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STABILITY AND SECULAR HEATING OF GALACTIC DISKS

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CHAPTER 1. INTRODUCTION

1.1 MOTIVATIONS AND OVERVIEW OF THE PROBLEM

It is a well-known fact that the kinematical properties of nearby disk stars are sistematically related to their spectral class, and in particular that the components of the stellar velocity dispersion show a tendency to increase with increasing spectral type; this has been interpreted in terms of a corresponding increase with age. Such a strong correlation between kinematical and physical properties of disk stars is indeed at the basis of their subdivision into archetype population groups, which are generally referred to as the spiral-arm population, the young disk population, the intermediate disk population and the oldest disk population. It should be born in mind, however, that globally the disk-component population exhibts much more drastically different features with respect to the spheroidal-component population, which reflect their different cosmological origin.

From a theoretical point of view, many efforts have been addressed to explain the observed increase of the components of the stellar velocity dispersion with age. The most diffused and currently accepted class of explanations is based on the existence of relaxation mechanisms which lead to a secular heating of galactic disks; the basic physical source responsible for such a heating process is still under debate. In this context it should be noted that observations do not put yet any stringent constraint on the age-velocity dispersion relation, although the opposite is often claimed by observers; this is due to the large experimental uncertainties and to the presence of statistical biases ingrained in the sample selection, which often cannot properly be estimated. For this reason only few theoretical models have definitely been ruled out, while a lot of speculations involving hypothetical massive perturbers have been made mostly to find upper and lower bounds for the mass of such objects; because of the large number of free parameters involved these theories, although they are appealing due to their connection with the problem of dark matter in the universe, have a low level of predictability. In all these approaches the restriction to nearly integrable situations is tacitly assumed; for strong departures from the integrability condition the relaxation is governed by the effects of Lyapunov (orbital) instability.

Even though it is often left out, the *secular evolution* of galactic disks, of of which the increase of the stellar velocity dispersion with age is the most striking expression from a kinematical point of view, is closely related to their *stability properties*. This fact stresses the *crucial* role that *collective effects* play in stellar systems, or more in general in systems whose dynamics is governed by long-range interactions. Such role is often underestimated in stellar dynamics, while perhaps too much emphasis is given to relaxation processes involving binary encounters alone. It is indeed a well-known fact that in (electromagnetic) plasmas the rate of relaxation towards the equilibrium state can considerably be enhanced by collective effects; when such collisionless relaxation mechanisms occur, more effective heating processes become operative leading to a rapid but usually incomplete randomization of particle velocities. It is just after such a collisionless collective phase that binary encounters become effective, and lead to a slow evolution of the partially relaxed system towards the final state of thermodynamical equilibrium. In virtue of the dynamical similarity between ordinary (electromagnetic) plasmas and "gravitational plasmas", the same phenomenon is expected to occur in stellar systems as well and to be competitive, if not dominant, with respect to other more commonly invoked relaxation mechanisms. However, the analysis required to describe quantitavely the relevant heating process would generally be much more complicated because of the natural inhomogeneity of stellar systems, which in particular makes the usual quasi-linear *local* approach no more suitable.

To avoid the difficulties connected with a *global* modal analysis, externally imposed and thus non-self-sustained perturbations of spiral form have generally been considered together with a local treatment in the action-angle canonical representation; collective effects are thus not taken into account in this simplified approach. An effective horizontal heating of galactic disks is then produced provided these spiral waves are assumed to be recurrent transient large-scale phenomena. Despite the formal elegance of the action-angle canonical representation and the relative simplicity inherent in a local analysis, two defects characterize this approach:

- The interpretation of the theoretical predictions in terms of observable phenomena might not be straightforward; this lack of physical intuition somewhat lowers the predictability level of the theory.
- The most drastic consequence that arises from neglecting the self-consistency of the perturbations lies in the fact that internal (to the system) excitation and feedback mechanisms, crucial for the maintenance of global spiral modes, are not taken into account; hence, most of the physics is missed.

For this reason it is worth formulating a *global quasi-linear theory of spiral structure*, in which the role of resonances is properly taken into account. To the above-mentioned difficulties one has to add also those deriving from the consideration of the *cold interstellar gas*, whose damping role in non-linear regimes cannot be disregarded as it contributes together with non-linear effects to saturate otherwise exponentially growing spiral overstabilities. Although the importance that such *self-regulation mechanisms* may have in connection with the secular evolution of galactic disks has long been recognized, no quantitative theory free from the above-mentioned defects has been developed yet. This thesis is just devoted to lay the foundations for such an attempt.

1.2 REVIEW OF OBSERVATIONS

In this section we shall discuss the main observational surveys which have recently been performed to determine the age-velocity dispersion relation for disk stars, and it will be shown that some of them are inconsistent with each other. This fact, which essentially is due to the unavoidable use of biased samples of stars, shows that observations do not put yet any stringent constraint on the age-dependence of the components of the stellar velocity dispersion, although the opposite is often claimed by observers. Other factors which contribute to such indetermination are the large experimental uncertainties, which are difficult to estimate properly, and the indirect estimates of stellar age. Moreover, some sample-selection criteria imply a contamination by spheroidal-component stars, which have a different cosmological origin and hence should not be included; however, they are spectroscopically so distinctive that generally it is not difficult to exclude them from the analysis. The fol lowing discussion does not pretend to be exhaustive at all; a more detailed report and comparison with other observational surveys can be found in the references cited.

In the 1970's three important observational surveys were performed by Byl (1974), Mayor (1974) and Wielen (1974). In particular, Wielen's (1974) analysis (see also Wielen 1967) is based on about 1000 stars contained in Gliese's (1969) catalogue of stars within 20 pc of the sun, for which trigonometric parallaxes accurate to $\pm$ 10% and accurate radial velocities and proper motions are known. This sample can be plotted directly in an H-R diagram, and hence it can be divided into unambiguous age groups by choosing stars found in definite color intervals along the main sequence or near the positions of the subgiants or giant branches of clusters of known age. For each main-sequence group the average age is assumed to be about half the main-sequence lifetime of the proper stellar type (i.e., a constant star-formation rate is assumed), and to the giants are assigned the ages of the clusters along whose giant branches they most closely lie. The sample includes a large number of McCormick K + M dwarfs with known CaII emission-line intensities, for which mean ages can be derived statistically from their relative abundances by assuming a constant star-formation rate over the lifetime of the Galaxy. These estimates can be checked by using observed average emission-line strengths in clusters of known ages; the two sets of age turn out to be in good agreement. The final results of this analysis are presented in Fig. 1.1 and Table 1.1; in particular, it is found that the age-velocity dispersion relation follows a $(1/3 \div 1/2)$-power-law.

Wielen's (1974, 1977) estimates of the age-dependence of the components of the stellar velocity dispersion has been questioned by very recent observational surveys (Carlberg et al. 1985; Knude, Schnedler Nielsen and Winther 1987; Strömgren 1987), which however are also inconsistent with each other.

Carlberg et al. (1985) combined Twarog's (1980) sample (suitably reduced to about 250 F stars within 100 pc of the sun), for which ages and photometric distances can be determined, with astrometric data to obtain tangential velocities of a set of stars with a large age-range. The stellar age was estimated by means of a new set of stellar evolutionary sequences and isochrones incorporating substantial improvements in the imput stellar physics. The resulting age-velocity dispersion relation rises fairly steeply for stars less than 6 Gyr old, thereafter becoming nearly constant with age (see Fig. 1.2 and Table 1.2).

Knude, Schnedler Nielsen and Winther (1987) considered the sample of stars obtained from the intersection of a photometric catalogue of A and F stars at the North Galactic Pole (Knude 1987) with the AGK 3 catalogue of proper motions (Dieckvoss et al. 1975). Due to the high latitude of these stars ($b > 70°$), an accurate estimate of the plane-parallel velocities was obtained from proper motions and distances alone (i.e., without considering radial velocities). Complete subsamples of about 550 unevolved and slightly evolved F stars of solar composition roughly within 200 pc of the sun were used to study the variation of the velocity dispersions σ_U and σ_V with age. Both dispersions are found to follow power-laws very closely, but the two laws have significantly different powers, $0.53 \simeq 1/2$ and $0.27 \simeq 1/4$ respectively; the total planar velocity dispersion is found to obey roughly a 1/2 -power-law (see Fig. 1.3 and Table 1.3). The most immediate consequence of this result would be a considerable change of the shape of the velocity ellipsoid with age; more precisely, the axial ratio would change from 1 to its equilibrium value of about 0.5 during a period lasting 5 Gyr. The observed relaxation time seems to be much longer than that suggested from studies of early-type stars, which is of the order of the epicyclic period.

Strömgren (1987) considered a sample of about 2300 A5 to G Population I stars within 100 pc of the sun, belonging to Olsen-Perry's (1984) photometric catalogue and for which a reliable determination of radial velocities was possible. Since for all these stars adequate photometric distances and proper motions were available, galactic velocity components relative to the sun of satisfactory accuracy were derived. It is found that while the plane-parallel components of the velocity dispersion σ_U and σ_V increase markedly throughout the range $3 \div 9$ Gyr, their ratio $\sigma_V/\sigma_U \simeq 0.6$ showing no appreciable variation, the perpendicular component of the velocity dispersion σ_W stops at a nearly constant value for stellar ages larger than 5 Gyr (see Table 1.4).

The apparent mutual inconsistency of the results of these observational surveys shows that we are still far away from a satisfactory knoledge of the age-velocity dispersion relation. In this context a more careful estimate of selection (and probably also contamination) effects might indeed raise the confidence level of observations.

Some indirect constraints on the age-dependence of the components of the stellar velocity dispersion can be obtained by constructing consistent kinematical and chemical models (e.g., Vader and de Jong 1981; Lacey and Fall 1983, 1985) or dynamical models (e.g., Bienaymé, Robin and Crézé 1987) of the Galactic disk in a solar neigh bourhood.

1.3 PLAN OF THE THESIS

This thesis is divided into two parts. PART I is devoted to a review of the results already known in the literature and of the methods employed to tackle the problem of the heating of galactic disks. This review does not pretend to be exhaustive, mainly because the interest in this specific problem arose at the beginning of the 1950's and since then a conspicuous number of papers have been written to shed light on it. However, among these only a few contributions have been so important as to represent real turning points in the understanding of this problem; it is just to such efforts that the review is mainly devoted. It should be born in mind, anyway, that even more fundamental results have been found in other branches of physics, which could suitably be extended to stellar dynamics and in particular applied to the specific problem of the heating of galactic disks. Often the importance of such suggestions is underestimated in favour of more standard viewpoints. The first part of this thesis is indeed devoted also to clarify the crucial role that some not properly appreciated effects (i.e., collective effects in nearly integrable systems and the effects of Lyapunov instability in non-integrable systems) can have in the secular evolution of stellar systems. The content of the various chapters will be summarized in the corresponding introductory sections; in any case it should appear clear even from their titles alone, which have been chosen in such a way as to result as pregnant as possible.

