ISAS - INTERNATIONAL SCHOOL
FOR ADVANCED STUDIES

Thesis submitted for the degree

of

"MAGISTER PHILOSOPHIAE"

The rôle of infinite-dimensional groups in 2-D critical phenomena

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Academic Year 1986/87

TRIESTE

SISSA - SCUOLA INTERNAZIONALE SUPERIORE DI STUDI AVANZATI

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The rôle of infinite-dimensional groups in 2-D critical phenomena[1]

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[1]Magister philosophiae thesis

To F.

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ACKNOWLEDGEMENTS

I would like to thank Prof. R.Jengo for his advice, supervision and, especially, patience. He has been a constant source of encouragement and ideas, not always readily exploited.

A particular thank is due to Prof. G.Sotkov, for his constant attention, collaboration and, above all, friendship.

I would like also to thank Dr. M.Staniskhov for our scientific collaboration and his hospitality in Sofia.

I have benefitted from interesting and stimulating conversations on conformal quantum field theory and related topics with E.Gava, T. Jayaraman, J.B.Zuber, S.Shenker and V.Fateev.

My friend and collegue G.Falqui deserves special thanks for his often ironic criticism and for his great patience in reading the manuscript, written in my personal English style.

Introduction

Recently there has been progress in understanding two-dimensional critical phenomena based on the existence of infinite dimensional groups of symmetry [10,11,19,13,14,15,32,33,31].In the past, the renormalization group was a good paradigma for explaning the occurance of scale invariant point: they are just the fixed points under renormalization group transformations [1,2,3,4,5,7,6].But this approach works only when it is given a specific model and it does not provide general procedure to classify all possible fixed points.

The observation,due to Polyakov [8], that the fluctuations of the order parameter fields at the second-order phase transition have conformal as well as scale invariance, has shed light on this problem. In fact, the task of classification of all types of universal critical behaviours can be formulated as the problem of finding the conformal invariant solutions of quantum field theory.The Polyakov suggestion [9] is to construct these solutions by the requirement of conformal invariance with the ansatz that the field operators form an algebra [12].

d=2 is a prominent place to pursue this aim since the 2-dimensional conformal group is infinite-dimensional one: its algebra is the Virasoro algebra [42]

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}$$

generated by the stress-energy tensor and the fields which enter in the operator algebra can be classified completely in terms of its representations.

In this way an infinite set of exactly solvable conformal models was found [10].Each of these models is parametrized by two relatively prime numbers p,q and corresponds to a value of the central charge c of the Virasoro algebra

$$c = 1 - \frac{6(p-q)}{pq}$$

The analysis of ref [11] shows that only the 'principal series' ($q = p+1, p \geq 3$) satisfies the unitarity condition.In these minimal unitarity models,for each value of the central charge c, there exists a finite set of fields which close the algebra, whose anomalous dimensions are rational numbers.The first models of this series are the Ising model, Z_3 Potts model [13],in addition to the corresponding tricritical models [11,15].The detailed discussion of all these subjects is developed in chapter 2.We shall see also that there exists a interesting Coulomb gas representation for these models [14,18] which gives an integral representation of the multipoint correlation functions and allows the computation of the structure constants of the operator algebra.

In 2-dimension one can use the powerful methods of complex analysis and a crucial property for the solvability of the models is the factorization of the amplitudes in an analytic part and an antianalytic one.In this way one can restrict to one of them (say the analytic part),solve it and finally combine with the other to obtain the physical quantities.But it needs a principle for extracting the correct combination:this can be the crossing symmetry for the amplitudes [10] or equivalently one can require a modular invariant theory [20]

The idea, due to Cardy, is to consider the system into a finite region of the complex plane,in particular a torus,and to demand the independence of the physical results from the particular parametrization chosen to describe this region.In [20] it is shown that the requirement of modular invariance for the partition function is sufficient for extracting the operatorial content of the theory.Furthemore,analysing the dependence on the boundary conditions it is possible to investigate the internal symmetry of the minimal models [25,26].All these questions will be discussed in the chapter 3.

However,the minimal models certanly do not exhaust the class of all the solutions of conformally field theories for d=2.For instance, a series with superconformal symmetry [19,15,16] and,also, conformal invariant solutions of the Wess-Zumino model [38,35,36,37] are known to exist. In these cases the conformally invariant field theory possesses a more general symmetry which is generated by local currents .Following the analysis by Zamolodchikov [31] in chapter 4 we discuss these additional symmetries associated with conformal fields of integer and half- integer spin .It turns out that in the case of higher spin ($s \geq \frac{5}{2}$) they do not form Lie algebra but more general algebra with quadratic relations among the generators.Moreover this structure can be enlarged to include conformal field theory with fractional spin,the so called *parafermionic currents* [32,33,34]. These parafermionic currents form closed operator algebra which is associated in a natural way with commutative groups,the simplest ones are the cyclic group Z_N [32,33,34,62,63,64,65,67,66].

The core of this work, to which chapter 5 is devoted, is the analysis of one of the most interesting extensions of the Virasoro algebra, namely the superconformal algebra. Supersymmetry, first invented in string theory [43,44] ,gives remarkable relations between the bosonic and fermionic sectors of quantum system.It is very attractive that there exist supersymmetric statistical models. The N=1 superconformal algebra

$$\begin{aligned}[L_n, L_m] &= (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m} \\ \{G_\alpha, G_\beta\} &= 2L_{\alpha+\beta} + \frac{c}{3}(\alpha^2 - \frac{1}{4})\delta_{\alpha+\beta} \\ [L_n, G_\alpha] &= (\frac{n}{2} - \alpha)G_{n+\alpha}\end{aligned}$$

has two different sectors,depending if α and β are either half-integer or integer:in the first case it is called the Neveu-Schwarz sector [44], in the second case the Ramond sector [43].

As it happens for the Virasoro algebra, the unitarity constraint gives a discrete series of models for central charge $c \leq \frac{3}{2}$ and correspondingly there exist finite sets of operators which close the algebra [19].

It turns out that the construction of the Neveu-Schwarz (NS) sector requires only slight modification of the Virasoro algebra methods.It consists only in using superfields instead of the ordinary fields. The task of a satisfactory description of the Ramond sector is far more suble. The problems are connected with double valuedness of the fermionic partner of the stress-energy tensor,$G(z)$, (anomalous dimension $\Delta_G = \frac{3}{2}$) in the presence of the Ramond fields (in the ref [19,15] called spin fields).It is this non-locality which makes the usual methods non-effective .The solution of this problem, discussed by the author in the ref [60,61] consists in the construction of a Coulomb gas representation for the Ramond fields.It occurs that all the elements of this construction are written in terms of the critical Ising model variables (order-disorder fields,free Majorana fermion) and of a free bosonic field.In this way it is possible to calculate the multipoint function of the order parameters and the structure constants of the operator algebra thus getting the full description of the models. Finally, to obtain a local theory including both sectors,one has to do a projection: since the spin fields are non-local with respect to the fermionic part of the superfields and the fermionic parity $\Gamma = (-1)^F$ is multiplicatively conserved, we can project on the Hilbert space with even fermionic number.This operation selects the bosonic part of the superfields and a subset of the spin fields.The theory so constructed is a new local bosonic field theory,called spin model,and is a direct generalization of the construction of the superstring from the Neveu-Schwarz Ramond models [48].

The first member of the supersymmetric discrete series, has $c = \frac{7}{10}$, and corresponds to the universality class of tricritical Ising model [15], which describes phenomena like superconductor mixture of He_3 and He_4, [59] or the adsorbed phase of the He on kripton surface [58] .These are the first examples of experimentally observed supersymmetric critical system.

Interesting features are also present in the second model of the superconformal series,with $c = 1$. There are some reasons for subspecting that the $c = 1$ supersymmetric model is a gaussian one [75,70,71,72,73,74,54]. First,the gaussian model and its equivalent (Ashkin-Teller [68] and the 8-vertex model [69,54]) are the only known models with $c = 1$.Second, the presence of a field with anomalous dimension 1 resembles the U(1) current of the gaussian model.In addition this U(1) supercurrent can be combined with the super-stress energy tensor to produce N=2 superconformal symmetry [76,77,80,81,82,83,84,85].The fields content of this model can be conveniently organized in N=2 supermultiplet.This new symmetry simplifies the computation of the physical correlators and gives new insight on the operator content of this system.

However, despite of the great deal of work in progress in various research groups, there still exist many open problem. Among these, the most relevant to the author, are

- The understanding of the region in which the central charge is not quantized [95]

- How to go out from the critical point,i.e. how to use the huge informations present in the conformal regime for analyzing the system in the neighbourhood of point in which correlation lenght is infinite [90,91]

- What is the meaning,(if any),of considering these conformal statistical models on a general Riemann surface [94,92,93]

- The clarification of the role of the parafermionic current in critical phenomena

To summarize,the organization of this work is as follows.In chapter 1 we discuss the basic facts about conformal invariance and Virasoro algebra.The chapter 2 deals with the modular invariance in minimal models.In chapter 3 it is discussed the infinite additional symmetries present in the conformal quantum field theories.In chapter 4 we formulate the superconformal formalism and give the original solution of the problem connected with the Ramond sector of theese models. In the appendices one can find the basic definitions in the critical phenomena,the detailed discussion of the fermionic representation of the Ising model and some mathematical properties of hypergeometric functions.

Chapter 1

Conformal Invariance-Virasoro algebra

1.1 Conformal invariance and bootstrap program in critical phenomena

When a physical system is near its critical point the most important parameter is the correlation lenght ξ which is much larger than all other microscopic lenghts.These are irrelevant scales in the description of the critical regime so that many systems, with the same internal symmetry and the same dimensionality of the space, appear identical: the principle of universality says that the singular thermodynamical quantities are related to the correlation lenght by a set of critical exponents (App.1) [1,2,3,4].At the critical point ξ is infinite:the statistical system can be described in terms of a massless euclidian quantum field theory since it has lost its only relevant scale,becoming globally scale invariant.Under a dilatation of the unit of lenght

$$a \to \lambda a$$

the fields transform as

$$\Phi_{\Delta_i} \to \lambda^{\Delta_i} \Phi_{\Delta_i}$$

where Δ_i are called the anomalous dimensions and their computation is the most important problem of the theory since these quantities determine the critical exponents.

The bootstrap program , as formulated by Polyakov [9] consists in solving the theory using two hypothesis: conformal invariance and operator algebra.

Conformal transformations are local scale transformations, i.e.they preserve angles but change locally the lenghts.A local field theory, invariant under translations, rotations and dilatations, automatically is also invariant under the conformal group [40,41]. For proving this, let's consider a transformation of coordinates

$$\delta x^\mu = \xi^\mu(x) \tag{1.1}$$

with the condition that the metric transforms in the following way

$$(ds')^2 = \rho(x)(ds)^2 \simeq (1 + 2p(x))(ds)^2 \tag{1.2}$$

Since

$$(ds')^2 = (ds)^2 + 2\xi^\mu \xi^\nu \partial_\mu \xi_\nu \tag{1.3}$$

The (1.3) implies

$$\partial_\mu \xi_\nu = p(x) g_{\mu\nu} + \epsilon_{\mu\nu} \tag{1.4}$$

where $g_{\mu\nu}$ is the metric tensor and $\epsilon_{\mu\nu}$ is an antisymmetric one.Considering the eq. (1.4) with $\mu \leftrightarrow \nu$ and summing the two equations,we obtain

$$\partial_\mu \xi_\nu + \partial_\nu \xi_\mu = 2p(x) g_{\mu\nu} \tag{1.5}$$

and taking the trace,finally we obtain the conformal Killing equations

$$\partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \frac{2}{d} g_{\mu\nu} \partial_\alpha \xi_\alpha = 0 \tag{1.6}$$

(d is the dimensionality of the space)

Under the symmetries above mentioned we have the following equations for the stress-energy tensor

$$\partial_\mu \Theta_{\mu\nu} = 0 \qquad (transl.inv.) \tag{1.7}$$

$$\Theta_{\mu\nu} = \Theta_{\nu\mu} \qquad (rotat.inv) \tag{1.8}$$

$$\Theta^\mu_\mu = 0 \qquad (scaling\ inv.) \tag{1.9}$$

Then,using the Killing vector we can construct the following extra conserved currents

$$j^\mu = \Theta^{\mu\nu}\xi_\nu \tag{1.10}$$

Their number is equal to the number of the linearly independent solutions of the Killing equation 1.6: in the case $d > 2$ they are $\frac{(d+1)(d+2)}{2}$ and generate the following transformations

$$x'_i = a_{ik}x_k + b_i \tag{1.11}$$

$$x'_i = \lambda x_i \tag{1.12}$$

$$\frac{x'_i}{x'^2_i} = \frac{x_i}{x^2} + a_i \tag{1.13}$$

The firsts are rotations and translations, the second scale transformations and the last ones are the so called special conformal transformations.They form the group SO(d+1,1).When $d = 2$ the conformal group can be enlarged to become infinite-dimensional since it coincides with the analytic mappings of the complex plane.This topic will be investigated in the next paragraph.

The hypothesis of conformal invariance consists in classifying the fields entering the theory in terms of representation of the conformal group. For what concern the completeness of the operator algebra, we make the hypothesis that there exists a denumerable complete set of fields $A_n(x)$, eigenvectors of the dilatation operator with eigenvalues Δ_n

$$A_n(\lambda x) \longmapsto \lambda^{-\Delta_n} A_n(x) \tag{1.14}$$

in such a way that any local function can be expanded with respect this basis

$$O(x) = \sum_{n=0}^{\infty} \mu_n A_n(x) \tag{1.15}$$

For instance, in a free field theory a basis is given by : $\phi^n(x)$; with

$$\Delta_n = n\Delta_1 = n\frac{d-2}{2}$$

In the general case the various Δ_n are not expressed in a simple way in terms of a minimal dimension Δ_1.The expansion 1.15 is intended to hold in a weak sense, i.e. when it is inserted into a correlation function

$$< O(x)B(y) >= \sum_{n=0}^{\infty} \mu_n < A_n(x)B(y) > \tag{1.16}$$

The 1.15 is a local algebra of the fields.

Furthermore, we make the hypothesis that we can construct a bilocal algebra taking the product of two fields A(x), B(y) in the limit $x \simeq y$: in this case we can consider this product as a local object and by the completeness of the A_n,one has the following expansion

$$A(x)B(y) \simeq \sum_{n=0}^{\infty} \mu_n(x,y)A_n(y) \tag{1.17}$$

Specializing this expansion on the basis one gets the following algebra

$$A_i(x)A_j(y) \simeq \sum_k C_{ij}^k(\mid x-y \mid)A_k(y) \tag{1.18}$$

Then, by conformal invariance the $C_{ij}^k(\mid x-y \mid)$ are homogeneous functions of $\mid x-y \mid$ of degree $\Delta_k - \Delta_i - \Delta_j$ so they are fixed up numerical structure constants.The most rigid requirement, considered as the main dynamical principle of this approach, is associativity of the operator algebra (1.17).This gives a system of non-linear equations which in principle determine the anomalous dimensions and the numerical value of the structure constants. This is the bootstrap program [9].However for $d > 2$ these equations are too complicated to be solved exactly,while in d=2 the situation changes drastically,allowing their exact solvability.

To end this section we discuss the deep consequences of the conformal invariance on the correlation functions.Polyakov [8] had pointed out that conformal invariance fixes the 2 and 3-point functions and severe constraints the other correlators.In particular the 2-point function of fields with different dimensions is zero (orthogonality relation).In fact the correlator $K_{AB}(x-y) = < A(x)B(y) >$ has the following behaviour

$$K_{AB} \sim \mid x-y \mid^{-(\Delta_A+\Delta_B)}$$

Under the special conformal transformation we have

$$\delta \mid \vec{x}-\vec{y} \mid = \frac{1}{2}[(\vec{a}\cdot\vec{x}) + (\vec{a}\cdot\vec{y})] \mid \vec{x}-\vec{y} \mid \tag{1.19}$$

Then

$$\delta K_{AB}(x-y) = -\frac{(\Delta_A+\Delta_B)}{2}[(\vec{a}\cdot\vec{x}) + (\vec{a}\cdot\vec{y})]K_{AB}(x-y) \tag{1.20}$$

But we can compute this variation using the conformal transformations of the fields

$$\begin{aligned} \delta A(x) &= -\Delta_A(\vec{a}\cdot\vec{x})A(x) \\ \delta B(y) &= -\Delta_B(\vec{a}\cdot\vec{y})B(y) \end{aligned}$$

so

$$\delta K_{AB}(x-y) = [\Delta_a(\vec{a}\cdot\vec{x}) + \Delta(\vec{a}\cdot\vec{y})]K_{AB}(x-y) \tag{1.21}$$

Comparing with (1.20) we see that K_{AB} is different from zero only when $\Delta_A = \Delta_B$

Considering the 3-point function of scalar fields

$$K_3(x_1, x_2, x_3) =< A_1(x_1)A_2(x_2)A_3(x_3) >,$$

it depends on the differencies $\mid x_1 - x_2 \mid, \mid x_1 - x_3 \mid, \mid x_2 - x_3 \mid$. Conformal invariance implies

$$\frac{1}{2}\sum_{i<k} x_{ik}\frac{\partial K_3}{\partial x_{ik}}[(\vec{a}\cdot\vec{x_i}) + (\vec{a}\cdot\vec{x_k})] = -\sum_{n=1}^{3}\Delta_n(\vec{a}\cdot\vec{x_n})K_3 \tag{1.22}$$

We have three differential equations whose solution is

$$K_3(x_1, x_2, x_3) = C_{123} \mid x_{12} \mid^{\Delta_3-\Delta_2-\Delta_1} \mid x_{13} \mid^{\Delta_2-\Delta_1-\Delta_3} \mid x_{23} \mid^{\Delta_1-\Delta_2-\Delta_3} \tag{1.23}$$

The constant C_{123} is an unknown parameter and coincides with the structure constant of the OPE of the three fields A_i (we use the orthogonality condition).

The 4-point function K_4 depends on the six relative distances $x_{ik}(i,k = 1, ..., 4)$ .The conformal equations

$$\frac{1}{2}\sum_{i\neq k}\frac{\partial \ln K_4}{\partial \ln x_{ik}} = -\Delta_i \tag{1.24}$$

are not sufficient to determine completely the solution since we have four equations and six variables .The solution of the above equations can be determined up an arbitrary function of two variables .Putting K_4 in the form

$$\begin{aligned} K_4 &= \prod_{ik} x_{ik}^{\frac{1}{3}\Delta-\Delta_i-\Delta_k} f \\ \Delta &= \sum_{i=1}^{4}\Delta_i \end{aligned} \tag{1.25}$$

The function f satisfies the following equation

$$\sum_{i\neq k} \frac{\partial \ln f}{\partial \ln x_{ik}} = 0 \tag{1.26}$$

i.e. f is a conformal invariant function with zero dimension.From the six relative distances we can construct two invariant parameters (anharmonic ratios)

$$\xi = \frac{x_{12}x_{34}}{x_{13}x_{24}}; \qquad \eta = \frac{x_{12}x_{34}}{x_{14}x_{23}} \tag{1.27}$$

Then f is, at this stage, an arbitrary function of ξ,η, fixed by further physical requirement.

1.2 Stress-energy tensor and Ward identity in 2-dimension

Consider a correlation function

$$< A_1(x_1)\cdots A_n(x_n) >= \frac{\int \mathcal{D}\mathcal{A} A_1\cdots A_n e^{-S(A)}}{\int \mathcal{D}\mathcal{A} e^{-S(A)}} \tag{1.28}$$

and perform an infinitesimal coordinate transformation

$$x^a \rightarrow x^a + \epsilon^a(x) \tag{1.29}$$

As is well known in QFT, it holds the following equation (Ward identity)

$$\begin{aligned}\sum_{k=1}^{n} < A_1(x_1)..\delta A_k(x_k)..A_n(x_n) > + \frac{1}{2}\int d\vec{\xi}\delta g_{ij} < T_{ij}(\xi)A_1..A_n >= \\ \sum_{k=1}^{n} < A_1(x_1)..\delta A_k(x_k)..A_n(x_n) > + \int d\vec{\xi}\partial_i\epsilon_j < T_{ij}(\xi)A_1..A_n >= 0\end{aligned} \tag{1.30}$$

where T_{ij} is the stress-energy tensor

$$T_{ij} = \frac{\delta S}{\delta g_{ij}} \tag{1.31}$$

and we have used the infinitesimal change of the metric

$$\delta g_{ij} = \partial_i\epsilon_j + \partial_j\epsilon_i \tag{1.32}$$

In the eq.(1.30) the variations of the fields are local expressions formed with $\epsilon(\xi)$ ant its derivatives taken at $\xi = x_k$.If $\epsilon(\xi)$ is a function with compact support,integrating by part and taking the functional derivative respect $\epsilon(\xi)$ we get

$$\partial_i < T_{ij}(\xi)A_1(x_1)\cdots A_n(x_n) >= 0 \tag{1.33}$$

everywhere but at the points $\xi = x_1 \ldots x_n$ where there are δ singularities.This is the quantum part of the conservation law for the stress-energy tensor.It is easy to show,using the eq. (1.30),the symmetry of the theory under rotation and scaling transformations.In fact, for the rotations

$\partial_i \epsilon_j = \lambda_{ij}$,an antisymmetric tensor.We have invariance if the stress-energy tensor is symmetric, $T_{ij} = T_{ji}$.If we consider dilatations $\partial_i \epsilon_j = \mu \delta_{ij}$ and the last term in the eq. (1.30) is zero if T_{ij} is traceless.

In two dimension the conformal group is $SO(3,1) \simeq SL(2,C)$.It is convenient to use the complex coordinates

$$\begin{aligned} z = & \ x + iy \\ \overline{z} = & \ x - iy \end{aligned}$$

In this basis the transformations of the global conformal group are the Moebius transformations

$$z \rightarrow w(z) = \frac{az+b}{cz+d}; \qquad ad - bc = 1 \tag{1.34}$$

and correspondingly for $\overline{z}$.Infinitesimally we have

$$dz = \epsilon_{-1} + \epsilon_0 z + \epsilon_1 z^2 \tag{1.35}$$

But ,as we said before, on the complex plane the conformal transformatios degenerate to all analytic mapping of the plane

$$(z, \overline{z}) \rightarrow (f(z), \overline{f(z)}) \tag{1.36}$$

with non vanishing derivative.

The variation of the metric is

$$dz d\overline{z} \rightarrow \mid f'(z) \mid^2 dz d\overline{z} \tag{1.37}$$

in which the angles are preserved and the lenghts dilatated by the factor $\mid f'(z) \mid$.Except for the Moebius transformations, which are biunivocal transformations of the Riemann sphere, all the others are no longer 1-1 mapping and can have singularities as poles and cuts. The infinitesimal coordinate transformation

$$z \rightarrow z + \epsilon(z) \tag{1.38}$$

can be expanded as a Laurent serie

$$\epsilon(z) = \sum_{-\infty}^{\infty} \epsilon_n z^{n+1} \tag{1.39}$$

and the Lie algebra of the group Γ is given by the differential operators

$$l_n = z^{n+1}\frac{d}{dz} \tag{1.40}$$

with

$$[l_n, l_m] = (n-m)l_{n+m} \tag{1.41}$$

The generators of the global conformal group are identified with $l_{\pm 1}, l_0$.

The quantum extension of this Lie group is obtained considering the stress- energy tensor [10]. With the symmetric, traceless stress-energy tensor we can form only two independent components

$$T(z,\overline{z}) = \frac{1}{2}(T_{11} - iT_{12}) \tag{1.42}$$

$$\overline{T}(z,\overline{z}) = \frac{1}{2}(T_{11} + iT_{12}) \tag{1.43}$$

Using (1.33) it is easy to see that the correlation functions of these components satisfy the Cauchy-Riemann equations

$$\partial_{\overline{z}} < T(z,\overline{z})A_1(z_1,\overline{z}_1 \cdots A_n(z_n,\overline{z}_n) > = 0 \tag{1.44}$$

$$\partial_z < \overline{T}(z,\overline{z})A_1(z_1,\overline{z}_1) \cdots A_n(z_n,\overline{z}_n) > = 0 \tag{1.45}$$

everywhere except the point $z = z_k$, where there are δ singularities. This means that the function

$$< T(z)A_1(z,\overline{z} \cdots A_n(z,\overline{z}_n) > \tag{1.46}$$

is an analytic function of z, regular everywhere but at the points $z = z_k$ where it has poles, the order and the residues being determined by the conformal properties of the fields [10].Analyticity of T(z) and correspondingly antianalyticity of $\overline{T(z)}$ means that z and $\overline{z}$ parts decouple. We can perform the analysis considering just one one of them, say z-part, suppressing the $\overline{z}$ dependence of correlation functions. However, the dependence on the latter coordinate must be restored when extracting the physical content of the theory.This reduction to 1-dimensional relation, instead of 2D one, is crucial since it leads to the integrability of the theory.

The Ward identity can be rewritten in the following way

$$\delta < A_1 \cdots A_n >= \oint_C d\xi \epsilon(\xi) < T(\xi)A_1 \cdots A_n > \tag{1.47}$$

where C is a contour enclosing all singular points $z_k, k = 1, \dots n$ Equivalently

$$\delta A_j(z) = \oint_C d\xi \epsilon(\xi) < T(\xi) A_j(z) > \tag{1.48}$$

This equation shows that T(z) is the generator of the conformal group Γ in the quantum theory.

The transformation laws for the fields will be discussed in the next paragraph: here we want to consider the conformal properties of T(z) and to recover the algebra.T(z) is a field with dimension 2; from euclidean invariance, i.e. translation invariance and correct scaling dimension, one finds the following operator product expansions (OPE)

$$T(z)T(w) \simeq \frac{c}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w} + \cdots \tag{1.49}$$

$$\overline{T}(z)\overline{T}(w) \simeq \frac{c}{2(\overline{z}-\overline{w})^4} + \frac{2\overline{T(w)}}{(\overline{z}-\overline{w})^2} + \frac{\partial_{\overline{w}} \overline{T(w)}}{\overline{z}-\overline{w}} + \cdots \tag{1.50}$$

$$\overline{T(z)}T(w) \simeq \textit{non sing.} \tag{1.51}$$

The field T(z) also satisfies a condition of regularity at infinity

$$T(z) \sim \frac{1}{z^4} \tag{1.52}$$

Inserting the operator expansion (1.51) into (1.48) we find

$$\delta T = \epsilon(z)T' + 2\epsilon'(z)T + \frac{1}{12}c\epsilon'''(z) \tag{1.53}$$

the last term is the Schwinger term, that one has to insert for having consistent quantum field theory . The finite transformation of T(z) is

$$T(z) \to \left(\frac{d\xi}{dz}\right)^2 T(\xi) + \frac{1}{12}c\{\xi, z\} \tag{1.54}$$

where $\{\xi, z\}$ is the Schwartz derivative

$$\{\xi, z\} = \left(\frac{\frac{d^3\xi}{dz^3}}{\frac{d\xi}{dz}}\right) - \frac{3}{2}\left(\frac{\frac{d^2\xi}{dz^2}}{\frac{d\xi}{dz}}\right)^2 \tag{1.55}$$

with the following composition law

$$\{w,z\} = \left(\frac{d\xi}{dz}\right)^2 \{w,\xi\} + \{\xi,z\} \tag{1.56}$$

Note that for the Moebius transformation the Schwartz derivative is zero. The parameter c is a real, positive number called central charge.

The central charge c corresponds to an anomaly of the theory, in the following sense.By translation invariance, the expectation value of T is zero, $< T(z) >= 0$. Under the transformation

$$z \to z + \epsilon(z)$$

we have

$$T \to T + \delta T \tag{1.57}$$

where

$$\delta T = \oint d\xi \epsilon(\xi) < T(\xi)T(z) >= \frac{c}{2}\oint \epsilon(\xi)\frac{1}{(z-w)^4} = \frac{c}{12}\epsilon'''(z) \tag{1.58}$$

We see that for transformations not belonging to the global conformal mapping, eq (1.35), we have a term which breaks the condition $< T(z) >= 0$, i.e. there is an anomaly for the diffeomorphism.

It is possible to give an interesting interpretation of the central charge, considering a conformal map of the plane into a strip of width L with periodic boundary condition (cylinder) [30]

$$z = \exp\left[\frac{2\pi i w}{L}\right] \tag{1.59}$$

Using the transformation law of T we get

$$[T(w)]_{cyl} = -\left(\frac{2\pi}{L}\right)^2 [T_{pl}(z)z^2 - \frac{c}{24}] \tag{1.60}$$

Taking the expectation value, we obtain

$$< T(w) >_{cyl}= \left(\frac{2\pi}{L}\right)^2 \frac{c}{24} \tag{1.61}$$

i.e. there is a density of energy connected with the finite geometry of the system (Casimir effect).

It is useful to introduce the operator L_n as the coefficients of Laurent serie of T(z)

$$T(z) = \sum_{-\infty}^{\infty} \frac{L_n}{z^{n+2}} \tag{1.62}$$

$$L_n = \oint dz z^{n+1} T(z) \tag{1.63}$$

The operator product expansion (1.51) can be expressed in terms of the algebra of $L_n(\overline{L}_n)$

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0} \tag{1.64}$$

$$[\overline{L}_n, \overline{L}_m] = (n-m)\overline{L}_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0} \tag{1.65}$$

$$[L_n, \overline{L}_m] = 0 \tag{1.66}$$

This algebra of L_n is the central extension of the algebra (1.41) and it is called Virasoro algebra [42].As the first one, it contains a subalgebra SL(2,C) formed by $L_{\pm 1}, L_0$. L_{-1} generates translation and L_0 infinitesimal dilatation.If we use the coordinate τ, σ defined by

$$z = e^{i\sigma + \tau} \tag{1.67}$$

we see that L_0 is a 'time' generator, playing a rôle of Hamiltonian: $\tau = -\infty$ corresponds to the origin $z = 0$ and $\tau \to +\infty$ to $|z| = \infty$. A 'radial' quantization scheme has been pursued in the ref. [39]. They have shown that it is possible to introduce a vacuum state $|0>$ as the ground state of L_0.It has to satisfy the equations

$$L_n |0> = 0, \qquad n \geq -1 \tag{1.68}$$

for having regularity of the stress-energy tensor at $z = 0$.These equations are nothing but conformal invariance of the vacuum. By the regularity of T(z) at infinity, we obtain for the 'out' vacuum state

$$<0| L_n = 0, \qquad n \leq 1 \tag{1.69}$$

However, from the reality of T(z) in the Minkowski space, one obtain

$$L_n^\dagger = L_{-n} \tag{1.70}$$

One can use this equation with the commutation relations of L_n for computing correlation function of T(z), as for example

$$< T(z)T(0) >= \sum_{n=2}^{\infty} < 0 \mid L_n L_{-2} \mid 0 > \frac{1}{z^{n+2}} = \frac{c}{2z^4} \tag{1.71}$$

In the following we shall fix our attention only on the L_n's algebra, since the $\overline{L}_n$'s algebra is just a copy of the first one.

