof to the vacuum representation of suitable local algebras an then use the above argument in order to extend the class of representations.

q.e.d.

II.2. Local fermion fields as strong limit of extended Bose fields.

This section contains the proof of **Prop. II.1.2** stated in the preceding section: we construct canonical Fermi fields as strong limits of extended Bose fields, in the representation defined by the vacuum state of the massless scalar field in 1+1 dimensions. This fact gives an answer to a question which was unsettled. It was indeed clear how to construct "approximated fermion fields" as in the preceding section, where we made this by introducing the extended CCR algebra $\mathcal{A}$. In order to recover canonical fermions, for instance by giving a precise meaning to formulas like Mandelstam's one (II.0.1), it is necessary to take the ultaviolet limit discussed in the introduction to this part. The question is -in suitable representations of $\mathcal{A}$, does this limit exists in the strong operator topology? Up to now, the question was unsettled, with propehension for a negative answer (see for instance [**STR2**] or [**RUIJ**]).

Since canonical fermions are bounded operators, all algebraic manipulations (like constructing anticommutators) we do on the approximating fermion-like fields commute with the strong limit (see [**DIXV**]). Hence, if we want to multiply our fermions, one needs only to multiply the approximating fields and to take the limit, with gain in simplicity. Since the strong and the ultrastrong operator topology coincide on the unit ball in $\mathcal{B}(\mathcal{H})$, the existence of the limit in the former implies the existence of the limit in the latter. We'll use always the word "strong" in this section.

In order to prove **Prop. II.1.2** we start by collecting our notations. Our "approximated fermi fields" are suitable elements in the CCR C^*-algebra $\mathcal{A} := \overline{\mathcal{A}(\mathcal{S} \times \partial^{-1}\mathcal{S}, \sigma_1)}$. this algebra is an extension in a CCR ambit of the CCR C^*-algebra $\mathcal{A}_0 := \overline{\mathcal{A}(\partial\mathcal{S} \times \mathcal{S}, \sigma_1)}$ (the current algebra). We take the strong limit using first the representation of $\mathcal{A}_0$ defined by the vacuum of the massless scalar field, called ω_0. We have seen in **Lemma 5.1,I** that ω_0 admits an unique extension to $\mathcal{A}$, called Ω.

We construct our approximating fermion fields by introducing

$$\hat{\rho}(x) :=< \rho(x), (\theta * \rho)(x) >\in \mathcal{S} \times \partial^{-1}\mathcal{S}$$

with $\rho(x)$ a nonnegative and symmetric function in $\mathcal{D}(\mathbb{R})$, and $Supp\rho(x) \subset [-\frac{1}{2}, \frac{1}{2}]$. The charge of our fermion is determined by $\tilde{\rho}(0) = \frac{1}{\sqrt{2\pi}} \int dx \rho(x) = -\frac{1}{\sqrt{2}}$. Following the interpretation of the preceding section (with the degree of arbitrariness there explained), we can say that $\delta(\hat{\rho})$ has unit chiral and fermionic charge.

Remark. We notice that the above value of the charge implies the approximated anticommutation relations of **Prop. II.1.1**. Fermions with higher charge are constructed, as strong limits, by taking products of charge one fields. Here it is *essential* the already mentioned joint continuity of the product in the strong operator topology, on uniformly bounded operators. Furthermore, it will be clear from our proofs that we are able to control also the case in which the two charges differ by a sign.

In order to take the ultraviolet limit, let $\rho_a(x) := \frac{1}{a}\rho(\frac{x}{a})$, $a \in \mathbb{R}^+$, and let $\hat{\rho}_a(x)$ correspond to it, in the same way as before. Clearly, $\lim_{a\to 0^+} \rho_a(x) = \delta(x)$ in $\mathcal{S}'$.

We set then, fixed $a \in \mathbb{R}^+$,

$$\psi_a(x) := \frac{C}{a^{\frac{1}{2}}} \pi_\Omega(\alpha_x(\delta(\hat{\rho}_a))) \tag{2.1}$$

where C is a suitable constant, to be determined, and α_x is the standard space translation automorphism. This operator is the "approximated canonical fermion localized in x". It has unit chiral and electric charge. It clearly belongs to $\pi_\Omega(\mathcal{A})$. The object we want to estimate is the following

$$\psi_a(f) := \int \psi_a(x) f(x) dx \tag{2.2}$$

with $f \in \mathcal{S}(\mathbb{R})$. This expression is well defined as Bochner integral, since α_x is strongly continuous in x. It follows that $\psi_a(f) \in \mathcal{B}(\mathcal{H}_\Omega)$. We then define $\psi_a(f)^\dagger$ as the operator obtained, in the same way as before, by using $-\hat{\rho}(x)$ instead of $\hat{\rho}(x)$. This is natural in view of the unitarity properties of the generators of $\mathcal{A}$: we simply take the adjoint of the Weyl operator.

We want to show that $s - \lim_{a \to 0^+} \psi_a(f)$ exists on $\mathcal{H}_\Omega$, so that it defines a bounded operator in $\pi_\Omega(\mathcal{A})''$. This is done in the following three steps.

Step 1. $\|\psi_a(f)\|$ is bounded, uniformly in $a \in \mathbb{R}^+$.

Step 2 The strong convergence of $\psi_a(f)$ on the dense set $\mathcal{A}\psi_\Omega$ in $\mathcal{H}_\Omega$ is *implied* by the strong convergence of the vector $\psi_a(f)\psi_\Omega$ in $\mathcal{H}_\Omega$.