PART II expresses my own point of view about the effectiveness of the heating mechanisms described in chapter 5 in situations of astrophysical interest; in particular, it will be stressed the fact that no such approaches take collective effects properly into account, which instead are expected to drive the dynamical evolution of galactic disks. A global collective heating mechanism is then proposed in strict analogy with the quasi-linear theory of plasma waves, which predicts the occurrence of the so-called turbulent heating whenever an initial overstability is saturated or damped by non-linear effects. Before tackling such a complicated global non-linear analysis, in which the cold interstellar gas plays a damping role, simpler self-regulation mechanisms in the linear regime will be considered, in which such a cold component has instead a destabilizing role. Finally, some preliminary results are given concerning the effects that the finite thickness of galactic disks has on their stability properties, to which the above-mentioned self-regulation mechanisms are intimately related. The originality of this analysis on finite-thickness corrections to the local dispersion relation lies in the fact that the stellar and the gaseous components are self-consistently taken into account in a more rigorous way than in previous works.

FIGURE AND TABLE CAPTIONS

Fig. 1.1 Age-dependence of the components of the stellar velocity dispersion and resulting age-(total) velocity dispersion relation derived by Wielen (1974); the theoretical 1/3- and 1/2-power-laws are also shown. (Readapted from Mihalas and Binney 1981).

Fig. 1.2 Age-stellar velocity dispersion relation derived by Carlberg et al. (1985).

Fig. 1.3 Age-dependence of the plane-parallel components σ_U ($\diamond$) and σ_V ($\square$) of the stellar velocity dispersion and resulting age-planar velocity dispersion ($\times$) relation derived by Knude, Schnedler Nielsen and Winther (1987).

Table 1.1 Data relative to Fig. 1.1 (from Wielen 1977).

Table 1.2 Data relative to Fig. 1.2.

Table 1.3 Data relative to Fig. 1.3.

Table 1.4 Age-dependence of the components of the stellar velocity dispersion and horizontal axial ratio of the velocity ellipsoid derived by Strömgren (1987).

Fig. 1.1

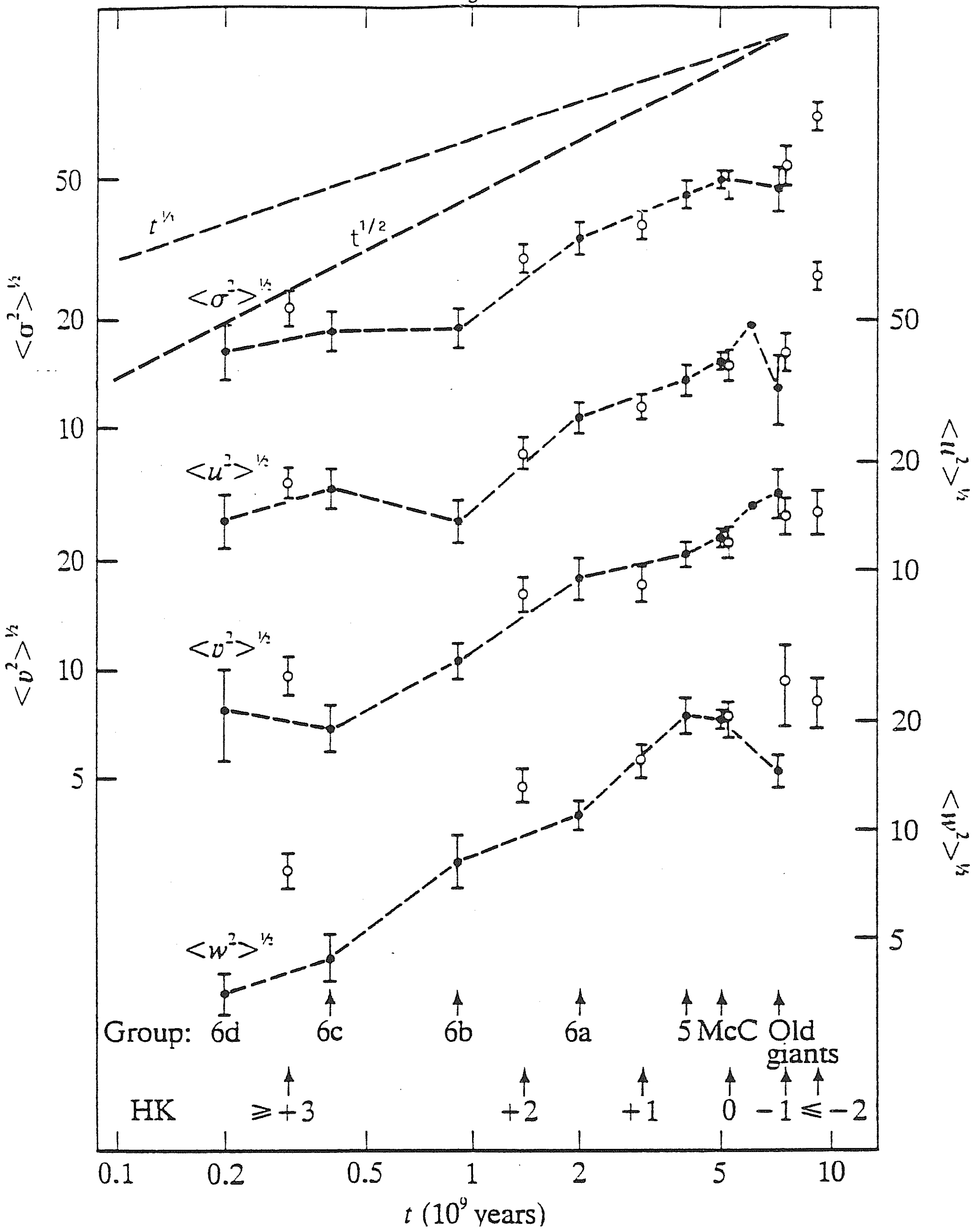

Table 1.1

Group of stars		At $z\sim0$ $\sigma_U^{(0)}$ km s^{-1}	$\sigma_V^{(0)}$	$\sigma_W^{(0)}$	Integrated over z σ_U km s^{-1}	σ_V	σ_W	σ_v	Age $\langle\tau\rangle$ 10^9 yrs
Classical Cepheids					8	7	5	12	0.05
Nearby	*6d*	14	8	4	14	8	3	16	0.21
stars	*6c*	17	7	4	20	7	4	21	0.47
on or near	*6b*	14	11	8	15	12	8	21	1.0
the main	*6a*	27	18	11	31	20	11	39	2.3
sequence	5	34	21	21	42	26	25	56	5.0
McCormick	HK +8/+3	18	10	8	20	10	6	23	0.3
K+M	HK +2	21	16	13	22	17	13	31	1.4
dwarfs	HK +1	29	17	15	30	16	15	37	3.0
	HK 0	38	23	20	40	21	21	50	5.2
	HK −1	40	27	26	40	34	34	63	7.2
	HK −2/−5	66	27	23	67	29	25	77	9.0
All McCormick stars		39	23	20	48	29	25	62	5.0

Fig. 1.2

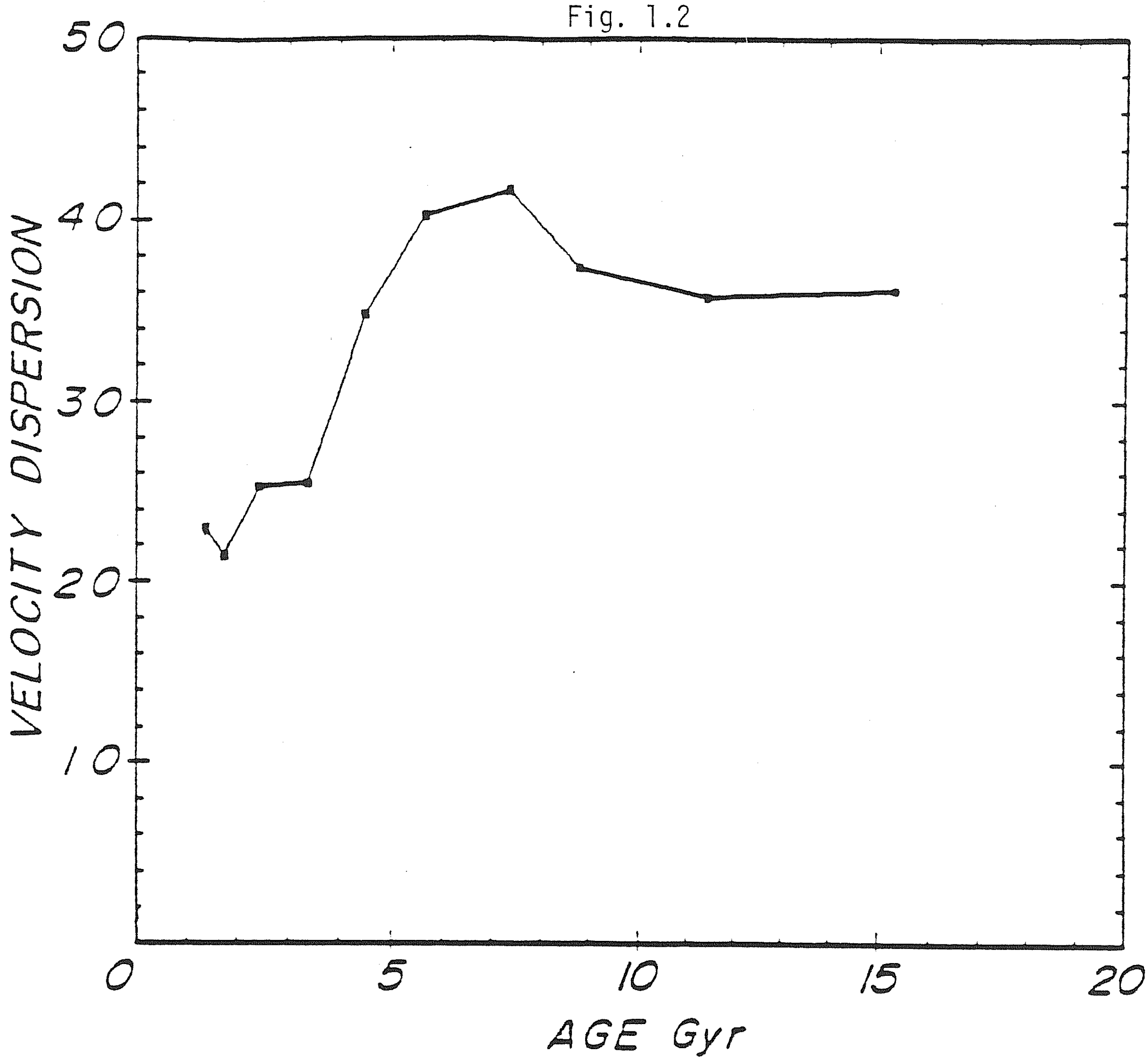

Table 1.2

t_{low}	t_{high}	t_{ave}	N	⟨[Fe/H]⟩	$\sigma_{[Fe/H]}$	$\sigma_{\perp}$
0.0	1.0	0.8	1	...	...	...
1.0	1.5	1.4	13	+0.11	0.06	23.0
1.5	2.0	1.7	37	+0.04	0.10	21.4
2.0	3.0	2.4	37	+0.01	0.10	25.3
3.0	4.0	3.4	20	−0.02	0.09	25.5
4.0	5.0	4.5	14	−0.04	0.14	34.8
5.0	6.5	5.7	40	−0.04	0.16	40.3
6.5	8.0	7.3	38	−0.15	0.17	41.7
8.0	10.0	8.8	20	−0.17	0.15	37.4
10.0	13.0	11.5	17	−0.11	0.11	35.7
13.0	20.0	15.3	18	−0.24	0.15	36.1

