SISSA - SCUOLA
INTERNAZIONALE
SUPERIORE
STUDI AVANZATI
TRIESTE
Strada Costiera 11

ISAS - INTERNATIONAL SCHOOL
FOR ADVANCED STUDIES

A Sample of Algebro-Geometrical Techniques in (Super)String Theory

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Candidate

Gregorio Falqui

Supervisor

Prof. Cesare Reina

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1.Introduction

1.1 General overlook

String theory, albeit originally arisen from attempts to justifying some peculiarities of high energy physics, soon became a candidate for unification of all forces in nature, including gravity, a task which, up to now, seems hardly achievable in the framework of ordinary quantum field theory. Moreover, the rich mathematical structure underlying string theory has opened new channels of interplay between physics and various branches of current mathematical research, such as infinite dimensional global analysis, including Lie groups and algebras, algebraic geometry in dimension 1,2 and 3, theory of modular forms ... In some sense the study of string theory has also prompted the unexpected unravelling of connections between such different areas (see, e.g., the work of Arbarello-De Concini, Kac and Procesi [ADKP] on the link between the topology of moduli spaces and the cohomology of the Virasoro algebra).

We will be mainly interested in algebro geometrical techniques involved in the developments of string theory, and the plan of the work is first to describe how algebraic geometry provides a natural arena for setting and possibly solving problems of the "bosonic" string, and then to try to extend them to the case of the fermionic string in the spirit advocated by Manin [M1]. According to this line, we will only study the case of the so-called closed string, where the powerful methods of complex algebraic geometry can be (at least in the bosonic case) applied.

More specifically in the first part of this thesis work, after having sketched how string theory involves the geometry of moduli space of Riemann surfaces , we will review in some detail the main properties of the latter, with special attention paid to "global" aspects, such as its Picard group and its compactification(s). Then we report how this techniques can be used to investigate physical issues, such as the holomorphic factorization theorem and the boundary behaviour of the string partition function.

Having in mind fermionic string theories, a section is subsequently devoted to the description of a compactification of moduli space of θ-characteristics as recently investigated by M. Cornalba.

Chapter 3. is finally devoted to the discussion about a similar setting for Superstring theories. Namely, we will try to clarify a little bit the arena in which one should move when arguing about such topics in this framework. In particular, we will discuss in some details integration theory on superspaces and the supermoduli problem, and finally, give an outlook on the perspectives of this research programme.

1.2 The Polyakov path-integral as an integral over moduli spaces of Riemann Surfaces

In the standard Polyakov formulation [Pol1](which is, *ab initio*, Euclidean in the sense of quantum field theory and is, essentially, a theory of random surfaces), the degrees of freedom of a string are described by a mapping X from a two-dimensional riemannian manifold (Σ, g) to a D-dimensional "target manifold"(M^D, h), whose dynamics is dictated by the following action

$$\mathcal{S} = \frac{1}{2\pi\alpha'} \int_\Sigma *\|dX\|^2 \tag{1}$$

which, in local coordinates reads

$$\mathcal{S} = \frac{1}{2\pi\alpha'} \int_\Sigma d^2\xi \sqrt{(g)} g^{ij} \partial_i X^\mu \partial_j X^\nu h_{\mu\nu}$$

where α' is the so-called string tension, a dimensioned parameter which, from now on, will be set equal to $\frac{1}{2\pi}$.

Polyakov's proposal for the quantization of the theory was to define the partition function as

$$Z = \int_{Met \times Emb} \mathcal{D}g \mathcal{D}X e^{-\mathcal{S}}$$

where the domain of integration $Met \times Emb$ is defined as

$Met := \{$space af all metrics on $\Sigma\}$

$Emb := \{$space of C^∞-maps from Σ to $M^D\}$

As all derivations in this section will be heuristic in spirit, we will not bother in defining correctly the topology of the infinite dimensional manifolds we will encounter. It suffices to say that, in any case, a nice mathematical structure of differentiable manifold can be given to them [EE].

The action (1) has a huge symmetry group $\mathcal{G}$. Namely, $\mathcal{G}$ contains

a) The isometry group of the target manifold (M^D, h)

b) The group $Diff(\Sigma)$ of diffeomorphisms of the surface Σ acting on $X \in Emb$ by composition and on $g \in Met$ by pull-back

c) The group $Weyl$ of conformal rescalings of the metric g, acting as $g \rightsquigarrow e^{\sigma} \cdot g$, where σ is a real valued C^{∞}-function.

Forgetting about the finite dimensional group of isometries of (M^D, h), which, in the case of a flat target manifold, reduces to the euclidean group, one is left with the semidirect product $Diff \ltimes Weyl$. Now, as it is customary in perturbative quantum field theory, such "gauge degrees of freedom" must be removed in doing Feynman path integral in order to get sensible answers in the computation of n-point functions. This can be done in different ways, none of which has a sound mathematical status due to the still incomplete understanding of path integrals. Anyhow, they are at least useful heuristic tools in passing from

$$Z = \int_{Met \times Emb} \mathcal{D}g \mathcal{D}X e^{-\mathcal{S}}$$

to

$$Z = \int_{Met \times Emb / Diff \ltimes Weyl} \mathcal{D}g \mathcal{D}X e^{-\mathcal{S}} \tag{2}$$

i.e. to integrate only over the space of gauge non-equivalent points $([g], [X])$, or to "mod out the volume of the gauge group". Now, in string theory, there is a great merit in doing so. Namely we will be able to give a precise meaning to (2), as the space on which one ultimately lands is a finite dimensional space order by order in string perturbation theory.

Before sticking to the manipulation of the integrand, we feel necessary to describe a little bit more [EE]the geometry of the fibration

$$\begin{array}{ccc} Diff \ltimes Weyl & \rightarrow & Met_{(p)} \\ & & \downarrow \pi \\ & & M_p \end{array}$$

where the subscript p refers to the genus of the surface.

First of all the group of conformal transformations acts freely on $Met_{(p)}$, and the quotient is the space of conformal structures on Σ. A section of the fibration (which is topologically trivial, as $Weyl$ is contractible) is given, e.g., for $p > 1$, by $[g] \rightsquigarrow \hat{g}$, where $\hat{g}$ is the metric with constant curvature $R_{\hat{g}} = -1$.

The local description of this fact relies on the possibility of introducing isothermal coordinates for any metric g on the surface Σ, i.e. a sufficiently fine covering $\{\mathcal{U}_\alpha\}$ of Σ and local coordinates $\{x_\alpha, y_\alpha\}$ such that the metric takes the form

$$g_\alpha = e^{\varphi_\alpha} \left(dx_\alpha \otimes dx_\alpha + dy_\alpha \otimes dy_\alpha \right)$$

or, introducing complex coordinates

$$z_\alpha = x_\alpha + i y_\alpha$$

$$\bar{z}_\alpha = x_\alpha - i y_\alpha$$

$$g_\alpha = e^{\varphi_\alpha(z_\alpha, \bar{z}_\alpha)} \cdot (dz_\alpha \otimes d\bar{z}_\alpha)$$

Now, fixing a conformal structure on Σ is the same thing as fixing a complex structure. In fact it can be easily seen that requiring that a coordinate transformation preserves the above form of the metric is tantamount to requiring that it is an analytic (or antianalytic) function when expressed in terms of the complex parameterization. Notice also that, because of dimensional reasons, any almost complex structure on Σ is integrable and so this local argument is straightforwardly globalized. Now one is left to discussing the quotient $Conf(\Sigma)/Diff^+(\Sigma)$ of conformal structures on Σ modulo orientation preserving diffeomorphisms.

As we shall discuss in greater detail in the next chapter, an oriented surface Σ together with a fixed conformal structure μ is, by definition, a Riemann surface. Equivalently, Riemann surfaces can be defined as connected 1-dimensional complex manifolds. A conformal map f between two Riemann surfaces $(\Sigma, \mu) \xrightarrow{f} (\Sigma', \mu')$ is an orientation preserving diffeomorphism which is conformal w.r.t. the given conformal structures $\mu$and μ'. Two Riemann surfaces connected by a conformal map will be called equivalent.

Now, let $\mu \in Conf(\Sigma)$ be a conformal structure and $f : \Sigma \to \Sigma$ an orientation preserving diffeomorphism; $f^*\mu$ will denote the unique conformal structure ν on Σ obtained by fixing a riemannian metric γ in the conformal class $[\mu]$ and taking $\nu := [f^*(\gamma)]$. Now, take another Riemann surface (Σ, ϱ) (notice that the underlying C^∞-manifolds are the same). Then, if $\varrho = f^*(\mu)$ then $(\Sigma, \mu)$and(Σ, ϱ) are equivalent and conversely. This argument proves the following

Fact: let $Diff^+(\Sigma)$ act on $Conf(\Sigma)$ by means of $f \cdot \mu := f^*(\mu)$. Then the quotient space is (in canonical one to one correspondence with) the set of equivalence classes of genus p Riemann surfaces, i.e. the so-called moduli space of genus p Riemann surfaces.

After having pointed out what will be the domain of integration for the string partition function, we have to discuss the actual reduction of the integrand to the quotient space $M_p = Met/Diff \ltimes Weyl$.

To see what is going on we will consider a finite dimensional analogue, namely the integration on a finite dimensional quotient space [Bo].

Given an m-dimensional vector space F, consider the line $\wedge^m(F^*)$. This can be identified with the set of densities over F, i.e. the set of scalar valued functions σ defined on the set $\mathcal{B}(f)$ of bases in F such that for any $T \in GL(F)$

$$\sigma(Tb) = detT \cdot \sigma(b)$$

We shall denote with $[v_1 \wedge \cdots \wedge v_m]^{-1}$ the unique density σ s.t. $\sigma(v_1 \cdots v_m) = 1$

The following construction will be of prominent interest in the sequel.

Given an exact sequence of vector spaces

$$0 \to F_0 \xrightarrow{T_0} F_1 \cdots \longrightarrow \xrightarrow{T_{n-1}} F_n \to 0$$

there is a canonical isomorphism

$$I : \otimes_{j=0}^{[n/2]} \wedge^{max} F_{2j}^* \quad \xrightarrow{\sim} \quad \otimes_{j=0}^{[\frac{n+1}{2}]} \wedge^{max} F_{2j+1}^*$$

A sketch of the proof of this can be given by the following argument. Let us decompose $F_i = Ker(T_i) \oplus E_i$ and let us fix a basis $\{v'_{i,n_i}\}$ in E_i. By the exactness of the sequence above, $Ker(T_i) = Im(T_{i-1})$ and $T_i(F_i) = T_i(E_i)$. Then a basis in F_i will be written as

$$\{v_{i,1}, \cdots v_{i,m_i}\} = \{T_{i-1}(v'_{i-1,1}), \cdots T_{i-1}(v'_{i-1,n_i}), v'_{i,n_i+1}, \cdots v'_{i,m_i}$$

so that one can define

$$I(\otimes_{j=0}^{[n/2]}[v_{2j,1} \wedge \cdots \wedge v_{2j,m_{2j}}]^{-1}) = \otimes_{j=0}^{[\frac{n+1}{2}]}[v_{2j+1,1} \wedge \cdots \wedge v_{2j+1,m_{2j+1}}]^{-1}$$

and the exactness of the sequence ensures that I is an isomorphism.

Let then V be a differentiable manifold carrying of a smooth action of a Lie group G. Suppose moreover that E is a vector space on which G acts linearly on the right and the following set of data is provided:
(1) a map $Q : V \rightarrow$ positive quadratic forms on E
(2) a map $\lambda : V \rightarrow$ positive measures on E
(3) a map $H : V \rightarrow$ positive measures on $Lie(G)$
(4) a C^∞-measure μ on V (in this framework a positive section of $\wedge^{max} T^* V$)
satisfying the following compatibility conditions:
(a) $\forall v, x, h \in V \times E \times G \quad Q_{v \cdot h}(h \cdot x) = Q_v(x)$
(b) $\forall v \in V \quad \lambda_v$ is invariant under the action of the little group $K_v \subset G$ of v over E.
Then one can define a C^∞-density on V/G, via an integration over E modulo G of the following volume form over $E \times V$

$$\sigma(x, v) = e^{-\frac{1}{2}Q_v(x)} \cdot \lambda_v(x)\mu(x)$$

by means of the following construction. Let $\tau \in V/g$ and $v \in V$be such that $\tau = v \cdot G$. The preimage of τ under

$$\begin{array}{ccc} (E \times V)/G & \longrightarrow & V/G \\ {[x, v]} & \rightsquigarrow & [v] \end{array}$$

is the quotient E/K_v. Notice that any density on $\wedge^{max} Lie K_v^*$ determines an invariant measure ω on K_v and so, as λ_v is K_v-invariant, a transverse measure λ_v/ω on E/K_v.

So we can define $\int_{E/K_v} e^{-\frac{1}{2}Q_v(x)}\lambda_v(x)$ as an element of $\wedge^{max} Lie K_v^*$ by means of

$$\int_{E/K_v} e^{-\frac{1}{2}Q_v(x)}\lambda_v(x) := \int_E e^{-\frac{1}{2}Q_v(x)}\lambda_v(x) \left[\int_{K_v} \omega\right]^{-1} \cdot \omega$$

We have at our disposal the exact sequence

$$0 \to LieK_v \to LieG \xrightarrow{a_v} T_v V \xrightarrow{b_v} T_\tau V/G \to 0 \tag{3}$$

where a_v is the tangent map at the origin of the group action $g \rightsquigarrow v \cdot g$ and b_v is the tangent to the canonical projection $V \to V/G$ Then, as discussed above, we have a canonical isomorphism of determinants

$$I_v : \wedge^{max}(T^*_\tau(V/G)) \otimes \wedge^{max}(LieG)^* \xrightarrow{\sim} \wedge^{max}(T^*_v V) \otimes \wedge^{max}(LieK_v)^*$$

which can be recasted in the form

$$\tilde{I}_v : \wedge^{max}(T^*_\tau(V/G)) \xrightarrow{\sim} \wedge^{max}(T^*_v V) \otimes \wedge^{max}(LieK_v)^* \otimes (\wedge^{max}(LieG)^*)^*$$

From eq. (6) we had that

$$\int_{E/K_v} e^{-\frac{1}{2}Q_v(x)} \lambda_v(x)$$

was an element of $\wedge^{max}(LieK_v)^*$ and hence we can consistently define

$$\nu(v) := \tilde{I}^{-1}\left[\int_{E/K_v} e^{-\frac{1}{2}Q_v(x)} \lambda_v(x) \otimes \mu(v) \otimes H(v)^{-1}\right] \tag{4}$$

For any $v \in V$ this is an element of $\wedge^{max}(T_\tau V/G)^*$. Thus, supposing μ G-invariant, the equivariance and the positivity property of λ_v and $H(v)$ yield that $\nu(v)$ is a measure on V/G.

Suppose now we are given, for any $v \in V$ scalar products $< \cdot, \cdot >_v$ on E and $(\cdot,\cdot)_v$ on $LieG$, and a riemannian metric g on V. Then we have:

(i)$\lambda_v(x_1, \cdots x_m) = [\det < x_i, x_j >]^{1/2}$ $\quad x_i \in E$
(ii)$H(v)(X_1, \cdots X_M) = [\det(X_I, X_J)]^{1/2}$ $\quad X_I \in LieG$
(iii)$\mu_v(Y_1, \cdots Y_d) = [\det g(Y_\alpha, Y_\beta)]^{1/2}$ $\quad Y_\alpha \in T_v V$

and, if T_v is the symmetric operator such that $Q_v(x) =< x, T_v x >_v$, then

(iv) $\int_E e^{-\frac{1}{2}Q_v(x)} \lambda_v(x) = (\det \quad T_v)^{-1/2}$

Let $\{\varphi_1, \cdots \varphi_k\}$ be a basis in $LieK_v$ and $\{\psi_1, \cdots \psi_n\}$ a basis in $T_\tau(V/G)$. Furthermore, due to the exactness of the sequence (3), b_v is an isomorphism between $(Ima_v)^\perp \subset T_v V$ and $T_\tau V/G$ so that in $(Ima_v)^\perp$ there are unique n vectors $\tilde{\psi}_1, \tilde{\psi}_n$ such that $\tilde{\psi}_i = b_v^{-1}\psi_i$. Then

$$\tilde{I}_v^{-1}([\varphi_1 \wedge \cdots \wedge \varphi_k]^{-1} \otimes \mu_v \otimes H(v)^{-1}) = \left[\det{}' a_v^* a_v \cdot \det < \tilde{\psi}_i, \tilde{\psi}_j >_v\right]^{1/2}$$

$$\left[\det(\varphi_r, \varphi_s)^{-1}]^{1/2}\right] \cdot [\psi_1 \wedge \cdots \wedge \psi_n]^{-1}$$

where det' means the product of nonzero eigenvalues.

Finally one gets

$$\nu(v) = (detT_v)^{-1/2}\left[\int_{K_v}[\varphi_1\wedge\cdots\wedge\varphi_k]^{-1}\right]\left[\det{}' a_v^* a_v\cdot\det<\tilde{\psi}_i,\tilde{\psi}_j>_v\right]^{1/2}\cdot$$

$$\left[\det[(\varphi_r,\varphi_s)]^{-1}\right]^{1/2}\cdot|\psi_1\wedge\cdots\wedge\psi_n|^{-1}$$

The last purpose of this section is to apply, at least at a formal level, the above results to get an expression for the string partition function. Let us then start by identifying the various building blocks of the construction.
We will set
$V = Met_{(p)}$
$G = Diff^+ \ltimes Weyl$
$E = Emb(\Sigma, \mathbb{R}^D)/\mathbb{R}^D$.
Their tangent spaces are easily identified as follows:
$T_g(Met_{(p)}) = C^\infty(\Sigma, S^2T^*_{\mathbb{R}}\Sigma)$ is the space of sections of the symmetric tensor product of the real cotangent bundle of Σ.
$\mathcal{L}(Diff^+) = C^\infty(\Sigma, T_{\mathbb{R}}\Sigma)$
$\mathcal{L}(Weyl) = C^\infty(\Sigma, \mathbb{R})$

Remark As the space of riemannian metrics is an open cone in a vector space its tangent bundle is trivial (whence no remnant of g in the l.h.s. of the above expression). What actually depends on g is the decomposition $S^2T^*_{\mathbb{R}}(\Sigma) = C_g \oplus N_g$ where N_g is the vector space of g-traceless quadratic forms.

Let p denote the projection on N_g. Then the infinitesimal action of $Diff^+$ on $Met_(p)$ is given by[D'H-P,Bo]

$$\begin{array}{ccc} C^\infty(\Sigma, T_{\mathbb{R}}\Sigma) & \xrightarrow{P_g} & C^\infty(\Sigma, S^2T^*_{\mathbb{R}}\Sigma) \\ \xi & \rightsquigarrow & p(\mathcal{L}_\xi g) \end{array}$$

where $\mathcal{L}$ here stands for the Lie derivative.
One has the following commutative diagram

$$\begin{array}{ccc} C^\infty(\Sigma, T_{\mathbb{C}}\Sigma) & \xrightarrow{\bar\partial} & C^\infty(\Sigma, T_{\mathbb{C}}\Sigma\otimes\overline{K}) \\ \downarrow & & \downarrow \\ C^\infty(\Sigma, T_{\mathbb{R}}\Sigma) & \xrightarrow{P_g} & C^\infty(\Sigma, N_g) \end{array}$$

Considering the Teichmüller space $\mathcal{T}_{(p)}$, which can be realized as the quotient of the space of metrics by the semidirect product of the Weyl group with the group of diffeomorphisms homotopic to the identity (see Ch. 2 for a thorough discussion of this and the following topics), one can establish the following cartesian diagram:

$$\begin{array}{ccc} C^\infty(\Sigma, T_{\mathbb{C}}\Sigma \otimes \overline{K}) & \longrightarrow & H^1(\Sigma, T_{\mathbb{C}}(\Sigma) \\ \downarrow & & \downarrow \\ C^\infty(\Sigma, N_g) & \longrightarrow & T_{[g]}\mathcal{T}_{(p)} \end{array}$$

The stabilizer K_g of g in $Diff_0^+ \ltimes Weyl$ can be identified with the group of conformal automorphisms of (Σ, g), i.e. the group of holomorphic diffeomorphisms of (Σ, g). Summing up one has the following commutative diagram, the rows of which are exact [Bo]

$$\begin{array}{ccccccccccc} 0 & \to & H^0(\Sigma, T_{\mathbb{C}}\Sigma) & \to & \Gamma(\Sigma, T_{\mathbb{C}}\Sigma) & \xrightarrow{\bar\partial} & \Gamma(\Sigma, T_{\mathbb{C}}\Sigma \otimes \bar\omega) & \to & H^1(\Sigma, T_{\mathbb{C}}\Sigma) & \to & 0 \\ & & \downarrow{\scriptstyle l} & & \downarrow{\scriptstyle l} & & \downarrow & & \downarrow{\scriptstyle l} & & \\ 0 & \to & \mathcal{L}Aut\Sigma & \to & \mathcal{L}(Diff^+) & \xrightarrow{P_g} & \Gamma(N_g) & \xrightarrow{T_g\pi} & T_{[g]}\mathcal{T}_{(p)} & \to & 0 \\ & & \uparrow{\scriptstyle l} & & \uparrow{\scriptstyle pr_2} & & \uparrow{\scriptstyle p} & & \| & & \\ 0 & \to & \mathcal{L}K_g & \to & \mathcal{L}(Diff^+ \ltimes Weyl) & \xrightarrow{A_g} & T_g Met & \xrightarrow{T_g\pi} & T_{[g]}\mathcal{T}_{(p)} & \to & 0 \end{array}$$

where $\Gamma(\cdot)$ stands for $C^\infty(\Sigma, \cdot)$ and A_g is the differential at the identity of $Diff^+ \ltimes Weyl$ of the action of this gauge group on Met_Σ.

Notice also that every space in the diagram above naturally has or inherits an "L^2" inner product.

The integration over $E = Emb$ is gaussian and hence gives

$$\int_E e^{-S(g,X)} = (\|1\|^2)^{D/2} \cdot (det'\triangle_g^0)^{-D/2}$$

Suppose now $\{\varphi_1, \cdots \varphi_k\}$ is a basis for $LieK_g$, $\{\psi_1, \cdots \psi_n\}$ a basis for $T_{[g]}\mathcal{T}_{(p)}$ and $\tilde\psi_i$ the preimages under $T_g\pi$ of the ψ_i's into $KerA_g^* = Im(A_g)^\perp$. Then the naive application of eq.(4) gives

$$\nu(g) = \left(\frac{det'\triangle_g^0}{\|1\|_g^2}\right)^{-D/2} \int_{K_g} |\varphi_1 \wedge \cdots \wedge \varphi_k|^{-1} \left[det' A_g^* A_g \frac{det(\tilde\psi_i, \tilde\psi_j)_g}{det(\varphi_r, \varphi_s)_g}\right]^{1/2} |\psi_1 \wedge \cdots \wedge \psi_n|^{-1}$$

By chasing around the exact diagram above one can prove that the determinant of $A_g^* A_g$ can be substituted with $det' P_g^* P_g$, thus getting for the integrand

$$\left(\frac{det'\triangle_g^0}{\|1\|_g^2}\right)^{-D/2} \cdot \left[det' P_g^* P_g \cdot \frac{det(\tilde\psi_i, \tilde\psi_j)_g}{det(\varphi_r, \varphi_s)_g}\right]^{1/2}$$

We are now in a position to discuss the invariance properties of the integrand w.r.t. the action of the conformal group. We prefer to skip the somewhat tedious and nowadays well known computations [Alv,Pol1,D'H-P]. We just report that by passing to the complex analytic counterparts of the quantities appearing in the above expression one gets that the integrand is turned into

$$\left(\frac{det'(\overline{\partial}_0^*\overline{\partial}_0)_g}{\|1\|_g^2}\right)^{-D/2} \cdot det'\overline{\partial}_{-1}^*\overline{\partial}_{-1} \cdot \frac{det(\Psi_i, \Psi_j)_g}{det(\phi_r, \phi_s)_g} \tag{15}$$

where Ψ_i and ϕ_r are the complexification of the ψ_i and φ_r and $\overline{\partial}_j$ denotes the Dolbeault operator coupled to j-tensors, i.e. the $\overline{\partial}$-operator with values in $(T_{\mathbb{C}}^*\Sigma)^j$.
Polyakov's conformal anomaly formulas give[Pol1,Alv]

$$\frac{det'(\overline{\partial}_0^*\overline{\partial}_0)_{g'}}{\|1\|_{g'}^2} = e^{L(g,R)} \cdot \frac{det'(\overline{\partial}_0^*\overline{\partial}_0)_g}{\|1\|_g^2}$$

$$det'\overline{\partial}_{-1}^*\overline{\partial}_{-1} \cdot \frac{det(\Psi_i, \Psi_j)_{g'}}{det(\phi_r, \phi_s)_{g'}} = e^{13L(g,R)} \cdot det'\overline{\partial}_{-1}^*\overline{\partial}_{-1} \cdot \frac{det(\Psi_i, \Psi_j)_g}{det(\phi_r, \phi_s)_g}$$

Here $g' = e^\rho g$ is conformally equivalent to g, R is the curvature (1,1)-form of $T_{\mathbb{C}}\Sigma$ and

$$L(\rho, R) = \frac{1}{2\pi i}\int_\Sigma \partial\rho \wedge \overline{\partial}\rho + 2\rho \cdot R$$

is the Liouville action. So, if $D = 26$ the dependence on the conformal factor ρ actually drops out and so one is left with the following finite dimensional integral as the p-loop contribution to the string partition function

$$Z_p = \int_{M_p} |\Psi_1 \wedge \cdots \wedge \overline{\Psi}_{3g-3}|^{-1} \left(\frac{det'(\overline{\partial}_0^*\overline{\partial}_0)_g}{\|1\|_g^2}\right)^{-D/2} \cdot det'\overline{\partial}_{-1}^*\overline{\partial}_{-1} \cdot \frac{det(\Psi_i, \Psi_j)_g}{det(\phi_r, \phi_s)_g}$$

This formula, albeit derived in a heuristic way, will be the starting point for the analysis to be surveyed in the next chapter. In particular we are going to give a sound meaning to the regularized determinants and to describe the geometry of the "integration domain" of eq. (17). This will enable us to draw some physical consequence out of analytic geometry and, furthermore, will introduce and somehow justify the approach to the superstring problem that will be pursued in the last chapters.

2.The bosonic string from an algebro-geometrical standpoint

2.1 Determinants and determinant bundles

As we have sketched in chapter 1, determinants of differential operators on Riemann surfaces play a dominant rôle in string theory. This section is devoted to give a clear characterization of their definition and to the study of the geometrical features they induce on a family of vector bundles parameterized by some variety B. As it is well known in physical applications, one has to face the following twofold problem. First, one has to deal with operators that are unbounded when defined in the physical Hilbert spaces of the theory, and second, in most significant cases, the relevant operators (such as the chiral Dirac operator in Yang-Mills theory) map their domain in a different vector space, so that the notion of a determinant for them must be somehow supplied.

To begin with, let us follow the strategy of chapter 1 and stick to a finite dimensional example. Let $D : V \to W$ be an invertible linear operator between vector spaces of the same dimension n. Its determinant $\det D$ is defined as the unique linear map making the following diagram

$$\begin{array}{ccc} V & \xrightarrow{D} & W \\ \downarrow & & \downarrow \\ \wedge^n V & \xrightarrow{\det D} & \wedge^n W \end{array}$$

commutative.
As noticed in chapter 1, $\det D$ is an element in $(\wedge^n V)^* \otimes \wedge^n W$ which, choosing a basis $\{v_i\}$ in V (and hence getting the dual basis $\{\alpha_i\}$ in V^*), is given by

$$\det D = \alpha_1 \wedge \cdots \wedge \alpha_n \otimes Dv_1 \wedge \cdots \wedge Dv_n .$$

Sometimes we will not making explicit reference to the dual basis $\{\alpha_i\}$ but will use (as in chapter 1) the notation

$$|v_1 \wedge \cdots \wedge v_n|^{-1} := \alpha_1 \wedge \cdots \wedge \alpha_n$$

From the above equation it is clear that, actually, the definition of det$D$ does not depend on the choice of the basis in V.
In the cases in which D fails to be invertible, its determinant vanishes. Actually one can take care of zero modes, by means of the procedure described in chapter 1 (which is very well known from the theory of anomalies in quantum field theory) which we now specialize to the case of complexes of length 4. Given $D \in Hom(V, W)$ one can set up the following exact sequence

$$0 \to \mathrm{Ker}D \xrightarrow{i} V \xrightarrow{D} W \xrightarrow{p} \mathrm{Coker}D \to 0$$

thus getting an isomorphism

$$(\wedge^{max} V)^* \otimes \wedge^{max} W \simeq (\wedge^{max}\mathrm{Ker}D)^* \otimes \wedge^{max}\mathrm{Coker}D$$

Then one defines [F1]

$$\det D := \det{}' D \; |v_1^0 \wedge \cdots \wedge v_l^0|^{-1} \otimes w_1^0 \wedge \cdots \wedge w_k^0$$

Here det'D is the determinant of D considered as an isomorphism between $V/\det D$ and $W/\mathrm{Coker}D$, l = dim Ker$D$, k = dim Coker$D$, and $\{v_i^0\}$ (resp. $\{w_i^0\}$) is a basis in Ker$D$ (resp. Coker$D$). Notice that in this case the prescription is ambiguous, due to the choice of a basis in Ker$D$ and Coker$D$, which amount in a multiplicative factor in front of det$D$. This is a feature one has to take into account when dealing with families of operators, as one should control this ambiguity as a "function" of the parameter space B.

Let us now discuss some issues related to infinite dimensionality. Let $E_1 \xrightarrow{\pi_1} M$ and $E_2 \xrightarrow{\pi_2} M$ be smooth vector bundles over a compact riemannian manifold M. Suppose we are given metrics $h_{1,2}$ along the fibers of E_1 and E_2. Then the completion of the spaces $\Gamma(E_1), \Gamma(E_2)$ in the obvious L^2 metrics, denoted by $\mathcal{H}_1$and $\mathcal{H}_2$ are separable Hilbert spaces. Let $\mathcal{H}_1 \xrightarrow{T} \mathcal{H}_2$ be an elliptic Fredholm operator. Then we can construct self adjoint operators $T^\dagger T \in End(\mathcal{H}_1)$, $TT^\dagger \in End(\mathcal{H}_2)$ such that

$$\mathcal{H}_i = \oplus_{j\in\mathbb{Z}} \mathcal{H}_i^j, i = 1,2$$

where $\mathcal{H}_1^j$ is an eigenspace of the nonnegative operator $T^\dagger T$ relative to the eigenvalue λ_j (and the same for $TT^\dagger$in $\mathcal{H}_2$). The ellipticity of T insures that
i) the decomposition above exists, i.e. the spectrum of $TT^\dagger$ and $T^\dagger T$ is discrete.
ii) the nonzero eigenvalues of $TT^\dagger$ and $T^\dagger T$ agree and T and $T^\dagger$ are isomorphisms between the corresponding eigenspaces.
One the can proceed in the following way, in order not to be troubled too much by infinite dimensional subtleties [BF,F2]. Let $a \notin Sp(T^\dagger T)$ and let $\mathcal{H}_i^{(a)} = \oplus_{\lambda_j < a} \mathcal{H}_i^j$ be the direct

sum of eigenspaces with eigenvalue λ_j less than a. Then, again because of ellipticity, $\mathcal{H}_i^{(a)}$ is finite dimensional and consists of smooth sections. The exact sequence

$$\begin{array}{ccccccccccc} 0 & \to & \mathrm{Ker}T^\dagger T & \to & \mathcal{H}_1^{(a)} & \xrightarrow{T} & \mathcal{H}_2^{(a)} & \to & \mathrm{Ker}TT^\dagger & \to & 0 \\ & & \| & & & & & & \| & & \\ & & \mathrm{Ker}T & & & & & & \mathrm{Ker}T^\dagger & & \end{array}$$

induces an isomorphism

$$\left(\det\mathcal{H}_1^{(a)}\right)^* \otimes \det\mathcal{H}_2^{(a)} \simeq (\det\mathrm{Ker}T)^* \otimes \det\mathrm{Coker}T$$

Notice that, whenever $a < b$ and both do not belong to $Sp(T^\dagger T)$ we have isomorphisms between

$$\mathcal{H}_1^{(a,b)} \equiv \mathcal{H}_1^{(b)})/\mathcal{H}_1^{(a)} \xrightarrow{T_{ab}} \mathcal{H}_2^{(b)}/\mathcal{H}_2^{(a)} \equiv \mathcal{H}_2^{(a,b)}$$

Then what is left to do is "to let a go to infinity". Obviously to have a sensible answer we must give a prescription for regularizing the infinite product of the eigenvalues which is going to occur. A natural way to do so is the use of Seeley's [RS] ζ-function regularization. This scheme works in the following way. Let D be an elliptic differential operator of order p ($p > 0$) on the space of C^∞-sections of a vector bundle E over a compact manifold M. Then it is essentially self-adjoint, the spectrum of its closure is a closed and discrete set of the real numbers and any eigenvalue has a finite multiplicity. One defines the ζ-function of D, ζ_D by means of the formal power series

$$\zeta_D = \sum_{n=0}^{+\infty} (\lambda_n)^{-s}$$

where the λ_n's are the non-zero eigenvalues of D, counted with their multiplicity. It is a classical result that this Dirichlet series converges for $Re(s) > \dfrac{\mathrm{dim}M}{\mathrm{ord}D}$ and defines a holomorphic function on the right complex half plane which admits a meromorphic continuation on the whole $\mathbb{C}$ which is holomorphic in $s = 0$. A more explicit expression for ζ_D is given the in terms of the Mellin transform

$$\zeta_D(s) = \frac{1}{\Gamma(s)} \cdot \int_0^\infty \left[tre^{-tD} - \mathrm{dimKer}D\right] \cdot t^{s-1}\, dt$$

Recall that the identification of the regularized determinant with zero-modes omitted with $\exp \zeta'_D(0)$ is prompted by the following (formal!) computation:

$$\frac{\mathrm{d}}{\mathrm{ds}}\zeta'_D\Big|_{s=0} = \frac{\mathrm{d}}{\mathrm{ds}}\sum_{n=0}^{+\infty}(\lambda_n)^{-s}\Big|_{s=0} = \frac{\mathrm{d}}{\mathrm{ds}}\sum_{n=0}^{+\infty}\exp(-s\log\lambda_n)\Big|_{s=0} =$$

$$-\sum_{n=0}^{+\infty}\log\lambda_n\exp(-s\log\lambda_n)\Big|_{s=0} = \sum_{n=0}^{+\infty}\log\lambda_n = \log\prod_{n=0}^{+\infty}\lambda_n$$

and by the fact that in finite dimensions the manipulations above are actually rigorous. Furthermore, the ζ-function regularization is well-behaving w.r.t. "finite dimensional shifts" in the following sense.
Let $\mathcal{H}_{\lambda>a}$ and $\mathcal{H}_{\lambda>b}$ be the complements in $\mathcal{H}$ of $\mathcal{H}^{(a)}$ and $\mathcal{H}^{(b)}$ introduced above. Then, if det' is defined by means of the ζ-function it holds

$$\det{}' D\restriction_{\mathcal{H}_{\lambda>a}} = \det{}' D\restriction_{\mathcal{H}_{\lambda>b}} \cdot \prod_{a<\lambda_i<b} \lambda_i$$

Let us now discuss the family version of this construction. Suppose we are given the following geometric data;
i) a smooth fibration $Z \xrightarrow{X} Y$
ii) a complex vector bundle $E \longrightarrow Z$ endowed with an hermitean metric $< \cdot, \cdot >_E$
iii) a metric along the fibers $g^{Z/Y}$
iv) a family of elliptic Fredholm operators D_y parameterized by Y acting on the sections of E
Then [BF-Q] one has the following
Theorem . The geometric data above determine a smooth line bundle $\det D \longrightarrow Y$ together with a natural hermitean metric $< \cdot, \cdot >_Q$ on it.
We will give just a sketch of the proof of the theorem, referring to [BF] for the details. What is relevant for us is the following construction. Let us consider the spaces $\mathcal{H}_i^{(a)}$ $\mathcal{H}_i^{(a,b)}$ as constructed above (notice that, typically, the subscript 1 is referred to E, and the subscript 2 to $E \otimes T^*_{Z/Y}$). Consider the open set

$$U_a := \{y \in Y \text{s.t. } a \notin Sp(D_y)\}$$

As D_y is elliptic, the finite dimensional spaces $\mathcal{H}_i^{(a)}$ glue to give a smooth vector bundle over U_a. But it is clear from the above construction that also

$$\mathcal{L}^{(a)} = \left(\det \mathcal{H}_1^{(a)}\right)^* \otimes \left(\det \mathcal{H}_2^{(a)}\right)$$

is a well defined line bundle on U_a. On $U_a \cap U_b$ we can glue $\mathcal{L}^{(a)}$and $\mathcal{L}^{(b)}$ by means of the transition functions $\det D_{(a,b)}$, and, as the U_a are a covering of Y the line bundle $\mathcal{L} \equiv \det D \longrightarrow Y$ is well defined. Notice that the infinite dimensionality of the spaces of sections is completely circumvented with this construction.