Step 3. The vector $\psi_a(f)\psi_\Omega$ converges strongly in $\mathcal{H}_\Omega$, as $a \to 0^+$.

Remark. As it will be clear from the proofs, the first two steps are really independent of the representation one chooses. They are then intrinsic to the structure of $\mathcal{A}$.

Remark. It is very important to notice that one needs only to verify the existence of the $s - \lim_{a \to 0^+}$ for *real* testfunctions f in (2.2). One then extend the result to complex (as it is generally required) f by linearity.

Remark. We have in this way constructed a right spinor. In order to construct left spinors, one needs only to use the function

$$\hat{\rho}_l(x) := < -\rho(x), (\theta * \rho)(x) >,$$

as we have seen in the previous section.

We then explicitely verify that our limiting operator satisfies canonical anticommutation relations. Furthermore, when the dynamics α_0^t is applied to it, it satisfies Dirac's equation for the massless fermionic field. This last point is made simpler by the observation that we are really constructing left and right movers field, since the chiral charge assumes a definite value. We also identify ψ_Ω as the fermionic vacuum. The gauge invariance of fermionic correlations will be a direct consequence of the particular structure of π_Ω, Ω being nonregular: charged products (in one of the two charges) of "approximated fermions" have zero expectation on ψ_Ω, which goes through the strong limit. By **Prop. 1.3** in the previous section, the strong limit exists, once it exists in π_Ω, in all representations of $\mathcal{A}_0$ arising from states which are locally normal with respect to ω_0.

Step 1. We start by noticing that, fixed anyhow $a \in \mathbb{R}^+$, one has

$$\{\psi_a(x), \psi_a(y)\} = 0 \qquad if\ |x - y| > a \tag{2.3}$$

as it follows from **Prop II.1.1**. Hence we have

Lemma 2.1 *$\forall f \in \mathcal{S}$, Then $\|\psi_a(f)\|$ is bounded, uniformly in $a \in \mathbb{R}^+$.*

Proof. We have already noticed that, at fixed a, $\psi_a(f) \in \mathcal{B}(\mathcal{H}_\Omega)$. It follows that

$$\|\psi_a(f)\|^2 \leq \|\psi_a(f)^\dagger \psi_a(f) + \psi_a(f)\psi_a(f)^\dagger\|$$

since both summands in the righthand side are bounded and positive operators on $\mathcal{H}_\Omega$. Hence

$$\|\psi_a(f)\|^2 \leq \|\psi_a(f)^\dagger \psi_a(f) + \psi_a(f)\psi_a(f)^\dagger\| = \| \int dxdy|f(x)f(y)|\{\psi_a(x)^\dagger \psi_a(y)\}\|$$

$$\leq \int_{|x-y|<a} dxdy f(x)f(y)\|\{\psi_a(x)^\dagger, \psi_a(y)\}\| \leq \frac{2C^2}{a}\int_{|x-y|<a} dxdy|f(x)f(y)|$$

where we have used equation (2.3). This expression is finite if $a \in \mathbb{R}^+$ and it is a continuous function of a. Hence our statement follows from

$$\lim_{a\to 0^+} \frac{2C^2}{a}\int_{|x-y|<a} dxdy f(x)f(y) = 4C^2\|f\|_2^2.$$

q.e.d.

It is clear that the proof of this lemma is *independent of the representation we used.* It is a consequence of formula (2.3), which is true in $\mathcal{A}$.

Step 2. The uniform (in a) boundedness of the family $\{\psi_a(f)\}_{a\in\mathbb{R}}$ of operators in $\mathcal{B}(\mathcal{H}_\Omega)$ has a simple consequence: one needs only to show that the strong limit exists on a dense set D in $\mathcal{H}_\Omega$. We select as our D the set of the finite linear combinations of vectors in

$$D_0 := \{\psi \in \mathcal{H}_\Omega : \psi = \pi_\Omega(\delta(G))\psi_\Omega,\ G \in \mathcal{S}\times\partial^{-1}\mathcal{S}\}.$$

This set is dense in $\mathcal{H}_\Omega$ by the very definition of GNS construction. Then our second step will be completed, by finite additivity, with the following

Lemma 2.2 *$\forall G \in \mathcal{S}\times\partial^{-1}\mathcal{S}$, and for any state Ω such that $s-\lim_{a\to 0^+}\psi_a(f)\psi_\Omega$ exists $\forall f \in \mathcal{S}$, $s-\lim_{a\to 0^+}\psi_a(f)\pi_\Omega(\delta(G))\psi_\Omega$ exists $\forall f\in\mathcal{S}$.*

Proof. We show that, as $a \to 0^+$, $\{\psi_a(f)\pi_\Omega(\delta(G))\psi_\Omega\}$ is a Cauchy net. Indeed, we make the crucial observation that

$$\|\{\psi_{a'}(f) - \psi_a(f)\}\pi_\Omega(\delta(G))\psi_\Omega\| = \|\psi_{a'}(e^{i\Phi_{a'}}f) - \psi_a(e^{i\Phi_a}f)\psi_\Omega\|$$

where $\Phi_a(x) := \sigma_1(G, \hat{\rho}_a^x)$, with $\rho_a^x(y) := \rho_a(y-x)$. It is immediately seen that $\Phi_a(x)$ is a C^∞ function in x. The same property holds for $\Phi_0(x) := \lim_{a\to 0^+}\Phi_a(x) = (g_1 * \theta)(x) - g_2(x)$. From this property it follows that $s-\lim\psi_a(e^{i\Phi_0}f)$ exists.