1.3 Primary fields, conformal families, unitary representations

A prominent role in the classification of the representations of the Virasoro algebra is played by the so called *primary fields*. They are fields which transform as 'tensorial' quantities of degree Δ_n

$$\phi_n(z)(dz)^{\Delta_n} = \phi_n(\xi)(d\xi)^{\Delta_n} \tag{1.72}$$

under the change of variable $\xi = \xi(z)$.Infinitesimally we have

$$\begin{aligned} \phi_n(z) &= \phi_n(\xi(z))\left(\frac{d\xi}{dz}\right)^{\Delta_n} \simeq \phi(1+\epsilon(z))\left(1+\Delta_n\frac{d\epsilon}{dz}\right) \simeq \\ &\simeq \left(\phi_n(z)+\epsilon(z)\frac{d\phi_n}{dz}\right)\left(1+\Delta_n\frac{d\epsilon}{dz}\right) = \\ &\simeq \phi_n + \Delta_n\frac{d\epsilon}{dz}\phi_n + \epsilon\frac{d\phi_n}{dz} \end{aligned} \tag{1.73}$$

so

$$\delta\phi_n = \left(\Delta_n\frac{d\epsilon}{dz} + \epsilon(z)\frac{d}{dz}\right)\phi_n \tag{1.74}$$

Using the (1.48), this equation is equivalent to the following OPE

$$T(z)\phi_n(w) = \frac{\Delta_n}{(z-w)^2} + \frac{1}{z-w}\partial\phi_n(w) \tag{1.75}$$

i.e. to the following commutation relations with the L_m's

$$[L_m, \phi_n] = z^{m+1}\partial\phi_n(z) + \Delta_n(m+1)z^m\phi_n(z) \tag{1.76}$$

The theory of the representations of the infinite-conformal group can be carried on in a way that resembles the theory of the representations of compact Lie algebras [86]. So, as easily seen from (1.64) the lowering operators for L_0 are the $L_n, n > 0$ .An eigenvector of L_0, annihilated by all the lowering operators is called *highest-weight vector*(HWV). Explicity, it satisfies

$$L_0 \mid \Delta > \; = \; \Delta \mid \Delta > \tag{1.77}$$

$$L_n \mid \Delta > \; = \; 0 \qquad n > 0 \tag{1.78}$$

The vacuum $\mid 0>$ is a HWV because it has the lowest eigenvalue for the 'hamiltonian' L_0.There is a one-to-one correspondence between the conformal primary fields and the HWV's.Namely, the state

$$\mid \Delta_n >= \phi_{\Delta_n}(0) \mid 0> \tag{1.79}$$

is a HWV, with eigenvalue Δ_n, as it can be seen using the eq. (1.76)

The space of states is a sum of irreducible representations of the algebra of L_n, each generated from one of the HWV's.A representation space of the Virasoro algebra is built from a HWV by applying the raising operators, i.e. the $L_{-n}, n \geq 1$.A state is said to be in the k-th level if its L_0 eigenvalue is $\Delta + k$.The k-th level is spanned by the vectors

$$L_{-j_1} \ldots L_{-j_m} \phi_{\Delta}(0) \mid 0> \tag{1.80}$$

with

$$j_1 \geq j_2 \geq \cdots j_m; \qquad \sum_{i=1}^{m} j_i = k \tag{1.81}$$

There are P(n) such states, where P(n) is the number of ways of writing an integer n as a sum of positive integers. These higher level states correspond to operators of higher scaling dimension obtained by the OPE of stress-energy tensor with the primary field ϕ_{Δ_n}: they are called descendent fields [10].These fields, together with the primary field ϕ_{Δ_n} constitute a *conformal family*$[\phi_n]$. A distinguished feature of a conformal family is that under conformal transformations every member of the family is mapped into a representative of the same conformal family. So, each conformal family is a representation of the conformal algebra.

The fundamental requirement to be fulfilled by any physically sensible QFT is unitarity, which, together with the spectral condition, is equivalent to reflection positivity of statistical system (i.e.EQFT) [11].This means that the inner product on the space of states is positive definite, i.e. the state space is a Hilbert space. The inner product of any two states in the basis (1.80) can be computed using the conjugation rule of L_n, i.e. $L_n^\dagger = L_{-n}$ and the commutation relations. The analysis is enormously simplified by the fact that we can impose the positivity constraint level by level because different levels have different L_0 eigenvalues and hence are orthogonal.The unitarity constraint is that the matrix of inner product should have no negative eigenvalues.

Let us shetch some of the consequences of this requirement of unitarity. At level 1 there is a single state $\mid 1 >= L_{-1} \mid \Delta >$ and

$$< 1 \mid 1 >= 2\Delta$$

Then positivity at level 1 imposes $\Delta > 0$. At level n the state

$$\mid n >= L_{-n} \mid \Delta >$$

has

$$< n \mid n >= 2n\Delta + \frac{c}{12} n(n^2 - 1)$$

If $c < 0$ this inner product is negative for large n.Hence we can limit our attention to the region $c \geq 0, \Delta > 0$.

An instructive example is at level 2: there are two states $L_{-1}^2 \mid \Delta >$ and $L_{-2} \mid \Delta >$ and the inner product matrix is

$$M_2(\Delta, c) = \begin{pmatrix} 4\Delta(2\Delta + 1) & 6\Delta \\ 6\Delta & 4\Delta + \frac{1}{2}c \end{pmatrix} \tag{1.82}$$

the determinant is

$$det M_2(\Delta, c) = 4\Delta(8\Delta^2 + \Delta(c-5) + \frac{1}{2}c) = 4\Delta(\Delta - \Delta_{1,2})(\Delta - \Delta_{2,1}) \tag{1.83}$$

where $\Delta_{1,2}, \Delta_{2,1}$ are the roots of the quadratic equation. If $\Delta = \Delta_{1,2}$ or $\Delta = \Delta_{2,1}$ there exists some linear combination of the two states at level 2 which has zero norm.

The correct combination is

$$\mid null >= \left(L_{-2} - \frac{3\Delta}{2(2\Delta + 1)} L_{-1}^2 \right) \mid \Delta > \tag{1.84}$$

This state has zero norm when $\Delta = \Delta_{1,2}$ or $\Delta = \Delta_{2,1}$.This state is orthogonal to both level-two states generated from the HWV $\mid \Delta >$.Since states at different levels are orthogonal to each other, $\mid null >$ is orthogonal to all the states of the space. Then it is a zero vector in the Hilbert space.This fact allows to derive a differential equation for the correlation function of the field $\phi_\Delta(z)$, when $\Delta = \Delta_{1,2}, \Delta_{2,1}$ [10]. In fact, one insert the combination (1.84) into correlation functions and since this is a nul vector we have

$$< 0 \mid \phi_n(z_n) \dots \phi_2(z_2) \left(L_{-2} - \frac{3\Delta}{2(2\Delta + 1)} L_{-1}^2 \right) \phi_\Delta(0) \mid 0 >= 0 \tag{1.85}$$

Using the commutation relation (1.76) ,global conformal invariance and the fact that L_{-2}, L_{-1} annihilate the state $< 0 \mid$, we deduce the following differential equation

$$\mathcal{D} < 0 \mid \phi_n(z_n) \ldots \phi_1(z_1) \mid 0 >= 0 \tag{1.86}$$

where $\mathcal{D}$ is the differential operator

$$\mathcal{D} = \frac{3\Delta}{2(2\Delta+1)} \frac{\partial^2}{\partial z_1^2} - \sum_{i=2}^{n} \left[\frac{\Delta_i}{(z_1 - z_i)^2} - \frac{1}{z_1 - z_i} \frac{\partial}{\partial z_i} \right] \tag{1.87}$$

The analysis for an arbitrary level n has been performed by Kac [87], who has given a formula for the determinant of the matrix of inner products of the states:

$$det M_n(\Delta, c) = const \prod_{pq \leq n} [\Delta - \Delta_{p,q}(c)]^{P(n-pq)} \tag{1.88}$$

where p,q range over the positive integers and the quantities $\Delta_{p,q}$ are defined by

$$\Delta_{p,q} = \frac{[p(m+1) - qm]^2 - 1}{4m(m+1)} = \Delta_{m-p,m+1-q} \tag{1.89}$$

$$c = 1 - \frac{6}{m(m+1)} \tag{1.90}$$

and P(i) is the classical partition function for the integers defined by

$$\prod_{i=1}^{\infty} \frac{1}{(1-t^n)} = \sum_{n=0}^{\infty} P(n) t^n \tag{1.91}$$

Friedan, Qiu and Shenker [11], have shown that unitarity restricts the allowed values of c, for $c < 1$ to the (1.90) with

$$m = 3, 4, 5 \ldots \tag{1.92}$$

$$p \leq m \tag{1.93}$$

$$q \leq m - 1. \tag{1.94}$$

In these unitarity series, the allowed anomalous dimensions are rational numbers given by. (1.90).The conformal fields $\phi_{p,q}$ are operators of dimensions $\Delta_{p,q}$ and it exists a null state at level pq generated from HWV $\mid \Delta_{p,q} >$

because there is a zero eigenvalue in Kac's determinant at that level. The null state at that level is of the form

$$\sum_{\sum_i \lambda_i = pq} a_{\lambda_1\lambda_2..} L_{-1}^{\lambda_1} L_{-2}^{\lambda_2} \ldots \mid \Delta_{p,q} >$$

where $a_{\lambda_1\lambda_2..}$ are constants which can be determined by the condition that the null state is orthogonal to all the other states at that level.Using the procedure explained for level 2, one can show that the correlation functions of the field $\phi_{p,q}$ satisfy differential equation of order pq.

The finite set of fields, identified by their anomalous dimensions, eq.(1.90), form a close operatorial algebra [10].In fact, by the characteristic equations associated to the differential equations, we get the following OPE (in ref.[10] called 'fusion rules')

$$[\phi_{p_1,q_1}][\phi_{p_2,q_2}] \sim \sum_{k_1=|p_1-q_1|+1}^{p_1+q_1-1} \sum_{k_2=|p_2-q_2|+1}^{p_2+q_2-1} C_{(pqk)}[\phi_{k_1,k_2}] \tag{1.95}$$

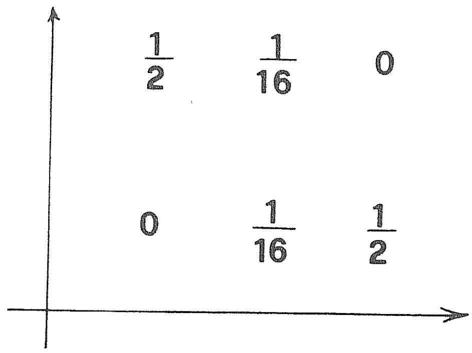
where the constant $C_{(pqk)}$ are the only nonzero operator product coefficients and the coordinate dependence is determined by dimensional arguments.The constant $C_{(pqk)}$ of the primary fields coincides with the one which appears in the 3-point function

$$< \phi_p \phi_q \phi_k >$$

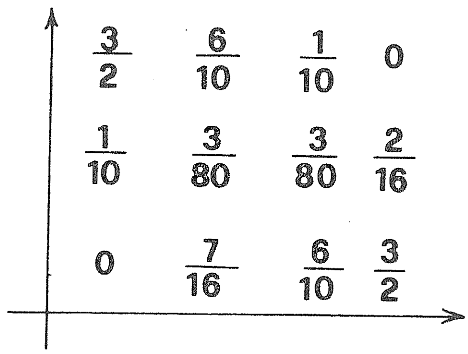
and those of the descendent fields are determined by recursive equations [10].

These OPE coefficients, together with the dimensions of the conformal fields, are the parameters that specify the conformal theory. We shall see later, using the Coulomb gas approach [14] how to compute the structure constants $C_{(pqk)}$.

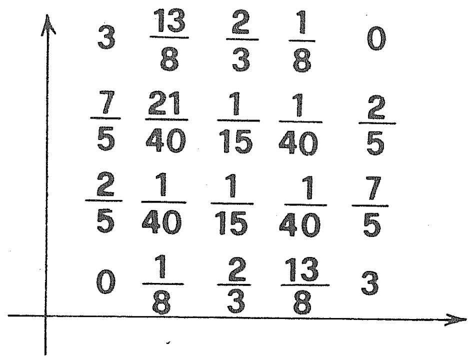
It is useful to introduce the *conformal grid* of the different models, identified by the value of c: this is the diagram of dimensions in which the vertical an horizontal axes correspond to the value of the parameters p and q in the Kac's formula.The first ones are



$n = 3, \qquad c = \frac{1}{2}$



$m = 4, \qquad c = \frac{7}{10}$



$m = 5, \qquad c = \frac{4}{5}$

They were identified respectively with the Ising model ($c = \frac{1}{2}$) [10], the tricritical Ising model ($c = \frac{7}{10}$) [11,19,15] and the 3-state Potts model [65], as we discuss in the next section. As a final remark note that the physical anomalous dimension involve also transformation properties under the 'antianalytic' algebra generated by the $\overline{L}$'s. Namely, taking in account the full dependence on z and $\overline{z}$ of ϕ, from the 2-point function

$$< \phi_{\Delta,\overline{\Delta}}(re^{i\theta})\phi(0) >= r^{-2(\Delta+\overline{\Delta})}e^{-2i\theta(\Delta-\overline{\Delta})}$$

we see that the field $\phi_{\Delta,\overline{\Delta}}$ has scaling dimension $d = \Delta + \overline{\Delta}$ and spin $s = \Delta - \overline{\Delta}$

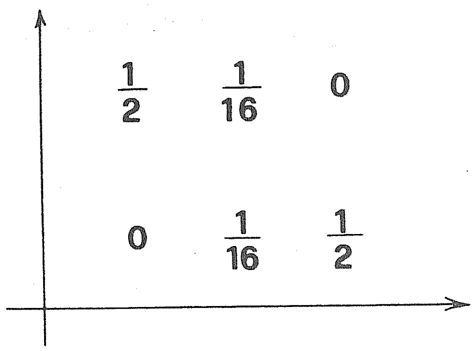
The operator with dimension zero specifies the identity family.

1.4 Classification of the minimal models

We have seen that a special set of theories with $c < 1$ are picked out by the unitarity condition.The corresponding scaling operators $\phi_{p,q}$ have scaling dimensions given by the Kac's formula and are degenerated at level pq.The identification of each model is based on the knowledge of the critical exponents and the operator algebra which one can recover in the field theoretic limit of the statistical system, as it is illustrated in the following examples.

1.4.1 Ising model

The first minimal model, $m = 3$, has central charge $c = \frac{1}{2}$ and the conformal grid



As it is shown in the appendix (A.2), the two-dimensional Ising model is equivalent to the theory of free massless Majorana fermion described in the continuum limit by the Lagrangian

$$\mathcal{L} = \psi \frac{\partial}{\partial \bar{z}} \psi + \overline{\psi} \frac{\partial}{\partial z} \overline{\psi} \tag{1.96}$$

The equation of motions are

$$\frac{\partial}{\partial \bar{z}} \psi = \frac{\partial}{\partial z} \overline{\psi} = 0 \tag{1.97}$$

Then ψ is an analytic field and $\overline{\psi}$ an antianalytic one. Their 2-point correlation functions are

$$< \psi(z_1)\psi(z_2) >= \frac{1}{z_1 - z_2} \qquad\qquad < \overline{\psi(z_1)\psi(z_2)} >= \frac{1}{\overline{z}_1 - \overline{z}_2} \tag{1.98}$$

From the expression of the stress-energy tensor

$$T(z) = -\frac{1}{2} : \psi(z)\frac{\partial}{\partial z}\psi(z) : \tag{1.99}$$

and its 2-point function

$$< T(z_1)T(z_2) >= \frac{1}{4}\frac{1}{(z_1 - z_2)^4} \tag{1.100}$$

we recover that $c = \frac{1}{2}$

The dimension $\Delta_{1,2} = \Delta_{3,1} = \frac{1}{2}$ is correctly identified with the Onsager spinor of the Ising model.The correlation functions involving the degenerate field $\psi(z)$ satisfy the differential equation [10]

$$\left\{\frac{3}{4}\frac{\partial^2}{\partial z^2} - \sum_{i=1}^{n}\frac{\Delta_i}{(z - z_i)^2} - \sum_{i=1}^{n}\frac{1}{z - z_i}\frac{\partial}{\partial z_i}\right\} < \psi(z)\phi_1(z_1)\dots\phi_n(z_n) >= 0 \tag{1.101}$$

where $\phi_i(z)$ are arbitrary primary fields (local by themselves but not necessarly local with respect to $\psi(z)$).In particular, it can be seen that the function

$$< \psi(z_1)\psi(z_2)\dots\psi(z_n) >$$

which can be computed using the Wick-theorem, satisfies the eq.(1.101).

The order-parameter fields of the Ising model is the magnetization operator $\sigma(z,\overline{z})$; however, there exists also the disorder-parameter field $\mu(z,\overline{z})$.These fields have zero spin, i.e.

$$\Delta_\sigma = \overline{\Delta}_\sigma, \qquad\qquad \Delta_\mu = \overline{\Delta}_\mu$$

and the Kramers-Wannier symmetry fixes their anomalous dimensions to be equal. Explicity

$$\Delta_\sigma = \Delta_\mu = \frac{1}{16} \tag{1.102}$$

as it can be seen from the 2-point correlation function of the magnetization fields

$$< \sigma(z,\bar{z})\sigma(0,0) >= \frac{1}{\mid z \mid^{\frac{1}{4}}} \tag{1.103}$$

known from the exact solution of the Ising model.

The fields σ, μ, ψ are not mutually local : in fact the correlation function

$$< \psi(z)\sigma(z_1)\ldots\sigma(z_n)\mu(z_{n+1})\ldots\mu(z_m) > \tag{1.104}$$

is a double-valued analytic function of z which acquires a phase factor (-1) after the analytic continuation around any singular point z_k [12]. This is equivalent to the following operator product expansions

$$\psi(\xi)\sigma(z) = \frac{1}{\sqrt{\xi - z}}\mu(z) \tag{1.105}$$

$$\psi(\xi)\mu(z) = \frac{1}{\sqrt{\xi - z}}\sigma(z) \tag{1.106}$$

The remaining relevant operator of Ising model is the energy operator

$$\Phi_{\frac{1}{2},\frac{1}{2}} = \epsilon =: \overline{\psi}\psi :\sim \sum_{i=1}^{2} \sigma(x)\sigma(x+e_i) \tag{1.107}$$

From its 2-point function $< \epsilon(x)\epsilon(0) >$, we extract the critical exponent of specific heat, $\alpha = 0$.

The complete operator algebra is

$$\begin{array}{lll} [\psi][\psi] \sim 1 & [\psi][\overline{\psi}] \sim [\epsilon] & [\epsilon][\epsilon] \sim 1 \\ (\epsilon][\psi] \sim [\overline{\psi}] & [\epsilon][\overline{\psi}] \sim [\psi] & [\psi][\mu] \sim [\sigma] \\ (\psi][\sigma] \sim [\mu] & [\epsilon][\sigma] \sim [\sigma] & [\sigma][\sigma] \sim 1 + [\epsilon] \end{array}$$

We can extract three different sets of fields for describing the Ising model

$$\{A_1\} = \{1, \psi, \overline{\psi}, \epsilon\} \tag{1.108}$$

$$\{A_2\} = \{1, \sigma, \epsilon\} \tag{1.109}$$

$$\{A_3\} = \{1, \mu, \epsilon\} \tag{1.110}$$

Each of these sets forms a closed operator algebra and all the fields entering one set are mutually local whereas the fields entering different sets are in general non-local with respect to each-other.

We can compute the 4-point correlation function of the spin fields using the null-vector method. With the parametrization

$$< \sigma(\infty)\sigma(1)\sigma(z)\sigma(0) >= z^{-\frac{1}{8}}(z-1)^{-\frac{1}{8}}f(z) \tag{1.111}$$

(we can fix 3 points by the SL(2,C) invariance) the function f(z) satisfies the hypergeometric differential equation

$$\left\{ z(z-1)\frac{d^2}{dz^2} + \left(\frac{1}{2} - z\right)\frac{d}{dz} + \frac{1}{16} \right\} f(z) = 0 \tag{1.112}$$

whose solutions are elementary hypergeometric functions

$$f(z) = \begin{cases} F(\frac{1}{4}, -\frac{1}{4}, \frac{1}{2}, z) & = \sqrt{\frac{1+\sqrt{1-z}}{2}} \\ \\ \sqrt{z}F(\frac{3}{4}, \frac{1}{4}, \frac{3}{2}, z) & = \sqrt{\frac{1-\sqrt{1-z}}{2}} \end{cases} \tag{1.113}$$

We will continue the discussion on the Ising model in the chapter devoted to the Ramond sector of the superconformal models, since there exists a deep connection between the two problems.

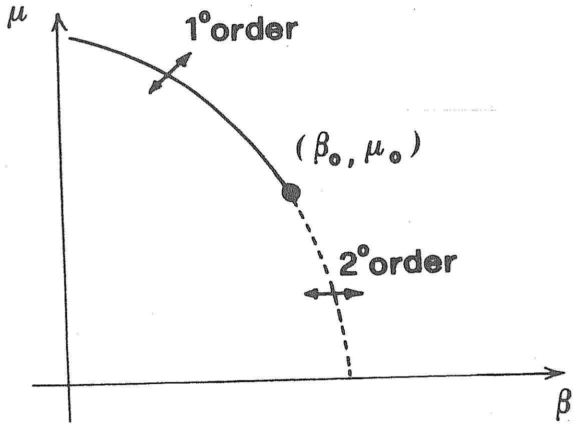
1.4.2 Tricritical Ising model

The next model has $m = 4, c = \frac{7}{10}$.This can be identified with the *tricritical Ising model* (TIM).Let us shortly remind some basic notions and properties of TIM.It is a Ising model with vacancies, which means that some of the sites of the lattice are not occupied by the Ising spins $\sigma_i(\sigma_i = \pm 1)$.So we can write the lattice hamiltonian of this system in the form

$$\mathcal{H} = -\beta \sum_{<ij>} \sigma_i \sigma_j t_i t_j - \mu \sum_i t_i \tag{1.114}$$

where as usually the summation $< ij >$ is over all the nearest neighbors, β is the inverse temperature, μ is the chimical potential and the vacancy density t takes values 0,1.

The model presents a tricritical point (β_0, μ_0) in which a critical line of second order phase transition meets a line of first order jumps (fig.3)



This model parametrize the behaviour of superconducting mixture of He_3 and He_4 [59] and the adsorbed phase of He on Kripton surface [58].

At its tricritical point (β_0, μ_0) TIM can be described by the following fields: energy density $\epsilon(z, \bar{z})$ with $(\Delta, \overline{\Delta}) = (\frac{1}{10}, \frac{1}{10})$, vacancy operator $t(z, \bar{z})$ with $(\frac{3}{5}, \frac{3}{5})$, magnetization (or order parameter) $\sigma(z, \bar{z})$ with $(\frac{3}{80}, \frac{3}{80})$ and the so-called subleading magnetization $\alpha(z, \bar{z})$ with anomalous dimensions $(\frac{7}{16}, \frac{7}{16})$.

The Z_2 symmetry group splits the operators in two sectors: odds and even.To the first belong the spin operator and to the latter the energy operator and the vacancy operator.But, the peculiar feature of TIM is the presence of a higher symmetry, i.e. N=1 supersymmetry [19,15]. We postpone the detail discussion of this model to the next chapters, devoted to additional symmetries of minimal models (chap.3) and to superconformal invariance (chap.4).

1.4.3 Other models

The third model of the minimal series, $m = 5, c = \frac{4}{5}$, was identified with the continuum limit of the 3-state Potts model [13].The partition function of the 2-d Z_3 model is

$$\begin{aligned} Z(\beta) &= \sum_{\{\sigma\}} \exp\left\{\beta \sum_{<i,j>} \frac{1}{2}(\sigma_i\overline{\sigma}_j + \overline{\sigma}_i\sigma_j)\right\} \\ &= \sum_{\{\sigma\}} \exp\left\{\beta \sum_{i,j} \cos(\phi_i - \phi_j)\right\} \end{aligned} \tag{1.115}$$

Its discrete spin variables $\{\sigma = \exp(i\phi), \overline{\sigma} = \exp(-i\phi); \phi = 0, \pm\frac{2}{3}\pi\}$ interacts only with nearest neighbours.This model is known to be self-dual, the same as the 2-d Ising model.At its self-dual point $\beta_c = \frac{2}{3}\ln(\sqrt{3}+1)$ it undergoes a second order phase transition [54,55].We refer to the original paper of Dotsenko [13] for the detailed discussion of this model.

The next model, with $m = 6, c = \frac{6}{7}$ is the tricritical 3-state Potts model and the remaining ones, $m \geq 7$ were shown by Huse [57] to fall into a class of statistical models recently solved by Andrew, Baxter and Forrester [56].These are the Restricted Solid on Solid models (RSOS).We do not enter here into the explanation of these systems, but refer at once to the original literature.

1.5 Conformal Coulomb gas

It was known from long time that many 2-d spin models can be describe as a Coulomb gas system with logarithmic interaction [70,73,74]. In some sense, the Coulomb gas picture represents a general language to discuss problems in 2-d statistical mechanics.Dotsenko and Fateev [14] introduce the Feigin-Fuchs representation [18] in the Virasoro case to compute the n-point correlation function of the fields as a special average of the appropriate vertex operator.

In this approach the basic conformal operator is

$$\mathcal{V} = \exp[i\alpha\phi(z,\bar{z})] \tag{1.116}$$

where $\phi(z,\bar{z})$ is a free bosonic field whose action is

$$S = 2\int \frac{dzd\bar{z}}{2\pi}\partial_z\phi\partial_{\bar{z}}\phi \tag{1.117}$$

and therefore with propagator

$$< \phi(z_1,\bar{z}_1)\phi(z_2,\bar{z}_2) >= -2\ln\frac{\mid z_{12}\mid}{R} \tag{1.118}$$

where R is an infrared cut-off.Then, for the n point function of the vertices $\mathcal{V}_{\alpha_i}(z,\bar{z})$, with real charge α_i, we get

$$\begin{aligned} < \prod_{i=1}^{n}\mathcal{V}_{\alpha_i}(z_i,\bar{z}_i) > &= \int \mathcal{D}\phi\prod_{i=1}^{n} e^{i\alpha_i\phi(z_i,\bar{z}_i)}e^{-S(\phi)} = \\ &= \left(\frac{a}{R}\right)^{(\sum_i \alpha_i)^2}\prod_{i<j=2}\mid\frac{z_{ij}}{a}\mid^{2\alpha_i\alpha_j} \end{aligned} \tag{1.119}$$

(a is an ultraviolet cut-off).

In order to eliminate the cut-offs dependence of the function (1.119) in the limit $R\to\infty, a\to 0$, we have to introduce the renormalized vertex

$$V_\alpha(z) = \lim_{a\to 0} a^{-\frac{\alpha^2}{2}}e^{i\alpha\phi} =: e^{i\alpha\phi}: \tag{1.120}$$

and to impose the neutrality condition

$$\sum_{i=1}^{N}\alpha_i = 0 \tag{1.121}$$

Then, restricted to the analytic part, the vertex propagator takes the form

$$< V_\alpha(z_1)V_{-\alpha}(z_2) >= (z_{12})^{-\alpha^2} \tag{1.122}$$

and consequently the fields $V_{\pm\alpha}$ have dimension $\Delta(\pm\alpha) = \frac{\alpha^2}{2}$ From the action (1.117), one gets the expression of the stress-energy tensor

$$T(z) = -\frac{1}{2} : (\partial\phi)^2 : \tag{1.123}$$

All the conformal properties of the vertex V_α can be verified explicity using the Wick-theorem and the propagator (1.118).

The defect of this naive construction is that the central charge is $c = 1$ as it can be seen, for example, from the two point function of T

$$< T(z_1)T(z_2) >= \frac{1}{2(z_1 - z_2)^4} \tag{1.124}$$

The construction which admits an anomalous central charge, i.e. the one given in (??), is based on the modified partition function

$$Z(-2\alpha_0) = \int \mathcal{D}\phi \lim_{|z|\to\infty} e^{-2i\alpha_0(\phi(z)+\overline{\phi(z)})+2i\alpha_0 \ln|z|} e^{-S(\phi)} \tag{1.125}$$

in which it is inserted a charge $-2\alpha_0$ to infinity [14].

The correlation function calculated with the new partition function should satisfy the following neutrality condition

$$\sum_{i=1}^{N} \alpha_i = 2\alpha_0 \tag{1.126}$$

Because of the logarithmic interaction, putting a charge at infinity has the effect to renormalize the dimensions of the fields, since the new non-zero two point function is

$$\ll V_\alpha(z_1)V_{2\alpha_0-\alpha}(z_2) \gg= (z_{12})^{-\alpha(\alpha-2\alpha_0)} \tag{1.127}$$

where $\ll \ldots \gg$ means average with the partition function $Z(-2\alpha_0)$.Therefore the vertices V_α and $V_{2\alpha_0-\alpha}$ have the equal dimension

$$\Delta(\alpha) = \Delta(2\alpha_0 - \alpha) = \frac{\alpha(\alpha - 2\alpha_0)}{2} \tag{1.128}$$

The insertion of a charge at infinity corresponds to a change of the boundary condition and of the transformation properties of the field $\phi(z)$ whioh is now

$$\phi(z) \rightarrow \phi(f(z)) + 2i\alpha_0 f'(z) \tag{1.129}$$

and it implies also a redefinition of the expression of the stress-energy tensor

$$T = -\frac{1}{2} : (\partial\phi)^2 : +i\alpha_0\partial^2\phi \tag{1.130}$$

The central charge is given by

$$c = 1 - 12\alpha_0^2 \tag{1.131}$$

The Kac formula for the dimensions of the fields is obtained analysing the 4-point function of the field ϕ_Δ represented by the vertex operator V_α.Naively this can be expressed in three different way

$$\ll V_\alpha V_\alpha V_\alpha V_\alpha \gg \tag{1.132}$$

$$\ll V_\alpha V_\alpha V_\alpha V_{2\alpha_0-\alpha} \gg \tag{1.133}$$

$$\ll V_\alpha V_\alpha V_{2\alpha_0-\alpha} V_{2\alpha_0-\alpha} \gg \tag{1.134}$$

but it is impossible to satisfy the neutrality condition, eq (1.126), unless $\alpha = \frac{1}{2}\alpha_0$ in the first case, or $\alpha = 0$ in the second case, while the third correlation function is zero if $\alpha_0 \neq 0$

The solution of this problem consists in introducing a non-local operator $J_\pm$ with anomalous dimension $\Delta = 0$ (in order to not change the conformal properties of correlators) but with a non-zero charge: pictorially the rôle of this operator is to screen the extra charge present in the correlation function of the vertex operators [14]. Their expressions are

$$J_\pm = \oint dz V_{\alpha\pm}(z) \tag{1.135}$$

where the charges $\alpha_\pm$ are determined by the condition $\Delta(J) = 0$, i.e.

$$\frac{\alpha(\alpha - 2\alpha_0)}{2} = 1 \tag{1.136}$$

$$\alpha_\pm = \alpha_0 \pm \sqrt{\alpha_0^2 + 2} \tag{1.137}$$

Let us look at the 4-point correlator 1.133.This function has a charge surplus equal to 2α and this can be cancelled by adding $J_{\pm}$ if α is quantized

$$\alpha = \alpha_{n,m} = \frac{1}{2}\{[1-n]\alpha_- + [1-m]\alpha_+\} \tag{1.138}$$

Inserting the (1.138) into the formula given the anomalous dimension one gets

$$\Delta_{n,m} = \frac{\alpha_{n,m}(\alpha_{n,m} - 2\alpha_0)}{2} = \frac{[(n\alpha_- + m\alpha_+)^2 - (\alpha_- + \alpha_-)^2]}{8} \tag{1.139}$$

which is precisely the Kac's formula expressed in terms of α_- and α_+.