Closely tied to the argument above is the construction of the Quillen metric $< \cdot, \cdot >_Q$ on $\det D$. Let us consider the fiber bundle $\mathcal{H} \longrightarrow Y$ and its subbundles having as fiber $\mathcal{H}^{(a)}$. The natural L^2 on $\mathcal{H}$ metric induces, by linear algebra, a metric on $\left(\det \mathcal{H}_1^{(a)}\right)^* \otimes \det \mathcal{H}_2^{(a)}$. The point is that this metric is not even continuous on Y, because of possible jumps in the dimension of the kernel and the cokernel of D.

To remedy this situation, Quillen's proposal [Q] was to insert a regularizing term taking into account non-zero eigenvalues by means of the ζ-function regularization, thus yielding a smooth metric on $\det D$. To grasp how this works, let us denote by $g^{(a)}$ the L^2 metric induced on U_a and see what happens in the intersection $U_a \cap U_b$. The glueing of two local sections s_a and s_b is done by identifying

$$s_a \sim s_b \cdot \det D^{(a,b)}$$

so that

$$\det D^{(a,b)} = (\psi_1^* \wedge \cdots \wedge \psi_N^*) \otimes (D\psi_1 \wedge \cdots \wedge D\psi_M)$$

and

$$\|\det D^{(a,b)}\|^2 = \prod_{i=1}^{N} \|\psi_i^*\|^2 \cdot \|D\psi_i\|^2 = \prod_{a<\lambda_i<b} \lambda_i$$

so that $g^{(b)} = g^{(a)} \cdot \prod_{a<\lambda_i<b} \lambda_i$.

By the abovementioned properties of the ζ-function regularization,the definition of the Quillen metric as

$$\| \cdot \|_Q^2 = \det{}' D^\dagger D \cdot \| \cdot \|_{L^2}^2$$

gives a C^∞ metric on $\mathcal{L} \equiv \det D$.

But this is not the whole story. In fact, as we will see later on, the Quillen metric satisfies another very important property, in relation to the Atiyah-Singer family index theorem. Before considering that we need to state the holomorphic version of the results above.

Theorem. Let $\pi : Z \xrightarrow{X} Y$ be a holomorphic fibration with smooth fibers. Suppose Z admits a closed (1,1)-form τ which restricts to a Kähler form on each fiber X. Let $E \to Z$ be a holomorphic hermitean vector bundle with its hermitean connection. Then the determinant line bundle $\mathcal{L} \longrightarrow Y$ of the relative $\bar{\partial}$ complex coupled to E admits a holomorphic structure. The canonical connection of $\mathcal{L}$ is the hermitean connection for the Quillen metric.

We are not giving the proof of this theorem, which has been recently generalized by Bismut and Bost to the case of families admitting singular fibres, but simply notice that in our context the additional hypothesis of the existence of a relative Kähler form will be always satisfied.

2.2 The Grothendieck-Riemann-Roch Theorem

In view of the applications to be done in the sequel. we are going to discuss the Grothendieck-Riemann-Roch theorem with relation to determinant bundles switching from the previous differential geometrical approach to a truly algebraic geometrical one. To do so we have to list some terminology and properties of complex spaces and sheaves [see, e.g., Hart].

Let X be a complex manifold of dimension n and let $\mathcal{O}$ denote its structural sheaf, i.e. the sheaf of local analytic functions on X.

Definition. A sheaf of abelian groups on X is called analytic if

i) the stalks $\mathcal{S}_x$ are $\mathcal{O}_x$-modules

ii) the map $\bigcup_{x\in X} \mathcal{S}_x \times \mathcal{O}_x \longrightarrow \mathcal{S}_x$ defined by the module operation is continuous.

Definition. An analytic sheaf S is called coherent if $\forall x \in X$ there is a neighbourhood U_x of x and a short right exact sheaf sequence

$$\mathcal{O}^p\restriction_{U_x} \to \mathcal{O}^q\restriction_{U_x} \to \mathcal{S}\restriction_{U_x} \to 0$$

Here $\mathcal{O}^p = \mathcal{O} \oplus \cdots \oplus \mathcal{O}$ p-times

What matters for us is the following

Fact. if E is a complex analytic vector bundle on X, then the sheaf $\mathcal{E}$ of local holomorphic sections of E is coherent analytic [Hirz].

In order to take into account families of holomorphic structures, one has to define the so-called direct image sheaves as follows.

Let $X \xrightarrow{f} Y$ be a holomorphic map and $\mathcal{S}$ an analytic sheaf over X. One defines the q^{th} – direct image sheaf $R^q f_*(\mathcal{S})$ by means of a suitable presheaf in the following way. Let V be open in Y. Then the cohomology space $H^q(f^{-1}(V), \mathcal{S})$ is an $\mathcal{O}_X\restriction_{f^{-1}(V)}$-module. By composition of maps it is also an $\mathcal{O}_Y\restriction_V$-module. The map $V \rightsquigarrow H^q(f^{-1}(V), \mathcal{S})$ defines a presheaf on Y whose associated sheaf is, by definition, $R^q f_*(\mathcal{S})$. Naively, the stalk at $y \in Y$ of $R^q f_*(\mathcal{S})$ can be identified with $H^q(f^{-1}(y), \mathcal{S})$.

Coherent analytic sheaves have 'simple' cohomological properties. Namely

Proposition 1. if $\mathcal{S}$ is a coherent analytic sheaf over an n-dimensional manifold X, then

$$H^q(X, \mathcal{S}) = 0 \quad \text{for } q > n$$

Proposition 2. If $X \xrightarrow{f} Y$ is a proper holomorphic map,

$$R^q f_*(\mathcal{S}) = 0 \quad \text{for } q > \dim X$$

and $R^q f_*(\mathcal{S})$ is coherent for $q \geq 0$.

One can define the set of virtual objects in this framework. Let $Coh(X)$ denote the set of isomorphism classes of coherent analytic sheaves over a complex manifold X, and let $F(X)$ denote the free abelian group generated by $Coh(X)$. If $R(X)$ is the subgroup generated by all elements of the form $\mathcal{S} - \mathcal{S}' - \mathcal{S}"$ where

$$0 \to \mathcal{S}' \to \mathcal{S} \to \mathcal{S}" \to 0$$

is short exact, one defines the Grothendieck group of coherent analytic sheaves over X as

$$K_\omega(X) := F(X)/R(X)$$

Given a proper holomorphic map $X \xrightarrow{f} Y$ one gets an homomorphism

$$\begin{array}{ccccc} F(X) & \xrightarrow{\tilde{f}_!} & F(Y) & & \\ \mathcal{S} & \rightsquigarrow & \tilde{f}_!(\mathcal{S}) & := & \sum_{q=0}^{n}(-)^q R^q f_*(\mathcal{S}) \end{array}$$

As $\tilde{f}_!$ maps $R(X)$to $R(Y)$ it induces an homomorphism

$$f_! : K_\omega(X) \longrightarrow K_\omega(Y)$$

The Grothendieck-Riemann-Roch theorem for analytic sheaves is an equality between Chern characters of coherent analytic sheaves.
This notion is introduced by means of the following
Lemma . Let $\mathcal{S}$ be a coherent analytic sheaf aver an n-dimensional algebraic manifold X. Then there are complex vector bundles $W_0 \cdots W_n$ over X and an exact sequence (called resolution by vector bundles)

$$0 \to \mathcal{W}_0 \to \cdots \to \mathcal{W}_n \to \mathcal{S} \to 0$$

of analytic sheaves over X.(where $\mathcal{W}_i$ denotes the sheaf of local holomorphic sections of W_i). Then the Chern character $Ch(\mathcal{S})$ is defined as

$$Ch(\mathcal{S}) := \sum_{i=0}^{n}(-)^i\ Ch(W_i)$$

Now, as if

$$0 \to \mathcal{S}' \to \mathcal{S} \to \mathcal{S}" \to 0$$

is short exact then

$$Ch(\mathcal{S}) = Ch(\mathcal{S}') + Ch(\mathcal{S}")$$

the Chern character is a homomorphism

$$Ch : K_\omega(X) \longrightarrow H^*(X, \mathbb{Q})$$

Theorem (Grothendieck-Riemann-Roch) Let $b \in K_\omega(X)$ and $X \xrightarrow{f} Y$ a proper holomorphic map between algebraic varieties. Then

$$Ch(f_!(b)) = f_*\left(Ch(b) \cdot Td(X) \cdot (f^*(Td(Y)))^{-1}\right)$$

Here $Td(\cdot)$ is the total Todd class of the tangent sheaf to $\cdot$ and f_* is the so -called Gysin homomorphism (represented, in the smooth case, by integration along the fibers in De Rham cohomology).

The formula above is clarified to a great extent when working with smooth objects by means of the Chern-Weil construction, which gives explicit expressions for characteristic classes in terms of polynomial invariants built out of the curvature of the relevant bundles. Recall that [MK] given a holomorphic vector bundle $E \xrightarrow{\pi} X$ with a hermitean structure $< \cdot, \cdot >_E$, there is a unique unitary connection ∇_E which is compatible with the holomorphic structure, in the sense that the (0,1)-component of ∇_E coincides with $\bar{\partial}_E$.
The Chern-Weil construction associates (in a functorial way) to $(E \quad, < \cdot, \cdot >_E, \nabla_E)$ a set of distinguished differential forms which represent in De Rham cohomology the Chern classes of E (and hence any characteristic class) and are built out of the curvature R_E of ∇_E. In particular, the relevant polynomials entering the $\bar{\partial}_E$-complex are the Todd genus of the tangent bundle TX and the Chern character of E, given by

$$Ch(E) = \mathrm{tr}\ \mathrm{e}^{\frac{i}{2\pi} R_E}$$

$$Td(X) = \sqrt{\det \frac{R_{TX}/4\pi}{\sinh R_{TX}/4\pi}} \quad \mathrm{tr}\ \mathrm{e}^{\frac{i}{2\pi} R_{TX}}$$

By means of the splitting principle one can deduce the following formulas:

$$Ch(E) = \sum_{j=1}^{rk\ E} \mathrm{e}^{x}_{j} = \mathrm{rk}\ E + c_1(E) + \frac{1}{2}\left(c_1(E)^2 - 2c_2(E)\right) + \cdots$$

$$Td(X) = \prod_{j=1}^{dim X} \frac{y_j}{1 - \mathrm{e}^{-y_j}} = 1 + \frac{1}{2} c_1(TX) + \frac{1}{12}\left(c_1(TX)^2 + c_2(TX)\right) + \cdots$$

Now, let us specialize this construction to the case in which X is a holomorphic family of Riemann surfaces and $\mathcal{E}$ is the sheaf of sections of a holomorphic vector bundle on X. In this case $f_!(\mathcal{E})$ is the formal difference $H^0\left(f^{-1}(s), \mathcal{E}\right) \ominus H^1\left(f^{-1}(s), \mathcal{E}\right)$. A simple computation based again on the splitting principle yields immediately

$$c_1\left(f_!(\mathcal{E})\right) = c_1\left(\det f_!(\mathcal{E})\right)$$

Notice also that, by their very definitions, $H^0\left(f^{-1}(s), \mathcal{E}\right)$ and $H^1\left(f^{-1}(s), \mathcal{E}\right)$ fit into the Dolbeault complex as the kernel and cokernel of $\bar{\partial}_E$.

$$0 \to H^0\left(f^{-1}(s), E\right) \longrightarrow \mathcal{A}^{0,0}\left(f^{-1}(s), E\right) \xrightarrow{\bar{\partial}_E} \mathcal{A}^{0,1}\left(f^{-1}(s), E\right) \longrightarrow H^1\left(f^{-1}(s), E\right) \to 0$$

Then, extracting the 2-form part of the Grothendieck-Riemann-Roch formula one gets

$$c_1(\det \bar{\partial}_E) = \int_{fibers} \left(Ch(E) \cdot Td(X) \cdot \left(f^*(Td(Y))\right)^{-1}\right)_{(4)} \qquad (*)$$

This equality, in general, is true at the level of cohomology classes. The real beauty of the Quillen metric on determinant line bundles can be read off the following proposition[BF]

Theorem. Equality $(*)$ holds at the level of differential forms, provided that the Chern forms appearing there are computed by means of
i)the curvatures of the given metrics on X, Y, and E
ii)the Quillen metric on $\det\bar{\partial}_E$.

2.3 Algebraic curves

This section is devoted to a short review of the most significant properties of Riemann surfaces , in order to clarify some of the steps done in chapter 1 and to outline the set up in which we will work in the last chapters.

Definition (A). A (smooth) Riemann surface is a complex 1-dimensional manifold

Definition (B). A (smooth) Riemann surface is a 2-dimensional real orientable manifold together with a conformal class of metrics

The equivalence between the two definitions has been proven in chapter 1. We need also a suitable notion of "possibly singular" Riemann surfaces , which is best given in terms of algebraic geometrical structures.
Let U be an open set in $\mathbb{C}^p$ and let $f_1, \cdots, f_h$ be holomorphic functions on U. Let $X \subseteq U$ be the set of common zeroes of the f_i's. Given an open set $V \subset X$ (in the induced topology) a complex valued function $g : V \longrightarrow \mathbb{C}$ will be called holomorphic if it is locally the restriction of some holomorphic function $\tilde{g} : U \longrightarrow \mathbb{C}$. The set $\mathcal{O}$ of holomorphic functions on V is naturally a ring and the map $V \rightsquigarrow \mathcal{O}_V$ defines a sheaf $\mathcal{O}_X$, naturally called the structure sheaf of X.
Definition. The couple $(X, \mathcal{O}_X)$ is called a (reduced) complex space patch.

Let Y be a Hausdorff paracompact topological space and let $\mathcal{F}$ be a sheaf of rings which is a subsheaf of the sheaf $\mathcal{C}_Y$ of continuous functions on Y.
Definition. $(Y, \mathcal{F}_Y)$ is called a (reduced) complex space iff it is locally isomorphic to reduced complex space patches. $\mathcal{F}$ will be denoted by $\mathcal{O}_Y$ and called the structure sheaf of Y.
Remark 1. The definitions above show how complex spaces are a generalization of complex manifolds allowing for singularities of algebraic type.
Remark 2. The notion of complex space will be further generalized, in chapter 3, to the one of ringed space which is needed in the Kostant - Leites approach to supermanifolds.
Definition. A compact node curve is a compact complex space locally isomorphic to one of the following complex space patches:
i) the disk $(D, \mathcal{O}_D)$ where $D = \{z \in \mathbb{C} : |z| < 1\}$
ii) the 'double cone' $(C, \mathcal{O}_C)$ where $C = \{(z, w) \in \mathbb{C}^2 \mid zw = 0, |z| < 1 \text{ and } |w| < 1\}$

Notice that a node is the mildest kind of singularity one can envisage; fortunately, as will be discussed in great detail in the sequel, considering only such singularities is enough for our purposes.

To discuss the most important geometrical features of algebraic curves we will follow the strategy of defining them in the smooth case and then consider the modifications needed to give them precise meaning also on singular curves. A word of warning: we will use interchangeably the expressions 'line bundle" and "invertible sheaf".

From the differentiable point of view, any Riemann surface C is isomorphic to a sphere with a certain number of handles. This number is called the (topological) genus of the curve. The first homology group $H_1(C, \mathbb{Z})$ is a free abelian group generated by $2g$ elements which are canonically chosen as in fig.1

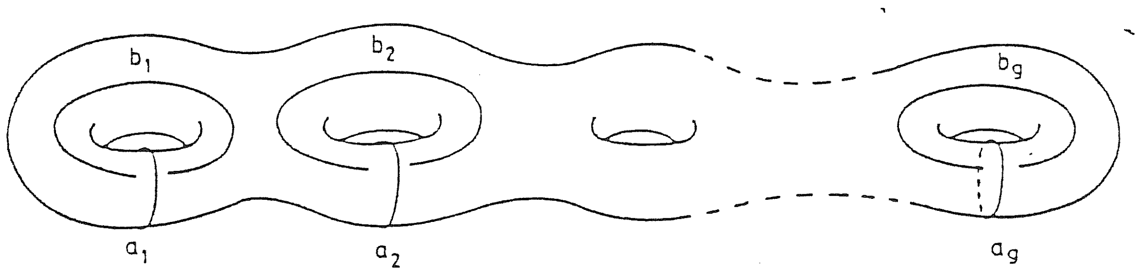


fig. 1

Such a choice is called *canonical* since the intersection matrix of the basis (a_i, b_i) is the $2g \times 2g$ symplectic matrix $\begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix}$ The Picard group Pic(C) of isomorphism classes of line bundles on C can be read off the exact sequence

$$0 \to \mathcal{Z} \longrightarrow \mathcal{O}_C \overset{\exp\ 2\pi i}{\longrightarrow} \mathcal{O}_C^* \to 1$$

where $\mathcal{O}_C^*$ is the sheaf of nowhere vanishing holomorphic functions and $\mathcal{Z}$ is the sheaf of $\mathbb{Z}$-valued continuous functions. The associated long exact cohomology sequence reads

$$\cdots \to \check{H}^1(C, \mathcal{Z}) \to \check{H}^1(C, \mathcal{O}_C) \to \check{H}^1(C, \mathcal{O}_C^*) \overset{\delta^*}{\longrightarrow} \check{H}^2(C, \mathcal{Z}) \to \check{H}^2(C, \mathcal{O}_C) \to \cdots$$

which can be shrunk to

$$\begin{array}{ccccccc} 0 & \to & \check{H}^1(C, \mathcal{O}_C)/H_1(C, \mathbb{Z}) & \to & \check{H}^1(C, \mathcal{O}_C^*) & \overset{\delta^*}{\longrightarrow} & H_2(C, \mathbb{Z}) & \to 0 \\ & & & & & & \| & \\ & & & & & & \mathbb{Z} & \end{array}$$

as the Čech cohomology groups of constant sheaves are isomorphic to the singular homology groups of the topological space and $\check{H}^2(C, \mathcal{O}_C)$ vanishes as $\mathcal{O}_C$ is coherent analytic on a 1-dimensional space. Then $\mathrm{Pic}(C) \equiv \check{H}^1(C, \mathcal{O}_C^*)$ is the semidirect product $\mathbb{Z} \ltimes \mathrm{Pic}_0(C)$ and $\mathrm{Pic}_0(C)$ is a complex g-dimensional torus which can be geometrically realized in the following way.
Pick a basis $\omega_1, \ldots, \omega_g$ of the space $H_{\bar\partial}^{(0,1)}(C, K_C)$ of abelian differentials (K_C is the canonical bundle), normalized according to $\oint_{a_i} \omega_j = \delta_{ij}$. Then

$$\oint_{b_i} \omega_j = \Omega_{ij}$$

is a symmetric matrix with positive definite imaginary part, called the period matrix of the Riemann surface C and, if Λ denotes the lattice in $\mathbb{C}^g$ generated by the columns of the $g \times 2g$ matrix $(\delta_{ij}|\Omega_{ij})$, then $\mathrm{Pic}_0(C) \simeq \mathbb{C}/\Lambda$. The latter is customary called the *Jacobian* variety of C and denoted $J(C)$

Let us now consider the case of node curves. First of all, let us qualify a node curve C by $(C, p_1, \ldots, p_r)$, the p_i's are the nodes (in a compact curve this number is always finite). A node curve comes equipped with "two" smooth curves, called its normalization N_C and its (one of its) desingularization $\tilde{C}$. Desingularizing the curve simply means fattening the nodes, i.e. replacing each patch $z_i w_i = 0$ with $z_i w_i = t_i, t_i \neq 0$. This procedure is clearly highly non unique and hence not canonical.
On the contrary, the normalization of a node curve is uniquely defined by the request that, in the normalization process, meeting branches are pulled apart . This is achieved by glueing $Y \setminus \{p_1, \ldots, p_r\}$ with the disjoint union of $2r$ disks (See fig. 2).

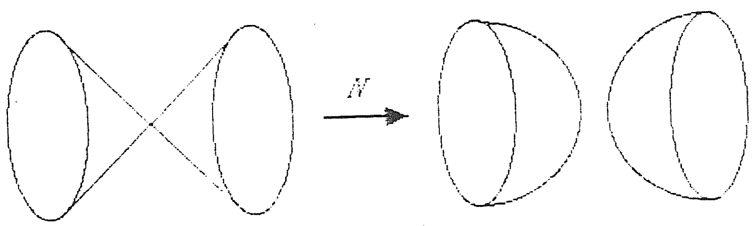


fig. 2

There is a natural map $\alpha : N_C \longrightarrow C$ which is holomorphic . Notice that whilst a desingularization $\tilde{C}$ of a node curve is connected iff C is connected, the normalization of C may very well be non connected even if C is so. In fact nodes can be subdivided into dividing and non-dividing ones, according to the connectedness properties of the normalization of C

at those points. A useful criterion is the following: a node p_i is dividing iff it can be obtained by shrinking a homologically trivial cycle. The Jacobian of a node curve can be described by means of the following argument [ACGH]. The normalization procedure yields the exact sequence

$$0 \to \mathcal{O}_C \longrightarrow \alpha_* \mathcal{O}_{N_C} \longrightarrow \oplus_{p_i \in C_{sing}} \mathcal{S}^{\mathbb{C}}_{p_i} \to 0$$

Where C_{sing} is the set of singular points of C and $\mathcal{S}^{\mathbb{C}}_{p_i}$ is the skyscraper sheaf with stalk $\mathbb{C}$ centered at P_i. By exponentiating one gets

$$1 \to \mathcal{O}^*_C \longrightarrow \alpha_* \mathcal{O}^*_{N_C} \longrightarrow \prod_{p_i \in C_{sing}} \mathcal{S}^{\mathbb{C}}_{p_i} \to 1$$

and hence, supposing C connected

$$1 \to H^0(C, \alpha_* \mathcal{O}^*_{N_C}) \longrightarrow H^0(C, \prod_{p_i \in C_{sing}} \mathcal{S}^{\mathbb{C}}_{p_i}) \longrightarrow H^1(C, \mathcal{O}_C) \longrightarrow H^1(\alpha_* \mathcal{O}^*_{N_C}) \to 0$$

As α is finite, the cohomology groups of $\mathcal{O}_C$ and $\alpha_* \mathcal{O}^*_{N_C}$ coincide, so that the latter turns into

$$1 \to (\mathbb{C}^*)^\nu \to (\mathbb{C}^*)^r \to \mathrm{Pic}(C) \to \mathrm{Pic}(N_C) \to 0$$

where ν is the number of connected components of N_C and r is the number of nodes in C. Collapsing it one gets

$$1 \to (\mathbb{C}^*)^{\nu - r} \to \mathrm{Pic}(C) \to \mathrm{Pic}(N_C) \to 0$$

which shows that the Picard group of C is a $\mathbb{C}^*$-extension of $\mathrm{Pic}(N_C)$. Finally notice that the dimension of the extension equals the number of non-dividing nodes in C_{sing}.

In more geometrical terms, the above discussion can be rephrased by saying that giving a "line bundle" L on C is tantamount to giving its pull-back $\tilde{L}$ on N_C (this is a true line bundle as N_C is smooth) plus descent data, i.e. giving, for any node p of C an identification $\varphi_p : \tilde{L}_{q_1} \longrightarrow \tilde{L}_{q_2}$ of the fibres over the preimages of the node p. Also, whenever $\tilde{L}$ is trivial, a choice of trivialization identifies each φ_p with a nonzero complex number.

Definition. A (Cartier) divisor on a smooth curve C is the datum of an open cover $\{U_\alpha\}_{\alpha \in I}$ and, $\forall \alpha \in I$ a holomorphic function f_α such that, if $U_\alpha \cap U_\beta \neq \emptyset$ $\frac{f_\beta}{f_\alpha}$ is holomorphic and nowhere vanishing.

Again by the compactness of C, the collection of f_α defines a finite set of points counted with multiplicities, and so one gets the notion of a (Weyl) divisor as an element of the free abelian group generated by the points of C. It is straightforward to check that a divisor defines a line bundle. Also the converse is true, i.e.

Proposition. On an algebraic curve every line bundle is the line bundle associated to a divisor.

In particular every line bundle admits a meromorphic section and hence its first Chern class can be computed as

$$c_1(L) = \int_C \partial \bar{\partial} \log |s|^2$$

where s is *any* meromorphic section of L and $|\cdot|^2$ is *any* hermitean metric along the fibres.

Two divisors $D$and D' are called linearly equivalent whenever $D - D'$ is the divisor of a global meromorphic function. The set Div$C/\sim$ of divisors on C modulo linear equivalence is actually isomorphic to the Picard group of C, and, furthermore, if $L = [D]$ is a line bundle in the equivalence class of $D = \sum n_i p_i$ its first Chern class is given by

$$c_1(L) \equiv \deg\, L = \deg\, D = \sum n_i$$

The cornerstones of the theory of algebraic curves are the Riemann-Roch theorem and the Serre duality theorem.

The Riemann-Roch theorem computes the Euler characteristics of any invertible sheaf on a (smooth) algebraic genus g curve C as

$$\chi(L) = dim H^0(C, L) - dim H^1(C, L) = \deg\, L + 1 - g$$

Serre duality assumes, in one complex dimension, the following simple form:

$$H^0(C, L^{-1} \otimes K_C) \simeq H^1(C, L)^*$$

where $'*'$ means the dual vector space and the duality is given (in Dolbeault cohomology), by integrating the product of a holomorphic (1,0)-form with values in L^{-1} and a non-$\bar{\partial}$-exact (0,1)- form with values in L.

The Riemann-Roch theorem generalizes straightforwardly to the case of a node curve, provided one takes as g the so called arithmetic genus $p_a(C)$ of C, which can be read off the exact sheaf sequence associated to the map $\alpha : N_C \longrightarrow C$ as $p_a(C) = p_a(N_C) + r$, or, more explicitly,

$$p_a(C) = \sum_{i=1}^{\nu} g(N_i) \quad + 1 - \nu + r$$

where the N_i's are the connected components of the normalization N_C. The Serre duality theorem is more subtle, for reasons we will explain in detail farther on, after having introduced the notion of family of Riemann surfaces and discussed the moduli problem.
The classical results above, together with a Kodaira vanishing theorem give an almost complete solution to the so called Riemann-Roch problem, i.e. to the computation of the dimension $h^0(C, L)$of $H^0(C, L)$. In fact, one gets "for free" the following results:

$$\deg\, L < 0 \Rightarrow h^0(C, L) = 0$$

$$\deg\, L <= 0 \Rightarrow h^0(C, L) = \begin{cases} 1 & \text{and} \quad L = \mathcal{O}_C \\ 0 & \text{otherwise;} \end{cases}$$

$$\deg\, L = 2g - 2 \Rightarrow h^0(C, L) = \begin{cases} g & \text{and} \quad L = K_C \\ g - 1 & \text{otherwise;} \end{cases}$$

$$\deg L > 2g - 2 \Rightarrow h^0(C, L) = \deg L - g + 1$$

For L in the unstable range $0 < \deg L < 2g - 2$ Riemann-Roch gives only the lower bound $h^0(C, L) \geq \deg L - g + 1$. An upper bound is given by Clifford's theorem, stating that

$$h^0(C, L) \leq \frac{1}{2}(\deg L + 1)$$

with equality reached only if $L = \mathcal{O}_C$, $L = K_C$ or C is hyperelliptic, i.e. a double covering of the rational curve $\mathbb{P}^1$ (See fig. 3).

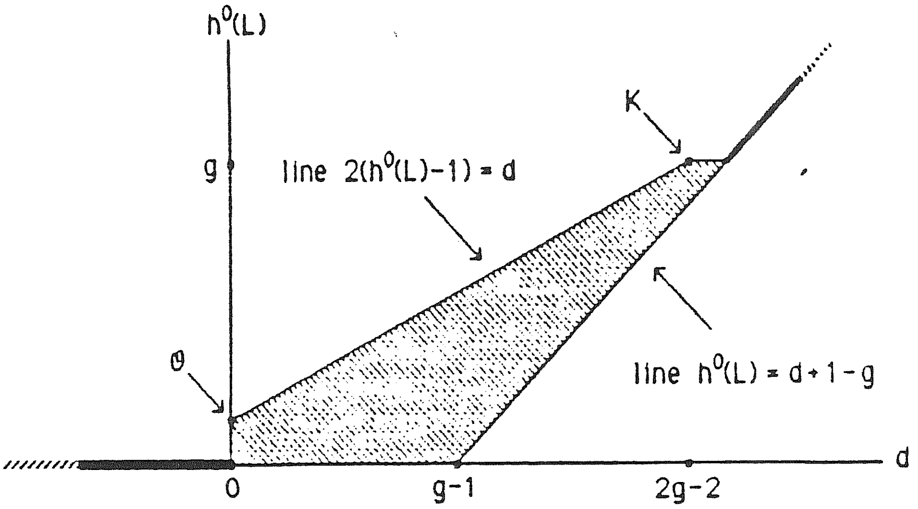


fig.3

The last topic we mean to cover is the notion of θ-characteristics .

Definition. A θ-characteristics on a smooth curve C is a line bundle $\mathcal{L}$ s.t. $\mathcal{L} \otimes \mathcal{L} \simeq K_C$. The defining equation makes sense as $\deg K_C = 2g - 2$ is always even. In particular the degree of a θ-characteristics is $g - 1$. In terms of divisors the defining relation becomes $2[D] = [K_C]$, so that the quickest way of computing the number of θ-characteristics is the following. Let us fix a θ-characteristics $\mathcal{L}_0$. Then for any θ-characteristics $\mathcal{L}$ it holds

$$\left(\mathcal{L} \otimes \mathcal{L}_0^{-1}\right)^2 = (\mathcal{L})^2 \otimes \left(\mathcal{L}_0^{-1}\right)^2 \simeq K_C \otimes K_C^{-1} = \mathcal{O}_C$$

so that θ-characteristics are in 1-1 (non canonical) correspondence with points of order two in the Jacobian $J(C)$. Then, as $J(C)$ is a g-dimensional torus, the number of such points, i.e. the number of non-isomorphic θ-characteristics is 2^{2g}. As $\deg\mathcal{L} = g-1$, $\mathcal{L}$ lies in the unstable range, so that one should not expect $h^0(C, \mathcal{L})$ to be independent of $\mathcal{L}$. In fact the best one can do is to divide θ-characteristics according to their parity, i.e. to $h^0(\mathcal{L})$mod 2. Actually there is some merit in doing so, because the parity of a θ-characteristics is invariant under deformations of the curve, and the number of even (resp. odd)θ-characteristics is known to be [M1] $2^{g-1}(2^g+1)$ (resp. $2^{g-1}(2^g-1)$).