Furthermore, since $\rho_a(x)$ is a C^∞ approximation of the deltafunction, $\Phi_a(x)$ converges uniformly to $\Phi_0(x)$.

By an $\frac{\epsilon}{3}$ argument, one needs then only to show that $\lim_{a\to 0^+}\|\psi_a((e^{i\Phi_a} - e^{i\Phi_0})f)\psi_\Omega\| = 0$. This follows from the previous lemma.

Indeed, let's define $g_a := (e^{i\Phi_a} - e^{i\Phi_0})f$. Then, independently of ψ_Ω,

$$\begin{aligned}
&\|\psi_a(f)((e^{i\Phi_a} - e^{i\Phi_0})f)\psi_\Omega\| \leq \\
&\|\psi_a(f)((e^{i\Phi_a} - e^{i\Phi_0})f)\| \leq \\
&\leq \frac{2C^2}{a}\int_{|x-y|<a} dxdy|g_a(x)g_a(y)| \leq \\
&\leq \frac{2C^2}{a}\sup|e^{i\Phi_a} - e^{i\Phi_0}|^2 \int_{|x-y|<a} dxdy|f(x)f(y)| \longrightarrow 0.
\end{aligned}$$

q.e.d.

As in the previous step, it is clear that this lemma does not involve any property of the chosen representation, apart from the existence of $s-\lim_{a\to 0+}\psi_a(f)\psi_\Omega$. Collecting our first two lemmas, we can reduce the proof of the convergence of $\psi_a(f)$ in the strong operator topology on $\mathcal{H}_\Omega$ to the proof of the strong convergence of $\psi_a(f)\psi_\Omega$ in $\mathcal{H}_\Omega$.

Step 3. It comes now into play the representation π_Ω. In order to show that the family of vectors $\psi_a(f)\psi_\Omega$ is strongly convergent in $\mathcal{H}_\Omega$ we argue in two steps.

a. We show that $\psi_a(f)\psi_\Omega$ converges weakly with respect to the dense set D. Then Lemma **2.1** implies that that it converges weakly.

b. Once $\psi_a(f)\psi_\Omega$ converges weakly, it is well known that strong convergence in $\mathcal{H}_\Omega$ is implied by

$$\lim_{a\to 0+}\|\psi_a(f)\psi_\Omega\| = \|\lim_{a\to 0+}\psi_a(f)\psi_\Omega\|.$$

We notice that this last equation is like to say that

$$\lim_{a\to 0+}(\psi_a(f)\psi_\Omega, \psi_a(f)\psi_\Omega) = \lim_{a\to 0+}\lim_{b\to 0+}(\psi_a(f)\psi_\Omega, \psi_b(f)\psi_\Omega). \tag{2.4}$$

a. We calculate first

$$\lim_{b\to 0+}(\psi_a(f)\psi_\Omega, \psi_b(f)\psi_\Omega). \tag{2.5}$$

It easily follows from (2.2), the rules for the product of elements in $\mathcal{A}$ and the explicit form of Ω that

$$(\psi_a(f)\psi_\Omega, \psi_b(f)\psi_\Omega) = \frac{C^2}{(ab)^{\frac{1}{2}}}\int dxdyf(x)f(y)\exp\left(-T_{a,b}(x-y)\right)\exp\left(iS_{a,b}(x-y)\right)$$

where

$$\begin{aligned}
T_{a,b}(x-y) &= \frac{1}{2}\int\frac{dk}{|k|}|\tilde{\rho}(ak) - \tilde{\rho}(bk)\exp\left(ik(x-y)\right)|^2 \\
S_{a,b}(x-y) &= \int\frac{dk}{k}\sin(k(x-y))\tilde{\rho}(ak)\tilde{\rho}(bk).
\end{aligned}$$

The kernel $(\exp(-T_{a,b}+iS_{a,b}))(x,y)$ is a polinomially bounded and continuous function and hence it defines a distribution. We take now the $b\to 0^+$ limit. As an application of the dominated convergence theorem we obtain

$$\lim_{b\to 0+} S_{a,b}(x-y)=\frac{1}{2}\int\frac{dk}{k}\sin(k(x-y))\tilde{\rho}(ak). \tag{2.6}$$

Moreover, it is immediate that, introducing a change of variables $k\to ak$

$$T_{a,b}(x-y)=\int\frac{dk}{|k|}\{\tilde{\rho}(k)^2-\cos(k(\frac{x-y}{a}))\tilde{\rho}(k)\tilde{\rho}(\frac{b}{a}k)\}+\frac{1}{2}\int\frac{dk}{|k|}\{\tilde{\rho}(\frac{b}{a}k)^2-\tilde{\rho}(k)^2\}.$$

The second integral is easy to calculate by parts and it gives

$$\frac{1}{2}\int\frac{dk}{|k|}\{\tilde{\rho}(\frac{b}{a}k)^2-\tilde{\rho}(k)^2\}=\frac{1}{2}\log\frac{a}{b}.$$

The first integral is easily controlled, in the $b\to 0^+$ limit, by using again the dominated convergence theorem. We obtain as a final result

$$\lim_{b\to 0+}\frac{C^2}{(ab)^{\frac{1}{2}}}\exp(-T_{a,b}(x-y)+iS_{a,b}(x-y))=\frac{C^2}{a}\exp R_{a,0}(x-y)$$

where

$$R_{a,0}(x-y):=2\int_0^{+\infty}\frac{dk}{k}\{\frac{1}{\sqrt{2}}\tilde{\rho}(k)\exp(ik(\frac{x-y}{a}))-\tilde{\rho}(k)^2\}. \tag{2.7}$$