Thus we can compute 4-point function by means of

$$< \prod_{i=1}^{4} \phi_\Delta(z_i) >= \oint \prod_{i=1}^{n-1} dv_i \oint \prod_{j=1}^{m-1} dw_j \ll V_\alpha(z_1) V_\alpha(z_2)$$
$$V_\alpha(z_3) V_{2\alpha_0-\alpha}(z_4) \prod_{i=1}^{n-1} V_{\alpha_-}(v_i) \prod_{j=1}^{m-1} V_{\alpha_+}(w_j) \gg \tag{1.140}$$

Now we want to sketch our method [61] to recover the fusion rules of the theory i.e. the formula (1.95). A more detailed discussion will be given (for the superconformal case) in chapter 5. The problem is, given two primary fields $\phi_{n_1,m_1}, \phi_{n_2,m_2}$, to find which fields $\phi_{x,y}$ enter their OPE. To find them, it is necessary to analyse the non-zero 3-point functions. There exists three different ways to construct them in terms of vertex representation, depending which field is represented by the vertex with the conjugate charge $2\alpha_0 - \alpha$. For each case the number of screening operators which one has to put to screen the extra charge is different, and hence

$$< \phi_{n_1,m_1}(z_1)\phi_{n_2,m_2}(z_2)\phi_{x,y}(z_3) >=$$
$$\left\{ \begin{array}{l} \ll V_{\alpha_{n_1,m_1}}(z_1) V_{\alpha_{n_2,m_2}}(z_2) V_{2\alpha_0-\alpha_{x,y}}(z_3) \overbrace{J_- \dots J_-}^{k-1} \overbrace{J_+ \dots J_+}^{l-1} \gg \\ \ll V_{\alpha_{n_1,m_1}}(z_1) V_{2\alpha_0-\alpha_{n_2,m_2}}(z_2) V_{\alpha_{x,y}}(z_3) \overbrace{J_- \dots J_-}^{p-1} \overbrace{J_+ \dots J_+}^{q-1} \gg \\ \ll V_{2\alpha_0-\alpha_{n_1,m_1}}(z_1) V_{\alpha_{n_2,m_2}}(z_2) V_{\alpha_{x,y}}(z_3) \overbrace{J_- \dots J_-}^{a-1} \overbrace{J_+ \dots J_+}^{b-1} \gg \end{array} \right. \tag{1.141}$$

From the neutrality contition, we get the following expressions for the unknown charge $\alpha_{x,y}$

$$\begin{aligned} \alpha^1_{x,y} &= \frac{1}{2}\{[1-(n_1+n_2-2k+1)]\alpha_- + [1-(m_1+m_2-2l+1)]\alpha_+\} \\ \alpha^2_{x,y} &= \frac{1}{2}\{[1-(n_2-n_1+2p-1)]\alpha_- + [1-(m_2-m_1+2q-1)]\alpha_+\} \\ \alpha^3_{x,y} &= \frac{1}{2}\{[1-(n_1-n_2+2a-1)]\alpha_- + [1-(m_1-m_2+2b-1)]\alpha_+\} \end{aligned}$$

with the conditions

$$n_1+n_2-2k+1>0$$

$$n_2-n_1+2p-1>0 \tag{1.142}$$

$$n_1-n_2+2a-1>0 \tag{1.143}$$

and analogous for the α_+ part.

The common solution of these three series gives us the only non-zero three point function and then the fusion rules of the models

$$x = \mid n_1-n_2\mid +1, \mid n_1-n_2\mid +3, \ldots, n_1+n_2-1 \tag{1.144}$$

$$y = \mid m_1-m_2\mid +1, \mid m_1-m_2\mid +3, \ldots, m_1+m_2-1 \tag{1.145}$$

$$[\phi_{n_1,m_1}][\phi_{n_2,m_2}] = \sum_{x=|n_1-n_2|+1}^{n_1+n_2-1} \sum_{y=|m_1-m_2|+1}^{m_1+m_2-1} [\phi_{x,y}] \tag{1.146}$$

1.5.1 Integral representation for conformal correlators

Using the screening procedure one obtains the multipoint correlators in terms of closed integral on the complex plane

$$\begin{aligned} <\phi_{n,m}(z_1)\phi_{n,m}(z_2)\phi_{n,m}(z_3)\phi_{n,m}(z_4)> \sim \prod_{i=1}^{n-1}\oint_{C_i} du_i \prod_{j=1}^{m-1}\oint_{S_i} dv_j \ll V_{\alpha_{n,m}}(z_1) \\ V_{\alpha_{n,m}}(z_2)V_{\alpha_{n,m}}(z_3)V_{2\alpha_0-\alpha_{n,m}}(z_4)\prod_{i=1}^{n-1}V_{\alpha_-}(u_i)\prod_{j=1}^{m-1}V_{\alpha_+}(v_j) \gg \end{aligned} \tag{1.147}$$

Since all the integrations are analytic, the integrals do not depend on the precise form of the contours $\{C_1 \ldots C_{n-1}, S_1 \ldots S_{m-1}\}$. But they must wind around the point z_1, z_2, z_3, z_4, to do not shrink to a point, resulting in the integral being zero.

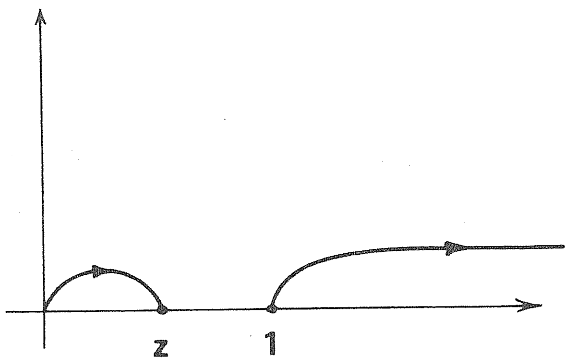
A simple expression is obtained for the correlators containing the operator $\phi_{1,2}(\phi_{2,1})$ since in this case it is sufficient to introduce only one screening operator $J_+(J_-)$. For example the correlator

$$< \phi_{n,m}(\infty)\phi_{1,2}(z)\phi_{1,2}(1)\phi_{n,m}(0) >$$

has the following expression

$$\begin{aligned} &< \phi_{n,m}(\infty)\phi_{1,2}(z)\phi_{1,2}(1)\phi_{n,m}(0) >= \\ &\oint dv \ll V_{2\alpha_0-\alpha_{n,m}}(\infty)V_{\alpha_{1,2}}(z)V_{\alpha_{1,2}}(1)V_{\alpha_{n,m}}(0)V_+(v) \gg= \\ &= z^{\alpha_{1,2}\alpha_{n,m}}(z-1)^{\alpha_{1,2}^2}\oint dv v^{\alpha+\alpha_{n,m}}[(v-1)(v-z)]^{\alpha+\alpha_{1,2}} \end{aligned} \quad (1.148)$$

(1.148) is the integral representation of the hypergeometric function which is a solution of the second order differential equation. There are two independent choices of the contour C and they correspond to the two independent solutions of the hypergeometric differential equation.



$$I_1(a,b,c,z) = \int_1^\infty dv\ v^a(v-1)^b(v-z)^c =$$

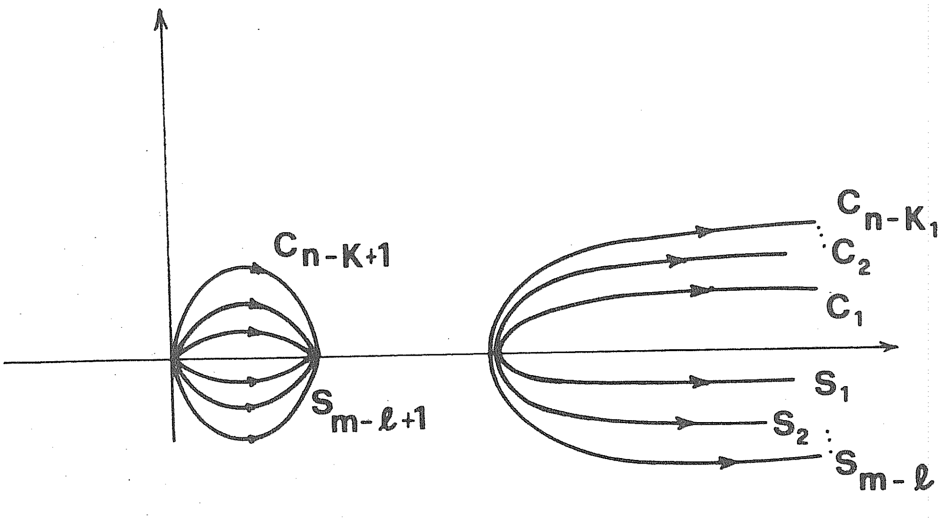
$$B(-a-b-c-1, b+1)F(-c, -a-b-c-1, -a-c, z) \quad (1.149)$$

$$I_2(a,b,c,z) = \int_0^z dv\ v^a(1-v)^b(z-v)^c =$$

$$B(a+1, c+1)z^{1+a+c}F(-b, a+1, a+c+2, z) \quad (1.150)$$

where B(x,y) is the beta-function and $F(\alpha, \beta, \gamma, z)$ the hypergeometric function [96].

In the general case there are n∗m independent contours which give n∗m solutions of the differential equation corresponding to the operator $\phi_{n,m}$. A possible choice of the integration contours is shown in fig.2



Dotsenko and Fateev have analyzed in detail this integral representation of the multipoint correlation function in ref [14], thus extracting all the analytic properties of the solutions.

1.5.2 The Monodromy problem and the structure constants of operator algebra

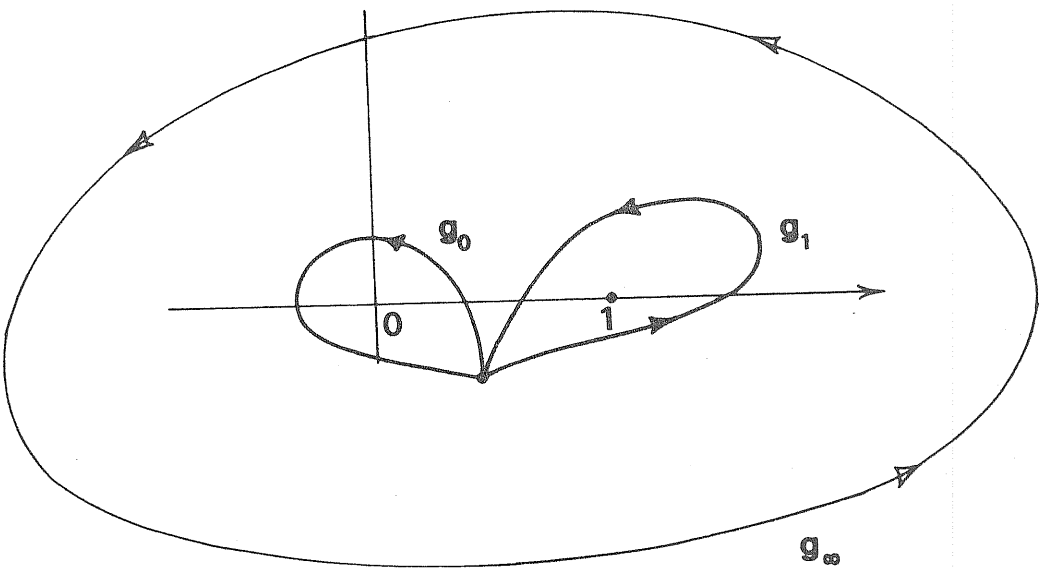
To obtain the physical correlators of scalar fields ($\Delta = \overline{\Delta}$) one has to combine the analytic part with the antianalytic one

$$G(z,\bar{z}) = \sum_{i,j} I_{ij} W_i(z) \overline{W_j(z)} \tag{1.151}$$

where $\{W_i\}$ is the set of functions coming from the independent integration contours of screening procedure. But these functions are not uniquely defined on the complex plane and if they are analitically continued along a closed curve enclosing the singular points, they transfor linearly into themselves

$$W_i(z) \to (g_l)_{ik} W_k(z) \tag{1.152}$$

The matrices g_l generate the monodromy group of the functions $W_i(z)$. In the case of the 4-point functions, our choice of the singular points is $0, 1, \infty$ (fig.) and we have correspondingly g_0, g_1, g_∞.



The correlation functions of scalar fields should be uniquely defined in the 2-d space, i.e. they have to be monodromy invariant [14]

$$G(z,\bar{z}) = \sum_{i,j} I_{ij} W_i \overline{W}_j = \sum_{ij} \sum_{k,l} I_{ij} (g_l)_{ik} W_k (\overline{g}_l)_{jp} \overline{W}_p =$$

$$= \sum_{k,l} \left(\sum_{i,j} (g_l^t)_{ki} I_{ij} (\overline{g}_l)_{jl} \right) W_k \overline{W}_p \qquad (1.153)$$

We have the following homogeneous equations for the unknown coefficient I_{ij}

$$I_{kp} = \sum_{ij} (g_l^t)_{ki} I_{ij} (\overline{g}_l)_{jp} \qquad (1.154)$$

These equations determine $\{I_{ij}\}$ up to an overall factor related to the normalization of the 2-point function.

A condition equivalent to the monodromy invariance is to require that the 4-point function

$$< \phi_k(\infty)\phi_l(1,1)\phi_n(z,\overline{z})\phi_m(0,0) >$$

satisfies crossing symmetry, i.e.[10]

$$G_{nm}^{lk}(z,\overline{z}) = G_{nl}^{mk}(1-z, 1-\overline{z}) = \left(\frac{1}{z}\right)^{2\Delta_n} \left(\frac{1}{\overline{z}}\right)^{2\overline{\Delta}_n} G_{nk}^{lm}\left(\frac{1}{z}, \frac{1}{\overline{z}}\right) \qquad (1.155)$$

As an illustration of this topic, we simply discuss here two examples, and refer at once to the original literature for the general case[14].

The first one is the 4-point function of the Ising spins

$$< \sigma(\infty)\sigma(1)\sigma(z)\sigma(0) >$$

The two independent solutions are (up to a factor $z^{-\frac{1}{8}}(z-1)^{-\frac{1}{8}}$)

$$W_1(z) = \sqrt{1 + \sqrt{1-z}} \qquad (1.156)$$

$$W_2(z) = \sqrt{1 - \sqrt{1-z}} \qquad (1.157)$$

The monodromy matrices are

$$g_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad (1.158)$$

$$g_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad (1.159)$$

Imposing invariance under g_0 implies the following expression for I_{ij}

$$\begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix} \tag{1.160}$$

and invariance under g_1 fixes .

$$x_1 = x_2 \tag{1.161}$$

The physical correlator is hence

$$G(z,\bar{z}) =< \sigma(\infty)\sigma(1,1)\sigma(z,\bar{z})\sigma(0,0) >= \lambda[W_1\overline{W}_1+W_2\overline{W}_2] \mid z \mid^{-\frac{1}{4}} \mid z-1 \mid^{-\frac{1}{4}} \tag{1.162}$$

Choosing the normalization

$$< \sigma(z,\bar{z})\sigma(0,0) >= \frac{1}{\mid z \mid^{\frac{1}{4}}} \tag{1.163}$$

the factor λ is fixed looking at the identity channel in the OPE of two spins

$$\sigma(z,\bar{z})\sigma(0,0) \simeq \frac{1}{\mid z \mid^{\frac{1}{4}}} + \cdots \tag{1.164}$$

Then

$$\lambda = \frac{1}{4} \tag{1.165}$$

The next example is the 4-point function of fields $\phi_{1,2}$ which needs only one screening operator

$$< \phi_{nm}(\infty)\phi_{12}(z)\phi_{12}(1)\phi_{nm}(0) >= z^{\alpha_{12}\alpha_{nm}}(z-1)^{\alpha_{12}^2}$$
$$\begin{cases} B(-a-b-c-1,b+1)F(-c,-a-b-c-1,-a-c,z) & = W_1 \\ B(a+1,c+1)z^{1+a+c}F(-b,a+1,a+c+2,z) & = W_2 \end{cases} \tag{1.166}$$

where

$$\begin{aligned} a &= \alpha_+\alpha_{nm} \\ b &= \alpha_+\alpha_{12} \\ c &= \alpha_+\alpha_{12} \end{aligned}$$

Using the monodromy matrices for the hypergeometric function (App.C), the physical correlator is

$$G(z,\overline{z}) = \mu \mid z \mid^{2\alpha_{12}\alpha_{nm}} \mid z-1 \mid^{2\alpha_{12}^2} \{s(a+b+c)s(b) \mid W_1(z) \mid^2 + s(a)s(c) \mid W_2(z) \mid^2\} \tag{1.167}$$

$(s(x) = \sin(\pi x))$

The structure constants of the operator algebra can be represented through the coefficients S^f_{abcd} at the leading singularities

$$(r)^{-2(\Delta_a+\Delta_b+\Delta_c+\Delta_d)} \left(\frac{r}{R}\right)^{4\Delta_f}$$

in the correlator

$$< \phi_a(z_1,\overline{z}_1)\phi_b(z_2,\overline{z}_2)\phi_c(z_3,\overline{z}_3)\phi_d(z_4,\overline{z}_4) >$$

Namely, if we take $\mid z_{12} \mid \simeq \mid z_{34} \mid \simeq r$ and $\mid z_{13} \mid \simeq \mid z_{24} \mid \simeq R, R \gg r$, then from the OPE

$$\phi_a(z,\overline{z})\phi_b(0,0) = \sum_f D^f_{ab} \mid z \mid^{-2(\Delta_a+\Delta_b-\Delta_f)} \phi_f(0,0) + \cdots \tag{1.168}$$

(in which $\Delta = \overline{\Delta}$ and the $\cdots$ represent the contribution of the conformal descendent operators), we obtain

$$< \phi_a(z_1,\overline{z}_1)\phi_b(z_2,\overline{z}_2)\phi_c(z_3,\overline{z}_3)\phi_d(z_4,\overline{z}_4) >\simeq$$

$$\sum_{f=1}^N S^f_{abcd} r^{-2(\Delta_a+\Delta_b+\Delta_c+\Delta_d-2\Delta_f)} < \phi_f(z_1,\overline{z}_1)\phi_f(z_3,\overline{z}_3) > \tag{1.169}$$

$$S^f_{abcd} = \lambda D^f_{ab} D^f_{cd} \tag{1.170}$$

For obtaining this equation we have used the orthogonality condition of the 2-point function. The proportionality factor λ is connected with the normalization freedom of correlation functions. We normalize our correlators by the condition that the coefficient of the contribution of the identity operator ($f = 1, \Delta_1 = 0$) i.e. at the singularity $\mid z \mid^{-2(\Delta_a+\Delta_b)}$ in (1.170), is equal to one.

In this way Dotsenko and Fateev have found the structure constants of the Virasoro algebra for each minimal model. Their cumbersome expressions are too long to be reported here, and can be found on the original papers [14].

Chapter 2

Modular invariance in minimal models

The physical operators of each minimal mode is charaterized by the couple of eigenvalues of L_0 and $\overline{L}_0$, $(\Delta, \overline{\Delta})$. The anomalous dimension is $d = \Delta + \overline{\Delta}$ and its spin $s = \Delta - \overline{\Delta}$.

One needs a principle to select the correct operatorial content of the theory, i.e. to combine the analytic part with the antianalytic one and to find the universality classes.

By a conformal transformation we can map all the complex plane into a finite width strip and the quantum field theory can be analyzed with the method of transfer matrix along the strip and of the finite size scaling [28,29]. Moreover, Cardy [20] has proposed to consider the system restricted to a strip with periodic boundary condition (a torus on the complex plane). If l and l' are the dimensions of this region, the partition function has the form

$$Z(l, l') = tr\ e^{-l'H} \tag{2.1}$$

A non-trivial constraint on the eigenvalues of H and their degeneracy arises if we impose the condition

$$Z(l, l') = Z(l', l) \tag{2.2}$$

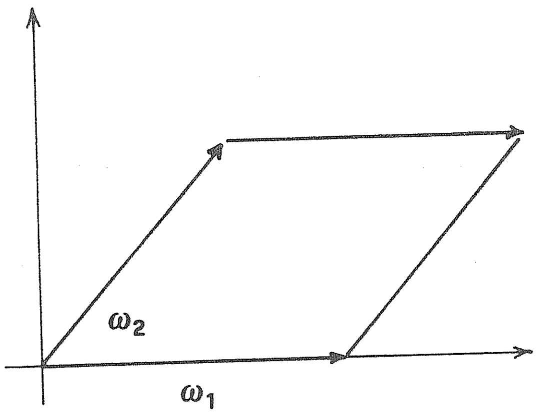
which means to exchange the role of the axis of propagation. This condition selects a set of couples $(\Delta, \overline{\Delta})$, closed under the OPE, and this set can be associated with a universality class of some statistical model.

Cappelli, Itzykson and Zuber [22] have analyzed in detail the problem to generalize eq.(2.2) to the invariance of the partition function under the reparametrization group of the torus, i.e. the Modular group.

The final result is that for each value of c, there exist at most three universality classes.Two of them appear for each value of c, while some exceptional solutions are present for some particular values of c. In this chapter we introduce the basic notions of the theory and present the final classification of the conformal models. We do not enter into the detailed calculation concerning the number-theory and diophantine equation [22,20].With the same methods, it is possible also to classify the superconformal models, which will be discussed in the chapter 5 [23,24].

2.1 The Partition function

Let us consider, in the complex plane, the region defined by the vectors $\omega_1 = L, \omega_2 = A + iB$ (fig.)



The partition function of the system, with periodic boundary condition, is

$$Z = tr\left(e^{\omega_2 L_{-1}^{(strip)} + \overline{\omega}_2 \overline{L}_{-1}^{(strip)}}\right) \tag{2.3}$$

where $L_{-1}^{(strip)}$ and $\overline{L}_{-1}^{(strip)}$ are the translation operators along the strip. They can be expressed in terms of the plane operators, using the transformation properties of the stress-energy tensor under the mapping

$$z = \exp\left[\frac{2iw}{L}\right] \tag{2.4}$$

The result is

$$L_{-1}^{(strip)} = \frac{2\pi i}{L}\left(L_0^{(plane)} - \frac{c}{24}\right) \tag{2.5}$$

Then

$$Z = tr\left(q^{(L_0-\frac{c}{24})}\overline{q}^{(\overline{L}_0-\frac{c}{24})}\right) \tag{2.6}$$

$$q = e^{2\pi i\frac{\omega_2}{\omega_1}} \tag{2.7}$$

The trace over the states can be expressed in terms of the irreducible representations of the conformal group $G = \Gamma \otimes \overline{\Gamma}$. For $c < 1$, the unitarity representations (for which Z is well defined) are given by the dimensions $(h, \overline{h})$, hence

$$Z = \sum_{h,\overline{h}} N_{h,\overline{h}}\chi_h(\tau)\overline{\chi_{\overline{h}}(\tau)} \tag{2.8}$$

where

$$\tau = \frac{\omega_2}{\omega_1}, \qquad (Im\tau > 0)$$

and $\chi_h(\tau)$ is the character of the irreducible representation h of Γ

$$\chi_h(\tau) = tr\left(q^{(L_0-\frac{c}{24})}\right)|_h = e^{2\pi i\tau(h-\frac{c}{24})}\sum_{n=0}^{\infty} d(n)e^{2\pi i n} \tag{2.9}$$

(d(n) is the number of states at level n).

The unknown parameters $N_{h,\overline{h}}$ are the multiplicity of the representation (h,$\overline{h}$): they are positive integer numbers with the normalization $N_{00} = 1$, by the uniqueness of the vacuum state. They give us the operatorial content of the theory. To fix them, it is sufficient to impose modular invariance for the partition function $Z(\tau)$.

First, one needs the expressions of the characters. For $h > 0, c > 1$, all the states are linear independent, so $d(n) = P(n)$, the partition function of integers and

$$\chi_h = \frac{q^{h-\frac{c}{24}}}{\prod_{n=1}^{\infty}(1-q^n)} \tag{2.10}$$

For the minimal models their expression is [27]

$$\chi_{r,s} = \frac{q^{-\frac{c}{24}}}{\prod_{n=1}^{\infty}(1-q^n)} \sum_{k=-\infty}^{\infty} \left\{ q^{\frac{(2km(m-1)+r(m-1)-sm)^2-1}{4m(m-1)}} - (s \to -s) \right\} \tag{2.11}$$

where m parametrizes the value of the central charge

$$c = 1 - \frac{6}{m(m+2)}$$

The formula of the characters has the following symmetries

$$\chi_{r,s} = \chi_{-r,-s} = \chi_{m-1+r,m+s} = -\chi_{r,-s} \tag{2.12}$$

In the next section we discuss the properties of the modular group, the transformation laws of the characters and the modular invariant solutions of the partition function.

2.2 Modular group and modular invariant solutions

The torus T is defined as a quotient space T/Λ

$$\Lambda = \{z = z_{n_1,n_2}, |\, z_{n_1,n_2} = n_1\omega_1 + n_2\omega_2, n_1, n_2 \in Z\} \tag{2.13}$$

The lattice Λ is invariant under a redefinition of elementary cell

$$\omega_2 \to a\omega_2 + b\omega_1 = \omega_2' \tag{2.14}$$

$$\omega_1 \to c\omega_2 + d\omega_1 = \omega_1' \tag{2.15}$$

$a, b, c, d \in Z$ with

$$ad - bc = 1 \tag{2.16}$$

This condition leaves the area of the elementary cell invariant. To count the number of parameters which describe the inequivalent tori, we first note that by an automorphism (rotation) we can choose axes for the lattice Λ so that ω_1 is along the positive real axis. Furthermore by another automorphism (rescaling) we can choose $\omega_1 = 1$. Finally by a reflection we can arrange that $Im\tau > 0$. Thus we have a family of tori realized as region in the complex plane C with corners $0, \tau, \tau+1, 1$.

To determine completely the set of inequivalent tori, we have to know the action of the reparametrization group on τ. This is given by the modular group M, described by matrices $A \in SL(2, Z)$

$$\tau \quad \to \tau' = \frac{a\tau + b}{c\tau + d} \tag{2.17}$$

$$A \quad = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{2.18}$$

M is a non compact discret group [88,89]. The generators are

$$T: \qquad \tau \to \tau + 1 \tag{2.19}$$

$$S: \qquad \tau \to -\frac{1}{\tau} \tag{2.20}$$

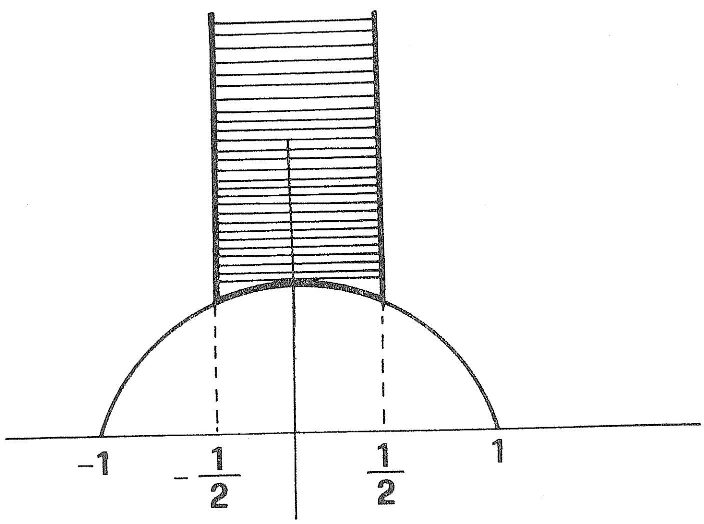
where T is an elementary translation $\omega_2 \to \omega_2 + \omega_1$

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{2.21}$$

and S is an inversion $\omega_1 \to \omega_2, \omega_2 \to -\omega_1$ (the minus sign preserves $Im\tau > 0$)

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{2.22}$$

The set of inequivalent tori is therefore obtained as a fundamental region of $SL(2, Z)$ on the upper half-plane, region which can be choosen to be $-\frac{1}{2} \leq Re\tau \leq \frac{1}{2}$ and $|\tau| \geq 1$



The partition function must satisfy the following conditions

$$T: \quad Z(\tau) = Z(\tau + 1) \tag{2.23}$$

$$S: \quad Z(\tau) = Z(-\frac{1}{\tau}) \tag{2.24}$$

Under these transformations, the characters transform with a unitary representation of the projective finite group $M_{2N} = PSL(2, Z, 2NZ)$ defined by the following matrices

$$A = \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{2.25}$$

where

$$\begin{array}{l} a, b, c, d \in Z \ mod.\ 2N \\ ad - bc = 1 \ mod.\ 2N \quad N = 2m(m+1) \end{array}$$

that is

$$A: \qquad \chi_h(\tau) \rightarrow \chi_h(\tau_A) \quad = D_{h,h'}(A)\chi_{h'}(\tau) \tag{2.26}$$

$$D(A)^\dagger = [D(A)]^{-1} \tag{2.27}$$

Then the $N_{h,\bar{h}}$ has to satisfy the equations

$$D(T)N \ = ND(T) \tag{2.28}$$

$$D(S)N \ = ND(S) \tag{2.29}$$

with the conditions that $N_{h,\bar{h}}$ is positive integers [20,22].

