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# Quantum Origin of Hawking Universe


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To My Parents

and

To The Mamory of My Brother Isaac

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# Quantum Origin of Hawking Universe

# I. Introduction

One of most ambitious aims of theoretical physics today is to understand the origin of the universe; the classical spacetime together with its matter content. Any attempt to understand the origin of spacetime with its indefinite Lorentzian metric must start from something more fundamental than the concept of spacetime itself. In classical general relativity and therefore in classical cosmology spacetime is a fundamental concept; therefore one can not ask questions about the origin of spacetime in the context of classical cosmology. Moreover a series of theorems by Hawking and Penrose show that any classical cosmological model, with a reasonable matter content must start with a big bang singularity. At this singularity, both spacetime curvature and matter energy density become infinite. This is a disaster to physics, since all the laws of physics will breakdown at the singularity [1,2]. Singularity theorems can be reinterpreted such that they imply that at very early time in the evolution of the universe the curvature of spacetime was so large (in Planck units) that quantum gravitational effects were very important and classical cosmology breakdown. As the consistent and complete theory of quantum gravity is yet to be found; main while, one can tackle certain special cases with the formalism already existed. Quantum cosmology, the subject of this thesis, is a special case of quantum gravity, where one is preliminary interested in the quantum dynamics of certain cosmological models for the universe. In quantum cosmology, the quantum state of the universe is represented by a wave functional, which is a functional of the geometry of three-space and the matter configurations on it, therefore, spacetime is not a fundamental concept, and the more fundamental variables are the geometry of three-space and the matter configurations on it. This makes quantum cosmology a suitable theory for addressing questions about the origin of spacetime as a fundamental concept in classical cosmology. Any answer for a question about the origin of the universe can not avoid addressing the issue of the initial state of the universe. Fortunately the initial state or the boundary condition of the universe plays a central role in quantum cosmology. Hawking has recently proposed a boundary conditions for the universe [3]. The idea was further studied in the famous paper by Hartle and Hawking [4]. They have found that Hawking boundary conditions could lead to a universe that was similar to our universe. In this thesis the universe is assumed to be in the "No boundary" state proposed by Hawking. Further explanation of Hawking boundary condition is given in section II.3.

Chapter.II in general, will deal with formalism of classical and quantum cosmology. The canonical formulation of classical cosmology is recapitulated in section.II.1. The next two sections deals with the formalism of quantum cosmology. In section II.2, both the canonical and path integral formulations of quantum cosmology are discussed, while in section II.3, Hawking boundary conditions are introduced. These boundary conditions are most easily spelled in terms of path integral over Euclidean histories of the universe. A universe in the quantum state singled out by Hawking boundary conditions is called Hawking universe, while the quantum state it self is called Hartle-Hawking wave function of the universe, since this wave function was first calculated in the joined paper of Hartle and Hawking [4]. The central idea of the thesis is whether H-universe in the classical limit resemble our own universe in its known part of history. This question is studied in chapters III, IV. In chapter III, a minisuperspace model is introduced, it is Friedman universe which contains a massive scalar field as a matter source. The general properties of this model are discussed in section III.1. In section III.2, the interpretation of the semiclassical wave function of the universe corresponding to the above model is discussed. It is found that the wave function in some regins in the superspace is oscillatory in behaviour. Only in this regins the concept of Lorentzian time is meaningful, this implies that the origin of Lorentzian 4-geometry of spacetime is the oscillatory behaviour of the wave function which may be considered as more fundamental fact.

The other question addressed here about the origin of the the matter content of the universe is discussed in section III.3, this question was first addressed in context of inflationary models [5,6,7]. In inflationary models, spacetime is usually taken to be classical, governed by Einstein equations, while matter source is taken to be quantum fields. The idea of inflation [5,6], is that at an early stage in the evolution of the universe, its expansion was dominated by the by the vacuum energy of the matter fields, which acts as a cosmological constant, causing the universe to enter a de Sitter stage of exponential expansion. According to inflationary models all matter content of the present universe was created at the end of inflation from the vacuum energy of matter fields [8]. Thus, inflation gives a definite answer to the above question but at the expense of assuming initial state of the universe for which, at least inflation occurs with sufficiently long period. Furthermore, in order for inflation to be able to cure all the diseases of classical cosmology, several strong constraints has to be satisfied by the parameters of both cosmology and particle physics [9]. Thus, in the absence of proper understanding of the initial state of of the universe, one can not claim that inflation occurs naturally. For one can conceive a large class of universes for which the above constraints were not satisfied, and inflation dose not take place or was not long enough to solve all the problems of classical cosmology, like (horizon,flatness .etc). In quantum cosmology one has a precise description of a boundary conditions for the universe, which lead to an initial state of the universe, in which inflation occurs naturally, at least in certain reasonable models of the universe, such as, the model of chapter III.

The last question addressed in this thesis is a bout the origin of large scale structures in the universe, such as, galaxies, and superclusters, this question is discussed in chapter IV. This question was also first addressed in the context of inflation [7,15]. According to inflation, the present structures have their seeds in the ground state quantum fluctuations of matter field which derive inflation, they were sufficiently amplified by inflation so that they lead to a density perturbations which can have both the correct amplitude and spectrum to evolve into the structures that we see now provided certain conditions were satisfied. In many inflationary models this leads to strong conditions on the parameters of the particle theory involved in the model, such as, the the self interaction coupling constant of the scalar field which often used to derive inflation has to be far smaller than its typical values in a realistic particle physics theory [9]. Moreover, one can conceive an initial state of the universe for which the quantum fluctuations in the matter field did not start in their ground state and they give rise to a density perturbations with the wrong amplitude or the wrong spectrum to explain both the observed structures [20], and the isotropy of microwaves background radiation. Again quantum cosmology provides us with more comprehensive understanding of the initial state of the universe, because it seems that the inhomogeneous and anisotropic modes always started out in their ground state in Hawking universe. The reason for this, is that, Hawking boundary conditions constraint these modes to be very small when the size of the universe was very small [10]. Clearly the origin of structure has to do with the evolution of the inhomogeneous modes, for if the early universe was strictly homogeneous then it is not possible for it to develop the structures that we see now, because if the history of the universe contains one homogeneous and isotropic three-space section (cuachy hypersurface) then, by Einstien's field equations, every space section in the history of the universe is homogeneous and isotropic. This means that it is not possible to address the question about the origin of structure in the context of the minisuperspace model of chapter III. Therefore in chapter IV, following reference [10] the model of chapter III is generalized so that the superspace contains in addition to the homogeneous and isotropic modes of the minisuperspace model of chapter III , infinite number of inhomogeneous and anisotropic modes. This extension is done in section IV.1. The wave functions for this modes are found in section IV.2, in the limit when the homogeneous modes become semiclassical, that is, when the concept of Lorentzian time becomes meaningful, these wave functions obey time dependent Schrödinger equations [10,20]. The most important result of section IV.2 is that, the inhomogeneous and anisotropic modes started out in their ground states, this result agree with that of reference [10]. The relevance of the result of section IV.1 to the origin of structures in the universe is discussed in section IV.3. In chapter V we comment on the validity of the ideas of the thesis, because the models discussed here were in some sense restrictive, other possible generalizations are discussed very briefly. We also comment briefly on the validity of using Einstien's theory of gravity as a basis for quantum cosmology.

# II. Formalism of Classical and Quantum cosmology

## II.1- Canonical Formulation of Classical Cosmology.

## II.2- The Quantum Dynamics of the Universe.

## II.3- Hawking Boundary Conditions of the Universe.

# II.1 Canonical Formulation of Classical Cosmology.

A classical cosmological model, is a model for the evolution of the universe, in which spacetime is represented by a 4-manifold M, with a metric $g_{\mu\nu}$, which is governed by Einstien equations, and a matter source, usually classical fields, governed by a classical field equations, according to their spin. Clearly the concept of spacetime plays a fundamental role in classical cosmology. The canonical formulation of classical cosmology begins with the recognition that spacetime is the trajectory or history of three-space. thus, the dynamical quantity is the geometry of three- space, represented by a metric $h_{ij}$ [11]. Similarly the dynamical quantity of matter field (assumed to be scalar field here $\Phi(x)$), is the configuration of the field $\varphi(X)$ on th space-like hypersurface S of constant time. In order to study the classical dynamics of the universe in the Hamiltonian form , one has to define more precisely the canonically conjugate variables for the dynamics of the universe. This is also very crucial for obtaining the corresponding quantum dynamics.

The starting point is to foliate spacetime 4-manifold $(M, g)$ into a one-parameter family of 3-spacelike hypersurfaces S. The parameter which label the surfaces, can be taken as a time coordinate. If one introduces another three coordinates, $X^i$, to label the points of S; then the geometry of three-spaces S can be represented by a metric $h_{ij}$ on S.

The metric of spacetime takes the following (3+1) form [11]:

$$dS^2 = -(N^2 - N_i N^i)dt^2 + 2N_i dX^i dt + h_{ij} dX^i dX^j, \quad \text{II}\cdot 1\cdot 1$$

where N is the lapse function, measuring the proper time separation between successive surfaces of constant coordinate time t, $N_i$ is the shift vector which generates a coordinate on all successive surfaces from a coordinate that was given on an initial surface S. This means that the functions N and $N_i$ are not dynamical variables; in fact they can be chosen quite arbitrary. This arbitraryness reflects the invariance under general coordinate transformations which will be seen to lead to constraints of the classical dynamics. These constraints will play a fundamental role at the quantum level. The action for matter field is that of a massive minimally coupled scalar field, (this is the matter field relevant to subsequent discussions):

$$I_m = -\frac{1}{2}\int_M [\partial_\mu \Phi \partial^\mu \Phi + m^2\Phi^2]\sqrt{-g}d^4x \quad \ldots\ldots \text{II}\cdot 1\cdot 2$$

The action for gravity is :

$$I_g = \frac{1}{16\pi G}[\int_M [R - 2\Lambda]\sqrt{-g}d^4x + 2\int_{\partial M} K\sqrt{h}d^3x] \quad \ldots \text{II}\cdot 1\cdot 3$$

where :

$g$ is the determinant of the spacetime metric ($g = N\sqrt{h}$).
$h$ is the determinant of the three-space metric $h_{ij}$.
$\Lambda$ is a cosmological constant.
$K$ is the trace of the second fundamental form $K_{ij}$.
The surface term was added by Gibbons and Hawking to insure that $I_g$ leads to Einstein's equations under variations of the metric $g_{\mu\nu}$ which vanish on the boundary , but whose normal derivatives do not vanish there [12]. The classical cosmological action [the action of the universe] is :

$$I = I_g + I_m \quad \ldots\ldots \text{II}\cdot 1\cdot 4$$

The next step is to break I into its (3+1) from, as we did for the metric $g_{\mu\nu}$

$$I = \int [L_g + L_m]d^3X dt \quad \ldots \text{II}\cdot 1\cdot 5$$

Using the inverse of spacetime metric $g^{\mu\nu}$ :

$$[g^{\mu\nu}] = \begin{pmatrix} -\frac{1}{N^2} & \frac{N^i}{N^2} \\ \frac{N^i}{N^2} & h^{ij} - \frac{N^iN^j}{N^2} \end{pmatrix} \quad \ldots \text{II}\cdot 1\cdot 6$$

The matter lagrangian becomes :

$$L_m = -\tfrac{1}{2}N\sqrt{h}[g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi + m^2\Phi^2]$$

$$= \frac{1}{2}N\sqrt{h}[N^{-2}\left(\frac{\partial\Phi}{\partial t}\right)^2 - \frac{2N^i}{N^2}\frac{\partial\Phi}{\partial t}\frac{\partial\Phi}{\partial X^i} - (h^{ij} - \frac{N^iN^j}{N^2})\frac{\partial\Phi}{\partial X^i}\frac{\partial\Phi}{\partial X^j} - m^2\Phi^2] \quad \cdots II.1.7$$

Similarly one could write $L_g$ in its $(3+1)$ form :

$$L_g = \frac{m_p^2}{16\pi}N[G^{ijkl}K_{ij}K_{kl} + \sqrt{h}(R_3 - 2\Lambda)] \quad \cdots II.1.8$$

where $m_p$ is Planck mass ($m_p = G^{-\frac{1}{2}}$ in units $\hbar = c = 1$), $K_{ij}$ is the second fundamental form of the surface S.