It is clear that this limit exists at least in $\mathcal{S}(\mathbb{R}^2)'$. Indeed, all expressions we use are continuous and polinomially bounded on the whole plane, uniformly in b, and this is true also for the limiting function. Hence, fixed a, we obtain that

$$\lim_{b\to 0+}(\psi_a(f)\psi_\Omega,\psi_b(f)\psi_\Omega)=\frac{C^2}{a}\int dxdyf(x)f(y)\exp R_{a,0}(x-y). \tag{2.8}$$

These calculation are substantially what is needed to prove

Lemma 2.3 *Chosen anyhow $\psi\in D$ and $f\in\mathcal{S}$, there exists the limit*

$$\lim_{a\to 0+}(\psi,\psi_a(f)\psi_\Omega),$$

that is, $\psi_a(f)\psi_\Omega$ is weakly convergent with respect to the dense set D.

Proof. By the definition of D, one needs only to prove our statement for vectors in it of the form $\psi=\pi_\omega(\delta(G))\psi_\Omega$ with $G=<g_1,\theta*g_2>\in\mathcal{S}\times\partial^{-1}\mathcal{S}$. In fact, the fermion number and chiral charge selection rules in π_Ω say that we obtain an identically zero value for this correlation unless $\tilde{g}_1(0) = \tilde{\rho}(0) = \tilde{g}_2(0)$. In this case the calculations are absolutely analogous to those just made and we obtain the well defined (and nonzero) limit

$$\lim_{a\to 0^+}(\pi_\Omega(\delta(G))\psi_\Omega, \psi_a(f)\psi_\Omega) =$$
$$= C\int dx f(x)\exp\Big(\int_0^{+\infty}\frac{dk}{k}\{-[\tilde{\rho}(k)^2 + \frac{1}{2}(|\tilde{g}_1(k)|^2 + |\tilde{g}_2(k)|^2)] + \tag{2.9}$$
$$+\frac{1}{4}\exp(-ikx)(\overline{\tilde{g}}_1(k) + \overline{\tilde{g}}_2(k))\}\Big)$$

so that our statement follows.

q.e.d.

Hence

Lemma 2.4 *Chosen anyhow $f \in \mathcal{S}$, $\psi_a(f)\psi_\Omega$ is weakly convergent in $\mathcal{H}_\Omega$, as $a \to 0^+$.*
Proof. It is a direct consequence of **Lemma 2.1** and **Lemma 2.3.**

q.e.d.

Remark. We will use these two lemmas to prove that, if there exists the strong limit of the family of operators $\psi_a(f)$, it is *independent* of the choice of the function $\rho(x)$. This appears to be surprising, in view of the explicit ρ dependence of equation (2.9). But, integrating by parts the *whole* exponent in it, the ρ contribution to it is given by

$$C' := \int_0^{+\infty} dk \log k \frac{d}{dk}\tilde{\rho}(k)^2.$$

At the end of this section, when dealing with anticommutators, we will show that one *needs* to choose $C = (\frac{e^{-C'}}{2\pi})^{\frac{1}{2}}$. From this it follows immediately that the whole expression (2.9) is really independent of the choice of $\rho(x)$, and hence the limit operator itself is independent of it.

This completes step **a.**

b. We come now to the crucial point of our job. A calculation analogous to the previous ones gives

$$(\psi_a(f)\psi_\Omega, \psi_a(f)\psi_\Omega) = \frac{C^2}{a}\int dx dy f(x) f(y)\exp R_a(x-y)$$

where

$$R_a(x-y) = 2\int_0^{+\infty}\frac{dk}{k}(\exp(ik(\frac{x-y}{a})) - 1)\tilde{\rho}(k)^2. \tag{2.10}$$

We have to show that the $a \to 0^+$ limits of the expressions appearing in equations (2.7) and (2.10) are equal. We start with an observation.

The function f is real. Hence, only $Re\exp R_a(x-y)$ and $Re\exp R_{a,0}(x-y)$ contribute to the respective integrals in (2.8) and (2.10). Indeed, one has only to notice that, since $\tilde{\rho}(k)$ is real, $R_a(x-y) = \overline{R_a(y-x)}$ and $R_{a,0}(x-y) = \overline{R_{a,0}(y-x)}$.

We want to show that, in the $a \to 0^+$ limit, these expressions give rise to well defined distributions and that these distributions are equal. We prove this by showing that

A. For every $a > 0$, both $\frac{1}{a} Re \exp R_a(x)$ and $\frac{1}{a} Re \exp R_{a,0}(x)$ are *positive* distributions.

B. Their supports shrink to the point $x = 0$.

C.

$$\lim_{a \to 0^+} \frac{1}{a} \int dx Re \exp R_a(x) = \lim_{a \to 0^+} \frac{1}{a} \int dx Re \exp R_{a,0}(x).$$

These three points imply that the limit distribution exists and it is proportional to $\delta(x-y)$ for both our expressions and so we would have completed **b.** and **Step 3.**.