Being $\mathcal{L}$ a "square root" of the canonical bundle, one naturally thinks of its sections as of spinor fields on the curve C. Now, in a more classical setting, recall that, on an m-dimensional real riemannian manifold M a spin structure is defined to be a principal fiber bundle $\tilde{P} \xrightarrow{\tilde{\pi}} M$ with structure group Spin(n) such that, if $P \xrightarrow{\pi} M$ is the bundle of orthonormal frames on M, and α is the non trivial double covering Spin$(n) \xrightarrow{\alpha} \mathrm{SO}(n)$ there exists a commutative diagram

$$\begin{array}{ccccccc} \mathrm{Spin}(n) & \times & \tilde{P} & \longrightarrow & \tilde{P} & & \\ & & & & & \stackrel{\tilde{\pi}}{\searrow} & \\ \downarrow{\scriptstyle\alpha} & & \downarrow & & & & M \\ & & & & & \stackrel{\pi}{\nearrow} & \\ \mathrm{SO}(n) & \times & P & \longrightarrow & P & & \end{array}$$

It is known that such a commutative diagram exists iff the obstruction class (the 2^{nd} Stiefel-Whitney class) $w_2 \in H^2(M, \mathbb{Z}_2)$ vanishes, and the number of inequivalent diagrams is the order of $H^1(M, \mathbb{Z}_2)$. Actually, due to the evenness of the first Chern class of the tangent bundle, every Riemann surface is a spin manifold, and the following theorem holds, relating θ-characteristics to spin-structures [A]

Theorem. The spin structures on a compact complex spin manifold correspond bijectively to the isomorphism classes of holomorphic line bundles $\mathcal{L}$ with $\mathcal{L}^2 \simeq K$, where K is the canonical bundle, i.e. the top exterior power of the cotangent bundle.

2.4 The moduli problem

The moduli problem is a central one in algebraic geometry, as it consists in a part of the classification problem. In fact, roughly speaking, algebraic varieties are classified by some discrete invariant (such as the genus for curves) and some continuous invariants which, for historical reasons are called *moduli*.

Definition. a family of compact complex manifolds is a proper surjective holomorphic map $\pi : X \longrightarrow S$ between two complex analytic manifolds such that $\forall s \in S$ the fiber $\pi^{-1}(s)$ is a compact complex manifold.

Theorem. [MK] A family of compact complex manifolds is differentiably locally trivial, i.e. locally isomorphic, in the C^∞ category, to the product $C_* \times S$.

Then, considering a family of Riemann surface s over a connected base S, the genus of the fibers is constant and, more generally, all discrete topological invariants of the fibers will not vary over S.

Given a family $\pi : X \longrightarrow S$ and a map $f : S' \longrightarrow S$ one can define the pull back family as the fibered product $f^*(X) = X \times_f S'$. It is a family over S' and comes equipped with a commutative diagram

$$\begin{array}{ccc} f^*(X) & \xrightarrow{\phi} & X \\ \downarrow & & \downarrow \\ S' & \xrightarrow{f} & S \end{array}$$

This notion leads to define in a natural way a contravariant functor from the category of complex manifolds and holomorphic maps to the category of sets and maps by sending:

$$S \rightsquigarrow \mathcal{M}_g(S)$$

where $\mathcal{M}_g(S)$ is the set of isomorphism classes of smooth algebraic varieties parameterized by S, with "topological" invariant g and

$$f \in Hom(S, S') \rightsquigarrow \mathcal{M}_g(f) = [\phi] \in Hom(\mathcal{M}_g(S'), \mathcal{M}_g(S))$$

where ϕ is the map defined in the diagram above up to isomorphisms.

Suppose now $\mathcal{M}_g$ is a representable functor, i.e. there exists a complex manifold M_g such that $\mathcal{M}_g(S)$ is isomorphic to the functor $S \rightsquigarrow Hom(S, M_g)$

Definition If $\mathcal{M}_g$ is representable, then M_g is called a *fine* moduli space for $\mathcal{M}_g$.

Unfortunately, for the case of algebraic curves, the functor $\mathcal{M}_g$ is not representable (here g is the only topological invariant of a connected curve, i.e. its genus) as the following example shows [Hu].

Let us consider the family defined in $\mathbb{C}^2 \times \mathbb{C} \setminus \{0\}$ by:

$$X = \{(x, y, t) \in \mathbb{C}^2 \times \mathbb{C} \setminus \{0\} \mid y^2 = x^{2g+2} - t$$

Then compactify to $\mathbb{P}^2$ and define $\pi : X \longrightarrow \mathbb{C} \setminus \{0\}$ as $\pi(x, y, t) = t$

It is easy to see that each curve C_t is isomorphic to a fixed one, say C_1, nonetheless X is not topologically a product (the locally trivializing map $(x, y, t) \rightsquigarrow (xt^{\frac{1}{2g+2}}, yt^{\frac{1}{2}}, t)$ is homotopically non trivial.) This is incompatible with the representability of the Riemann functor $\mathcal{M}_g$, as the classifying map for "constant" families sends the base into a point of M_g and hence a pull-back family under a constant map should be a product.

The non existence of a fine moduli space for curves of genus g can be circumvented in at least two ways. The first is to consider the moduli space of curves together with some additional structures, which leads, for instance, to the notion of Teichmüller space, which will be described later on. The second is to relax the assumption of representability of the functor $\mathcal{M}_g$ thus getting the notion of *coarse* moduli space as follows

Definition. M_g is called a coarse moduli space for genus g curves if there is a morphism of functors

$$\Phi : \mathcal{M}_g \longrightarrow Hom(\cdot, M_g)$$

satisfying

i) if B is a point, $\Phi(B)$ is an isomorphism

ii) for any other morphism of functors $\xi : \mathcal{M}_g \longrightarrow Hom(\cdot, X_g)$ there is a unique map $\varrho : M_g \longrightarrow X_g$ for which the corresponding morphism of functors $[\varrho] : M_g \longrightarrow X_g$ satisfies

$$\xi = [\varrho] \circ \Phi.$$

In practice, when considering a coarse moduli space one gives up uniqueness of the classifying map and simply requires its nonuniqueness to be under control.

A coarse moduli space does exist for genus g curves. The failure of the existence of a fine moduli space can be ascribed to the fact that some curves have automorphisms, and, moreover, certain ones have more automorphisms than neighbouring others. In fact , if the Riemann functor were representable, its moduli space would come equipped with a "universal" family $\mathcal{C}_g \xrightarrow{\pi} M_g$ from which any other family $X \xrightarrow{f} B$ could be obtained by means of a unique diagram like

$$\begin{array}{ccc} X & \xrightarrow{F} & \mathcal{C}_g \\ \downarrow f & & \downarrow \pi \\ B & \xrightarrow{\varphi} & M_g \end{array}$$

Now, suppose $C \longrightarrow p$ is a curve with a non trivial automorphism γ over a point p. Then one can get a different diagram from the one above simply by twisting F by γ, thus loosing uniqueness.

So one is prompted, in order to end up with a fine moduli space, to seek for objects without automorphisms . For instance one can consider the moduli space of triples $\{C, p, \omega\}$ where $p \in C$ and ω is a non-zero (co)tangent vector [ADKP].

In some sense, the classical Teichmüller theory can be described as originating from the following analysis of the automorphisms group of a smooth algebraic curve [ACGH].

The cases of genus 0 and 1 are easily dealt with. For the Riemann sphere $\mathbb{P}^1$, $Aut(\mathbb{P}^1)$ is $PGL(1, \mathbb{C})$. For $g = 1$, $Aut(C_1)$ is the extension by C itself of a group F of order $2, 4,$ or 6 and, actually, there are unique tori for which $ord\ F = 4$ or 6. In the cases $g \geq 2$ things are much sharpened as the following proposition holds

Proposition Let C be a smooth genus g curve. Then $Aut(C)$ is a finite group of order at most $84(g-1)$.

Furthermore Hurwitz's theorem holds:

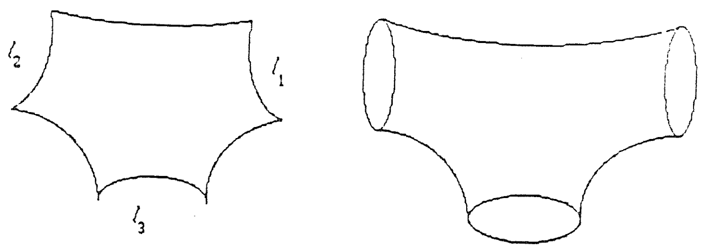


fig 4

Theorem. In the above hypothesis, if $\varphi \in Aut(C)$ and φ is homotopic to the identity, then φ is the identity.

The above considerations lead naturally to the following

Definition. Let Σ be a closed oriented 2-dimensional manifold of genus g. A *marked Riemann surface* is a pair $(C, [f])$ where C is a Riemann surface , $f : C \longrightarrow \Sigma$ is a homeomorphism and $[f]$ denotes the homotopy class of f.
Two marked Riemann surface $(C, [f])$and $(C', [f'])$ are called equivalent iff there is a conformal map $C \xrightarrow{h} C'$ such that $[f' \circ h] = [f]$.

The family version of this construction can be defined as follows [Hu]. Let $\pi : X \longrightarrow S$ be a family of curves. A Teichmüller structure of type Σ over X is the datum of an equivalence class of diffeomorphisms $[\Psi] : S \times \Sigma \longrightarrow X$ commuting with π where two diffeomorphisms are said to be equivalent if they are homotopic via a fiber map over S. Then, by a fiberwise argument, if $\Psi : S \times \Sigma \longrightarrow X$ is a representative of a Teichmüller structure, then any automorphism of X over S preserving $[\Psi]$ is the identity. In simpler words two curves related by a conformal automorphism not homotopic to the identity are to be considered as *different* in Teichmüller theory. Let $Diff^+(\Sigma)/Diff_0^+ \equiv \Gamma_\Sigma$ the mapping class group of the topological surface Σ. Given any family of curves $\pi : X \longrightarrow S$ one can consider the following principal fiber bundle

$$\begin{array}{ccc} \Gamma_\Sigma & \longrightarrow & \Gamma_X \\ & & \downarrow \\ & & S \end{array}$$

Definition. The Teichmüller functor of type Σ, $\mathcal{T}_g$ associates to an analytic space S the set of isomorphism classes of family of curves $\pi : X \longrightarrow S$ equipped with a section of Γ_X

By the previous discussion one expects good properties for the Teichmüller functor. In fact it holds [Hu]

Theorem. The Teichmüller functor $\mathcal{T}_g$ is representable by means of a Stein variety of

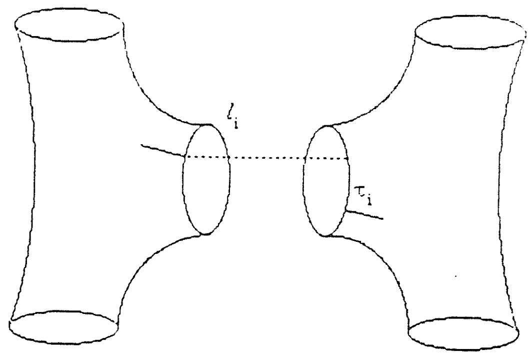


fig 5

dimension $3g-3$ isomorphic to an open ball in $\mathbb{C}^{3g-3}$, which is called the Teichmüller space T_g.

A real coordinatization of T_g is better achieved by means of the Fenchel-Nielsen coordinates, defined as follows. Consider an hexagon in the hyperbolic plane, which is determined, up to isometries, by the lengths l_1, l_2, l_3 of alternating sizes (see fig. 4) Considering its double across the remaining sizes we got a pair of pants, which are building blocks for a Riemann surface in the sense that a genus g Riemann surface can be obtained by glueing $2g-2$ pair of pants. This works as follows: fix a collection $\{\gamma_1 \dots \gamma_{3g-3}\}$ of disjoint simple closed curves such that $\Sigma \setminus \{\gamma_i\}$ is the disjoint union of pair of pants . Then Σ can be completely reconstructed by attaching these pair of pants along the $\{\gamma_i\}$'s. The Fenchel Nielsen coordinates are the free parameters in this construction: they are the geodesic lengths l_i of the γ_i and the hyperbolic distances τ_i between the feet of perpendiculars to γ_i dropped from fixed boundary points [Ha](fig. 5).

The Teichmüller space T_g admits a natural geometric structure called the Weil-Petersson form. The cotangent space $T^*_{[C]}(T_g)$ at $[C]$ is naturally identified with the space of holomorphic quadratic differentials on C. Then one defines the Weil-Petersson metric via the following hermitean form on $H^0(C, K_C)$

$$<\varphi_1, \varphi_2>= \int_C \varphi_1 \overline{\varphi}_2 \lambda^{-2}$$

where λ is the line element on C. One has the following results[Wo]
i) The Weil-Petersson metric is Kähler
ii) In Fenchel-Nielsen coordinates its Kähler form is expressed as

$$\omega_{WP} = -\sum d\tau_i \wedge dl_i$$

The study of the geometry of the Teichmüller space goes a great deal the study of the geometry of the (coarse) moduli space of curves. In fact, recalling that T_g can be thought of as $Conf(\Sigma)/Diff_0(\Sigma)$, while M_g is given by $Conf(\Sigma)/Diff^+(\Sigma)$, one can use the topological triviality of T_g to get some insight into the topology of M_g. In fact, intuitively, M_g should be T_g/Γ_g, where Γ_g is the mapping class group of the Riemann surface . In fact one has the following celebrated results [Ha]

Proposition. If $g \geq 3$ the action of Γ_g over T_g is properly discontinuous but not free. Its fixed points correspond to algebraic curves with non trivial automorphisms group. Correspondingly, the moduli space M_g has the structure of a complex space and a complex V-manifold (or *orbifold*). The lower cohomology groups of M_g are computed as follows

$$H^0(M_g, \mathbb{Z}) = \mathbb{Z}$$

$$H^1(M_g, \mathbb{Z}) = 0$$

$$H^2(M_g, \mathbb{Z}) = \mathbb{Z}$$

A deeper understanding of moduli space and of its complex structures is better achieved by means of deformation theory, which we are going to outline in the sequel. To begin with, let X be a smooth algebraic curve.

Definition. A deformation of X, parameterized by a pointed analytic space (Y, y_0) is a proper holomorphic map

$$\varphi : \mathcal{X} \longrightarrow Y$$

plus a given isomorphism $\psi : X \to \varphi^{-1}(0)$ between X and the central fiber $\varphi^{-1}(0)$.

The notion of deformation thus differs from the one of family by the prescribed identification of the central fiber with the object to be deformed. A it first order deformation of X is a deformation of X parameterized by $S = Spec\mathbb{C}[\epsilon]$, the spectrum of the dual numbers. In the sequel we will pursue the Kodaira-Spencer approach to deformation theory, in which on thinks of the curve C as being qualified by patching data $\{U_\alpha, z_\alpha, f_{\alpha\beta}\}$, and thinks of deforming it by deforming the patching data. Here

$\{U_\alpha\}_{\alpha \in I}$ is a finite covering of C

z_α is a holomorphic coordinate in U_α

$z_\alpha = f_{\alpha\beta}(z_\beta)$ in $U_\alpha \cap U_\beta$

In any triple intersection $U_\alpha \cap U_\beta \cap U_\gamma$ the cocycle rule holds

$$f_{\alpha\beta}(f_{\beta\gamma}(z_\gamma)) = f_{\alpha\gamma}(z_\gamma)$$

A first order deformation, can then be thought of as being given by glueing products $U_\alpha \times S$ by means of

$$z_\alpha = \tilde{f}_{\alpha\beta}(z_\beta, \epsilon) \equiv f_{\alpha\beta}(z_\beta) + \epsilon \cdot b_{\alpha\beta}(z_\beta)$$

For any fixed ϵ the $\tilde{f}_{\alpha\beta}$'s must be transition functions for a curve, so that they must satisfy the cocycle rule. This latter translates into the above condition for the $f_{\alpha\beta}$'s and the following condition for the $b_{\alpha\beta}$'s

$$b_{\alpha\beta} + \frac{\partial f_{\alpha\beta}}{\partial z_\beta} \cdot b_{\alpha\beta} = b_{\beta\gamma}$$

But, by the chain rule $\frac{\partial f_{\alpha\beta}}{\partial z_\beta}\frac{\partial}{\partial z_\alpha} = \frac{\partial}{\partial z_\beta}$ so that, putting $X_{\alpha\beta} := b_{\alpha\beta}\frac{\partial}{\partial z_\alpha}$ yields

$$X_{\alpha\beta} + X_{\beta\gamma} - X_{\alpha\gamma} = 0$$

i.e. $X_{\alpha\beta}$ defines a class

$$[X_{\alpha\beta}] \in \check{H}^1(C, T_C)$$

which is called the Kodaira-Spencer class of the first order deformation φ.

The "sheafified" version of this construction is given as follows. For any first order deformation $\varphi : \mathcal{X} \to S$ one gets the following exact sequence of $\mathcal{O}_C$-modules

$$0 \to T_C \longrightarrow T_{\mathcal{X}} \xrightarrow{\varphi_*} \varphi^* T_S \to 0$$

which induces the exact cohomology sequence

$$\cdots \to \check{H}^0(C, T_{\mathcal{X}}) \xrightarrow{\varphi_*} \check{H}^0(C, \varphi^*(T_S) \xrightarrow{\delta^*} \check{H}^1(C, T_C) \to \cdots$$

Then $X_{\alpha\beta}$ is just $\delta^*\left([\varphi^*\frac{\partial}{\partial\epsilon}]\right)$. Given two isomorphic deformations $\varphi : \mathcal{X} \longrightarrow S$ and $\varphi' : \mathcal{X}' \longrightarrow S$ the commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & \check{H}^0(C, T_C) & \longrightarrow & \check{H}^0(C, T_{\mathcal{X}}) & \xrightarrow{\varphi_*} & \check{H}^0(C, \varphi^*(T_S)) & \xrightarrow{\delta^*} & \check{H}^1(C, T_C) & \to \cdots \\
 & & \| & & \downarrow & & \downarrow & & \| & \\
0 & \to & \check{H}^0(C, T_C) & \longrightarrow & \check{H}^0(C, T'_{\mathcal{X}}) & \xrightarrow{\varphi'_*} & \check{H}^0(C, \varphi'^*(T_S)) & \xrightarrow{\delta^*} & \check{H}^1(C, T_C) & \to \cdots
\end{array}$$

insures that $[X_{\alpha\beta}]$and $[X'_{\alpha\beta}]$ are the same class in $\check{H}^1(C, T_C)$.
Conversely, taking two cocycles in the same class, the difference between the two infinitesimal deformations they induce is simply a holomorphic coordinate change, so that the two are really indistinguishable.

Now, given an arbitrary deformation $\phi : \mathcal{X} \longrightarrow (Y, y_0)$ and a map $(S, s_0) \xrightarrow{f} (Y, y_0)$ the pull-back $f^*(\mathcal{X})$ is the first order approximation of $\phi : \mathcal{X} \longrightarrow (Y, y_0)$ in the direction of the tangent vector corresponding to f - notice that the spectrum of dual numbers embodies the notion of tangent vector to an algebraic space, in the sense that $T_{y_0}Y \simeq Hom((S, s_0), (Y, y_0))$. Thus we get a homomorphism

$$\rho_\phi : T_{y_0}(Y) \longrightarrow H^1(C, T_C)$$

called the Kodaira-Spencer homomorphism associated to $\phi : \mathcal{X} \longrightarrow (Y, y_0)$.
Definition. Let $\pi : \mathcal{X} \longrightarrow (Y, y_0)$ be a deformation of C. $\mathcal{X}$ is said to be complete at y_0 (or *versal*) at y_0 if for any other deformation of C,

$$\pi' : \mathcal{Y} \longrightarrow (Y', y'_0)$$

there exists a neighbourhood $V' \ni y_0'$ and a holomorphic map $g : V' \longrightarrow Y$ sending y_0' to y_0 such that the restricted family $\mathcal{Y}\lceil V'$ is isomorphic to $f^*(\mathcal{X})$ over (V', y_0').

Definition. Let $\pi : \mathcal{X} \longrightarrow (Y, y_0)$ be a versal deformation of C. We say it is *universal at* y_0 if the germ of the classifying map above is unique.

Remark (Uni)versality is a local property, in the sense that if $\pi : \mathcal{X} \longrightarrow (Y, y_0)$ is (uni)versal at y_0, then it is (uni)versal in a whole neighbourhood of y_0.

The rôle of the Kodaira-Spencer map is clarified by the following
Theorem. Let $\pi : \mathcal{X} \longrightarrow (Y, y_0)$ be a deformation of C and suppose that
i)The Kodaira-Spencer map ρ_π is an isomorphism
ii)$H^2(C, T_C) = \{0\}$
Then $\pi : \mathcal{X} \longrightarrow (Y, y_0)$ is universal.

A deformation satisfying the above requirements will also be called a *Kuranishi* deformation.
The theorem above allows one to compute in a very elegant way the dimension of moduli space of genus g curves. In fact, if $g \geq 2$ the tangent sheaf T_C has negative degree, so the dimension of $H^1(C, T_C)$ is read off the Riemann-Roch theorem as

$$dimH^1(C, T_C) \equiv dimM_g = 3g - 3.$$

Before describing how the this formalism endows the set M_g of a complex structure by (roughly speaking) glueing together bases of universal deformations of nearby curves, we will discuss in some details the problem of compactifying M_g.
Needless to say, this issue is of overwhelming relevance as, for instance, most of the known algebro-geometrical techniques one could hope to use in studying M_g work only in the case of complete varieties. Also, given a family of curves $\pi : \mathcal{X} \longrightarrow B$ having a singular fiber, say $\pi^{-1}(0)$, one would like to describe it in terms of a map into a compactification of M_g, rather than considering the family of smooth curves $\pi : \mathcal{X}^* \longrightarrow (B \setminus 0)$ and then "taking the limit". As there are several compactifications, we will spend some time in describing them. A first attempt to compactifying M_g stems from the following construction.

Let C be a smooth curve and consider a canonical basis $\{a_1 \dots a_g, b_1 \dots b_g\}$ of the first homology group $H_1(C, \mathbb{Z}$ (see fig. 1) as in §2.3 and the Jacobian torus $J(C) = \mathbb{C}^g/\Lambda$, $\Lambda_{ij} = (\delta_{ij}|\omega_{ij})$. Besides being a complex torus, $J(C)$ is an algebraic variety and carries an ample line bundle L such that $dimH^0(J(C), L) = 1$. Summing up
Proposition. $J(C)$ is a *principally polarized abelian variety* (p.p.a.v.).
Let us now consider the generalized Siegel upper half space, i.e. the set $\mathcal{H}_g$ of all $g \times g$ symmetric matrices with positive definite imaginary part. Given $\tau \in \mathcal{H}_g$ we can consider the complex torus $X_\tau = \mathbb{C}^g/\Lambda_\tau$where $\Lambda_\tau = (\mathbb{I}|\tau)$

Considering coordinates $z^i$and τ^{ij} in $\mathbb{C}^g$and $\mathcal{H}_g$, one defines the Riemann's theta-"function" as

$$\theta(z, \tau) = \sum_{p \in \mathbb{Z}} \exp 2\pi i\{\frac{1}{2}{}^tp \cdot \tau \cdot p + {}^tp \cdot z\}$$

The quasi-periodicity property of the theta-function

$$\theta(z+n+{}^t m\cdot\tau,\tau)=\exp 2\pi i\{-\frac{1}{2}{}^t m\cdot\tau\cdot m-{}^t m\}\theta(z,\tau),\quad m,n\in\mathbb{Z}$$

insures that it defines a divisor on X_τ - the so-called Θ-divisor. Correspondingly let L_τ be the Θ-line bundle. One has the following property
Theorem . Any principally polarized abelian variety (X,L) is of the form (X_τ,L_τ) for some $\tau\in\mathcal{H}_g$.

One declares two principally polarized abelian variety $(X,L)$and (X',L') to be isomorphic iff there is an isomorphism $\phi: X\longrightarrow X'$ such that $\phi^*L'=L$.

Theorem. Given $\tau,\tau'\in\mathcal{H}_g$ the corresponding abelian varieties (X_τ,L_τ), (X'_τ,L'_τ) are isomorphic iff there exists a matrix

$$A=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in Sp(2g,\mathbb{Z})$$

such that

$$\tau'=\frac{\alpha\cdot\tau+\beta}{\gamma\cdot\tau+\delta}$$

Consequently, the moduli space of principally polarized abelian variety is given by

$$\mathcal{A}_g=\mathcal{H}_g/Sp(2g,\mathbb{Z})$$

Now it turns out [Po] that $\mathcal{A}_g$ is a quasi projective variety of dimension $\frac{g(g+1))}{2}$ which admits a natural compactification (called the Satake compactification) $\bar{\mathcal{A}}^s_g$, in which the boundary $\partial\bar{\mathcal{A}}^s_g$ has codimension 2.

What matters for us are the following considerations. Let $\tilde{M}_g$, the moduli space of pairs $(C,\{a_1,\ldots,a_g,b_1,\ldots,b_g\})$ of (smooth) curves and symplectic bases for $H_1(C,\mathbb{Z})$. Then the Jacobi map

$$\begin{array}{ccc} J:\tilde{M}_g & \longrightarrow & \mathcal{H}_g\\ [(C,\{a,b\}] & \rightsquigarrow & \tau=\left(\oint_{b_j}\omega_i\right)\end{array}$$

(where the ω_i's are a basis in $H^0(C,K_C)$ normalized according to $(\oint_{a_j}\omega_i=\delta_{ij})$ sends $\tilde{M}_g$ into a subvariety $\mathcal{J}_g$of $\mathcal{H}_g$ called the jacobian locus, and there is a commutative diagram

$$\begin{array}{ccc} \tilde{M}_g & \xrightarrow{J} & \mathcal{H}_g\\ \downarrow & & \downarrow\\ M_g & \xrightarrow{T} & \mathcal{A}_g & =\mathcal{H}_g/Sp(2g,\mathbb{Z})\end{array}$$

T is called the *Torelli* map and is injective.
Hence, by identifying M_g with $T(M_g)$ one gets the Satake compactification of the moduli space of genus g curves simply by taking $\overline{M}^s_g:=\overline{T(M_g)}^s$ in $\overline{\mathcal{A}}^s_g$.
Obviously, $\partial\bar{M}^s_g$ is of codimension 2 in $\bar{M}^s_g$. This fact has a powerful consequence: a simple

application of the classical Hartog's theorem yields that M_g is quasi-compact, i.e. any holomorphic function on M_g (regular at the boundary) is a constant.

But, by the same reason, (i.e. the impossibility of controlling singularities at the boundary due to the "thinness" of it) the Satake compactification is not a completely satisfactory one.

A different compactification scheme of M_g was proposed by Deligne Mumford and Mayer in the late seventies, yielding a thicker boundary to M_g and a complete understanding of the moduli space of curves. Roughly speaking the idea goes as follows: one compactifies M_g by adding curves with singularities. The hard task is to decide *which* singular curve one has to add, as the following toy example shows [H1]
Consider the genus 1 case. We obviously would like to add to M_1 a point C_* corresponding to the to the node curve $y^2 = x^2(x+1)$. Actually, in the family

$$y^2 = x^2(x+t), \quad t \in \mathbb{C}$$

all the fibers are isomorphic to C_* for $t \neq 0$ whilst the curve C_0 is the cuspidal curve $y^2 = x^3$. Then, in order for the induced map $\mathbb{C} \xrightarrow{\phi} \overline{M}_1$ to be continuous, the point $[C_0]$ must lie in the closure of the point $[C_*]$. Take now the family $y^2 = x^3 - t$. Here, again, all curves are isomorphic to one another except C_0 which is again the cuspidal one, and in particular, they are all isomorphic to the smooth curve C_1. This shows that the cuspidal curve must lie in the closure of any point of $\overline{M}_1$, so that such an $\overline{M}_1$ would be non separated!

Mumford's compactification scheme solves this trouble by choosing a specific set of singular curves to be plugged into $\bar{M}_g$ as its boundary, the so called *stable* ones. A word of warning: even though such curves have nice geometrical properties, what actually dictates their choice is geometric invariant theory, which is a major tool in Mumford's construction.

Definition. A stable curve (resp. a semistable one) is a connected reduced node curve of genus $g > 1$ such that any of its rational components meets the rest of the curve in at least three (resp. 2) points.
A satisfactory deformation theory can be settled up for such curves by using a particular class of Kuranishi families, i.e. the so-called *Schiffer variations.* Let p be a generic point of C. and consider the following exact sequence

$$0 \to T_C \longrightarrow T_C(p) \longrightarrow \mathcal{S}_p^{\mathbb{C}} \to 0$$

The coboundary map δ_p sends $H^0(C, \mathcal{S}_p^{\mathbb{C}}) = \mathbb{C} \longrightarrow H^1(C, T_C)$. In terms of Čech cocycles, considering the acyclic cover $\{U, V\}$, where $V = C \setminus p$ and U is a small disk centered at p and parameterized by z then a representative of $\delta_p(1)$ is

$$X_{UV} = \frac{1}{Z}\frac{\partial}{\partial z}$$

By Serre duality the map

$$\begin{array}{ccc} C & \longrightarrow & \mathbb{P}H^1(C, T_C) \\ p & \rightsquigarrow & [\delta_p] \end{array}$$

is the bicanonical map, so that one gets that Schiffer variations generate $H^1(C, T_C)$. Schiffer variations have the advantage of being easily integrated. Namely one can find a deformation

$$\Psi : \mathcal{D} \longrightarrow \Delta_\epsilon = \{t \in \mathbb{C} | |t| < \epsilon\}$$

whose Kodaira-Spencer map is δ_p. In fact, such a $\mathcal{D}$ can be obtained, if Δ' is another small disk of radius ϵ', by glueing

$$C \setminus \{p \in C s.t. |z(p)| < \frac{\epsilon}{2}\} \times \Delta_\epsilon \text{ with } \Delta' \times \Delta_\epsilon$$

via the glueing map

$$\begin{cases} t = \tilde{t} \\ w = z + \frac{t}{z} \end{cases}$$

In general, by choosing $3g-3$ distinct general points p_i and removing $3g-3$ small disks around them one gets a family parameterized by $3g-3$-dimensional polydisk, defined by the glueing law

$$\begin{cases} t_i = \tilde{t}_i \\ w = z_i + \frac{t_i}{z_i} \end{cases}$$

The Kodaira-Spencer map of this $3g-3$-dimensional Schiffer variation is the coboundary map δ in the exact sequence

$$\cdot \to \longrightarrow H^0(C, T_C(\textstyle\sum_1^{3g-3} p_i)) \stackrel{\oplus_i res_{p_i}}{\longrightarrow} \oplus_{j=1}^{3g-3} \mathcal{S}_{p_j}^{\mathbb{C}} \stackrel{\delta}{\longrightarrow} H^1(C, T_C) \longrightarrow H^1(C, T_C(\textstyle\sum_1^{3g-3} p_i)) \to \cdot$$

Now, denoting as usual $h^i(C, L) := dim H^i(C, L)$, one has

$$h^0(C, T_C) = h^1(C, T_C) = h^0(C, K_C^2(-\textstyle\sum_1^{3g-3} p_i))$$

where the second equality is just Serre duality, while the first comes from Riemann-Roch noticing that $deg(T_C(\sum_1^{3g-3} p_i)) = g-1$. Then δ will be an isomorphism, and hence the Schiffer variation above a Kuranishi family if and only if $h^0(C, K^2(\sum_1^{3g-3} p_i)) = 0$. But this can be achieved by choosing the points p_i according to the following strategy.
Let s_1 be a non-zero holomorphic section of K^2 and let p_1 be such that $s_1(p_1) \neq 0$ so that $h^0(C, K^2(-p_1)) = h^0(C, K^2) - 1$. Then let s_2 be a non-zero section of $K^2(-p_1)$ and choose p_2 where s_2 is not vanishing. Again

$$h^0(C, K^2(-p_1 - p_2)) = h^0(C, K^2) - 2$$

Inductively, one can find $p_1, \ldots, p_j$ such that $h^0(C, K^2(-\sum_i = 1^j p_i)) = h^0(C, K^2) - j$. Obviously this process ends after $3g-3$ steps as, by Riemann-Roch, $h^0(C, K^2) = 3g-3$. Notice that there is no flaw in this argument, as, at each step we are dealing with line bundles of degree $\geq g$ and again Riemann-Roch insures that they have at least one non-zero holomorphic section.

A similar procedure can be applied to the study of deformations of stable curves [ACGH]. In fact a stable curve C can be thought of as being qualified by its normalization N_C and the identification of the preimages of the nodes $\tilde{p}_i$ as $p_1 \sim q_1, \ldots, p_r \sim q_r$. Then we start from a universal deformation of N_C thus getting $\sum_{i=1}^{\nu} m(N_i)$-parameters, where the N_i's are the connected components of N_C and

$$m(N_i) = \begin{cases} 0 & \text{if} N_i \simeq \mathbb{P}^1 \\ 1 & \text{if} g(N_i) = 1 \\ 3g(N_i) - 3 & \text{otherwise.} \end{cases}$$

Taking into account that $\mathbb{P}^1$ has $PGL(2, \mathbb{C})$ and an elliptic curve a complex torus as automorphisms groups, we see that the total contribution of any component N_i is $3g(N_i) - 3$. Then we can deform by identifying a point near p_i to a point near q_i getting $2r$ more parameters, and r additional parameters occur as a result of smoothing the nodes (these are the transversal parameters to the boundary.) Summing up we have that the dimension of the base space T of a universal deformation of C, $\mathcal{Y} \xrightarrow{\psi} (T, t_0)$ is

$$dim\ T = \sum_{i=1}^{\nu} (3g(N_i) - 3) + 3r = 3\left(\sum_{i=1}^{\nu} g(N_i) \ - \nu + r + 1\right) - 3 = 3g(C) - 3$$

To complete the discussion, let us sketch how a natural complex structure can be given to the set $\overline{M}_g$. Natural here means that it must be induced by the notion of family, in a sense that we are going to describe.