$$K_{ij} = \frac{1}{2N}\left[-\frac{\partial h_{ij}}{\partial t} + 2N_{(i|j)}\right]$$

where (|) denotes covariant differentiation with respect the 3- metric $h_{ij}$, $R_3$ is the curvature scalar of the three- metric $h_{ij}$,

$$G^{ijkl} = \frac{1}{2}h^{\frac{1}{2}}[h^{ik}h^{jl} + h^{il}h^{jk} - 2h^{ij}h^{kl}] \quad \cdots\cdots II.1.9$$

$G_{ijkl}$ is the De Witt metric of the superspace, the space of all three-metric that can be put on a 3-surfce S . Now, one can obtain the momentum conjugate to the dynamical variables $(\varphi, h_{ij})$

$$\Pi_\varphi = \frac{\delta L_m}{\delta\dot{\varphi}} = N^{-1}h^{\frac{1}{2}}[\dot{\varphi} - N^i\frac{\partial\varphi}{\partial x^i}] \quad \cdots\cdots II.1.10$$

$$\Pi^{ij} = \frac{\delta L_g}{\delta\dot{h}_{ij}} = \frac{m_p^2}{16\pi}G^{ijkl}.K_{kl} = \frac{-m_p^2}{16\pi}h^{\frac{1}{2}}[K^{ij} - h^{ij}K] \quad \cdots II.1.11$$

The Hamiltonian of the universe is:

$$H = \int (\Pi^{ij} \dot{h}_{ij} + \Pi_\varphi \dot{\varphi} - L_g - L_m)\, d^3x$$

$$= \int \Big[ \frac{16\pi}{m_p^2} N G_{ijkl} \Pi^{ij} \Pi^{kl} + \frac{1}{2} h^{-\frac{1}{2}} N \Pi_\varphi^2 - h^{\frac{1}{2}} N \left(\frac{m_p^2}{16\pi}\right)(R_3 - 2\Lambda) +$$

$$h^{\frac{1}{2}} N h^{ij} \frac{\partial\varphi}{\partial x^i} \cdot \frac{\partial\varphi}{\partial x^j} + N m^2 h^{\frac{1}{2}} \varphi^2 - 2N_i \Pi^{ij}{}_{|j} + N_i h^{ij} \frac{\partial\varphi}{\partial x^j} \Pi_\varphi \Big] d^3x \cdots II \cdot 1 \cdot 12$$

Clearly it is linear in both N and $N_i$, thus it can be written as:

$$H = \int [N H_0 + N_i H^i]\, d^3x \quad \ldots\ \ldots\ II \cdot 1 \cdot 13$$

where:

$$H_0 = \frac{16\pi}{m_p^2} G_{ijkl} \Pi^{ij} \Pi^{kl} + \frac{1}{2} h^{-\frac{1}{2}} \Pi_\varphi^2 + \frac{1}{2} h^{\frac{1}{2}} \Big[ \frac{-m_p^2}{8\pi} R_3 +$$

$$h^{ij} \frac{\partial\varphi}{\partial x^i} \cdot \frac{\partial\varphi}{\partial x^j} + m^2 \varphi^2 \Big] \quad - - - - - \quad II \cdot 1 \cdot 13a$$

$$H^i = -2\, \Pi^{ij}{}_{|j} + h^{ij} \frac{\partial\varphi}{\partial x^j} \cdot \Pi_\varphi \quad \cdots II \cdot 1 \cdot 14$$

Here $G_{ijkl}$ is the inverse of the superspace metric :

$$G_{ijkl} = \frac{1}{2} h^{-\frac{1}{2}} [h_{ik} h_{jl} + h_{il} h_{jk} - h_{ij} h_{kl}] \cdots II \cdot 1 \cdot 15$$

In obtaining $H^i$ we have discarded a surface term arising from changing $N(i/j)\Pi^{ij}$ into $N_i\Pi^{ij}|j$, this is justified only if the the three-surface S has no boundary, this will be the case in all subsequent discussions. Thus, the canonical variables are identified, they are : $(h_{ij}, \varphi, \Pi^{ij}, \Pi_\varphi)$. Now it is more clear that the dynamics of the universe, [11, 12], is the dynamics of of the geometry of the three-space, and the matter configuration on it. Thus, any history of the three-space and matter configuration on it, represents a possible classical cosmological model for the evolution of the universe. In order to be able to choose a unique history to represent the actual history of our universe one needs:

In order to be able to choose a unique history to represent the actual history of our universe one needs:

- Dynamical equations.
- Initial conditions.

The dynamical equations in our case are the space-space part of Einstein's equations and the classical field equation for the scalar field, they can be obtained from the Hamiltonian II.1.13 :

$$\dot{h}_{ij} = \frac{\delta H}{\delta \Pi^{ij}} \quad ; \quad \dot{\Pi}_{ij} = -\frac{\delta H}{\delta h_{ij}} \quad \ldots \text{II.1.16}$$

$$\dot{\varphi} = \frac{\delta H}{\delta \Pi_\varphi} \quad ; \quad \dot{\Pi}_\varphi = -\frac{\delta H}{\delta \varphi} \quad \ldots \text{II.1.17}$$

By varying the Hamiltonian with respect to N and $N_i$ one gets respectively the Hamiltonian and the momentum constraints of the dynamical system:

$$H_0 = 0 \quad \ldots \text{II.1.18}$$

$$H^i = 0 \quad \ldots \text{II.1.19}$$

These constraints are conserved by the classical dynamical evolution, they are satisfied at all times, once they are satisfied at a given time.

The initial conditions amount to specifying the canonical variables$(h_{ij}, \varphi, \Pi^{ij}, \Pi_\varphi)$, on an initial three-surface such that they satisfies the above constraints.

# The Quantum Dynamics of the Universe

In the Schrödinger picture, the quantum state of the universe is represented by a wave functional $\Psi(h_{ij}, \varphi)$, it is a functional over the extended superspace, the later is the space of all possible three-metrics $h_{ij}$ and matter field configurations $\varphi$ that can be put on a three-surface S. (from now on it is called just superspace). In order to obtain the quantum dynamics from the classical canonical dynamics, one promotes the classical canonical variables into a quantum operators. Thus in Schrödinger picture the momentum conjugate to a dynamical variable is represented by a functional derivative with respect to this variable, it acts on the wave functional.

$$\Pi_{\varphi} \Longrightarrow -i\frac{\delta}{\delta\varphi} \quad \cdots\cdot \text{II}.2.1a$$

$$\Pi^{ij} \Longrightarrow -i\frac{\delta}{\delta h_{ij}} \quad \text{II}.2.1b$$

Thus, substituting this into the classical Hamiltonian will promote it into a functional differential operator. In order to be a physical quantum state for the universe, the wave functional, must describe a quantum dynamics which is consistent with the constraints. This is only possible if the wave functional is annihilated by the quantum Hamiltonian. Therefore in order to implement the classical constraints at the quantum level one is lead to require that the quantum state of the universe should satisfy the following set of functional differential equations [4, 13, 14, 15]:

$$\hat{H}^{i}\left(\varphi, h_{ij}; \frac{-i\delta}{\delta\varphi}, \frac{-i\delta}{\delta h_{ij}}\right)\Psi(\varphi, h_{ij}) = 0 \quad \cdots \text{II}.2.2$$

$$\hat{H}_{0}\left(\varphi, h_{ij}; -\frac{i\delta}{\delta\varphi}, \frac{-i\delta}{\delta h_{ij}}\right)\Psi(\varphi, h_{ij}) = 0 \quad \cdot\cdot \text{II}.2.3$$

These equations turn out to be highly non trivial requirements, in fact equation II.2.3 turns out to determine the whole quantum dynamics of the universe. Equation II.2.2 is a set of momentum constraints. They imply that

$\Psi$ is is a functional of the geometry of the the three-surface S; but not the particular coordinates in which this geometry is given, therefore this equation has no dynamical content. The non trivial equation II.2.3, is the Wheeler-De Witt equation [13], it governs the evolution of the wave functional of the universe in the superspace. If one looks at this equation as a time dependent Schrödinger equation, then it implies that the wave functional of the universe is not explicit function of time. While as an eigen value equation, it implies that the total energy of the universe is zero. This is reasonable for a spatially closed universe without boundary, like Hawking universe, because for such a universe all strictly conserved quantum numbers should vanish.

In writting down Wheeler-De Witt equation explicitly, there exist a non-trivial problem of factor ordering; essentially because in the classical Hamiltonian $H_0$, the kinetic term for the gravitational field contains $G_{ijkl}$, the later depends on $h_{ij}$, this leads to ambiguity in ordering $G_{ijkl}$ with the functional derivatives corresponding to the $\Pi^{ij}$s.This issue was discussed by Hawking and Page [17]. A part from chapter IV, H-P factor ordering is used in the rest of the thesis. With this ordering Wheeler-De Witt equation takes the following explicit form:

$$\left[-\frac{16\pi}{m_p^2} G_{ijkl}\cdot\frac{\delta}{\delta h_{ij}}\cdot\frac{\delta}{\delta h_{kl}} - \frac{1}{2} h^{-\frac{1}{2}} \frac{\delta^2}{\delta\varphi^2}\right]\Psi(h_{ij},\varphi)=0 \quad \cdot \text{II}\cdot 2\cdot 4$$

where,

$$V(h_{ij},\varphi) = \frac{-m_p^2}{16\pi} h^{\frac{1}{2}}(R_3 - 2\Lambda) + h^{\frac{1}{2}}\left(h^{ij}\frac{\partial\varphi}{\partial x^i}\cdot\frac{\partial\varphi}{\partial x^j} + m^2\varphi^2\right) \cdots \text{II}\cdot 2\cdot 5$$

Thus, the factor ordering is chosen so that the kinetic term in the Hamiltonian operator $H_0$, is a covariant laplacian in the superspace metric [17]. Since the superspace metric can be made hyperbolic by a suitable choice of gauge (choice of independent dynamical variables by solving the momentum constraints). This choice will also makes W-D equation hyperbolic with respect to the superspace metric, with $h^{1/2}$ playing the role of time [14]. So the quantum dynamics of the universe, in the Schrödinger representation, can be thought of as the evolution of the wave functional in the superspace in the direction of $h^{1/2}$, the evolution is governed by W-D equation. Any solution of W-D equation represents a possible quantum state of the universe.

So far the quantum dynamics of the universe is presented in the Schrödinger picture, an other very useful representation is, in terms of Feynman idea of sum over histories. This representation is also used in many occasions in the text. In this formulation the basic object is the quantum amplitude to go from one three-surface $S_1$, with a three-metric $h^1_{ij}$ and matter configuration $\varphi^1(X)$ to an other three-surface $S_2$, with a three- metric $h^2_{ij}$ and matter configuration $\varphi^2(X)$ Fig.1.This amplitude is given by path integral over all Lorentzian 4-metrics $g_{\mu\nu}$and matter configurations $\Phi(x)$ on 4-manifold M, bounded by $S_1$ and $S_2$, that can induce, the three-metric $h^1_{ij}$ and the matter configuration $\varphi^1(X)$ on $S_1$ and, the three-metric $h^2_{ij}$ and the matter configuration $\varphi^2(X)$ on $S_2$, [12].

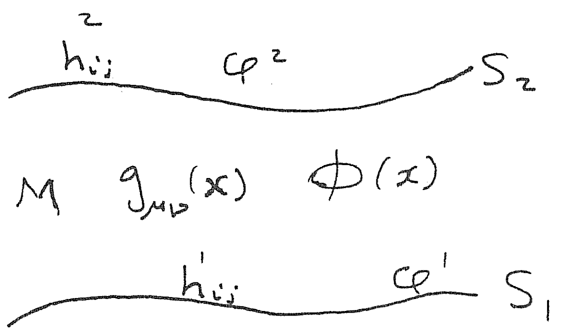


figure . 1

Explicitly, the amplitude may be written as :

$$\langle h^2_{ij}, \varphi^2 / h^1_{ij}, \varphi^1 \rangle =$$

$$\int d[g_{\mu\nu}]\, d[\Phi] \exp i\left(I(g_{\mu\nu}, \Phi)\right) \quad \cdots \text{II}.2.6$$

Clearly, time dose not appears explicitly in the above amplitude, actually this is not true if M is not compact, for example if $S_2$, $S_2$ were asymptotically flat, then their time positions in M will be invariant, and both the wave functional and the quantum amplitude would depend explicitly on time. The wave functional can be recovered from the quantum amplitude in the following way:

$$\Psi(h^2_{ij}, \varphi^2) = \int \Psi(h^1_{ij}, \varphi^1)\, \langle h^2_{ij}, \varphi^2 | h^1_{ij}, \varphi^1 \rangle\, d[h^1_{ij}]\, d[\varphi^1] \quad \cdots \text{II}.2.7$$

where the integral is over all the three-metrics and matter configuration that can be put on $S_1$. Alternatively one may write the wave functional as :

$$\Psi(h_{ij}, \varphi) = \int_C d[g_{\mu\nu}]\, d[\Phi] \exp i\left(I(g_{\mu\nu}, \Phi)\right) \quad \cdots \text{II}.2.8$$

here C is the class of Lorentzian 4-metrics $g_{\mu\nu}$ and matter configurations on a 4-manifold M, that are allowed in the functional integral. Clearly specifying C would lead through II.2.8 to a unique wave functional. In this sense, specifying C is equivalent to specifying an initial wave functional $\Psi(h^1_{ij}, \varphi^1)$ in equation II.2.6. Moreover, equation II.2.8 shows clearly that in quantum cosmology the geometry of spacetime is not an a observable, indeed one has to sum over all possible geometries of spacetime to get the quantum state. Therefore, spacetime is not a fundamental object in quantum cosmology. This brings the question addressed in the introduction , of how and when spacetime becomes a fundamental object, this question is discussed in the next chapter.