A. We already observed that both $\frac{1}{a} Re \exp R_a(x-y)$ and $\frac{1}{a} Re \exp R_{a,0}(x-y)$ are polinomially bounded continuous functions, for every $a > 0$. It is then clear from (2.7) and (2.10) that one needs only to show that the following functions are positive, for $a > 0$:

$$F_{a,0}(x) = \cos \frac{1}{2\sqrt{2}} \int \frac{dk}{k} \sin (k(\frac{x}{a})) \tilde{\rho}(k)$$

and

$$F_a(x) = \cos \frac{1}{2} \int \frac{dk}{k} \sin (k(\frac{x}{a})) \tilde{\rho}(k)^2.$$

This point will be completed if we show that the argument of the cosines in both these expressions is a (nondecreasing) continuous function of x, with range contained in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

Now, it is clear that both arguments are differentiable functions in the variable x, and that

$$\frac{d}{dx} Arg\ F_{a,o}(x) = \frac{1}{2a\sqrt{2}} \int dk \cos (k(\frac{x}{a})) \tilde{\rho}(k) = \frac{\sqrt{\pi}}{2a} \rho(\frac{x}{a}) \geq 0$$

$$\frac{d}{dx} Arg\ F_a(x) = \frac{1}{2a} \int dk \cos (k(\frac{x}{a})) \tilde{\rho}(k)^2 = \frac{1}{2a} (\rho * \rho)(\frac{x}{a}) \geq 0$$

Hence both arguments are nondecreasing. Finally, one easily observes that

$$Arg\ ReF_{a,o}(a) = -Arg\ ReF_{a,0}(-a) =$$
$$= \frac{1}{2\sqrt{2}} Im P \int \frac{dk}{k} e^{ik} \tilde{\rho}(k) = \frac{\sqrt{\pi}}{2} Im(-i) \int \epsilon(1-y) \rho(y) dy = \frac{\pi}{2},$$

having taken into account our convention in the constant of the Fourier transform (chosen so that it is isometric) and the support properties of ρ, togheter with the fact that $\int \rho(x)dx = \sqrt{\pi}$. We notice also that the value of the argument remains $\pm\frac{\pi}{2}$ for $|x-y| > a$ and hence $F_{a,0}(\cdot)$ is zero. In the very same way one shows that, for $z \geq a$,

$$Arg\ ReF_a(z) = \frac{\pi}{2}$$

and the the opposite for $z < -a$. Hence point **A.** is completed.

B. It is an immediate consequence of the results of the previous point.

C. Thanks to the explicit a dependence of our expressions in (2.7) and (2.10) one has only to show that $I_d = I_{nd} \in \mathbb{R}$, with

$$I_d := \int dx \exp 2(\int_0^{+\infty} \frac{dk}{k}(e^{ikx} - 1)\tilde{\rho}(k)^2)$$

and

$$I_{nd} := \int dx \exp 2(\int_0^{+\infty} \frac{dk}{k}(\frac{1}{\sqrt{2}}e^{ikx}\tilde{\rho}(k) - \tilde{\rho}(k)^2)).$$

By integrating by parts, one reduces these integrals to the following ones:

$$I_d = \int dx \exp(-2\int_0^{+\infty} dk \log k \frac{d}{dk}(e^{ikx}\tilde{\rho}(k)^2))$$

and

$$I_{nd} = \int dx \exp(-\sqrt{2}\int_0^{+\infty} dk \log k \frac{d}{dk}(e^{ikx}\tilde{\rho}(k)))$$

up to a common constant $2C' = 2\int_0^{+\infty} dk \log k \frac{d}{dk}\tilde{\rho}(k)^2$. Now, taking into account the constants arising from our definition of Fourier transform, and using the well known Fourier transform of the distribution $\log k \frac{d}{dk}$, we finally obtain

$$I_d = \int dx \exp -2(G * \frac{\rho}{\sqrt{2\pi}} * \frac{\rho}{\sqrt{2\pi}})(x)$$

and

$$I_{nd} = \int dx \exp -\sqrt{2}(G * \frac{\rho}{\sqrt{2\pi}})(x)$$

with $G(x) = \log(-i(x-y)+\epsilon)$. Both integrals can be evaluated easily in the complex x-plane. One has to notice that, thanks to the analiticity properties of the logarithm, the integrand is an analytic function in the upper halfplane $Imx > 0$. We can thus close the contour in it with a big semicircle and do the integral on this curve. Since $\rho(x)$ has compact support, the integrand has a leading term, obtained by taking $\log x$ instead of $\log(x-y)$ in the convolution with $\rho(y)$, with behaviour $\frac{1}{x}$ as $|x| \to \infty$. The error in the exponent is bounded by $O(\frac{1}{R})$ uniformly on a semicircle of radius R and therefore it contributes with a factor which goes to one as we let the contour go to infinity.

The coefficient of the leading therm in the exponent is respectively

$$\int (\frac{\rho}{\sqrt{2\pi}} * \frac{\rho}{\sqrt{2\pi}})(x)dx = 1$$

and

$$\int (\frac{\rho}{\sqrt{2\pi}})dx = 1.$$

The final result is then

$$I_d = I_{nd} = \pi.$$

The result is a real number and hence coincides with the value obtained when we take the real part of the integrand. Thus also point **C.** is complete. As a consequence, we have shown that formula (2.4) is true so that we can state

Lemma 2.5 *Chosen anyhow $f \in \mathcal{S}$, the vector $\psi_a(f)\psi_\Omega$ is strongly convergent in $\mathcal{H}_\Omega$, for $a \to 0^+$.*

q.e.d.

In this way also **Step 3.** is completed.