First of all, notice that every genus g curve C can be holomorphically embedded in $\mathbb{P}^{5g-6}$ by the means of the tricanonical map. Every automorphisms γ of C will induce an automorphisms of $H^0(C, K^3)$ but, as C is embedded in $\mathbb{P}H^0(C, K^3)$, γ is the restriction of an automorphisms of $\mathbb{P}H^0(C, K^3)$ so that $Aut(C)$ is a discrete algebraic subgroup of $PGL(5g-6, \mathbb{C})$ and hence is finite.

Now let C be a genus g stable curve and suppose

$$\begin{array}{ccc} C & \xrightarrow{\Psi} & \mathcal{C} \\ & & \downarrow \pi \\ & & (S, s_0) \end{array}$$

is its universal deformation; as was noticed when dealing with deformation theory, one can assume that it is a universal deformation for every $s \in S$. Taking $\gamma \in Aut(C)$ and replacing Ψ with $\Psi \circ \gamma$ one gets another universal deformation of C, say $\mathcal{C}'$.
But, via the defining property of universal deformations, there exist unique automorphisms a_γ and b_γ respectively of S and $\mathcal{C}$ such that

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{b_\gamma} & \mathcal{C}' \\ \downarrow \pi & & \downarrow \pi \\ (S, s_0) & \xrightarrow{a_\gamma} & (S, s_0) \end{array}$$

commutes, $a_\gamma(s_0) = s_0$ and $\Psi \circ \gamma = b_\gamma \circ \Psi$

If some other element $\gamma' \in Aut(C)$ carries s to s', then $\pi^{-1}(s) \simeq \pi^{-1}(s)$ so that the map from S to $\overline{M}_g$ factors through $S/Aut(C)$, i.e.

$$\begin{array}{ccc} S & & \\ & \overset{\delta}{\searrow} & \\ \downarrow \chi & & \overline{M}_g^0 \\ & \overset{\eta}{\nearrow} & \\ S/G & & \end{array}$$

Then, by means of geometric invariant theory, one can show that [Mu2,DM] S/G has a natural complex structure under which χ is holomorphic that can be transported via η to an open subset of $\overline{M}_g$. Namely, a local continuous function on $\overline{M}_g$ will be called holomorphic iff its composition with δ is holomorphic , or, in other words, we can give $\overline{M}_g$ the unique complex structure obtained by glueing together bases of universal families of isomorphic curves.

Final step is to show that the procedure of adjoining to M_g stable curves is "exhaustive" and gives rise to a compact space. This is achieved by means of the stable reduction theorem which, roughly speaking, asserts that every family of algebraic curves admits a stable limit. Without entering the details - actually the hard part of the proof is covered by a semi-stable reduction theorem and uniqueness of the limit requires stability - we can state it in the following form.

Theorem. Let X be a complex space and $X \xrightarrow{\pi} \Delta$ a proper map such that $\pi^{-1}(t)$ is a smooth algebraic curve $\forall t \neq 0$.

Then there are an integer n, a family $\Xi \xrightarrow{\sigma} \Delta$ and a commutative diagram

$$\begin{array}{ccc} \Xi \setminus \sigma^{-1}(0) & \xrightarrow{\beta} & X \setminus \pi^{-1}(0) \\ \downarrow \sigma & & \downarrow \pi \\ \Delta \setminus \{0\} & \xrightarrow{\alpha} & \Delta \setminus \{0\} \end{array}$$

where $\alpha(t) = t^n$ and$\beta\restriction_{\sigma^{-1}(t)}$ is an isomorphism $\forall t \neq 0$.

Most of the geometric structures we encountered when dealing with the Teichmüller theory carry over to the moduli space. Namely, for some of the computations to be done in the sequel, we need the following results on the extension of the Weil-Petersson metric to the boundary of moduli space [Wo]. Recall that, in the Teichmüller case, this metric was expressed as

$$< X, Y >= \frac{1}{2} \int_C \lambda^{-2} X \cdot \overline{Y} \qquad X, Y \in H^0(C, K_C^2)$$

If ω_{WP} is the Kähler form associated to the Weil-Petersson metric and ω_{FN} is its expression in Fenchel-Nielsen coordinates, then it holds:

i)ω_{WP} and ω_{FN} define two smooth forms on M_g.

ii)ω_{WP} extends to a closed current on $\overline{M}_g$, while ω_{FN} extends to a smooth symplectic form

on $\overline{M}_g$.
iii) $\omega_{FN} \in H^2_{DR}(\overline{M}_g)$.
iv) ω^k_{FN} is a nontrivial class in $H^{2k}_{DR}(\overline{M}_g)$.
v) The Weil-Petersson volume of $\overline{M}_g$ is finite
vi) ω_{FN} and ω_{WP} define the same Čech cohomology class in $\check{H}^2(\overline{M}_g, \mathcal{R})$.

2.5 Mumford's theorem and applications of the Grothendieck-Riemann-Roch theorem

Let us now describe divisor theory on $\overline{M}_g$, following works by Mumford, Harris, Arbarello, Cornalba and others. What one wants to end up with, is not only a formal description of $Pic(\overline{M}_g)$ but a classification of line bundles on $\overline{M}^0_g$ built by means of geometrically significant objects. First of all, we have to describe how to extend to singular curves abelian differentials and other sheaves [See e.g. Ba]. We have already remarked that, on smooth curves, the abelian differentials play a twofold rôle
(a) they are holomorphic sections of the cotangent sheaf of C
(b) they enter Serre duality theorem.
Actually, this is an accident due to the fact that a curve is a 1-dimensional complex manifold. More in general, let X be a smooth projective variety, and let K_X its canonical sheaf. Then, K_X is an object that "allows" to do duality in the sense that
i) $H^n(X, K_X) \simeq H^{n,n}_{\bar\partial}(X) \simeq H^{2n}(X, \mathbb{Z}) \otimes \mathbb{C} \simeq \mathbb{C}$
and the isomorphisms above are all natural
ii) Given any vector bundle $E \xrightarrow{\pi} X$ a canonical isomorphism is given

$$H^{i,0}_{\bar\partial}(X, E) \xrightarrow{\sim} H^{n-i,i}_{\bar\partial}(X, K_X \otimes E^*)^*$$

Then, given a possibly singular projective variety X one is naturally lead to the following
Definition . A dualizing sheaf ω_X for X is a coherent sheaf together with an explicit homomorphism $H^n(X, \omega_X) \to \mathbb{C}$, called the *trace* homomorphism, such that for all coherent sheaves $\mathcal{F}$ on X the pairing

$$Hom(\mathcal{F}, \omega_X) \times H^n(X, \omega_X) \longrightarrow H^n(X, \omega_X) \longrightarrow \mathbb{C}$$

is non-degenerate.

The following facts are known
i) such an ω_X, if existing, is unique.
ii) ω_X exists and is locally free if X is a projective variety, subjected to the technical condition

of being locally a complete intersection.
iii) if $\mathcal{F}$ is locally free and coherent

$$H^{n-i}(X,\mathcal{F}) \xrightarrow{\sim} \left(H^i(X,\omega_X \otimes \mathcal{F}^*)\right)^*$$

iv) if X is smooth, then ω_X coincides with the canonical sheaf.

Now consider a proper smooth morphism of algebraic varieties $\pi : \mathcal{X} \longrightarrow B$ and the associated sheaf exact sequence

$$0 \to \pi^*\Omega^1_B \longrightarrow \Omega^1_{\mathcal{X}} \longrightarrow \Omega^1_{\mathcal{X}/B} \to 0$$

where $\Omega^1_{\mathcal{X}/B} \simeq \Omega^1_{\mathcal{X}}/\pi^*\Omega^1_B$ is the sheaf of one-forms along the fibers. Taking the maximum wedge product in the exact sequence above one gets

$$\Omega^n_{\mathcal{X}/B} = \omega_X \otimes (\pi^*\omega_B)^*$$

so that
Definition . The relative dualizing sheaf of the family $\pi : \mathcal{X} \longrightarrow B$ is the invertible sheaf

$$\omega_{\mathcal{X}/B} = \omega_{\mathcal{X}} \otimes (\pi^*\omega_B)^*$$

The ultimate reason for doing so relies in the following
Proposition . The restriction of $\omega_{\mathcal{X}/B}$ to any fiber F of π is the dualizing sheaf ω_F.

On the other hand, the generalization of the sheaf of (relative) Kähler differentials to the case of (families with) singular fibres is done by looking at the singular variety X as embedded in a bigger smooth one V and then considering the restriction to X of Kähler differentials on V and imposing Leibnitz rule. It turns out [Mu2] that the relation between the relative dualizing sheaf and the sheaf of relative Kähler differentials is

$$\Omega^1_{C/B} = I_{Sing} \cdot \omega_{C/B}$$

where I_{Sing} is the ideal sheaf of the singular locus.

In more direct terms, let us consider a family $C \xrightarrow{\pi} D$ over a disk D such that $\pi^{-1}(0)$ is a noded curve and the local equation defining the node is $xy = t \quad t \in D' \subseteq D$. Then the restriction of the relative dualizing sheaf to C_0 (which is, by the proposition above, the dualizing sheaf ω_{C_0}) is the sheaf whose local generator is a nonvanishing abelian differential at smooth points, and $\frac{dx}{x}$ on the branch $y = 0$ ($\frac{dy}{y}$ on the branch $x = 0$) near the node $xy = 0$. By contrast, in a neighbourhood of the node, the sheaf of Kähler differentials is generated by dx and dy subjected to the relation $ydx + xdy = 0$. In particular, the latter sheaf is not locally free.

We are now in a position to discuss Mumford's theorem and the Picard group of moduli spaces. First of all we have to introduce two more varieties, namely

$$\overline{M}_g \supsetneq \overline{M}_{g,reg} \supsetneq \overline{M}_g^0$$

where $\overline{M}_{g,reg}$ is the open set of smooth points of $\overline{M}_g$ and $\overline{M}_g^0$ is the set of automorphism-free curves. We are restricting our discussion to $\overline{M}_g^0$, because when dealing with $\overline{M}_g$ the nonexistence of the universal curve

$$\mathcal{C} \xrightarrow{\pi} \overline{M}_g$$

generates subtleties to be treated with more sophisticated techniques. As we have outlined in §2.4 the Deligne-Mumford compactification of M_g amounts to adding the divisor Δ of one-noded genus g curves which can be expressed as

$$\Delta = \sum_{i=0}^{[g/2]} \delta_i$$

where the $C \in \delta_i, i > 0$ iff its normalization N_C is the disjoint union $N_C = N_1 \sqcup N_2$ with $g(N_1) = i \quad g(N_2) = g - i$ and $C \in \delta_0$ iff N_C has genus $g-1$ i.e. if it is obtained from a genus g smooth surface by pinching a handle.

Let us consider the universal curve $\mathcal{C} \longrightarrow \overline{M}_g^0$ the relative dualizing sheaf $\omega_{\mathcal{C}/\overline{M}_g^0}$ and its powers $\omega^n_{\mathcal{C}/\overline{M}_g^0}$. The Grothendieck-Riemann-Roch theorem for n-canonical relative forms gives

$$Ch(\pi_! \omega^n_{\mathcal{C}/\overline{M}_g^0}) = \pi_* \left(Ch\omega_{\mathcal{C}/\overline{M}_g^0} \cdot Td\Omega^1_{\mathcal{C}} \cdot \pi^*(Td\Omega^1_{\overline{M}_g^0})^{-1} \right)$$

which can be simplified , by means of the following exact sequence

$$0 \to \Omega_{\mathcal{C}/\overline{M}_g^0} \longrightarrow \Omega^1_{\mathcal{C}} \xrightarrow{\pi_*} \Omega^1_{\overline{M}_g^0} \to 0$$

which yields

$$Td\Omega^1_{\mathcal{C}} \cdot (\pi^* Td\Omega^1_{\overline{M}_g^0})^{-1} = Td\Omega^1_{\mathcal{C}/\overline{M}_g^0}$$

to

$$Ch(\pi_! \omega_{\mathcal{C}/\overline{M}_g^0}) = \pi_* \left[Ch\omega^n_{\mathcal{C}/\overline{M}_g^0} Td\Omega^1_{\mathcal{C}/\overline{M}_g^0} \right]$$

Now one has that $\pi_! \omega^n_{\mathcal{C}/\overline{M}_g^0} = \pi_* \omega^n_{\mathcal{C}/\overline{M}_g^0} \ominus R^1\pi_*\omega^n_{\mathcal{C}/\overline{M}_g^0}$ as $\omega^n_{\mathcal{C}/\overline{M}_g^0}$ is invertible and so its higher direct image sheaves vanish. Also, $\omega^n_{\mathcal{C}/\overline{M}_g^0}$ restricts to each fiber C_t of the family to $\omega^n_{C_t}$ so that, for $n > 1 \quad deg_{rel}\omega^n_{\mathcal{C}/\overline{M}_g^0} < 0$ and hence $R^1\pi_*\omega^n_{\mathcal{C}/\overline{M}_g^0} = 0$ and for $n = 1 \quad R^1\pi_*\omega_{\mathcal{C}/\overline{M}_g^0} \simeq \mathcal{O}_{\overline{M}_g^0}$ and hence does not affect $\pi_!$ so that one can rewrite the formula above as

$$Ch(\pi_* \omega_{\mathcal{C}/\overline{M}_g^0}) = \pi \left[Ch\omega^n_{\mathcal{C}/\overline{M}_g^0} Td\Omega^1_{\mathcal{C}/\overline{M}_g^0} \right]$$

Expanding both sides one gets

$$Ch(\pi_* \omega^n_{C/\overline{M}^0_g}) = \pi_* \left[\left(1 + c_1(\omega^n_{C/\overline{M}^0_g}) + \frac{c_1(\omega^n_{C/\overline{M}^0_g})^2}{2} + \cdots \right) \cdot \right.$$
$$\left. \cdot \left(1 - \frac{c_1(\Omega^1_{C/\overline{M}^0_g})}{2} + \frac{c_1(\Omega^1_{C/\overline{M}^0_g})^2 + c_2(\Omega^1_{C/\overline{M}^0_g})}{12} + \cdots \right) \right]$$

so that, by extracting the right codimension piece one has

$$c_1(\pi_* \omega^n_{C/\overline{M}^0_g}) = c_1(\text{ det } \pi_* \omega^n_{C/\overline{M}^0_g}) = \pi_* \left[\frac{c_1(\Omega^1_{C/\overline{M}^0_g})^2 + c_2(\Omega^1_{C/\overline{M}^0_g})}{12} + \right.$$
$$\left. - \frac{c_1(\omega^n_{C/\overline{M}^0_g}) c_1(\Omega^1_{C/\overline{M}^0_g})}{2} + \frac{c_1(\omega^n_{C/\overline{M}^0_g})^2}{2} \right]$$

Let us now forget for a moment the boundary of moduli space $\partial \overline{M}^0_g$ and work on M_g. There one has that the dualizing sheaf and the sheaf of Kähler differentials coincide, so that, denoting $\lambda_n := \pi_*(\text{ det } \omega^n_{C/M_g})$ one has

$$c_1(\lambda_n) = \frac{1}{12}(6n^2 - 6n + 1)\pi_*[c_1(\omega^n_{C/M_g})^2]$$

and, in particular,

$$c_1(\lambda) = \frac{1}{12}\pi_*[c_1(\omega_{C/M_g})^2]$$

so that one arrives at the celebrated Mumford's formula

$$c_1(\lambda_n) = (6n^2 - 6n + 1)c_1(\lambda)$$

Taking into account the boundary requires a careful use of the relation between the sheaf of Kähler differentials and the dualizing sheaf in the form given by the following exact sequence

$$0 \to \Omega^1_{C/\overline{M}^0_g} \longrightarrow \omega_{C/\overline{M}^0_g} \stackrel{\otimes \mathcal{O}_{C_{sing}}}{\longrightarrow} \omega_{C/\overline{M}^0_g} \otimes \mathcal{O}_{C_{sing}} \to 0$$

The Whitney product formula of the total Chern class applied to this sequence gives

$$1 + c_1(\omega_{C/\overline{M}^0_g}) = \left(1 + c_1(\Omega^1_{C/\overline{M}^0_g}) + c_2(\Omega^1_{C/\overline{M}^0_g}) + \cdots\right) \cdot \left(1 + c_1(\mathcal{O}_{C_{sing}}) + c_2(\mathcal{O}_{C_{sing}}) + \cdots\right)$$

One can deduce what are the first Chern classes of $\Omega^1_{C/\overline{M}^0_g}$ by means of the following

Lemma . Let X be a smooth variety, Y an r-codimensional subvariety of X and $\mathcal{F}$ a coherent sheaf on Y. Then, considering $\mathcal{F}$ as a sheaf on X it holds

$$c_i(\mathcal{F}) = \begin{cases} 0 & 1 \leq i \leq r-1 \\ ((-)^{r-1}(r-1)!rk_Y\mathcal{F})Y & i = r \end{cases}$$

Then, as C_{sing} is of codimension 2 in $\mathcal{C}$, one gets

$$c_1(\Omega^1_{\mathcal{C}/\overline{M}^0_g}) = c_1(\omega_{\mathcal{C}/\overline{M}^0_g})$$

$$c_2(\Omega^1_{\mathcal{C}/\overline{M}^0_g}) = [Sing_{\mathcal{C}}]$$

The Grothendieck-Riemann-Roch formula (on $\overline{M}^{0^0}_g$) thus gives

$$c_1(\lambda_n) = \pi_* \left[\frac{c_1(\omega_{\mathcal{C}/\overline{M}^0_g})^2 + [C_{sing}]}{12} + (n^2 - n)\frac{c_1(\omega_{\mathcal{C}/\overline{M}^0_g})^2}{2} \right]$$

Now, $\pi_*([C_sing]) = \Delta$, the locus of singular curves, and the above formula, taking $n = 1$ gives

$$c_1(\lambda) = \left(\frac{\pi_*(c_1(\omega_{\mathcal{C}/\overline{M}^0_g})^2) + [\Delta]}{12} \right)$$

so that one obtains the extension of the above Mumford formula as

$$c_1(\lambda_n) = c_1(\ \det \pi_* \omega^n_{\mathcal{C}/\overline{M}^0_g}) = \binom{n}{2} (12c_1(\lambda) - [\Delta]) + c_1(\lambda)$$

The equality between Chern classes can be transplanted to an equality between line bundles by means of the following remarkable result [AC]

Theorem . For $g \geq 3$ the Picard group of the moduli "space" $\overline{M}^0_g$ is freely generated by the classes

$$\lambda,\ \delta_0,\ \cdots\ \delta_{[g/2]}$$

Thus, setting $\mu = \lambda^{12} \otimes \Delta^{-1}$, Mumford's theorem can be expressed as

$$\lambda_n = \mu^{\binom{n}{2}} \otimes \lambda.$$

In the applications we will also need the boundary behaviour of the canonical bundle $K_{\overline{M}^0_g}$. This can be obtained again from the Grothendieck-Riemann-Roch theorem, by means of the following argument [HM]

We consider the sheaf $\mathcal{F} = \Omega^1_{\mathcal{C}/\overline{M}^0_g} \otimes \omega_{\mathcal{C}/\overline{M}^0_g}$ Then, $\pi_*\ \mathcal{F} \simeq T^*_{\overline{M}^0_g}$ and $R^1\pi_*\mathcal{F} = 0$. As observed before

$$c_1(K_{\overline{M}^0_g}) = c_1(\ \det T^*_{\overline{M}^0_g}) = c_1(T^*_{\overline{M}^0_g}) = c_1(\pi_*\mathcal{F}) = c_1(\pi_!\mathcal{F})$$

so that Grothendieck-Riemann-Roch gives

$$c_1(K_{\overline{M}^0_g}) = \pi_* \left[\frac{c_1(\mathcal{F})^2}{2} - c_2(\mathcal{F}) - \frac{c_1(\mathcal{F})c_1(\Omega^1_{\mathcal{C}/\overline{M}^0_g})}{2} + \frac{c_1(\Omega^1_{\mathcal{C}/\overline{M}^0_g})^2 + c_2(\Omega^1_{\mathcal{C}/\overline{M}^0_g})}{12} \right]$$

and taking into account the relations $c_1(\mathcal{F}) = 2c_1(\omega_{C/\overline{M}_g^0})$, $c_2(\mathcal{F}) = [Sing_C]$ one gets

$$c_1(K_{\overline{M}_g^0}) = \pi_* \left[13 \frac{c_1(\omega_{C/\overline{M}_g^0})^2 + [Sing_C]}{12} - 2[Sing_C] \right]$$

and hence

$$K_{\overline{M}_g^0} = 13\lambda - 2\delta$$

2.6 Some physical applications

The (heuristic) computations reported in chapter 1 lead to the following expression for the genus g contribution (or g-loop summand) to the Polyakov's string partition function

$$\mathcal{Z}_g = const \cdot \int_{M_g} \left[\frac{\det'_\zeta \triangle_0}{\|1\|} \right]^{-13} \cdot \det'_\zeta \triangle_2 \; d\nu$$

where $d\nu$ is the Weil -Petersson volume form and the meaning of the ζ-function regularization has been discussed in §2.1.
Let us now consider a sufficiently fine covering $\{\mathcal{U}_\alpha\}$ of the moduli space of smooth curves M_g. Then, considering the universal family (pretending it exists) $\mathcal{C} \xrightarrow{\pi} M_g$, one can find a local holomorphic frame for $\pi_* \omega_{C/\overline{M}_g^0}$, i.e. a basis $\varphi_1, \ldots, \varphi_g$ for the space of abelian differentials on $C_s = \pi^{-1}(s)$ depending holomorphically on $s \in \mathcal{U}_\alpha$. Let also $t_1, \ldots, t_{3g-3}$ be coordinates in $\mathcal{U}_\alpha$ and let $\psi_1, \ldots, \psi_{3g-3}$ be relative quadratic differentials dual to the coordinate vector fields $\frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_{3g-3}}$. Then the integrand for $\mathcal{Z}_g$ can be rewritten (locally) as

$$\frac{\prod_1^{3g-3} dt_i \wedge \bar{t}_i}{(\det < \varphi_i, \varphi_j >)^{13}} \cdot \left[\frac{\det'_\zeta \triangle_0}{\|1\| \cdot \det < \varphi_i, \varphi_j >} \right]^{-13} \cdot \left[\frac{\det'_\zeta \triangle_2}{\det < \psi_A, \psi_B >} \right]$$

Let us denote

$$F_\alpha := \left[\frac{\det'_\zeta \triangle_0}{\|1\| \cdot \det < \varphi_i, \varphi_j >} \right]^{-13} \cdot \left[\frac{\det'_\zeta \triangle_2}{\det < \psi_A, \psi_B >} \right]$$

and

$$\Psi_\alpha := \frac{\prod_1^{3g-3} dt_i \wedge \bar{t}_i}{(\det < \varphi_i, \varphi_j >)^{13}}.$$

Then, F_α is the product of two Quillen norms, and, precisely,

$$F_\alpha = \|1 \otimes \det(\varphi_1, \ldots, \varphi_g)\|_Q^{26} \cdot \|\det(\psi_1, \cdots, \psi_{3g-3})\|_Q^{-2}$$

where $1 \otimes \det(\varphi_1, \ldots, \varphi_g)$ is a section of $\mathcal{L}_1 = \det\pi_* \mathcal{O} \otimes \det\pi_* \omega_{C/\overline{M}_g^0}$ and $\det(\psi_1, \cdots, \psi_{3g-3})$ is a section of $\mathcal{L}_2 = \det\pi_* \omega^2_{C/\overline{M}_g^0}$.

Quillen's theorem gives then an equality between Chern forms

$$\frac{1}{2\pi i} \partial\bar{\partial} \log F_\alpha = 13 c_1(\lambda) - c_1(\lambda_2)$$

But then, by the Mumford formula described in the preceeding section,

$$c_1(\lambda_n) = (6n^2 - 6n + 1) c_1(\lambda) \text{ so that } \frac{1}{2\pi i} \partial\bar{\partial} \log F_\alpha = 0$$

Thus there exists on every open set $\mathcal{U}_\alpha$ a local holomorphic function f_α such that

$$F_\alpha = |f_\alpha|^2$$

Summing up one has the algebro geometrical proof of the so-called Belavin-Khniznik local holomorphic factorization theorem, asserting that locally the string integrand is the modulus squared of a holomorphic function. The physical relevance of this fact can be deduced from the following argument [BK]. Viewing string theory as a conformal field theory of matter fields plus ghost fields, the second variation of the effective action $\partial\bar{\partial} \log F_\alpha$ can be expressed as the correlation function of the total energy-momentum tensor $T = T^X + T^{gh}$ as

$$\partial\bar{\partial} \log F_\alpha = \int d^2\xi d^2\xi' \psi(\xi) \bar{\psi}(\xi') \ < T(\xi)\ \bar{T}(\xi') >$$

so that requiring conformal invariance (in this picture the decoupling of left and right movers) is tantamount to requiring the vanishing of $\partial\bar{\partial} \log F_\alpha$, i.e. local holomorphic factorization.

Moreover, a holomorphic factorization holds also globally, (at least for $g \geq 3$) in the following sense. Let s be a local section of $\lambda^{-13} \otimes \Omega^{3g-3}_{M_g}$ and define the following operator[C1]:

$$\begin{array}{llll} \mathbb{V} & : \Gamma(\mathcal{U}_\alpha, \lambda^{-13} \otimes \Omega^{3g-3}_{M_g}) & \longrightarrow & \Gamma(\mathcal{U}_\alpha, \Omega^{3g-3,3g-3}_{M_g}) \\ & s = \frac{\xi}{\sigma} & \rightsquigarrow & (-)^{\frac{g^2-3g}{2}} (i/2)^{3g-3} \frac{\xi \wedge \bar{\xi}}{\|\sigma\|^2} \end{array}$$

where $\| \ \|$ here means the 13^{th} power of the norm on λ defined by

$$\|\det(\varphi_1, \ldots, \varphi_g)\|^2 = \det(< \varphi_i, \varphi_j >)$$

A glance at the integrand of the string partition function shows that, in any coordinate chart $\mathcal{U}_\alpha$ it can be expressed as

$$\Psi_\alpha \cdot F_\alpha \equiv A_\alpha = \mathbb{V}(a_\alpha)$$

where $a_\alpha = \frac{dt_1 \wedge \cdots \wedge dt_{3g-3}}{\det(\varphi_1, \ldots, \varphi_g)} \cdot f_\alpha$.
In each overlap $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$ then it will hold

$$A_\alpha = \mu_{\alpha\beta} A_\beta$$

where $|\mu_{\alpha\beta}| = 1$ as the A_α's are squares of norms. By a classical theorem of complex analysis, provided the covering $\{\mathcal{U}_\alpha\}$ is sufficiently fine, the $\mu_{\alpha\beta}$'s are constant and so define a Čech cocycle

$$[\mu_{\alpha\beta}] \in \check{H}^1(M_g, \mathcal{U}(1))$$

which can be considered as a cocycle in the singular homology group $H_1(M_g, U(1))$ as Čech cohomology with values in constant sheaves is isomorphic to the singular homology with values in the same group. The universal coefficient theorem thus gives

$$H_1(M_g, U(1)) \simeq H_1(M_g, \mathbb{Z}) \otimes U(1)$$

and Harer's results on the homology of M_g proves that, actually, $\mu_{\alpha\beta}$ is a coboundary, $\mu_{\alpha\beta} = \nu_\alpha / \nu_\beta$.
Putting things together one gets global holomorphic factorization in the form [CCMR]
Theorem . If $g \geq 3$ there a global nowhere vanishing section $a \in H^0(M_g, \lambda{-13} \otimes \Omega_{M_g}^{3g-3})$ such that

$$\mathcal{Z}_g = \int_{M_g} \mathrm{V}(a)$$

As a last topic we want to discuss the singularities of the integrand near the boundary $\partial \overline{M}_g^0$. The integrand A constructed above can be thought of as the modulus squared of a holomorphic section a of $L = \lambda^{-13} \otimes \Omega_{\overline{M}_g^0}^{3g-3}$, defined everywhere except along the boundary $\Delta = \sum \delta_i$.
Recall that the Grothendieck-Riemann-Roch theorem applied to the canonical bundle $\Omega_{\overline{M}_g^0}^{3g-3}$ yielded $\Omega_{\overline{M}_g^0}^{3g-3} \simeq \lambda^{-13} \otimes \mathcal{O}(-2\Delta)$ so that

$$L \simeq \mathcal{O}(-2\Delta)$$

Being $\overline{M}_g^0$ projective, L admits a meromorphic section ϕ with a double pole along Δ and no zeroes, so that the quotient $f = a/\phi$ is a holomorphic function on all of $\overline{M}_g^0$. But as $\overline{M}_g^0$ is quasi-compact, a/ϕ is constant so that the results gotten by means of Selberg trace formula techniques [GIJR] can be recovered in this algebro-geometrical setting, namely:
Theorem . a is a meromorphic section of $\lambda^{-13} \otimes \Omega_{\overline{M}_g^0}^{3g-3}$ with a double pole along Δ.

For a complete understanding of the boundary behaviour of the string integrand the only piece that is missing is thus the analysis of the norm $\| \; \|^{13}$ on Δ. Let us work in the neighbourhood of a general curve $C_p \in \Delta$, (i.e. C_p is a curve with one node in p) and let, as usual, $\varphi_1, \ldots, \varphi_g$ be a local holomorphic frame for $\pi_* \omega_{C/\overline{M}_g^0}$.

If $C_p \in \delta_i$ $i > 0$ the problem is straightforwardly dealt with. In fact the normalization N_c of C has two components, N_1 and N_2, and each differential φ_i restricts to a differential having at most a simple pole on N_h, $h = 1, 2$. But, by the residue theorem, a holomorphic 1-form having at most a simple pole has no poles at all, so that

$$< \varphi_i, \varphi_j >= \int_C \varphi_i \wedge \bar{\varphi}_j$$

is finite and so $\|\det(\varphi_1, \ldots, \varphi_g)\|^2$ extends continuously across $\sum_{i>0} \delta_i$. It also has no zeroes, as one can choose the first i differentials to give a basis in $H^0(N_1, K_{N_1})$ and vanishing on N_2 (and conversely for the remaining $g - i$ so that the matrix $< \varphi_i, \varphi_j >$ is block diagonal and its determinant is non-zero.

The case $C_p \in \delta_0$ is somehow different. Namely, a convenient basis for $H^0(C_p, \omega_{C_p})$ can be chosen as follows
φ_i is regular on N_C if $i \geq 2$;
φ_1 has two simple poles with opposite residues at the preimages of the node p.
Let us consider what happens along a small disk transverse to δ_0 (i.e. what happens when "pinching a handle"). The relevant deformation will be $C \xrightarrow{\pi} D$, $D := \{t \in \mathbb{C}, \ |t| < r\}$, and the equation defining the node is $xy = t$. Moreover, let

$$B := \{(x, y) \in \mathbb{C}, \ |x| < 1, \ |y| < 1\}$$

Then

$$< \varphi_1, \varphi_k >_{C_t} = \int_{C_t} \varphi_1 \wedge \bar{\varphi}_k = \int_{C_t \cap B} \varphi_1 \wedge \bar{\varphi}_k + \text{ finite terms}$$

But, near p, $\varphi_1 = \dfrac{a\ dx}{x} + \dfrac{b\ dy}{y}$, $a + b = 0$ and $\varphi_{h>2} = c\ dx + e\ dy$ so that

$$\int_{C_t \cap B} \varphi_1 \wedge \bar{\varphi}_1 \sim |a|^2 \int_{|t|<|x|<R} \frac{dx \wedge \bar{x}}{|x|^2} \sim \log|t|$$

while

$$\int_{C_t \cap B} \varphi_1 \wedge \bar{\varphi}_h \propto \int_{|t|<|x|<R} \frac{dx \wedge \bar{x}}{x} \text{ which is finite.}$$

As before the determinant of $< \varphi_h, \varphi_k >$, $h, k \geq 2$ is non vanishing as they restrict to a basis for the abelian differentials on N_{C_p}.

Summing up one has the following boundary behaviour for the integrand of the string partition function

$$\mathbb{V}(a) = \begin{cases} \text{const } |t|^{-4} & \text{on } \delta_i, \ i > 0 \\ \text{const } |t|^{-4} \log|t|^{-13} & \text{on } \delta_0 \end{cases}$$

This analysis has a nice physical picture. In fact the power expansion of the integrand near the boundary is the device for the bookkeeping of what in an operator formalism is interpreted as the mass spectrum of the string modes (particles). In particular the mass formula associates to a fourth-order pole the propagation of a tachyon and to a pole of order 2 the propagation of a massless particle (the dilaton). Notice that by formula above, they are both divergent after integration in the case of dividing nodes ($\delta_i,\ i > 0$), while, in the nondividing case, a term growing like $1/|t|^2(\log|t|)^{13}$ is integrable, a fact that can be interpreted as signalling the propagation of a massless particle around a tadpole, which, in two dimensions, gives rise mild logarithmic divergences in Feynman amplitudes.

2.7 Moduli of θ-characteristics

In this section we want to describe the moduli space of θ-characteristics and their compactification. Given a family of curves $\mathcal{C} \xrightarrow{\pi} S$, a (relative) θ-characteristics is an invertible sheaf $\mathcal{L}$ over $\mathcal{C}$ which is a "square root" of the relative canonical sheaf $\omega_{\mathcal{C}/S}$ i.e. such that, $\forall s \in S \quad L^2\restriction_{\pi^{-1}(s)} \simeq \omega_{\pi^{-1}(s)}$. A θ-characteristics is called even or odd according to the parity of $dim\ H^0(C, L)$. There are $2^{g-1}(2^g+1)$ even and $2^{g-1}(2^g-1)$ odd θ-characteristics , adding up to a total of 2^{2g}.