The solutions of these diophantine equations are discussed in detail in reference [22] and the final result is that they are in correspondence with the simple laced Lie algebras, that is the partition functions which are modular invariant are classified by a couple of Lie algebras, according to the following scheme

- The so called 'diagonal' solutions (A_{m-1}, A_m)
- The so called 'complementary' solutions, given by

$$(A_{4p}, D_{2p+2}) \qquad m = 4p + 1 \geq 5 \tag{2.30}$$

$$(D_{2p+2}, A_{4p+2}) \qquad m = 4p + 2 \tag{2.31}$$

$$(A_{4p+2}, D_{2p+3}) \qquad m = 4p + 3 \tag{2.32}$$

$$(D_{2p+3}, A_{4p+4}) \qquad m = 4p + 4 \tag{2.33}$$

- Finally, there exist some exceptional solutions for particular value of m

$$(A_{10}, E_6) \qquad m = 11 \tag{2.34}$$
$$(E_6, A_{12}) \qquad m = 12 \tag{2.35}$$
$$(A_{16}, E_7) \qquad m = 17 \tag{2.36}$$
$$(E_7, A_{18}) \qquad m = 18 \tag{2.37}$$
$$(A_{28}, E_8) \qquad m = 29 \tag{2.38}$$
$$(E_8, A_{30}) \qquad m = 30 \tag{2.39}$$

In this way one can list the conformal partition functions which are modular invariant.

$$(A_{m-1}, A_m) \qquad \frac{1}{2}\sum_{r=1}^{m-1}\sum_{s=1}^{m} |\chi_{r,s}|^2$$

$$(D_{2p+2}, A_{4p}) \qquad \frac{1}{2}\sum_{s=1}^{4p}\{\sum_{r\ odd=1, r\neq 2p+1}^{4p+1} |\chi_{r,s}|^2 + 2|\chi_{2p+1,s}|^2 + \\ + \sum_{r\ odd=1}^{2p-1}(\chi_{r,s}\chi^*_{4p+2-r,s} + c.c)\}$$

$$(D_{2p+2}, A_{4p+2}) \qquad \frac{1}{2}\sum_{s=1}^{4p+2}\{\sum_{r\ odd=1}^{4p+1} |\chi_{r,s}|^2 + 2|\chi_{2p,s}|^2 + \\ + \sum_{r\ even=2}^{2p-2}(\chi_{r,s}\chi^*_{4p-r,s} + c.c)\}$$

$$(E_6, A_{p-1}) \qquad \frac{1}{2}\sum_{s=1}^{p-1}\{|\chi_{1,s} + \chi_{7,s}|^2 + \\ + |\chi_{4,s} + \chi_{8,s}|^2 + |\chi_{5,s} + \chi_{11,s}|^2\}$$

$$(E_7, A_{p-1}) \qquad \frac{1}{2}\sum_{s=1}^{p-1}\{|\chi_{1,s} + \chi_{17,s}|^2 + |\chi_{5,s} + \chi_{13,s}|^2 + \\ |\chi_{7,s} + \chi_{11,s}|^2 + |\chi_{9,s}|^2 + [(\chi_{3,s} + \chi_{15,s})\chi_{9,s} + c.c.]\}$$

$$(E_8, A_{p-1}) \qquad \frac{1}{2}\sum_{s=1}^{p-1}\{|\chi_{1,s} + \chi_{11,s} + \chi_{19,s} + \chi_{29,s}|^2 + \\ + |\chi_{7,s} + \chi_{13,s} + \chi_{17,s} + \chi_{23,s}|^2\}$$

An analogous classification can be done for the superconformal minimal models [23,24].

Furthermore, Cardy [25] and Zuber [26] have analized the effect of different boundary conditions chosen for the torus on the operator content of the 2-d conformal field theories. The different choices excite different states of the system, which propagate by the transfer matrix, and, however, they are tied with the internal symmetries of the models.The detailed analysis of this point is too long to be discussed here.

Chapter 3

Infinite additional symmetries in 2-D conformal models

In each model of the unitarity series, characterized by the integer m, there exists a field whose dimension is integer or half-integer, precisely

$$s = \Delta_{m-1,1} = \Delta_{1,m} = \frac{(m-1)(m-2)}{4} \tag{3.1}$$

If among the primary fields in the theory there is a field Q_s with dimensions $(\Delta, \overline{\Delta}) = (s, 0)$ (and then spin s), one can speak of an additional infinite symmetry of such field theory [31].

Indeed, such a field satisfies the equation

$$\partial_{\overline{z}} Q_s = 0 \tag{3.2}$$

i.e. is an analytic field; therefore we have an infinite number of conserved currents of the form

$$\begin{aligned} J^s_\omega(z) &= \omega(z) Q_s(z) && (3.3) \\ \partial_{\overline{z}} J^s_\omega(z) &= 0 && (3.4) \end{aligned}$$

where $\omega(z)$ is an arbitrary analytic function. These currents can be regarded as the generators of the additional symmetry. The main requirement imposed on the field theory is the associativity of the operator algebra [10]. We shall see that in the case $s \leq 2$ there is no essential new algebra, since $s = \frac{1}{2}$ corresponds to the free fermions, $s = 1$ is just a general case of Kac-Moody algebra, $s = \frac{3}{2}$ corresponds to supersymmetry and $s = 2$ splits into a direct products of Virasoro algebra.

A new ones are obtained with spin $s = \frac{5}{2}$ and $s = 3$, since in these cases the generators Q_s do not form Lie algebra but more general algebras with quadratic relations between the generators.

Since Q_s is a primary field, it satisfies the OPE

$$T(z)Q_s(w) = \frac{s}{(z-w)^2}Q_s(w) + \frac{1}{z-w}\partial Q_s(w) \tag{3.5}$$

This analytic property allows the computation of any correlation function of the form

$$< T(z_1)T(z_2)\ldots T(z_n)Q_s(z_{n+1})\ldots Q_s(z_m) > \tag{3.6}$$

in terms of the correlation function

$$< Q_s(z_{n+1})\ldots Q_s(z_m) > \tag{3.7}$$

For computing these ones, it is necessary to specify the following operator product expansion

$$Q_s(z)Q_s(w) = \sum_{k=1}^{2s-1} a_k \frac{R_k(w)}{(z-w)^{2s-k}} \tag{3.8}$$

where $R_k(w)$ are local fields with dimension and spin k and the coefficient a_k are the structure constants of the algebra. R_0 is the identity operator 1, and a_0 is related to the normalization of the fields Q_s. The current choice is $a_0 = \frac{c}{s}$, where c is the central charge of the Virasoro algebra.

We restrict our attention to the case in which the OPE contains only the identity and the field Q_s families, i.e.

$$[Q_s][Q_s] = a_0[1] + b[Q_s] \tag{3.9}$$

Let us consider in detail each case.

i)Spin $\frac{1}{2}$.

This is the case of the Ising model. The generator of additional symmetry is the Majorana spinor, which has the following operator product expansion

$$\psi(z)\psi(w) = \frac{1}{z-w} \tag{3.10}$$

We can expand it into Laurent modes

$$\psi(z) = \sum_{-\infty}^{\infty} \frac{\psi_k}{z^{k+\frac{1}{2}}} \tag{3.11}$$

where k runs over the half-integers if the field $\psi(z)$ has periodic boundary condition

$$\psi(e^{2\pi i}z) = \psi(z) \tag{3.12}$$

and over the integers if $\psi(z)$ has antiperiodic boundary condition

$$\psi(e^{2\pi i}z) = -\psi(z) \tag{3.13}$$

The full algebra of the Ising model is the direct sum of Virasoro and canonical anticommutation relation of spinor field:

$$\begin{aligned}
[L_n, L_m] &= (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0} \\
[L_n, \psi_m] &= -(\frac{n}{2}+m)\psi_{n+m} && (3.14) \\
\{\psi_n, \psi_m\} &= \delta_{n+m,0} && (3.15)
\end{aligned}$$

In the antiperiodic sector, called in ref.[60,61] the 'Ramond sector' of Ising algebra, there are zero-modes, i.e.

$$\begin{aligned}
[L_0, \psi_0] &= 0 && (3.16) \\
\psi_0^2 &= \frac{1}{2} && (3.17)
\end{aligned}$$

The lowest energy state of this algebra, $\mid \sigma^{\pm} >$ is doubly degenerate and has dimension $\Delta^{\pm} = \frac{1}{16}$, as it can be seen if one expresses T(z) as a quadratic function of ψ

$$T(z) =: \psi(z)\partial_z\psi(z) : \tag{3.18}$$

and correspondingly

$$L_n = \sum_{l\geq -[\frac{n}{2}]} (l+\frac{n}{2})\psi_{-l}\psi_l, \qquad n\neq 0 \tag{3.19}$$

$$L_0 = \frac{1}{2}[L_1, L_{-1}] = \frac{1}{8}\psi_0^2 + \sum_{l=1}^{\infty} l\psi_{-k}\psi_k \tag{3.20}$$

($[\frac{n}{2}]$ is the largest integer which does not exceed $\frac{n}{2}$) The 'spin fields' corresponding to these 'Ramond' states

$$\mid \sigma^{\pm} >= \sigma^{\pm}(0) \mid 0> \tag{3.21}$$

produce a branch-cut singularity of the antiperiodic fermionic field

$$\psi(z)\sigma^{\pm}(w) = \frac{1}{\sqrt{2(z-w)}}\sigma^{\mp}(w) \tag{3.22}$$

We shall use also the diagonal basis

$$\tilde{\sigma} = \frac{\sigma^+ \mp \sigma^-}{\sqrt{2}} \tag{3.23}$$

with the following OPE

$$\psi(z)\tilde{\sigma}(w) = \mp\frac{1}{\sqrt{z-w}}\tilde{\sigma}(w) \tag{3.24}$$

This OPE, together with (3.10) leads to the recursive equation for the following correlation function [60,61]

$$G^N(z,v_i) =< \sigma(\infty)\sigma(z)\sigma(1)\prod_{i=1}^N \psi(v_i)\sigma(0) >=$$

$$\sqrt{\tfrac{(1-v_1)v_1}{z-v_1}}[\tfrac{1}{\sqrt{z}}\left(\tfrac{\sqrt{z-1}}{1-v_1} + (-1)^{N+1}\tfrac{\sqrt{z}}{v_1}\right) < \sigma(\infty)\sigma(z)\sigma(1)\prod_{i=2}^N \psi(v_i)\sigma(0) >$$

$$+\sum_{k=2}^N \tfrac{(-1)^k}{v_{1k}}\sqrt{\tfrac{z-v_k}{(1-v_k)v_k}} < \sigma(\infty)\sigma(z)\sigma(1)\prod_{j=2,j\neq k}^N \psi(v_j)\sigma(0) >] \tag{3.26}$$

whose solution can be written in terms of the 4-point function of σ's

$$< \sigma(\infty)\sigma(z)\sigma(1)\sigma(0) >= \frac{z^{\frac{1}{8}}(z-1)^{-\frac{1}{8}}}{\sqrt{2}}\sqrt{1+\sqrt{1-\frac{1}{z}}} \tag{3.27}$$

A recursive equation can be written also in the case in which there are two $\tilde{\sigma}$ fields, that is for the function

$$\tilde{G}^N(z, v_i) =< \tilde{\sigma}(\infty)\sigma(z)\tilde{\sigma}(1)\prod_{i=1}^{N}\psi(v_i)\sigma(0) > \qquad (3.28)$$

and the solution is expressed in terms of the following 4-point function

$$< \tilde{\sigma}(\infty)\sigma(z)\tilde{\sigma}(1)\sigma(0) >= \frac{z^{\frac{1}{8}}(z-1)^{-\frac{1}{8}}}{\sqrt{2}}\sqrt{1-\sqrt{1-\frac{1}{z}}} \qquad (3.29)$$

ii)Spin 1.

This case does not belong to the unitarity minimal models and this is the reason of the absence of continuum internal symmetry of these models [26].

Considering the case of a multicomponent field Q_1, denoting by

$$Q_1^a(z) = (\frac{c}{k})^{\frac{1}{2}}J^a(z), \qquad a = 1, \dots D$$

where k is a new constant, the OPE has the general form

$$J^a(z)J^b(w) = \frac{k\delta^{ab}}{(z-w)^2} + f_c^{ab}\frac{J^c(w)}{z-w} \qquad (3.30)$$

Since $J^a(z)$ are Bose fields, the symmetry under the change $z \leftrightarrow w, a \leftrightarrow b$, implies

$$f_c^{ab} = -f_c^{ba}$$

However from the 3-point function

$$< J^a(z_1)J^b(z_2)J^c(z_3) >$$

one gets that $f^{abc} \equiv f_c^{ab}$ is an antisymmetric tensor in all indices and imposing the crossing symmetry to the 4-point function

$$< J^a(z_1) \dots J^d(z_4) >$$

one obtains that the structure constants f have to satisfy the Jacobi identities

$$f^{a_1a_2b}f^{a_3a_4b}+f^{a_2a_3b}f^{a_1a_4b}+f^{a_3a_1b}f^{a_2a_4b}=0 \tag{3.31}$$

We have a D-dimensional Lie algebra. Then the case s=1 is related to a semidirect product of Kac-Moody algebra and Virasoro algebra.

iii)Spin $\frac{3}{2}$.

This is the case of the tricritical Ising model (TIM). The OPE of the conserved current is

$$S(z)S(w)=\frac{2}{3}\frac{c}{(z-w)^3}+\frac{2T(w)}{(z-w)} \tag{3.32}$$

which can be expressed in terms of the algebra of the Laurent modes S_k

$$S(z)=\sum_{-\infty}^{\infty}\frac{S_k}{z^{k+\frac{3}{2}}} \tag{3.33}$$

given the following (anti)commutation relations

$$[L_n,S_k] \;=(\frac{1}{2}n-k)S_{n+k} \tag{3.34}$$

$$\{S_k,S_l\} \;=2L_{k+l}+\frac{c}{3}(k^2-\frac{1}{4})\delta_{k+l,0} \tag{3.35}$$

Together with the Virasoro algebra, these (anti)commutation relations forms the supersymmetry algebra. Thus TIM is a superconformal field theory. We shall analize in detail the representations and the realizations of the superconformal algebra in the next chapter.

iv)Spin $\frac{5}{2}$.

This case, like that of spin 1, does not belong to the minimal unitarity series. Since this field is a fermionic one, the OPE contains only the identity family

$$U(z)U(w)=\frac{2}{5}\frac{c}{(z-w)^5}+\frac{2T(w)}{(z-w)^3}+\frac{1}{(z-w)^2}\partial T(w)+\frac{1}{z-w}[\frac{3}{10}\partial^2T(w)+2\gamma\Lambda(w)] \tag{3.36}$$

where

$$\gamma = \frac{(\frac{25}{2}+1)}{22+5c} \tag{3.37}$$

and the field $\Lambda(z)$ is determined by the relation

$$T(z)T(w) = \frac{c}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{1}{z-w}\partial T(w) + \frac{3}{10}\partial^2 T(w) + \Lambda(w) + \cdots \tag{3.38}$$

The 4-point function of the field U(z) has the expression [31]

$$< U(z_1)U(z_2)U(z_3)U(z_4) >= [(z_1-z_4)(z_3-z_2)]^{-5}G(x) \tag{3.39}$$

where x is the anharmonic ratio

$$x = \frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_3-z_2)} \tag{3.40}$$

and G(x) is

$$G(x) = \frac{4c^2}{25}\left[\frac{1}{x^5} + \frac{1}{(1-x)^5} - 1\right] +$$

$$+2c\left[\frac{1}{x^3} + \frac{1}{(1-x)^3} + \frac{1}{x^2} + \frac{1}{(1-x)^2} + \frac{\lambda}{x} + \frac{\lambda}{1-x}\right] \tag{3.41}$$

(3.42)

(3.43)

$$\lambda = \frac{9}{10}\left[1 + \frac{81}{2(22+5c)}\right] \tag{3.44}$$

If one requires crossing symmetry for this 4-point function, the final condition is $\lambda = 3$, whence

$$c = -\frac{13}{14} \tag{3.45}$$

The negative value of the central charge c prevents the identification of this model with a model of quantum field theory, since the positivity condition cannot be satisfied [11]. But, in any case, the solutions of the bootstrap equations with $c < 0$ can describe phase transitions in two-dimensional statistical system with non-Gibbs distribution (like self-avoid random walk or system with random interaction). The algebra of the Laurent modes U_k, defined by

$$U(z) = \sum_{-\infty}^{\infty} \frac{U_k}{z^{k+\frac{5}{2}}} \qquad k \in Z + \frac{1}{2} \tag{3.46}$$

is

$$[L_n, U_k] = (\frac{3}{2}n - k)U_{n+k} \tag{3.47}$$

$$\{U_l, U_k\} = \frac{14}{9}\Lambda_{k+l} + [\frac{3}{10}(k+l+2)(k+l+3) + $$
$$- \quad (k+\frac{3}{2})(l+\frac{3}{2})]L_{k+l} - \frac{13}{840}(k^2 - \frac{9}{4}(k^2 - \frac{1}{4})\delta_{k+l} \tag{3.48}$$

where Λ_n are quadratic in L_m

$$\Lambda_n = \quad d_n L_n + \sum_{-\infty}^{\infty} : L_m L_{n-m} : \tag{3.49}$$

$$d_{2m} = \quad \frac{1}{5}(1 - m^2) \tag{3.50}$$

$$d_{2m-1} = \quad \frac{1}{5}(1+m)(2-m) \tag{3.51}$$

Then the current of spin $\frac{5}{2}$ does not form Lie algebra but more general algebra with quadratic relations.

The space of local fields occuring in a conformal field theory with additional symmetry corresponds to some representation of the symmetry algebra. If one requires that the spectrum of dimensions is bounded from below, then every irreducible representation is a HWV representation .

In the case of spin $\frac{5}{2}$ this means that there exists a field satisfying the equations

$$L_n \phi_\Delta = 0, \qquad U_k \phi_\Delta = 0, \qquad n, k > 0 \tag{3.52}$$

$$L_0 \phi_\Delta = \Delta \phi_\Delta \tag{3.53}$$

The theory with $c = -\frac{13}{14}$ is a minimal theory even if it is not in the unitarity series [10,31]. The spectrum of dimensions of the primary fields is

$$\Delta_{n,m} = \frac{(7m-4n)^2-9}{112}, \qquad n=1,2; m=1,2,..6 \tag{3.54}$$

The field $\phi_{1,6}$ has dimension $\Delta_{1,6}=\frac{5}{2}$ and from its fusion rule

$$[\phi_{1,6}][\phi_{1,6}] = 1$$

it can be consistently identified with the symmetry generator.

The fields $\psi_{1,2}=\phi_{-\frac{1}{14}}$ and $\psi_{1,3}=\phi_{\frac{1}{7}}$ are HWV of the algebra. In these case the representation of the algebra, given by the vectors of the forms

$$L_{-n_1}L_{-n_2}\dots L_{-n_k}U_{-m_1}\dots U_{m_j}\phi_\Delta$$

contain null vector at the level $\frac{1}{2}$ and $\frac{3}{2}$, respectively

$$\Delta = -\frac{1}{14} \qquad U_{-\frac{1}{2}}\phi_{-\frac{1}{14}} = 0 \tag{3.55}$$

$$\Delta = \frac{1}{7} \qquad (U_{-\frac{3}{2}} - \frac{7}{3}L_{-1}U_{-\frac{1}{2}})\phi_{\frac{1}{2}} = 0 \tag{3.56}$$

Finally, the primary fields $\psi_{1,4}$ and $\psi_{1,5}$ with

$$\Delta_{1,4} = \tfrac{9}{14} = \tfrac{1}{7}+\tfrac{1}{2}$$
$$\Delta_{1,5} = -\tfrac{1}{14}+\tfrac{3}{2}$$

can be identified with the state

$$\psi_{1,4} = U_{-\frac{1}{2}}\phi_{\frac{1}{7}} \tag{3.57}$$

$$\psi_{1,5} = U_{-\frac{3}{2}}\phi_{-\frac{1}{14}} \tag{3.58}$$

(we have used the (3.56)). In this minimal model there exists a subalgebra containing the primary fields $\psi_{1,m}, m=1,..6$, which realizes the symmetry given by the eq.(3.48).

After this short discussion on higher symmetry in conformal field theory, in the next chapter we shall point out in detail the superconformal invariance, their unitarity representation and their physical realizations.

Chapter 4

Superconformal symmetry

The natural way to discuss supersymmetric theories is the superspace formalism: in N=1, two-dimensional space-time, one has fermionic coordinates $\theta, \overline{\theta}$ in addition to the usual ones z and $\overline{z}$. Let us briefly review some formalism. The supersymmetric transformations are

$$z \longrightarrow z - \epsilon\theta \qquad \overline{z} \longrightarrow \overline{z} - \overline{\epsilon}\overline{\theta} \tag{4.1}$$
$$\theta \longrightarrow \theta + \epsilon \qquad \overline{\theta} \longrightarrow \overline{\theta}\overline{\epsilon} \tag{4.2}$$

where $\epsilon, \overline{\epsilon}$ are Grassmann parameters. Correspondingly, we have the following differential expression for the generators and the covariant derivatives

$$Q = \frac{\partial}{\partial\theta} - \theta\frac{\partial}{\partial z} \qquad \overline{Q} = \frac{\partial}{\partial\overline{\theta}} - \overline{\theta}\frac{\partial}{\partial\overline{z}} \tag{4.3}$$
$$D = \frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial z} \qquad \overline{D} = \frac{\partial}{\partial\overline{\theta}} + \overline{\theta}\frac{\partial}{\partial\overline{z}} \tag{4.4}$$
$$\{D, D\} = 2\frac{\partial}{\partial z} \qquad \{\overline{D}, \overline{D}\} = 2\frac{\partial}{\partial z} \tag{4.5}$$

Under the (4.1)(4.2), a superfield

$$\Phi(z,\theta;\overline{z},\overline{\theta}) = \phi(z,\overline{z}) + \theta\psi(z,\overline{z}) + \overline{\theta\psi}(z,\overline{z}) + \theta\overline{\theta}F(z,\overline{z})$$

transforms as

$$\Phi(z,\theta;\overline{z},\overline{\theta}) \longrightarrow \Phi(z-\epsilon\theta, \theta+\epsilon; \overline{z}-\overline{\epsilon}\overline{\theta}, \overline{\theta}+\overline{\epsilon})$$

or in components

$$\delta\phi \ = \epsilon\psi + \overline{\epsilon}\overline{\psi} \tag{4.6}$$

$$\delta\psi = \epsilon\partial_z\phi + \bar{\epsilon}F \tag{4.7}$$
$$\delta\overline{\psi} = -\bar{\epsilon}\partial_{\bar{z}} - \epsilon F \tag{4.8}$$
$$\delta F = -\epsilon\partial_z\overline{\psi} + \bar{\epsilon}\partial_{\bar{z}}\psi \tag{4.9}$$

Since the (z,θ) and $(\bar{z},\bar{\theta})$ part have the same structure, usually one considers only one of them, having the so- called $N = \frac{1}{2}$ supersymmetry. The superfield associated to this one-dimensional system described by $\xi = (z,\theta)$, is

$$\Phi(z,\theta) = \phi(z) + \theta\psi(z) \tag{4.10}$$

with trasformation laws

$$\delta\phi = \epsilon\psi \tag{4.11}$$
$$\delta\psi = \epsilon\partial_z\phi \tag{4.12}$$

Under the general super-analytic map

$$\xi \to \tilde{\xi}(\xi) = (\tilde{z}(z,\theta), \tilde{\theta}(z,\theta))$$

the covariant derivative D trasforms as

$$D = (D\tilde{\theta})\tilde{D} + (D\tilde{z} - \tilde{\theta}D\tilde{\theta})\tilde{D}^2 \tag{4.13}$$

The maps restricted by the condition

$$D\tilde{z} = \tilde{\theta}D\tilde{\theta} \tag{4.14}$$

(superconformal trasformations), ensures that D transforms homogeneously i.e.

$$D = (D\tilde{\theta})\tilde{D} \tag{4.15}$$

We can introduce the supersymmetric generalization of the differential dz, denoted by dZ, starting from the exterior derivative

$$d = dz\frac{\partial}{\partial z} + \overline{d}\theta\frac{\partial}{\partial\theta} = dZ\frac{\partial}{\partial z} + \overline{d}\theta D \tag{4.16}$$
$$dZ \equiv dz + \theta\overline{d}\theta \tag{4.17}$$

($\overline{d}\theta$ has the same dimension of θ, the opposite of $d\theta$ which appears into the supervolume element $dV = d\theta dz$)

Using the eq.(4.14), it is easy to show the following transformation law

$$d\tilde{Z} = (D\tilde{\theta})^2 dZ \tag{4.18}$$

and then the square-root of dZ transforms inversely to the covariant derivative. Following the analogy with the usual conformal fields, we introduce the superconformal primary fields requiring the invariance of the form $\Phi(z,\theta)(dZ)^{\Delta}$:

$$\tilde{\Phi}(\tilde{z},\tilde{\theta})(d\tilde{Z})^{\Delta} = \Phi(z,\theta)(dZ)^{\Delta} \tag{4.19}$$

i.e. they transform as tensors of order Δ, where Δ is the anomalous dimension of the field Φ. The infinitesimal version of (4.19) it is obtained considering a infinitesimal superconformal vector field v, with the same properties of dZ

$$v(z,\theta) = dZ = dz + \theta \bar{d}\theta \tag{4.20}$$

then

$$\tilde{z} = z + v - \theta \bar{d}\theta \tag{4.21}$$

$$\tilde{\theta} = \theta + \bar{d}\theta \tag{4.22}$$

The superconformal condition (4.14) implies

$$\bar{d}\theta = \frac{1}{2} Dv \tag{4.23}$$

so, we have

$$\begin{aligned}\Phi(z,\theta) &= \tilde{\Phi}(\tilde{z},\tilde{\theta})(D\tilde{\theta})^{\Delta} \\ &\simeq (\tilde{\Phi}(z,\theta) + dz\partial_z\tilde{\Phi} + \bar{d}\theta\partial_\theta\tilde{\Phi})(1 + 2\Delta D(\bar{d}\theta))\end{aligned} \tag{4.24}$$

that is

$$\delta_v \Phi(z,\theta) = (v\partial_z + \frac{1}{2}(Dv)D + \Delta\partial v)\Phi(z,\theta) \tag{4.25}$$

The generators of the superconformal transformations is the super-stress energy tensor

$$W(z,\theta) = \frac{1}{2}G(z) + \theta T(z)$$

in terms of which the variation of the local superfield is

$$\delta_v \Phi(z,\theta) = \oint_C d\xi d\eta\, v(\xi,\eta) W(\xi,\eta) \Phi(z,\theta) \tag{4.26}$$

where the contour C is around the point z and the θ integration is done with the usual Grassmann rules. Eqs. (4.25) and (4.26) imply the following operator product expansion

$$W(z_1,\theta_1)\Phi_\Delta(z_2,\theta_2) = \left(\frac{\Delta\theta_{12}}{z_{12}^2} + \frac{1}{2z_{12}}D_2 + \frac{\theta_{12}}{z_{12}}\partial_2\right)\Phi(z_2,\theta_2) \qquad (4.27)$$

where

$$\begin{aligned} z_{12} &= z_1 - z_2 - \theta_1\theta_2 \\ \theta_{12} &= \theta_1 - \theta_2 \end{aligned}$$

The dimension of $W(z,\theta)$ is $\frac{3}{2}$ and the most general OPE of $W(z,\theta)$ with itself is

$$W(z_1,\theta_1)W(z_2,\theta_2) = \frac{c}{6z_{12}^3} + \left(\frac{3}{2}\frac{\theta_{12}}{z_{12}^2} + \frac{1}{2z_{12}}D_2 + \frac{\theta_{12}}{z_{12}}\right)W(z_2,\theta_2) \qquad (4.28)$$

which implies the following finite transformation property

$$W(z,\theta) = \tilde{W}(\tilde{z}\tilde{\theta})(D\tilde{\theta}) + \frac{c}{6}S(\xi,\tilde{\xi}) \qquad (4.29)$$

where $S(\xi,\tilde{\xi})$ is the super-Schwartz derivative

$$S(\xi,\tilde{\xi}) = \frac{D^4\tilde{\theta}}{D\tilde{\theta}} - 2\frac{(D^3\tilde{\theta})(D^2\tilde{\theta})}{(D\tilde{\theta})^2} \qquad (4.30)$$

$W(z,\theta)$ is not a primary superfield for the presence of the real parameter c, the central charge.

All the operator product expansions can be equivalently expressed in terms of (anti)commutation relations. In fact, using the Laurent series

$$v(z,\theta) = \epsilon(z) + \theta\alpha(z) = \sum_{-\infty}^{\infty}\epsilon_n z^{-(n-1)} + \theta\sum_{-\infty}^{\infty}\alpha_{n-\frac{1}{2}}z^{-(n-1)} \qquad (4.31)$$

$$W(z,\theta) = \frac{1}{2}\sum_{-\infty}^{\infty}\frac{G_{n+\frac{1}{2}}}{z^{n+2}} + \theta\sum_{-\infty}^{\infty}\frac{L_n}{z^{n+2}} \qquad (4.32)$$

From eq. (4.27) we recover the following commutation relations

$$[L_n, \phi_\Delta(z,\theta)] = \left(z^{n+1}\partial_z + (n+1)z^n(\Delta + \frac{1}{2}\theta\frac{\partial}{\partial\theta})\right)\Phi(z,\theta) \quad (4.33)$$

$$[\epsilon G_{n+\frac{1}{2}}, \Phi(z,\theta)] = \epsilon z^n \left(z(\frac{\partial}{\partial\theta} - \theta\frac{\partial}{\partial\theta}) - 2\Delta(n+1)\theta\right)\Phi(z,\theta) \quad (4.34)$$

(ϵ is a Grassmann variable)

and from eq. (4.28) we get the algebra of superconformal transformations (super-Virasoro algebra)

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0} \quad (4.35)$$

$$[L_n, G_\alpha] = (\frac{n}{2} - \alpha)G_{n+\alpha} \quad (4.36)$$

$$\{G_\alpha, G_\beta\} = 2L_{\alpha+\beta} + \frac{c}{3}(\alpha^2 - \frac{1}{4})\delta_{\alpha+\beta,0} \quad (4.37)$$

The largest finite dimensional subalgebra, spanned by $L_{\pm1}, L_0, G_{\pm\frac{1}{2}}$ (known as Osp(2,1)) is the supersymmetric extension of the SL(2,R) algebra of fractional linear transformations. To find the corresponding global superconformal transformation, we impose the condition of vanishing of the super-Schwartz derivative, $S(\xi,\tilde{\xi}) = 0$. The solution is

$$\tilde{z} = \frac{az+b}{cz+d} + \frac{\theta\tilde{\theta} - \alpha\beta}{cz+d} \quad (4.38)$$

$$\tilde{\theta} = \alpha + \frac{\theta+\beta}{cz+d} \quad (4.39)$$

(a,b,c,d $\in$ R, with $ad - bc = 1$ and α, β Grassmann variables)

and the associated global defined superconformal vector field is

$$v(z,\theta) = v_{-1} + v_0 z + v_1 z^2 + \theta(v_{-\frac{1}{2}} + v_{\frac{1}{2}}) \quad (4.40)$$

Until now, we have consider only one sector of the superconformal algebra, which has the periodic boundary condition for the fermionic part of the super- stress energy tensor, that is

$$G(e^{2\pi i}z) = G(z)$$

This is called the Neveu-Schwartz sector (NS).