We have thus shown that it exists

$$\psi(f) := s - \lim_{a \to 0^+} \psi_a(f).$$

Remark. It is important to notice that our whole proof can be applied also to the case in which the chiral and electric charge of the approximating fields $\psi_a(f)$ are equal up to a sign. Indeed, the only change that occurs is a minus sign in front of $S_{a,b}(\cdot)$ and of $S_{a,a}(\cdot)$.

We want briefly to analyze some properties of this operator on $\mathcal{H}_\Omega$.

Anticommutators. In order to treat anticommutators, one first notices that, again by linearity, one needs only to use real testfunctions f. We start by studying the anticommutator of the above constructed right spinor $\psi_a(f)$ with its adjoint. It will be clear from our analysis that the same procedure, with the same results, can be applied to the left spinors. We want then to estimate

$$\{\psi(f)^\dagger, \psi(g)\} \qquad f, g \in \mathcal{S}.$$

Since the field $\psi(f)$ is obtained as strong limit, we can calculate the above anticommutator by using the approximating fields $\psi_a(f)$, and then take the $a \to 0^+$ limit of the result. But this has been already done in our last step **b.**, since it is only the real part of the two point function that gives contributions to the anticommutator. Indeed, we showed that, in $\mathcal{S}'$,

$$\lim_{a \to 0^+} (\psi_a(x)\psi_\Omega, \psi_a(y)\psi_\Omega) = C^2 \pi e^{C'} \delta(x-y).$$

This, togheter with the equation for $\psi_a(x)^\dagger$, implies that, again in $\mathcal{S}'$,

$$\lim_{a \to 0^+} (\psi_\Omega, \{\psi_a(x)^\dagger, \psi_a(y)\}\psi_\Omega) = \delta(x-y)$$

if we choose $C = (\frac{e^{-C'}}{2\pi})^{\frac{1}{2}}$. This is the canonical result for the anticommutator if we are able to show that it is a c-number in π_Ω: in this case one can well calculate its value on the vacuum state and this gives the exact result. We remember to this end that Ω is a pure state on $\mathcal{A}$ and hence π_Ω gives an irreducible representation of this algebra. Our aim is then to show that

$$\{\psi(f)^\dagger, \psi(g)\} \in \pi_\Omega(\mathcal{A})'.$$

Since $\{\psi(f)^\dagger, \psi(g)\}$ is obtained as a strong limit (and hence coincides with the result obtained by taking the weak limit of $\{\psi_a(f)^\dagger, \psi_a(g)\}$) and given the structure of $\mathcal{A}$, one needs only to show that

$$w - \lim_{a \to 0+} [\{\psi_a(f)^\dagger, \psi_a(g)\}, \pi_\Omega(\delta(G))] = 0$$

for every $G \in \mathcal{S} \times \partial^{-1}\mathcal{S}$. Again, by **Lemma 2.1** it is enough to show this on dense set in $\mathcal{H}_\Omega$ and an argument like that in **Lemma 2.2** (this time applied to $[\{\psi_a(f)^\dagger, \psi_a(g)\}, \pi_\Omega(\delta(G))]$) implies that we can restrict ourselves to use the vacuum vector. But then, moving on the left the operator $\pi_\Omega(\delta(G))$ in the first summand of the above commutator, we obtain that we have to show that, with the *same* notations as in **Lemma 2.2**,

$$\lim_{a \to 0+} (\psi_\Omega, \pi_\Omega(\delta(G))\{\psi_a(e^{i\Phi_a} f)^\dagger, \psi_a(e^{-i\Phi_a} g)\}\psi_\Omega) =$$
$$= \lim_{a \to 0+} (\psi_\Omega, \pi_\Omega(\delta(G))\{\psi_a(f)^\dagger, \psi_a(g)\}\psi_\Omega)$$

which is true since

$$\lim_{a \to 0+} (\psi_\Omega, \{\psi_a(x)^\dagger, \psi_a(y)\}\psi_\Omega) = \delta(x-y)$$

in $\mathcal{S}'$ and hence the two phases in the left hand side of the above formula cancel, in the limit. Using exactly the same procedure we obtain the same result for the anticommutator of the left spinor with its adjoint.

The other anticommutators are controlled in the following way. We remember again that they all are obtained as strong limits and so one needs only to study the corresponding approximated expressions. Now, consider, in order to recover (1.14), the expression

$$\{\psi_a^r(x), \psi_a^l(y)\}.$$

It is then easily shown that it is identically zero as a consequence of the fact that

$$\sigma(\hat{\rho}_a(z-x), \hat{\rho}_{a,l}(z-y)) = \pi$$

for every $a > 0$, for every $x, y \in \mathbb{R}$. Hence also the limiting expression is zero, proving (1.14).

In order to recover (1.13), one notices that it is enough, since it is obtained as strong limit and hence coincides with the weak limit, to show that (by arguments we already exposed)

$$\lim_{a \to 0+} (\psi_\Omega, \pi_\Omega(\delta(G))\{\psi_a(f), \psi_a(g)\}\psi_\Omega) = 0$$

where G is chosen so as to obtain a nonzero expression. We do not include here the explicit calculation showing that this is indeed true: it is a consequence of the fact that

$$\Omega(\delta(G)\alpha_x(\delta(\hat{\rho}_a))\alpha_y(\delta(\hat{\rho}_a))) \to 0$$

as $a \to 0^+$.

In this way the proof of **Prop. II.1.2** is completed.