Mimicking deformation theory of stable curves, one can define a deformation of a (smooth) θ-characteristics (C, L) to be a relative θ-characteristics [H2], i.e. a diagram

$$\begin{array}{ccccc} L & & \mathcal{L}_\pi & & \\ & \searrow & & \searrow & \\ & C & \xrightarrow{i} & & \mathcal{X} \\ & \downarrow & & & \downarrow \pi \\ & p & \longrightarrow & & S \end{array}$$

such that the isomorphism $i : C \to \pi^{-1}(s_0)$ induces an isomorphism of L and $i^*\mathcal{L}_\pi$. More generally, one can deform all θ-characteristics $(C, L_1, ...L_m)$, $(m = 2^{2g})$ on C, by giving 2^{2g} sheaves $\mathcal{L}_{\pi_1}, ..., \mathcal{L}_{\pi_m}$ on $\mathcal{X}$ satisfying the above property. In this way we get a 2^{2g}-fold covering of the base space S of $\mathcal{X}$.

Recalling that θ-characteristics on an algebraic smooth curve are in a (non-natural) one-to-one correspondence to points of order 2 on the Jacobian $J(C)$, a more concrete way of describing such a covering is to consider the family $\tau : \mathcal{J} \to S$ of Jacobians associated to the deformation $\pi : \mathcal{X} \to S$ of C, whose fibre $\tau^{-1}(s)$ is precisely the Jacobian of $\pi^{-1}(s)$. The choice of (C, L_1) gives us its deformation $\mathcal{L}_{\pi_1}$ over X and 2^{2g} sections $\sigma_1, ..., \sigma_m$ of $\mathcal{J}$ over S gotten by setting $\sigma_i = \mathcal{L}_{\pi_i} \otimes \mathcal{L}_{\pi_1}^{-1}$. Their image is the desired covering of S. When dealing with smooth curves , this local covering extends by isomorphisms to the whole M_g, thus realizing the moduli space of θ-characteristics over smooth curves S_g as a 2^{2g}-fold of

M_g. The problem is that, when considering singular curves besides the smooth ones, trouble can arise, as the following example shows.

Let us consider a family of elliptic curves parameterized by a small disk $\Delta \in \mathbb{C}$ as follows. Set $\tau = ln(b)/2\pi i$, $b \in \Delta$, and consider the lattice $\Lambda_\tau \subset \mathbb{C}$ generated by 1 and τ. This acts holomorphycally on $\Delta \times \mathbb{C}$ by translations on the second factor. The quotient $X = \Delta \times_{\Lambda_\tau} \mathbb{C}$ is a family of tori degenerating to a single-node curve for $b = 0$. At genus one all θ-characteristics have degree 0 and one of them is isomorphic to the structure sheaf $\mathcal{O}_{c_\tau}$. So the other three naturally corresponds to points of order two on the Jacobian, which in turn coincides with the torus itself. So, on $\Delta \setminus \{0\}$ we get the following sections of $J = X \to \Delta$

$$\sigma_1 = 0; \qquad \sigma_2 = 1/2$$
$$\sigma_3 = \tau/2; \qquad \sigma_4 = \tau/2 + 1/2 \qquad (\mathrm{mod}\,\Lambda_\tau)$$

where $\tau = \tau(b)$ as above.

We can now clearly see three phenomena. First of all, we have monodromy in the covering, because a rotation around $b = 0$ exchanges the two sections σ_3 and σ_4. Second, these two sections are 'asymptotic' for $b \to 0$ ($|\tau| \to \infty$), meaning that there is branching in the covering (recall that the Jacobian of a torus with one node can be compactified getting again the same torus; being asymptotic here means that the two section above go to the node in the limit.) Finally, this limit point cannot be interpreted any more as an invertible sheaf, but corresponds to a more general coherent sheaf.

If we abstract from the peculiarities of genus 1, the picture we get from this example is general. In particular, the three phenomena mentioned above, i.e. monodromy, branching and the appearance of more general sheaves than sheaves of sections of line bundles reproduce themselves at all genera. For instance, such sheaves occur in the compactification of the moduli of θ-characteristics recently constructed by Deligne [D]. A different way for getting a compactified moduli space of θ-characteristics has been given by Cornalba [C1]. This involves the addition to the moduli space of smooth curves of a wider class of singular curves, (namely a certain subclass of *semistable* ones), but have the desirable feature of yielding invertible sheaves as "limits of θ-characteristics ".

In a certain sense, the whole construction stems from the observation that the appearance of monodromy and of non locally free sheaves are somewhat related. In fact, let $\pi : \mathcal{X} \to \Delta$ be a family of stable curves with smooth fibres $\pi^{-1}(t)$, ($t \in \Delta \setminus \{0\}$) and assume for simplicity that C_0 itself has a single node. In other words, $t \in \Delta$ is a local coordinate transversal to some component δ_i of the boundary of the Deligne-Mumford compactified moduli space $\overline{M}_g$. The local equation of $\mathcal{X}$ near the node of the central fibre can be written as $xy = t$. It follows that, in spite the central fibre is singular, the (2-complex dimensional) surface $\mathcal{X}$ is smooth. Next, assume a family of θ-characteristics $\mathcal{L}'_\pi$ is given on $\mathcal{X} \setminus \pi^{-1}(0)$, and ask whether it can be extended to the whole of $\mathcal{X}$. We can get rid of monodromy, if present, by double covering the base of $\mathcal{X}$, i.e. by setting $t = f(q) = q^2$ and pulling-back $\mathcal{X}$ to get a deformation $\mathcal{Y} = f^*\mathcal{X}$ over another disk Q. The local equation for the singular point now reads $xy = q^2$, which clearly shows that $\mathcal{Y}$ is singular at the node on the central fibre. So $f^*\mathcal{L}'_\pi$ cannot be extended as an invertible sheaf. To get such an extension, one first smooths out $\mathcal{Y}$ by blowing up the singular point. The family $\mathcal{Z} \to Q$ gotten in this way is the same as $\mathcal{Y}$ off the singular point, while this latter has been substituted by an entire line E (a copy of the Riemann sphere), called the exceptional line. Thus the central fibre is now a semistable curve C_0. Its normalization has two components, C' and E given respectively by the normalization of C and by the exceptional line E. On C_0, E and C' intersect in two points p_1, p_2 given by the preimages on C' of the node on C. If $a, b \quad (a = 1/b)$ are local coordinates on E, the blow up is given by

$$\begin{cases} ax = q \\ by = q \end{cases}$$

which shows the presence of two nodes at $q = 0$. In spite that C has been replaced by an even more singular curve C_0, now $\mathcal{Z}$ is smooth and $\mathcal{L}'_\pi$ can be extended to $\mathcal{L}_\pi$ on the whole of $\mathcal{Z}$; we denote by $\mathcal{L}_0$ the sheaf we get in this way on the central fibre.

Clearly enough, such an extension $\mathcal{L}_\pi$ is not unique, because by tensoring with any sheaf of the form $\mathcal{O}(nE)$ one gets another extension. The basic fact which matters for us is that one can judiciously choose the extension $\mathcal{L}_\pi$ so that the restriction $\mathcal{L}_0\restriction_E$ of $\mathcal{L}_\pi$ to E is isomorphic to $\mathcal{O}(n)$ with n either 0 or 1. To see why this is so, assume that $\mathcal{L}_\prime\restriction_E$ was $\mathcal{O}(s)$, then $\mathcal{L}_\pi(nE)$ restricts to $\mathcal{L}_0\restriction_E(-np_1 - np_2)$ which is then isomorphic to $\mathcal{O}(s - 2n)$. Therefore, by suitably choosing n the degree of $\mathcal{L}_0\restriction_E$ can be adjusted to be either 0 or 1.

Let's now see the relations between $\mathcal{L}_\pi$ and θ-characteristics . For $q \neq 0$, $\mathcal{L}_q^2 = \omega_q$, where as usual the subscript q indicates the restriction to the fibre of $\mathcal{Z}$ over $q \in Q$. So $deg\mathcal{L}_q = g - 1$ and the same is true for $\mathcal{L}_0$. We have thus two cases

Proposition . Let ω_0 be the dualizing sheaf of C_0.
a) if $deg\mathcal{L}_0\restriction_E = 0$ (and then $deg\mathcal{L}_0\restriction_{C'} = g - 1$) we have that $\mathcal{L}_0^2 = \omega_0$.
b) if $deg\mathcal{L}_0\restriction_E = 1$ (and then $deg\mathcal{L}_0\restriction_{C'} = g - 2$) we have that $\mathcal{L}_0^2(E) = \omega_0$

Proof. We first recall the intersection properties of the divisors C' and E on the surface $\mathcal{Z}$[Hart]. Since $C' + E$ is homologically equivalent to a generic fibre $\mathcal{Z}_q$ which does not intersect either C' or E, we have $0 = E.(C' + E) = E.C' + E.E$ and $0 = C'.(C' + E) = C'.C' + C'.E$. As by construction C' and E intersect in two points (i.e. $C'.E = 2$), it follows that $C'.C' = E.E = -2$. Notice also that, being $\mathcal{Z} \to Q$ a family over a polydisk Q, $\mathcal{F}(C' + E) = \mathcal{F}$ for any sheaf $\mathcal{F}$. The tensor product $\omega_\pi \otimes \mathcal{L}_\pi^{-2}$ is trivial off the central fibre and therefore we must have

$$\omega_\pi \otimes \mathcal{L}_\pi^{-2} = \mathcal{O}(mC' + nE)$$

for some integers m, n. From the relations above, it is easy to compute the degrees

$$d_{C'} := deg\mathcal{O}(mC' + nE)\restriction_{C'} = mC'.C' + nE.C' = -2m + 2n$$

$$d_E := deg\mathcal{O}(mC' + nE)\restriction_E = mC'.E + nE.E = 2m - 2n$$

To prove a), notice that $deg\omega_\pi = 2g - 2 = deg\mathcal{L}^2_{C'}$ yielding $d_{C'} = 0$, that is $m = n$, and $\mathcal{O}(m(C' + E))$ is trivial. As for b) the same reasoning leads to $d_{C'} = 2$, $d_E = -2$, i.e. $n = m + 1$, and $\mathcal{O}(m(C' + E) + E) = \mathcal{O}(E)$■

This result generalizes quite nicely what is usually meant by 'plumbing fixture' in the physical literature. In fact, sticking to the case of a single separating node, we have the following situation. The normalization of C has two components C_i of genera g_i, $(i = 1, 2)$ with $g_1 + g_2 = g$ and the dualizing sheaf of C restricts to $\omega_i(p_i)$, on C_i. As these have odd degree, only b) applies in this case. In particular $\mathcal{L}^2(E)\restriction_{C'} = L^2(p_1 + p_2)$ restricts to C_i to $L^2(p_i)$ which is isomorphic to $\omega_i(p_i)$. Hence, giving such a limit θ-characteristics on C_0 is tantamount to choosing θ-characteristics on the components C_i. We have then $2^{2g_1}.2^{2g_2} = 2^{2g}$ inequivalent choices.

A less common picture arises for a single non-separating node, where both cases a) and b) apply. This is to be expected as the genus of C' is $g-1$ and the number of θ-characteristics on it is only a quarter of what one would like to have. The correct number is restored on C_0 in the following way. If $\mathcal{L}_\pi\restriction_E$ is trivial, $\mathcal{L}_\pi\restriction_{C'}$ is one of the $2^{2(g-1)}$ square roots of $\omega\restriction_{C'}(p_1 + p_2)$. Notice that these do not come from θ- characteristics on the normalization of C. An extra factor of two is given by the two different identifications between the stalks on the points p_i's, yielding in total a half of what we need. The rest comes in the same way when $\mathcal{L}_\pi\restriction_E$ is $\mathcal{O}(1)$.

Remark - The proposition above tells us that we can get a line bundle as a limit of a family of θ-characteristics by simply blowing up the nodes on a family of stable curves. Actually this is not always necessary because, when $\mathcal{L}_\pi\restriction_E$ is trivial, one can safely blow down the exceptional component E, reverting to the previous family of stable curves. These are precisely the θ-characteristics which have already a limit as line bundles. In general, however, one has to deal with families of semi- stable curves. Luckily enough they enter the game with extra data, leading to the notion of 'spin-curves' [C2] as triples (C, L, ϕ), where C is a semistable curve with disjoint rational components E_i (briefly speaking a *decent* curve), L is the sheaf of sections of a line bundle of degree $g - 1$ on C such that $L\restriction_{E_i} = \mathcal{O}(1)$, $\phi : L^2 \longrightarrow \omega_C$ is a homomorphism vanishing on all E_i's. These generalize the one-node case and allow a compactification $\overline{S}_g$ of the moduli space S_g of θ-characteristics on smooth curves, much alike the Deligne-Mumford compactification of ordinary moduli spaces.

Without entering too much the details of this compactification scheme, we simply quote the following results [C2];

1) $\overline{S}_g$ has a natural structure of a normal projective variety, $\partial\overline{S}_g = \overline{S}_g \setminus S_g$ is a closed proper analytic subvariety of $\overline{S}_g$, and therefore S_g is an open subvariety.

2) The natural map $\chi : \overline{S}_g \longrightarrow \overline{M}_g$ given by forgetting spin structures and reverting to stable models (i.e. blowing down all exceptional components) is finite.

3) Since the parity of a θ-characteristics is invariant under deformations, $\overline{S}_g$ is the disjoint union

$$\overline{S}_g = \overline{S}_g^+ \sqcup \overline{S}_g^-$$

of the two closed irreducible subvarieties of even and odd spin curves of genus g.

From the topological point of view, $\overline{S}_g^+$ and $\overline{S}_g^-$ are much alike, the only difference being the order of the covering

$$\chi^{+(-)} : \overline{S}_g^{+(-)} \longrightarrow \overline{M}_g$$

so that, for a closer description of their boundaries one can consider just one of them, say $\overline{S}_g^+$. Then, the previous introductory analysis shows that $\partial \overline{S}_g^+$ consists of the following divisors (see fig. 6):

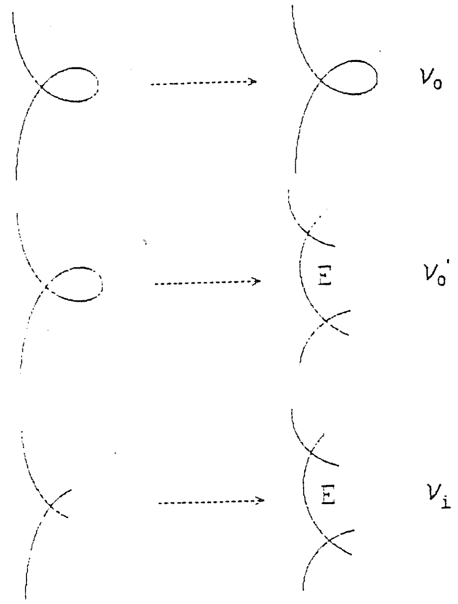


fig. 6

- ν_0, made of one-node curves with a square root of the canonical bundle (case a of the Proposition above).

- ν_0', consisting of classes of semistable curves with one-node irreducible model and with an invertible free sheaf (corresponding to case b).

- ν_i, $i > 0$, parameterizing classes of semistable curves with stable model consisting of two components of genus i and $g - i$ and with an L as in case b).

In the sequel, the boundary classes of odd spin moduli spaces, $\overline{S}_g^-$ will be denoted for the sake of simplicity in the same way although one should actually distinguish between e.g. ν_0^+ and ν_0^-. For instance in the even (odd) case, ν_i consists of semistable curves with an L restricting on the two components to both even or odd (one even and one odd) θ-characteristics. Also, denoting with δ_i the pull-back to $\overline{M}_g^{spin\pm}$ of the boundary classes of $\overline{M}_g$ consisting of stable curves with components of genera i and $g - i$, one has [C2]:

$$\delta_0 = \nu_0 + 2\nu_0'$$

$$\delta_i = 2\nu_i$$

To grasp these relations, notice that $\delta = \sum \delta_i$ coincides with the image of the nodes of the 'universal curve' [HM]. This has precisely one node over ν_0 and two nodes over all the other boundary components.

As we discussed in §2.5, the major tool in controlling the behaviour of determinants of $\overline{\partial}$-operators in the Polyakov bosonic string was the Grothendieck-Riemann-Roch theorem. We will now briefly see how this can be applied in the present situation. As usual we will pretend that there exists the universal curve $\mathcal{C}$ over spin moduli spaces. What we are going to say is actually rigorous if one restricts himself to work on the open and dense subvariety $\overline{S}_g^0$ made of spin curves without automorphisms, which obviously cover the subvariety $\overline{M}_g^0$ under χ. Let then $\pi : \mathcal{C} \to \overline{S}_g^0$ be the 'universal' spin curve of genus g. This comes together with an invertible sheaf $\mathcal{L}_\pi$, fitting the diagram

$$\begin{array}{ccc} \mathcal{L}_\pi & & \\ & \searrow & \\ & & \mathcal{C} \\ & & \downarrow \pi \\ & & \overline{S}_g^0 \end{array}$$

which represents the 'universal' spin structure. On $\mathcal{C}$ we have as well the relative structure sheaf $\mathcal{O}_\pi$ and the relative dualizing sheaf ω_π. Recall that, if we have a family of relative $\overline{\partial}$-operators coupled to an invertible sheaf $\mathcal{F}$ on C, its determinant $\det\overline{\partial}$ is a section of a line bundle $\det \pi_! F$ on $\overline{S}_g^0$ with first Chern class

$$c_1(\pi_! F) = \lambda + \pi_* \left[\frac{1}{2} c_1(F).c_1(F)\right] - \pi_* \left[\frac{1}{2} c_1(F).c_1(\omega_\pi)\right]$$

where '.' denotes intersection in homology (or better in the Chow ring), π_* is the Gysin homomorphism and we have set

$$\lambda := c_1(\pi_! \omega_\pi)$$

for the Hodge class of $\overline{S}_g^0$. As in §2.5, one finds that

$$\pi_*(c_1(\omega_\pi).c_1(\omega_\pi)) = 12\lambda - \delta$$

where $\delta = \sum \delta_i$ is the boundary class.

In fermionic string theories, one is interested in computing Chern classes of integral powers of L. Mumford's formula still applies, yielding the following relation

$$c_1(\pi_! \mathcal{L}_\pi^{2s})) = (6s^2 - 6s + 1)\lambda - \frac{1}{2}(s^2 - s)\nu_0 - (2s^2 - s)\nu'$$

where $\nu' = \nu_0' + \sum \nu_i$ is the boundary class corresponding to semistable spin curves with exceptional components.

3. 'Super'-algebraic Geometry and fermionic string theories

3.1- Two dimensional supergravity and Super Riemann surfaces

In the last chapter we will consider fermionic string theories, with the aim of testing how far the algebro-geometrical techniques discussed in the previous chapters can be pushed on in the analysis of such theories. In this introductory section we want to give a sketchy account of which are the steps that lead [Pol2, D'H-P] to the notion of super Riemann surface and to the expression of the superstring partition function as an integral over "supermoduli space".

The starting point is the supersymmetric extension of the Polyakov action for the Bose string, i.e.

$$S = \frac{1}{2}\int_\Sigma d^2\xi\sqrt{g}g^{\alpha\beta}\partial_\alpha x\cdot\partial_\beta x + \overline{\psi}i\gamma^\alpha\cdot\partial_\alpha\psi + \overline{\chi}_\alpha\gamma^\alpha\gamma^\beta(\partial_\beta x + \frac{1}{2}\chi_\beta\psi)\psi$$

where $g_{\alpha\beta}$ is the metric tensor on the surface Σ, the x's describe the embedding of Σ in $\mathbb{R}^D$, ψ denotes a collection of two-dimensional spinors (whose precise specification differs according to the model an the signature of the base space) and χ is a spin $3/2$ "gravitino" field.

The action above can be studied also by means of superspace techniques, i.e. by introducing two "anticommuting" coordinates ϑ^1, ϑ^2 (see §3.2 for precise mathematical definitions of this kind of procedure) besides the ξ's. In analogy with the usual case, one introduces complex coordinates

$$z = \xi_1 + i\xi_2 \qquad \theta = \vartheta^1 + i\vartheta^2$$

and their conjugates.

The symmetries of the above action amount to the so-called *Superdiffeomorphism* group $SDiff$, to the *SuperWeyl* group $SWeyl$ and, having to deal with local frames, to the frame group (i.e. a local $U(1)$ group). The $N = 1$ supergravity multiplet consists of the superzweibein $E_A^{\ M}$ and a $U(1)$-superconnection Ω_M, from which the covariant superderivative $\mathcal{D}_M$ can be constructed. Then, from these building blocks, one defines curvature and torsion and [Ho] imposes some constraints in order to get rid of the extra degrees of freedom involved in locally supersymmetric theories. In this sense the above symmetries can be viewed also as the local transformations which leave such constraints invariant. One defines then the flat $N = 1$ superspace to be given (in real coordinates) by the following superzweibein

$$E_m^{\ a} = \delta_m^{\ a} \qquad E_M^{\ \alpha} = 0$$

$$E_\mu^{\ a} = (\gamma^a)_\mu^{\ \beta}\vartheta_\beta \quad E_\mu^{\ \alpha} = \delta_\mu^{\ \alpha}$$

(here greek letters refer to anticommuting coordinates) yielding the following simple form for the superderivative

$$\mathcal{D} = \frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial z}$$

The fundamental result by Howe, which can be considered as the supersymmetric generalization of the existence theorem of isothermal coordinatization for orientable surfaces, is the following:

Proposition . Every two dimensional supermanifold is locally superconformally flat.

The expression of the Polyakov supersymmetric action in the superfield formalism is obtained by pasting the x's and the ψ's in a multiplet of scalar superfields X^μ as

$$S = \frac{1}{8\pi}\int_{\mathcal{F}} d^2z d\theta d\bar{\theta} E \overline{\mathcal{D}} X^\mu \mathcal{D} X_\mu$$

where E is the *Berezinian* of the superzweibein (see §3.2). In a complete analogy with the bosonic case then the h-loop vacuum-to-vacuum amplitude for the fermionic string will then be given by

$$\mathcal{Z}_h = \int [DE_M^A][D\Omega_M][DX^\mu]\exp(-S[X^\mu, E_M^A])$$

the integral running on the space of field configurations modulo the action of the symmetry group $\mathcal{G}$ of S.

As should be clear from the discussion above, the ultimate domain of integration will be the space of all non-equivalent superconformal structures on the topological surface Σ, a space which will be discussed in greater detail in §3.5. For what the actual reduction of the integrand to that space is concerned, one can proceed as in the case of the bosonic string theory, even if some peculiarities of the action S render the steps a little more awkward.

We just sketch the major steps in the construction without entering too much the details, which can be found, f.i. in [D'H-P], and signalling some topics that will be the objects of future investigations. The basic idea is to choose local slices of the action of $\mathcal{G}$, and to factor out its volume, thus getting a Faddeev-Popov determinant. Formally proceeding in this way, one can show that (in the critical dimension D=10 in which anomalies are cancelled) the partition function reduces to

$$\mathcal{Z} = \int_{\mathcal{F}} |\det{}'\Box_0(\Sigma)|^{-5} |\det{}'\Box_2(\Sigma)| d\nu$$

where $d\nu$ is a superanalogue of the Weil-Petersson metric and $\Box_m$ are the so-called *superlaplacians* (see §3.5). Notice that the integration runs over the (yet to be defined) supermoduli space, and the subscript $\mathcal{F}$ means that it involves also odd coordinates.

Passing from this expression to an ordinary integral over moduli space of Riemann surface is a delicate question. In fact, not only this involves e careful analysis of the integration procedure over supermanifolds, but also, with respect to the bosonic case, has an unpleasant feature due to the following fact.

Along a slice of the "fermionic" part of the group $SDiff \ltimes SWeyl$, the Polyakov action reduces to

$$S\restriction_{\text{slice}} = \int_{\Sigma} d^2\xi \sqrt{g} g^{\alpha\beta} \partial_\alpha x^\mu \partial_\beta x_\mu - \frac{1}{2} i \psi^\mu \gamma^\alpha D_\alpha \psi_\mu - \psi^\mu \chi^\alpha \partial_\alpha x_\mu + \frac{1}{4} \psi^\mu \chi^\alpha \psi_\mu \chi_\alpha$$

so that in the bosonic matter integration $\int [Dx] e^{-S}$, the exponent must be "gaussianized" in order to yield the desired determinant of the $\bar{\partial}_0$-operator. This can be achieved [D'H-P] at the price of introducing, in the fermionic integration, an extra piece, in the sense that

$$\int [Dx] e^{-S} = \left(\frac{\det{}'\triangle_g}{\int_\Sigma d^2\xi \sqrt{g}} \right)^{-5} \cdot \exp\left(\frac{1}{2} i \int_\Sigma d^2\xi \sqrt{g} \psi^\mu \gamma^\alpha D_\alpha \psi_\mu \right)$$

$$\exp\left(\frac{1}{2} \sum_{j,k=1}^{4h-4} a_j a_k W_{jk} \right)$$

where the a_j's are essentially odd coordinates on supermoduli space and W_{ij} is a matrix of correlation functions over Σ of momenta of the scalar fields x^μ and their fermionic counterparts ψ^ν. This fact is clearly annoying, because, such correlation function will develop an explicit dependence on the points on which they are evaluated, and so, when discussing the behaviour of the string integrand on (ordinary) moduli space, one should be very careful in determining what happens when "moving" such points.

In the next few sections we will try to discuss some features of the issues we have very informally introduced here, beginning by fixing some definitions and peculiarities of complex (possibly singular) supermanifolds.

3.2- Complex Superanalytic Spaces

There are several geometric structures which are commonly called supermanifolds both in the physical and in the mathematical literature. In this thesis work we will follow the Berezin-Kostant-Leites approach to "supergeometry"[Be,L], and this section is devoted to a collection of the basic definitions and results concerning this framework. The main motivations for this choice are the following. First, in this picture the 'anticommuting coordinates' will emerge as local generators of the 'minimal' extension of the structure sheaf of an ordinary manifold thus relying as close as possible to the framework suggested by the works on supersymmetry in physics. Second, from a mathematical standpoint, as will be apparent in the sequel, this category is very close to the one of ordinary complex spaces, thus allowing the use of powerful techniques of sheaf theory and complex geometry.

Namely, in this scheme, complex superspaces can be seen as topological spaces X together with a structure sheaf which is a $\mathbb{Z}_2$-graded extension of the ordinary structure sheaf $\mathcal{O}_X$.

Before entering the details of this construction, we feel necessary to recall some properties of $\mathbb{Z}_2$-graded rings and algebras.

Definition. Let $A = A_0 + A_1$ be a $\mathbb{Z}_2$-graded ring . We will denote by $\tilde{a}$ the degree of any of its homogeneous elements. Given a pair (a, b) of homogeneous elements of A, their *supercommutator* is defined to be

$$\lceil a, b \rfloor = a \cdot b - (-1)^{\tilde{a}\tilde{b}} b \cdot a$$

The definition of supercommutator is then extended to arbitrary elements $x, y \in A$ by linearity. A $\mathbb{Z}_2$-graded ring A is called supercommutative (or graded commutative) iff

$$\forall a, b \in A \quad \lceil a, b \rfloor = 0.$$

In a complete analogy one can introduce the notion of $\mathbb{Z}_2$-graded algebra and of supercommutative $\mathbb{Z}_2$-graded algebra . Notice that in both cases the supercommutator satisfies the following fundamental identities:

$$(1) \lceil a, b \rfloor = -(-1)^{\tilde{a}\tilde{b}} \lceil b, a \rfloor$$

$$\lceil a, \lceil b, c \rfloor \rfloor + (-1)^{\tilde{a}(\tilde{b}+\tilde{c})} \lceil b, \lceil c, a \rfloor \rfloor + (-1)^{\tilde{c}(\tilde{a}+\tilde{b})} \lceil c, \lceil a, b \rfloor \rfloor = 0$$

Given a $\mathbb{Z}_2$-graded ring A, one can define right and left A-modules, which, consistently, will be $\mathbb{Z}_2$-graded abelian groups. What actually happens is that every (say right) A-module is a bimodule, whenever left multiplication by $a \in A$ is defined taking into account a "sign rule", i.e. if M is a right A-module we will define the left action as

$$m \cdot a \overset{\text{def}}{=} (-1)^{\tilde{a}\tilde{m}} a \cdot m$$

Tensor products and the usual operations of linear algebra carry over to the graded case with the only precaution of taking correctly into account the sign rule.

Definition. An additive map $f : S \to T$ between two A-modules is called a homomorphism whenever it is A-linear and *preserves* grading.

Definition. Let S be an A-module. We define the A-module ΠS by means of the following prescriptions:

i) $\Pi S_0 = S_1 \qquad \Pi S_1 = S_0$

ii) $\Pi S \simeq S$ *qua* abelian groups

iii) right multiplication differs by a sign factor:

$$a \cdot \Pi s = (-1)^{\tilde{a}} \Pi(a \cdot s)$$

Example . The prototypical $\mathbb{Z}_2$-graded rings A we will deal with are the following:

(1) The Grassmann algebra $\wedge^* V$ of a n-dimensional vector space V

(2) The ring of "regular" functions on a domain in $\mathbb{C}^m$ with values in $\wedge^* V$. This second ring can be thought of as generated by considering it as the quotient of the polynomial ring in $m+n$ indeterminates $x_1, \cdots, x_m; \xi_1, \cdots, \xi_n$ by the ideal generated by the following relations:

$$\begin{cases} x_i x_j = x_j x_i \\ x_i \xi_\alpha = \xi_\alpha x_i \\ \xi_\alpha \xi_\beta = -\xi_\beta \xi_\alpha \end{cases}$$

An A-module S is said to be free of rank $p|q$ iff it is isomorphic to the A-module $A^{p|q} := A^p \oplus (\Pi A)^q$. Notice that $A_0^{p|q} = A_0^p \oplus (\Pi A_1)^q$ and conversely. The rank of a free A-module shares (thanks to graded-commutativity) with the dimension of vector spaces the property of being uniquely defined, in the sense that, two free A-modules S and S' will be isomorphic iff they have the same rank. This property enables one to discuss of matrices as representative of (even) homomorphisms between free A-modules. An $(m|n \times p|q)$ matrix with entries in A will be said to be in standard form if it is in block form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with $a_{ir}, d_{\alpha\beta} \in A_0$ and $c_{i\alpha}, b_{\beta r} \in A_1$. The set of matrices in standard form with entries in A is commonly denoted by

$$M(m|n, p|q; A)$$

It is a $\mathbb{Z}_2$-graded algebra naturally isomorphic to $Hom(A^{p|q}, A^{m|n})$, via the usual isomorphism given by considering the natural bases in $A^{p|q}$ and $A^{m|n}$. Given an element $X \in M(m|n, m|n; A)$ one defines its *Supertrace* to be

$$StrM = TrA_M - TrD_M.$$

Definition. A derivation in A is an additive map $X : A \longrightarrow A$ satisfying the graded Leibnitz rule:

$$X(ab) = (Xa)b + (-1)^{\tilde{a}\tilde{X}} a(Xb)$$

where $\tilde{X}$ is the parity of X *qua* additive map.

If A is an algebra over a field F, we will say that X is a derivation over F if

$$Xf = 0 \quad \forall\, f \in F.$$

The set of F-derivations in A are made into a Lie $\mathbb{Z}_2$-graded algebra by defining

$$[X, Y] = X \circ Y - (-1)^{\tilde{X}\tilde{Y}} y \circ X.$$

which naturally has the structure of A-module.

The last definition we want to recall here is the one of *Berezinian* or *Superdeterminant.* Let $B \in GL(p|q; A)$ an even automorphisms of $A^{p|q}$. Writing B in standard form

$$B = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}$$

one defines

$$BerB = det\ (B_1 - B_2 B_4^{-1} B_3)/det\ B_4$$

The meaning of this definition and the reason why it is the right "super"-generalization of the notion of determinant is clarified by the following

Proposition . $Ber : GL(p|q; A) \longrightarrow Gl(1|0; A_0)$ is the unique group homomorphism satisfying

$$Ber(\exp M) = \exp(StrM)$$

Berezin-Kostant-Leites supermanifolds are substantially complex spaces together with a sheaf of $\mathbb{Z}_2$-graded rings , i.e. they are built by pasting together collections of the objects we have described above.

Definition . (i) A ringed space $(X, \mathcal{A}_X)$ is a topological space X together with a sheaf of rings $\mathcal{A}_X$ over it. X is commonly called the underlying space and $\mathcal{A}_X$ the structure sheaf. (ii) a map between two ringed spaces $(X, \mathcal{A}_X)$ and $(Y, \mathcal{A}_Y)$ is a map $f : X \to Y$ and a sheaf homomorphism $f_\sharp : \mathcal{A}_Y \to f_* \mathcal{A}_X$ (or equivalently $f^\sharp : f^* \mathcal{A}_Y \to \mathcal{A}_X$)

Definition A2. (i) Let F be a field. An F-ringed space is a ringed space $(X, \mathcal{A}_X)$ such that the restriction $\mathcal{A}_X\restriction_U$ of the structure sheaf to any open set $U \subset X$ has the structure of an F-algebra with unity. One also assumes that for all stalks $\mathcal{A}_p$ of $\mathcal{A}_X$ a morphism of F-algebras $c_p : \mathcal{A}_p \to F$ is given. Its kernel is then a maximal two-sided ideal I_p
(ii) a map of F-ringed spaces is a map of ringed spaces such that $f_\sharp$ is a morphism of sheaves of F-algebras making the diagram

$$\begin{array}{ccc} & F & \\ {}^{c_{f(p)}}\nearrow & & {}^{c_p}\nwarrow \\ \mathcal{A}_{Y_{f(p)}} & \xrightarrow{f_\sharp} & f_*\mathcal{A}_{X_p} \end{array}$$

commutative.