Fig. 1.3

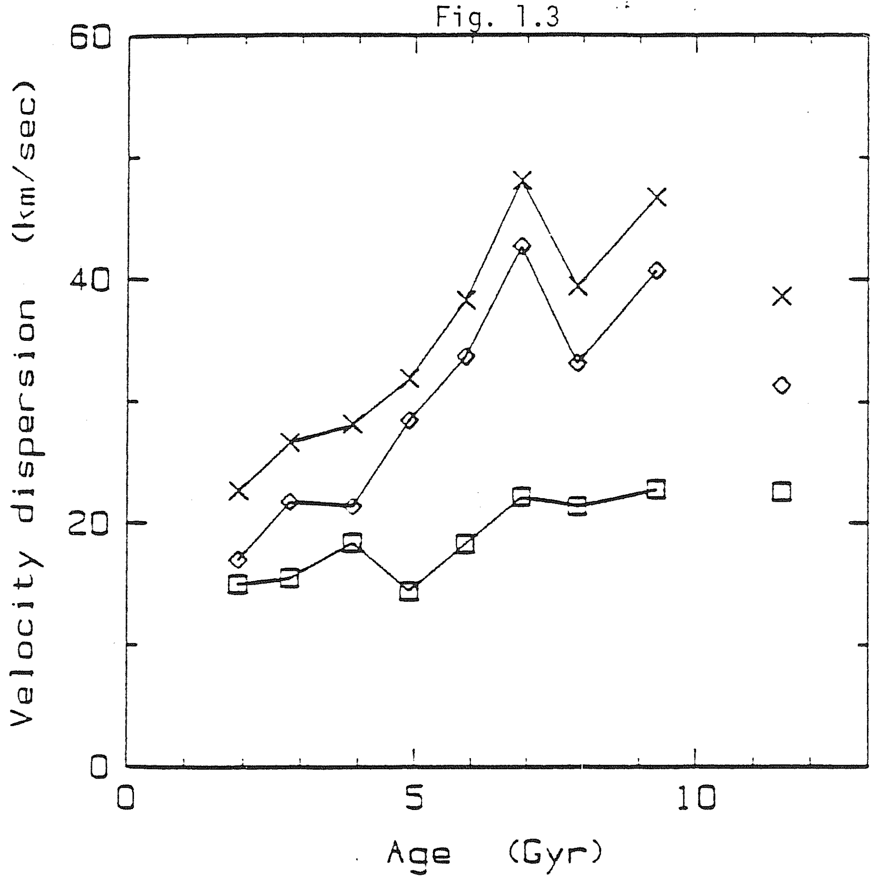

Table 1.3

	$\bar{A}$	σ_A	$\bar{U}$	σ_U	$\bar{V}$	σ_V	N	$\frac{\sigma_V}{\sigma_U}$	$\sigma\left(\frac{\sigma_V}{\sigma_U}\right)$
	(10^9 yr)		(km s^{-1})		(km s^{-1})				
I	1.9	0.3	8.8	17.3	−14.2	15.3	51	0.882	0.076
II	2.8	0.3	1.8	22.1	−14.1	15.8	64	0.715	0.049
III	3.9	0.3	5.0	21.7	−17.0	18.6	60	0.857	0.055
IV	4.9	0.3	6.6	28.7	−21.3	14.7	46	0.512	0.040
V	5.9	0.3	9.3	33.9	−19.1	18.5	38	0.546	0.038
VI	6.9	0.3	13.0	42.8	−25.4	22.4	28	0.523	0.035
VII	7.9	0.3	16.8	33.3	−20.1	21.6	28	0.649	0.047
VIII	9.0	0.3	21.0	34.4	−12.5	20.3	13	0.590	0.066
IX	9.9	0.3	4.0	51.8	−30.1	24.6	7	0.475	0.057
X	11.5	0.7	−28.4	31.5	−22.3	22.8	5	0.724	0.123

Table 1.4

Age unit 10^9y	$\bar{A}$ unit 10^9y	N	σ_U	σ_V	σ_W	[Fe/H]	$\frac{\sigma_V}{\sigma_U}$
2.0- 3.9	2.8	262	24.1±1.1	14.9±0.7	10.9±0.5	-0.02	0.62±0.04
4.0- 5.9	4.7	160	25.5±1.4	14.8±0.8	14.0±0.8	-0.04	0.58±0.05
6.0- 7.9	6.9	85	32.4±2.5	18.7±1.4	15.6±1.2	-0.04	0.58±0.07
8.0- 9.9	8.7	35	34.8±4.2	22.0±2.7	15.0±1.8	-0.03	0.63±0.11
10.0-12.9	11.3	16	26.6±4.9	20.2±3.7	15.3±2.8	-0.07	0.76±0.19

PART I

GENERAL BACKGROUND AND CRITICAL REVIEW OF PREVIOUS ANALYSES

CHAPTER 2. RELAXATION PROCESSES IN DYNAMICAL SYSTEMS: CLASSICAL ESTIMATES AND THEIR VALIDITY●

2.1 INTRODUCTION

It is a well-known fact that many-particle systems tend asympotically to a state of thermodynamical equilibrium characterized by a Maxwellian distribution function, provided some general assumptions on the nature of the collision processes between particles are fulfilled (see, e.g., Huang 1963). The relaxation towards this equilibrium state is governed by a characteristic timescale which generally is referred to as the relaxation time of the system. In this particular context ordinary binary collisions give the dominant contribution to the relaxation process leading to such a randomization of particle velocities. In many situations of physical interest this binary relaxation time turns out to be extremely long compared to the dynamical time scale or even to any other "observable" characteristic time; systems whose dynamics is governed by long-range interactions, such as plasmas and stellar systems, can indeed exhibit such a peculiar feature. If this is the case, other relaxation processes toward approximate equilibrium states (stationary states)[1] are possible on the intermediate timescales of interest; this kind of relaxation mechanisms can actually be much more effective than ordinary two-body encounters (which need not be "physical collisions" in the case of long-range interactions), because they arise from the collective nature of long-range interactions.

We shall now stress the main differences and similarities between binary and collective relaxation processes:

- Ordinary binary encounters can be viewed as short-wave fluctuations of the interaction field.[2] They are essentially random, and produce small effects which accumulate slowly in time. Each encounter can lead in two directions, increasing or decreasing the energy of one of the particles, so that the cumulative effect is a

●In this introductory chapter a number of concepts and results will be given without a satisfactory discussion, that can instead be found in chapters 3. and 4. which are intended to be a continuation of section 2.2.

random walk of each particle in velocity space, which gradually takes the whole system towards thermal equilibrium.

- Collective encounters are long-wave fluctuations of the interaction field.[2] They are completely analogous to binary encounters, except that one particle collides simultaneously with many particles collected together by some coherent process such as a wave. Observe that in this context the impact parameter can never be taken much smaller than the characteristic wavelength of the perturbation, and the collected bunch of particles moves with the group velocity of the wave rather than with the typical particle velocity. Again the process is random (generally it is not a random walk, as in the previous case, because memory effects cannot be disregarded) but usually much stronger, and rapidly takes the whole system towards a stationary state.

In this chapter we shall analyze in some detail the assumptions which are at the basis of classical estimates of the relaxation time in many-particle dynamical systems, with particular reference to plasmas and stellar systems. Often some of these assumptions are tacitly taken for granted, or even worse the results obtained in this context are claimed to be more general than they really are. A crucial point that must indeed be stressed right now is that all classical estimates of the relaxation time and related quantities apply to integrable sistems alone; in non-integrable systems, in fact, these calculations lose their validity because of the existence of extremely rapid phase-mixing mechanisms which make the orbits very sensitive to the initial conditions and to perturbations (Lyapunov instability). General reference is made to Chandrasekhar (1960), who studied extensively in the 1940's the role of binary encounters in the relaxation of stellar systems, Gurzadyan and Savvidy (1986), Pfenniger (1986), who stressed the crucial point mentioned above, and to books on dynamical systems and ordinary differential equations where rigorous formulations of the ergodic theory and of the Lyapunov stability are given (e.g., Arnold and Avez 1968; Arnold 1974, 1980a,b; Arnold and Wihstuts 1986; Bergé, Pomeau and Vidal 1984; Contopoulos 1966, 1973, 1985; Galiullin 1984; Gallavotti 1986; Lindblad 1983; Moser 1973; Schuster 1984; Starzhinskii 1980; Voronov 1985; Wightman 1985). See also the review papers by Eckman and Ruelle (1985), Escande (1985), Vivaldi (1984). General reference is made also to the books on plasma physics and stellar dynamics listed in section 3.1.

2.2 THE GLOBAL RELAXATION TIME OF INTEGRABLE SYSTEMS: COLLISIONAL VS COLLISIONLESS PROCESSES

From the discussion made in the previous section it appears that systems whose dynamics is governed by long-range interactions are characterized by two relaxation times: the binary relaxation time τ_{bin} associated with the relaxation towards the final thermodynamical-equilibrium state, and the collective relaxation time τ_{coll} associated with the relaxation towards an intermediate stationary state. It is implicitly assumed that the systems under consideration are close to a situation of integrability (and quasi-stationariety), because otherwise other relaxation mechanisms occurring on shorter timescales are to be considered; a discussion of the effects of such an orbital instability is deferred to the next section.

Another important fact to bear in mind is the characterization of the collision processes in which binary encounters and collective effects are involved. Due to the long-range nature of the interactions involved, a test particle during its motion interacts simultaneously with all the other field particles of the system, so that the question arises whether a similar distinction between binary and collective encounters does make sense. This question can be answered by observing that the main effect of the collective property of these systems consists in the presence of large-scale <u>self-consistent</u> mean fields (only in the gravitational case) and in the possibility of exciting <u>self-sustained</u> oscillation modes[3]; in this respect, differences in the large-scale dynamics between plasmas and stellar systems arise from the fact that in the former screening effects are present which make them locally neutral. The role of collective effects can thus be singled out in a first approximation by formally dividing the potential into two parts: a part consisting in the mean field itself plus possible self-sustained perturbations of relatively long wavelengths, both produced by the "smoothed out" distribution of matter, and a part which takes into account the fluctuations of the interaction field of relatively short and intermediate wavelengths arising from the "discrete" distribution of matter. With such a decomposition the bulk of collective effects is contained in the first part, but a non-negligible contribution (with respect to binary encounters) is given also by the intermediate-wavelength fluctuations, in which particle correlations are taken into account. The following characterization hence follows: <u>collective effects</u> are mainly associated with collisionless relaxation processes, even though they contribute also to collisional relaxation processes, whereas binary encounters are relevant to collisional relaxation processes alone. It turns out that binary and collective encounters give comparable contributions to collisional relaxation processes in several situations of physical interest, as will be discussed later on in this section; the non-Markovian character of collective encounters, however, makes them difficult to treat. In this context it should be mentioned that several authors have tried to evaluate such a contribution by calculating the relaxation time or the dynamical friction in idealized models of stellar systems (e.g., Julian and Toomre 1966; Julian 1967; Thorne 1968; Kalnajs 1972). Although they clame to have taken collective effects fully into account, actually their analyses are restricted to neutral fluctuations alone, and thus disregard the most essential contribution which comes indeed from overstabilities of the system.