Equations of motion. We consider the natural extension (namely the one defined by applying the matrix T_0^t in (4.16) to the function in V_1 that labels the generic generator of $\mathcal{A}$) of α_0^t (the massless scalar field time evolution) to $\mathcal{A}$. In order to see which time evolution this dynamics induces on $\psi(f)$, one needs only to study what happens to the approximating fields $\psi_a(x)$, since $\alpha_0^t(\cdot)$ is strongly continuous. Now, thanks to our choice for the charge density function $\hat{\rho}(x)$, $\psi_a(x)$ is really a right or left mover, respectively if the chiral charge has the same or the opposite sign with respect to the electric charge.

Indeed, the solutions of the wave equation with initial conditions $< \pm\dot{\rho}(x), \rho(x) >$, as it is in our case, is of the form

$$\rho(x,t) = \rho(x \pm t).$$

Since α_0^t precisely induces such a behaviour on the labelling functions, it follows that

$$\alpha_0^t \psi_a(x) = \psi_a(x \pm t)$$

with the same sign as in

$$< \pm\rho(x), (\theta * \rho)(x) > .$$

By the strong continuity of α_0^t the operator valued distribution $\psi(x)$ defined by the existence of $s - \lim_{a \to 0^+} \psi_a(f)$ for every $f \in \mathcal{S}$ satisfies the massless Dirac's equation

$$\frac{d}{dx}\psi(x,t) = \pm\frac{d}{dt}\psi(x,t)$$

where $\psi(x,t) := \alpha_0^t \psi(x)$. The pure, gauge and chiral invariant state Ω is invariant under α_0^{t*} and hence can be interpreted as the massless fermionic vacuum.

II.3. Fermionic currents and bosonic variables.

Our aim in this section is to identify the algebra generated by the chiral current and by the gauge current associated to the canonical fermionic field constructed, as an element of $\pi_\Omega(\mathcal{A})''$, in the previous section. We work indeed, in this section, only in the representation defined by the vacuum state ω_0. In agreement with the motivations of our construction, this algebra results to coincide with the CCR *-algebra $\mathcal{A}(\partial\mathcal{S} \times \mathcal{S}, \sigma)$.

Remembering that, in 1+1 dimensions, $\gamma_5\gamma^\mu = \epsilon^{\mu\nu}\gamma_\nu$, it is enough to study the gauge current j^μ. We define a "regularized gauge current"

$$j^\mu_{\varepsilon,a}(x) := :\overline{\psi_{\mathbf{a}}(\mathbf{x}+\varepsilon)}\gamma^\mu\psi_{\mathbf{a}}(\mathbf{x}): \qquad \mu = 0,1 \quad \varepsilon > 0$$

where

$$\psi_{\mathbf{a}}(\mathbf{x}) = < \psi^l_a(x), \psi^r_a(x) >$$

with the superscript l (left) associated to the case in which we have a left spinor ($q_{chiral} = -q_{fermion}$) and r (right) associated to the opposite one ($q_{chiral} = q_{fermion}$).

The canonical point splitting as been introduced and : : means subtraction of the expectation value over the vector ψ_{ω_0}; it is essential to notice that $j^\mu_{\varepsilon,a}(x) \in \pi_{\omega_0}(\mathcal{A}_0)$. We show that, *following this order in the limits,*

$$j^\mu(f) := \lim_{\varepsilon\to 0}\lim_{a\to 0} j^\mu_{\varepsilon,a}(f) = -\frac{1}{\sqrt{\pi}}\epsilon^{\mu\nu}\partial_\nu\phi(f) \qquad \forall f \in \mathcal{S} \tag{3.1}$$

where the limit is taken in the sense of the weak operator topology on $D_1 \times D_1$, with D_1 a dense set in $\mathcal{H}_{\omega_0}$. By definition,
$\partial^0\phi(f)$ is Stone's generator of $\pi_{\omega_0}(\delta(< 0, f >))$
$\partial^1\phi(f)$ is Stone's generator of $\pi_{\omega_0}(\delta(< -\partial f, 0 >))$.

It follows that the algebra generated by the (exponentials of the) currents j^μ and j^μ_5 as defined in (3.1) coincides with $\mathcal{A}(\partial\mathcal{S} \times \mathcal{S}, \sigma)$.
We come to the proof of (3.1). We choose $D_1 = Span[\mathcal{A}(\partial\mathcal{S} \times \mathcal{S}, \sigma)\psi_{\omega_0}]$: it is dense in $\mathcal{H}_{\omega_o}$ by definition of GNS construction. We notice that D_1 is in the domain of $\partial_\mu\phi(f)$, $f \in \mathcal{S}$. Hence, chosen anyhow ψ_1 and ψ_2 contained in D_1, we want to show that

$$j^\mu(f) := \lim_{\varepsilon\to 0}\lim_{a\to 0}(\psi_1, j^\mu_{\varepsilon,a}(f)\psi_2) = -\frac{1}{\sqrt{\pi}}\epsilon^{\mu\nu}(\psi_1, \partial_\nu\phi(f)\psi_2) \qquad \forall f \in \mathcal{S} \tag{3.2}$$