The wave function II.2.8, can be shown, at least formally, to satisfy W-D equation, so it represents a possible quantum state of the universe [14,15]. This can be done by varying II.2.8, with respect to the lapse function N. on the left hand side the effect is to push the boundary of M, on which $h_{ij}$ and $\varphi$ have the values specified by the argument of the wave functional, forward or backward in time. Since the wave functional is not an explicit function of time (at least for a closed universe ), the left hand side give zero. While the right hand side gives:

$$\frac{\delta\Psi}{\delta N} \propto \frac{\partial\Psi}{\partial t} = 0 = i\int_C d[g_{\mu\nu}]\,d[\Phi]\,\frac{\delta I}{\delta N}\,e^{iI(g_{\mu\nu},\Phi)}$$

$$= \int_C d[g_{\mu\nu}]\,d[\Phi]\,H_0\,e^{iI(g_{\mu\nu},\Phi)} = H_0\Psi \quad \ldots \quad II.2.9$$

Similarly the momentum constraint can be derived by varying with respect to $N_i$. In this case the left hand side will vanish because, a variation with respect to $N_i$ will only change the coordinates in which $h_{ij}$, $\varphi$ are given.

An other representation of $\Psi$ is in terms of path integral over Euclidean histories [12, 14, 15].

$$\Psi(h_{ij},\varphi) = \int_{C'} d[g_{\mu\nu}]\,d[\Phi]\,e^{-\hat{I}(g_{\mu\nu},\Phi)} \quad \ldots \quad II.2.10$$

where $\hat{I}$ is the Euclidean action of the history $(g_{\mu\nu}(X,T), \Phi(X,T)$ obtained from I by taking the lapse function to be negative imaginary, $N \to -iN$

Interestingly enough the wave functional II.2.10 obeys the same W-D equation [15], and so it is automatically ( without analytic continuation back to Lorentzian time) a possible quantum state for the universe. Here the Euclidean path integral is used in all the text, since in this case the integrand is formally exponentially damped, (a part from the integral over the conformal fluctuations which need a special choice of the contour in the complex conformal functional space so that also this integral converges [12, 15]), this make the functional integral, at least formally, convergent [12]. This is to be contrasted with the Lorentzian functional integral which oscillates but dose not converge.

## II.3-Hawking Boundary Conditions for the Universe.

W-D equation is a functional differential equation, therefore it has infinite number of solutions. The question is which of these solutions represents the actual quantum state of the universe, this quastion can only be answered if one has a theory or a proposal for the boundary conditions of the universe. Hawking [3] has proposed that the boundary conditions of the universe is that it has " no boundary ". The implications of these boundary conditions are studied in detail by Hartle and Hawking in [4]. They have calculated the wave function of the universe for certain quantum cosmological models. Further studies of this boundary conditions have shown that the wave function singled out by Hawking boundary conditions describes a universe similar to ours [4,14 ,15, 20]. As in classical dynamics, the most convenient way to incorporate the initial conditions is in terms of an action principle. Here too the most convenient way to spell Hawking boundary conditions is in terms of path integral, moreover since the idea involved summing over compact 4-metrics, it was not possible to give the boundary conditions in terms of Lorentzian path integral. The reason is that any compact Lorentzian metric must be singular, no body no how to put boundary conditions on the spacetime singularities [1]. On the other hand a compact Euclidean 4-manifold is not necessarily singular [1], a clear example is the 4-sphere, the north and the south poles of a four sphere is as regular as any other point of this sphere. With this in mind the boundary conditions is reduced to specification of the class $\grave{C}$ of Euclidean histories in equation II.2.10 Hawking boundary conditions a mount to taking the class $\grave{C}$ to be the class of all compact Euclidean positive definite 4-metrics and regular matter configurations on a 4-manifold M, whose only boundary is a compact three- surface S, the 4-metrics and the matter configurations in $\hat{C}$ must induce the argument of the wave functional $\Psi(h_{ij}, \varphi)$ on S [3,4]. Clearly, the three-surface S must also be compact without any boundary in order to be a section of compact 4-manifold without boundary, this the origin of the statement that the boundary conditions of the universe is that it has "no boundary" [1]. This choice of boundary conditions would give a unique solution to W-D equation, if definite choice of the measure of the functional integral is made. It turns out that the ambiguity in the choice of measure in the functional integral is a reflection of the problem of the factor ordering of W-D equation [21, 22]. Here we shall assume that the measure of the functional integral was chosen so that II.2.10 satisfies W.D with the factor ordering of discussed in section II.2. A universe in the unique quantum state singled out by the above boundary conditions is called here Hawking universe. Although other kinds boundary conditions have been suggested [18,19]; Hawking boundary conditions are the most comprehensive, therefore it is studied in more details in literature [4, 14, 15, 16, 17, 20, 21]. Here too, we shall be discussing Hawking boundary conditions. Clearly, Hawking boundary condition is just a proposal it can not be derived from any thing else [1], the only way to prove or falsify it is by comparing Hawking universe with the universe in which we live. As our universe is very big in size now $10^{60}$ Planck lengths, we are fairly sure that it is to a very high accuracy a classical object, indeed it is the most classical object one can conceive. Thus one should study the semiclassical approximation of H-H wave function of the universe, in order to see whether Hawking universe in its later stages of evolution (when it becomes large) becomes similar to our own universe in its recent known history. It turns out that H-H wave function in the semiclassical limit dose not describe only one classical universe, rather it describe a family of classical universes to see this, in the semiclassical limit (and when $Psi$ oscillates rapidly) H-H wave function can be written as:

$$\Psi(h_{ij},\varphi) = Rel\, C(h_{ij},\varphi)\, e^{iS(h_{ij},\varphi)} \quad \ldots \text{II.3.1}$$

where C is a slowly varying function, given by path integral over quadratic quantum fluctuations around the classical histories defined by the Hamilton principle function $S(h_{ij},\varphi)$, the latter obey Hamilton-Jacobi equation.

$$\left[\frac{-16\pi}{m_p^2} G_{ijkl} \frac{\delta S}{\delta h_{ij}} \cdot \frac{\delta S}{\delta h_{kl}} - \frac{1}{2} h^{-\frac{1}{2}} \left(\frac{\delta S}{\delta \varphi}\right)^2 + V(h_{ij},\varphi\right] = 0$$
$$= 0 \quad \ldots \text{II.3.2}$$

where $V(h_{ij}, \varphi)$ is given by equation II.2.5 Now Hawking boundary conditions, chooses a unique solution to II.3.2, through II.3.1, this solution define a family of classical histories of the universe given by:

$$\Pi^{ij} = \frac{\delta S_{H\text{-}H}}{\delta h_{ij}} \quad ; \quad \Pi_{\varphi} = \frac{\delta S_{H\text{-}H}}{\delta \varphi} \quad \cdots \text{II.3.3}$$

Thus, if the superspace has n dimensions, then the solution of II.3.3 has n arbitrary constants, while the general solution of the classical field equations would have (2n-1) arbitrary constants [20,24]. This clearly shows that Hawking boundary conditions for the universe, gives initial conditions for the classical field equations. This fact is due to the requirement of compactness and regularity, which must be satisfied by the classical Euclidean histories.

# III-Origin of Classical Spacetime and Matter from Minisuperspace Model.

## III.1-Minisuperspace Model, Hawking Massive Scalar Field.

## III.2-Interpretation of the Wave Function and Origin of the Lorentzian Geometry of spacetime.

## III.3-Inflation from Quantum Cosmology and Origin of Matter Content of the Universe.

## III.1-Minisuperspace Model, Hawking Massive Scalar Field.

It is well known that the gravitational field, or the geometry of the three-space has two dynamical degrees of freedom per point of three-space. Similarly the matter source (in our case massive scalar field) has one degree of freedom per point of three-space. Thus, the superspace has got infinite dimensions. this means that the general W-D equation of section II.2 is a differential equation over a space of infinite dimensions.It is very difficult if not impossible to solve such an equation in its most general form [14]. In order to proceed further, it has been suggested by many authors [4,13], that in the case of quantum cosmology one can use a minisuperspace approximation of the superspace for which all but finite number of the gravitational and matter degrees of freedom are frozen. Then one solves the resulting W-D equation on the minisuperspace, has finite number of dimensions. In order this approximation to be a reasonable one, the degrees of freedom which play important role in the dynamics of the universe must be identified, the minisuperspace, should consist of such a degrees of freedom. Classically one can use the cosmological principle, which says that the three-space sections of spacetime must be homogeneous and isotropic spaces, the principle is supported by large scale observations, such as the isotropy of the microwaves background radiations on scales ($L \geq 50Mps$). This means that at least classically, the large scale dynamics of the universe can be described by Robertson-Walker model, and the most important degrees of freedom, are the homogeneous and isotropic ones. The matter source also has to be homogeneous in order to be consistent with the homogeneity of the three-space. This together with the fact that we are interested solely in closed universes (because of the boundary conditions) define a minisuperspace model which is a good candidate for the description of the quantum dynamics of the universe. In this model spacetime is described by a closed Friedman metric [14]:

$$ds^2 = \sigma^2\left[-N^2(t)\,dt^2 + e^{2\alpha(t)} \cdot dX^i \cdot dX^j\right] \quad \ldots\ \text{III}\cdot 1\cdot 1$$

where $\sigma$ is a constant of dimension of length (it is of the order of Planck length), ( $\sigma^2 = \frac{2}{3\pi m_p^2}$ ), $e^{\alpha(t)} = a(t)$ the scale factor of the universe, $\Omega ij$ is the metric on unit three-sphere.