But since the physical quantities is at least bilinear in the fermionic fields, one can choose also antiperiodic boundary condition for G, i.e.

$$G(e^{2\pi i}z) = -G(z)$$

obtaining the so-called Ramond sector of the algebra. In this case the mode expansion is

$$G^R(z) = \sum_{-\infty}^{\infty} \frac{G_n}{z^{n+\frac{3}{2}}} \tag{4.41}$$

The algebra has the same form as in the case of NS, but we have integer mode for the fermionic generators.

Notice that the most general superconformal field theory must have both sectors.

The representations of the NS algebra is a direct generalization of the procedure for the Virasoro algebra: the Hilbert space is constructed starting from the HWV $\mid \Delta >$, which satisfies the equations

$$\begin{aligned} L_0 \mid \Delta >&= \Delta \mid \Delta > \\ L_n \mid \Delta >&= 0, \qquad n > 0 \\ G_\alpha \mid \Delta >&= 0, \qquad \alpha > 0 \end{aligned} \tag{4.42}$$

acting on it by the raising operators $L_{-n}, G_{-\alpha}(n, \alpha > 0)$. The states

$$L_{-n_1} \dots L_{-n_p} G_{-\alpha_1} \dots G_{-\alpha_l} \mid \Delta > \tag{4.43}$$

$$0 < n_1 \leq n_2 \cdots \leq n_p \tag{4.44}$$

$$0 < \alpha_1 \leq \alpha_2 \cdots \leq \alpha_l \tag{4.45}$$

$$\sum_{i=1}^{p} n_i + \sum_{i=1}^{l} \alpha_i = k \tag{4.46}$$

are eigenvectors of L_0 with eigenvalue $\mid \Delta + k >$ and they span the space of the k-level states. States of different levels are orthogonal.

The vacuum $\mid 0 >$ is the lowest HWV. From the non-singular behaviour of the super stress-energy tensor at the origin, z=0, we get

$$G_\alpha \mid 0 > \ = 0 \qquad \alpha \geq -\frac{1}{2} \tag{4.47}$$

$$L_n \mid 0 > \ = 0 \qquad n \geq -1 \tag{4.48}$$

so the vacuum is invariant under the global superconformal group Osp(2,1). We can obtain the HWV Δ applying the superconformal fields Φ_Δ to the vacuum

$$| \Delta >= \Phi_\Delta(0,0) \mid 0 > \tag{4.49}$$

Using the commutation relations (4.33)(4.34), it is easy to show that the vector so constructed satisfies the conditions (4.42).

The representations of the Ramond algebra is more subtle, for the presence of the zero-mode. In fact, G_0 commutes with L_0 and there is a degeneration of HWV, since $\mid \Delta^+ >$ and $\mid \Delta^- >= G_0 \mid \Delta^+ >$ have the same dimension.

From the algebra

$$[L_0, G_0] = 0, \qquad G_0^2 = L_0 - \frac{c}{24} \tag{4.50}$$

we see that we have no supersymmetry breaking if there exists a HWV with dimension

$$\Delta = \frac{c}{24}$$

In this case we have a singlet as a ground state in the Ramond sector, since it is consistent to put

$$\mid (\frac{c}{24})^- >= G_0 \mid (\frac{c}{24})^+ >= 0 \tag{4.51}$$

We can obtain the Ramond states applying to the NS vacuum ordinary conformal fields, called spin-fields

$$\mid \Delta^\pm >_R = R^\pm_\Delta(0) \mid 0 >_{NS} \tag{4.52}$$

The OPE with fermionic part of super stress-energy tensor is

$$G(z)R^\pm_\Delta(w) = \frac{a^\pm(\Delta)}{(z-w)^{\frac{3}{2}}} R^\mp_\Delta(w) \tag{4.53}$$

$$a^+(\Delta) = 1 \tag{4.54}$$

$$a^-(\Delta) = \Delta - \frac{c}{24} \tag{4.55}$$

This non-locality of the spin-fields with G(z) is a general fact, since it happens with all the fermionic fields. So it becomes important to characterize the fermionic content of the Hilbert space, introducing a chiral projection operator Γ which satisfies

$$[\Gamma, L_n] = 0 \qquad (4.56)$$

$$\{\Gamma, G_n\} = 0 \qquad (4.57)$$

and

$$\Gamma \mid \Delta^{\pm} >_R = \pm \mid \Delta^{\pm} >_R \qquad (4.58)$$

$$\Gamma \mid \Delta >_{NS} = \mid \Delta >_{NS} \qquad (4.59)$$

The usual conformal fields and L_n operators are bosonic ones while the Ramond HWV are doublet with different chirality.

The more interesting way to obtain a local theory is to project out all the fermionic parts of the superconformal fields and half of the Ramond fields, restricting to those states which satisfy the condition $\Gamma = 1$ (*Spin model*).

As in Virasoro case, the unitarity condition for the field theory puts severe constraints on the possible values of the central charge in the region $0 < c < \frac{3}{2}$ and on the dimensions of the primary fields. The final results are [19,15]

$$c = \frac{3}{2}\left(1 - \frac{8}{p(p+2)}\right) \qquad p = 3, 4, \ldots \qquad (4.60)$$

$$\Delta_{n,m} = \frac{[(p+2)n - mp]^2 - 4}{8p(p+2)} + \frac{1}{16}\left(\frac{1 - (-1)^{n-m}}{2}\right) \qquad (4.61)$$

(n-m $\in$ 2Z for NS and n-m $\in$ 2Z+1 for the Ramond sector)

and we have the conformal grid shown in figure for the first two models of the c_p series, identified respectively as the tricritical Ising model (TIM) and the Kosterlitz-Thouless model (KT). Note the presence of a field with dimension $\Delta = \frac{1}{24}$ in the second conformal grid, associated with the Ramond field $\mid \Delta_{3,2} >$, which is a singlet.

4.1 Superconformal formalism of the NS sector

The representation fields of the NS sector of the algebra is organized in superfield multiplet. The 2-point function is fixed by the Osp(2,1) invariance of the vacuum. If ϕ is a scalar operator, that is $\Delta = \overline{\Delta}$ the expression is

$$< 0 \mid \Phi(z_1, \theta_1)\Phi(z_2, \theta_2) \mid 0 >=\mid z_{12} \mid^{-4\Delta} \tag{4.62}$$

and using the component expression

$$\Phi(z, \theta) = \phi(z, \overline{z}) + \theta\psi(z, \overline{z}) + \overline{\theta\psi}(z, \overline{z}) + \theta\overline{\theta}F(z, \overline{z})$$

all the 2-point functions of the component fields can be computed

$$< 0 \mid \phi(z_1)\phi(z_2) \mid 0 > \; =\mid z_1 - z_2 \mid^{-4\Delta} \tag{4.63}$$

$$< 0 \mid F(z_1)F(z_2) \mid 0 > \; = -4\Delta^2 \mid z_1 - z_2 \mid^{-4\Delta-2} \tag{4.64}$$

$$< 0 \mid \psi(z_1)\psi(z_2) \mid 0 > \; = -2\Delta \mid z_1 - z_2 \mid^{-4\Delta} (z_1 - z_2)^{-1} \tag{4.65}$$

We can use again the Osp(2,1) invariance to determine the 3-point function of the NS superfields.This has six independent variables (z_i, θ_i) and there are five global symmetric transformation.There is one combination which is invariant under all the five transformation: this is the anti-commuting variable

$$\tilde{\eta} = (z_{12}z_{13}z_{23})^{-\frac{1}{2}}(\theta_1 z_{23} + \theta_2 z_{13} + \theta_3 z_{12} + \theta_1\theta_2\theta_3)$$

Since the square of $\tilde{\eta}$ is zero, the general expression of 3-point function is

$$< 0 \mid \Phi(z_1, \theta_1)\Phi(z_2, \theta_2)\Phi(z_3, \theta_3) \mid 0 >= \tag{4.66}$$

$$\mid z_{12} \mid^{2(\Delta_3-\Delta_2-\Delta_1)}\mid z_{13} \mid^{2(\Delta_2-\Delta_1-\Delta_3)}\mid z_{23} \mid^{2(\Delta_1-\Delta_2-\Delta_3)} [a_1 + a_2 \mid \tilde{\eta} \mid^2]$$

(a_1 ordinary variable, a_2 a Grassmanian one)

We introduce the following definition: an expression is said to be even (odd) if it contains an even (odd) number of θ's or even (odd) number of $\overline{\theta}$'s. So the 3-point function has an even and an odd part. However, since there are two independent constants in its expression, there are in general two OPE structure constants to be calculated [15,17].

In the NS sector, a powerful method to compute the correlation functions is the 'null-vector' method. This is a straighforward generalization of the Virasoro case. In the region $c < \frac{3}{2}$ correlation functions of superconformal fields satisfy superdifferential equations: these are a generalization of an ordinary differential equations to superspace and it is equivalent to a set of differential equations in component form.

To derive these equations one consider the 'null-vector' expression. The first non-trivial null-state is at level $\frac{3}{2}$ [15]

$$| \, null >= \left(G_{-\frac{3}{2}} - \frac{2}{2\Delta + 1} L_{-1} G_{-\frac{1}{2}} \right) | \, \Delta > \tag{4.67}$$

where $\Delta = \Delta_{1,3}$ or $\Delta = \Delta_{3,1}$. If Φ_1 is a superfield associated to one of these dimensions and $\Phi_2 \ldots \Phi_N$ are other arbitrary primary superfields, the correlation function

$$< 0 \, | \, \Phi_N(z_N, \theta_N) \ldots \Phi_1(z_1, \theta_1) \, | \, 0 >$$

satisfies the equation

$$< 0 \, | \, \Phi_N(z_N, \theta_N) \ldots \Phi_2(z_2, \theta_2) \left(G_{-\frac{3}{2}} - \frac{2}{2\Delta + 1} L_{-1} G_{-\frac{1}{2}} \right) \Phi_1(z_1, \theta_1) \, | \, 0 >= 0 \tag{4.68}$$

Using the commutation relations of G's and L's, we can move these operators to the left and obtain the superdifferential equation

$$\mathcal{D} < 0 \, | \, \Phi_N \ldots \Phi_1 \, | \, 0 >= 0 \tag{4.69}$$

where

$$\mathcal{D} = \frac{2}{2\Delta + 1} \frac{\partial}{\partial z_1} \left(\frac{\partial}{\partial \theta_1} - \theta_1 \frac{\partial}{\partial z_1} \right) - \sum_{i=2}^{N} \left[\frac{1}{z_{1i}} \left(\frac{\partial}{\partial \theta_i} - \theta_{i1} \frac{\partial}{\partial z_i} - 2\Delta_i \theta_{i1} z_{i1}^{-2} \right] \tag{4.70}$$

In general there is a null-vector at level $\frac{1}{2}pq$ in the NS sector corrisponding to the HWV $| \, \Delta_{pq} >$. We can use this null vector to derive superdifferential equations for the correlation functions containing the primary superfield Φ_{pq}, of order $\frac{1}{2}pq$. Qiu [15] has used these tecnique to discuss the NS sector of TIM, determining the OPE and the structure constants.

The HWV for this model, fixed by the unitarity and Virasoro invariance, are $\Delta = 0, \frac{3}{80}, \frac{1}{10}, \frac{7}{16}, \frac{6}{10}, \frac{3}{2}$. Remember that this critical system falls into both unitary series, Virasoro and superconformal one.

In terms of the classification of superconformal algebra, the NS states are

$$\Delta_{1,1} = 0, \qquad \Delta_{2,2} = \Delta_{3,1} = \frac{1}{10}$$

They decompose into irreducible representations of the Virasoro algebra

$$[0]_{NS} = [0]_{Vir} \oplus [\frac{3}{2}]_{Vir} \tag{4.71}$$

$$[\tfrac{1}{10}]_{NS} = [\frac{1}{10}]_{Vir} \oplus [\frac{6}{10}]_{Vir} \tag{4.72}$$

These operators belong to the Z_2 even sector of the model and they are identify as energy operator ϵ $(\frac{1}{10}, \frac{1}{10})$, vacancy operator t $(\frac{6}{10}, \frac{6}{10})$ and an irrelevant operator $T_F\overline{T}_F$ $(\frac{3}{2}, \frac{3}{2})$, respectively.

The Z_2 odd sector describe the order parameter sector: there are a magnetic spin operator σ with dimensions $(\frac{3}{80}, \frac{3}{80})$ and a so-called sub-leading magnetization operator α with $(\frac{7}{16}, \frac{7}{16})$. These operators are in the Ramond sector of the superconformal algebra.

The Z_2 even operators of TIM can be conveniently written as a superfield with weight $(\frac{1}{10}, \frac{1}{10})$, i.e.

$$\Phi(z,\theta) = \epsilon(z,\overline{z}) + \theta\psi(z,\overline{z}) + \overline{\theta\psi}(z,\overline{z}) + \theta\overline{\theta}t(z,\overline{z}) \tag{4.73}$$

and the super stress-energy tensor

$$W(z) = \frac{1}{2}G(z) + \theta T(z) \tag{4.74}$$

$$\overline{W}(\overline{z}) = \frac{1}{2}\overline{G}(\overline{z}) + \overline{\theta T}(\overline{z}) \tag{4.75}$$

The 3-point function of the superfield Φ has to satisfy the differential equation (4.69): substituting one finds that the even part of the 3-point function is zero, so

$$< 0 \mid \Phi(z_1,\theta_1)\Phi(z_2,\theta_2)\Phi(z_3,\theta_3) \mid 0 >= (z_{12}z_{13}z_{23})^{-\frac{3}{5}}[\theta_1 z_{23} + \theta_2 z_{13} + \theta_3 z_{12} + \theta_1\theta_2\theta_3] \tag{4.76}$$

This implies the following OPE

$$\epsilon\epsilon \sim 1 - 25Ct \qquad \epsilon\psi \sim 5C\overline{\psi} \qquad \epsilon\overline{\psi} \sim 5C\psi \tag{4.77}$$

$$\epsilon t \sim C\epsilon \quad t\psi \sim -C\psi \qquad tt \sim -\frac{1}{5}1 + Ct \tag{4.78}$$

$$\psi\psi \sim -\frac{1}{5}1 - 5Ct \qquad \psi\overline{\psi} \sim -C\epsilon \qquad t\overline{\psi} \sim C\overline{\psi} \tag{4.79}$$

where C is an unknown constant. For fixing it one has to compute the 4-point function of the superfield Φ. Using the 'null-vector' method and the monodromy invariance, the final expression is [15]

$$< 0 \mid \Phi(z_1,\theta_1)\dots\Phi(z_4,\theta_4) \mid 0 >=\mid z_{12}z_{23}z_{14}z_{34} \mid^{-\frac{1}{5}}$$
$$\{\mid S_1(\eta,\xi) \mid^2 + A \mid S_2(\eta,\xi) \mid^2\} \tag{4.80}$$

where

$$\eta = \frac{z_{12}z_{34}}{z_{13}z_{24}} \qquad \xi = \frac{z_{14}z_{23}}{z_{13}z_{24}} - (1-\eta) \tag{4.81}$$

and

$$S_1(\eta,\xi) = (1+\eta\xi\frac{d}{d\eta})\left([\eta(1-\eta)]^{-\frac{1}{10}}F(\frac{1}{5},-\frac{2}{5},\frac{2}{5},\eta)\right)$$
$$+\frac{3}{5}\xi[\eta(1-\eta)]^{\frac{9}{10}}F(\frac{8}{5},\frac{11}{5},\frac{12}{5},\eta) \tag{4.82}$$
$$S_2(\eta,\xi) = (1+\xi\eta\frac{d}{d\eta})\left([\eta(1-\eta)]^{\frac{1}{2}}F(\frac{7}{5},\frac{4}{5},\frac{8}{5},\eta)\right)$$
$$\frac{1}{5}\xi[\eta(1-\eta)]^{-\frac{1}{2}}F(-\frac{6}{5},-\frac{3}{5},-\frac{2}{5},\eta) \tag{4.83}$$
$$A = \frac{4}{9}\frac{\Gamma(\frac{4}{5})\Gamma^3(\frac{2}{5})}{\Gamma(\frac{1}{5})\Gamma^3(\frac{3}{5})} \tag{4.84}$$

We can use this result to find the unknown constant C, doing the appropriate limit in the 4-point function and using the OPE.

The value is

$$C = i\frac{1}{15}\sqrt{\frac{\Gamma(\frac{4}{5})\Gamma^3(\frac{2}{5})}{\Gamma(\frac{1}{5}\Gamma^3(\frac{3}{5})}} \tag{4.85}$$

This ends the complete understanding of the Z_2 even sector of TIM. For a full picture one needs to discuss the Ramond sector of the superconformal algebra. To this problem is devoted the next sections.

4.2 Analytic properties and null-vector method in Ramond sector

In the NS sector the null-vector method, together with the Ward identities, lead to differential equations for the n-point function of the superfields and one can find explicity, from these equations, the corresponding fusion rules and the structure constants.

The difficulties with the application of this method for the Ramond fields come from the branch-cut singularity in the OPE (4.53). In this section we describe a modification of the null-vector method based on the specific analytic properties of the Ramond fields.

From the boundary conditions, we have the following qualitative description of the N=1 supersymmetric OPE algebra of the fields

$$[R][R] \sim [NS]$$
$$[R][NS] \sim [R]$$
$$[NS][NS] \sim [NS]$$

In order to find the fusion rules for the first level degenerated R fields (i.e. nm=2) we take the first level null vector

$$\mid \xi^{\pm}_{\tilde{\Delta}} >= \left(L_{-1} - \frac{4\tilde{\Delta}}{3(\tilde{\Delta} - \frac{c}{24})} G_{-1} G_0 \right) R^{\pm}_{\tilde{\Delta}}(0) \mid 0 > \qquad (4.86)$$

and we have to analyse the solutions of the following equation

$$< 0 \mid N_{\Delta_z}(z_1) R^{\pm}_{\tilde{\Delta}}(z_2) \left(L_{-1} - \frac{4\tilde{\Delta}}{3(\tilde{\Delta} - \frac{c}{24})} G_{-1} G_0 \right) R^{\pm}_{\tilde{\Delta}}(0) \mid 0 >= 0 \quad (4.87)$$

Because of the Γ parity properties of the fields in the OPE of two R fields with equal parities only the bosonic components of the NS superfields contribute. It is almost evident that the first term in the eq. (4.87) can be written in the form

$$< 0 \mid N_{\Delta_z}(z_1) R^{\pm}_{\tilde{\Delta}}(z_2) L_{-1} \mid \tilde{\Delta}^{\pm} >= -(\partial_1 + \partial_2) < 0 \mid N_{\Delta_z}(z_1) R^{\pm}_{\tilde{\Delta}}(z_2) \mid \tilde{\Delta}^{\pm} > \quad (4.88)$$

To obtain the explicit expression for the second term in the eq. (4.87) we consider the following auxiliary function

$$F^{(I)}(v \mid z_1, z_2) = \sqrt{v(z_1 - z_2)}(z_1 - v) < 0 \mid N_{\Delta_z}(z_1) R^{\pm}_{\Delta}(z_2) G(v) \mid \tilde{\Delta}^{\pm} > \tag{4.89}$$

The Ward identities and the asymptotic behaviour of the fields allow us to express this function in terms of the 3-point functions

$$f^{\pm} =< 0 \mid N_{\Delta_z}(z_1) R^{\pm}_{\Delta}(z_2) \mid \tilde{\Delta}^{\pm} >$$

that is

$$F^{(I)}(v \mid z_1, z_2) = \frac{A(z_1, z_2)}{z_2 - v} + \frac{B(z_1, z_2)}{v} \tag{4.90}$$

where

$$A = \oint_{z_2} F dv = \sqrt{z_2}(z_1 - z_2) f^{\mp} a^{\pm}(\Delta) \tag{4.91}$$

$$B = \oint_0 F dv = \sqrt{z_2} z_1 f^{\pm} a^{\mp}(\Delta) \tag{4.92}$$

Using the OPE (...) in the form

$$G(v) \mid \tilde{\Delta}^{\pm} >= \left(\frac{G_0}{v^{\frac{3}{2}}} + \frac{G_{-1}}{v^{\frac{1}{2}}} + \cdots \right) \mid \tilde{\Delta}^{\pm} \tag{4.93}$$

and the eq.(4.90) we can calculate in two different ways the coefficient C_{-1} in the Laurent expansion of the function $F^{(I)}(v \mid z_1, z_2)$:

$$\oint_0 F \frac{dv}{v} = \frac{A(z_1, z_2)}{z_2} \tag{4.94}$$

and

$$\oint_0 F \frac{dv}{v} = -\frac{z_1 + 2z_2}{2\sqrt{z_2} a^{\mp}}(\tilde{\Delta}) f^{\pm}(z_1, z_2) + \sqrt{z_2} z_1 < 0 \mid N_{\Delta_z}(z_1) R^{\pm}(z_2) G_{-1} \mid \tilde{\Delta} > \tag{4.95}$$

Finally, the eq.(4.87) takes the form

$$(\partial_1 + \partial_2) f^{\pm}(z_1, z_2) = \frac{4\tilde{\Delta}}{3(\tilde{\Delta} - \frac{c}{24})} a^{\pm}(\tilde{\Delta}) [\frac{z_1 - z_2}{z_1 z_2} a^{\pm}(\Delta) f^{\mp} + \frac{z_1 + 2z_2}{2z_1 z_2} a^{\mp}(\tilde{\Delta}) f^{\pm}] \tag{4.96}$$

Taking into account the explicit conformal invariant form of the 3-point function $f^{\pm}$

$$f^{\pm}(z_1, z_2) = h^{\pm}(z_1 - z_2)^{\tilde{\Delta}-\Delta_1-\Delta_x} z_1^{\Delta-\Delta_x-\tilde{\Delta}} z_2^{\Delta_x-\Delta-\tilde{\Delta}}$$

we obtain the following system of algebraic equations

$$(\Delta - \Delta_x + \tfrac{1}{3}\tilde{\Delta})h^+ - \frac{4\tilde{\Delta}}{3(\tilde{\Delta}-\frac{c}{24})}h^- \quad = 0 \qquad (4.97)$$

$$\tfrac{4}{3}\tilde{\Delta}(\Delta - \tfrac{c}{24})h^+ - (\Delta - \Delta_x + \tfrac{1}{3}\tilde{\Delta})h^- \quad = 0 \qquad (4.98)$$

The condition of consistence of this system gives us the unknown dimension Δ_x that we looked for

$$\Delta_x = \Delta + \frac{\tilde{\Delta}}{3} \mp \frac{4}{3}\tilde{\Delta}\sqrt{\frac{\Delta - \frac{c}{24}}{\tilde{\Delta} - \frac{c}{24}}} \qquad (4.99)$$

For TIM ($c = \frac{7}{10}$) the first level degenerated R fields are with dimensions

$$\Delta_{1,2} = \frac{3}{80} \qquad \Delta_{2,1} = \frac{7}{16}$$

Then according to (4.99) we get the following fusion rules

$$\begin{aligned}
[\tfrac{3}{80}]_R[\tfrac{3}{80}]_R &= [0]^1_{NS} + [\frac{1}{10}]^1_{NS} \\
[\tfrac{3}{80}]_R[\tfrac{7}{16}]_R &= [\frac{1}{10}]^1_{NS} + [\frac{4}{5}]^1_{NS} \\
[\tfrac{7}{16}]_R[\tfrac{3}{80}]_R &= [\frac{1}{10}]^1_{NS} + [\frac{4}{15}]^1_{NS} \\
[\tfrac{7}{16}]_R[\tfrac{7}{16}]_R &= [0]^1_{NS} + [\frac{7}{6}]^1_{NS}
\end{aligned}$$

where $[\Delta]^1$ denotes the family of the first component of the NS superfield. In order to obtain the true fusion rules we have to take the intersection of the two OPE relations

$$[\frac{3}{80}][\frac{7}{16}] \qquad [\frac{7}{16}][\frac{3}{80}]$$

that is

$$[\frac{3}{80}]_R[\frac{7}{16}]_R = [\frac{7}{16}]_R[\frac{3}{80}]_R = [\frac{1}{10}]^1_{NS} \qquad (4.100)$$

The restricted fusion rules

$$[\frac{7}{16}]_R[\frac{7}{16}]_R = [0]^1_{NS} \qquad (4.101)$$

is a consequence of the intersection of the corresponding fusion rules given above and those coming from the analysis of the equation generated by the second level null vector

$$\mid \chi^{\pm}_{\tilde{\Delta}_{1,4}} >= \{\frac{1}{15}(4\tilde{\Delta} + \frac{11}{4})(\frac{2}{5}\tilde{\Delta} + \frac{c}{4} + \frac{7}{5})L_{-2} - \frac{1}{3}(\frac{2}{5}\tilde{\Delta} + \frac{c}{4} + \frac{7}{5})L^2_{-1}$$
$$+\frac{4}{5}(\tilde{\Delta} + 1)G_{-2}G_0 + L_{-1}G_{-1}G_0\} \mid \tilde{\Delta}^{\pm} > \qquad (4.102)$$

since $\Delta_{2,1} = \Delta_{1,4} = \frac{7}{16}$ has this degeneracy, too.

In the case $c = 1$, $\Delta_{1,2} = \frac{3}{8}, \Delta_{2,1} = \frac{1}{16}$ we have

$$\begin{aligned} [\tfrac{1}{16}]_R[\tfrac{3}{80}]_R &= [\frac{1}{16}]^1_{NS} \\ [\tfrac{1}{16}]_R[\tfrac{1}{16}]_R &= [0]^1_{NS} + [\frac{1}{16}]^1_{NS} \\ [\tfrac{3}{8}]_R[\tfrac{3}{8}]_R &= [0]^1_{NS} + [1]^1_{NS} \end{aligned}$$

We use the same procedure to calculate the FR's for the Ramond fields of different Γ parity. The parity conservation implies that only the fermionic component of the NS superfields contribute to these FR's. In this case we introduce a new auxiliary function

$$F^{(II)}(v \mid x,z) = \sqrt{v(z-v)}(x-v)^2 < 0 \mid N_{II}(x)R^{\pm}_{\tilde{\Delta}}(z)G(v) \mid \tilde{\Delta}^{\pm} > \quad (4.103)$$

As before we insert the square-root in order to avoid the branch-cut of the correlation function when $v \to 0$ and $v \to z$ and the factor $(x-v)^2$ for cancelling the second order pole at $v \to x$:

$$G(v)N_{II}(x) = \frac{2\Delta}{(v-x)^2}N_I(x) + \cdots$$

Then $F^{(II)}$ has only simple pole at $v = z$ and at the origin and constant asymptotics at infinity, i.e.

$$F^{(II)}(v \mid x,z) = \frac{A(x,z)}{z-v} + \frac{B(x,z)}{v} + C(x,z) \qquad (4.104)$$

The coefficients A,B,C computed by the Cauchy theorem and the Ward identity have the form

$$A(x,z) = a^{\pm}(\Delta)\sqrt{z}(x-z)^2\tilde{f}^{\pm} \tag{4.105}$$

$$B(x,z) = a^{\pm}(\tilde{\Delta})\sqrt{z}x^2\tilde{f}^{\pm} \tag{4.106}$$

$$C(x,z) = \sqrt{z}x^2 < 0 \mid N_{II}(x)R^{\pm}(z)G_{-1} \mid \tilde{\Delta}^{\pm} > + \\ -\frac{a^{\pm}(\tilde{\Delta})}{2\sqrt{z}}(x^2+4zx)\tilde{f}^{\pm} - \frac{a^{\pm}(\Delta)}{\sqrt{z}}(x-z)^2\tilde{f}^{\mp} \tag{4.107}$$

where

$$\tilde{f}^{\pm} =< 0 \mid N_{II}(x)R^{\pm}_{\tilde{\Delta}}(z) \mid \tilde{\Delta}^{\mp} >= \tilde{h}^{\pm}(x-z)^{\tilde{\Delta}-\Delta-\Delta_x}x^{\Delta-\Delta_x-\tilde{\Delta}}$$

Taking the integral

$$\oint_0 \frac{F^{(II)}}{v^2}dv$$

in two different ways and using the identity

$$G_{-2} \mid \tilde{\Delta}^{\pm} >= \left(\frac{3}{2\tilde{\Delta}}a^{\pm}(\tilde{\Delta})L_{-1}^2 - \frac{8}{3}\frac{\tilde{\Delta}}{a^{\mp}(\tilde{\Delta})}L_{-2}\right) \mid \tilde{\Delta}^{\mp} > \tag{4.108}$$

(valid only for $\Delta = \Delta_{1,2}$ or $\Delta = \Delta_{2,1}$) we reduce the eq.(4.87) to the system of algebraic equations with the following consistency condition

$$\frac{1}{8} + \frac{3}{8\tilde{\Delta}}(\Delta_x - \Delta - \tilde{\Delta}) + \frac{3}{2\tilde{\Delta}}(\Delta_x - \Delta - \tilde{\Delta})(\Delta_x - \Delta - \tilde{\Delta} - 1) + \\ -\frac{8\tilde{\Delta}}{\tilde{\Delta} - \frac{c}{24}}(2\Delta - \Delta_x + \tilde{\Delta}) = \pm\sqrt{\frac{\Delta - \frac{c}{24}}{\tilde{\Delta} - \frac{c}{24}}} \tag{4.109}$$

The solutions of these equations give the dimensions of the second components of the NS fields, together with a few "wrong" dimensions, which can be eliminated considering the null vectors at the higher levels of degeneracy. We omit discussion of the higher level fusion rules since the Coulomb gas representation developed in the next section allows us to avoid the hard computations in the modified null-vector method described above.

Our experience in the application of this method to the computation of the 4-point function of the Ramon fields showed that there exist techical obstruction which make it a non-effective one. The only case in which these null-vector techniques work well is in the calculation of the Ising model's 4-point function as we have seen in the previous section.