In order to do that, we first notice that it is enough, by finite additivity, to verify (3.2) only in the case

$$\psi_1 = \pi_{\omega_0}(\delta(-G^{(1)}))\psi_{\omega_0}; \quad \psi_2 = \pi_{\omega_0}(\delta(G^{(2)}))\psi_{\omega_0} \tag{3.3}$$

with $G^{(1)}, G^{(2)} \in \partial\mathcal{S} \times \mathcal{S}$. It is then easy to calculate the right hand side of (3.2) and the result is
$\mu = 1$

$$\begin{aligned} -\frac{1}{\sqrt{\pi}}\omega_0(\delta(G^{(1)})\delta(G^{(2)}))\{\frac{i}{2}\int dp|p|Re[\tilde{f}(p)(\bar{\tilde{g}}^{(1)}_2(p) + \bar{\tilde{g}}^{(2)}_2(p))]+ \\ \frac{1}{2}\int dp\tilde{f}(p)(\bar{\tilde{g}}^{(1)}_1(p) + \bar{\tilde{g}}^{(2)}_1(p))\}. \end{aligned} \tag{3.4}$$

$\mu = 0$

$$-\frac{1}{\sqrt{\pi}}\omega_0(\delta(G^{(1)})\delta(G^{(2)}))\{\frac{i}{2}\int dp|p|Re[ip\tilde{f}(p)(\overline{\tilde{g}}_1^{(1)}(p)+\overline{\tilde{g}}_1^{(2)}(p))]+ \frac{i}{2}\int dpp\tilde{f}(p)(\overline{\tilde{g}}_2^{(1)}(p)+\overline{\tilde{g}}_1^{(2)}(p))\}. \tag{3.5}$$

We calculate now the left hand side of (3.2).

$\mu = 0$.

We notice that

$$j^0_{\varepsilon,a}(x) = :\psi^l_a(x+\varepsilon)^\dagger\psi^l_a(x) + \psi^r_a(x+\varepsilon)^\dagger\psi^r_a(x):$$

and we remember from the preceding section that

$$\omega_0(\psi^l_a(x+\varepsilon)^\dagger\psi^l_a(x)) = \frac{C^2}{a}\exp 2\int_0^{+\infty}\frac{dp}{p}(e^{-ip\frac{\varepsilon}{a}}-1)\tilde{f}(p)^2 \quad (\equiv S_a(\varepsilon))$$

and

$$\omega_0(\psi^r_a(x+\varepsilon)^\dagger\psi^r_a(x)) = S_a(-\varepsilon).$$

The left spinors contribution to the left hand side of (3.2) is then (including the vacuum subtraction)

$$\begin{aligned}
&\int \omega_0(\delta(G)^{(1)})\psi^l_a(x+\varepsilon)^\dagger\psi^l_a(x)\delta(G^{(2)}))f(x)dx = \\
&= \omega_0(\delta(G^{(1)})\delta(G^{(2)}))\omega_0(\psi^l_a(y+\varepsilon)^\dagger\psi^l_a(y))\cdot\int dxf(x) \\
&\{\exp(\frac{1}{2}\int\frac{dp}{|p|}\tilde{\rho}(ax)Re[(\overline{\tilde{g}}_1^{(1)}(p)+\overline{\tilde{g}}_1^{(2)})e^{-ipx}(1-e^{-ip\varepsilon})]) && (\equiv e^{I^a_1(\varepsilon,x)}) \\
&\exp(-\frac{1}{2}\int\frac{dp}{p}|p|\tilde{\rho}(ax)Re[-i(\overline{\tilde{g}}_2^{(1)}(p)+\overline{\tilde{g}}_2^{(2)})e^{-ipx}(1-e^{-ip\varepsilon})]) && (\equiv e^{I^a_2(\varepsilon,x)}) \\
&\exp(\frac{1}{2}\int\frac{dp}{p}\tilde{\rho}(ax)(-\overline{\tilde{g}}_1^{(1)}(p)+\overline{\tilde{g}}_1^{(2)})e^{-ipx}(1-e^{-ip\varepsilon})) && (\equiv e^{I^a_3(\varepsilon,x)}) \\
&\exp(+\frac{i}{2}\int dp\tilde{\rho}(ax)(-\overline{\tilde{g}}_2^{(1)}(p)+\overline{\tilde{g}}_2^{(2)})e^{-ipx}(1-e^{-ip\varepsilon}))-1\} && (\equiv e^{I^a_4(\varepsilon,x)}).
\end{aligned}$$

The expression for the right spinors contribution is given by the same formula with opposite sign in front of $I^a_1(\varepsilon,a)$ and $I^a_4(\varepsilon,a)$. We now take the $a \to 0^+$ limit. By the dominated convergence theorem

$$I_i(\varepsilon,x) := \lim_{a\to 0+} I^a_i(\varepsilon,x) \qquad i = 1,2,3,4$$

is well defined, and it is a C^∞ function in the variable ε, Since we identified, in the preceding section, $s-\lim_{a\to 0^+}\psi_a(f)$ as the massless fermionic field, it follows in particular that, $\forall\varepsilon > 0$,

$$\lim_{a \to 0+} \omega_0(\psi_a^l(y+\varepsilon)\psi_a^l(y)) = \frac{i}{2\pi\varepsilon}. \tag{3.6}$$

We now sum the left and right contributions to (3.2), subtracting the vacuum expectation value. We can safely take the $\varepsilon \to 0$ limit, thanks to the dominated convergence theorem, since the product of the exponentials minus the vacuum subtraction gives a factor of order ε. One can easily check that all constants combine to give the right factor. The job to be done in the $\mu = 1$ case is absolutely analogous, since the only change with respect to the $\mu = 0$ one consist in subtracting the right spinors contribution from the left spinors one instead of summing it. The final result is that formula (3.2) is true, chosen anyhow $G^{(1)}$ and $G^{(2)}$ in $\partial\mathcal{S} \times \mathcal{S}$.

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