As in chapter II the matter source is assumed to be a massive scalar field $\Phi$, it is further restricted by homogeneity requirement to be function of time only. The action for $\Phi$ can be obtained from II.1.7 by substituting III.1.1 and integrating the spatial dependence in equation II.1.5 (it amount simply to multiplying by the volume of unit three-sphere ($2\pi^2$).

$$I_m = -\frac{1}{2}\int N e^{3\alpha}\left[-\left(\frac{\dot{\Phi}}{N}\right)^2 + m^2\Phi^2\right] dt \quad \ldots\ldots \text{III}.1.2$$

where $m$ is measured in units of $\sigma^{-1}$, $\Phi^2$ is measured in units of $\frac{1}{2\pi^2\sigma^2}$. Similarly, the action of the gravitational field is obtained from II.1.8 by substituting the values of $K_{ij}$,$R_3$ for the metric III.1.1 :

$$K_{ij} = -\frac{\dot{\alpha}\, h_{ij}}{\sigma N} = -\frac{\dot{\alpha}}{\sigma N}\left(\sigma^2 e^{2\alpha}\Omega_{ij}\right) \ldots \text{III}.1.3$$

$$R_3 = \frac{6}{[\sigma e^{\alpha}]^2} \quad \ldots\ldots\ldots \text{III}.1.4$$

and integrating the spatial dependence in equation II.1.5, the result is :

$$I_g = -\frac{1}{2}\int dt\, N e^{3\alpha}\left[\left(\frac{\dot{\alpha}}{N}\right)^2 - \left(e^{-2\alpha} - \Lambda'\right)\right. \ldots\ldots \text{III}.1.5$$

here $\Lambda'$ is measured in units of $\frac{1}{3\sigma^2}$, it is zero for the minisuperspace introduced above, but it is kept here just to show its analogy with the vacuum energy term $m^2\Phi^2$ in the matter action, this fact is relevant for the subsequent discussions. Thus the action for the model is :

$$I = \int [L_m + L_g]\, dt$$

$$= -\frac{1}{2}\int N e^{3\alpha}\left[-\left(\frac{\dot{\Phi}}{N}\right)^2 + m^2\Phi^2 + \left(\frac{\dot{\alpha}}{N}\right)^2 - e^{-2\alpha}\right] dt \ldots \text{III}.1.6$$

This model is called Hawking massive scalar field model, it was first introduced by Hawking in [14]. The classical equations of motion can be obtained by varying with respect to $\alpha$ and $\Phi$ :

$$N\frac{d}{dt}\left(\frac{\dot{\alpha}}{N}\right) + 3\dot{\Phi}^2 - N^2 e^{-2\alpha} = 0 \quad \ldots\ldots \text{III}.1.7a$$

$$N\frac{d}{dt}\left(\frac{\dot{\phi}}{N}\right)+3\dot{\alpha}\dot{\phi}+N^2m^2\phi=0 \quad \cdots \text{III}.1.7b$$

The superspace for the model has two degrees of freedom $(\alpha,\varphi)$, the momentum conjugate to these degrees of freedom are:

$$\Pi_\alpha=\frac{\partial L_m}{\partial\dot{\alpha}}=-Ne^{3\alpha}\frac{\dot{\alpha}}{N^2} \quad \cdots \text{III}.1.8.a$$

$$\Pi_\varphi=\frac{\delta L_g}{\delta\dot{\varphi}}=Ne^{3\alpha}\frac{\dot{\phi}}{N^2} \quad \cdots \text{III}.1.8.b$$

The Hamiltonian function for the model is

$$H=\dot{\alpha}\Pi_\alpha+\dot{\varphi}\Pi_\varphi-L_g-L_m$$

$$=\frac{1}{2}Ne^{-3\alpha}\left[\Pi_\varphi^2-\Pi_\alpha^2+e^{6\alpha}m^2\varphi^2-e^{4\alpha}\right] \quad \cdots \text{III}.1.9$$

The Hamiltonian constraint is :

$$\frac{\delta H}{\delta N}=H_0=\frac{1}{2}e^{-3\alpha}\left[\Pi_\varphi^2-\Pi_\alpha^2+e^{6\alpha}m^2\varphi^2-e^{4\alpha}\right]=0 \quad \cdots \text{III}.1.10$$

$H_0$ define the inverse of the superspace metric, while the metric itself is :

$$dS^2=e^{3\alpha}\left[d\varphi^2-d\alpha^2\right] \quad \cdots\cdots \text{III}.1.11$$

The quantum dynamics in the Schrödinger picture obtained by promoting ( $\Pi_\alpha$ ,$\Pi_\varphi$ )into a differential operators, and requiring the resulting Hamiltonian constraint to annihilate the quantum state of the universe, clearly in this case the quantum state becomes a wave function rather than a wave functional. This together with the factor ordering picked by III.1.11 specify the W-D equation completely :

$$\hat{H}_0\Psi=\left[\frac{\partial^2}{\partial\alpha^2}-\frac{\partial^2}{\partial\varphi^2}+V(\alpha,\varphi)\right]\Psi(\alpha,\varphi)=0 \quad \cdots \text{III}.1.12a$$

where

$$V(\alpha,\varphi)=m^2\varphi^2e^{6\alpha}-e^{4\alpha} \quad \cdots \text{III}.1.12b$$

Clearly, W-D equation has the form of the hyperbolic Klein-Gordan equation ,with a space-time dependent mass. It is hyperbolic with respect to the superspace metric III.1.11. The solution of III.1.12 which obey Hawking boundary conditions can be written as :

$$\Psi(\alpha,\varphi) = \int_C d[N]\, d[\alpha]\, d[\Phi]\; e^{-\hat{I}(\alpha,\varphi)} \quad \ldots\ \text{III}.1.13$$

where $\hat{I}$ is the Euclidean action :

$$\hat{I} = -iI = \frac{1}{2}\int dT\, e^{3\alpha}\left[-\left(\frac{d\alpha}{dT}\right)^2 - e^{-2\alpha} + \left(\frac{d\Phi}{dT}\right)^2 + m^2\Phi^2\right] \ldots \text{III}.1.14$$

here, T is the Euclidean time ($dT = -iNdt$), C is the class of configurations specified by Hawking boundary conditions H.b.c . In the case of the above particular model, it is the class of Euclidean 4-metrics that have the following functional form :

$$dS^2 = \sigma^2\left[N^2(T)\, dT^2 + e^{2\alpha(T)}\, d\Omega_3^2\right]$$

and the regular homogeneous matter configurations on a compact 4-manifold M , that induce the argument of $\Psi$ on the unique boundary of M, in our case III.1.15 restrict this boundary to be a three-sphere $S^3$. In practice it is very difficult to evaluate the functional integral III.1.13 exactly, since the Euclidean action is not quadratic in $\alpha$. The usual way to calculate H-H wave function is either to evaluate the path integral using saddle point approximation [4,14,15 ], or to evaluate the wave function from III.1.13 near $e^\alpha = 0$ and to use the result as boundary conditions to integrate W-D equation III.1.12. In this sense H.b.c on the path integral provide a boundary condition for W-D equation on the superspace [14,15,20,21]. We shall be using the second option, in this case in order to estimate the H-H wave function for small $e^\alpha$, one can estimate the Euclidean action along a history $(\alpha(T), \Phi(T))$, the history must satisfies H.b.c :

the 4-geometry has to be compact this implies :

$$T \to 0\ ;\ e^\alpha \to 0\ ;\ \frac{de^\alpha}{dT} \to 1 \quad \ldots\ \text{III}.1.16$$

the regularity of the matter configurations implies:

$$T \to 0\ ,\ \frac{d\Phi}{dT} \to 0\ ,\ |\Phi| = |\varphi_0| < \infty \ \ldots\ \text{III}.1.17$$

(notice that $|\varphi_0| = \infty$ would lead to a singularity of the 4-geometry at T = 0) With the above conditions, one can find the Euclidean action for any history connecting the points $(T = 0, e^\alpha = 0, \varphi = \varphi_0)$ , $(T, e^\alpha, \varphi)$, for $e^\alpha$ very small :

$$\hat{I}_{e^\alpha \to 0}(\alpha, \varphi) = \frac{1}{2}\int_0^T dT\,(-T^3 + T + 0 + T^3 m^2 \varphi_0^2) \quad \ldots\ III \cdot 1 \cdot 18$$

The Euclidean action of equation III.1.14 becomes :

$$\hat{I}(T) = \frac{1}{2}\left[-\frac{T^4}{4} - \frac{T^2}{2} + m^2 \varphi_0^2 \frac{T^4}{4}\right] \quad \ldots\ III \cdot 1 \cdot 19$$

If one substitutes III.1.16 and III.1.17, the action can be expressed in terms of the end points of the history :

$$\hat{I}_{e^\alpha \to 0}(\alpha, \varphi) = \frac{1}{2}\left[-\frac{e^{4\alpha}}{4} - \frac{e^{2\alpha}}{2} + \frac{m^2}{4}\varphi^2 e^{4\alpha}\right] \quad \ldots\ III \cdot 1 \cdot 20$$

This lead to the approximate H-H wave function at very small three-geometries [21] :

$$\Psi_{e^\alpha \to 0}(\alpha, \varphi) = \exp + \frac{1}{8}\left[2e^{2\alpha} + (1 - m^2\varphi^2)\, e^{4\alpha}\right] \quad \ldots\ III \cdot 1 \cdot 21$$

In the above evaluation we have neglected a possible contribution from a history of the form of Fig.1.b which also connect the same end points, it turns out that Fig.1.a dose indeed give dominant contribution , but the history of Fig.1.b could be complex, (has complex action) and therefore it will add oscillatory component to the wave function [21], this would change drastically the answer to the question of whether Q.C remove the singularities of classical cosmology, fortunately it dose not change the boundary conditions obtained from III.1.21.

In order to use III.1.21 as a boundary condition for W-D equation, it is useful to use an other variables, which display more clearly the casual structure of W-D equation Fig.2 [14]: These variables are :

$$y = e^\alpha \cosh\varphi \; ; \; X = e^\alpha \sinh\varphi \quad \ldots\ III \cdot 1 \cdot 22$$


Figure .1: Euclidean histories that contribute to the wave function at small 3-geometries

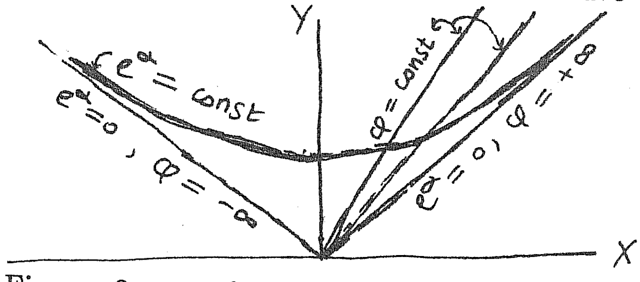


Figure .2: causal structure of minisuperspace

With this variables W-D equation takes the form :

$$[\partial_Y^2 - \partial_X^2 + V(X,Y)]\,\Psi(X,Y) = 0 \quad \cdots III \cdot 1 \cdot 23$$

where,

$$V(X,Y) = (Y^2 - X^2)^3\left[\left(m \tanh^{-1} \frac{X}{Y}\right)^2 - (Y^2 - X^2)^{-1}\right] \quad \cdots\cdot III \cdot 1 \cdot 24$$

Now the physical region of interest ($e^{\alpha} \geq 0$) is mapped by III.1.22 into the region $Y \geq |X|$. The the point $e^{\alpha} = 0$ is mapped into the forward null lines $Y = |X|$. thus in the variables$(X, Y)$,equation III.1.21 gives the wave function on and closed to the forward null lines ($Y = |X|$).This is sufficient initial value data for integrating III.1.23 in the whole physical region $Y \geq |X|$. In integrating III.1.23 one distinguishes two regions ; the first region is $V \leq 0$ and as equation III.1.21 indicates, the wave function is an exponential in behaviour in this region; while in the second region where, $V > 0$, the wave function is oscillatory in behaviour [ 21, 16].

In the second region an approximate form of the wave function which satisfy

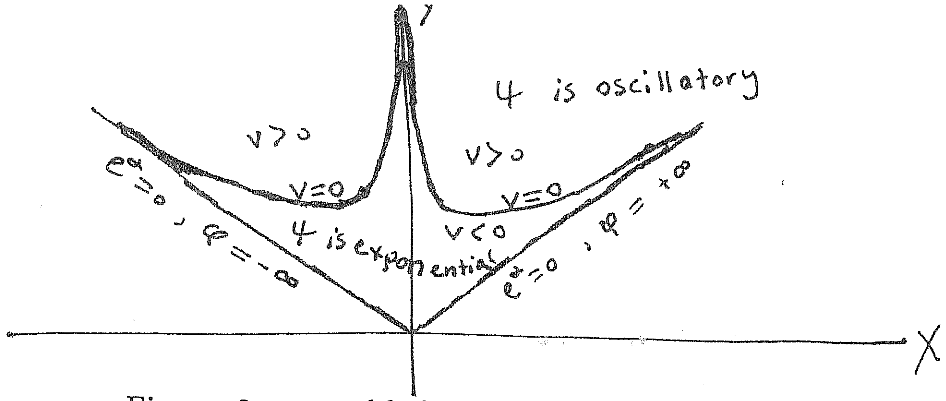


Figure .3: general behaviours of the wave function

H.b.c. may be obtained, it is valid for $V > 0$ and $|\varphi| >> 1$ [21]

$$\psi = J_0\left(\frac{1}{3} e^{3\alpha} \cdot m\, |\varphi|\right) \qquad \ldots\ldots\ \text{III}.1.25$$

$J_0$ is the zeroth order ordinary Bessel function.
Which has the asymptotic form :

$$\psi = \left(\frac{3}{2\pi}\right)^{\frac{1}{2}} \left(e^{3\alpha} m\, |\varphi|\right)^{-\frac{1}{2}} \cos\left(\frac{1}{3} e^{3\alpha} \cdot m \cdot |\varphi| - \frac{\pi}{4}\right)$$

$$\ldots\ldots\ \text{III}.1.26$$

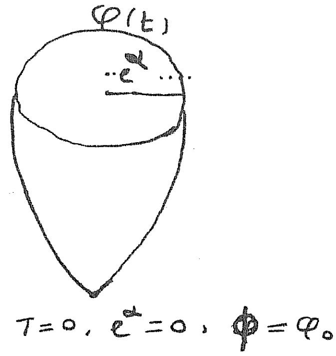


Figure .1: all the Euclidean histories must originate from zero three-geometry at T = 0

# III.2 Interpretation of the Wave Function and the Origin Lorentzian Spacetime.

In section II.2, it was noticed that the 4-geometry of spacetime is not an observable in Q.C, indeed one has to sum over all possible geometries of spacetime, in order to get the quantum state. This means that the 4-geometry of spacetime dose not play a fundamental role at the quantum level. Yet in classical general relativity, and therefore in classical cosmology the 4-geometry of spacetime plays the most fundamental role in the description of the universe. Since Q.C is expected to be more fundamental than classical cosmology, it must have a limit in which one recover classical cosmology, and the Lorentzian 4-geometry of spacetime becomes fundamental in the description of the universe. It turns out that in the semiclassical limit of Q.C, one recover the Lorentzian 4-geometry of spacetime, but only in certain regions of the superspace [15,16].