We shall now list the basic assumptions underlying classical estimates of the relaxation time of systems governed by long-range interactions (Chandrasekhar 1960, for the gravitational case); some of them will be discussed in some detail, together with their physical and mathematical implications, in chapter 3. The system is assumed to be homogeneous, and only instantaneous, distant, mutually independent, binary encounters in the impulse approximation (straight-line orbits) are considered. As all these work assumptions lead to an overidealized model of relaxation process, only the order of magnitude of the relaxation time so derived is thought to be meaningful. For this reason we prefer to give a rough expression, as derived by Hénon (1973) in the case of stellar systems, rather than the extremely precise formulae (different possible definitions of relaxation time can in fact be given) found in Chandrasekhar (1960):

$$\tau_{bin} = \frac{v^3}{8\pi G^2 \, m \varrho \ln N} \,, \tag{2.1}$$

where N is the total number of stars in the system of mass m, mean volume density ϱ and typical random velocity v; here the virial theorem[4] for stellar systems has been used to express the Coulomb logarithm $\ln (b_{max}/b_{min})$ in terms of the total number of particles in the system, b being the impact parameter. Note that the relaxation time of large stellar systems, as elliptical and spiral galaxies, largely exceeds even the age of the universe (Hubble time). An unphysical feature deriving from the use of drastic assumptions such as the restriction to instantaneous binary encounters and the impulse approximation consists in the fact that logarithmic divergences occur at small and large impact parameters, which are formally removed by introducing the long-range and short-range cutoffs b_{max} and b_{min}, respectively (b_{max} is different in the two cases of plasmas and stellar systems; see section 3.3). Several attempts have been made to relax some of the classical assumptions mentioned previously (e.g., Hénon 1958; King 1958; Lee 1968; Ostriker and Davidsen 1968; see also Horedt 1984) and to take collective effects into account (see the references cited in the previous paragraph). The order of magnitude of the relaxation time or related quantities turns out to be preserved, except in some "pathological" cases (e.g., Kalnajs 1972) which can be explained in the light of more sophisticated approaches (see the references listed at the end of section 3.3 concerning dynamical friction).

The use of the collisionless Boltzmann equation (Vlasov equation) for investigating galactic structure and dynamics has been justified by simple estimates of the ratio of the relaxation time by particle encounters τ_{bin} to the typical orbital time in the mean field, i.e. the crossing time T_{cross}. It is found that (see, e.g., King 1967; Hénon 1973)

$$\frac{\tau_{bin}}{T_{cross}} \sim \frac{N}{\ln N} , \qquad (2.2)$$

so that for large N the effects of particle encounters can be neglected on a dynamical timescale. Galactic disks, even though they may be considered very thin, are still 3-dimensional systems, and it is to 3-dimensional systems that these estimates of relaxation time apply. However, galactic disks are often approximated as strictly disk systems, in which stars are still assumed to interact by inverse-square forces but are constrained to move in a plane; therefore, it is of some interest to consider the problem of relaxation time for strictly disk systems. It is found that (Rybicki 1972)

$$\frac{\tau_{bin}}{T_{cross}} \sim \frac{\lambda}{2} \equiv \frac{v}{2\,V} , \qquad (2.3)^{5}$$

where V is the typical total particle velocity and v the typical random particle velocity relative to a local frame of rest, so that the relaxation time is at most of the same order of magnitude as the crossing time, independently of the number of par ticles. Therefore, the collisionless Boltzmann equation can never provide an adequate description of strictly disk systems, however large. It can be shown also that, in contrast to the 3-dimensional case, the relaxation is substantially due to close encounters, and that the cumulative effect of long-range encounters is of no more than the same order of magnitude; thus, the assumption of independence of encounters does not present any difficulty in this 2-dimensional case, since close encounters are indeed expected to occur independently. In this regard it is interesting to note that the long-range divergence occurring in the 3-dimensional case when simple derivations are used does not occur in the 2-dimensional case, so that there is no need to introduce a long-range cutoff. All these differences can be traced back to the different statistical weighting of impact parameters in the two cases. These arguments concerning strictly disk systems of course do not apply to actual galaxies, which are 3-dimensional systems; the validity of the Vlasov equation is rather well established in this case (see, however, section 2.3). However, strictly disk sistem approximations are commonly used in analytical and numerical treatments of disk galaxies, and hence it is necessary to judge these approximations in the light of the preceding results. For analytical treatments there is no such problem of the relaxation time at all. The use of Vlasov theory is first established in view of the finite thickness of the disk, and then it is simply a question whether a 2-dimensional form of the Vlasov equation is a good approximation to the 3-dimensional form; although this question is not trivial, at least it can be answered within the framework of Vlasov theory. The situation concerning numerical simulations, on the other hand, is not so straightforward: 2-dimensional models of disk galaxies are often used, and the relaxation results presented here apply. Therefore, 2-dimensional N-body codes, if performed in a sufficiently precise way, are not faithful simulation of the Vlasov equation and thus do not apply to actual disk galaxies; fortunately, numerical simulations are themselves subject to further approximations which tend to reduce the severity of this difficulty. For a more detailed discussion reference is made to Rybicki (1972).

2.3 THE ORBITAL RELAXATION TIME OF NON-INTEGRABLE SYSTEMS: THE EFFECTS OF LYAPUNOV INSTABILITY

Many numerical tests have been performed during the last two decades concerning the applicability of the classical expression for the relaxation time τ_{bin} given in section 2.2. Some of them (e.g., Miller 1964), although they were based on 3-dimensional models of stellar systems and thus were free from the criticism arisen by Rybicki (1972), showed a general and fast exponential divergence of systems starting very close to each other in the 6N-dimensional phase space; even when no close encounters occurred, such systems diverged exponentially at a rate much larger than that estimated by Chandrasekhar (1960). A rough explanation of this peculiar behaviour was put forward by Miller (1966) in terms of "polarization effects in the difference medium (system)". Put in another way, he gave an original formulation of the well-known problem of Lyapunov instability in dinamical systems, but a number of wrong conclusions were drawn (e.g., regular orbits do not show an exponential but rather a linear divergence; see below); the roughness of this formulation lies indeed in the fact that it cannot discriminate integrable from non-integrable systems. The aim of the forthcoming discussion is just to explain these concepts in some more detail and to describe the effects of Lyapunov instability on the relaxation of dynamical systems, with particular reference to stellar systems.

In the framework of the ergodic theory dynamical systems are divided into two classes:

- Integrable systems, for which the number of integrals of motion is equal to the number of degrees of freedom and the phase space trajectories lie on N-dimensional tori.
- Non-integrable systems, whose classification is given by increasing the degree of their statistical properties: dynamical systems with divided phase space (i.e., containing both motion on N-dimensional tori and chaotic motions), ergodic systems, systems with weak and n-fold mixing, K-systems and finally Bernoulli systems, which are a subclass of K-systems. More precisely, the classification criterion is the rate at which an initial cell of phase space tends to cover uniformly the energy hypersurface. In mixing systems an initial cell complicates its shape in such a way (i.e., preserving its volume) as to cover uniformly the energy asymptotically; in this sense, a mixing system in a non-equilibrium state tends asymptotically to equilibrium. K-mixing systems, which possess maximally strong statistical properties, tend to such microcanonical equilibrium state with an exponential rate, the relaxation time being proportional to the Kolmogorov

entropy; one of their main properties is, in fact, the decay of phase space trajectories into beams of exponentially approaching and expanding trajectories (transversal fibers).

Several attempts have been made to relate the exponential divergence observed in the above-mentioned numerical experiments to the peculiar behaviour characterizing strongly non-integrable systems. From the point of view of ergodic theory, this can be attained by reducing the problem of a self-gravitating N-body system to the investigation of the behaviour of a geodesic flow on a Riemannian manifold, making use of the Maupertuis principle. It is found that the negativity of the 2-dimensional curvature of this manifold is a sufficient condition for an exponential deviation of the geodesics, and the minimum of its absolute value defines an orbital relaxation time (e.g., Gurzadyan and Savvidy 1986; Gurzadyan and Kocharyan 1987a,b). Although this geometric method for investigating the stochasticity of dynamical systems is attractive from a formal point of view, other methods have been found to be more predictive from a numerical point of view; they are based on the calculation of the so-called Lyapunov characteristic exponents χ_i, which will now be discussed in some detail. An important property of non-integrable systems is to contain a definite fraction of irregular orbits, also qualified as stochastic, semi-ergodic etc., exhibiting an exponential sensitivity to the initial conditions and to perturbations (as, e.g., the granularity of the system), which thus are rapidly amplified; in contrast, regular orbits are only linearly sensitive[6]. This intrinsic sensitivity is measured, indeed, by the Lyapunov exponents. After a certain time the largest one, if positive, will dominate the divergence, and is therefore the best physically observable one; it determines an (individual) orbital relaxation time $\tau_{orb} \sim \chi_{max}^{-1}$[7], which is not a global relaxation time of the system (see also below; for a more detailed discussion see Pfenniger 1986). The other χ_i can be computed by various numerical techniques. For an autonomous (i.e., time-independent) Hamiltonian system with n degrees of freedom there exist 2n Lyapunov exponents, two of which vanish for every isolating integral of motion[8] and the others appear in pairs $(-\chi_i, \chi_i)$; each isolating integral, therefore, makes the motion robust in two directions of phase space, which are characterized by a simple linear divergence. As a consequence, actions characterize regular orbits, while the positive Lyapunov exponents characterize irregular orbits. The sum of these positive exponents turns out to be just the (specific) Kolmogorov entropy (see also the discussion made in the previous paragraph), which otherwise vanishes for regular orbits, so that only non-integrable systems evolve irreversibly.