As in the case of ordinary manifolds, we define a supermanifold as a space locally isomorphic to a prototypical one, a "model one". Such models are called superdomains. As in this thesis work we are primarily interested in holomorphic graded manifolds from now on we will stick to that case. Quite obviously, all what we are recalling now is true also for C^∞ real manifolds, provided one substitutes $\mathbb{C}$ with $\mathbb{R}$ and "'holomorphic" with "infinitely differentiable".
Definition . A superdomain $\overline{U}$ of dimension $m|n$ is the ringed space

$$\overline{U} := (U, O_U \otimes \wedge^*(\mathbb{C}^n))$$

where U is a domain in $\mathbb{C}^m$, O_U is the ring of holomorphic functions on U and $\wedge^*(\mathbb{C}^n)$ is the Grassmann algebra of $\mathbb{C}^n$. Notice that the ring $\mathcal{A} := O_U \otimes \wedge^*(\mathbb{C}^n)$, has a natural $\mathbb{Z}$ - grading, and hence inherits a rougher $\mathbb{Z}_2$ grading. This apparently trivial observation will be of primary interest in the discussion of splitness and projectedness of supermanifolds.
Remark. An ordinary domain (U, O_U) is thus naturally a superdomain of dimension $m|0$. Moreover, considering the subalgebra $\mathcal{N}$ generated by nilpotent elements in A and taking the quotient $\mathcal{A}/\mathcal{N}$ one gets the natural map

$$\mathcal{A}/\mathcal{N} \longrightarrow \mathcal{O}_U$$

i.e., (recalling the definition of maps between ringed spaces) an embedding

$$(U, \mathcal{O}_U) \longrightarrow (U, \mathcal{O}_U \otimes \wedge^*(\mathbb{C}^n))$$

Morphisms of superdomains can be characterized in the following way. Let $\overline{U} = (U, \mathcal{O}_U \otimes \wedge^*(\mathbb{C}^n))$ and $\overline{V} = (V, \mathcal{O}_V \otimes \wedge^*(\mathbb{C}^p))$ and let x^i, ξ^j be coordinates in $\overline{U}$ (this means that the x's are coordinates in U and the ξ's are a free set of odd generators for $\wedge^*(\mathbb{C}^n)$). Then, given m ($m = dimV$) even sections y^i and p odd sections η^k of the sheaf $\mathcal{O}_U \otimes \wedge^*(\mathbb{C}^n)$ such that $y^i(x, 0)$ lies in V, one defines a morphism of superdomains by means of

$$\mathcal{A}_{\overline{V}} \ni b(y, \eta) \rightsquigarrow b(y(x, \xi), \eta(x, \xi)) \in \mathcal{A}_{\overline{U}}$$

Conversely, any morphism of superdomains has this form.

Now we can give a definition (the constructive one) of a supermanifold as an object built up by glueing superdomains.

Definition . A complex supermanifold of dimension $m|n$ is a $\mathbb{C}$-ringed space $(X, \mathcal{A}_X)$ satisfying the following conditions

(i) Every point $p \in X$ has a neighbourhood U_p s.t. there exists an isomorphism of $\mathbb{C}$-ringed spaces

$$(f, f_\sharp) : (U, \mathcal{A}\restriction_U) \to (W, \mathcal{O}_W \otimes \wedge^*(\mathbb{C}^n))$$

where $(W, \mathcal{O}_W \otimes \wedge^*(\mathbb{C}^n))$ is an $(m|n)$ dimensional superdomain. Such an $\overline{f} = (f, f_\sharp)$ will be called a chart.

(ii) if $U \cap V \neq \emptyset$ and $(g, g_\sharp) : (V, \mathcal{A}\restriction_V) \to (Y, \mathcal{O}_Y \otimes \wedge^*(\mathbb{C}^n))$ is another chart, then the composite map $(f, f_\sharp) \circ (g, g_\sharp)^{-1}$ is an isomorphism of superdomains wherever defined. Such maps will be called transition functions.

Proposition .[Be] Suppose we are given a collection of superdomains $\overline{U}_\alpha = (U_\alpha, \mathcal{O}_{U_\alpha} \otimes \wedge^*(\mathbb{C}^n))$ and, $\forall$ ordered pair α, β of indices an open subspace $U_{\alpha\beta}$ of U_α together with morphisms

$$\overline{\varphi}_{\alpha\beta} \colon \overline{U}_{\alpha\beta} \longrightarrow \overline{U}_{\beta\alpha}$$

such that they satisfy the cocycle condition

$$\overline{\varphi}_{\alpha\beta} \circ \overline{\varphi}_{\beta\gamma} \circ \overline{\varphi}_{\gamma\alpha} = Id_{\overline{U}_{\alpha\beta\gamma}}$$

Then there exists a unique (up to isomorphisms) complex supermanifold $(X, \mathcal{A}_X)$ having the $\overline{\varphi}_{\alpha\beta}$'s as transition functions.

A more formal definition (which we will use in the sequel) which fits also the case of "singular supermanifolds" (also called superanalytic complex spaces) is the following

Definition . A complex supermanifold is a ringed space $(X, \mathcal{A}_X)$ such that

(i) X is Hausdorff with countable basis

(ii) $A_X = \mathcal{A}^0_X \oplus A^1_X$ is a sheaf of graded commutative $\mathbb{C}$-algebras

(iii) $(X, \mathcal{A}^0_X)$ is an analytic space

(iv) $\mathcal{A}^1_X$ is a coherent $\mathcal{A}^0_X$ - module

(v) if $\mathcal{N} \hookrightarrow \mathcal{A}_X$ is the ideal of nilpotents and $\mathcal{A}_{red} = \mathcal{A}_X/\mathcal{N}$, then $(X, \mathcal{A}_{red})$ is an ordinary complex space.

(vi) the $\mathcal{A}_{red}$-module $\mathcal{E} = N/N^2$ is locally free and $\mathcal{A}_X$ is locally isomorphic to its Grassmann algebra $\wedge^*(\mathcal{E})$.

Moreover, $(X, \mathcal{A}_X)$ is said to have dimension $m|n$ if $m = dim(X, \mathcal{A}_{red})$ and $n = rk_{\mathcal{A}_{red}}\, \mathcal{E}$. The ringed space $(X, \mathcal{A}_{red})$ is called the *underlying* or *reduced* complex space and often denoted X_{red} for the sake of brevity. Notice that, via the usual identification of analytic locally free sheaves of constant rank on X and (sheaves of local sections of) vector bundles on X, roughly speaking a supermanifold is a ringed space whose structure sheaf is *locally* isomorphic to the sheaf of sections of a Grassmann algebra of a vector bundle. This interpretation raises a natural question, i.e. how far this local isomorphism can be "globalized".

To be concrete, from the very definition of a supermanifold, one gets for free the following two sheaf exact sequences[R1]

$$0 \to N \hookrightarrow A \to \mathcal{A}/\mathcal{N} = \mathcal{O} \to 0 \qquad (a)$$

$$0 \to \mathcal{N}^2 \hookrightarrow \mathcal{A} \to \mathcal{A}/\mathcal{N}^2 = \mathcal{O} \oplus \mathcal{E} \to 0 \qquad (b)$$

If there is a splitting $0 \to \mathcal{O} \overset{i}{\hookrightarrow} \mathcal{A}$ of (a) the supermanifold is said to be *projected* and if (b) splits as $0 \to \mathcal{O} \oplus \mathcal{E} \overset{\mu}{\to} \mathcal{A}$ it is said to be *split*. Notice that μ must satisfy

$$\mu(f\xi) \;=\; \mu(f) \cdot \mu(\xi) \quad \forall f \in O \text{ and } \xi \in \mathcal{O} \oplus \mathcal{E}.$$

The terminology deserves a bit of explanation. For what projectedness is concerned, notice that, having an injective map $\mathcal{O} \overset{i}{\longrightarrow} \mathcal{A}$ gives a map (id, i) between the ringed spaces

$$(id, i) : (X, A) \longrightarrow (X, O)$$

so that the supermanifold "projects" down to the underlying manifold.

As for the splitness, suppose (b) splits. Then we can consistently extend $\mu : \mathcal{O} \oplus \mathcal{E} \to \mathcal{A}$ to $\hat{\mu} : \wedge^*(\mathcal{E}) \longrightarrow \mathcal{A}$ by means of

$$\hat{\mu}(f \cdot \xi_1 \wedge \cdots \wedge \xi_n) = \mu(f) \cdot \mu(\xi_1) \wedge \cdots \wedge \mu(\xi_n)$$

which is clearly a $\mathbb{Z}_2$ -ring isomorphism. Then splitness of a supermanifold means that the structure sheaf A is globally isomorphic to the sheaf of section of a Grassmann algebra of a vector bundle $\mathcal{E} :\overset{\pi}{\longrightarrow} X$.

Remark. Every $m|1$-dimensional supermanifold is trivially split. In fact in this case, $\mathcal{N}^2 = 0$ and hence the sequence (b) collapses to

$$0 \to \mathcal{A} \to \mathcal{O} \oplus \mathcal{E}(= \wedge^*(\mathcal{E})) \to 0.$$

We next want to enter in more details the issues of splitness and projectedness of supermanifolds. A keen starting point, due to Rothstein [R1], is to try to to characterize how far a given supermanifold is different from its "split counterpart". Namely, given $(X, \mathcal{A})$ and $(X, \wedge^*(\mathcal{E}))$ with $\mathcal{E} \simeq \mathcal{A}/\mathcal{N}$ we have two supermanifolds which agree, by construction, up the so-called first infinitesimal neighbourhood, and one wants to set up a machinery telling how far this isomorphism can be pushed on.

Let $Aut \wedge^* (\mathcal{E})$ denote the sheaf of parity preserving $\mathbb{C}$-linear automorphisms of $\wedge^*(\mathcal{E})$ and $\wedge_{(k)}(\mathcal{E}) := \sum_{j \geq k} \wedge^j(\mathcal{E})$. If $g : \wedge^*(\mathcal{E}) \longrightarrow \wedge^*(\mathcal{E})$ is an automorphism, then it induces naturally an automorphism $\tilde{g}$ of $\mathcal{E}$. Let then $Aut^+ \wedge^* (\mathcal{E})$ denote the subsheaf of $Aut \wedge^* (\mathcal{E})$ s.t. $\tilde{g} = id_{\mathcal{E}}$. $Aut^+ \wedge^* (\mathcal{E})$ can be identified with a more tractable object. In fact, let k be an even integer and let $Der_k \wedge^* (\mathcal{E})$ be the sheaf of derivations in $\wedge^*(\mathcal{E})$ which increase the degree by k. As above, let $Der^{(k)} := \sum_{j \leq 2k \leq n} Der_{2k} \wedge^* (\mathcal{E})$. The following holds [R1]

Proposition

$$\exp : Der^{(2)} \wedge^* (\mathcal{E}) \to Aut^+ \wedge^* (\mathcal{E})$$

is a bijection.

Now, let us consider an open cover $\{U_\alpha\}_{\alpha \in A}$ such that the sheaf $\mathcal{A}\restriction_{U_\alpha}$ is isomorphic to $O_{U_\alpha} \otimes \wedge^*(\mathbb{C}^n)$ and also U_α trivializes the vector bundle E. Let us denote

$$Gr(\mathcal{A}) := \mathcal{A}/\mathcal{N} \oplus \mathcal{N}/\mathcal{N}^2 \oplus \cdots \oplus \mathcal{N}^{(n-1)}/\mathcal{N}^n$$

and suppose that the supercoordinatization

$$\varphi_\alpha^\sharp : \mathcal{A}\restriction_{U_\alpha} \longrightarrow \wedge^*(\mathcal{E})\restriction_{U_\alpha}$$

is the identity when thought as being defined on $Gr(\mathcal{A})\restriction_{U_\alpha}$. Then the cocycle $\varphi_\alpha^\sharp \circ {\varphi_\beta^\sharp}^{-1}$ defines A up to isomorphisms and hence one gets the following

Proposition [R1,Gr] The isomorphism classes of supermanifolds $(X, \mathcal{A}_X)$ with underlying $\mathcal{O}$-module $\mathcal{E}$ are in a natural $1-1$ correspondence with the cohomology set

$$H^1(X, Aut^+ \wedge^* (\mathcal{E})) \simeq H^1(X, Der^{(2)} \wedge^* (\mathcal{E})).$$

Then a (smooth) supermanifold is characterized by the datum of

(i) a complex manifold X

(ii) a holomorphic vector bundle $E \xrightarrow{\pi} X$

(iii) a cohomology class $\tau \in H^1(X, Aut^+ \wedge^* (\mathcal{E}))$ (or $\log \tau \in \mathrm{H}^1(\mathrm{X}, \mathrm{Der}^{(2)} \wedge^* (\mathcal{E}))$.

Notice that, being $Aut^+ \wedge^* (\mathcal{E})$ non-abelian, $H^1(X, Aut^+ \wedge^* (\mathcal{E}))$ is a pointed set rather than a group, and its distinguished point labels the isomorphism class of the split supermanifold.

As observed in [Gr], the picture above gives immediately a nice proof of Batchelor's theorem which states that, in the category of C^∞ supermanifold, every object is the wedge product of a vector bundle. This follows at once by noticing that $Der^{(2)} \wedge^* (\mathcal{E})$ is a sheaf of $C^\infty(X)$-modules and hence, being $C^\infty(X)$ fine, its first cohomology vanishes. Coming back to the complex analytic case, to sharpen the analysis above, one can express the obstruction to splitness and projectedness by means of "a chain of obstructions".

Proposition . [Gr] For any holomorphic vector bundle $E \xrightarrow{\pi} X$, $Aut^+ \wedge^* (\mathcal{E})$ admits a decreasing filtration $Aut_k^+ \wedge^* (\mathcal{E})$ satisfying

(i) $Aut_2^+ \wedge^* (\mathcal{E}) = Aut_1^+ \wedge^* (\mathcal{E})$

(ii) if k is even $Aut_k^+ \wedge^* (\mathcal{E})/Aut_{k+1}^+ \wedge^* (\mathcal{E}) \simeq Der_{\mathcal{O}_X}(\mathcal{O}_X, \wedge^k(\mathcal{E})) \simeq TX \otimes \wedge^k(\mathcal{E})$

(iii) if k is odd $Aut_k^+ \wedge^* (\mathcal{E})/Aut_{k+1}^+ \wedge^* (\mathcal{E}) \simeq Hom_{\mathcal{O}_X}(\wedge^1(\mathcal{E}), \wedge^k(\mathcal{E})) \simeq \mathcal{E}^* \otimes \wedge^k(\mathcal{E})$

Then it is clear that the obstruction to splitness is given by cohomology classes $\tau_k \in H^1(X, Aut_k^+ \wedge^* (\mathcal{E})/Aut_{k+1}^+ \wedge^* (\mathcal{E}))$ and, in particular, the obstruction to projectedness is given by the even classes τ_{2k}.

This fact can be easily understood in terms of transition functions. Let $\{U_\alpha\}$ be a (locally finite) open covering of X and suppose we can express the cocycle $\varphi_\alpha^\sharp \circ {\varphi_\beta^\sharp}^{-1}$ by means of

$$\begin{cases} x_\alpha = f_{\alpha\beta}(x_\beta, \xi_\beta) \\ \xi_\alpha = g_{\alpha\beta}(x_\beta, \xi_\beta) \end{cases}$$

Expanding in power series in the odd generators ξ^i_β's one has

$$\begin{cases} x^\mu_\alpha = f^\mu_{\alpha\beta}(x_\beta) + f^\mu_{\alpha\beta ij}(x_\beta)\xi^i_\beta\xi^j_\beta + \cdots \\ \xi^i_\alpha = g^i_{\alpha\beta j}(x_\beta)\xi^j_\beta + g^i_{\alpha\beta jkl}\xi^j_\beta\xi^k_\beta\xi^l_\beta + \cdots \end{cases}$$

Whenever the supermanifold is (isomorphic to) $(X, \wedge^*(\mathcal{E}))$, one can find a refinement $\{V_\alpha\}$ of$\{U_\alpha\}$ such that the transition functions for the generators are those for a holomorphic vector bundle, i.e.

$$\begin{cases} x^\mu_\alpha = f^\mu_{\alpha\beta}(x_\beta) \\ \xi^i_\alpha = g^i_{\alpha\beta j}(x_\beta)\xi^j_\beta \end{cases}$$

which means that the "higher order terms" appearing in the above most general transformation law are cohomologuous to zero. Notice that, in this context, projectedness is realized when all the $f^\mu_{\alpha\beta ij\cdots}(x_\beta)$'s are Čech coboundaries.

To give a more concrete meaning to the preceeding discussion we show here that the set of non-split supermanifolds is non-void by constructing a (very elementary) example [Be]. Let us consider two copies of $(\mathbb{C}^*, \mathcal{O}_{\mathbb{C}^*} \otimes \wedge(\mathbb{C}^2))$ parameterized by (z, ξ_1, ξ_2) and (w, η_1, η_2) and glue them by means of the following cocycle:

$$\begin{cases} z = \frac{1}{w} + \frac{\eta_1 \cdot \eta_2}{w^3} \\ \xi_i = -\frac{1}{w^2}\eta_i \end{cases}$$

Thus the underlying manifold is the Riemann sphere $\mathbb{P}^1$ and $\mathcal{E} = K \oplus K$ is the direct sum of two copies of the canonical bundle K. The obstruction class reduces to τ_2 represented by the cocycle

$$\frac{1}{w^3} \cdot \frac{\partial}{\partial z} \otimes \eta_1 \wedge \eta_2 \text{ in } H^1(\mathbb{P}^1, K^{-1} \otimes \wedge^2(K \oplus K))$$

Now, under the identification $H^1(\mathbb{P}^1, K^{-1} \otimes \wedge^2(K \oplus K)) \simeq H^1(\mathbb{P}^1, K)$ the obstruction cocycle becomes dw/w which is precisely the generator of $H^1(\mathbb{P}^1, K) = \mathbb{C}$.

Finally, as it should be expected, most of the constructions of ordinary differential geometry carries over the graded-commutative case. For instance, one has a sound notion of what are to be considered the correct generalizations of the notion of vector bundle. Obviously, as Berezin-Kostant-Leites supermanifolds are introduced as ringed spaces, vector bundles are to be generically intended in sheaf-theoretic terms as follows.

Definition . Let $(X, \mathcal{A}_X)$ be a supermanifold. We define a rank $r|s$ super vector bundle over to be an $\mathcal{A}_X$ locally free sheaf $\mathcal{F}$ over X of rank $r|s$.

Definition . The tangent sheaf $\hat{T}X$ to $(X, \mathcal{A}_X)$ is the sheaf defined by the presheaf

$$U \longrightarrow \hat{D}er(\mathcal{A}_X)\restriction_U$$

where $\hat{D}er(\mathcal{A}_X)\restriction_U$ is the sheaf of graded derivations of the ring $\mathcal{A}_X)\restriction_U$.

To be definite, we will choose left derivations, and consequently $\hat{\mathcal{T}}X\restriction_U$ admits a natural structure of free left $\mathcal{A}_X$-module, of rank equal (by definition!) to the dimension of $(X, \mathcal{A}_X)$.

Remark $\hat{\mathcal{T}}X$ is locally isomorphic to

$$\mathcal{A}_X \otimes (\mathcal{T}X \oplus \Pi\mathcal{E}^*)$$

where $\mathcal{T}X$ is the tangent sheaf to the underlying manifold and $\mathcal{E} = \mathcal{A}_X/\mathcal{N}$ is the local model. This isomorphism is clearly globalized whenever $(X, \mathcal{A}_X)$ is split.

In the sequel we will be mainly interested in line bundles over supermanifolds. They are usually defined as rank $1|0$ vector bundles over $(X, \mathcal{A}_X)$, or equivalently, by means of the following construction.

Let $\mathcal{A}^*_{ev}$ denote the ring of invertible even local sections of the structure sheaf $\mathcal{A}_X$. Then a line bundle $\hat{L} \to \hat{X}$ is obtained by glueing products $U_\alpha \times \mathcal{A}_X$ where $\bigcup U_\alpha$ is a covering of X by means of a (equivalence class of) $\mathcal{A}^*_{ev}$-valued cocycle $g_{\alpha\beta}$. The usual classification of line bundles over a manifold carry over the graded commutative case. Namely, the set of equivalence classes of line bundles over $\hat{X}$ is in a 1–1 correspondence with the cohomology "group"

$$H^1(X, \mathcal{A}^*_{ev})$$

and one has at his disposal (the analogue of) the exponential sequence:

$$0 \to \mathbb{Z} \longrightarrow \mathcal{A}_X \longrightarrow \mathcal{A}^*_{ev} \to 1$$

from which

$$\cdots \to H^1(X, \mathbb{Z}) \to H^1(X, \mathcal{A}_X) \to H^1(X, \mathcal{A}^*_{ev}) \to H^2(X, \mathbb{Z}) \to \cdots$$

More interesting for us is the identification of the group of line bundles over $(X, \mathcal{A}_X)$ and the group of line bundles on the reduced manifold X as given by [M2]. Namely, $Pic(\hat{X}) \simeq Pic(X)$ via

$$\hat{L} \xrightarrow{\sim} L_{red} \cdot [\mathcal{A}_X]$$

3.3 Integration theory on supermanifolds

Integration theory over ringed spaces is an intriguing problem. In fact, remarkable differences exist between integration theory on a reduced ringed space (e.g. an ordinary manifold) and a more general ringed space, such as a graded manifold. Namely, most of the peculiarities of integration theory on supermanifolds rely on the properties of the graded generalization of the determinant functor, the Berezinian.

Let us first define integration theory on a real C^∞ supermanifold of dimension $p|q$. Generically speaking, an integration theory on a ringed space $(X, \mathcal{A}_X)$ consists in the individuation of a suitable sheaf of $\mathcal{A}_X$-modules , the *would-be* sheaf of volume forms and the prescription of a linear functionals $\int_{\mathcal{F}}$ on its global sections.

Given a $(p|q)$-dimensional superdomain $(U, C^\infty(U) \otimes \wedge^*(\mathbb{R}^q)$, with a fixed coordinatization (x_i, ξ_α) one has at its disposal Berezin's integration theory [Be], which stems from a direct transposition of ordinary integration theory to the "super" case. Namely, just like volume forms (i.e. C^∞-measures) are obtained on manifolds by applying the determinant functor to frames in the cotangent sheaf $\mathcal{T}M$, one takes as a definition of *super volume forms* the result of applying the Berezinian functor to a set of free generators for the graded cotangent sheaf $\hat{\mathcal{T}}\hat{M}$. We recall that the compelling motivation for which the Berezinian must be taken as the determinant is that it is the only group homomorphism well behaving with respect to traces (see the discussion above).

We recall that the Berezin integral is defined, in this context, in the following way. The berezinian sheaf is locally (i.e. in any coordinate chart) generated by $[d\underline{x}/d\underline{\xi}]$, more commonly written as

$$[d\underline{x}/d\underline{\xi}] \equiv dx_1 \wedge \cdots \wedge dx_p \frac{\partial}{\partial \xi_q} \cdots \frac{\partial}{\partial \xi_1}$$

and, writing a section of Ber as $f(\underline{x}, \underline{\xi}) \cdot [d\underline{x}/d\underline{\xi}]$, the berezinian integral picks out the upper ξ-degree part of f and integrates it over the underlying domain U, as the local form of the generator for the Berezinian sheaf dictates.

Actually, as it was already clear to Berezin himself, this procedure is well defined only when integrating compactly supported forms, i.e. when integrating over compact supermanifolds . The classical counterexample is the following.

Consider the supermanifold $((0,1), C^\infty(0,1) \otimes \wedge^*(\mathbb{R}))$, with coordinates x, θ_1, θ_2. Considering the coordinate transformation

$$\begin{cases} y = x + \theta_1\theta_2 \\ \eta_1 = \theta_1 \\ \eta_2 = \theta_2 \end{cases}$$

Then, as the berezinian of this transformation law is 1, one gets a contradiction, since

$$\int_B [dy/d\eta] y = \int_0^1 dy \frac{\partial}{\partial \eta_2} \frac{\partial}{\partial \eta_1} y = 0$$

whereas

$$\int_B [dx/d\theta] y = \int_0^1 dx \frac{\partial}{\partial \theta_2} \frac{\partial}{\partial \theta_1} [x + \theta_1\theta_2] \ = \ 1$$

A simple computation shows that the difference between the integrands can be written as a total differentials, and so, in some sense, the two differ by a boundary term. Notice that, when integrating over a non flat manifold, such trouble will arise in each overlap between coordinate charts. Nonetheless, whenever the underlying space is compact and without boundary, the berezinian integration prescription is well defined, as all discrepancies will add up to zero. In any case, such a prescription has two annoying features.

i) Integration over non-compact supermanifold is not well defined (in this case we do not have finite subcoverings of the underlying space, so that we can no more add up harmlessly the "boundary" contributions).

ii) The volume form resulting from integration over odd variables is ill-defined.

The solution to this twofold problem can be obtained by pursuing the following "radical" idea [R2]. The failure for Berezin integration theory to be a consistent integration theory lies in the fact that the Berezinian sheaf is too "small" a sheaf to be the good candidate for the sheaf of super-volume forms. Then, as intuitively, integration over odd "directions" must be a derivation, one is naturally lead to consider the sheaf $\Omega^p_X \otimes_{\mathcal{A}_X} \mathcal{D}$ of linear differential operators on $\mathcal{A}_X$ with values in the sheaf of (ordinary) volume forms on X.

The sheaf $\Omega^p_X \otimes_{\mathcal{A}_X} \mathcal{D}$ is a locally free $\mathcal{A}_X$-module of infinite rank. In local coordinates, its generic section can be written as

$$\Phi = \sum_{I,\mu} dx_1 \wedge \cdots \wedge dx_p \frac{\partial}{\partial \theta_\mu} \frac{\partial}{\partial x^I} \circ f^{I,\mu}$$

where I is a multiindex with values in $\mathbb{N}^+$ and μ is a $\mathbb{Z}_2$-valued multiindex. Notice that $\circ$ means composition of operators, i.e., if g is a local super holomorphic function, then

$$\Phi(g) = \sum_{I,\mu} dx_1 \wedge \cdots \wedge dx_p \frac{\partial}{\partial \theta_\mu} \frac{\partial}{\partial x^I} (f^{I,\mu} \cdot g)$$

The sum above is not free. Nonetheless a free set of generators for $\Omega^p_X \otimes_{\mathcal{A}_X} \mathcal{D}$ is readily exhibited as

$$D_I(x,\theta) = dx_1 \wedge \cdots \wedge dx_p \frac{\partial}{\partial \theta_q} \cdots \frac{\partial}{\partial \theta_1}$$

Now, a section Φ of $\Omega^p_X \otimes_{\mathcal{A}_X} \mathcal{D}$ define a linear functional on $\mathcal{A}_X$ by

$$< \Phi, f > = \int_X \Phi(f)$$

so that it is natural to define the integral for such a Φ to be

$$\int_{\mathcal{F}} := \int_X \Phi(1)$$

Notice that, whenever $\Phi = \sum_I D_I(x,\theta) f^I$ is such that $f^I = 0 \quad I > 0$, the above definition coincides locally with the one of Berezin integral. In some sense, the sheaf $\Omega^p_X \otimes_{\mathcal{A}_X} \mathcal{D}$ is too "big" to give a completely satisfactory integration theory on $(X, \mathcal{A}_X)$, but rather one should seek for a procedure yielding locally the usual fermionic integration theory. Define the subsheaf $(\Omega^p_X \otimes_{\mathcal{A}_X} \mathcal{D})^+ \subset \Omega^p_X \otimes_{\mathcal{A}_X} \mathcal{D}$ by means of

$$\omega \in (\Omega^p_X \otimes_{\mathcal{A}_X} \mathcal{D})^+ \restriction_U \;\Leftrightarrow\; f^0_\omega = 0$$

Then it is clear that if h is a compactly supported section of $\mathcal{A}_X$, with support contained in U, then $\int_U \omega(h) = 0$. Globalizing this observation we have the following

Proposition . Let ω be a global section of $(\Omega^p_X \otimes_{\mathcal{A}_X} \mathcal{D})^+$. Then $\int_F \omega$ is an exact p-form.

The construction above comes equipped with an exact sequence

$$0 \to (\Omega^p_X \otimes_{\mathcal{A}_X} \mathcal{D})^+ \hookrightarrow \Omega^p_X \otimes_{\mathcal{A}_X} \mathcal{D} \to \Omega^p_X \otimes_{\mathcal{A}_X} \mathcal{D}/(\Omega^p_X \otimes_{\mathcal{A}_X} \mathcal{D})^+ \to 0$$

What is relevant in the whole construction is that $\Omega^p_X \otimes_{\mathcal{A}_X} \mathcal{D}/(\Omega^p_X \otimes_{\mathcal{A}_X} \mathcal{D})^+ \simeq Ber$ so that the sequence above can be rewritten as

$$0 \to (\Omega^p_X \otimes_{\mathcal{A}_X} \mathcal{D})^+ \hookrightarrow \Omega^p_X \otimes_{\mathcal{A}_X} \mathcal{D} \to Ber \to 0$$

Then, a satisfactory integration theory will be gotten once choosing a splitting of this sequence, thus regarding Ber as embedded in $\Omega^p_X \otimes_{\mathcal{A}_X} \mathcal{D}$, and then transforming its sections properly when changing the coordinates. Namely, any coordinate system endows $\mathcal{A}_X \restriction_U$ of a $\mathbb{Z}$-grading, while general coordinate transformations will only preserve the natural $\mathbb{Z}_2$-grading. A coordinate transformation preserving the $\mathbb{Z}$ grading will be called a split transformation. A key observation is that any coordinate change can be written as the composition of a split one times one generated by a degree increasing even derivation. More explicitly we have the following [R2]

Proposition . Let (x, θ) and (y, η) two coordinate systems for $\mathcal{A}_X$. Then there is a unique coordinate system (w, λ) and a unique degree increasing even derivation Y such that

i) $(x, \theta) \rightsquigarrow (w, \lambda)$ is a split transformation

ii) $(y, \eta) = \exp(Y)(w, \lambda)$.

Considering the canonical basis $D_I(x, \theta)$ of $\Omega^p_X \otimes_{\mathcal{A}_X} \mathcal{D}$, one has that

$$D_I(y, \eta) = D_J(x, \theta)\Psi^J_I$$

and, while in the case of split transformations Ψ^J_I reduces to the usual transformation law for differential operators, and, in particular Ψ^0_0 is the Berezinian transformation law, when considering non-split transformations $D_I(y, \eta) = D_J(x, \theta)\exp(-Y)$. In particular it holds

Proposition . Let

$$\exp(Y) = \sum g^I \frac{\partial}{\partial X^I} + \kappa^\mu \frac{\partial}{\partial \theta^\mu}$$

Then

$$D_I(y, \eta) = \sum_J D_{I+J}(x, \theta) Ber\left(\frac{\partial(x, \theta)}{\partial(y, \eta)}\right) g^J$$

and specifically

$$D_0(y, \eta) = \sum_J D_J(x, \Theta) Ber\left(\frac{\partial(x, \theta)}{\partial(y, \eta)}\right) g^J$$

Even without entering too deep the details of the computations, one can immediately see that the appearance of these extra terms in the transformation law for the image of the berezinian under the splitting of the sequence above are exactly the one needed to cancel the "anomalous" transformation law for the berezinian, thus yielding a well definite super-integration prescription.

To grasp how the scheme works, let us come back to the example recalled at the beginning, namely to the problem of integrating $F(y, \eta_1, \eta_2)$ over the interval $(0, 1)$. The most general coordinate transformation law (up to split ones) is

$$\begin{cases} y = x + \theta_1\theta_2 g(x) \\ \eta_i = \theta_i \qquad i = 1, 2. \end{cases}$$

so that the jacobian is

$$\frac{\partial(y,\theta)}{\partial(x,\theta)} = \begin{pmatrix} 1 + \theta_1\theta_2 g'(x) & \theta_2 g(x) & -\theta_1 g(x) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and its berezinian is

$$Ber\left(\frac{\partial(y,\theta)}{\partial(x,\theta)}\right) = 1 + \theta_1\theta_2 g'(x)$$

We have $(y, \eta) = \exp\{\theta_1\theta_2 g(x)\frac{\partial}{\partial x}\}(x, \theta)$ and hence

$$\begin{aligned} D_0(y,\eta) &= D_0(x,\theta)(1 + \theta_1\theta_2 g'(x)) + D_1(x,\theta)(1 + \theta_1\theta_2 g'(x))(\theta_1\theta_2 g(x)) \\ &= D_0(x,\theta)(1 + \theta_1\theta_2 g'(x)) + D_1(x,\theta)(\theta_1\theta_2 g(x)) \end{aligned}$$

Thus

$$\int D_0(y,\eta)F(y,\eta) = \int D_0(y,\eta)[f_0(y) + \eta_1\eta_2 f_2(y)] = \int dy\ f_2(y) = \int dx\ f_2(x)$$

on the other hand

$$\int [D_0(x,\theta)(1 + \theta_1\theta_2 g'(x)) + D_1(x,\theta)(\theta_1\theta_2 g(x))][f_0(x, \theta_1\theta_2 g(x) + \theta_1\theta_2 f_2(x + \theta_1\theta_2 g(x)] =$$

$$= \int D_0(x,\theta)\theta_1\theta_2(f_0 g' + f_0' g + f_2) + D_1(x,\theta)\theta_1\theta_2 f_0 g =$$

$$= \int D_0(x,\theta)\theta_1\theta_2(f_0 g' + f_0' g + f_2 - \frac{d}{dx}(f_2 g)) = \int dx\ f_2(x)$$

In more intuitive terms, one can proceed as follows. Fix a local section of the projection $\pi : \Omega^p_X \otimes_{\mathcal{A}_X} \mathcal{D} \to Ber$, say $i([dx/d\theta])$, and promote it to be the local representative for $D_0(x,\theta)$. Then the transformation law for such an object will be the ones for a generic element of $\Omega^p_X \otimes_{\mathcal{A}_X} \mathcal{D}$. Notice that the procedure apparently suffers by the non naturality in the choice of the splitting of the sequence

$$0 \to (\Omega^p_X \otimes_{\mathcal{A}_X} \mathcal{D})^+ \xrightarrow{j} \Omega^p_X \otimes_{\mathcal{A}_X} \mathcal{D} \xrightarrow{\pi} Ber \to 0$$

but, giving two different splittings i_1 and i_2, their difference is in the kernel of the projection π, so that, being in the image of j, produces a global exact form after integration over the odd variables.