Similar ideas about the origin of the universe were discussed in th context of inflationary models [8]. In the model discussed by Velinken, the universe was created from "nothing" in a quantum tunneling event. The tunneling took place from a region in the configuration space where Lorentzian spacetime is not defined to a region where it becomes Lorentzian de Sitter spacetime. In Q.C one could also say that Hawking universe originates from "nothing", but in different sense than that of Velinken. In the latter case, "nothing" means every 4-geometry allowed in the path integral must be compact, and has only one boundary specified by the argument of the wave function. This means that every Euclidean history in the sum must originate from "nothing" at zero Euclidean time Fig.1.

If one wants to extract more useful informations from the wave function obtained in the last section one has to chose definite interpretation of the H-H wave function of the universe. The interpretation of the full wave function is difficult at least for two reasons. The first is that we live in one classical universe now, while the statistical interpretation of quantum mechanics require the existence of ensample of identical systems. The second reason is that the wave function is often not normalizable, that is, its square integral over the superspace diverges. The above difficulties makes any standard probabilistic interpretation of Q.C pathological. Even though some progress has been achieved in the interpretation of the wave function of the universe, for example Hartle has recently proposed that in the spirit of relative state interpretations of quantum mechanics, due to Everett and Wheeler, one should interpret a peak in the wave function of the universe as a definite prediction of Q.C [23, 24] . Here we are more interested in the interpretations of the semiclassical wave function since the questions addressed in the introduction concerns the semiclassical behaviour of the wave function of the H-I modes. Furthermore the wave function of last section III.1.25, III.1.26 just takes the semiclassical form II.3.1 , so they are valid approximation only in this limit. In the semiclassical case, Hawking has suggested that one can use the square of the trace second fundamental form $K^2$ to interpret the wave function of the universe [14 ,15 ].

From equation II.1.11 one has :

$$h_{ij}\pi^{ij}\Psi = -\frac{m_p^2}{16\pi}h^{\frac{1}{2}}\left[h_{ij}\hat{K}^{ij}-3\hat{K}\right] = \frac{m_p^2 h^{\frac{1}{2}}}{8\pi}\left[\hat{K}\right]\Psi \quad \ldots \text{III.2.2}$$

$$\hat{K}^2\Psi = \frac{-64}{m_p^4}\left[h^{-\frac{1}{2}}h_{ij}\cdot\frac{\delta}{\delta h_{ij}}\cdot h^{-\frac{1}{2}}h_{kl}\cdot\frac{\delta}{\delta h_{kl}}\right]\Psi \quad \ldots \text{III.2.3}$$

For the metric III.1.1 :

$$\hat{K}^2\Psi = -9\,e^{-6\alpha}\cdot\frac{\partial^2}{\partial\alpha^2}\Psi \quad \ldots \text{III.2.4}$$

Now Hawking suggestion is :

Case a- $\frac{\hat{K}^2\Psi}{\Psi} > o$ ; the wave function describes a family of classical histories with Lorentzian spacetime geometry. These histories describe a static, expanding or contracting universe according to whether $\frac{K\Psi}{\Psi}$ is zero, negative or positive. The Lorentzian histories corresponding to the wave function of last section will be studied in more details in the next section.

case b- $\frac{\hat{K}^2\Psi}{\Psi} < 0$; the wave function describes a family of Euclidean histories of the universe ; or in this region of the superspace the concept of spacetime with Lorentzian 4-geometry cease to exist.

The above interpretation is reasonable because, in the first case the observable K is real and classically the 4-geometry is Lorentzian when the rate of expansion is real, furthermore, the universe is expanding or contracting according to whether the trace of the second fundamental form is negative or positive. In the second case the observable K is imaginary, according to equation III.1.3, this is possible only if either N or t is imaginary, in both cases the metric III.1.1 becomes Euclidean.

Now it is important to note that the wave function calculated from the Euclidean functional integral III.1.21, III.1.25 automatically describes a real quantum state of the universe without the need for analytic continuation. As was noted in the last section this wave function has two different kinds of general behaviour, in the region close to the null lines $Y = |X|$ the wave function is exponential in behaviour leading to negative $\frac{K^2\Psi}{\Psi}$, in this region the geometry of spacetime is Euclidean. On the other hand in the region of validity of equation III.1.25 or III.1.26 the wave function oscillate leading to positive values of $\frac{K^2\Psi}{\Psi}$, in this region the geometry of spacetime is Lorentzian. Numerical calculation confirms the above behaviours, and shows that in general the wave function oscillates in the region $V > 0$ [16]. Only in this region the concept Lorentzian spacetime exist.

Although the above interpretation was very successful in providing a deeper understanding of the origin Lorentzian metric of spacetime. the second piece of the interpretation can not be claimed equally successful , for according to this interpretation, in the region $V < 0$ the 4-geometry of spacetime is Euclidean. The problem is that in ordinary quantum mechanics in all tunneling processes similar thing happens, in latter case there are regions in the configuration space of the tunneling particle where the kinetic energy operator has negative expectation values, yet one dose not speak of spacetime becoming Euclidean in that region of the configuration space. Rather one usually interpret this region as classically forbidden. Now I think that the situation in Q.C is very similar to that of particle tunneling. What happens in the region where the expansion rate is imaginary is very similar to what happens in the region where the particle momentum is imaginary. In the case of tunneling there are no classical trajectories that pass through the forbidden region. By analogy one can says that in the region where the wave function is exponential in behaviour Lorentzian spacetime (the trajectory of the three-space) dose not exist.

# Inflation from Quantum Cosmology and the Origin of Matter Content of the Universe.

According to the discussion of section II.3, in the semiclassical limit the wave function of the universe III.1.26 must have the general form:

$$\Psi(\alpha,\varphi) = Rel\, C(\alpha,\varphi)\, e^{iS(\alpha,\varphi)} \quad \ldots\ \text{III}\cdot 3\cdot 1$$

here C is a slowly varying function, it is given as a path integral over quadratic quantum fluctuations $\delta\alpha, \delta\Phi$ a way from the classical histories defined by Hamilton principle function S. Furthermore, $|C|^2$ represents a conserved probability measure over the set of classical histories [24]. S is a rapidly varying phase satisfying Hamilton-Jacobi equation, this equation can be obtained from the classical Hamiltonian constraint :

$$-\left(\frac{\partial S}{\partial\alpha}\right)^2 + \left(\frac{\partial S}{\partial\varphi}\right)^2 + V(\alpha,\varphi) = 0 \quad \ldots\ \text{III}\cdot 3\cdot 2$$

Of the many solutions of equation III.3.2, H.b.c single out the function S corresponding to the wave function III.1.26, as this wave function takes exactly the form III.3.1.

$$S_{H-H}(\alpha,\varphi) = -\frac{1}{3} e^{3\alpha}\cdot m\cdot |\varphi| \quad \ldots\ \text{III}\cdot 3\cdot 3$$

where the negative sign corresponds to expanding phase according to the discussion of last section.

This action function corresponds to one classical trajectory in the phase space of the system, or to a two parameters family of classical trajectories in the minisuperspace,they are the solutions of the two first order differential equations [24] :

$$\begin{aligned} \Pi_\alpha &= -\frac{e^{3\alpha}}{N}\dot{\alpha} = \frac{\partial S_{H\cdot H}}{\partial\alpha} = -e^{3\alpha} m|\varphi| \\ \Pi_\varphi &= e^{3\alpha}\frac{\dot{\varphi}}{N} = \frac{\partial S_{H\cdot H}}{\partial\varphi} = \mp\frac{1}{3} e^{3\alpha} m \end{aligned} \quad \ldots\ \text{III}\cdot 3\cdot 4$$

Clearly H.b.c provides initial conditions for classical equations of motion , because the general solution of equation III.1.7, together with the Hamiltonian constraint involves three arbitrary constants. Equation III.3.3 can be further simplified by substituting the action function III.3.2 and choosing a coordinate time such that $N = 1$ :

$$\dot{\alpha} = m|\varphi| \quad , \quad |\dot{\varphi}| = -\frac{1}{3}m \quad \cdots \quad \text{III}\cdot 3\cdot 5$$

In the region of validity of equation III.1.26 ( $V > 0, |\Phi| >> 1$), the rate of expansion $\alpha^{\cdot}$ is much larger than the rate of change $\Phi^{\cdot}$, for some time T, $\Phi$ stay almost constant. As it was noticed in section III.1, when $\Phi$ constant the term $m^2\Phi^2$ in the action act as a cosmological constant causing the universe to expand exponentially (inflating). For wide range of the parameters $m, \Phi_0$, T is long enough to solve the problems of horizon, flatness, large scale homogeneity and primordial magnetic monopoles. Clearly, the problem of cosmological constant can not be solved in the frame work of the model of section III.1, because Hawking boundary conditions dose not say any thing about the initial and final value of $\Phi$, and therefore the initial and final value of the vacuum energy $m^2\Phi^2$. This leads one to speculate that the problem of cosmological constant (why the cosmological constant now is many orders of magnitude smaller than the typical values of the vacuum energy, say during inflation) can not be solved in the frame work of Hawking boundary conditions.

Clearly, when $\Phi$ changes appreciably, say when it reach the value$|\Phi|= 1$ the wave function III.1.25, is not a good approximation, the time when this take place may be estimated from III.3.4 :

$$T = 3m^{-1}(|\varphi_0| - 1) \quad \cdots \quad \text{III}\cdot 3\cdot 6$$

At this time the rate of change of $\Phi$ is no more small compared to the expansion rate and equations III.3.4 is no more a valid approximation. Fortunately, by this time the universe is very large in Planck units and the classical equations III.1.7 are valid, when the choice of coordinate $N = 1$ is made these equations take the form of :

$$\ddot{\alpha} + 3\dot{\phi}^2 - e^{-2\alpha} = 0 \quad \cdots \quad \text{III}\cdot 3\cdot 7a$$

$$\ddot{\phi} + 3\dot{\alpha}\dot{\phi} + m^2\phi = 0 \quad \cdots \quad \text{III}\cdot 3\cdot 7b$$

So far the matter field is assumed to be coupled only to gravity, in a realistic particle physics theory, there are many kinds of matter fields which are self interacting, or interact with each other. In order to take into account the interaction of the other fields with $\Phi$, one can add by hand a term proportional to $\dot{\Phi}$ [9]. this will modify the equation of motion for the $\Phi$ :

$$\ddot{\phi} + 3\dot{\alpha}\dot{\phi} + \Gamma\dot{\phi} + m^2\phi^2 = 0 \quad \cdots \quad III \cdot 3 \cdot 8$$

here $\Gamma^{-1}$ is the life time of the scalar particles. When $(\Gamma^{-1} << \dot{\alpha}^{-1}$, the solution of equation III.3.7 is

$$\phi = \phi_0 \exp{-\Gamma t} \cos(mt + \delta) \quad \cdots \quad III \cdot 3 \cdot 9$$

This means that the $\Phi$ field starts oscillating when it cross zero. Quantum mechanically, these oscillations represent a coherent quantum state of the homogeneous mode of the scalar field. During the above oscillations the vacuum energy of the scalar field is being converted into all sort of particles that were coupled to the scaler particle through the $\Gamma$ term. Furthermore, during this time the universe expands as a matter dominated Friedman universe,($a = t^{2/3}$, The reason for this is that the average pressure over a cycle $< 1/2\dot{\Phi}^2 - 1/2m^2\Phi^2 >$ vanishes, and the curvature term in III.3.4a was washed by inflation.