We now turn to analyze the most direct physical implications of Lyapunov instability. The extreme sensitivity of irregular orbits to the initial conditions and to perturbations makes estimates of the binary relaxation time clearly meaningless. This difficulty also occurs when collective effects are taken into account, because the rapid phase-mixing mechanisms associated with this orbital instability can damp self-sustained oscillations of the system on timescales much shorter than the dynamical timescale, which in turn is generally comparable to the inverse of the growth rates of these oscillations (i.e., no coherent process such as a self-sustained wave can be maintained over the relevant timescales). A global relaxation time is thus no more meaningful because, apart from the impossibility of defining a binary or a collective relaxation time, one has to consider also the fact that the Lyapunov relaxation time τ_{orb} can be very different from orbit to orbit since different values of χ_{max} are involved. On the other hand, binary encounters and collective effects are the most effective relaxation mechanisms in integrable systems, or in those regions of nearly integrable systems where regular orbits are dominant. Moreover, classical estimates of the relaxation time still hold in non-integrable systems, provided it is defined in such a way as to refer to the exchange of isolating integrals alone. The same considerations apply as regards the formal validity of the stochastic equations which will be investigated in section 3.2; in non-integrable systems the diffusion in velocity space produced by Lyapunov instability cannot in fact be disregarded (see also below). In non-integrable systems the non-uniform coverage of phase space by irregular orbits makes the use of the Vlasov equation and the applicability of Jeans theorem questionable as well (cf. Binney 1982). The question thus naturally arises how often the departure from integrability of observed stellar systems can be neglected; but since all kind of systems exist, from systems far from integrability as small open clusters up to nearly integrable systems as spherical globular clusters, no general rule can be given. Analytical methods usually apply to nearly integrable problems, so that the successful models are strongly biased toward "nicely symmetric situations. Attention must be paid, however, not to extrapolate superficially the results so obtained to real stellar systems; from KAM (Kolmogorov, Arnold, Moser) theorem it follows, in fact, that asymmetries can generally destroy the principal isolating integrals of motion, since stochasticity invades phase space in a complicated manner as a perturbation grows, sometimes abruptely (Arnold diffusion). Note, however, that Nekhoroshev theorem on Arnold diffusion shows that under quite mild assumptions this is a very slow phenomenon (see, e.g., Benettin, Galgani and Giorgilli 1985; Benettin and Gallavotti 1986; Benettin 1986, 1987; Galgani 1985).

So far we have more or less tacitly assumed to be in time-independent or at least in weakly time-dependent situations. Indeed, the "autonomous" assumption is not restrictive at all; we can, in fact, always transform a time-dependent Hamiltonian system into an autonomous Hamiltonian system by extending its phase space in such a way as to include the time coordinate. The considerations made in the previous paragraph can thus be applied even to stellar systems in a collapse phase; We shall now discuss some of their physical implications in relation to the theory of violent relaxation (Lynden-Bell 1967; see also section 4.2). In some cases it may happen that the collapsing system is integrable, so that some non-classical individual stellar integrals are conserved; this situation would not be radically different from a steady-state system. But what makes the concept of violent relaxation nevertheless mostly correct is that integrable systems are very rare, so that a spherical collapse is expected to produce a large fraction of stochastic orbits; as a consequence of their exponential sensitivity to perturbations, strong phase-mixing[9] mechanisms become operative and lead to an efficient relaxation of the system. The violence of the relaxation is thus the consequence of the strongly non-integrable situations considered. For a more detailed discussion reference is made to Pfenniger (1986).

FOOT-NOTES

1 The term "equilibrium" is properly referred to the thermodynamical state which is ultimately attained in the process of randomization of particle velocities; in any other time-independent situation the term "stationary" is commonly used. In the following we shall drop this distinction whenever no ambiguity arises, bearing in mind, however, that such a distinction does indeed exist.

2 The fluctuations of the interaction field that a particle experiences during its

motion among the other particles of the system can be Fourier analyzed (under the assumption of local homogeneity) into wave components with different wave-vector $\underline{k}$. The distinction between binary and collective encounters is based, indeed, on the different typical wavelengths involved in the two cases, which are to be compared to the mean interparticle distance.

3 The self-consistency property concerning the mean fields and the self-sustenance property concerning the oscillation modes express the fact that these large-scale phenomena are produced by the distribution of matter in the system (Poisson equation), and thus are not externally imposed.

4 Bear in mind that the virial theorem implies a relaxation in the configuration space, which generally is attained in a dynamical timescale $T_{cross} \ll \tau_{bin}$, the latter being associated with the relaxation in velocity space (see the discussion on the validity of the collisionless Boltzmann equation in strictly disk systems). Large stellar systems, for instance, have a stationary shape but have not yet reached a state characterized by a complete randomization of particle velocities.

5 This result can be expressed in a compact form relating the relaxation time τ_{bin} to the epicyclic frequency $\varkappa$ and to the local stability parameter Q of infinitesimally thin, one-component, self-gravitating disk systems in differential rotation (for the definition of Q see Toomre 1964):

$$\varkappa \, \tau_{bin} \sim (\pi/2) \, Q \; . \qquad (2.3)'$$

6 The exponential or the linear character of this sensitivity is restricted to the linear regime. In a non-linear regime, in fact, saturation effects occur due to the presence of damping terms neglected in the linear treatment.

7 If χ_{max} is close to zero (nearly regular orbits), then a more detailed analysis, as for instance that performed by Chandrasekhar (1960), is required since the relaxation time turns out to be directly dependent on the characteristics of the system.

8 The number of isolating integrals of motion is connected with the symmetry properties of the system, and therefore depends on the form of the (self-consistent) potential (see, e.g., Freeman 1975; Woltjer 1967).

9 This extremely rapid phase mixing is not to be confused with the ordinary phase mixing occurring in quasi-stationary nearly integrable systems, which instead proceeds only linearly in time (see, e.g., Frieman 1975).

CHAPTER 3. COLLISIONAL RELAXATION PROCESSES: THE FOKKER-PLANCK APPROACH AND OTHER ALTERNATIVE DESCRIPTIONS

3.1 INTRODUCTION

The Fokker-Planck equation has widely been applied to the study of plasmas and stellar systems for describing the evolution of the one-particle distribution function when collisional effects are taken into account. Its derivation is based on a number of assumptions which in general are not clearly specified or are tacitly taken for granted. In this chapter we shall inquire into the validity of this equation by analyzing the underlying assumptions in some detail. The following discussion does not pretend to be exhaustive, because most of the mathematical concepts and techniques inherent in this description are subtle and cannot thus properly be expressed and discussed in this context. Reference is made to books on probability theory and stochastic processes, where rigorous derivations of the Fokker-Planck equation (cf. Kolmogorov forward equation) are given (e.g., Cox and Miller 1965; Feller 1968, 1971; Friedman 1975, 1976; Kac and Logan 1987; Montroll and West 1987; Nelson 1967; Ventsel 1983; Wax 1954); see also the review papers by Haken (1975), Li (1986), Spohn (1980). General reference is made also to books on plasma physics (e.g., Boyd and Sanderson 1969; Hinton 1983; Ichimaru 1980; Krall and Trievelpiece 1973; Schmidt 1979; Sivukhin 1966) and stellar dynamics (e.g., Binney and Tremaine 1987; Saslaw 1985). Some other more general and/or correct, but less predictive, approaches will also be described.

The evolution of the one-particle distribution function $f(\underline{x},\underline{v};t)$ in the 6-dimensional phase space μ is described by the Boltzmann equation

$$\frac{df}{dt} \equiv \frac{\partial f}{\partial t} + [f,H] = \left(\frac{\partial f}{\partial t}\right)_{coll} , \qquad (3.1)$$

where H is the Hamiltonian of the system, self-consistently related to f via the Poisson equation, the symbol [..,..] denotes the Poisson brackets, and the term $(\partial f/\partial t)_{coll}$ represents the contribution of particle encounters to the time-variation of f. In general this is an integro-differential equation, which can be reduced to a differential equation only by making certain assumptions on the collision processes.

For instance, if in the relevant timescale collisional effects can be neglected (as in high-temperature plasmas and large stellar systems, where the relaxation time largely exceeds the dynamical timescale), we recover the Vlasov equation

$$\frac{\partial f}{\partial t} + [f,H] = 0 ; \tag{3.2}$$

it states that f is a conserved quantity along the particle orbit. For steady-state systems this implies that f is a function of the isolating integrals of motion alone (Jeans theorem; see, e.g., Chandrasekhar 1960; Lynden-Bell 1962; for some controversial points see section 2.3). The collisionless Boltzmann equation should not be confused with the Liouville equation

$$\frac{\partial f^{(N)}}{\partial t} + [f^{(N)},H] = 0 , \tag{3.3}$$

which describes the evolution of the N-particle distribution function $f^{(N)}(\underline{x}_1,\ldots,\underline{x}_N;\underline{v}_1,\ldots,\underline{v}_N;t)$ in the 6N-dimensional phase space Γ without any assumption on the collisional nature of the system.

3.2 MARKOVIAN STOCHASTIC APPROACHES: THE CONCEPT OF DYNAMICAL FRICTION

When the effect of encounters is taken into account, the only way to reduce the Boltzmann equation to a differential equation, i.e. to make it operate locally in time, is to require that the system has no memory in the collision processes (ergodic assumption), so that a test particle suffers random displacements in velocity space generated by the fluctuating part of the interaction field in a manner that can be described in terms of a random walk; this is equivalent to state that the increments of velocities are regarded as stochastically independent in disjoint time intervals. In systems whose dynamics is governed by long-range interactions, such as indeed plasmas and stellar systems, this assumption may not be well justified because correlations among particles cannot be disregarded a priori; collective effects, in fact, always play an important or even dominant role in these two kinds of systems (in the gravitational case even more than in the electromagnetic case because no Debye shielding length, i.e. no local neutrality, exists). We then keep this as a working assumption, bearing in mind that the resulting evolution equation neglects collisional collective effects; a different approach which takes them fully into account will be dis cussed in section 3.3.

The standard approach consists in deriving a diffusion process in velocity space (see, e.g., Chandrasekhar 1943a,b,c,d). The evolution of the distribution function $f(\underline{x},\underline{v};t)$ is then written in the form of a Fokker-Planck-type equation:

$$\frac{\partial f}{\partial t} + [f, H] = \nabla_{\underline{v}} \cdot \left(q \nabla_{\underline{v}} f + \eta f \underline{v} \right), \tag{3.4}$$

where we recall that H is the Hamiltonian related to the smoothed out distribution of matter, $q = q(\underline{v})$ is the diffusion coefficient and $\eta = \eta(\underline{v})$ is the coefficient of dynamical friction appearing in the Langevin equation[1]; these two coefficients are related by the condition that a given Maxwellian distribution function f_M remains invariant in time, i.e. $(\partial f_M/\partial t)_{coll} \equiv 0$, so that η turns out to be connected with the reciprocal of the relaxation time of the system. More standard forms of the Fokker-Planck equation are the following:

$$\frac{\partial f}{\partial t} + [f, H] = - \frac{\partial}{\partial v_i} \left(f \frac{\langle \Delta v_i \rangle}{\Delta t} \right) + \frac{1}{2} \frac{\partial^2}{\partial v_i \partial v_j} \left(f \frac{\langle \Delta v_i \Delta v_j \rangle}{\Delta t} \right) \tag{3.4$'$}$$

(see, e.g., Hénon 1973), where the averages are taken with respect to a transition probability distribution of gaussian type, and

$$\frac{\partial f}{\partial t} + [f, H] = - \frac{\partial}{\partial v_i} (A_i f) + \frac{1}{2} \frac{\partial^2}{\partial v_i \partial v_j} (D_{ij} f) \tag{3.4$''$}$$

found in most plasma physics textbooks, where A_i and D_{ij} are referred to as the dynamical-friction vector and the diffusion tensor, respectively[2]. Sometimes it is more useful, especially in the case of stellar systems, to use the action-angle variables as the proper canonical coordinates in virtue of their physical meaning (the actions correspond to adiabatic invariants); the Fokker-Planck equation retains exactly the same form with the velocities v_i replaced by the actions J_i.