4.3 Supersymmetric Coulomb gas

The generalization of the Coulomb gas to the case of N=1 superconformal models [16] is based on the properties of the free dimensionless superfield

$$S(z,\theta) = \phi(z) + \theta\psi(z)$$

(and $\overline{S}(\overline{z},\overline{\theta})$) with action

$$A(S,\overline{S}) = \frac{2}{\pi}\int dzd\overline{z}(\frac{1}{2}\partial\phi\overline{\partial\phi} - \psi\overline{\partial\psi}) \tag{4.110}$$

As follows from this expression, the propagator of the field is

$$< 0 \mid S(z_1,\theta_1)S(z_2,\theta_2) \mid 0 >= -\ln\frac{z_{12}}{R} \tag{4.111}$$

(R is an infrared cut-off). We note that the eq. (4.111) takes place only when ϕ and ψ are both periodic fields

$$\phi(e^{2\pi i}z) = \phi(z) \qquad \psi(e^{2\pi i}z) = \psi(z) \tag{4.112}$$

In order to construct the superfiels of the conformal grid we consider the so-called NS vertices

$$\mathcal{V}_{\alpha_j}(z_j,\theta_j) = e^{i\alpha_j S(z_j,\theta_j)} \qquad \alpha_j \in R$$

Since their N-point functions have both infrared and ultraviolet singularities

$$\begin{aligned} <\prod_{i=1}^{N}\mathcal{V}_{\alpha_j}(z_j,\theta_j) >&= \int \mathcal{D}S\prod_{i=1}^{N} e^{i\alpha_i S(z_i,\theta_i)}e^{-A(S,\overline{S})} \\ &= \left(\frac{a}{R}\right)^{(\sum_k \alpha_k)^2}\prod_{i<j=2}\left(\frac{z_{ij}}{a}\right)^{\alpha_i\alpha_j} \end{aligned} \tag{4.113}$$

(a is an ultraviolet cut-off), we have to introduce the renormalized vertex

$$V_\alpha(z,\theta) = \lim_{a\to 0} a^{-\frac{\alpha^2}{2}}e^{i\alpha S(z,\theta)} \equiv: e^{i\alpha S(z,\theta)} : \tag{4.114}$$

and to imply the neutrality condition

$$\sum_k \alpha_k = 0 \tag{4.115}$$

In this way we eliminate the cut-off dependence of the N-point function. Then the vertex propagator takes the form

$$< V_\alpha(z_1,\theta_1)V_{-\alpha}(z_2,\theta_2) >= (z_{12})^{-\alpha^2} \tag{4.116}$$

and consequently the fields $V_{\pm\alpha}$ have dimension

$$\Delta(\pm\alpha) = \frac{\alpha^2}{2}$$

Since from the action we get

$$T = \frac{1}{2} : (\partial\phi)^2 : + : \psi\partial\psi : \qquad G = i : \partial\phi\psi : \tag{4.117}$$

all the properties of the NS superfields can be verified explicity using the Wick theorem and the free propagator (4.116).Therefore this vertex represents aprimary superfield with dimension $\alpha^2/2$. The only defect of this construction is that the central charges of the model is $c = \frac{3}{2}$.

The construction which leads to the anomalous central charges

$$c = \frac{3}{2} - \frac{12}{p(p+2)}$$

are generated by the following improved action [61]

$$\mathcal{A}(S,\overline{S},\tilde{R}) = \frac{2}{\pi}\int dzd\overline{z}d\theta d\overline{\theta}(\frac{1}{2}DS\overline{DS} - 2\alpha_0\tilde{R}(S+\overline{S})) \tag{4.118}$$

where

$$\tilde{R} = \tau + \theta\chi + \overline{\theta}\overline{\chi} + \theta\overline{\theta}R$$

plays a role of supercurvature ($\Delta_{\tilde{R}} = 1$).

Taking the last component of $\tilde{R}$ to be a curvature of the sphere

$$R = \partial\overline{\partial}\ln g = -\pi\delta(z-z_\infty)\delta(\overline{z}-\overline{z}_\infty) \tag{4.119}$$

the superfield $\tilde{R}$ will represent the curvature of the "super-sphere" and the action (4.118) can be rewritten in the form

$$\mathcal{A}(S,\overline{S},\tilde{R}) = 2\alpha_0(S(z_\infty,\theta) + \overline{S}(\overline{z}_\infty,\overline{\theta})) + A(S,\overline{S}) \tag{4.120}$$

Therefore the "super-curvature" term in the eq. (4.118) introduces a vertex

$$e^{-2\alpha_0 i S}$$

at infinity in the new partition function (with charge $-2\alpha_0$)

$$Z(-2\alpha_0) = \int \mathcal{D}S e^{-A} \equiv \int \mathcal{D}S \lim_{|z|\to\infty} e^{-2\alpha_0 i(S(z,\theta)+\overline{S}(\overline{z},\overline{\theta})}e^{-A}\,|_{\theta=\overline{\theta}=0} \qquad (4.121)$$

Thus the correlation functions calculated with the new action (4.118) should satisfy a new neutrality condition

$$\sum_{i=1}^{N} \alpha_i = 2\alpha_0 \qquad (4.122)$$

For instance, we obtain that the only non-zero 2-point function is

$$\ll V_\alpha(z_1)V_{2\alpha_0-\alpha}(z_2) \gg = (z_{12})^{-\alpha(\alpha-2\alpha_0)} \qquad (4.123)$$

and therefore the vertices V_α and $V_{2\alpha_0-\alpha}$ have equal dimensions

$$\Delta(\alpha) = \Delta(2\alpha_0 - \alpha) = \frac{\alpha(\alpha-2\alpha_0)}{2} \qquad (4.124)$$

The action (4.118) gives a new expression for the super stress energy tensor

$$T(z) \;= -\frac{1}{2} : (\partial\phi)^2 : + : \psi\partial\psi : + i\alpha_0\partial^2\phi \qquad (4.125)$$

$$G(z) \;= i : \partial\phi\psi : + 2\alpha_0 : \partial\psi : \qquad (4.126)$$

and the central charge is a function of the charge at infinity α_0:

$$c = \frac{3}{2} - 12\alpha_0^2 \qquad (4.127)$$

Thus the different superconformal minimal models are parametrized by their charges at infinity

$$\alpha_0^2 = \frac{1}{p(p+2)}$$

The 4-point functions of the NS fields can be constructed in terms of the vertices (3.114) by the following procedure [16]

$$< \prod_{k=1}^{4} \phi_\Delta(z_k,\theta_k) >= \oint_{C_i} \prod_{i=1}^{n-1} d\varsigma_i dv_i \oint_{C_j} \prod_{j=1}^{m-1} d\eta_j dw_j \ll V_\alpha(z_1,\theta_1)$$
$$V_\alpha(z_2,\theta_2 V_{2\alpha_0-\alpha}(z_3,\theta_3)V_\alpha(z_4,\theta_4) \prod_{i=1}^{n-1} V_{\alpha_-} v_i,\varsigma_i) \prod_{j=1}^{m-1} V_{\alpha_+}(w_j,\eta_j) \gg \tag{4.128}$$

The superinvariant dimensionless screening operators

$$J_\pm = \oint_{C_\pm} d\theta dz V_{\alpha_\pm}(z,\theta) = \oint_{C_\pm} dz \psi(z) e^{i\alpha\phi(z)} \tag{4.129}$$

with charges and dimensions

$$\alpha_\pm = \alpha_0 \pm \sqrt{\alpha_0^2+1} \qquad \Delta(\alpha_\pm) = \frac{\alpha_\pm(\alpha_\pm - 2\alpha_0)}{2} = \frac{1}{2} \tag{4.130}$$

are introduced in the eq. (3.128) in order to screen the extra charge 2α. They generate non-trivial solutions of the neutrality condition (3.122)

$$\alpha_{n,m} = \frac{[(1-n)\alpha_- + (1-m)\alpha_+]}{2} \tag{4.131}$$

This quantization of the charges of the superfields leads to the well-known quantization of the dimension of the minimal models

$$\Delta_{n,m} = \frac{(n\alpha_- + m\alpha_+)^2 - (\alpha_- + \alpha_+)^2}{8} \tag{4.132}$$

which exactly coincides with the Kac's formula (4.61) if $(n-m) \in 2Z$.

The Ramond fields of the minimal models should have the same stress-energy tensor T and supercurrent G as the NS fields. The only difference is that in this case G(z) has to be an antiperiodic field and therefore we have to impose antiperiodic boundary conditions on the free Majorana field $\psi(z)$. Then the fields $\phi(z)$ and $\psi(z)$ cannot be combined in a superfield multiplet. An important role in the construction of the Ramond vertices is played by the "spin field" σ and $\tilde{\sigma}$, which are the lowest energy state for $\psi(z)$.

In fact, following the analogy with the Ramond sector of the superstring models [45,47], we define these vertices as follows [60,61]

$$R_\alpha(z) = \sigma(z) : e^{i\alpha\phi(z)} : \tag{4.133}$$

A direct inspection, based on the expression of the super stress energy tensor and on the Wick theorem for free fields, shows that the Ramond vertices satisfy the eq.(4.53) and their dimensions are

$$\Delta_R(\alpha) = \frac{1}{16} + \frac{\alpha(\alpha - 2\alpha_0)}{2} \tag{4.134}$$

In fact, since we have

$$G(z_1)R_\alpha(z_2) = \frac{\alpha - \alpha_0}{\sqrt{2}(z_1 - z_2)^{\frac{3}{2}}} R_\alpha(z_2) + \cdots \tag{4.135}$$

simple algebra gives

$$\frac{\alpha - \alpha_0}{\sqrt{2}} = \pm\sqrt{\Delta_R(\alpha) - \frac{c}{24}} \tag{4.136}$$

Therefore the vertices (3.133) form a correct representation of the Ramond algebra.

Accepting that the screening operators $J_\pm$ are the same as for the NS sector

$$J_\pm = \oint_{C_\pm} dz \psi(z) e^{i\alpha_\pm\phi(z)} \equiv \oint_{C_\pm} R_\pm(z) dz \tag{4.137}$$

(with antiperiodic $\psi(z)$)

we can construct the correlation functions of the Ramond fields, introducing the appropriate number of screening operators.

Considering the 4-point function of Ramond fields we get the charge quantization as for the NS case and that the dimensions are quantized in accordance with the Kac's formula (4.61)

$$\Delta^R_{n,m} = \frac{1}{16} + \frac{[(n\alpha_- + m\alpha_+)^2 - (\alpha_- + \alpha_+)^2]}{8} \tag{4.138}$$

$(n - m \in 2Z + 1)$

This screening procedure works well also in the case of mixed R-NS correlation functions.

4.4 Fusion rules and 4-point functions of Ramond fields

The screening procedure and the neutrality condition (4.122) applied to the 3-point functions generate the fusion rules for the fields of a given minimal model: in fact the primary field $\phi_{x,y}$ which enter in the OPE of two given fields ϕ_{n_1,m_1} and ϕ_{n_2,m_2} should have a non-zero 3- point function

$$< \phi_{n_1,m_1}(z_1)\phi_{n_2,m_2}(z_2)\phi_{x,y}(z_3) >$$

As we said before, the qualitive composition rules are

$$\begin{aligned} [R][R] &\sim [NS] \\ [R][NS] &\sim [R] \\ [NS][NS] &\sim [NS] \end{aligned}$$

Let us start with fusion rules in the pure NS sector considering the correlation function of 3 superfields: it is known that there exist two different structure in it, an even part and an odd one, eq. (4.66). This structure gives rise to different fusion rules in the odd and even parts of the fields, as clarify in ref [17].

In the Coulomb gas picture with the screening operators, three different ways exists for constructing 3-point function, depending on which field is put the conjugate charge: in each case there exists a number of screening operators which assures the neutrality condition (4.122).

$$\prod_{i=1}^{k-1}\prod_{j=1}^{l-1}\oint_{C_i} d\epsilon_i dv_i \oint_{C_j} d\eta_j dw_j \ll V_{\alpha_{n_1,m_1}}(z_1,\theta_1)$$
$$V_{\alpha_{n_2,m_2}}(z_2,\theta_2)V_{2\alpha_0-\alpha_{x,y}}(z_3,\theta_3)V_{\alpha_-}(v_i,\epsilon_i)V_{\alpha_+}(w_j,\eta_j) \gg \qquad (4.139)$$
$$\prod_{i=1}^{p-1}\prod_{j=1}^{q-1}\oint_{C_i} d\epsilon_i dv_i \oint_{C_j} d\eta_j dw_j \ll V_{\alpha_{n_1,m_1}}(z_1,\theta_1)$$
$$V_{2\alpha_0-\alpha_{n_2,m_2}}(z_2,\theta_2)V_{\alpha_{x,y}}(z_3,\theta_3)V_{\alpha_-}(v_i,\epsilon_i)V_{\alpha_+}(w_j,\eta_j) \gg \qquad (4.140)$$
$$\prod_{i=1}^{a-1}\prod_{j=1}^{b-1}\oint_{C_i} d\epsilon_i dv_i \oint_{C_j} d\eta_j dw_j \ll V_{2\alpha_0-\alpha_{n_1,m_1}}(z_1,\theta_1)$$
$$V_{\alpha_{n_2,m_2}}(z_2,\theta_2)V_{\alpha_{x,y}}(z_3,\theta_3)V_{\alpha_-}(v_i,\epsilon_i)V_{\alpha_+}(w_j,\eta_j) \gg \qquad (4.141)$$

From the neutrality condition (4.122) we get the following expressions for the charge $\alpha_{x,y}$

$$\begin{aligned}\alpha^1_{x,y} &= \frac{1}{2}\{[1-(n_1+n_2-2k+1)]\alpha_- + [1-(m_1+m_2-2l+1)]\alpha_+\}\\ \alpha^2_{x,y} &= \frac{1}{2}\{[1-(n_2-n_1+2p-1)]\alpha_- + [1-(m_2-m_1+2q-1)]\alpha_+\}\\ \alpha^3_{x,y} &= \frac{1}{2}\{[1-(n_1-n_2+2a-1)]\alpha_- + [1-(m_1-m_2+2b-1)]\alpha_+\}\end{aligned}$$

with the conditions

$$\begin{aligned}n_1+n_2-2k+1>0\\ n_2-n_1+2p-1>0\\ n_1-n_2+2a-1>0\end{aligned}$$

and analogous for the α_+ part.

We have to take the common solutions of these equations, that is

$$\begin{aligned}x &= \mid n_1-n_2\mid+1, \mid n_1-n_2\mid+3, \cdots n_1+n_2-1\\ y &= \mid m_1-m_2\mid+1, \mid m_1-m_2\mid+3, \cdots m_1+m_2-1\end{aligned}$$

and this gives the fusion rules for the fields ϕ_{n_1,m_1}, ϕ_{n_2,m_2}

$$[\phi_{n_1,m_1}][\phi_{n_2,m_2}] = \sum_{x=|n_1-n_2|+1}^{n_1+n_2-1} \sum_{y=|m_1-m_2+1}^{m_1+m_2-1} [\phi_{x,y}] \tag{4.142}$$

The even fusion rules are recovered when there is an even number of screening operators, while the odd ones when there is an odd number, as can be seen from the following examples [61]

First let us consider the top term of the sum (4.142) when there is no screening operator at all: we have

$$\begin{aligned}&\ll V_{\alpha_1}(z_1,\theta_1)V_{\alpha_2}(z_2,\theta_2)V_{2\alpha_0-\alpha_3}(z_3,\theta_3)\gg=\\ &= (z_{12})^{\alpha_1\alpha_2}(z_{13})^{\alpha_1(2\alpha_0-\alpha_3)}(z_{23})^{\alpha_2(2\alpha_0-\alpha_3)}\end{aligned} \tag{4.143}$$

From the neutrality condition we have

$$\alpha_1+\alpha_2=\alpha_3 \tag{4.144}$$

ant it is easy to rewrite this 3-point function in terms of the anomalous dimensions

$$\ll V_{\alpha_1}(z_1,\theta_1)V_{\alpha_2}(z_2,\theta_2)V_{2\alpha_0-\alpha_3}(z_3,\theta_3) \gg =$$
$$= (z_{12})^{\Delta_3-\Delta_1-\Delta_2}(z_{13})^{\Delta_2-\Delta_1-\Delta_3}(z_{23})^{\Delta_1-\Delta_2-\Delta_3} \qquad (4.145)$$

where the absence of the odd term is evident.

Now consider the case in which there is only one screening operator, say J_+. For semplicity we put $z_1 \to \infty, z_2 = z, z_3 = 0$ and $\theta_3 = 0$. This choice is always possible due to the Osp(2,1) invariance. The 3-point function is given by

$$\oint d\theta dw \lim_{z_1\to\infty} z_1^{2\Delta_1} \ll V_{\alpha_1}(z_1,\theta_1)V_{\alpha_2}(z_2,\theta_2)V_{2\alpha_0-\alpha_3}(z_3,0)V_{\alpha_+}(w,\theta) \gg =$$
$$= (z_{23})^{\alpha_2(2\alpha_0-\alpha_3)} \oint d\theta dw (z_2 - w - \theta_2\theta)^{\alpha_2\alpha_+} w^{\alpha_+(2\alpha_0-\alpha_3)} \qquad (4.146)$$

Taking the θ-integral, we get

$$< \Phi_1(\infty)\Phi_2(z,\theta_2)\Phi_3(z_3,0) >= \theta_2(z_{23})^{\alpha_2(2\alpha_0-\alpha_3)} \oint dw w^{\alpha_+(2\alpha_0-\alpha_3)}(z-w)^{\alpha_+\alpha_2-1} \qquad (4.147)$$

and the remaining integral is a particular case of integral representation of the hypergeometric function

$$I(a,b,c,z) = \int_0^z dv v^a(1-v)^b(z-v)^c = const\ z^{1+a+c}$$

with b=0, so

$$< \Phi_1(\infty)\Phi_2(z,\theta_2)\Phi_3(z_3,0) >=$$
$$= \theta_2(z_{23})^{\alpha_2(2\alpha_0-\alpha_3)+(2\alpha_0-\alpha_3)\alpha_+ + \alpha_2\alpha_+} =$$
$$= \theta_2(z_{23})^{\Delta_1-\Delta_2-\Delta_3-\frac{1}{2}} \qquad (4.148)$$

The presence of θ_2 signals that we have the odd part of the 3-point function.

Applying the (4.142), implemented with the symmetry of the formula of dimension

$$\Delta(\alpha) = \Delta(2\alpha_0 - \alpha)$$

we have the following fusion rules for the first two models of the unitary series in the NS sector

$$p = 3 \qquad c = \frac{7}{10} \qquad (TIM)$$

$$[\frac{1}{10}]_{NS}[\frac{1}{10}]_{NS} = [0]^{even}_{NS} + [\frac{1}{10}]^{odd}_{NS}$$

$$p = 4 \qquad c = 1 \qquad (KT)$$

$$\begin{array}{ll} [1]_{NS}[1]_{NS} = [0]^{even}_{NS} & ,[\frac{1}{6}]_{NS}[\frac{1}{6}]_{NS} = [0]^{even}_{NS} + [\frac{1}{6}]^{odd}_{NS} + [1]^{even}_{NS} \\ [1]_{NS}[\frac{1}{6}]_{NS} = [\frac{1}{6}]^{even}_{NS} & ,[\frac{1}{6}]_{NS}[\frac{1}{16}]_{NS} = [\frac{1}{16}]^{even}_{NS} + [\frac{1}{16}]^{odd}_{NS} \\ [1]_{NS}[\frac{1}{16}]_{NS} = [\frac{1}{16}]^{odd}_{NS} & ,[\frac{1}{16}]_{NS}[\frac{1}{16}]_{NS} = [\frac{1}{6}]^{even}_{NS} + [\frac{1}{6}]^{odd}_{NS} + [0]^{even}_{NS} + [1]^{odd}_{NS} \end{array}$$

To find the fusion rules of two given Ramond fields R_{n_1,m_1} and R_{n_2,m_2} we have to look at the non-zero 3-point functions with the NS superfield $\Phi_{x,y}$,

$$< R_{n_1,m_1}(z_1) R_{n_2,m_2}(z_2) \Phi_{x,y}(z_3, \theta_3) >$$

The same procedure as before leads to an analogous formula

$$[R_{n_1,m_1}][R_{n_2,m_2}] = \sum_{x=|n_1-n_2|+1}^{n_1+n_2-1} \sum_{|m_1-m_2|+1}^{m_1+m_2-1} [\Phi_{x,y}] \qquad (4.149)$$

The set of these fusion rules completes the structure of the associative OPE algebra of the fields of the corresponding supersymmetric minimal models. In the case p=3 ($c = \frac{7}{10}$) our formula (4.149) reproduces the well-known fusion rules of TIM [10].

$$\begin{array}{l} [\frac{3}{80}]_R[\frac{3}{80}]_R = [0]_{NS} + [\frac{1}{10}]_{NS} \\ [\frac{7}{16}]_R[\frac{7}{16}]_R = [0]_{NS} \\ [\frac{3}{80}]_R[\frac{7}{16}]_R = [\frac{1}{10}]_{NS} \end{array}$$

while in the case p=4 (c=1) the structure of the Ramond sector is determined by the following fusion rules

$$
\begin{array}{llll}
[\frac{1}{16}]_R[\frac{1}{16}]_R & = [0]_{NS} + [\frac{1}{6}]_{NS} & [\frac{1}{16}]_R[\frac{3}{8}]_R = [\frac{1}{16}]_{NS} \\
[\frac{3}{8}]_R[\frac{3}{8}]_R & = [0]_{NS} + [1]_{NS} & [\frac{9}{16}]_R[\frac{1}{16}]_R = [\frac{1}{6}]_{NS} + [1]_{NS} \\
[\frac{9}{16}]_R[\frac{9}{16}]_R & = [0]_{NS} + [\frac{1}{6}]_{NS} & [\frac{1}{24}]_R[\frac{1}{16}]_R = [\frac{1}{16}]_{NS} \\
[\frac{1}{24}]_R[\frac{1}{24}]_R & = [0]_{NS} + [1]_{NS} + [\frac{1}{6}]_{NS} & [\frac{9}{16}]_R[\frac{1}{24}]_R = [\frac{1}{16}]_{NS}
\end{array}
$$

Now we want to discuss in detail the computation of the 4-point functions of the Ramond fields. Using the vertex representation and the screening procedure, we have [60,61]

$$
\begin{aligned}
&< R_{n_1,m_1}(z_1)R_{n_2,m_2}(z_2)R_{n_3,m_3}(z_3)R_{n_4,m_4}(z_4) >= \\
&= \oint_{C_i} dv_i \oint_{C_j} dw_j < \sigma(z_1)\sigma(z_2)\sigma(z_3)\prod_{i=1}^{n-1}\psi(v_i)\prod_{j=1}^{m-1}\psi(w_j)\sigma(z_4) > \\
&\ll e^{i\alpha_{n_1,m_1}\phi(z_1)}e^{i\alpha_{n_2,m_2}\phi(z_2)}e^{i\alpha_{n_3,m_3}\phi(z_3)}e^{i(2\alpha_0-\alpha_{n_4,m_4})\phi(z_4)} \\
&\prod_{i=1}^{n-1} e^{i\alpha_-\phi(v_i)}\prod_{j=1}^{m-1} e^{i\alpha_+\phi(w_j)} \gg
\end{aligned}
\qquad (4.150)
$$

where the second factor in the integrand is the well-known multipoint function of the modified Coulomb system

$$
\ll \prod_{k=1}^{N} e^{i\alpha_k\phi(z_k)} \gg = \prod_{l<n=2}^{N}(z_{ln})^{\alpha_n\alpha_l} \qquad \sum_{i=1}^{N}\alpha_i = 2\alpha_0 \qquad (4.151)
$$

and the first factor is calculated by the recursive equation for the Ising model, eq.(3.25). Then the 4-point function for the same field takes the form

$$
\begin{aligned}
&< R_{n_1,m_1}(\infty)R_{n_2,m_2}(z)R_{n_3,m_3}(1)R_{n_4,m_4}(0) >= z^{\alpha^2}(z-1)^{\alpha(2\alpha_0-\alpha)} \\
&\oint_{C_i} dw_i \oint_{C_j} dv_j < \sigma(\infty)\sigma(z)\sigma(1)\prod_{i=1}^{n-1}\psi(w_i)\prod_{j=1}^{m-1}\psi(v_j)\sigma(0) > \\
&\prod_{l<k=2}^{n-1} w_{lk}^{\alpha_-^2}\prod_{s<t=2}^{m-1} v_{st}^{\alpha_+^2}\prod_{i=1}^{n-1}\prod_{j=1}^{m-1}[((w_i-z)w_i)^{\alpha_-\alpha}(w-1)^{\alpha_-(2\alpha_0-\alpha)} \\
&((v_j-z)v_j)^{\alpha_+\alpha}(v_j-1)^{\alpha_+(2\alpha_0-\alpha)}(w_i-v_j)^{\alpha_-\alpha_+}]
\end{aligned}
\qquad (4.152)
$$

The integration contours C_i are fixed by the branch cut singularities of the integrand. Thus for the general expression of the 4-point function of the Ramond fields we should take a linear combination of all 4-point functions corresponding to the possible independent choices of the contours C_i.

Consider, for semplicity, the case in which only one screening operator is present, say J_+. We get from (4.152) (taking the sum of the functions $< RRRR >$ and $< \tilde{R}R\tilde{R}R >$)

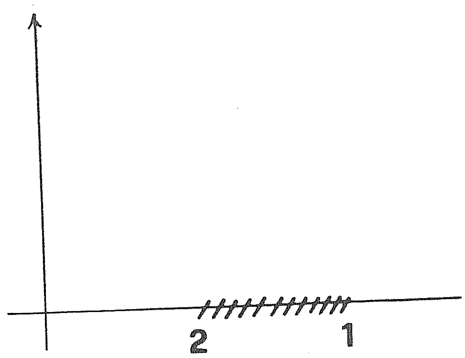
$$G^p_{1,2}(z) = z^{\alpha_{12}(2\alpha_0-\alpha_{12})+\frac{1}{8}}(z-1)^{\alpha_{12}^2-\frac{1}{8}} \left[\sqrt{1+\sqrt{1-\frac{1}{z}}}\sum_{a=1}^{2} A^a(\tilde{I}_a + I_a) + \sqrt{1-\sqrt{1-\frac{1}{z}}}\sum_{a=1}^{2}(\tilde{I}_a - I_a)\right] \tag{4.153}$$

where

$$\tilde{I}_a = \sqrt{z}\oint_{C_a} dv v^{\frac{3}{p}}(1-v)^{-\frac{1}{p}}(z-v)^{-(1+\frac{1}{p})} \tag{4.154}$$

$$I_a = \sqrt{z-1}\oint_{C_a} dv v^{1+\frac{3}{p}}[(1-v)(z-v)]^{-(1+\frac{1}{p})} \tag{4.155}$$

We see that only in the case of TIM (p=3), $\Delta_{1,2} = \frac{7}{16}$, we have a unique branch-cut from 1 to z, as shown in the figure below

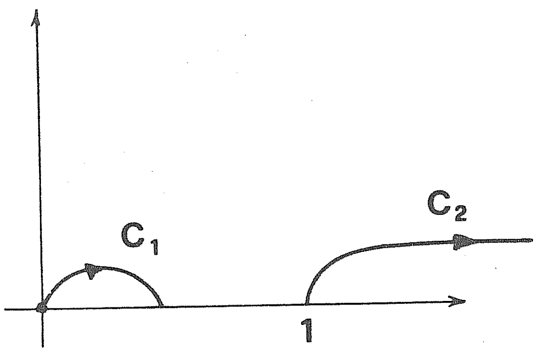


and correspondingly only one independet integration contour. This case is a good test for our construction since the expression for the 4-point function of the sub-leading magnetization field $\Delta = \frac{7}{16}$ can be calculated from the "null-vector" method of the Virasoro algebra. The solution of the corresponding differential equation is

$$< R_{\frac{7}{16}}(\infty)R_{\frac{7}{16}}(z)R_{\frac{7}{16}}(1)R_{\frac{7}{16}}(0) >= z^{-\frac{5}{8}}(z-1)^{-\frac{7}{8}}$$
$$\{C_1\sqrt{1+\sqrt{1-\frac{1}{z}[2(z^2-z+1)-(2z-1)\sqrt{z(z-1)}]}}+$$
$$C_2\sqrt{1-\sqrt{1-\frac{1}{z}[2(z^2-z+1)+(2z-1)\sqrt{z(z-1)}]}}\} \quad (4.156)$$

The comparison of this expression with the corresponding one from eq.(4.153) shows their full coincidence.