If the interaction between matter fields was sufficiently strong, the matter content of the universe will soon reach thermal equilibrium at a temperature of about :

$$T_H = \left[\frac{30 \cdot m^2 \cdot \phi_0^2}{\pi^2 g_*}\right]^{1/4} \quad \cdots\cdots \quad III \cdot 3 \cdot 10$$

$$g_* = n_b + 7/8\, n_f$$

where $n_b$, $n_f$ are the effective number of degrees of freedom of bosonic and fermionic fields respectively [9].

Clearly, in Q.C the universe started cold, all the energy content is in the vacuum form $(m^2\Phi^2)$, the total a mount of matter energy increased tremendously during inflation, this was on the expense of the negative gravitational potential energy [1]. At the end of inflation almost all the vacuum energy was used to creat the matter content of the universe.This took place during the oscillation of the scalar field, these oscillations can be understood as a decay of the scalar field into all sort of particles to which it was coupled, the decay life time is $(\Gamma^{-1})$. The created particles reach thermal equilibrium at temperature $T_H$, and the universe proceeds along the lines of closed Friedman model of a hot Big Bang.

# IV-Quantum Origin of Large Scale Structures in the Universe

## IV.1-Infinite Dimensional Extension Of the Minisuperspace.

## IV.2- The Wave Function of the Inhomogeneous and Inisotropic Modes.

## IV.3- Quantum Origin of Structures.

# IV.1 Infinite Dimensional Extension of the Minisuperspace Model

In the last chapter we were interested in a minisuperspace approximation of the full superspace. All the inhomogeneous (IH) and anisotropic (ANS) degrees of freedom of gravitational and matter fields were frozen. The question of how reasonable is this approximation can not be answered in the context of the previous model. Specifically the question of quantum stability of the minisuperspace model has to be answered. It is well known that according to the uncertainty principle it is not possible, quantum mechanically, to put any dynamical variable together with its conjugate momentum to zero. This means that the IH,ANS gravitational and matter modes can be small but can not be exactly zero. Now the question of stability can be rephrased as, do these modes remain small for some time during which the homogeneous and isotropic modes (H,IS) played dominant role in the quantum dynamics of the universe. This question was answered by Halliwell and Hawking in [10]. Fortunately the answer was yes, before the H ,IS modes become classical and during inflation the IH and ANS perturbations with short wavelengths remain small in quantum sense, i.e they remain in their ground state. This makes the dynamics of the universe, during the time when the IH, ANS modes were small, is correctly represented by the minisuperspace model of last chapter. Furthermore, this IH, ANS modes will be seen in section IV.3 to account for the origin of structure that we see in the universe.

Following Halliwell and Hawking [10], the minisuperspace model of last chapter is extended to the full infinite dimensional superspace by retaining the IH, ANS degrees of freedom. however, as they did, those modes are treated as a perturbations around the minisuperspace model discussed in chapter III. In this way one is probing a small region of infinite dimensions of the superspace around the minisuperspace . In the perturbed form of the Friedman model with the massive scalar field of the last chapter the metric of spacetime takes the general form II.1.1 with :

$$N = \sigma N_0(t)\left[1 + \delta N(x,t)\right] \quad \text{IV.1.1}$$

$$N_i = \sigma \cdot \left[0 + \delta N_i(x,t)\right] \quad \text{IV.1.2}$$

$$h_{ij} = \sigma^2 e^{2\alpha(t)}\left[\Omega_{ij}(x) + \delta h_{ij}(x,t)\right] \quad \text{IV.1.3}$$

Where $\delta N$, $\delta N_i$, $\delta h_{ij}$ are small IH or ANS perturbations. Similarly the the perturbed scalar field becomes :

$$\Phi = \sigma^{-1}\left[\frac{1}{\sqrt{2\pi^2}}\,\phi(t) + \delta\phi(X,t)\right] \quad \cdots\cdots \quad IV.1.4$$

Where $\delta\Phi$ is inhomogeneous perturbation of the scalar field $\Phi$. Since the background H and IS modes still represent Friedman universe, with 3-spheres spatial sections on which the metric is $\sigma^2 e^{2\alpha}\Omega_{ij}$; one can expand the above perturbations in terms of harmonics on three-sphere $S^3$:

$$\delta h_{ij} = \sum_{nlm}\Big[6^{\frac{1}{2}} a_{nlm}(t)\cdot\frac{1}{3}\Omega_{ij}\, Q^{n}_{lm} + 6^{\frac{1}{2}} b_{nlm}(t)\,(P_{ij})^{n}_{lm} + 2^{\frac{1}{2}} c^{o}_{nlm}(t)\,(S^{o}_{ij})^{n}_{lm}$$

$$+\, 2^{\frac{1}{2}} c^{e}_{nlm}(t)\,(S^{e}_{ij})^{n}_{lm} + 2\, d^{o}_{nlm}(t)\,(G^{o}_{ij})^{n}_{lm} + 2\, d^{e}_{nlm}(t)\,(G^{e}_{ij})^{n}_{lm}\Big] \quad \cdots \quad IV.1.5$$

$$\delta N = 6^{-\frac{1}{2}} \sum_{n,l,m} g_{nlm}(t)\; Q^{n}_{lm} \quad \cdots\cdots \quad IV.1.6$$

$$\delta N_i = e^{\alpha(t)}\Big(\sum_{n,l,m} k_{nlm}(t)\; P^{n}_{i\,lm} + 2^{\frac{1}{2}} j_{nlm}(t)\; S^{n}_{i\,lm} \quad \cdots\cdots \quad IV.1.7$$

$$\delta\Phi = \sum_{n,l,m} f_{nlm}(t)\cdot Q^{n}_{lm} \quad \cdots\cdots \quad IV.1.8$$

Where $Q_n(X)$, $P_n(X)$ are scalar harmonics on $S^3$,

$S^e_n and S^o_n$ are even and odd vector harmonics on $S^3$,

$G^e_n and G^o_n$ are even and odd tensor harmonics on $S^3$.

The properties of these harmonics can be found in [20], as an example we have collected the important properties of scalar harmonics in an appendix. Clearly the superspace has the following degrees of freedom:

$\alpha(t), \varphi(t), a_n(t), b_n(t), c_n(t), d_n(t), f_n(t)$, where n stands for n,l,m, e, o indices. So the superspace has countably infinite number of degrees of freedom; this is due to the fact that the background three-space is compact, it is $S^3$. So far the perturbations are not restricted to be small, but in practice it is not possible to treat $\delta h, \delta N, \delta N_i$ non-perturbatively, since the lagrangians in II.1.7, II.1.8 depend on these perturbations in a complicated manner. This leads Halliwell and Hawking to expand the action II.1.5 for the lagrangians II.1.7, II.1.8 in the perturbations $\delta h_{ij}, \delta N, \delta N_i, \delta\Phi$, keeping terms up to second order in these perturbations, the result is that one can do all the integrals in II.1.5 using the properties of harmonics (essentially the orthonormality of these harmonics). The resulting action may be written as:

$$I = I_0(\alpha, \phi) + \sum_n I_n \quad \cdots\cdots\cdots \quad IV.1.9$$

here $I_0$ is the action for the minisuperspace model III.1.5 without the $\Lambda$ term. While $I_n$ can be further decomposed into:

$$I_n = \int dt \left[ L_g^n + L_m^n \right] \qquad \text{IV.1.10}$$

here$L_g^n$ comes from the nth order term in the expansion of II.1.8 up to second order in $\delta h_{ij}, \delta N, \delta N_i$ and then substituting IV.1.5, IV.1.6, IV.1.7. While $L_m^n$ comes from expanding II.1.7 up to the second order in $\delta h_{ij}, \delta N, \delta N_i, \delta \Phi$, and substituting the expansion of these quantities in terms of harmonics. Clearly, in order to obtain IV.1.10 the spatial integrals over a three-sphere are done using the orthonormality of the harmonics.

With the lagrangians $L_o, L_g, L_m$ one could proceed along the lines of section II.1, finding the momentum conjugate to each of the degrees of freedom. Then one can find the Hamiltonian of the system it takes the following form [20] :

$$H = N_0 \left[ H_{|0} + \sum_n H^n_{|2} + \sum g_n H^n_{|1} \right] + \sum_n \left[ k_n {}^S H^n_{-1} + j_n {}^V H^n_{-1} \right] \qquad \text{IV.1.11}$$

The subscripts 0,1,2 on $H_|$, $H_-$ denote the orders of the quantities in the perturbations, and S, V denote the scalar and vector parts of the shift of the shift parts of the Hamiltonian. Clearly their will be a Hamiltonian constraint for each $g_n$, apart from the one corresponding to $N_0$. Moreover their will be scalar momentum constraint for each $k_n$ and a vector momentum constraint for each $j_n$. At the classical level the scalar momentum constraints imply that not all the scalar modes $(a_n, b_n, f_n)$ are independent. In fact only one of the sets $(a_n), (b_n)(f_n)$, can be independent. The vector momentum constraints imply that the vector modes $c_n$, are purely gauge variables. On the other hand, all tensor modes are truly dynamical modes classically, they represent a gravitational waves, with wave lengths that depend on the order of the mode.

# IV. 2 The Wave Function of the Inhomogeneous and Anisotropic Modes

The quantum dynamics of the perturbations is most easily obtained in Schrödinger picture, in this picture the universe is described by a wave function, $\Psi$ which is a function of all the modes, $\Psi(\alpha, \varphi, a_n, b_n, c_n, d_n, f_n)$. The dynamical equation is obtained by promoting the momentum conjugate to each of the superspace variables into a differential operator with respect to that variable :

$$(\Pi_\alpha, \Pi_\varphi, \Pi_{a_n}, \Pi_{b_n}, \Pi_{c_n}, \Pi_{d_n}, \Pi_{f_n}) \Rightarrow$$

$$\left(-i\frac{\partial}{\partial\alpha}, -i\frac{\partial}{\partial\varphi}, -i\frac{\partial}{\partial a_n}, -i\frac{\partial}{\partial b_n}, -i\frac{\partial}{\partial c_n}, -i\frac{\partial}{\partial d_n}, -i\frac{\partial}{\partial f_n}\right) \quad \ldots \text{IV.2.1}$$

and substituting the resulting operators into the classical Hamiltonian IV.1.11, the resulting Hamiltonian is required to annihilate the quantum state of the universe :

$$\hat{H}\Psi(\alpha, \varphi, a_n, b_n, c_n, d_n, f_n) = 0 \quad \ldots \text{IV.2.2}$$

Now, the sets $(N_0), (g_n), (k_n), (j_n)$, are independent Lagrange multipliers, since both the lapse and the shift functions can be arbitrary chosen. Therefore equation IV.2.1 can be further decomposed into a set of momentum constraints, for each $k_n$ and $j_n$ :

$$^{S}\hat{H}^{n}_{-1}\Psi = 0 \quad \ldots \text{IV.2.3a}$$

$$^{V}\hat{H}^{n}_{-1}\Psi = 0 \quad \ldots \text{IV.2.3b}$$

and a set of W-D equations for each $g_n$ and for $N_0$ :

$$\hat{H}^{n}_{|1}\Psi = 0 \quad \ldots \text{IV.2.4.a}$$

$$(\hat{H}_{|0} + \hat{H}_{|2})\Psi = 0 \quad \ldots \text{IV.2.4b}$$

where $H_{|0}$ is just the Hamiltonian of the minisuperspace model of section III.1, $H_{|2}$, can be further decomposed into :

$$\hat{H}_{12} = \sum_n \hat{H}^n_{12} = \sum_n {}^S\hat{H}^n_{12} + {}^V\hat{H}^n_{12} + {}^T\hat{H}^n_{12} \quad \ldots \text{IV.2.5}$$

where the superscripts S, V, T denote quantities that contains scalar vector and tensor modes respectively.

Here, we shall be interested only in the scalar modes though, the treatment of the tensor modes is very similar [10, 20]. The vector modes $c_n s$ are not interesting because they are pure gauge, they can be given any value by gauge transformations parameterized by $j_n$.