Now it should be born in mind that the sample paths of every diffusion process are continuous (with probability one), so that the random velocity of the test object varies continuously in the course of time. Because of this fact, the Fokker-Planck approach seems not to be suitable to describe systems governed by the electromagnetic or the gravitational interaction, since a close encounter of a test par ticle with a field particle is able to produce a large change of velocity within a small time interval, clearly contradicting the notion of continuity. This difficulty arises because the ergodic assumption is indeed more general than the choice of a diffusion process; in other words, the phrase "stochastically independent events in disjoint time intervals" is not equivalent to the property "diffusion", since there exist infinitely many Markov processes which share only the first property but not the second (these concepts will be explained in some more detail in the forthcoming discussion). If the stochastic variations in velocity space can approximately be described as a Markov process on the whole, the question arises whether it is uniquely determined by the properties of the fluctuating part of the interaction field, and whether the jump phenomena mentioned above can be explained by an analysis of its sample paths alone without employing additional assumptions. Several attempts have been made to answer this question, and more in general to formulate a statistical theory in the framework of stellar dynamics, but we are still far away from a satisfactory understanding (e.g., Chandrasekhar 1941, 1943a,b,c,d; Chandrasekhar and von Neumann 1942, 1943 and Chandrasekhar 1944a,b; see also Chandrasekhar 1960 for a review; Camm 1963; Lee 1968; Tscharnuter 1972).

Bearing this fact in mind, we shall now briefly discuss some basic ideas which lead to the derivation of another, still not completely satisfactory, stochastic differential equation for plasmas and stellar systems. Let us first reformulate more definitely the conditions of a random walk: the total increment of velocity within the time interval (0,t), where t is much larger than the characteristic time T during which an elementary fluctuation of the random interaction field takes place, can be written as a sum of a great number of independent[3] random variables representing the (at least on the average) small displacements in velocity space after the small amount of time T has passed. Now the crucial point is the determination of the distribution law of this sum. This problem, however, is solved exhaustively by the so-called extended central limit theorems of probability theory, which were essentially established by Lēvi and Khintchine in the 1930's. In general, a convergence to the normal (i.e., gaussian) distribution and hence a diffusion is expected, but the probability distribution of the random gravitational field is shown to be asymptotically the Holtsmark distribution, whose characteristic function (Fourier transform) is $h(\underline{\varrho}) = \exp(-a|\underline{\varrho}|^{3/2})$ (however, even this distribution contains some unphysical features; see, e.g., Chandrasekhar 1941, 1943a, 1960; Feller 1971). The Holtsmark distribution is a symmetric stable distribution[4] and belongs therefore to its own domain of attraction. This leads necessarily to a distribution law for the total increment of velocity within the time interval (0,t) with characteristic function $\varphi(\underline{\omega}) = \exp(-\sigma t|\underline{\omega}|^{3/2})$, where $\sigma = aT^{1/2}$[5]; φ is the Fourier transform of the transition function belonging to the Markov process which is called the stable process with characteristic exponent 3/2. From general theorems on Markov processes it follows that its sample paths are right-continuous (i.e., jump phenomena occur). It can also be shown that the mean number of jumps increases to infinity as their heights converge to zero, and conversely; this is a very important property because, if one identifies these jumps as the results of far and close encounters respectively, the importance of far encounters is emphasized on the one hand, but spontaneous large changes in velocity due to close encounters are also possible on the other hand.

So far dynamical friction acting in a purely systematic manner has been ignored; taking it into account, a stochastic equation for the stable Markov process with characteristic exponent 3/2, analogous to the diffusion equation derived by Chandrasekhar, can be written down (Tscharnuter 1972):

$$\frac{\partial f}{\partial t} + [f,H] = -\left(-\nabla_{\underline{v}}^2\right)^{3/4}(\sigma f) + \nabla_{\underline{v}}\left(\eta f \underline{v}\right), \tag{3.5}$$

where σ appears to play a role similar to the diffusion coefficient q in Chandrasekhar diffusion equation. The 3/4 -power of the Laplace operator $\nabla_{\underline{v}}^2$ is uniquely defined in the sense of its spectral representation: this elliptic pseudo-differential operator (see, e.g., Hormander 1976, 1985; Taylor 1981) acts on a given function f in such a way that the Fourier transform of $(-\nabla_{\underline{v}}^2)^{3/4} f(\underline{v})$ is the function $|\underline{\omega}|^{3/2}\hat{f}(\underline{\omega})$, where $\hat{f}$ denotes the Fourier transform of f. The correct derivation of this term in the previous evolution equation is not simple and requires sophisticated functional analysis techniques (semi-group theory). Since it seems impossible to solve Tscharnuter stochastic equation in the whole 6-dimensional phase space analytically as well as numerically, its investigation relies on the assumption of spatial homogeneity, i.e. $[f,H] \equiv 0$, which is a quite drastic assumption for stellar systems; bear in mind, however, that this assumption is already inherent in the derivation of the

Holtsmark distribution, and is used also to calculate explicitly the coefficients of diffusion and dynamical friction in the Fokker-Planck equation. In contrast to the Fokker-Planck equation, it can be shown that a Maxwellian distribution function f_M is not an invariant distribution of the given Markov process, i.e. $(\partial f_M/\partial t)_{coll} \neq 0$; this fact causes troubles since the relation between σ and η cannot directly be established, and might have important physical implications.

Now the question arises which of the two Markovian stochastic approaches described here is more correct from a physical point of view. Both of them, in fact, seem to give rise to unphysical features: the Fokker-Planck approach predicts continuous sample paths, whereas the Tscharnuter approach, although it is characterized by right-continuous sample paths (jump phenomena), predicts too high probabilities for high field values (the Holtsmark distribution has infinite variance); in other words, they seem to overestimate the effect of distant and close encounters, respectively. The unphysical features present in the two cases are intimately related and seem unavoidable; in fact, every finite-variance distribution for the interaction field gives rise to a normal distribution for the total increment of velocity (central limit theorem), which is at the basis of the Fokker-Planck approach for diffusion processes. Any attempt to regularize the Holtsmark distribution for avoiding high-field divergences falls therefore in this case. It is difficult to judge whether these are real unphysical features because, for instance, the notion of continuity (with probability one) of the sample paths is not intuitive at all (it implies the existence of a stochastic process with continuous sample paths equivalent to that physically observed). Apart from these difficulties, which however should be born in mind, the Fokker-Planck equation has widely been used in view of its higher level of predictability.

3.3 GENERAL STATISTICAL APPROACH: THE CONCEPT OF DYNAMICAL FRICTION REVISED

As mentioned in section 3.2, an unphysical feature inherent in the ergodic assumption, used to reduce the Boltzmann equation to a differential equation, lies in the fact that this approach does not take collisional collective effects into account, which cannot be disregarded a priori in systems whose dynamics is governed by long-range interactions. A different description which overcomes this difficulty and is not restricted by the assumption of spatial homogeneity, inherent in the two previous approaches, will now be discussed.

Statistical correlations among particles arise both from the initial probability distribution and the dynamics. It seems plausible that in most cases the disorganized motions of the particles will disrupt groups which were initially nearby, quickly erasing the original correlations; the resulting correlations will then be determined by the dynamics and the single-particle distribution function alone. In order to obtain a closed theory in which the one-particle distribution function is the only variable, Gilbert (1968) (see also Gilbert 1972) made the basic assumption that the probability distributions have evolved from initially uncorrelated states (although it is possible to imagine also quite different situations). The strategy he adopted in its theory on collisional collective processes in stellar systems consists in a decoupling of the BBGKY (Bogolioubov, Born, Green, Kirkwood, Ivon) hierarchy[6] accomplished by a perturbation series expansion in powers of $1/N$, the inverse of the total number of stars in the system, with the aid of certain combinations of the distribution functions called the correlation functions $g^{(s)}$ (they represent multi-particle correlations). A system of two coupled evolution equations of integro-differential type for $f^{(1)}$ and $g^{(2)}$ is thus obtained, which in principle may simultaneously be solved. The simplest situation occurs when the system is in equilibrium with respect to purely collective motions, and the only time-dependence is through the slow, secular effects of stellar encounters; in that case a kinetic equation for $f^{(1)}$ alone can formally be derived (the corresponding equation of plasma physics is the Balescu-Lenard equation).

The two coupled evolution equation for $f^{(1)}$ and $g^{(2)}$ derived for stellar systems are similar but not identical to the corresponding plasma equations. The differences come about because the latter are based upon a perturbation series expansion in powers of the inverse of the number of electrons contained in a Debye sphere (see, e.g., Rostoker and Rosenbluth 1960), which is independent of the total number of electrons in the system, usually taken as infinite. On the other hand, the role of the Debye screening length for a stellar system is played by the linear dimensions of the system itself; the number of stars in a Debye sphere is thus equal to the total number of stars, so that N has a dual meaning. Another point which should be stressed in this context is the fact that, while it is reasonable to assume spatial homogeneity in plasmas (not subject to strong external fields), the same is not true in self-gravitating systems, because the absence of screening effects makes them naturally inhomogeneous on large scales. The mathematical counterpart of this different physical feature is expressed by the fact that an explicit elimination of $g^{(2)}$ in terms of $f^{(1)}$ cannot be achieved in the gravitational case, but it is still possible to construct a formal solution (determining $g^{(2)}$ as a functional of $f^{(1)}$) and to interpret it in terms of the underlying physical processes.

The physical content of this formal solution can more easily be understood in terms of the auxiliary concept of *gravitational polarization* (for the plasmistic analogue of this effect see, e.g., Balescu 1960). It represents the response of the system to the gravitational field of a selected star moving in a specified orbit. In calculating this response one ignores collisional effects entirely and treats the field of a selected star as a small externally applied perturbation; the polarization is the change in the single-particle distribution function that this perturbation induces. The final result of this analysis is that collisional effects in stellar systems, i.e. dynamical effects of order 1/N (this ordering holds provided the system is in equilibrium with respect to purely collective motions), may be divided into two distinct phenomena:

- The gravitational force exerted on each star by the polarization (wake) it induces, which may be termed *polarization drag* (it represents a more precise formulation of the concept of *dynamical friction*[7], first introduced by Chandrasekhar 1943b). It is expected to retard the motion of a test star, the deceleration being directly related to its velocity; moreover, since the polarization induced by a given star is proportional to its mass, we expect heavy stars to be slowed more effectively than light stars. It may be worth observing also that, since the characteristic distance over which the gravitational polarization extends is of the same order as the linear dimensions of the stellar system, a given star cannot be thought of as being affected only by stars in its immediate neighbourhood.
- The effect upon each star of the random fluctuating field resulting from the superposition of the fields of the other stars, each modified by its own polarization; these stars are to be considered to move in unperturbed orbits and not to respond to the influence of the test star under consideration. It may be termed *statisti-*

cal acceleration. The statistical acceleration acting on a star increases on the average its energy and, because of the identity between the inertial and the gravitational mass, affects all stars in the same way.