The situation is a little bit more embarrassing when dealing with complex supermanifolds . In fact, in this situation, one would like to retain the usual identification of the sheaf of volume forms as the tensor product of the canonical sheaf times the anticanonical sheaf. More interesting for us is the fact that issues like holomorphic factorization can be discussed only when such structures are well settled. To be more specific, the complex counterpart of the Berezinian sequence should be a sequence built with "half volume forms", i.e. top exterior powers of the complex cotangent sheaf. Namely in this case we are interested in the following sequence

$$0 \to (\Omega_X^{p,0} \otimes_{\mathcal{A}_X} \mathcal{D})^+ \xrightarrow{j} \Omega_X^{p,0} \otimes_{\mathcal{A}_X} \mathcal{D} \xrightarrow{\smallfrown} Ber \to 0$$

where now $(\Omega_X^{p,0} \otimes_{\mathcal{A}_X} \mathcal{D})^+$ is defined by $f_0 = 0$ and "Ber" is the holomorphic square root of the usual berezinian.

Given an exact sequence of sheaves $\mathcal{A}_X$-modules

$$0 \to \mathcal{E} \to \mathcal{G} \to \mathcal{F} \to 0$$

we recall that the sequence is said to be split iff there exists a $\mathcal{A}$-homomorphism

$$i : \mathcal{F} \longrightarrow \mathcal{G} \text{ such that } \quad \pi \circ i = id_{\mathcal{F}}.$$

When the sequence above is a sequence of analytic sheaves over a complex (super)manifold its failure to splitting is measured by a cohomology class, as it turns out of the following argument.
By tensoring with the dual sheaf $\mathcal{F}^*$ one deduces the following exact sequence

$$0 \to Hom(\mathcal{F},\mathcal{E}) \to Hom(\mathcal{F},\mathcal{G}) \xrightarrow{\pi} Hom(\mathcal{F},\mathcal{F}) \to 0$$

and hence the associated long cohomology sequence reads

$$\cdots \to H^0(Hom(\mathcal{F},\mathcal{G})) \xrightarrow{\pi^*} H^0(Hom(\mathcal{F},\mathcal{F})) \xrightarrow{\delta_*} H^1(Hom(\mathcal{F},\mathcal{E})) \to \cdots$$

where δ_* is the coboundary map. It is immediate to see that the obstruction class to the splitting of the sequence above is $\delta_*(id_{\mathcal{F}})$ since, if it is non vanishing, the equation

$$\pi \circ i = id_{\mathcal{F}} \quad \Leftrightarrow \quad \pi_*(i) = id_{\mathcal{F}}$$

cannot be solved. Notice that, as usual, no obstruction to the splitting of the Berezinian sequence

$$0 \to (\Omega_X^p \otimes_{\mathcal{A}_X} \mathcal{D})^+ \xrightarrow{j} \Omega_X^p \otimes_{\mathcal{A}_X} \mathcal{D} \xrightarrow{\smallfrown} Ber \to 0$$

is present in the C^∞-case because of the existence of partitions of unity for the structure sheaf $\mathcal{A}_X$

In our case, being Ber a rank $0|1$ sheaf, $Hom(Ber, Ber) = \mathcal{A}_X$ and the long cohomology sequence looks like

$$\cdots \to H^0(Hom(Ber, \Omega_X^{p,0} \otimes_{\mathcal{A}_X} \mathcal{D})) \to H^0(\mathcal{A}_X) \to H^1(Hom(Ber, (\Omega_X^{p,0} \otimes_{\mathcal{A}_X} \mathcal{D})^+)) \to \cdots$$

so that the obstruction to the splitting is $\delta^*(1) \in H^1(Hom(Ber, (\Omega_X^{p,0} \otimes_{\mathcal{A}_X} \mathcal{D})^+))$.

As a final remark notice that, whenever the supermanifold is split no such trouble can arise, as one has at his disposal a split atlas in which the glueing of local sections of the Berezinian sheaf is well defined also when they are considered as local sections of $\Omega_X^{p,0} \otimes_{\mathcal{A}_X} \mathcal{D}$.

3.4 Super Riemann surfaces

As discussed in §3.1, the analysis of two-dimensional superconformal supergravity lead to the remarkable observation that real 2-dimensional supermanifolds are locally superconformally flat, i.e. the superzweibein $E_A^{\ M}$ can be put in a definite form and the superderivative $\mathcal{D}$ looks locally like

$$\mathcal{D} = \frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial z}$$

By analogy with ordinary complex manifolds theory, one then is naturally lead to the definition of a Super Riemann surface as a supermanifold locally built with coordinate patches that preserve in some sense such a structure.

Namely, considering a mere $1|1$-dimensional complex manifold, we see that, considering holomorphic transition functions

$$\begin{cases} \tilde{z} = \tilde{z}(z,\theta) \\ \tilde{\theta} = \tilde{\theta}(z,\theta) \end{cases}$$

the superderivative transforms as [Fr]

$$\mathcal{D} = \left[\mathcal{D}\tilde{\theta}\right]\tilde{\mathcal{D}} + \left[\mathcal{D}\tilde{z} - \tilde{\theta}\mathcal{D}\tilde{\theta}\right]\tilde{\mathcal{D}}^2$$

A holomorphic map of the above form will be called a *superconformal* transformation iff the transformation law for the superderivative $\mathcal{D}$ is homogeneous, i.e. if

$$\mathcal{D} = \left[\mathcal{D}\tilde{\theta}\right]\tilde{\mathcal{D}} \quad \Leftrightarrow \left[\mathcal{D}\tilde{z} - \tilde{\theta}\mathcal{D}\tilde{\theta}\right]\tilde{\mathcal{D}}^2 = 0$$

Notice that $\mathcal{D}^2 = \frac{1}{2}[\mathcal{D},\mathcal{D}] = \frac{\partial}{\partial z}$ and $(\mathcal{D}, \frac{\partial}{\partial z})$ span the tangent space $\hat{T}_{\mathbb{C}}\Sigma$. Accordingly, one is lead to the following

Definition . A super-curve is the datum of

a) an algebraic curve C_{red}, with structure sheaf $\mathcal{O}$,

b) a sheaf $\mathcal{A}$ over C_{red} of super-commutative $\mathbb{C}$-algebras (with nilpotent ideal $\mathcal{N}$) such that

i) $\mathcal{A}/\mathcal{N} = \mathcal{O}$,

ii) the $\mathcal{O}$−module $\mathcal{N}$ is locally free of rank 1.

Definition . A Super Riemann surface (also called, especially in the Russian literature a *SUSY-curve*) $\hat{C}$ is a supercurve, together with

c) a locally free rank $0|1$ subsheaf $\mathcal{D}$ of the tangent sheaf $\hat{T}C$

$$0 \to \mathcal{D} \to \hat{T}C \to \hat{T}C/\mathcal{D} \to 0$$

such that

iii) the commutator (mod $\mathcal{D}$)

$$[\ ,\ ]_{\mathcal{D}} : \mathcal{D} \otimes \mathcal{D} \longrightarrow \hat{T}C/\mathcal{D}$$

is an isomorphism, and so $\mathcal{D}$ and $[\mathcal{D},\mathcal{D}]_{\mathcal{D}}$ generate $\hat{T}C$.

To make contact with the previous coordinate approach to superconformal field theory one can argue in the following way. Considering obviously the smooth case, one can identify a local generator for the distribution $D\restriction_{U_\alpha}$ with the coordinate expression $\partial/\partial\theta_\alpha + \theta_\alpha \otimes \partial/\partial z_\alpha$. The equivalence between the two definitions is given by the following

Proposition . Any Super Riemann surface admits a canonical atlas [LB-R].

Proof . Let (w,ϕ) be a coordinate system. Then, as $\mathcal{D}$ is locally free of rank $0|1$, it has a generator of the form $\frac{\partial}{\partial\phi} + h\frac{\partial}{\partial w}$, with h odd. Then $[\mathcal{D},\mathcal{D}]_{\mathcal{D}}$ looks locally like $\frac{\partial h}{\partial \phi}\cdot\frac{\partial}{\partial z}$ so that for the commutator to be an isomorphism, $\frac{\partial h}{\partial\phi}$ must be invertible. Let us introduce coordinates (z,η) with $\eta = \phi$. Now the local generator for $\mathcal{D}$ looks like

$$\frac{\partial}{\partial\phi} + h\frac{\partial}{\partial w} = \frac{\partial}{\partial\eta} + (h\frac{\partial z}{\partial w} + \frac{\partial z}{\partial\phi})\frac{\partial}{\partial z}$$

so that one has to solve the equation $h\frac{\partial z}{\partial w} + \frac{\partial z}{\partial\phi} = \eta$. Expanding both z and h in powers of ϕ as

$$z = z_0 + \phi z_1 \qquad h = h_0\phi h_1$$

and equating terms of the same degree in ϕ one obtains the equations

$$\begin{cases} z_1 + h_1\frac{\partial z_0}{\partial w} \\ h_0\frac{\partial z_0}{\partial w} + h_1\frac{\partial z_1}{\partial w} = 1 \end{cases}$$

As $h_1 \equiv \frac{\partial h}{\partial\phi}$ is invertible this system has solutions.

Remark . The steps done in the proof are actually redundant. In fact, noticing that we have at our disposal only one odd coordinate, things simplify notably, as, for instance, in the above expressions, $z \equiv z_0$ $h \equiv \phi_1$ so that the first equation is identically satisfied, and, as for the second, one has as a ready-made solution $\frac{\partial z_0}{\partial w} = \frac{1}{h_0}$. Nonetheless we have preferred to give such a redundant proof since it will apply, with only lexical modifications to the case of families of Super Riemann surfaces .

As a consequence on each intersection $U_\alpha \cap U_\beta$ we have that both $\partial/\partial\theta_\alpha + \theta_\alpha \otimes \partial/\partial z_\alpha$ and $\partial/\partial\theta_\beta + \theta_\beta \otimes \partial/\partial z_\beta$ generate $\mathcal{D}$ and therefore should be proportional. An easy computation yields the following clutching functions

$$\begin{cases} z_\alpha = f_{\alpha\beta}(z_\beta) \\ \\ \theta_\alpha = g_{\alpha\beta}(z_\beta)\,\theta_\beta \end{cases}$$

with $g^2_{\alpha\beta} = f'_{\alpha\beta}$, $(f'_{\alpha\beta} = df_{\alpha\beta}/dz_\beta)$, showing that the $g_{\alpha\beta}$ are transition functions for a θ-characteristics $\mathcal{L}$ on C_{red}. Conversely, given a pair $(C_{red}, \mathcal{L})$ we can construct a SUSY-curve $\hat{C}$ just setting $\mathcal{A} = \mathcal{O} \oplus \Pi\mathcal{L}$, where Π is the parity changing functor, whose effect is to make sections of $\mathcal{L}$ anticommute.

Summing up we have the

Proposition . There are as many SUSY-curves C on a fixed smooth algebraic curve C_{red} as reduced space as θ- characteristics on C_{red}.

Remark. From the discussion above one can grasp another virtue of the compactification of moduli space of θ-characteristics discussed in §2.7. Namely, having at our disposal a sound notion of a θ-characteristics on a (mildly) singular algebraic curve as an invertible sheaf, we can transplant the above definitions and theorems without any modifications to the case of decent spin curves, thus getting a clear notion of a SUSY-curve having as reduced space a semistable curve. Notice also that the property of Cornalba's compactification of yielding the *same* number of θ-characteristics in the singular case as in the smooth one, fits naturally into the proposition above.

Before dealing with the problem of moduli of Super Riemann surfaces , we feel necessary to deep a little bit into the geometry of SUSY-curves and discuss some line bundle theory on such objects [GN].

In the ordinary case, a complex 1-dimensional manifold comes equipped with the operators ∂ and $\bar{\partial}$ which have the following significant features.

i) ∂ sends the structure sheaf into the canonical sheaf, which can be seen as the holomorphic square root of the sheaf *vol* of volume forms.

ii)by the Dolbeault lemma, $\bar{\partial}$ fits into a fine resolution of $\mathcal{O}_C$ as

$$0 \to \mathcal{O}_C \longrightarrow \mathcal{C}^\infty(C) \xrightarrow{\bar{\partial}} \mathcal{C}^\infty(C, T^*_{\bar{\mathbb{C}}}C) \to 0$$

The suitable generalization of these features in the case of SUSY-curves can be deduced from the following arguments. Firstly, being SUSY-curves split and compact, there is no difficulty in outlining the Berezinian sheaf as the holomorphic square root of the sheaf of volume forms, i.e.

$$\hat{vol} = Ber(\hat{T}^*C) \otimes \overline{Ber}(\hat{T}^*C) \equiv \hat{k} \otimes \bar{\hat{k}}$$

The key observation is that [D] the defining exact sequence for $\mathcal{D}$

$$0 \to \mathcal{D} \longrightarrow \hat{T}C \longrightarrow \hat{T}C/\mathcal{D} \to 0, \qquad \hat{T}C/\mathcal{D} \wr \mathcal{D}^{\otimes^2}$$

dualizes into

$$0 \to \mathcal{D}^{*\otimes^2} \longrightarrow \Omega^1_{\hat{C}} \longrightarrow \mathcal{D}^* \to 0$$

and hence one gets, taking Berezinians and noticing that $\mathcal{D}^*$ is odd

$$Ber\Omega^1_{\hat{C}} \simeq \mathcal{D}^*$$

Thus one can define the fundamental operator

$$\delta : \mathcal{A}_C \longrightarrow Ber\hat{C}$$

by defining locally

$$\begin{array}{rcccl} \delta : & \mathcal{A}_X\restriction_U & \longrightarrow & BerC\restriction_U & \\ & f & \rightsquigarrow & \delta(f) := & (\partial_\theta + \theta\partial_z)f \cdot [dz/d\theta] \end{array}$$

where $[dz/d\theta]$ is the local generator for $BerC\restriction_U$. The fact that this procedure is well defined can be grasped from the following coordinate computation [GN]. Let $(z,\theta)\rightsquigarrow(w,\varphi)$ be a superconformal transformation. Then the superderivative $\partial_\theta + \theta\partial_z$ will transform as $\partial_\varphi + \varphi\partial_w = (D\varphi)^{-1}(\partial_\theta + \theta\partial_z)$ and the Berezinian as $[dw/d\varphi] = Ber[\partial(w,\varphi)/\partial(z,\theta)]\cdot[dz/d\theta]$. But, using the fact that the coordinate change is a superconformal transformation, one finds that

$$Ber\begin{pmatrix} \partial_z w & \partial_z\varphi \\ \partial_\theta w & \partial_\theta\varphi \end{pmatrix} = Ber\left[\begin{pmatrix} 1 & 0 \\ -\theta & 1 \end{pmatrix} \cdot \begin{pmatrix} \partial_z w + \varphi\partial_z\varphi & \partial_z\varphi \\ 0 & D_{z\theta}\varphi \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ \varphi & 1 \end{pmatrix}\right] =$$

$$= \frac{\partial_z w + \varphi\partial_z\varphi}{D\varphi} = D\varphi$$

so that the R.H.S. of the above defining equation for δ has the correct glueing properties on overlaps.

Furthermore, δ can be used to define a resolution of the structure sheaf $\mathcal{A}_C$. Namely, thanks to the fact that $\mathcal{D}$ is 'as much as non-integrable as possible',

$$\overline{D}f = 0 \Rightarrow \overline{D}^2 = 0 \text{ i.e. } \bar{\partial}f = 0$$

Hence, considering a smooth 'superfunction' σ, it will be holomorphic iff it is in the kernel of $\bar{\delta}$, so that $\bar{\delta}$ fits the exact sequence

$$0 \to \mathcal{A}_C \longrightarrow \hat{C}^\infty(C) \xrightarrow{\bar{\delta}} \hat{C}^\infty(C) \otimes Ber \to 0$$

(Here exactness at the last step is proven by combining the above observation about holomorphicity of local sections plus the ordinary Dolbeault lemma).

This fact has a nice consequence. We have seen in §3.2 that the group of super line bundles on a supermanifold $\hat{X}$ is (isomorphic to) the first cohomology group $H^1(X, \mathcal{A}^*_{ev})$ of cocycles with values in the ring of even invertible super holomorphic functions. Now as a consequence of the existence of the fine resolution above, the long cohomology sequence associated to the exponential sequence stops at $H^2(C, \mathcal{A}_C)$, thus giving

$$\cdots \to H^1(C, \mathbb{Z}) \to H^1(C, \mathcal{A}_C) \to H^1(C, \mathcal{A}^*_{C\ ev}) \to H^2(C, \mathbb{Z}) \to 0$$

and exhibiting the group $Pic(\hat{C})$ as the semidirect product

$$H^2(C, \mathbb{Z}) \ltimes H^1(C, \mathcal{A}_C)/H^1(C, \mathbb{Z}).$$

We want to finish this section mentioning how one can geometrically extend the formalism of conformal field theory to the super case, i.e. identifying the super-analogue of 'fields of type $p, \bar{q}$' (tensor products of p-canonical forms with q-anticanonical in the algebro-geometrical language).

The hint comes, once again, by noticing that the sheaf playing the rôle of the canonical sheaf here is the dual $\mathcal{D}^*$. Then, one will define [Fr,BMFS] superfields of type $p, \bar{q}$ as sections of the sheaf $\mathcal{D}^{(p,q)} := \mathcal{D}^p \otimes \overline{\mathcal{D}}^q$. To be more specific, notice that, giving a section of $\mathcal{D}^{(p,q)}$ is tantamount to giving a coordinate covering U_α of C and local sections ϕ_α of $\mathcal{A}_C$ satisfying, in each overlap $U_\alpha \cap U_\beta$ the patching relation

$$\phi_\alpha = \mathcal{D}_\alpha \theta^p_\beta \overline{\mathcal{D}_\alpha \theta_\beta}^q \phi_\beta$$

Notice, by the way, that $g_{\alpha\beta} = \mathcal{D}_\alpha \theta^p_\beta \overline{\mathcal{D}_\alpha \theta_\beta}^q$ are $\mathcal{A}^*_{C\ ev}$-valued and satisfy the cocycle condition, so that they actually define a line bundle over $\hat{C}$.

Furthermore, a "scalar" product can de introduced in $\mathcal{D}^{(p,q)}$ by considering local sections e of $\mathcal{D}$ and putting, for $X, Y \in \mathcal{D}^{(p,q)}$

$$< X, Y > = \int_C \widehat{dvol} X \overline{Y} (e\bar{e})^{p+q}$$

where $\widehat{dvol}$ is the canonical volume form associated to D, i.e. $\widehat{dvol} = Ber D^* \cdot \overline{Ber D^*}$. One can then define, in the spaces $\mathcal{D}^{(p,0)}$ and $\mathcal{D}^{(p,0)}$ differential operators $\overline{D}_p$, D_p together with their formal adjoints (with respect to the bilinear form introduced above) $\overline{D}_p^\dagger$, D_p and, finally, the superlaplaceans

$$\Box_p := \overline{D}_p^\dagger \overline{D}_p : \mathcal{D}^{(p,0)} \to \mathcal{D}^{(p,0)}$$

$$\overline{\Box}_p := D_p^\dagger D_p : \mathcal{D}^{(0,p)} \to \mathcal{D}^{(0,p)}$$

3.5 Deformation theory of complex superspaces and supermoduli spaces

The aim of this section is to give some definitions of the deformation theory of complex superspaces, and to show that, the usual technology of the ordinary case (discussed, for the case of curves, in chapter 2) can be carried over to the graded commutative category.

By a complex analytic superspace we mean, in analogy with ordinary reduced complex spaces, a ringed space $(X, \mathcal{A}_X)$, where $\mathcal{A}_X$ is a sheaf of graded commutative $\mathbb{C}$-algebras, which is locally isomorphic to a complex analytic superspace patch, where the latter is defined as follows.

Let us consider a superdomain $\hat{U} := (U, \mathcal{A}_U) = (U, \mathcal{O}_U \otimes \wedge^*(\mathbb{C}^q))$, a set $\{f_1, \cdots, f_k\}$ of sections of $\mathcal{A}_U$ and the ideal $\mathcal{J}$ they define in $\mathcal{A}_U$. The reduction modulo nilpotents defines a complex analytic space patch V in the sense of §2.3, so that one defines the complex analytic superspace patch (defined by the f_i's) as the ringed space

$$\hat{V} = (V, \mathcal{A}_U / \mathcal{J})$$

Let $\hat{V}$ and $\hat{W}$ two complex analytic space patches, both subsuperspaces of $\mathbb{C}^{p|q}$. They are called equivalent at $x \in \mathbb{C}^p$ iff there is a neighbourhood U of x such that $\mathcal{A}_V \restriction_{U \cap V} \tilde{\rightarrow} \mathcal{A}_W \restriction_{U \cap W}$ are isomorphic.

Definition . A *germ* of complex superspace at x is an equivalence class of complex superspaces.

Morphisms between such objects are defined by taking representatives and morphisms between them and requiring the correspondent equivalence condition.

Definition . Let $(\hat{S}, s)$ be a germ of complex superspace at s. A deformation $(\hat{\mathcal{X}}, \hat{S})$ of a complex superspace $(X, \mathcal{A}_X)$ over $(\hat{S}, s)$ is a commutative diagram

$$\begin{array}{ccc} \hat{X} & \xrightarrow{i} & \hat{\mathcal{X}} \\ \downarrow & & \downarrow \pi \\ \{s\} & \longrightarrow & \hat{S} \end{array}$$

where $\pi : \hat{\mathcal{X}} \to \hat{S}$ is a flat complex superspace morphism and i is a fixed isomorphism between $\hat{X}$ and the central fiber $\pi^{-1}(s)$.

Remark . Flatness is a technical condition replacing the requirement of smoothness.

A morphism

$$(\hat{\mathcal{X}}, \hat{S}) \longrightarrow (\hat{\mathcal{Y}}, \hat{T}')$$

is a pair of complex superspace and germ morphisms $\mathcal{X} \xrightarrow{\Phi} \mathcal{Y}$ and $S \xrightarrow{f} T$ such that the following diagram

$$\begin{array}{ccccc}
\mathcal{X} & & \xrightarrow{\Phi} & & \mathcal{Y} \\
 & {}^{i}\nwarrow & & {}^{j}\nearrow & \\
\downarrow \pi & & X & & \downarrow \sigma \\
S & & \downarrow \xrightarrow{f} & & T \\
 & \nwarrow & & \nearrow & \\
 & & \{*\} & &
\end{array}$$

is commutative.

Given a deformation of a complex superspace $(\mathcal{X}, (S, s))$ and a germ morphism

$$(T, t) \xrightarrow{f} (S, s)$$

one defines the pullback deformation $f^*(\mathcal{X})$ over (T, t) as the fibered product $\mathcal{X} \times_f S$, meaning that, as topological spaces,

$$F^*(\mathcal{X}) = \{(x, t) \in \mathcal{X} \times T \,|\, f(t) = \pi(x)\}$$

and, as for the structure sheaf,

$$\mathcal{A}_{f^*\mathcal{X}} = \mathcal{A}_T \hat{\otimes} \mathcal{A}_{\mathcal{X}} / \left((id_T \times \pi)^* \mathcal{J}\right)$$

where $\mathcal{J}$ is the ideal defining the graph of f.

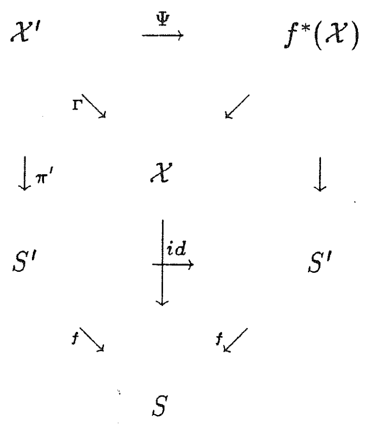
We denote by $Def(X, S)$ the set of isomorphism classes of deformations of X over S.

The pull-back deformation comes equipped with two morphisms p_1, p_2, mapping onto the first and second component of each pair (x, t), which make the diagram

$$\begin{array}{ccc}
f^*\mathcal{X} & \xrightarrow{p_1} & \mathcal{X} \\
\downarrow p_2 & & \downarrow \pi \\
T & \xrightarrow{f} & S
\end{array}$$

commutative.

Recall that the pull-back deformation has the following property: for any deformation $\pi' : \mathcal{X}' \to B'$ and any morphisms of deformations $(\Gamma, f) : \mathcal{X}' \to \mathcal{X}$ there exists a unique morphism $(\Psi, h) : \mathcal{X}' \to f^*\mathcal{X}$ such that the diagram



commutes.

Given a complex superspace $(X, \mathcal{A}_X)$, the pull-back makes $Def(X, .) : \mathcal{CS} \to Ens$ into a contravariant functor from the category $\mathcal{CS}$ of (germs of pointed) complex superspaces to the category Ens of sets, which assigns to each $B \in \mathcal{CS}$ the set $Def(X, B)$ of isomorphism classes of deformations of X over B.

As usual in deformation theory, we say that a deformation $\mathcal{X} \in Def(X, B)$ is
i) complete, if for any other deformation $\mathcal{X}' \in Def(X, B')$ there exists a morphism

$$f : B' \to B$$

such that $\mathcal{X}'$ is isomorphic to the pull-back deformation $f^*\mathcal{X}$,
ii) universal if such an f is unique or versal if all the morphisms f satisfying condition i) have the same differential.

The same definitions can be repeated when B is a purely even superspace. In this case one speaks of even completeness, even versality etc.. Notice that a purely even superspace is the same thing as a (non reduced) complex space.

Having introduced the moduli functor for complex superspaces, one can argue about moduli (super)spaces, defined as (see §2.5) spaces M realizing an isomorphism of functors $Def_X(\cdot, M) \simeq Hom(\cdot, M)$. The search for a fine moduli space in this category is almost hopeless. Namely, $\mathbb{Z}_2$-graded commutative algebras always admit the canonical involution

$$\alpha(x) = (-1)^{\tilde{x}} x$$

which play the rôle of non trivial automorphisms in the case of moduli spaces of Riemann surfaces . Nonetheless, what one is really interested in is the existence of a coarse moduli space for superspaces of a certain "topological" class. Solving this problem is tantamount to solving the problem of existence of versal deformations for $(X, \mathcal{A}_X)$. That they exist has been recently proven by Vaintrob [V] considering first evenversal deformations and then considering their extensions to "odd" directions.

What is outstandingly relevant is the extension to the super case of the Kodaira-Spencer formalism, and, namely the definition of the super Kodaira-Spencer map, $\widehat{KS}$. Mimicking what happens in standard deformation theory, one first studies infinitesimal deformations. To this purpose, one introduces the super-commutative ring of super-dual numbers $O_S = \mathrm{C}[t,\zeta]/(t^2, t\zeta)$, where $(t,\zeta) \in \mathrm{C}^{1,1}$, $\mathrm{C}[t,\zeta]$ is the polynomial ring and $(t^2, t\zeta)$ is the ideal generated by t^2 and $t\zeta$. Associated to this ring there is a superspace $S = (\{*\}, O_S)$, which embodies the idea of a super-tangent vector.

Definition . Let $(X, \mathcal{A}_X)$ be a complex superspace. A deformation of $(X, \mathcal{A}_X)$ over S will be called an infinitesimal deformation.

Given a complex superspace (B, b_0), the tangent space $T_{b_0}B$ at b_0 is isomorphic to the linear superspace $Mor(S, B) = \{f : S \to B \mid f(*) = b_0\}$ of superspace morphisms. Now, given a deformation $\mathcal{X} \to B$ of X, we can think of a tangent vector in $T_{b0}B$ as a map $f \in Mor(S, B)$ and the pull-back deformation $f^*\mathcal{X} \to S$ is a first order deformation of C. The Kodaira-Spencer class of is obtained by considering the exact sheaf sequence

$$0 \to f^*\mathcal{TX}_/ \longrightarrow f^*\mathcal{TX} \longrightarrow f^*(Der\hat{B}) \to 0$$

where $\mathcal{TX}_/$ is the relative tangent sheaf. Taking the coboundary map one has

$$\widehat{KS}_f : H^0(f^*(Der\hat{B})) \equiv A \in T_{b_0}B \to H^1(f^*\mathcal{TX}_/) \equiv T\hat{X}$$

and then, letting f vary one gets the $\widehat{KS}$ homomorphism

$$\widehat{KS} : T_{b_0}B \to H^1(X, \hat{T}X)$$

The fundamental theorem of the Kodaira-Spencer theory has the following graded-commu tative version [V]:

Theorem . A deformation of $(X, \mathcal{A}_X)$ such that $\widehat{KS}$ is surjective (an isomorphism) is complete (resp. versal).

Example . The above construction allows one to compute the dimension of the moduli space for $1|1$-dimensional compact supermanifolds $(C, \mathcal{A}_C)$. In fact such objects will be completely specified by the datum of

i) a smooth algebraic curve C

ii) an invertible sheaf $\mathcal{L}$ over C. Their tangent space is then (see §3.2)

$$\hat{T}C = (\mathcal{O} \oplus \Pi\mathcal{L}) \otimes (\mathcal{K}^{-1} \oplus \Pi\mathcal{L}^{-1})$$

and thus

$$H^1(\hat{T}C) \simeq H^1(\mathcal{K}^{-1} \oplus \mathcal{O}) \oplus \Pi H^1(\mathcal{L}^{-1} \oplus \mathcal{L}\mathcal{K}^{-1})$$

The dimension can be then computed by means of ordinary algebro-geometrical techniques (i.e. the Riemann-Roch theorem, provided $deg\mathcal{L}$ is fit). To make contact with some results recently appeared in the literature[GN], let us compute the dimension of the universal deformation $\hat{\mathcal{M}}$ space of such objects in the case in which $\mathcal{L}$ is a θ-characteristics . Then the equation above gives

$$dim\hat{\mathcal{M}} = h^0(\mathcal{K}) + h^0(\mathcal{O}) \mid 2h^0(\mathcal{KL}) = 4g - 3|4g - 4$$

Turning at this point to Super Riemann surfaces we have to notice that built into their definition was the existence of a non-integrable 0|1 dimensional distribution $\mathcal{D}$ in the tangent sheaf. Then one is naturally lead to the following

Definition -. A deformation of a SUSY-curve $\hat{C}$ over a germ of a complex superspace (B, b_0) at $b_0 \in B$ is a quadruple $(\mathcal{X}, \pi, D_\pi, i)$, where

i) $\pi : X \longrightarrow B$ is a proper flat morphism of complex superspaces,

ii) D_π is a subsheaf of the relative tangent sheaf T_π to X, such that

$$[\ ,\ ] : D_\pi \otimes D_\pi \longrightarrow T_\pi / D_\pi$$

is an isomorphism

iii) i is an isomorphism between C and the special fibre $\pi^{-1}(b_0)$.

By a first order (or infinitesimal) deformation of a SUSY-curve C we mean a deformation $\mathcal{X}$ of C over S. The extra datum of the odd distribution has notable consequences, which can be summarized in the following apparent paradox:

Supermoduli space is not the moduli space of Super Riemann surfaces !

The trick off the hook relies in a careful analysis of the meaning of Kodaira-Spencer deformation theory. In very general terms, the heart of Kodaira-Spencer deformation theory lies in the individuation of the sheaf of infinitesimal automorphisms S of the "structure" being deformed. This fundamental object, together with its subsheaf $\mathcal{S}_/$ of "vertical" automorphisms will fit the Kodaira-Spencer sequence

$$0 \rightarrow \mathcal{S}_/ \longrightarrow \mathcal{S} \xrightarrow{\pi} \pi^*\mathcal{T}B \rightarrow 0$$

from which the Kodaira-Spencer map is defined to be the first coboundary map. Now, for Super Riemann surfaces, one can identify [LB-R] these sheaves as

$$\mathcal{S} := \{Y \in \hat{D}er\mathcal{A}_\mathcal{X} \equiv \hat{\mathcal{T}\mathcal{X}} \mid [X, \mathcal{D}] \subset \mathcal{D}\}$$

$$\mathcal{S}_/ = \mathcal{S} \cap Der_/\mathcal{X}$$

which are strictly contained in the sheaves $\mathcal{TX}$.

To classify infinitesimal deformations we prefer to report an explicit computation in the spirit of the "original" Kodaira-Spencer approach. Namely we regard Super Riemann surfaces as built by patching together 1|1-dimensional superdomains by means of superconformal transformation, singling out moduli as non-trivial parameters in the transition functions. Namely, consider a canonical atlas $\{U_\alpha, z_\alpha, \theta_\alpha,\}$ with clutching functions

$$\begin{cases} z_\alpha = f_{\alpha\beta}(z_\beta) \\ \theta_\alpha = \pm\sqrt{f'_{\alpha\beta}}\theta_\beta \end{cases}$$

where, here and in the sequel $f'_{\alpha\beta}$means $\frac{\partial f_{\alpha\beta}}{\partial z_\beta}$ on $U_\alpha \cap U_\beta$. They obviously satisfy the cocycle condition $f_{\alpha\beta}(f_{\beta\gamma}(z_\gamma)) = f_{\alpha\gamma}(z_\gamma)$ on $U_\alpha \cap U_\beta \cap U_\gamma$.