The quantum Hamiltonian of the scalar modes has the following explicit form [10] :

$${}^S\hat{H}^n_{12} = \frac{1}{2} e^{-3\alpha}\Big[-\Big(\frac{1}{2}a_n^2 + 10\frac{(n^2-4)}{(n^2-1)} b_n^2\Big)\frac{\partial^2}{\partial\alpha^2} - \Big(\frac{15}{2}a_n^2 + 6\frac{(n^2-4)}{(n^2-1)} b_n^2\Big)\frac{\partial^2}{\partial\varphi^2}$$

$$+ \frac{\partial^2}{\partial a_n^2} - \frac{(n^2-1)}{n^2-4}\frac{\partial^2}{\partial b_n^2} - \frac{\partial^2}{\partial f_n^2} - 2a_n\cdot\frac{\partial}{\partial a_n}\cdot\frac{\partial}{\partial\alpha} - 8b_n\cdot\frac{\partial}{\partial b_n}\cdot\frac{\partial}{\partial\alpha}$$

$$+ 6a_n\frac{\partial}{\partial f_n}\cdot\frac{\partial}{\partial\varphi} - e^{4\alpha}\Big[\frac{1}{3}(n^2-\tfrac{5}{2})a_n^2 + \frac{(n^2-7)}{3}\frac{(n^2-4)}{n^2-1} b_n^2 + \frac{2}{3}(n^2-4)a_n b_n$$

$$- (n^2-1) f_n^2\Big] + e^{6\alpha} m^2\big[f_n^2 + 6a_n\cdot f_n\cdot\varphi\big] + e^{6\alpha} m^2\varphi^2\Big[\frac{3}{2}a_n^2$$

$$- \frac{6(n^2-4)}{(n^2-1)} b_n^2\Big]\Big] \quad \ldots\ldots \text{IV.2.6}$$

Since the total Hamiltonian constraint of equation IV.2.4b is a sum of contributions from scalar, vector, and tensor modes, and each contribution is a sum over the order of the modes, the wave function of the universe decouple into product over the kinds of the modes and over the orders of the modes, however, each of the perturbations mode continues to be coupled with the background H and IS modes, This leads to the following form of the wave function :

$$\Psi(\alpha,\varphi,a_n,b_n,c_n,d_n,f_n) =$$

$$\Psi_0(\alpha,\varphi)\prod_n {}^S\Psi_n(\alpha,\varphi,a_n,b_n,f_n)\cdot{}^V\Psi_n(\alpha,\varphi,c_n)\cdot{}^T\Psi_n(\alpha,\varphi,d_n) \quad \ldots \text{IV.2.7}$$

In the limit when the H and IS modes becomes semiclassical, the wave function for these modes takes the form of equation III.3.1, here $S(\alpha,\varphi)$ defines a family Lorentzian classical histories of the universe. The parameter along these histories is the Lorentzian cosmological time, it is defined in terms of S as :

$$\left(\frac{\partial}{\partial t}\right) = \nabla S \cdot \nabla = \left(e^{-3\alpha}\right)\left(\frac{\partial S}{\partial \varphi}\cdot\frac{\partial}{\partial \varphi} - \frac{\partial S}{\partial \alpha}\cdot\frac{\partial}{\partial \alpha}\right) \quad \ldots \mathrm{IV}.2.8$$

As it was discussed in section III.2 only in this limit the Lorentzian time becomes defined. The important point is that in this limit it can be shown that each of the perturbation wave functions obey a time dependent Schrödinger [10, 20], this can be done substituting the wave function IV.2.7 into the W-D equation IV.2.4b, and taking into account that $\Psi_0$ has the semiclassical form III.3.1, with S obeying the Hamilton-Jacobi equation III.3.2 .

As an example, the time dependent Schrödinger equations for the scalar modes take the form of :

$$-i\frac{\partial\, {}^{S}\Psi_n(\alpha,\varphi,a_n,b_n,f_n)}{\partial t} = {}^{S}\hat{H}^{n}_{12}\,\Psi(\alpha,\varphi,a_n,b_n,f_n) \quad \ldots\ldots \mathrm{IV}.2.9$$

here ${}^{S}H$ is the Hamiltonian of the scalar modes equation IV.2.6, with the momentum operators corresponding to the homogeneous modes changed into their classical form

$$-i\frac{\partial}{\partial \alpha} \Longrightarrow \frac{\partial S(\alpha,\varphi)}{\partial \alpha} \;, \quad -i\frac{\partial}{\partial \varphi} \Longrightarrow \frac{\partial S(\alpha,\varphi)}{\partial \varphi} \quad \ldots\ldots \mathrm{IV}.2.10$$

This equation can be further simplified if one solve the momentum constraints, IV.2.3a, IV.2.4a for the derivatives with respect to $b_n, f_n$ in terms of the derivative with respect to $a_n$, (the scalar modes dynamical variables are chosen to be the set $(a_n)$. The time dependent Schrödinger equations for the modes $a_n$, for large n becomes [20] :

$$i\frac{\partial\, {}^{S}\Psi(\alpha,\varphi,a_n,0,0)}{\partial t} = {}^{S}\hat{H}_{12}\,{}^{S}\Psi_n(\alpha,\varphi,a_n,0,0)$$
$$= \frac{1}{2}e^{-3\alpha}\Big[-Y^2\frac{\partial^2}{\partial a_n^2} + e^{4\alpha}(n^2-4)\Big[\tfrac{1}{Y^2} - \tfrac{1}{3}e^{4\alpha}\Big(\frac{\partial S}{\partial \alpha}\Big)^{-2}\Big]a_n^2\Big]\,{}^{S}\Psi_0^{\,n}(\alpha,\varphi,a_n) \quad \ldots\ldots \mathrm{IV}.2.11$$

where Y is an abbreviation for the expression :

$$Y = \frac{\partial S}{\partial \alpha}\left(\frac{\partial S}{\partial \varphi}\right)^{-1} \quad \ldots\ldots\ldots \mathrm{IV}.2.12$$

and $\Psi_0^n$ is equal to $\Psi^n$ up to un important phase.

The solution of this equation can be written as a Euclidean path integral :

$$ {}^{S}\psi_0^n(\alpha,\varphi,a_n;T) = \int_C d[a_n]\; e^{-\hat{I}_n(\alpha,\phi,a_n)} \quad \cdots\cdots \text{IV.2.13} $$

where $\hat{I}_n$ is the Euclidean action for the mode $a_n$, it can be calculated from the Hamiltonian of these modes defined by equation IV.2.11 :

$$ \hat{I}_n = \frac{1}{2} e^{3\alpha}\Big[\Big(\frac{1}{\gamma}\cdot\frac{da_n}{dT}\Big)^2 + e^{-2\alpha}(n^2-4)\Big[1/\gamma^2 - \frac{1}{3} e^{4\alpha}\Big(\frac{\partial S}{\partial\alpha}\Big)^{-2}\Big]a_n^2\Big] \quad \cdots\cdots \text{IV.2.14} $$

It is important to note that although $a_n$ are conformal modes, the action $I_n$ is positive,(the second term between brackets is always smaller than the first, except near the time of maximum radius of the background solution [20] ), this is due to the fact that the momentum constraints were already solved and the action contains only truly dynamical variables, furthermore this verifies the claim made in section II.2 that W-D equation can always be made hyperbolic by solving the momentum constraint.

Having wrote the wave function for the scalar modes as Euclidean path integral, one can now use Hawking boundary conditions to single out one quantum state for each mode. It turns out that Hawking boundary conditions constraint all the perturbation modes to be zero at zero Euclidean time of the background modes (T = 0 ). This means that the class C which define the H-H wave function for the scalar mode $a_n$ contains only the histories $a_n(T)$ that were zero at the zero Euclidean time.

$$ {}^{S}\psi_0^n(\alpha,\varphi,a_n;T) = \langle \alpha,\varphi,a_n,T \,/\, -\infty,\varphi_0,0;0\rangle \quad \cdots\cdots \text{IV.2.15} $$

On the other hand equation IV.2.11 for the wave function of a mode $a_n$ can be solved in the limit when the background modes are slowly varying functions of time relative to the time scale set by the frequency of that mode :

$$ \nu_n = e^{-\alpha}\Big[(n^2-4)\Big(1/\gamma^2 - \frac{1}{3} e^{4\alpha}\Big(\frac{\partial S}{\partial\alpha}\Big)^{-2}\Big)\Big]^{\frac{1}{2}} \xrightarrow{n\to\infty} \frac{n\, e^{-\alpha(T)}}{\gamma} \quad \cdots \text{IV.2.16} $$

In this situation the right hand side of equation IV.2.11 can be considered as time independent, the time dependence of the wave function is just a phase, when this phase is factored out the resulting equation becomes the time independent Schrödinger equation for a harmonic oscillator with a time dependent frequency $\nu_n(T)$.

$$\frac{1}{2} e^{-3\alpha}\Big[-\gamma^2 \frac{\partial^2}{\partial a_n^2} + e^{4\alpha}(n^2-4)\big[\tfrac{1}{\gamma^2} -$$

$$\frac{1}{3} e^{4\alpha}\Big(\frac{\partial S}{\partial \alpha}\Big)^{-2}\big] a_n^2\Big] \psi_{0n}^{m} = E_n^m \psi_{0n}^{m} \quad \ldots \text{IV.2.17}$$

Clearly the above approximation is valid for modes with large $\nu_n$, it is called the adiabatic approximation. When the left hand side of equation IV.2.17 is slowly varying functions of time the solution can be found, in fact it is just the quantum states of a harmonic oscillator with frequency $\nu_n$.

The general solution of the time dependent equation may be written as :

$${}^{S}\psi_0^{\,n} = \sum_m \beta_n^m \, e^{i(m+\frac{1}{2})\nu_n t} \, U_n^m(\alpha, \varphi, a_n) \ldots \text{IV.2.18}$$

here $U_n^m$ is the mth quantum state of the mode n, it is just mth quantum state of harmonic oscillator with a frequency $\nu_n$

This means that the path integral solution IV.2.12 can be expanded in terms the stationary states IV.2.18 :

$${}^{S}\psi_0^{\,n}(\alpha, \varphi, a_n; T) = \sum_m \gamma_n^m(\alpha, \varphi) \, e^{-(m+\frac{1}{2})\nu_n T} \, U_n^m \quad \ldots \text{IV.2.19}$$

where in the right hand side the time is Wick rotated to imaginary Euclidean values. As a part of renormalization, the ground state energies must be subtracted from the energy of each of the quantum states of the mode $a_n$, this dose not remove all the divergences of the corresponding field theory but at least it makes the contribution of the IH,ANS modes the total matter energy small, and consistent with treating these modes as a perturbations.

In equation equation IV.2.19 one can again take the adiabatic limit, this amount to taking the limit $(\nu_n \to \infty)$. In this limit it is clear that the first term gives the dominant contribution to the sum on the right hand side. This leads to the most important result of this section that, H-H wave function for modes with very large $\nu_n$ is a ground state. In fact this result is quite general, for example all the above steps can be repeated for the tensor modes and would clearly lead to the same result if the frequency of these modes is an increasing function as the time T decreases, which is indeed the case, all the frequences diverges exponentially as Euclidean time approaches zero. In all the steps we have only used the adiabatic approximation, which means that the frequency of the mode is much larger than the rate of change in the background variables $(\alpha(T), \Phi(T))$. In chapter III it was found that H.b.c lead to inflation in the semiclassical limit of the wave function of H and IS modes. Since we are interested here in precisely the same limit, the time scale for the background variables is set by $(H^{-1} = \dot{\alpha}^{-1})$, where $H^{-1}$ is the size of event horizon of the de Sitter universe during inflation.This means that the adiabatic approximation $(H << \nu_n(T))$ hold for any mode as far as the wave length of the mode dose not exceed the size of the horizon during inflation. This leads to the interpretation of the above result as, all the perturbations modes stated out in their ground state, they continue in their ground state, up to a time during inflation when their frequency was red shifted by the expansion of the background universe up to H. Or in terms of the wave lengths of the modes, one can say that all the modes started out in their ground state and continue in this state up to a time when their wavelengths became larger than the size of the horizon of the background de Sitter spacetime.

This result completely agrees with that of Halliwell and Hawking [10], the derivation of our result is based on more general arguments and can be applied to all the other modes. Often in literature similar arguments is used to define the ground state wave function of a quantum mechanical system using Euclidean path integral [15], their in order to pick the ground state wave function in expression like the right hand side of IV.2.19, one usually take the limit as the Euclidean time approaches infinity. In our case we have taken the limit as the energy or the frequency rather than the Euclidean time to approach infinity.