A nice description of these two effects can be found in Hénon (1973) as well. A similar decomposition exists also for the Fokker-Planck collisional term calculated under the assumptions of spatial homogeneity and binary encounters; there, however, the polarization term is incompletely calculated and the statistical term consists of a superposition of bare (i.e., not modified by polarization effects) interparticle forces. To stress the contribution of these two effects to the collisional relaxation of stellar systems, the resulting kinetic equation for the evolution of the single-particle distribution function can be written in the following form:

$$\frac{\partial f}{\partial t} + [f, H] \equiv \frac{\partial f}{\partial t} + \underline{v} \cdot \nabla_{\underline{x}} f + \left(1 - \frac{1}{N}\right) \underline{A} \cdot \nabla_{\underline{v}} f = \left(\frac{\partial f}{\partial t}\right)_{Pol} + \left(\frac{\partial f}{\partial t}\right)_{Stat} , \qquad (3.6)$$

where the factor (1 - 1/N) reflects the fact that the test star feels the average gravitational acceleration due to the other N-1 stars, and not the total average gravitational acceleration. The competition between polarization and statistical effects is expected to lead to a relative concentration of heavier stars in the central regions of stellar systems and lighter stars in their outer parts, i.e. an approach to equipartition (see, e.g., Hénon 1973; Lynden-Bell 1973).

The theory sketched here takes thoroughly into account the effects of collective interactions and spatial inhomogeneity which are absent from more elementary treatments; as a consequence, no long-range divergence appears as it does in the Fokker-Planck approach (when the coefficients of diffusion and dynamical friction are evaluated according to the standard treatment; see, e.g., Braginskii 1965). It is interesting to note that although the equations of plasma physics and stellar dynamics are very similar, the mechanism for the elimination of this divergence is different in the two cases. In plasmas the repulsive interparticle force results in Debye shielding which cuts the force off, eliminating the divergence. In stellar systems the attractive interstellar force results in anti-shielding or amplification of the "bare" gravitational force of a star; this tends to make the divergence worse, and it is only the limited spatial extent of the system that finally removes it. There is, however, in the present theory a divergence at small distances, arising because the perturbation series expansion in powers of $1/N$ is non-uniformly convergent. The physics behind this is quite simple, since this divergence is precisely equivalent to that occurring in elementary Fokker-Planck treatments, where the impulse approximation (straight-line orbits) fails at small impact parameters (see, e.g., Braginskii 1965); the corresponding failure in the Gilbert approach can easily be expressed in terms of $f^{(1)}$ and $g^{(2)}$. The suppression of this unphysical divergence can be achieved by retaining all terms in $g^{(2)}$ appearing in the two original exact coupled equations; simpler approaches, in which the true inverse-square force is replaced by an effective fictitious force, can also be used.

FOOT-NOTES

1 The *Langevin equation* represents an attempt to rewrite the equations of motion in many-particle systems in such a way as to split the contribution of the smoothed out distribution of matter from the effect of the fluctuating part of the interaction field (of the perturbers):

$$\underline{\dot{v}} = \underline{a} + \underline{R} - \eta \underline{v} \ , \qquad (3.7)$$

where $\underline{a}$ is the systematic acceleration produced by the former, $\underline{R}$ and $-\eta\underline{v}$ the *stochastic acceleration* and the *dynamical friction* due to the action of the latter, respectively.

2 This definition is *different* from that given by Chandrasekhar, apart from the obvious generalizations inherent in this last equation.

3 The *ergodic property* characterizing random walks "no memory of the initial state after a macroscopically small time intervall" or equivalently "stochastically independent events in disjoint time intervals" is stronger than the *Markovian property* "future development dependent only on the present state, and not on the past history of the process or on the manner in which the present state was reached".

4 Stable distributions play an important role in the theory of stochastic processes as a natural generalization of the normal distribution. The importance of the *normal distribution* N is largely due to the *central limit theorem*. Let $\underline{X}_1,..,\underline{X}_n$ be mutually independent variables with a common distribution F having zero expectation and unit variance, and define $\underline{S}_n \equiv \underline{X}_1 +..+ \underline{X}_n$; the central limit theorem

states that the distribution of $\underset{\sim}{S}_n n^{-1/2}$ tends asymptotically to N. For distribution without variance similar limit theorems (extended central limit theorems) can be formulated, but the norming constants must be chosen differently; the interesting point is that all stable distributions and no others occur as such limits.

5 Chandrasekhar's (1941) use of a gaussian distribution is justified by the fact that the modified Holtsmark distribution, that he derived for avoiding some unphysical divergences at small distances, has finite variance and hence falls into the cases considered by the central limit theorem; the corresponding evolution equation is thus of diffusion type.

6 The BBGKY hierarchy of equations is obtained by integrating the Liouville equation over the phase space of all but s particles ($1 \leq s \leq N-1$). It turns out that the evolution equation for $f^{(s)}$ involves $f^{(s+1)}$ as well, so that these N-1 equations are all coupled; the closure of the system is given by the Liouville equation itself, as it involves $f^{(N)}$ alone.

7 The important role played by dynamical friction especially in situations of astrophysical interest has stimulated many analytical works confined to nearly integrable systems (e.g., Tremaine 1981; Palmer and Papaloizou 1982, 1985; Palmer 1983; Tremaine and Weinberg 1984) in addition to those listed in section 2.2; they all stress the crucial contributions of the resonances where precisely in an integrable system slightly perturbed irregular orbits first appear. For a more exhaustive and detailed discussion reference is made to Manorama (1986).

FIGURE AND TABLE CAPTIONS

Fig. 3.1 Oversimplified hierarchical scheme for stochastic processes.

Table 3.1 Comparison between the three basic approaches discussed in this chapter.

Fig. 3.1

STOCHASTIC PROCESSES → MARKOV PROCESSES → RANDOM WALKS

MARKOV PROCESSES

- The future dependent only on the present and not on the past

RANDOM WALKS

- The future independent both of the present and the past

↑ **DIFFUSION PROCESS**

- Normal (gaussian) random walk →
- Continuous sample paths

↓ **STABLE PROCESS WITH CHARACTERISTIC EXPONENT 3/2**

- Random walk with a Holtsmark distribution →
- Right-continuous sample paths (jump phenomena)

Table 3.1

Approach (Type of evol. eq.)	FOKKER-PLANCK (Diff. eq.)	TSCHARNUTER (Diff. eq.)	GILBERT (Integro-diff. eq.)
Basic assumptions and (consequent assumptions)	- Ergodic assumption - (Binary encounters)	- Ergodic assumption - (Binary encounters)	- Initially uncorrelated states - Equilibrium with respect to purely collective motions
Type of stochastic process and (implicit assumptions)	- Diffusion process	- Stable Markov process with characteristic exponent 3/2 (Spatial homogeneity)	- Non-Markovian process of general type
Method	- Chapman-Kolmogorov equation + Langevin equation	- Semi-group theory + Langevin equation	- Decoupling of the BBGKY hierarchy
Assumptions for calculating explicitly the coefficients and (comments)	- Instantaneous distant binary encounters in the impulse approximation - Spatial homogeneity - (Starting diff. eq. $\rightarrow$ integro-diff. eq. $\rightarrow$ diff. eq. using $f \sim f_M$)	- (The relation between the two coefficients cannot directly be established; thus, no explicit calculation can be carried out)	- (The resulting kinetic equation can only formally, i.e. not explicitly, be derived)
Divergences inherent in the theory or (related to the explicit calculation of the coefficients)	- (Small distances) - (Large distances)	- Small distances	- Small distances

CHAPTER 4. COLLISIONLESS RELAXATION PROCESSES: THE ROLE OF COLLECTIVE EFFECTS IN ELECTROMAGNETIC AND GRAVITATIONAL PLASMAS●

4.1 INTRODUCTION

In the previous chapters we have often stessed the fact that collective effects always play a crucial role in systems whose dynamics is governed by long-range interactions, such as plasmas and stellar systems; large-scale organized motions and coherent processes such as self-sustained waves in isolated systems (e.g., differential rotation and spiral density waves in disk galaxies) are indeed expressions of the collective nature of such interactions. From a mathematical point of view, one of the main implications of this lies in the fact that a local analysis is often no more suitable for describing electromagnetic and gravitational plasmas, and a global analysis is thus required (i.e., boundary conditions are to be taken into account). In the gravitational case a further mathematical complication arises from the fact that stellar systems are naturally inhomogeneous, because the gravitational force is always attractive (there is, in fact, only one gravitational charge) and screening effects are thus absent; the presence of these large-scale inhomogeneities generally requires the use of certain asymptotic perturbation methods, whose validity depends upon the value of some local parameters characterizing the equilibrium state of the system.

In this chapter we shall restrict only to one particular role played by collective effects in electromagnetic and gravitational plasmas, namely the enhancement of relaxation processes by collective effects, a complete discussion of the general topic being extremely wide (see, e.g., Fridman and Polyachenko 1984) and not directly related to the argument of the thesis. To be more specific, we shall analyze collisionless relaxation processes alone (i.e., we shall adopt the Vlasov description),

●The term "gravitational plasmas" is often used to indicate stellar systems in virtue of the fact that they are in several respects dynamically similar to ordinary (electromagnetic) plasmas (see, e.g., Bertin 1980; Lin and Bertin 1981).

since the role of collective interactions on collisional relaxation processes has already been discussed in section 3.3; in the framework of the Gilbert (1968) approach, this corresponds to drop the assumption that the system is in equilibrium with respect to purely collective motions and to neglect particle correlations at all. Two limits of the weakly non-linear theory of plasma waves, which gives the correct framework for studying such collective relaxation processes, will finally be discussed. General reference is made to the review papers by Kulsrud (1972) and Sagdeev (1966), and to the books on plasma physics and stellar dynamics listed in section 3.1; other more specific references will be given later on.

4.2 ENHANCEMENT OF RELAXATION PROCESSES BY COLLECTIVE EFFECTS

It is a well-known fact that in a plasma the rate of relaxation towards the equilibrium state can be enhanced by collective effects; in virtue of the analogy between electromagnetic and gravitational plasmas, the question naturally arises whether a similar enhancement may be observed in stellar systems as well. From an observational point of view, the existence of rapid relaxation mechanisms can be inferred from the fact that galaxies seem well relaxed, as they exhibit well-developed velocity distributions, and from the consideration that in such systems ordinary two-body collision processes operate on timescales largely exceeding even the Hubble time (Zwicky's paradox). An important step to explain this phenomenon was made by Lynden-Bell (1967), who formulated the theory of collisionless violent relaxation (see also section 2.3). This theory, although it has a great heuristic advantage and has stimulated a lot of interest (e.g., Saslaw 1968, 1969, 1970; Goldstein, Cuperman and Lecar 1969; Cuperman, Goldstein and Lecar 1969; Shu 1978, 1987; Madsen 1987), nevertheless cannot avoid certain difficulties; one of these lies in the fact that it describe essentially the non-equilibrium phase of evolution (collapse) of stellar systems and not their quasi-equilibrium phase. Among many further attempts to understand the relaxation of collisionless stellar systems it is worth mentioning that of Severne and Luwel (1980), where the contribution of fluctuations of the self-consistent field to the relaxation process has been considered (see also section 4.4). Before considering the gravitational case in more detail, it is better to review such phenomena in plasmas, since they have long been studied just in this context. Because of the reasons mentioned at the biginning of this section, the extension of these results to stellar systems is not straightforward.