We can cover a first order deformation $\pi : X \to S$ of C glueing the $U_\alpha \times S$ via the identification

$$z_\alpha = f_{\alpha\beta}(z_\beta) + tb_{\alpha\beta}(z_\beta) + \zeta\theta_\beta g_{\alpha\beta}(z_\beta)F_{\alpha\beta}(z_\beta)$$

$$\theta_\alpha = F_{\alpha\beta}\theta_\beta + \zeta g_{\alpha\beta}$$

where $F_{\alpha\beta} = \sqrt{f'_{\alpha\beta}}(1 + tb_{\alpha\beta}/2)$, so that the clutching functions are superconformal for any t, ζ. The cocycle condition for these transformation rules reduce to the cocycle condition for the $f_{\alpha\beta}$'s as before, plus

$$b_{\alpha\beta} + f'_{\alpha\beta}\, b_{\beta\gamma} = b_{\alpha\gamma}$$

and

$$g_{\alpha\beta}\theta_\alpha + f'_{\alpha\beta}\, g_{\beta\gamma}\theta_\beta = g_{\alpha\gamma}\theta_\gamma$$

Taking the tensor product by $\partial/\partial z_\alpha$, one sees that the one cochains

$$v^0_{\alpha\beta} = \{b_{\alpha\beta}\partial/\partial z_\alpha\}$$

$$v^1_{\alpha\beta} = \{g_{\alpha\beta}\theta_\alpha \otimes \partial/\partial z_\alpha\}$$

are actually cocycles. They define a class in $H^1(C_{red}, \omega^{-1}) \oplus \Pi H^1(C_{red}, L^{-1}) \subset H^1(C, \hat{T}C)$, called the Kodaira- Spencer class of the first order deformation $\mathcal{X} \xrightarrow{\pi} S$. Here we obviously assume that C is smooth. Deformation theory of SUSY-curves with nodes requires the handling of θ-characteristics in the singular case (see §2.7).

A similar computation, considering local superconformal reparametrizations with local odd parameters λ_α, shows that they leave the cocycle v_0 invariant and send v_1 into

$$\tilde{v}^1_{\alpha\beta} = v^1_{\alpha\beta} + (\lambda_\alpha - \lambda_\beta)\theta_\alpha \frac{\partial}{\partial z_\alpha}$$

which is enough for us to conclude

Proposition . The set of equivalence classes of first order deformations of a SUSY-curve C is a linear complex superspace with dimension $3g - 3|2g - 2$.

Proof. It is enough to compute the dimensions of $H^1(C_{red}, \omega^{-1})$ and $H^1(C_{red}, L^{-1})$ by means of Riemann-Roch theorem.

We can interpret this result as saying that there are 3g-3 linearly independent variations of a θ-characteristic and that, on the corresponding versal first order deformation, there are

2g-2 independent variations of the supermanifold structure which keep the property of being a deformation of a SUSY-curve.

Notice that this coordinate results fits perfectly in the general scheme of Kodaira-Spencer theory as it was outlined above. Namely, the sheaf $\mathcal{S}_/$ of infinitesimal automorphisms of a super Riemann surface is actually isomorphic to a subsheaf of the tangent sheaf of the central fibre. In fact, it holds the following

Lemma [LB-R] .$\mathcal{S}_/ \simeq \mathcal{D}^2$

Proof. Consider canonical relative coordinates, in which $D = \frac{\partial}{\partial \eta} + \eta \frac{\partial}{\partial w}$. Then, writing $Y \in \mathcal{S}_/$ as $Y = a\frac{\partial}{\partial w} + bD$

$$[D, Y] \sim D \Leftrightarrow b = (-1)^{\tilde{Y}} Da$$

Then, as D^2 is the local generator for the tangent space to C_{red},

$$H^1(C, \mathcal{S}_/) = H^1(C, \mathcal{A}_C \otimes T_C) = H^1(\mathcal{K}^{-1}) \oplus \Pi H^1(C, \mathcal{L} \otimes \mathcal{K}^{-1})$$

With a slight abuse of language we will set

$$H^1(C, \mathcal{S}_/) \equiv H^1(C, T^0 C)$$

Since C is split, $H^1(C, T^0C)$ naturally splits into even and odd subspaces and we can speak about even and odd Kodaira-Spencer homomorphisms KS_0 and KS_1, by composing $\widehat{KS}$ with the projections of $H^1(C, T^0C) = H^1(C_{red}, \omega^{-1}) \oplus \Pi H^1(C_{red}, L^{-1})$ onto the first and second summand. It follows that, if B is a purely even superspace (i.e. an ordinary complex space, see [V]), $KS_1 = 0$ and $KS_0; T_{b0}B \to H^1(C_{red}, \omega^{-1})$ is the ordinary Kodaira-Spencer map. As we need the datum of a θ-characterisitics on C_{red}, the deformation $X \to B$ has to be be considered as a deformation of a θ-characteristics, whenever B is purely even. We have therefore the following

Proposition -. Even-versal deformations of a SUSY-curve exist and are in 1-1 correspondence with versal deformations of the underlying θ-characteristic.

This proposition suggests that the compactified underlying space of moduli of SUSY-curves can be given in terms of isomorphism classes of pairs $(C_{red}, \mathcal{L})$, where $\mathcal{L}$ is a spin structure in the sense of §2.7, by defining the graded structure sheaf to be $\mathcal{A} = \mathcal{O} \oplus \Pi\mathcal{L}$. Obviously, the odd part of supermoduli space still lacks. In fact, if one should insist in dealing with purely even objects, and so considers the transition functions of the *would-be* universal family to be

$$\begin{cases} z_\alpha = f_{\alpha\beta}(z_\beta, t) \\ \theta_\alpha = \sqrt{f'_{\alpha\beta}(z_\beta, t)}\theta_\beta \end{cases}$$

then (super-)Kodaira-Spencer map fails to be an isomorphism.

A very appealing hint for the complete solution of the problem is given by noticing that, in physical applications, world-sheet supersymmetry requires the presence of a gravitino field on a given spin curve. This can possibly vanish, giving us back the (special) SUSY-curves of §3.4. If not, one can fix local superconformal gauges [Ho], which amount to choosing local complex coordinates, a local holomorphic trivialization of L and to identifying a chiral piece of the gravitino field with a section χ of $A^{0,1}(C_{red}, L^{-1})$, i.e. with a smooth antiholomorphic one form with values in $\mathcal{L}^{-1}$. Notice that obviously $\overline{\partial}\chi = 0$. As is well known, there is a large 'symmetry group' which survives the superconformal gauge fixing. Indeed, Dolbeault's lemma tells us that χ is locally exact and one can set $\chi\restriction_{U\alpha} = \overline{\partial}\epsilon_\alpha$ for some $\epsilon_\alpha \in A^{0,0}(U_\alpha, L^{-1})$. Clearly, χ can be gauged away by means of a local SUSY-transformation generated by $-\epsilon_\alpha$ on U_α. The obstruction to gauge away χ globally is given by (the cohomology class defined by) the one-cocycle with values in $\mathcal{L}^{-1}$

$$\epsilon_{\alpha\beta} = \epsilon_\alpha - \epsilon_\beta$$

This is holomorphic, because χ was globally defined and hence $\overline{\partial}\epsilon_{\alpha\beta} = \chi\restriction_{U_\alpha} - \chi\restriction_{U_\beta} = 0$. Notice that the action of supersymmetry has no effect on the $\epsilon_{\alpha\beta}$, while we have a local symmetry generated by holomorphic sections η_α of L^{-1}acting via Čech coboundaries, i.e. as $\epsilon_{\alpha\beta} \to \epsilon_{\alpha\beta} + \eta_\alpha - \eta_\beta$. In other words, we can benefit of the isomorphism $H^{0,1}_{\overline{\partial}}(C_{red}, L^{-1}) = \check{H}^1(C_{red}, \mathcal{L}^{-1})$ to represent gravitino fields χ (up to supersymmetry) via Čech cocycles $\epsilon_{\alpha\beta}$ (up to coboundaries).

Then, by a (family of) Super Riemann surface(s) one can mean the data of a SUSY-curve $\hat{C} = (C_{red}, \wedge^*(\mathcal{L}))$ plus a fixed cohomology class $[\chi] \in H^{0,1}_{\overline{\partial}}(C_{red}, L^{-1})$. Special susy curves of §3.4 can be considered simply as being given by $[\chi] = 0$. The datum of $[\chi]$ can be encoded in an extension of the structure sheaf of C as follows. Forgetting about parity, we notice that the extensions $\mathcal{F}$ of $\mathcal{L}$ by $\mathcal{O}$,

$$0 \to \mathcal{O} \to \mathcal{F} \to \mathcal{L} \to 0$$

are precisely parameterized by $\mathrm{Ext}^1(\mathcal{L}, O) = H^1(C_{red}, \mathcal{L}^{-1})$. In other words, $\mathcal{F}$ is the sheaf of sections of a rank 2 vector bundle locally generated by $\theta_\alpha, \zeta_\alpha$ with transition functions

$$\begin{pmatrix} \theta_\alpha \\ \zeta_\alpha \end{pmatrix} = \begin{pmatrix} \sqrt{f_{\alpha\beta}} & \epsilon_{\alpha\beta} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \theta_\beta \\ \zeta_\beta \end{pmatrix}$$

The supermanifold $\tilde{C}_{[\chi]} = (C_{red}, \wedge^*(\mathcal{F}))$ is not yet a Super Riemann surface , but

Lemma . There is a unique (up to isomorphism) deformation $\mathcal{A}_{[\chi]}$ of the structure sheaf $\wedge^*(\mathcal{F})$ of $\tilde{C}_{[\chi]} = (C_{red}, \wedge^*(\mathcal{F}))$ such that

$$(C_{red}, \mathcal{A}_{[\chi]})$$

is a Super Riemann surface .

Proof . It is enough to find a superconformal coordinate patching

$$\begin{cases} z_\alpha = z_\alpha(z_\beta, \theta_\beta, \zeta) \\ \theta_\alpha = \theta_\alpha(z_\beta, \theta_\beta, \zeta) \end{cases}$$

which reproduces the transition functions above for $\mathcal{F}$ and to $z_\alpha = f_{\alpha\beta}(z_\beta) \bmod \mathcal{N}^2$. If $\epsilon_{\alpha\beta}$ is a representative of $[\chi]$, then the answer is uniquely given by

$$\begin{cases} z_\alpha = f_{\alpha\beta}(z_\beta) + \theta_\beta f'_{\alpha\beta}(z_\beta)\epsilon_{\alpha\beta}(z_\beta)\zeta \\ \theta_\alpha = \sqrt{f'_{\alpha\beta}(z_\beta)} \cdot (\theta_\beta + \epsilon_{\alpha\beta}(z_\beta)\zeta) \end{cases}$$

Notice that the family constructed above is parameterized by the odd nilpotent parameter ζ, and can be considered as the simplest example of a family of Super Riemann surfaces depending non-trivially on odd parameters.
When considering an arbitrary deformation of a SUSY-curve, things get more involved, since it is not difficult to prove that requiring that, for any fixed $(t,\zeta) \in U^{p|q}$ one has superconformal transition functions

$$\begin{cases} z_\alpha = z_\alpha(z_\beta, \theta_\beta; t, \zeta) \\ \theta_\alpha = \theta_\alpha(z_\beta, \theta_\beta; t, \zeta) \end{cases}$$

implies that these assume the form:

$$\begin{cases} z_\alpha = f_{\alpha\beta}(z_\beta;t,\zeta) + \theta_\beta f'_{\alpha\beta}(z_\beta;t,\zeta)\epsilon_{\alpha\beta}(z_\beta;t,\zeta)\zeta \\ \theta_\alpha = \sqrt{f'_{\alpha\beta}(z_\beta;t,\zeta)} \cdot \left(\theta_\beta + \epsilon_{\alpha\beta}(z_\beta;t,\zeta)\zeta + \frac{1}{2}\theta_\beta\epsilon_{\alpha\beta}(z_\beta;t,\zeta)\zeta\epsilon'_{\alpha\beta}(z_\beta;t,\zeta)\zeta\right) \end{cases} \qquad (*)$$

Notice that no "$\epsilon\epsilon'$"-terms entered the family of the above lemma since in that case ζ was a *single* odd parameter and hence its square was vanishing.
A direct "Kodaira-Spencer -like" analysis of the latter transition functions is overwhelmingly complicated by the fact that an "engeneering" computation of the cocycle conditions shows a highly non trivial interplay between the higher ζ-degree terms of $f_{\alpha\beta}$ and the other parameters entering $(*)$. It is then clear that a promising insight into the geometry of supermoduli space can only be gotten by choosing a very special class of versal deformation. The hint comes from a careful analysis of the non trivial odd deformation of the lemma above, and from the observation that, looking at the cocycle condition for $(*)$, one has that the explicit dependence of the transition functions on the odd parameters can be consistently taken to be linear as long as one can establish a vanishing condition for the product of two basis vectors in $R^1\pi_*(\mathcal{L}^{-1})$.

We star next to construct a very convenient class of family of Super Riemann surfaces [FMRT], which, as will be clear in the sequel, deserve the name of *Super Schiffer* variations, and have the remarkable property of being linearly dependent on modular parameters (see also [Bers]). First we give a concrete example of how one can construct the SUSY-curve $(C_{red}, \mathcal{A}_{[\chi]})$ of the previous lemma.

Example . Let $(C_{red}, \wedge_*(\mathcal{L}))$ be a SUSY-curve over a point. Given a generic point $p \in C_{red}$, we consider the cohomology sequence

$$0 \to H^0(C,\mathcal{L}^{-1}) \to H^0(C,\mathcal{L}^{-1}(p)) \to \mathbb{C} \to H^1(C,\mathcal{L}^{-1}) \to H^1(C,\mathcal{L}^{-1}(p)) \to 0$$

associated to the sheaf short exact sequence

$$0 \to \mathcal{L}^{-1} \to \mathcal{L}^{-1}(p) \to \mathcal{L}^{-1}(p)/\mathcal{L}^{-1} \to 0$$

As $\mathcal{L}^{-1}(p)/\mathcal{L}^{-1}$ is the skyscraper sheaf with group $\mathbb{C}$ at p, S_p, it follows that

$$H^0(C,\mathcal{L}^{-1}(p)/\mathcal{L}^{-1}) = \mathbb{C}.$$

Since $deg\ \mathcal{L}^{-1}(p) = 1 - g + 1 = 2 - g$ is negative for $g \geq 3$, $H^0(C,\mathcal{L}^{-1}(p)) = \{0\}$ and then the map

$$0 \to \mathbb{C} \xrightarrow{\delta_p} H^1(C,\mathcal{L}^{-1}) \to \cdots$$

is injective. (The same is true for even θ-characteristics at genus 2. Odd ones deserve a special treatment as $\mathcal{L}^{-1}(p)$ can be $\mathcal{O}$ and in that case $H^0(C, L^{-1}(p)) = \mathbb{C}$.)
This means that (up to multiplicative constants) there is a unique element of $H^1(C, L^{-1})$, given the image of $1 \in \mathbb{C}$ under the map δ_p. Now, if $U \subset C_{red}$ is a small neighbourhood of p and $V = C_{red} \setminus \{p\}$, the cocycle representing $\delta_p(1)$ w.r.t. the covering $\{U, V\}$ is precisely given by

$$\frac{1}{z} \cdot \theta \otimes \frac{\partial}{\partial z}$$

where θ is a local generator of $\mathcal{L}^{-1} \restriction_U$. We get in this way a particular infinitesimal deformation, which can be easily integrated by glueing a $1|1$ super disk with coordinates (z' , θ') with the $1|1$ superdisk with coordinates $(z , \theta)$with the transition functions

$$\begin{cases} z' = z + \frac{\theta\zeta}{z} \\ \theta' = \theta + \frac{\zeta}{z} \end{cases}$$

with $\zeta^2 = 0$ It follows that

$$KS_1(\frac{\partial}{\partial \zeta}) = [\frac{1}{z} \cdot \theta \otimes \frac{\partial}{\partial z}]$$

as it should.
Clearly, the SUSY-curve given by these transition functions represents $(C_{red}, \mathcal{A}_{[\chi]})$ for some $[\chi]$.To show that each $(C_{red}, \mathcal{A}_{[\chi]})$ can be gotten in this way, we let p move around the central fiber, getting a map

$$C_{red} \xrightarrow{\delta} \mathbb{P}H^1(C_{red}, L^{-1})$$

given by

$$p \mapsto [\frac{1}{z} \cdot \theta \otimes \frac{\partial}{\partial z}]_p$$

up to multiplicative constants. It is enough now to prove that the map is full, i.e. it is not contained in any hyperplane in $\mathbb{P}H^1(C_{red}, L^{-1})$. If this was not so, there would be an element $\varphi \in \mathbb{P}H^0(C_{red}, L^3)$ of the dual projective space such that

$$\langle \varphi, \delta(p) \rangle = 0 \text{ for all } p \in C_{red}$$

Here $\langle \cdot, \cdot \rangle$ is Serre duality, i.e.

$$\langle \varphi, \delta(p) \rangle = Res_p \varphi(z) \otimes \frac{1}{z} \cdot \theta \frac{\partial}{\partial z} = \varphi(p)$$

and therefore $\varphi \equiv 0$, an *absurdum*.

Summing up, given $[\chi] \in H^1(C_{red}, L^{-1})$ there exist $2g-2$ points $p_\mu \in C_{red}$ s.t.

$$\lambda([\chi]) = \sum_{\mu=1}^{2g-2} \delta_{p_\mu}([\chi]) \cdot a^\mu$$

We can now iterate $(2g-2)$-times the glueing procedure described above by glueing other superdisks with coordinates $(z'_\mu, \theta'\mu)$ with the old ones (centered in p_μ) with coordinates $(z_\mu, \theta\mu)$ by the transition functions

$$z'_\mu = z_\mu + \frac{\theta_\mu \cdot \zeta_\mu}{z_\mu}$$

$$\theta'_\mu = \theta_\mu + \frac{\zeta_\mu}{z_\mu}$$

This yields

$$\widehat{KS}_1(\frac{\partial}{\partial \zeta_\mu}) = [\frac{1}{z_\mu} \cdot \theta_\mu \otimes \frac{\partial}{\partial z_\mu}]$$

and, by the argument above, $\widehat{KS}_1$ is an isomorphism.

To get a versal deformation of the SUSY-curve $\hat{C}$ one can then proceed as follows. Fix a set of $\{(3g-3)+(2g-2)\}$ points p_i, p_μ on C_{red}. and consider the open covering

$$\{U_0 \equiv (C \setminus \{p_i, p_\mu\}) \cup \{D_i\} \cup \{D_\mu\}\}$$

where D_i (resp. D_μ) is a small disk centered at p_i (resp. p_μ)with $D_i \cap D_j = \emptyset$ if $i \neq j$ and so on. Then composing an ordinary Schiffer variation based on the points p_i with the odd-versal deformation described above one gets a deformation

$$\hat{C} \to \hat{D}^{3g-3|2g-2}$$

with basis a $3g-3|2g-2$ super polydisk whose Super Kodaira-Spencer map is an isomorphism $\mathbb{C}^{3g-3|2g-2} \xrightarrow{\sim} H^1(C, \hat{C}_/)$.

What is really relevant for us is the fact that a bais for the odd modular parameters is here realized as the set of cocycles $\epsilon_{0,\mu}$ having support in $\overset{\circ}{D}_\mu \equiv D_\mu \cap U_0$, so that

$$\epsilon_{0,\mu} \cdot \epsilon_{0,\nu} = 0 \text{ if } \mu \neq \nu$$

As a final remark we would like to add some comments on the issue of the splitness (or non splitness) of supermoduli space. At present, only very preliminary results are available. Namely, one has the two following propositions.

Proposition 1. Supermoduli space splits in genus $g = 1$.
Proof. On families of elliptic curves one either has no holomorphic 3/2-differentials (in the case of even θ-characteristics) or, when $\mathcal{L}_{/} \simeq \mathcal{O}_{C_t}$, only one, so that splitness is insured by dimensional reasons.

Proposition 2. Supermoduli space (for *smooth* curves) splits in genus $g = 2$.
Proof. Here, no matter which is the parity of the θ-characteristics , we have that the local model for supermoduli $\hat{S}_g$ is the rank two sheaf $\mathcal{E} \equiv R^1\pi_*(\mathcal{L}^{-1})$ over the moduli space of genus 2 spin curves S_2. Then (see §3.2) its obstruction to splitness is measured by a single class $\tau_2 \in H^1(S_2, \mathcal{T}S_2 \otimes \wedge^2(\mathcal{E}))$. Here we can argue as follows. The natural map

$$S_2 \xrightarrow{p} M_2$$

obtained by forgetting spin structures is finite. But, because of the existence of a peculiar relation between the canonical sheaf K_{S_2} and the Deligne-Mumford boundary classes δ_0 and δ_1, it can be proven [M3] that, actually, M_2 is an affine variety and hence is Stein. As being a Stein space is a property which is preserved in both senses under finite maps, then also S_2 is Stein and so, as $\mathcal{T}S_2 \otimes \wedge^2(\mathcal{E})$ is coherent analytic τ_2 vanishes.

As it is clear from the proofs above, the two results above, being essentially based on dimensional properties, cannot be generalized as they stand to the higher genus case. More specifically, the question of splitness of supermoduli space deserves a careful analysis of the procedure by which one glues together its local building blocks, i.e. the bases of versal deformations. This is not a harmless question, as it has been clearly pointed out in [LB-R]. In fact, although it is not so difficult to produce a very general versal deformation for a SUSY-curve, once a certain set of parameters - i.e. essentially, a basis for $H^1(C, \mathcal{L}^{-1})$ - is fixed, its form is going to depend in a highly non trivial way on this choice, so that, when discussing the glueing procedure, such "ambiguities" must be taken into account. Work is in progress in the study of the glueing properties of *SuperSchiffer* deformations involving the geometrical properties of local sections of the n^{th}-fold symmetric product of the (local) universal θ-characteristics , in order to give at least a more tractable form to the splitting cocycle for supermoduli space.

3.6 Conclusions and overlook

As a conclusion, we want to spend some words in a discussion on the perspectives of the approach to Superstring theory which has been pursued in this thesis work. As a matter of fact, due to the still incomplete understanding of both the structure of the ultimate integration domain (i.e. supermoduli space) and the reduction procedure of the superstring amplitude from a Feynman integral over the space of field configurations to an "ordinary" integral over supermoduli space, there are still some subtle points that deserve special attention. Nonetheless, we feel appropriate to give a taste of some possible applications of the machinery we described up to now, possibly skipping those slippery questions, i.e. assuming that everything works in the right way.

The superstring partition function, after Wess-Zumino gauge fixing and "fermionic integration" over the odd supermoduli is given by [VV,AMS]

$$Z = \int_{\overline{S}_g} d\mu(m^i) \wedge d\mu(\overline{m}^i)\mathcal{L}(\overline{m})\mathcal{R}(m) \tag{1}$$

where the "half string measure" $\mathcal{L}(\mathcal{R})$ is associated to the left (right) mover sector of the fermionic closed string. For the type II model, one has that $\mathcal{L} = \overline{\mathcal{R}}$. Let us concentrate only on a chiral part, say the right moving sector of (1).

As sketched in §3.1, in the case of a target flat space time $\mathbb{R}^{10}$, the half measure reads

$$[d\mu\mathcal{R}_\alpha](m_i) = d\mu(m_i)\frac{det\overline{\partial}_{-1}}{det\langle\psi^i|\psi^j\rangle} \cdot \Big(\frac{det_\alpha(\overline{\partial}_{\frac{1}{2}})}{det\langle\nu^A|\nu^B\rangle}\Big)^{-1} \cdot$$

$$\cdot(det_\alpha(\overline{\partial}_{\frac{1}{2}})^5 \cdot W_{Mat} \cdot \langle S_1 \cdots S_{2g-2}\rangle_\alpha \tag{2}$$

and

$$S_{A,\alpha} = \int_C (\chi_A T_A^{(F)})_\alpha$$

$$T_A^{(F)} = \Psi_A^\mu \cdot \partial_z X^\mu + (ghost\ contributions) \tag{3}$$

where $T_A^{(F)}$ is the world-sheet supercurrent, χ_A are the gravitino zero-modes and W_{Mat} is defined through

$$W_{Mat} \wedge \overline{W}_{Mat} = \Big(\frac{det(-\triangle_0)}{\int_C \sqrt{g} det Im(\Omega)}\Big)^{-5} \tag{4}$$

In Eq.(2), ψ^i (resp. ν^A) are a basis for the holomorphic $j = 2$ ($j = \frac{3}{2}$) differentials on the curve C. The finite dimensional determinants $det\langle\psi^i|\psi^j\rangle$ and $det\langle\nu^A|\nu^B\rangle$ essentially

represent the "volume of the zero-modes [D'H-P]" of the reparametrization and SUSY ghosts and the suffix α denotes the choice of a spin structure.
Finally, $\prod_{A=1}^{2g-2}\langle S_A\rangle_{z_A-fixed}$ is the correlator of the so-called changing picture operator $S_A(z_A)$ defined in Eq. (3).

The usual procedure in the so-called "picture changing formalism" [VV] is to show that, in the overlapping of two different patches in moduli space, a superconformal transformation of the gravitino field $\chi_A \to \chi_A + \overline{\partial}\epsilon$ modifies the local expression of the integrand by an amount of the form $\sum_i \partial_i N^i$. Here the N^i's are defined in each overlap and are generically singular when the overlaps are "near" the boundary of S_g. In our context supersymmetry transformations on the gravitino may give rise to non-split supercoordinate transformations $(m_i \to f(m_i) + o(\zeta^2))$ on supermoduli space, and, as we discussed in §3.3, the appearance of total derivative terms is essentially due to a too *naive* Berezin integration prescription which is inconsistent on a non-compact supermanifold. Note that the fact that we obtain a nice compactification of S_g does not avoid this problem, since we do not know *a priori* anything about the regularity of the integrand along the boundary ∂S_g, and, generically speaking, physical considerations require it to be divergent there. Let us stress that [AMS] those boundary terms are extremely dangerous, as they deprive [MM] any explicit computation of n-point correlation functions of global physical meaning. In particular, even if one is so clever in finding convenient parametrizations showing *e.g.* the local vanishing of the measure for the string partition function [GIS], the presence of ambiguities of the form $\partial_i N^i$ can spoil the result.

On the contrary, the careful handling of the transformation law for the "Berezin form" as given by Rothstein [R2] gets rid of total derivative ambiguities [MT]. Then, provided that integration on supermoduli space is done *a la* Rothstein, the local expression for the integrand in Z will be of the form (1), no matter how one chooses the gravitino slice [MT,D'H-P]. Now, as we saw in §3.3, the lacking of naturality in the splitting of Rothstein's sequence

$$0 \to (\Omega^{p,0} \otimes D)^+ \hookrightarrow \Omega^{p,0} \otimes D \overset{i}{\leftrightarrows} Ber \to 0$$

can give a contribution looking locally like $\partial_i q^i$, where now, q is a global holomorphic form. Namely, this term arises as the difference between two different choices of the splitting,

$$i_1(Ber) - i_2(Ber) = \partial_i q^i$$

However, we can express it as a true total derivative,

$$\partial_i q^i = (\partial_i + \overline{\partial}_i)\ q = d\ q$$

and hence, by the Stokes theorem for analytic varieties [GH], its total contribution is vanishing and so we do not get any ambiguities in the computation of, say, the cosmological constant.

Having at our disposal a nice compactification of the moduli space of θ-characteristics together with a sound divisor calculus on it [C2], we next want to use it to get some insight into the structure of string amplitudes, and namely to specialize the discussion to the zero-point function.

Let us consider, as a toy example, the case of genus 1. Here, incidentally, when restricting to the case of even θ-characteristics , we do not have to bother with the problem of Fermi integration, as supermoduli space is purely even and coincides with S_1^+. Then the fermionic part of the 1-loop superstring partition function is given by

$$(\det \bar{\partial}_{1/2})^4$$

We recall that in Cornalba's compactification scheme, S_1^+ is a triple covering of M_1, and the preimages under the natural projection

$$\overline{S}_1^+ \xrightarrow{p} \overline{M}_1$$

of the compactification divisor $\delta_0 \subset \overline{M}_g$ are three points $\{R, NS_1, NS_2\}$ where, following the notations of §2.7

$$R = \nu_0$$
$$E \equiv NS_1 \cup NS_2 = \nu_0'$$

Notice that, being our spin modular parameter the square of the usual modular parameter, one has

$$p^{-1}(\delta_0) \equiv \Delta_0 = 2E + 2R$$

Considering the "universal" θ-characteristics

$$\begin{array}{ccc} \mathcal{L} & \rightarrow & \mathcal{C} \\ & & \downarrow f \\ & & \bar{S}_g \\ & & \downarrow p \\ & & \bar{M}_g \end{array}$$

by means of the Grothendieck-Riemann-Roch theorem one can compute the Mumford class as

$$\lambda = \frac{1}{12} f_*(\omega_f.\omega_f) + \delta$$

and the θ-class as

$$c_1(f_! \mathcal{L}) = \lambda - \frac{1}{2} f_*(\mathcal{L}.\mathcal{L}) - \frac{1}{2} E$$

so that, taking into account the fundamental relation $12\lambda = \delta_0$ which stems from the triviality of the relative dualizing sheaf in genus 1 one gets that [FMRT] $(\det \bar{\partial}_{1/2})^4$ has divisor

$$4c_1(f_! \mathcal{L}) = 4\lambda - 2f_*(\mathcal{L}.\mathcal{L}) - 2E = 4\lambda - E = \frac{2}{3}R - \frac{1}{3}E$$

which is in perfect agreement with the results found in the literature [SW] by means of explicit computations.

Such computations can also be used to give a very appealing hint to a new proof of the vanishing of the cosmological constant at high genera. To this purpose, we compute the divisor associated to the section $\mathcal{R}$ defined in Eq. (2). We want to stress that we are still working on spin moduli space $\overline{S}_g$, i.e. we are not giving any prescription for a higher genus analogue of GSO-projection, and, consequently, we do not have to be bothered by such delicate question.
$\mathcal{R}$ gets contributions both from products of determinants,

$$D = (\det\,\bar{\partial}_{1/2})^5 \cdot (\det\,\bar{\partial}_{-1/2})^{-1} \cdot (\det\,\bar{\partial}_{-1}) \cdot (\det\,\bar{\partial}_0)^{-5}$$

and from the Pfaffian term

$$Pf = \langle S_1(z_1) \cdot S_{2g-2}(z_{2g-2}) \rangle_\alpha$$

Again by Grothendieck-Riemann-Roch gymnastics one can show that the Chern class of D adds up to give

$$c_1(D) = -\nu_0 - 3 \cdot \nu'$$

where we recall once again ν_0 is the boundary component of ∂S_g which projects to the usual Mumford - Deligne's δ_0 in ∂M_g "without the addition of exceptional $\mathbb{P}^1$'s and ν' is the rest.

Let us now study the asymptotics of the Pfaffian factor Pf. The leading contribution is the $(g-1)$-fold product of terms like

$$\langle \Psi^\mu(z_1)\Psi^\nu(z_2)\rangle \partial_{z_1}\partial_{z_2}\langle X^\mu(z_1)X^\nu(z_2) \;\sim\; S(z_1,z_2)\cdot \partial_{z_1}\partial_{z_2}\ln E(z_1,z_2)$$

where $S(z,w)$ and $E(z,w)$ are, respectively, the Szegö kernel and the prime form. One has that [Fay]

$$\begin{cases} S(z_1,z_2) \sim 1/E(z_1,z_2) \sim \sqrt{t} = q \\ \partial_{z_1}\partial_{z_2)} \ln E(z_1,z_2) \sim const \end{cases}$$

Now, the product $D \cdot Pf$ defines a divisor $(D \cdot Pf)$ in $\overline{S}_g$. The previous computations give

$$(D \cdot Pf) = -\nu_0 - 3\cdot\nu' + (g-1)\cdot\nu' = -\nu_0 + (g-4)\nu'$$

so that, when $g\; > 5$ the degree of $L \equiv (D \cdot Pf)$ is negative and hence its only holomorphic section is the vanishing one. Then, under the assumption of holomorphicity of $D \cdot Pf$ (on spin moduli space), one gets an algebro-geometrical direct proof of the vanishing of the cosmological constant.

The border case $g \;=\; 5$ is also easily dealt with. Here deg$L = 0$ so that $L \simeq O_{\overline{S}_g}$. Then, $D \cdot Pf$ is a global holomorphic function and so is a constant. A convenient point for evaluating it is $\overline{s} \in \pi^{-1}(\delta_1)$ where one can use factorization plus the proven vanishing of the cosmological constant at genus 1 to get its vanishing also for $g = 5$.
We stress that this result matches nicely both with the result of [GIS] based on explicit computations at genus 2 and with those of [MP], which rely on generalized Riemann identities holding for $g \leq 4$.

The above discussions shows that the "super-algebraic" geometrical approach to the Superstring theory, besides being mathematically sound-based, is computationally quite promising. Still, as we have repeatedly remarked, some problems remain open and deserve further study. In addition to the ones (like splitness of supermoduli space) that have been extensively discussed in the previous chapters, we would like to list the following ones

a) The extension of the above formalism to the case of the heterotic string, where left-right asymmetry must be taken into account from the very beginning and the definition of "Superdeterminants" is not so clear.

b) The correct generalization of the high-genus analogue of the GSO-projection, which is needed to get a physically reasonable fully supersymmetric particle spectrum.

c) The characterization of $N = 2$ Super Riemann surface in this formalism, in relation both with compactification schemes *à la* Calabi-Yau and, more generically, with $N = 2$ superconformal field theories.

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