For the other superconformal models, with $p > 3$, we have two independent contours, one from 0 to z ant the other from 1 to ∞, as in figure



and we have four independent solutions for the corresponding 4-point function

$$G_{12}^{p\neq 3}(z) = (z-1)^{\frac{p+4}{8p}} z^{-\frac{p+12}{8p}} \sum_{i=1}^{4} A^i W_i(z) \tag{4.157}$$

where, putting $h = \frac{1}{p}$

$$\begin{aligned} W_1(z) = \sqrt{1+\sqrt{1-\frac{1}{z}}}&[B(1-h,h)\sqrt{z}F(1+h,-h,1-2h,z)+ \\ &-B(-h,-h)\sqrt{z-1}F(1+h,-h,-2h,z)] \end{aligned} \tag{4.158}$$

$$\begin{aligned} W_2(z) = \sqrt{1+\sqrt{1-\frac{1}{z}}}&[B(1+3h,-h\sqrt{z}z^{2h}F(h,1+h,1+2h,z)+ \\ &+B(2+3h,-h)\sqrt{z-1}z^{-2-2h}F(1+h,2+3h,2+2h,z)] \end{aligned} \tag{4.159}$$

$$\begin{aligned} W_3(z) = \sqrt{1-\sqrt{1-\frac{1}{z}}}&[B(1-h,-h)\sqrt{z}F(1+h,-h,1-2h,z)+ \\ &+B(-h,-h)\sqrt{z-1}F(1+h,-h,-2h,z)] \end{aligned} \tag{4.160}$$

$$\begin{aligned} W_4(z) = \sqrt{1-\sqrt{1-\frac{1}{z}}}&[B(1+3h,-h\sqrt{z}z^{2h}F(h,1+3h,1+2h,z)+ \\ &-B(2+3h,-h)\sqrt{z-1}z^{-2-2h}F(1+h,2+3h,2+2h,z)] \end{aligned} \tag{4.161}$$

In the same way we can calculate the 4-point function of the Ramond field $\Delta_{2,1}$, since it needs only the insertion of one J_- screening operator

$$\begin{aligned} G_{21}^{p}(z) &= z^{\alpha_{21}(2\alpha_0-\alpha_{21})+\frac{1}{8}}(z-1)^{\alpha_{21}^2-\frac{1}{8}} \\ &[\sqrt{1+\sqrt{1-\frac{1}{z}}}\sum_{a=1}^{2} A^a(\tilde{K}_a+K_a)+\sqrt{1-\sqrt{1-\frac{1}{z}}}\sum_{a=1}^{2} B^a(\tilde{K}_a-K_a)] \end{aligned} \tag{4.162}$$

where

$$\tilde{K}_a = \sqrt{z}\oint_{C_a} dv v^{-\frac{3}{p+2}}(1-v)^{\frac{1}{p+2}}(z-v)^{\frac{1}{p+2}-1} \tag{4.163}$$

$$K_a = \sqrt{z-1}\oint_{C_a} dv v^{1-\frac{3}{p+2}}(1-v)^{\frac{1}{p+2}-1}(z-v)^{\frac{1}{p+2}-1} \tag{4.164}$$

From the branch-cut analysis of the integrand we that there exist two independent integration contours for each p. Putting $a = \frac{1}{p+2}$, we have four independent solutions

$$G^p_{21}(z) = z^{\frac{10-p}{8(p+2)}}(z-1)^{\frac{p-2}{8(p+2)}}\sum_{i=1}^{4} A^i Y_i(z) \tag{4.165}$$

where

$$Y_1(z) = \sqrt{1+\sqrt{1-\frac{1}{z}}}[B(a,1+a)\sqrt{z}F(1-a,a,1+2a,z)+ \\ -B(a,a)\sqrt{z-1}F(1-a,a,2a,z)] \tag{4.166}$$

$$Y_2(z) = \sqrt{1+\sqrt{1-\frac{1}{z}}}[B(1-3a,a)\sqrt{z}z_{-2a}F(-a,1-3a,1-2a,z)+ \\ +B(2-3a,a)\sqrt{z-1}z^{1-2a}F(1-a,2-3a,2-2a,z)] \tag{4.167}$$

$$Y_3(z) = \sqrt{1-\sqrt{1-\frac{1}{z}}}[B(a,1+a)\sqrt{z}F(1-a,a,1+2a,z)+ \\ +B(a,a)\sqrt{z-1}F(1-a,a,2a,z)] \tag{4.168}$$

$$Y_4(z) = \sqrt{1-\sqrt{1-\frac{1}{z}}}[B(1-3a,a)\sqrt{z}z_{-2a}F(-a,1-3a,1-2a,z)+ \\ -B(2-3a,a)\sqrt{z-1}z^{1-2a}F(1-a,2-3a,2-2a,z)] \tag{4.169}$$

According to this formula, the 4-point function of the magnetization field $\Delta_{2,1} = \frac{3}{80}$ of TIM is

$$< R_{\frac{3}{80}}(\infty)R_{\frac{3}{80}}(z)R_{\frac{3}{80}}(1)R_{\frac{3}{80}}(0) >= z^{\frac{7}{40}}(z-1)^{\frac{1}{40}} \\ [\sqrt{1+\sqrt{1-\frac{1}{z}}}\sum_{a=1}^{2} C^a L_a(z) + \sqrt{1-\sqrt{1-\frac{1}{z}}}\sum_{a=1}^{2} D^a \tilde{L}_a(z)] \tag{4.170}$$

where

$$\tilde{L}_1 = [B(\frac{1}{5},\frac{6}{5})\sqrt{z}F(\frac{4}{5},\frac{1}{5},\frac{7}{5},z)\pm \\ B(\frac{1}{5},\frac{1}{5})\sqrt{z-1}F(\frac{4}{5},\frac{1}{5},\frac{2}{5},z)] \tag{4.171}$$

$$\tilde{L}_2 = [B(\frac{2}{5}, \frac{1}{5})\sqrt{z} z^{-\frac{2}{5}} F(-\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, z) \mp$$
$$B(\frac{7}{5}, \frac{1}{5})\sqrt{z-1} z^{\frac{3}{5}} F(\frac{4}{5}, \frac{7}{5}, \frac{8}{5}, z)] \qquad (4.172)$$

Similarly the 4-point function of the order parameter field $R_{\frac{1}{16}}$ in the KT model (c=1) is

$$< R_{\frac{1}{16}}(\infty) R_{\frac{1}{16}}(z) R_{\frac{1}{16}}(1) R_{\frac{1}{16}}(0) >= z^{\frac{1}{8}}(z-1)^{\frac{1}{24}}$$
$$[\sqrt{1+\sqrt{1-\frac{1}{z}}} \sum_{a=1}^{2} C^a M_a(z) + \sqrt{1-\sqrt{1-\frac{1}{z}}} \sum_{a=1}^{2} D^a \tilde{M}_a(z)] \qquad (4.173)$$

where

$$\tilde{M}_1 = [B(\frac{1}{6}, \frac{7}{6})\sqrt{z} F(\frac{5}{6}, \frac{1}{6}, \frac{4}{3}, z) \pm$$
$$B(\frac{1}{6}, \frac{1}{6})\sqrt{z-1} F(\frac{5}{6}, \frac{1}{6}, \frac{1}{3}, z)] \qquad (4.174)$$
$$\tilde{M}_2 = [B(\frac{1}{2}, \frac{1}{6})\sqrt{z} z^{-\frac{1}{3}} F(-\frac{1}{6}, \frac{1}{2}, \frac{4}{3}, z) \mp$$
$$B(\frac{3}{2}, \frac{1}{6})\sqrt{z-1} z^{\frac{2}{3}} F(\frac{5}{6}, \frac{3}{2}, \frac{5}{3}, z)] \qquad (4.175)$$

4.5 Physical correlators and structure constants of OPE

As we have discussed in the Virasoro case, section (1.5.2), also in the superconformal models to obtain the physical correlators of scalar fields ($\Delta = \overline{\Delta}$) one has to combine the analytic part with the antianalytic one

$$G(z,\overline{z}) = \sum_{i,j} I_{ij} W_i(z) \overline{W(z)}_j \tag{4.176}$$

where $\{W_i\}$ is the set of functions coming from the independent integration contours of screening procedure.

To these functions it is associated a monodromy group related to their analytic continuation around the singular points, which we choose to be 0,1,∞. We have to calculate the monodromy matrices g_0, g_1 and g_∞ and to impose the condition of monodromy invariance

$$\begin{aligned} G(z,\overline{z}) &= \sum_{ij} I_{ij} W_i \overline{W}_j = \sum_{ij}\sum_{kp} I_{ij} (g_l)_{ik} (\overline{g})_{jp} W_k \overline{W}_p = \\ &= \sum_{kp} (\sum_{ij} (g_l)^t_{ki} I_{ij} (\overline{g}_l)_{jp}) W_k \overline{W}_p \end{aligned} \tag{4.177}$$

The homogeneous equations

$$I_{kp} = (g_l)^t_{ki} I_{ij} (\overline{g})_{jp} \tag{4.178}$$

determine I_{ij} up an overall factor related to the normalization of 2-point functions.

Let us consider, first, the 4-point function of the Ramond field of dimension $\Delta = \frac{7}{16}$ in the tricritical model

$$< R_{\frac{7}{16}}(\infty) R_{\frac{7}{16}}(z) R_{\frac{7}{16}}(1) R_{\frac{7}{16}}(0) >= z^{-\frac{5}{8}} (z-1)^{-\frac{7}{8}} [C_1 W_1 + C_2 W_2] \tag{4.179}$$

$$W_1 = \sqrt{1 + \sqrt{1 - \frac{1}{z}} [2(z^2 - z + 1) - (2z-1)\sqrt{z(z-1)}]} \tag{4.180}$$

$$W_2 = \sqrt{1 - \sqrt{1 - \frac{1}{z}} [2(z^2 - z + 1) + (2z-1)\sqrt{z(z-1)}]} \tag{4.181}$$

In this case it is sufficient to calculate only two monodromy matrices, say g_1 and g_∞

$$g_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad g_\infty = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.182}$$

The monodromy invariant correlator is

$$G(z,\bar{z}) = \lambda \mid z \mid^{-\frac{5}{4}} \mid z-1 \mid^{-\frac{7}{4}} (W_1\overline{W}_1 + W_2\overline{W}_2) \tag{4.183}$$

and the overall factor λ is fixed to be

$$\lambda = \frac{1}{8} \tag{4.184}$$

if we normalized

$$< R_{\frac{7}{16}}(\infty) R_{\frac{7}{16}}(0) >= 1 \tag{4.185}$$

One can works out, using the formulas reported in the appendix C, the general case of the 4-point function of the Ramond fields $R^p_{1,2}$, for each value of the central charge c_p

$$G^p_{1,2}(z,\bar{z}) = \lambda_{1,2}(p) \mid z \mid^{-\frac{p+12}{4p}} \mid z-1 \mid^{\frac{p+4}{4p}}$$
$$[W_1\overline{W}_1 + W_3\overline{W}_3 + (4\cos^2(\frac{\pi}{p}) - 1)(W_2\overline{W}_2 + W_4\overline{W}_4)] \tag{4.186}$$

where $W_i(z)$ are those of eq.(4.158)-(4.161).This formula includes also the precedent case of $\Delta = \frac{7}{16}$ (p=3), since

$$4\cos^2(\frac{\pi}{3}) - 1 = 0$$

and the hypergeometric functions W_1 and W_2 are elementary ones.

The same calculations for the 4-point functions of the Ramond fields $R^p_{2,1}$ give as final result

$$G^p_{2,1}(z,\bar{z}) = \lambda_{2,1}(p) \mid z \mid^{\frac{10-p}{4(p+2)}} \mid z-1 \mid^{\frac{p-2}{4(p+2)}}$$
$$[Y_1\overline{Y}_1 + Y_3\overline{Y}_3 + (a\cos^2(\frac{\pi}{p+2}) - 1)(Y_2\overline{Y}_2 + Y_4\overline{Y}_4)] \tag{4.187}$$

where $Y_i(z)$ are the functions defined in the equations (4.166)-(4.169).

Analogously the monodromy invariant expression for the correlator involving the Ramond fields $R^p_{1,2}$ and $R^p_{2,1}$

$$< R^p_{12}(\infty) R^p_{12}(z,\bar{z}) R^p_{21}(1,1) R^p_{21}(0,0) >$$

is

$$G^p_{12,21}(z,\bar{z}) = \frac{1}{8} \mid z \mid^{-\frac{3}{4}} \mid z-1 \mid^{-\frac{3}{4}} (N_1\overline{N}_1 + N_2\overline{N}_2) \quad (4.188)$$

$$N_1(z) = \sqrt{1+\sqrt{1-\frac{1}{z}}(\frac{1-2z+2\sqrt{z(z-1)}}{\sqrt{z}})} \quad (4.189)$$

$$N_2(z) = \sqrt{1-\sqrt{1-\frac{1}{z}}(\frac{1-2z-2\sqrt{z(z-1)}}{\sqrt{z}})} \quad (4.190)$$

with the usual normalization obtained by the identity channel ($z \to \infty$).

Looking at the singularities of the 4-point functions, in the limit $\mid z \mid \to 1$, we can recover the operator product expansion

$$[R^p_{21}][R^p_{21}] = 1 + c_1(p)\Phi^p_{31} \quad (4.191)$$

$$[R^p_{12}][R^p_{12}] = 1 + c_2(p)\Phi^p_{13} \quad (4.192)$$

$$[R^p_{12}][R^p_{21}] = c_3(p)\Phi^p_{22} \quad (4.193)$$

and the structure constants c_1, c_2, c_3.

In fact, using the OPE

$$\phi_{\Delta_1}(z,\bar{z})\phi_{\Delta_2}(1,1) \simeq \sum_k \frac{c_k}{\mid z-1 \mid^{2(\Delta_1+\Delta_2-\Delta_k)}} \phi_k(1,1) \quad (4.194)$$

with normalization fixed by the identity family, i.e.

$$c_0 = 1$$

we have

$$G(z,\bar{z}) =< R_\Delta(\infty)R_\Delta(z,\bar{z})R_\Delta(1,1)R_\Delta(0,0) >=$$
$$\simeq \sum_k \frac{c_k}{\mid z-1 \mid^{2(2\Delta-\Delta_k)}} < R_\Delta(\infty)\Phi_k(1,1)R_\Delta(0,0) >= \sum_k \frac{c_k^2}{\mid z-1 \mid^{2(2\Delta-\Delta_k)}} \quad (4.195)$$

and the contribution of the different conformal families is identified by the power singularities.

In the case $[R^p_{21}][R^p_{21}]$, taking the limit $|z| \to 1$ in eq.(4.187), we have

$$Y_1\overline{Y}_1 + Y_3\overline{Y}_3 \to 2\{| B(a,1+a)\frac{\Gamma(1+2a)\Gamma(2a)}{\Gamma(3a)\Gamma(1+a)} |^2 + \\ + | z-1 |^{4a-1} | B(a,a)\frac{\Gamma(2a)\Gamma(1-2a)}{\Gamma(1-a)\Gamma(a)} |^2\} \tag{4.196}$$

$$Y_2\overline{Y}_2 + Y_4\overline{Y}_4 \to 2\{| B(1-3a,a)\frac{\Gamma(1-2a)\Gamma(2a)}{\Gamma(a)\Gamma(1-a)} |^2 + \\ + | z-1 |^{4a-1} | B(2-3a,a)\frac{\Gamma(2-2a)\Gamma(1-2a)}{\Gamma(1-a)\Gamma(2-3a)} |^2\} \tag{4.197}$$

$(a = \frac{1}{p+2})$

The second block in both the expressions comes from the unit operator and from this we can normalize the function

$$\lambda_{2,1}(a) = \frac{1}{32\cos^2(\pi a)} | \frac{\Gamma(-a)}{\Gamma(a)\Gamma(-2a)} |^2 \tag{4.198}$$

while the first block in both the expressions are identified as the contribution of the $\Phi^p_{3,1}$ operator, so we obtain

$$c_1(p) = \frac{1}{2}\frac{\Gamma(\frac{2}{p+2})\Gamma(\frac{p-1}{p+2})}{\Gamma(-\frac{2}{p+2})\Gamma(\frac{p+3}{p+2})}\sqrt{4\cos^2(\frac{\pi}{p+2}) - 1} \tag{4.199}$$

Similarly in the case $[R_{12}][R_{12}]$ the results are

$$\lambda_{12}(p) = \frac{1}{32\cos^2(\frac{\pi}{p})} | \frac{\Gamma(\frac{1}{p})}{\Gamma(-\frac{1}{p})\Gamma(\frac{2}{p})} |^2 \tag{4.200}$$

$$c_2(p) = \frac{3}{2}\frac{\Gamma(\frac{3}{p})\Gamma(-\frac{2}{p})}{\Gamma(\frac{2}{p})\Gamma(-\frac{1}{p})}\sqrt{4\cos^2(\frac{\pi}{p}) - 1} \tag{4.201}$$

$c_2(3)$ is correcty zero, since in the OPE of the Ramond field $R_{\frac{7}{16}}$ exists only the identity family and the operator Φ_{13} decouples in this case from the conformal grid.

In the last case of OPE ,eq. (4.193), the final result is very simple

$$c_3(p) = \frac{1}{2} \tag{4.202}$$

We can also extract the structure constants of the NS fields Φ^p_{31} with itself, i.e. the constants which appear in the 3-point function of this field. To do this, we start computing the following correlation function

$$\begin{aligned}
&< \Phi^p_{31}(\infty)\Phi^p_{31}(1)R^p_{21}(z)R^p_{21}(0) >= \\
&\quad = \oint dv < V_{2\alpha_0-\alpha_{31}}(\infty)V_{\alpha_{31}}(1)V_{\alpha_{21}}(z)V_{\alpha_{21}}(0)V_{\alpha_-}(v) = \\
&\quad = z^{\alpha^2_{21}}(z-1)^{\alpha_{31}\alpha_{21}} \oint dv v^{\alpha-\alpha_{21}}(1-v)^{\alpha-\alpha_{31}}(z-v)^{\alpha-\alpha_{21}} < \sigma(z)\psi(v)\sigma(0) >= \\
&\quad = z^{\alpha^2_{21}+\frac{3}{8}}(z-1)^{\alpha_{31}\alpha_{21}} \oint dv v^{\alpha-\alpha_{21}-\frac{1}{2}}(1-v)^{\alpha-\alpha_{31}}(z-v)^{\alpha-\alpha_{21}-\frac{1}{2}} = \\
&\quad = z^{\frac{5p+6}{8(p+2)}}(z-1)^{\frac{p}{2(p+2)}} \left\{ \begin{array}{l} B(\frac{2p}{p+2},\frac{2}{p+2})F(\frac{p+1}{p+2},\frac{2p}{p+2},\frac{2p+2}{p+2},z) \\ B(\frac{1}{p+2},\frac{1}{p+2})z^{-\frac{p}{p+2}}F(\frac{p}{p+2},\frac{1}{p+2},\frac{2}{p+2},z) \end{array} \right.
\end{aligned} \tag{4.203}$$

The monodromy invariant solution with the usual normalization is

$$\begin{aligned}
G(z,\overline{z}) =& | z |^{\frac{5p+6}{4(p+2)}} | z-1 |^{\frac{p}{p+2}} \{| z^{-\frac{2p}{p+2}} | F(\frac{p}{p+2},\frac{1}{p+2},\frac{2}{p+2},z) |^2 + \\
&+\frac{s(\frac{2}{p+2})s(\frac{4}{p+2})}{s^2(\frac{1}{p+2})} \left(\frac{\Gamma^2(\frac{2}{p+2})\Gamma(\frac{2p}{p+2})}{\Gamma^2(\frac{1}{p+2})\Gamma(\frac{2p+2}{p+2})} \right)^2 | F(\frac{p+1}{p+2},\frac{2p}{p+2},\frac{2p+2}{p+2},z) |^2\}
\end{aligned} \tag{4.204}$$

(s(x)$\equiv$ sin(πx))

The first term, in the limit $z \to 0$ is the identity family contribution while the second is the contribution of the Φ_{31} operator. In this limit the function becomes

$$\begin{aligned}
G(z,\overline{z}) =&< \Phi^p_{31}(\infty)\Phi^p_{31}(1,1)R^p_{21}(z,\overline{z})R^p_{21}(0,0) >\simeq \\
&\simeq c_1(p) | z |^{-2(2\Delta_{21}-\Delta_{31}} < \Phi^p_{31}(\infty)\Phi^p_{31}(1)\Phi^p_{31}(0) >= \\
&= c_1(p)[a_1 + a_2 | \tilde{\eta} |^2] | z |^{-2(2\Delta_{21}-\Delta_{31}}
\end{aligned} \tag{4.205}$$

Since the value of $c_1(p)$ is known, eq.(4.199), we can extract the structure constant a_1 and a_2 of the even and odd part of the field Φ^p_{31}. From the singularities of the expression we obtain

$$a_1 = 0 \tag{4.206}$$

$$a_2(a) = \frac{\Gamma^3(2a)\Gamma^2(2-4a)}{\Gamma(4a)\Gamma(1-4a)\Gamma(2a-1)\Gamma^2(2-2a)}\sqrt{\frac{\Gamma(1-a)\Gamma(3a)}{\Gamma^3(a)\Gamma(1-a)}} \tag{4.207}$$

$$a \equiv \frac{1}{p+2} \tag{4.208}$$

In the case p=3 (TIM) these results coincides with yhe values obtained in ref.[15].

$$a_2(p=3) = \frac{1}{15}\sqrt{\frac{\Gamma(\frac{4}{5})\Gamma^3(\frac{2}{5})}{\Gamma(\frac{1}{5})\Gamma^3(\frac{3}{5})}} \tag{4.209}$$

The complete description of this model is obtained from the remaining operator product expansions

$$\begin{aligned}
(\tfrac{7}{16})_R(\tfrac{7}{16})_R &= [0]_{NS} \\
(\tfrac{3}{80})_R(\tfrac{7}{16})_R &= \frac{1}{2}[\frac{1}{10}]_{NS} \\
(\tfrac{3}{80})_R(\tfrac{3}{80})_R &= [0]_{NS} + \frac{5}{2}\frac{\Gamma^2(\frac{2}{5})}{\Gamma(-\frac{2}{5})\Gamma(\frac{1}{5})}\sqrt{4\cos^2(\frac{\pi}{5})-1}\;[\frac{1}{10}]_{NS}
\end{aligned}$$

Considering the appropriate correlation function and the analytic properties of the contours integral solutions, one can extract all the structure constants of the superconformal OPE, as in the Virasoro case. We list here some of them for the second model of the superconformal series, identified as the Ashkin-Teller model on the critical line [76]

$$\begin{aligned}
[\tfrac{1}{16}]_R[\tfrac{1}{16}]_R &= [0]_{NS} + \sqrt{\frac{3}{2\pi}\frac{\Gamma^2(\frac{1}{3})}{\Gamma(\frac{1}{6})}}[\frac{1}{6}]_{NS} \\
[\tfrac{3}{8}]_R[\tfrac{3}{8}]_R &= [0]_{NS} + \frac{3}{4}[1]_{NS} \\
[\tfrac{3}{8}]_R[\tfrac{1}{16}]_R &= \frac{1}{2}[\frac{1}{16}]_{NS} \\
[\tfrac{1}{16}]_R[\tfrac{9}{16}]_R &= \sqrt{\frac{2}{3}\frac{\Gamma^2(\frac{1}{3})}{\Gamma(\frac{1}{6})}}[\frac{1}{6}]_{NS} + \frac{1}{2}[1]_{NS} \\
[\tfrac{1}{24}]_R[\tfrac{1}{24}]_R &= [0]_{NS} + \frac{1}{2}[\frac{1}{6}]_{NS} + \frac{1}{144}[1]_{NS}
\end{aligned}$$

An important peculiarity of this model is its N=2 superconformal symmetry [17,81,83,76] generated by the N=1 superstress energy tensor and by the N=1 superfield

$$V = J(z) + \theta\overline{G}(z) \tag{4.210}$$

where $\Delta_J = 1$, as we discuss briefly in the next section.

One can use this symmetry to compute some 4-point functions of the Ashlin-Teller model.For example, the 4-point function of the Ramond fields with dimension $\Delta = \frac{1}{24}$ and U(1) charge equals to $q = \pm\frac{1}{12}$ is

$$\begin{aligned} &< R_+(z_1)R_-(z_2)R_+(z_3)R_-(z_4) >= (z_{13}z_{24})^{-\frac{1}{12}} \\ &[\eta(1-\eta)]^{-\frac{1}{12}}[c_1 + c_2\eta^{\frac{1}{6}} + c_3(1-\eta)^{\frac{1}{6}}] \\ &\eta = \frac{z_{12}z_{34}}{z_{13}z_{24}} \end{aligned} \tag{4.211}$$

The physical correlator of these fields, satisfying the crossing symmetry condition, is

$$G(z,\overline{z})_{\frac{1}{24}} =| z_{13}z_{24} |^{-\frac{1}{6}} \left[| \eta(1-\eta) |^{-\frac{1}{6}} + | \frac{\eta}{1-\eta} |^{\frac{1}{6}} + | \frac{1-\eta}{\eta} |^{\frac{1}{6}}\right] \tag{4.212}$$

from where one obtains the last expression in the OPE of the model.

4.6 Ashkin-Teller model and N=2 super-conformal symmetry

In course of examination of the N=1 superconformal models the natural question arises if the N=2 superconformal algebra appears as a subalgebra in some of the N=1 OPE algebras. The positive answer of this question is based on the properties of the superfields OPE's and fusion rules for the N=1 minimal model given by c=1.

Considering the ope algebra generated by the super stress energy tensor

$$W(z,\theta) = \frac{1}{2}G(z) + \theta T(z)$$

and by the superfield

$$V(z) = J(z) + \theta\overline{G}(z)$$

where J has dimension $\Delta = 1$:

$$W(z_1,\theta_1)W(z_2,\theta_2) = \frac{c}{6z_{12}^3} + (\frac{3}{2}\frac{\theta_{12}}{z_{12}^2} + \frac{1}{2z_{12}}D_2 + \frac{\theta_{12}}{z_{12}}\partial)W(z_2,\theta_2) \tag{4.213}$$

$$W(z_1,\theta_1)V(z_2,\theta_2) = (\frac{\theta_{12}}{z_{12}^2} + \frac{1}{2z_{12}}D_2 + \frac{\theta_{12}}{z_{12}}\partial)V(z_2,\theta_2) \tag{4.214}$$

$$V(z_1,\theta_1)V(z_2,\theta_2) = \frac{c}{3z_{12}^2} + 2\frac{\theta_{12}}{z_{12}}W(z_2,\theta_2) \tag{4.215}$$

From these singular terms we obtain the corresponding (anti)commutation relations for the operators of the Laurent expansions

$$W(z,\theta) = \frac{1}{2}\sum_{-\infty}^{\infty}\frac{G_{n+\frac{1}{2}}}{z^{n+2}} + \theta\sum_{-\infty}^{\infty}\frac{L_n}{z^{n+2}} \tag{4.216}$$

$$V(z,\theta) = \sum_{-\infty}^{\infty}\frac{J_n}{z^{n+1}} + \theta\sum_{-\infty}^{\infty}\frac{\overline{G}_{n+\frac{1}{2}}}{z^{n+2}} \tag{4.217}$$

The algebra is

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}$$

$$[L_n, G_\alpha] = (\frac{n}{2} - \alpha)G_{n+\alpha}$$

$$
\begin{aligned}
[L_n, J_m] &= -mJ_{n+m} \\
[J_n, G_\alpha] &= -\overline{G}_{n+\alpha} \\
[J_n, \overline{G}_\alpha] &= -G_{n+\alpha} \\
[J_n, J_m] &= \frac{c}{3} n\delta_{n+m,0} \\
\{G_\alpha, G_\beta\} &= 2L_{\alpha+\beta} + \frac{c}{3}(\alpha^2 - \frac{1}{4})\delta_{\alpha+\beta,0} \\
\{\overline{G}_\alpha, \overline{G}_\beta\} &= -2L_{\alpha+\beta} - \frac{c}{3}(\alpha^2 - \frac{1}{4})\delta_{\alpha+\beta,0} \\
\{G_\alpha, \overline{G}_\beta\} &= (\alpha - \beta)J_{\alpha+\beta}
\end{aligned}
$$

We can introduce a conveniently N=2 complex superspace

$$\xi = (z, \theta, \overline{\theta})$$

with the covariant derivatives

$$D = \frac{\partial}{\partial\theta} + \frac{1}{2}\overline{\theta}\partial_z \tag{4.218}$$

$$\overline{D} = \frac{\partial}{\partial\overline{\theta}} + \frac{1}{2}\theta\partial_z \tag{4.219}$$

$$\{D, \overline{D}\} = \partial_z \tag{4.220}$$

The N=2 superalgebra can be obtained in a compact way by the OPE of the N=2 super stress-energy tensor [85]

$$W(z, \theta, \overline{\theta}) = 2J(z) + \frac{i}{\sqrt{2}}(\theta\overline{G}(z) + \overline{\theta}G(z)) + \theta\overline{\theta}T(z) \tag{4.221}$$

that is

$$
\begin{aligned}
W(z_1, \theta_1, \overline{\theta}_1)W(z_2, \theta_2, \overline{\theta}_2) &= [(\theta_1 - \theta_2)(\overline{\theta}_1 - \overline{\theta}_2) + (\overline{\theta}_1 - \overline{\theta}_2)\overline{D} + \\
-(\theta_1 - \theta_2)D &+ \frac{(\theta_1 - \theta_2)(\overline{\theta}_1 - \overline{\theta}_2)}{z_{12}}]\frac{W(z_2, \theta_2, \overline{\theta}_2)}{z_{12}} + \frac{c}{3z_{12}^2}
\end{aligned} \tag{4.222}
$$

where the N=2 invariant distance is

$$z_{12} = z_1 - z_2 - \frac{1}{2}(\theta_2\overline{\theta}_1 + \overline{\theta}_2\theta_1)$$

A N=2 superfield $\Phi(\xi)$ has two bosonic and two fermionic components and is characterized by two parameters: a superconformal dimension h and a U(1) charge q. The Ward identity is

$$W(z_1,\theta_1,\overline{\theta}_1)\Phi_{h,q}(z_2,\theta_2,\overline{\theta}_2) = [(\overline{\theta}_1-\overline{\theta}_2)(\theta_1-\theta_2)+(\overline{\theta}_1-\overline{\theta}_2)\overline{D}+ \\ -(\theta_1-\theta_2)D + h\frac{(\overline{\theta}_1-\overline{\theta}_2)(\theta_1-\theta_2)}{z_{12}} - 2q]\frac{\Phi_{h,q}(z_2,\theta_2,\overline{\theta}_2)}{z_{12}} \qquad (4.223)$$

The N=2 superalgebra has three sectors corresponding to the different modes of the generators: in addition to the Neveu-Schwartz and Ramond sectors, there is also a twisted sector

$$NS \quad L_{n\in Z}, \quad J_{n\in Z} \quad ,G_{r\in Z+\frac{1}{2}}, \quad \overline{G}_{s\in Z+\frac{1}{2}} \qquad (4.224)$$

$$R \quad L_{n\in Z}, \quad J_{n\in Z} \quad ,G_{r\in Z}, \quad \overline{G}_{s\in Z} \qquad (4.225)$$

$$T' \quad L_{n\in Z}, \quad J_{n\in Z+\frac{1}{2}} \quad ,(G+\overline{G})_{r\in Z}, \quad (G-\overline{G})_{s\in Z+\frac{1}{2}} \qquad (4.226)$$

The hermiticity conditions are

$$L_n^\dagger = L_{-n}, \qquad G_r^\dagger = \overline{G}_{-r}, \qquad J_n^\dagger = J_{-n} \qquad (4.227)$$

In the various sectors there are vacuum states satisfying

$$\begin{array}{ll} NS & L_n \mid 0>= J_m \mid 0>= G_r \mid 0>= \overline{G} \mid 0>= 0 \\ & n \geq -1, \quad m \geq 0 \quad r \geq -\frac{1}{2} \\ R & L_n \mid 0>= J_m \mid 0>= G_r \mid 0>= \overline{G} \mid 0>= 0 \\ & n \geq 1, \quad m \geq 1 \quad r \geq 0 \\ & L_0 \mid 0>= \frac{c}{24} \mid 0> \\ T & L_n \mid 0>= J_m \mid 0>= (G+\overline{G})_r \mid 0>= (G-\overline{G})_s \mid 0>= 0 \\ & n \geq 1, \quad m \geq \frac{1}{2} \quad r \geq 0 \quad s \geq \frac{1}{2} \\ & L_0 \mid 0>= \frac{c}{24} \mid 0> \end{array}$$

HWV are obtained by acting with primary superfields on the vacua

$$\mid h,q>= \Phi_{h,q}(0) \mid 0> \qquad (4.228)$$

It satisfies

$$L_n \mid h,q >= G_r \mid h,q >= \overline{G}_s \mid h,q >= J_m \mid h,q >= 0 \quad (4.229)$$
$$n,r,s,m > 0$$

and (in the NS and R sectors)

$$L_0 \mid h,q >= h \mid h,q > \quad (4.230)$$
$$J_0 \mid h,q >= q \mid h,q > \quad (4.231)$$

In the T-sector there is no zero modes of J(z) and no eigenvalue q, while in the R and T sectors the existence of G(z)'s and $\overline{G}(z)$'s zero modes introduces a doubling of states. The representation of this algebra is constructed by the descendent state

$$L_{-n_1} \dots L_{-n_N} J_{-m_1} \dots J_{-m_M} G_{-r_1} \dots G_{-r_R} \overline{G}_{-s_1} \dots \overline{G}_{-s_S} \mid h,q > \quad (4.232)$$

For special values of c, h, q it is possible to find a linear combination of these vectors which is itself a HWV. It gives rise to the so-called degenerate representations. One can classify the N=2 superconformal models, [81,82,83,84]finding the values of quantized dimensions and charges. Moreover, it is possible to give a Feigin-Fuchs representation for the degenerate superfields and by screening procedure to compute the correlation functions. This is not the place to discuss in detail all these developments of the theory, which one can find in the original literature. We wish to point out some reasons of the interest in the investigations of the N=2 superconformal algebra.