The above result answer the question addressed in the previous section about the quantum mechanical stability of the minisuperspace model, the answer is positive because the modes started small (in their ground states) and continues to be small, at least up to the time when their wavelength leave the horizon during inflation.

# Quantum Origin of Structures.

The results of last section have lead to a picture of Hawking universe that was very smooth and symmetric when it was very small, yet it contains very small inhomogeneities that can be the seeds for the large scale structures that we see now in our universe like, the galaxies, clusters and superclusters. The outstanding question about the origin of these structures seems to have its answer in the fact that the early universe can not be absolutely homogeneous and isotropic, rather it must contains a small density fluctuations that can evolve to give rise to the structures that we see now [10]. This can not be considered as a satisfactory answer to the above question, because these density perturbations have to be assumed as initial conditions in the early universe.

Inflation makes an other step forward by identifying the seeds for the density fluctuation with the ground state quantum fluctuation of the matter field during inflation [7,15]. The latter gets sufficiently amplified by inflation so that they give rise to a density perturbations that can have the correct amplitude and spectrum to give rise to the structures in our universe, and to be consistent with the large scale isotropy of the microwaves background radiations.

This is still not completely satisfactory, since one would like to understand why the the quantum fluctuations have to be in their ground state during inflation.

Clearly if the universe really obey Hawking boundary conditions, then one can understand why the quantum fluctuations has to start out in their ground state. The reason is that H.b.c constraint the class C which define the quantum states of the modes. They constraint it to contain Euclidean histories of these modes that that were small near the origin of the Euclidean time.

In order to actually prove that the IH, ANS modes in H-H wave functions can give rise to the structures that we see now, one has to study the latter evolution of these modes. In fact it is the scalar modes that give rise to the density perturbations which lead to formation the large scale structures. The later evolution of all the IH, INS modes is governed by a time dependent Schrödinger equations, with the time being the cosmological time defined by semiclassical background modes. In particular the scalar modes evolve according to equation IV.2.11, the initial condition for this equation is the ground state wave function of the corresponding mode.

The evolution of these modes was studied in [10, 20], they have found that this evolution could lead to the correct amplitude of density perturbations provided the mass of the scalar field is about ($10^{14} GeV$). Clearly just as in the case of inflation the perturbations will have the correct spectrum (Zeldovich spectrum) because they originates from a causal microphysical processes (inside the horizon) in a time translation invariant universe, de Sitter spacetime [7].

# V.Comments and Discussion.

Summerizing what is done so far, one can say that in chapter III, it was found that the Lorentzian spacetime is a valid concept only in certain regions of the minisuperspace, essentially the region where the wave function is oscillatory in behavior. In the same region the wave function describes a family of classical Lorentzian universes, each has an initial period of inflation, at the end of inflation the vacuum energy of the matter field was converted into all sort of matter that we see now in our universe. Almost all this energy was borrowed from the negative gravitational energy during inflation [1].

In chapter IV, it was found that all the IH, ANS modes started out in their ground states, since these modes are believed to be responsible for the formation of the large scale structures in the universe, one can say that the large scale structures have their origin in the ground state quantum fluctuations which must be their by the uncertainty principle [1].

This justifies the title of the thesis "Quantum Origin of Hawking Universe", when this universe was small, it was described by a wave function, it contains no matter, the notion of Lorentzian spacetime was not their , the seeds for all the structures were also very small. This lead to the picture of Hawking universe that was very very smooth when it was very small, and it gradually loses its smoothness and order, through the growth of the IH, ANS modes. Hawking has argued that this can explain the thermodynamical arrow of time in our universe, entropy increase because it was initially small [1].

The above successes of Hawking boundary condition, leads one to believe that this boundary condition must have some truth in it, in fact this these boundary conditions may turn out to be a piece of the fundamental theory of gravity. In order to prove or falsify these boundary condition more work has to be done, The boundary conditions must be challenged with other more quantitative observations in our universe, so that one may say that Hawking universe is not only qualitatively similar to our universe but it is also similar in its details. On the theoretical side, the models studied in the text are in many ways a restrictive models. The matter field was always restricted to be a massive scalar field, although such a field appears quite naturally in unification theories, its existence lacks any experimental evidence, so far all the discovered elementary particles are fermions. A second theoretical restriction is the fact that in all the models discussed Einstein's action was used as the fundamental action for gravity, that is, it was used as a basis to obtain quantum gravity. In the current theories for quantum gravity like (superstring theory), it seems that the law energy limit of these theories gives in addition to Einstein's action, a terms which is quadratic in the curvature tensor. Thus, one could argue that if superstring theory gives the correct theory of Q.G, then it is not possible to use Einestein action as a basis for obtaining the quantum theory of gravity. However the models studied here, were used only in situations were any curvature square term is much smaller than Einstein action. This is because the regions in the superspace were the background wave function oscillates the three-geometry is large and the curvature of the universe is small in Planck units. Therefore the corrections added by this terms might affect drastically the behaviours of the wave function in the Euclidean region but not in the Lorentzian region.

The above leads to many ways for a generalization of the models studied here for one can study models with different matter content [15,20], or in which one add curvature square terms to Einstein action [25] . It is an open problem whether H.b.c also in the context of these models can lead to a universe similar to our own universe.

## Appendix

### Scalar harmonics on three-sphere

In this appendix we have collected the important properties of the scalar harmonics on three-sphere all these properties can be found in reference [20], the vector and the tensor harmonics satisfy similar properties [20]. The metric on $S^3$ can be written as :

$$dS^2 = \Omega_{ij}\, dx^i dx^j = d\chi^2 + \sin^2\chi\,(d\theta^2 + \sin^2\theta\, d\phi^2) \tag{.1}$$

In what follows X stands for the three coordinates of a point on a three- sphere, ($|$) stands for covariant differentiation with respect to $\Omega_{ij}$ .

The scalar spherical harmonics $Q^n_{lm}(X)$ are scalar eigenfunctions of the Laplacian operator on $S^3$, they satisfy the eigenvalue equation:

$$Q^{(n)}{}_{|K}{}^{|K} = -(n^2-1)\, Q^{(n)} \qquad n = 1, 2, 3 \cdots \tag{.2}$$

The most general solution of .2, for a given n is a sum of solutions :

$$Q^{(n)}(\chi, \theta, \phi) = \sum_{l=0}^{n-1} \sum_{m=-l}^{+l} A^n_{lm}\, Q^n_{lm}(\chi, \theta, \phi) \tag{.3}$$

here $A^n_{lm}$ are a set of arbitrary constants. The $Q^n_{lm}$ can be expressed in terms of the harmonics on two-sphere $S^2$ as :

$$Q^n_{lm}(\chi, \theta, \phi) = \Pi^n_l(\chi)\, Y_{lm}(\theta, \phi) \tag{.4}$$

where $Y_{lm}$ are the harmonics on two-sphere, and $\Pi_l^n$ are the Fock harmonics. The spherical harmonics $Q_{lm}^n$ make a complete orthogonal set for the expansion of any scalar field on the three-sphere.

Using the scalar harmonics $Q_{lm}^N$ one can construct a vector harmonics $P^i$ in the following way :

$$P_i = \frac{1}{(n^2-1)} Q_{|i} \qquad n = 2, 3, 4, \ldots \tag{.5}$$

It can be shown that these vector harmonics satisfy :

$$P_{i|k}{}^{|k} = -(n^2-3) P_i \,, \quad P_i{}^{|i} = -Q \tag{.6}$$

In the same way one can construct tensor harmonics from the Q's

$$Q_{ij}^{(n)} = \frac{1}{3} \Omega_{ij} Q^{(n)} \qquad n = 1, 2, 3 \tag{.7}$$

$$P_{ij}^{(n)} = \frac{1}{n^2-1} Q^{(n)}_{|ij} + \frac{1}{3} \Omega_{ij} Q^{(n)} \qquad n = 2, 3, 4 \ldots$$

The $P_{ij}$ are traceless, from the equation satisfied by the Q's, these harmonics must satisfy :

$$P_{ij}{}^{|j} = -\frac{2}{3} (n^2-4) P_i \tag{.8}$$

$$P_{ij|k}{}^{|k} = -(n^2-7) P_{ij}$$

$$P_{ij}{}^{|ij} = \frac{2}{3} (n^2-4) Q$$

If one denotes the integration measure on $S^3$ by $d\mu$ :

$$d\mu = d^3x \, (\det \Omega_{ij})^{\frac{1}{2}} = \sin^2\chi \, \sin\theta \, d\chi \cdot d\theta \cdot d\phi \tag{.9}$$

Then, the $Q^n_{lm}$, satisfy the orthonormality condition :

$$\int d\mu \, Q^n_{lm} \, Q^{n'}_{l'm'} = \delta^{nn'} \delta_{ll'} \delta_{mm'} \tag{.10}$$

The orthonormality condition for the Q's leads to the following normalization of the P's :

$$\int d\mu \, (P_i)^n_{lm} \, (P^i)^{n'}_{l'm'} = \frac{1}{n^2-1} \delta^{nn'} \delta_{ll'} \delta_{mm'} \tag{.11}$$

$$\int d\mu \, (P_{ij})^n_{lm} \, (P^{ij})^{n'}_{l'm'} = \frac{2(n^2-4)}{3(n^2-1)} \delta^{nn'} \delta_{ll'} \delta_{mm'}$$

All the spatial integrals needed to obtain the action IV.1.9, as integral over time can be done using the orthonormality of the scalar vector and tensor harmonics.

# Bibliography


[1] (a) S. W. Hawking " The origin of the universe", lecture given on the conference "three hundred years of gravitation" at the university of cambridge (1987). (b) S. W. Hawking " The ground state of the universe", in quantum cosmology workshop, university of camdridge (1987).

[2] see for example, S. W. Hawking and G. F. R. Ellis, The large scale structure of space-time (cambridge university press 1973), and Ref.'s therein.

[3] S. W. Hawking in: Astrophysical Cosmology, Pontificiae Academiae Scientiarum Varia **48**, 563 (1982).

[4] J. B. Hartle and S. W. Hawking, Phys. Rev. D **28**, 2960 (1983).

[5] A. H. Guth, Phys. Rev. D **23**, 347 (1981).

[6] A. D. Linde, Phys. Lett. **108B**, 389 (1982).

[7] (a) A. D. Linde, Rep. Prog. Phys. **47**, 925 (1984). (b) R. H. Brandenberger, Rev. Mod. Phys. **57**, 1 (1985).

[8] (a) L. P. Grishchuk and Y. A. B. Zeldowich, *in quantum structure space and time*, edited by M. J. Duff and C. J. ISham (1981). (b) A. Vilenkin, Phys. Rev. D **27**, 2848 (1983).

[9] P. J. Steinhordt and M. S. Turner, Phys. Rev. D **29**, 2162 (1984).

[10] J. J. Halliewll and S. W. Hawking, Phys. Rev. D **31**, 1777 (19

[11] see for example, C. W. Misner, K. S. Thorne and J. A. Wheeler, *Gravitation*, (Freeman 1973), Ch. 21.

[12] see for example, S. W. Hawking in: General relativity an Einestien centenary survey, edited by S. W. Hawking and W. Israel (1979), and Ref.'s therein.

[13] B. S. Dewitt, Phys. Rev. **160**, 1113 (1967).

[14] S. W. Hawking, Nucl. Phys. B 39, 257 (1984).

[15] S. W. Hawking, in *Relativity Groups and Topology*, Les Houches session XL (1984), North-Holland.

[16] S. W. Hawking and Z. C. Wu, Phys. Lett. **151B**, 15 (1985).

[17] S. W. Hawking and D. N. Page, Nucl. Phys. B **264**, 185 (1986).

[18] A. Vilenkin, Phys. Rev. D **33**, 3560 (1986).

[19] J. V. Narlikar and T. Padmanabhan, *Gravity, Gauge Theories and Quantum Cosmology*, (Dordrecht, Reidel 1986), p. 468.

[20] J. J. Halliwell, Ph.D thesis, Univ. of Cambridge (1986).

[21] D. N. Page, in *Quantum concepts in spacetime*, edited by C. Isham and R. Penrose (1986), Univ of Oxford.

[22] J. Louko, Semiclassical path measure and factor ordering in quantu cosmology, DAMTP preprint (1987).

[23] J. B. Hartle, in *Gravitation in astrophysics*, Corgese NATO Advanced Summer institute (1986).

[24] J. J. Halliwell, Correlations in the wave function of the univers DAMTP preprint (1987).

[25] S. W. Hawking and J. C. Luttrell, Nucl. Phys. B 47, 250 (1984