First, N=2 supersymmetry is the gauge algebra of the U(1) string [52]: whereas N=1 supersymmetry algebra is directly relevant for formulating superstring theory, it has been speculated that the N=2 structure could be relevant for understanding string compactification (see the bible ref.[47]). For our aims, it is of the grat importance the identification of the statistical models with this higher symmetry. The only known system is that with c=1, corresponding to the Ashkin-Teller model (AT). It consists of two Ising models coupled by a four-spin interaction [68] The Hamiltonian is

$$H = -\sum_{<ij>} [K_2(s_i s_j + t_i t_j) + K_4 s_i s_j t_i t_j] \quad (4.233)$$

where two Ising spins $s_i = \{\pm 1\}$ and $t_i = \{\pm 1\}$ are placed at each site on a square lattice. When $K_4 = 0$ we have two decouple Ising models.

The AT quantum chain is obtained by a highly anisotropic version of the above Hamiltonian [78,79]: the quantum Hamiltonian is

$$H = -\frac{1}{2(1+\epsilon)} \sum_{i=1}^{N} [(\sigma_i + \sigma_i^{\dagger} + \epsilon\sigma_i^2) + \lambda(\Gamma_i \Gamma_{i+1}^{\dagger} + \Gamma_i^{\dagger}\Gamma_{i+1} + \epsilon \Gamma_i^2 \Gamma_{i+1}^2)] \quad (4.234)$$

ϵ is a coupling constant, λ plays the role of the inverse temperature and

$$\sigma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}, \qquad \Gamma = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad (4.235)$$

The full structure of the phase diagramm is very rich. Restricting to the critical line

$$\lambda = 1$$
$$-1 \leq \epsilon \leq 1$$

the critical exponents vary continously in function of the value of the coupling constant. For the thermal exponent, the expression is

$$x_T(\epsilon) = \frac{1}{2}(1 - \frac{1}{\pi}\cos^{-1}(\epsilon)) \qquad \frac{1}{2} \leq x_T < \infty \quad (4.236)$$

The marginal scalar operator with dimensions (1,1) is responsable for the motion along the critical line . But in any case, the magnetic exponent is given by that of the Ising model, i.e.

$$x_M = \frac{1}{8} \quad (4.237)$$

along all the critical line.

Along this line, there are many interesting points

1. $x_T = \frac{1}{2}$ is the 4-state Potts model.
2. $x_T = \frac{2}{3}$ belongs to the Z_4 model studied by Zamolodchikov and Fateev [32,33]

3. $x_T = 1$ is the decoupling point into two Ising models.

4. $x_T = 2$ is the Kosterlitz-Thouless model.

Since all these models have c=1, according to Kadanoff et al.[72], the behaviour on the critical line is described by the gaussian theory with quantum field action

$$S = \frac{k}{2} \sum_{<ij>} (\phi_i - \phi_j)^2 \tag{4.238}$$

The primary fields in the gaussian theory have scaling dimensions

$$h_{n,m} = \frac{n^2}{8\pi k} + \frac{\pi k}{2} m^2 + \frac{nm}{2} \qquad n, m = 1, 2, \ldots \tag{4.239}$$

$$\overline{h}_{n,m} = h_{n,-m} \tag{4.240}$$

The energy density is identified with $\phi_{2,0}$ and hence

$$x_T = \frac{1}{\pi k} \tag{4.241}$$

Other parametrization of the scaling dimension is

$$h_{n,m} = \frac{1}{4} \left(\sqrt{\frac{x_T}{2}} n + \sqrt{\frac{2}{x_T}} m \right)^2 \tag{4.242}$$

In ref. [80] there is the identification of the operators of the N=2 superalgebra with the corresponding operators in N=1 system. The correspondence is as follows (subscript indicate N=1,2).

The unit operator $(0)_2$ decompose into $(0)_1$) and a primary field $(1)_1$ in the NS sector. The energy momentum tensor and one of the supercharges are contained into $(0)_1$, whereas the other supercharge are contained into $(1)_1$. The NS representation

$$(h, q)_2 = (\frac{1}{6}, \pm\frac{1}{3})_2$$

decomposes into $(\frac{1}{6})_1$ in the NS sector. In the Ramond sector, $(\frac{3}{8}, 0)_2$ decomposes into $(\frac{3}{8})_1$ in the Ramond sector of the N=1 superalgebra whereas $(\frac{1}{24}, \pm\frac{1}{3})_2$ and $(\frac{1}{24}, \pm\frac{2}{3})_2$ decompose into $(\frac{1}{24})_1$ in the Ramond sector. Finally $(\frac{1}{16})_2$ in the twisted sector of the N=2 theory decomposes into $(\frac{1}{16})_1$ in the NS sector of the N=1 theory. Of course N=2 superformalism provides a compact way in the computation of the correlation functions of the models and in their solvability.

Appendix A

Critical indices

In this appendix we give the definition of thermodynamical quantities and their parametrization near the critical point. We use the terminology of a magnetic system for semplicity.

The most important quantity is the partition function. In the thermodynamical limit (in a QFT description) we have

$$\begin{aligned} Z(T,B) &= \int \mathcal{D}M(x) e^{[-\beta(H(M)-B\int dx M(x))]} = \\ &= e^{-\beta F(T,B)} \end{aligned} \tag{A.1}$$

where H(M) is the Hamiltonian of the system and B is an external magnetic field. All the quantities are obtained from partition function by derivation.

The entropy is

$$S = -(\frac{\partial F}{\partial T}) \tag{A.2}$$

and the internal energy is

$$U = -(\frac{\partial \ln Z}{\partial \beta}) \tag{A.3}$$

From this one, it is possible to compute the specific heat

$$C = (\frac{\partial U}{\partial T}) = -T\frac{\partial^2 F}{\partial T^2} \tag{A.4}$$

The magnetization is obtained by

$$M = -\frac{\partial F}{\partial B} \tag{A.5}$$

and the magnetic susceptibility by

$$\chi = \frac{\partial M}{\partial B} \tag{A.6}$$

The 2-point correlation function

$$G(x-y) =< (M(x)- < M(x) >)(M(y)- < M(y) >) > \tag{A.7}$$

is linked to the magnetic susceptibility as follows in this formula

$$\begin{aligned}\chi = \frac{1}{\beta}\frac{\partial^2 F}{\partial B} = \frac{1}{\beta}[\frac{1}{Z}\frac{\partial^2 Z}{\partial B^2} - (\frac{1}{Z}\frac{\partial Z}{\partial B})^2] = \\ = \frac{1}{\beta}\int dxdyG(x-y)\end{aligned} \tag{A.8}$$

So, the divergencies of the susceptibility are due to the long-range behaviour of the 2-point function.

Near the critical point T_c the thermodynamical quantities have singularities which are usual parametrize as a power behaviour [5,4,2] Putting

$$t = \frac{T - T_c}{T_c}$$

we have the following definition

$$M(B=0,t) \quad \sim\mid t \mid^{\beta} \tag{A.9}$$

$$C(B=0,t) \quad \sim\mid t \mid^{-\alpha} \tag{A.10}$$

$$\chi(B=0,t) \quad \sim\mid t \mid^{-\gamma} \tag{A.11}$$

$$M(B,T=T_c) \quad \sim\mid B \mid^{\frac{1}{\delta}} \tag{A.12}$$

$$G(R,T=T_c) \quad \sim \frac{1}{R^{2-d+\eta}} \tag{A.13}$$

$$\xi(B=0,t) \quad \sim\mid t \mid^{-\nu} \tag{A.14}$$

where ξ is the correlation lenght of the system. Note that at the critical point the 2-point function have a power behaviour without any parameter which fixes the scale.

Not all the critical indeces so introduced are independent. With general thermodynamical considerations and scaling hypothesis, it is possible to prove the following identities

$$\alpha + 2\beta + \gamma = 2 \tag{A.15}$$
$$\gamma = \beta(\delta - 1) \tag{A.16}$$
$$\gamma = (2 - \eta)\nu \tag{A.17}$$
$$\nu d = 2 - \alpha \tag{A.18}$$

The problem is to compute only two of them, usually η and δ are chosen.

Appendix B

Fermionic representation of 2-d Ising model

Consider a square lattice of $N = n^2$ spins consisting of n rows and n colomns, with periodic boundary condition. Let μ_a $(a = 1, \ldots, n)$ denote the collection of all spin coordinates of the a-row

$$\mu_a \equiv \{\sigma_1, \ldots, \sigma_n\}_{a-row} \tag{B.1}$$

The boundary condition implies

$$\mu_{n+1} = \mu_1 \tag{B.2}$$

A configuration on lattice it is specified by a set of $\{\mu_1 \ldots, \mu_n\}$. The row a interacts only with the rows (a-1) and (a+1). Let us introduce $E(\mu_a, \mu_{a+1})$ as the interaction energy between the a-row and the (a+1)'s. Let $E(\mu_a)$ be the interaction energy of the spins within the a-row plus an eventual energy coming by coupling to a magnetic field B

$$E(\mu, \mu') = -\epsilon_1 \sum_{k=1}^{n} \sigma_k \sigma'_k \tag{B.3}$$

$$E(\mu) = -\epsilon_2 \sum_{k=1}^{n} \sigma_k \sigma_{k+1} - B \sum_{k=1}^{n} \sigma_k \tag{B.4}$$

The total energy of the lattice for the configuration

$$\{\mu_1, \ldots \mu_n\}$$

is given by

$$E(\mu_1 \dots \mu_n) = \sum_{a=1}^{n} [E(\mu_a, \mu_{a+1}) + E(\mu_a)] \tag{B.5}$$

and the partition function is

$$Z = \sum_{\mu_1} \sum_{\mu_2} \dots \sum_{\mu_n} \exp[-\beta E(\mu_1 \dots \mu_n)] =$$

$$= \sum_{\mu_1} \dots \sum_{\mu_n} \exp\{-\beta \sum_{a=1}^{n} [E(\mu_a, \mu_{a+1}) + E(\mu_a)]\} \tag{B.6}$$

Let a $2^n \times 2^n$ matrix P be defined by these matrix elements

$$< \mu \mid P \mid \mu' >= \exp[-\beta(E(\mu, \mu') + E(\mu))] \tag{B.7}$$

Then

$$Z = \sum_{\mu_1} \dots \sum_{\mu_n} < \mu_1 \mid P \mid \mu_2 >< \mu_2 \mid P \mid \mu_3 > \dots < \mu_n \mid P \mid \mu_1 >=$$

$$= \sum_{\mu_1} < \mu_1 \mid P^n \mid \mu_1 >= Tr(P^n) \tag{B.8}$$

For finding an explicit representation of P, we need some mathematical tools. Explicity the matrix elements of P are

$$< \sigma_1 \dots \sigma_n \mid P \mid \sigma'_1 \dots \sigma'_n >= \prod_{k=1}^{n} e^{\beta B \sigma_k} e^{\beta \epsilon_2 \sigma_k \sigma_{k+1}} e^{\beta \epsilon_1 \sigma_k \sigma'_k} \tag{B.9}$$

Define three matrices $2^n \times 2^n$ by this rule

$$< \sigma_1 \dots \mid V_1 \mid \sigma'_1 \dots \sigma'_n >= \prod_{k=1}^{n} e^{\beta \epsilon_1 \sigma_k \sigma'_k} \tag{B.10}$$

$$< \sigma_1 \dots \mid V_2 \mid \sigma'_1 \dots \sigma'_n >= \delta_{\sigma_1 \sigma'_1} \dots \delta_{\sigma_n \sigma'_n} \prod_{k=1}^{n} e^{\beta \epsilon_2 \sigma_k \sigma_{k+1}} \tag{B.11}$$

$$< \sigma_1 \dots \mid V_3 \mid \sigma'_1 \dots \sigma'_n >= \delta_{\sigma_1 \sigma'_1} \dots \delta_{\sigma_n \sigma'_n} \prod_{k=1}^{n} e^{\beta B \sigma_k} \tag{B.12}$$

It is easy to verify that

$$P = V_3 V_2 V_1 \tag{B.13}$$

in the usual matrix multiplication.

Now we define *direct product of matrices*: if A and B are two matices $m \times m$ with elements $< i \mid A \mid j >$, $< i \mid B \mid j >$, we define the direct product

$$A \times B$$

as a matrix $m^2 \times m^2$ with elements

$$< i, i' \mid A \times B \mid j, j' > \equiv < i \mid A \mid j >< i' \mid B \mid j' > \tag{B.14}$$

This definition can be immediately extended to the direct product of n matrices.

The multiplication rule for these matrices is

$$(A \times B)(C \times D) = (AC) \times (BD) \tag{B.15}$$

Now we introduce three $2^n \times 2^n$ matrices in terms of which V_1, V_2 and V_3 can be expressed.

$$\hat{\sigma}_1(a) = 1 \times 1 \times \ldots \times \overbrace{\sigma_1}^{a} \times 1 \ldots 1 \tag{B.16}$$

$$\hat{\sigma}_2(a) = 1 \times 1 \times \ldots \times \overbrace{\sigma_2}^{a} \times 1 \ldots 1 \tag{B.17}$$

$$\hat{\sigma}_3(a) = 1 \times 1 \times \ldots \times \overbrace{\sigma_3}^{a} \times 1 \ldots 1 \tag{B.18}$$

where σ_i are the usual Pauli's matrices.

For $a \neq b$ we can verify that

$$[\hat{\sigma}_i(a), \hat{\sigma}_j(b)] = 0 \tag{B.19}$$

while for the same a, the $\hat{\sigma}_i(a)$ satisfy the (anti)commutation relations

$$[\hat{\sigma}_i(a), \hat{\sigma}_j(a)] = 2i\epsilon_{ijk}\hat{\sigma}_k(a) \tag{B.20}$$

$$\{\hat{\sigma}_i(a), \hat{\sigma}_j(a)\} = 2\delta_{ij} \tag{B.21}$$

With these definitions it is clear that V_1 is a direct product of n 2×2 matrix a, whose elements are

$$< \sigma \mid a \mid \sigma' > = e^{\beta\epsilon_1\sigma\sigma'} \tag{B.22}$$

$$a = \begin{pmatrix} e^{\beta\epsilon_1} & e^{-\beta\epsilon_1} \\ e^{-\beta\epsilon_1} & e^{\beta\epsilon_1} \end{pmatrix} = e^{\beta\epsilon_1} + \hat{\sigma}_1 e^{-\beta\epsilon_1} \tag{B.23}$$

The following identity is true for each matrix whose square is the identity operator

$$e^{\alpha X} = \cosh\alpha + X \sinh\alpha \tag{B.24}$$

Then we have

$$a = \sqrt{2\sinh(2\beta\epsilon_1)}e^{\theta\hat{\sigma}_1} \tag{B.25}$$

$$\tanh\theta = e^{-2\beta\epsilon_1} \tag{B.26}$$

and

$$V_1 = a \times a \times a \ldots \times a = [2\sinh(2\beta\epsilon_1)]^{\frac{n}{2}} e^{\theta\hat{\sigma}_1(1)} e^{\theta\hat{\sigma}_1(2)} \ldots e^{\theta\hat{\sigma}_1(n)} = [2\sinh(2\beta\epsilon_1)]^{\frac{n}{2}} e^{\theta(\hat{\sigma}_1(1)+\hat{\sigma}_1(2)+\cdots\hat{\sigma}_1(n)} \tag{B.27}$$

In the same way we have

$$V_2 = \prod_{a=1}^{n} e^{\beta\epsilon_2\hat{\sigma}_3(a)\hat{\sigma}_3(a+1)} \tag{B.28}$$

$$V_3 = \prod_{a=1}^{n} e^{\beta B\hat{\sigma}_3(a)} \tag{B.29}$$

Then

$$P = [2\sinh(2\beta\epsilon_1)]^{\frac{n}{2}} \prod_{a=1}^{n} [e^{\beta B\hat{\sigma}_3(a)} e^{\beta\epsilon_2\hat{\sigma}_3(a)\hat{\sigma}_3(a+1)} e^{\theta\hat{\sigma}_1(a)}] \tag{B.30}$$

We can interpred P as the transfer matrix, i.e. we introduce an Hamiltonian H with this definition

$$P = ce^{aH} \tag{B.31}$$

where a is the lattice spacing an

$$c = [2\sinh(2\beta\epsilon_1)]^{\frac{n}{2}}$$

In the continuum limit $(a \to 0)$ we have (B=0)

$$H = \sum_{a=1}^{n} [\tilde{\eta}\hat{\sigma}_1(a) + \eta\hat{\sigma}_3(a)\hat{\sigma}_3(a+1)] \tag{B.32}$$

where $\tilde{\eta}, \eta$ are coupling constants functions of the previous ones.

We introduce the dual variables

$$\hat{\mu}(r+\tfrac{1}{2}) = \prod_{\rho=-\infty}^{r} \hat{\sigma}_1(\rho) \tag{B.33}$$

$$\hat{\mu}(r+\tfrac{1}{2}) = \hat{\sigma}_3(r)\hat{\sigma}_3(r+1) \tag{B.34}$$

and it is easy to check the following relations

$$\hat{\mu}_3^2 = \hat{\mu}_1^2 = 1 \tag{B.35}$$

$$\hat{\mu}_3(r-\frac{1}{2})\hat{\mu}_3(r+\frac{1}{2}) = \hat{\sigma}_1(r) \tag{B.36}$$

$$[\hat{\mu}_1(r+\frac{1}{2}), \hat{\mu}_3(r'+\frac{1}{2})] = 2\delta_{r,r'} \tag{B.37}$$

$$[\hat{\mu}_3(r+\frac{1}{2}), \hat{\mu}_3(r'+\frac{1}{2})] = 0 \tag{B.38}$$

$$[\hat{\mu}_3(r+\frac{1}{2}), \hat{\sigma}_1(r')] = 0 \tag{B.39}$$

The Hamiltonian can be expressed in terms of the dual variables

$$H = \tilde{\eta}\sum_r \hat{\mu}_3(r+\frac{1}{2})\hat{\mu}_3(r+\frac{1}{2}) + \eta\sum_r \hat{\mu}_1(r+\frac{1}{2}) \tag{B.40}$$

The Kramers-Wannier symmetry is expressed by the following equations

$$\hat{\mu}_1 \leftrightarrow \hat{\sigma}_1$$
$$\hat{\mu}_3 \leftrightarrow \hat{\sigma}_3$$
$$\eta \leftrightarrow \tilde{\eta}$$

and we can rewrite the Hamiltonian in such a way that this symmetry is evident

$$H = \frac{1}{2}\sum_r [\tilde{\eta}\hat{\sigma}_1(r) + \eta\hat{\mu}_1(r+\frac{1}{2})] \tag{B.41}$$

The equation of motion are

$$\frac{\partial\hat{\sigma}_3}{\partial t}(r) = [H, \hat{\sigma}_3(r)] = \tilde{\eta}\hat{\sigma}_1(r)\hat{\sigma}_3(r) \tag{B.42}$$

$$\frac{\partial\hat{\mu}_3}{\partial t}(r+\frac{1}{2}) = [H, \hat{\mu}_3(r+\frac{1}{2})] = \eta\hat{\sigma}_3(r)\hat{\sigma}_3(r+1)\hat{\mu}_1(r+\frac{1}{2}) \tag{B.43}$$

Introduce

$$u(r) = \hat{\mu}_3(r - \frac{1}{2})\hat{\sigma}_3(r) \tag{B.44}$$

$$v(r) = \hat{\mu}_3(r + \frac{1}{2})\hat{\sigma}_3(r) \tag{B.45}$$

then

$$\frac{du(r)}{dt} = \tilde{\eta}v(r) - \eta v(r-1) \tag{B.46}$$

$$\frac{dv(r)}{dt} = \tilde{\eta}u(r) - \eta u(r+1) \tag{B.47}$$

In the continuum limit these equations become

$$\frac{du(r)}{dt} = (\tilde{\eta} - \eta)v(r) + \eta\frac{dv(r)}{dr} \tag{B.48}$$

$$\frac{dv(r)}{dt} = (\tilde{\eta} - \eta)u(r) - \eta\frac{du(r)}{dr} \tag{B.49}$$

This two fields can be organized in a vector

$$\psi(r) = \begin{pmatrix} u(r) \\ v(r) \end{pmatrix} \tag{B.50}$$

and putting

$$m = \tilde{\eta} - \eta \tag{B.51}$$

we have the compact equation

$$(\gamma^0\frac{d}{dt} + \gamma^3\frac{d}{dr} - m)\psi(r) = 0 \tag{B.52}$$

where

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \gamma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{B.53}$$

are the Euclidean γ matrices.

Finally, note that holds the following anti-commutation relation

$$\{u(r), u(r')\} = 2\delta_{r,r'} \tag{B.54}$$

$$\{v(r), v(r')\} = 2\delta_{r,r'} \tag{B.55}$$

So, at the critical point, defined by

$$\tilde{\eta} = \eta \tag{B.56}$$

we have demonstrate that the Ising model in two dimension is equivalent to a free massless fermionic (Majorana-Weyl) system.

Appendix C

Hypergeometric functions and their monodromy matrices

The hypergeometric functions are solutions of the following second order differential equation

$$z(1-z)\frac{d^2u}{dz^2}+[c-(a+b+1)z]\frac{du}{dz}-abu=0 \tag{C.1}$$

where a,b,c are independent of z and are arbitrary complex numbers. The complete theory of this equation can be found in the ref.[96].

If none of the exponent differences 1-c, b-a, c-a-b is an integer, then

$$\begin{cases} u_1(z) & = F(a,b,c,z) \\ u_2(z) & = z^{1-c}F(a-c+1,b-c+1,2-c,z) \end{cases} \tag{C.2}$$

are two linearly independent solutions which are one valued and regular within the domain

$$\mid z-1\mid<\frac{1}{2}$$

but at each of the three points z=0,1,∞, at least one of them will have a branch point.Using some properties of these functions, it is possible to compute their monodromy matrices around the singular points, that is how change the functions if we continue them analitically along a loop enclosing the singular points.

$$g_0 \quad :\begin{cases} u_1 & \to u_1 \\ u_2 & \to e^{-2\pi i c}u_2 \end{cases} \tag{C.3}$$

$$(C.4)$$

$$g_1 \quad : \begin{cases} u_1 & \rightarrow B_{11}u_1 + B_{12}u_2 \\ u_2 & \rightarrow B_{21}u_1 + B_{22}u_2 \end{cases} \tag{C.5}$$

$$(C.6)$$

$$g_\infty \quad : \begin{cases} u_1 & \rightarrow L_{11}u_1 + L_{12}u_2 \\ u_2 & \rightarrow L_{21}u_1 + L_{22}u_2 \end{cases} \tag{C.7}$$

$$(C.8)$$

where

$$\begin{aligned}
B_{11} &= 1 - 2ie^{i\pi(c-a-b)} \frac{s(a)s(b)}{s(c)} \\
B_{12} &= -2i\pi e^{i\pi(c-a-b)} \frac{\Gamma(c)\Gamma(c-1)}{\Gamma(c-a)\Gamma(c-b)\Gamma(a)\Gamma(b)} \\
B_{21} &= 2i\pi e^{i\pi(c-a-b)} \frac{\Gamma(2-c)\Gamma(1-c)}{\Gamma(1-a)\Gamma(1-b)\Gamma(1+a-c)\Gamma(1+b-c)} \\
B_{22} &= 1 + 2ie^{i\pi(c-a-b)} \frac{s(a)s(c-b)}{s(c)}
\end{aligned}$$

$$\begin{aligned}
L_{11} &= \frac{s(b)s(c-a)e^{2\pi ia} - s(a)s(c-b)e^{2\pi ib}}{s(c)s(a-b)} \\
L_{12} &= \frac{\pi}{c-1} \frac{e^{2\pi ib} - e^{2\pi ia}}{s(a-b)} \frac{\Gamma^2(c)}{\Gamma(a)\Gamma(c-a)\Gamma(b)\Gamma(c-b)} \\
L_{21} &= \frac{\pi}{c-1} \frac{e^{2\pi ia} - e^{2\pi ib}}{s(a-b)} \frac{\Gamma^2(2-c)}{\Gamma(1-a)\Gamma(1+a-c)\Gamma(1-b)\Gamma(1+b-c)} \\
L_{22} &= \frac{s(b)s(a)e^{2\pi ib} - s(a)s(c-b)e^{2\pi ia}}{s(c)s(a-b)}
\end{aligned}$$

$$s(x) \equiv \sin(\pi x)$$

The 4-point functions W_i and Y_i of the Ramond field R_{12} and R_{21} are expressed in terms of the hypergeometric functions times a factor coming from the 'Ising' construction.Using the above monodromy transformations, we can compute the monodromy matrices of the Ramond fields.

We introduce the diagonal basis (so-called since g_0 is diagonal in this basis)

$$\begin{aligned} F_1 &= W_1 + W_3 \\ F_2 &= W_1 - W_3 \\ F_3 &= W_2 + W_4 \\ F_3 &= W_2 - W_4 \end{aligned}$$

On this basis the monodromy matrices for the Ramond field R_{12} have the following expressions

$$g_0 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -y & 0 \\ 0 & 0 & 0 & y \end{pmatrix} \qquad y = e^{-4\pi i \frac{1}{p}} \tag{C.9}$$

(C.10)

$$g_1 = \begin{pmatrix} r & 0 & t & 0 \\ 0 & -r & 0 & -t \\ s & 0 & u & 0 \\ 0 & -s & 0 & -u \end{pmatrix} \tag{C.11}$$

(C.12)

$$g_\infty = \begin{pmatrix} 0 & a & 0 & c \\ a & 0 & c & 0 \\ 0 & b & 0 & d \\ b & 0 & d & 0 \end{pmatrix} \tag{C.13}$$

where

$$\begin{aligned} r &= 1 - ie^{-2\pi ih} tg(\pi h) \\ t &= 2ie^{-2\pi ih} \frac{s(3h)s(h)}{s(2h)} \\ s &= -2ie^{-2\pi ih} \frac{s^2(h)}{s(2h)} \\ u &= 1 - 2ie^{-2\pi ih} \frac{s(3h)s(h)}{s(2h)} \end{aligned}$$

$$
\begin{aligned}
a &= -\frac{s(h)s(3h)e^{2\pi i h} + s^2(h)e^{-2\pi i h}}{s^2(2h)} \\
b &= -2i\frac{s^3(h)}{s^2(2h)} \\
c &= -2is(3h)[\frac{s(h)}{s(2h)}]^2 \\
d &= -\frac{s(h)s(3h)e^{-2\pi i h} + s^2(h)e^{2\pi i h}}{s^2(2h)} \\
h &\equiv \frac{1}{p}
\end{aligned}
$$

The invariance under g_0 restricts the matrix I_{ij} (in the diagonal basis) to be diagonal

$$
I_{ij} = \begin{pmatrix} x_1 & 0 & 0 & 0 \\ 0 & x_2 & 0 & 0 \\ 0 & 0 & x_3 & 0 \\ 0 & 0 & 0 & x_4 \end{pmatrix} \tag{C.14}
$$

Imposing invariance under the g_1 matrix we have the equations

$$
\begin{aligned}
\mid r \mid^2 x_1 + \mid s \mid^2 x_3 &= x_1 && \text{(C.15)} \\
\mid r \mid^2 x_2 + \mid s \mid^2 x_4 &= x_2 && \text{(C.16)} \\
\mid u \mid^2 x_3 + \mid t \mid^2 x_1 &= x_3 && \text{(C.17)} \\
\mid u \mid^2 x_4 + \mid t \mid^2 x_2 &= x_4 && \text{(C.18)} \\
r\bar{t}x_1 + s\bar{u}x_3 &= 0 && \text{(C.19)} \\
r\bar{t}x_2 + s\bar{u}x_4 &= 0 && \text{(C.20)} \\
\bar{r}tx_1 + \bar{s}ux_3 &= 0 && \text{(C.21)} \\
\bar{r}tx_2 + \bar{s}ux_4 &= 0 && \text{(C.22)}
\end{aligned}
$$

from which the x_i are real and it holds the following consistent equations

$$
\begin{aligned}
\frac{x_1}{x_3} = \frac{x_2}{x_4} = A = \frac{\mid s \mid^2}{1 - \mid r \mid^2} = \frac{1 - \mid u \mid^2}{\mid t \mid^2} = \\
-\frac{s\bar{u}}{r\bar{t}} = \frac{\Gamma(3h)\Gamma(1-3h)}{\Gamma(h)\Gamma(1-h)} = \frac{1}{4\cos^2(\pi h) - 1}
\end{aligned} \tag{C.23}
$$

Finally the last ratios

$$\frac{x_1}{x_2} = \frac{x_3}{x_4}$$

are fixed by the invariance under g_∞. The resulting equations are

$$\mid a \mid^2 Ax_4 + \mid b \mid^2 x_4 = Ax_3 \tag{C.24}$$
$$\mid a \mid^2 Ax_3 + \mid b \mid^2 x_3 = Ax_4 \tag{C.25}$$
$$\mid c \mid^2 Ax_3 + \mid d \mid^2 x_3 = Ax_4 \tag{C.26}$$
$$\mid c \mid^2 Ax_4 + \mid d \mid^2 x_4 = Ax_3 \tag{C.27}$$
$$\bar{a}cA + \bar{b}d = 0 \tag{C.28}$$

and thus

$$\frac{x_1}{x_2} = \frac{x_3}{x_4} = 1 \tag{C.29}$$

with the consisten equations

$$A = \frac{\mid b \mid^2}{1 - \mid a \mid^2} = \frac{1 - \mid d \mid^2}{\mid c \mid^2} = -\frac{\bar{b}d}{\bar{a}c} \tag{C.30}$$

which are satisfied. The resulting physical correlators are given in the section 4.5

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