Examples of rapid relaxation in plasmas and the corresponding relaxation times are the following:

- The confinement of a plasma in a mirror machine: $\tau_{coll} \sim (n\lambda_D^3)^{-1}\tau_{bin} \sim (10^{-6} \div 10^{-4})$ τ_{bin}. A similar situation in which no exponential relaxation occurs is provided by the particles of the van Allen's belt trapped (mirror effect) in the earth dipole magnetic field.
- The two-stream instability: $\tau_{coll} \sim (n_{beam}\,\lambda_D^3)^{-1}\tau_{bin}$.

Note the different orders of magnitude involved in the two cases of binary and collective relaxation processes. The Debye shielding length λ_D is defined as

$$\lambda_D \equiv \left(\frac{KT}{4\pi m e^2}\right)^{1/2} = \left(\frac{v_{th}}{\omega_p}\right), \tag{4.1}$$

where n is the number density of the relevant particles contributing to the relaxation (ions in the first example, electrons in the second example), ω_p their plasma frequency and v_{th} their one-dimensional thermal velocity. These examples are only two of the possible cases in which enhancement of relaxation can occur in a plasma, but in some sense they are typical and the following comments about them are relevant:

- These instabilities are in some sense weak. Strong instabilities tend to destroy the equilibrium; these weak instabilities work instead on a smaller scale, leading to a rapid relaxation towards a situation in which the equilibrium is no more unstable.
- These instabilities can occur in a time-dependent situation, in which case they usually lead to a relaxation at a rate comparable to the growth rate of the instability, or in a steady state, in which case they lead to a marginally stable equilibrium with a relaxation rate in balance with whatever external forces tend to disturb it.
- The process of growth of the waves and relaxation towards the equilibrium are intimately related. They can generally be interpreted as a maser in which the unstable equilibrium corresponds to an overpopulation of the emitting states for the wave; the induced emission process is then the scattering process, as well as

the process which makes the wave grow.

- The wave scattering is a collective process in which the particles collide with bunches of particles.

It turns out that these phenomena can no longer be described in the framework of a linear theory of plasma waves. There are two limits in which the weakly non-linear theory is tractable. In the first case, where there are only a few waves of finite amplitude, it is possible to treat each wave individually; this is called the theory of weak coherent waves. The second case concerns the situation in which so many waves are present that a statistical approach can be employed to find those features of the time evolution of the plasma state which do not depend on the details of the initial phase of the waves (random-phase approximation; see, e.g., Pines and Schrieffer 1962); other approaches resembling the van der Pol method used in non-linear mechanics (see, e.g., Starzhinskii 1980) can also be employed. This is called the theory of weak turbulence, in which three basic interactions are taken into account:

- The wave-particle interaction (quasi-linear effect), studied in the framework of the quasi-linear theory, which is particularly strong near the resonance $\omega = \underline{k}\cdot\underline{v}$ when no external magnetic field is applied.[1]
- The non-linear wave-wave interaction (second-order effect), also known as the resonant-mode coupling, characterized by the resonance condition $\omega_1 + \omega_2 + \omega_3 = 0$, $\underline{k}_1 + \underline{k}_2 + \underline{k}_3 = 0$.[2]
- The wave-particle-wave interaction (third-order effect), also known as the non-linear wave-particle interaction, characterized by the resonance condition $\omega_1 \pm \pm \omega_2 = (\underline{k}_1 \pm \underline{k}_2)\cdot\underline{v}$.

Both theories proceed essentially out of an iteration of the Vlasov equation, and fail when the wave amplitudes become so large that either the perturbation theory fails to converge, or the particle orbits become so distorted by the wave fields that the equilibrium distribution function can no longer be used to calculate accurately the linear wave properties of the plasma; the most usual example of such a distortion of orbits occurs when the particles become trapped in the troughs of plasma waves.

4.3 THE QUASI-LINEAR THEORY OF PLASMA WAVES

The quantitative theory which describes rapid relaxation processes of the type dis cussed in section 4.2 is the quasi-linear theory, first proposed by Vedenov, Velikhov and Sagdeev (1961, 1962), Romanov and Filippov (1961), Drummond and Pines (1962). For general reference and different formulations see, in addition to the references cited in section 4.1, also Akhiezer et al. (1975), Biskamp (1973), Dewar (1970), Drummond and Pines (1964), Drummond (1965), Drummond and Ross (1973), Frieman, Bodner and Rutherford (1963), Frieman and Rutherford (1964), Galeev and Sagdeev (1979, 1983), Goldman (1984), Lifshitz and Pitaevskii (1981), Pines and Schrieffer (1962), Sagdeev and Galeev (1969), Sitenko (1982), Tsytovich (1970, 1972), Vedenov (1967), Whitham (1965), Yasseen and Vaclavik (1983). We shall now sketch out the basic ideas underlying this theory. When studying small (linear) oscillations in a plasma the distribution function is taken to be split into two terms: a non-oscillating part (the initial distribution function) and a small correction to it which oscillates with the frequency of the plasma waves; the non-oscillating part is then assumed not to be connected at all with the oscillations. Actually, however, either the damping or the growth of plasma waves affects the unperturbed distribution function, and this in turn generally alters the stability properties of the plasma; this effect increases with increasing amplitude of the oscillations. When the amplitude of the oscillations increases the basic property of linear oscillations, i.e. the independence of the propagation of oscillations with different wave-vectors and frequencies (superposition principle), tends also to be violated since processes involving the interactions between different waves begin to play an ever more important role. The simplest among the non-linear processes which cannot be treated without taking into account the effect of plasma oscillations on the non-oscillating part of the distribution function, while the violation of the superposition principle is still neglected, is indeed the quasi-linear relaxation. In this process only the distribution of the resonant particles, whose number is assumed to be much smaller than the total number of particles (i.e., only sharp wave packets in $\underline{k}$-space are considered), is affected in a non-negligible manner by these weakly non-linear waves (quasi-linear diffusion); such particles, in fact, are involved in strong interactions with the plasma oscillations, which lead to damping (Landau damping) or amplification (inverse Landau damping) phenomena depending on the monotonic properties of their velocity distribution. Non-resonant particles do not exchange energy with the waves on the average, so that their distribution is almost insensitive to the effect of the oscillations (adiabatic quasi-linear diffusion).

Having stressed the main ideas which are at the basis of the quasi-linear theory, we now turn to discuss their implications in some more detail also from a quantitative point of view. As mentioned at the biginning of this section, different (almost equivalent) formulations can be given; in what follows we shall try to extract the common essential features of these approaches, avoiding any particular reference to specific physical situations. Taking into account the fact that two considerably different timescales are involved, one governing the relaxation towards the equilibrium state and the other associated with the plasma oscillations, we separate the distribution function into a slowly varying part f_0 and a rapidly varying part f_1:

$$f(\underline{x},\underline{v};t) = f_0(\underline{x},\underline{v};t) + f_1(\underline{x},\underline{v};t) , \qquad \left|\frac{\partial \ln f_0}{\partial t}\right| << \left|\frac{\partial \ln f_1}{\partial t}\right| . \tag{4.2}$$

The distribution function f is then taken to satisfy the system of the (coupled) Vlasov and Poisson equations (self-consistent description). By singling out the two contributions and performing a Fourier expansion of the perturbations, it can be shown that under the basic assumptions discussed previously (quasi-linear approximation) the time evolution of f_0 is described by the quasi-linear diffusion equation

$$\frac{\partial f_0}{\partial t} = \frac{\partial}{\partial v_i}\left(D_{ij}\,\frac{\partial f_0}{\partial v_j}\right) , \tag{4.3}$$ [3]

where the diffusion tensor $D_{ij} = D_{ij}(\underline{v},t)$ is of second order in the perturbations, being a linear functional of the energy density of the waves in the turbulent plasma; it is the main task of the quasi-linear theory to express it in terms of the wave spectrum. This evolution equation has exactly the same form as the Fokker-Planck equation with vanishing coefficient of dynamical friction, which is a higher (than second) order effect; it should be noted, however, that the quadratic form $D_{ij}v_iv_j$ is not necessarily positive definite as in the case of collisional relaxation processes, and this stresses the fact that in collisionless relaxation processes stochastic deceleration mechanisms can take place. Processes in which particles are scattered by plasma waves (wave-particle interaction)[4] are thus formally similar, but not identical, to ordinary particle-particle scattering processes. From a quantum-mechanical point of view, the resonance condition for this interaction expresses the conservation of energy and momentum in the elementary process involving the emission or absorption of a plasmon with energy $\hbar\omega$ and momentum $\hbar\underline{k}$ by a particle moving with velocity $\underline{v}$; thus it is not surprising that the wave-particle interaction conserves the total energy and momentum of the waves and particles, rather than the energy and momentum of the waves alone. The number of plasmons tends to be conserved and satisfies a continuity equation with a source term in the $(\underline{x},\underline{k})$ phase space:

$$\frac{\partial N_{\underline{k}}}{\partial t} + \left[N_{\underline{k}}, \omega_{\underline{k}}\right] = 2\gamma_{\underline{k}} N_{\underline{k}} \tag{4.4}$$

valid for inhomogeneous equilibrium states, provided the wavelengths are sufficiently short and the frequencies sufficiently large; $N_{\underline{k}}$ is the plasmon density in phase space (wave-action density), defined as

$$N_{\underline{k}} \equiv \frac{\varepsilon_{\underline{k}}}{\omega_{\underline{k}}} , \tag{4.5}$$

$\varepsilon_{\underline{k}}$ being the energy density of the waves in phase space, $\omega_{\underline{k}}$ and $\gamma_{\underline{k}}$ their frequency and growth (or damping) rate respectively. These two quantities are related to the wave-vector $\underline{k}$ and to the quasi-equilibrium distribution function f_0 by the same dispersion relation holding in the linear regime:

$$D(\omega, \gamma; k; f_0) = 0 . \tag{4.6}$$

The quasi-linear diffusion equation for f_0, the continuity equation for the number of plasmons, and the dispersion relation for ω and γ represent the complete set of equations of the quasi-linear theory.

In the forthcoming discussion we shall assume that no external magnetic field is applied; in the more general case in which an external magnetic field is present the results presented below are still roughly valid, even though a more detailed description is required. Particle diffusion resulting from the scattering by plasma waves leads to the establishment of an asymptotic stationary state, which is characterized by a fixed distribution of resonant particles and some definite spectral level; more precisely, either the oscillations are damped or a plateau is formed on the distribution function (i.e., f_0 is constant in the resonant interval $\Delta(\omega_{\underline{k}}/k)$ along the direction of wave propagation. It may be worth noting that if the phase-space volume oc cupied by resonant particles is rather large, the formation of a plateau in that volume becomes impossible as it would require too much energy; in that case either the oscillations are damped, or the spectrum becomes one-dimensional while along the direction of wave propagation a plateau is formed on the distribution function. As a consequence of the quasi-linear relaxation process in which an initial overstability[5] is finally damped (i.e., $\gamma(t)<0$ for $t \rightarrow \infty$) all the particles of the system (especially the resonant particles) undergo an effective collisionless stochastic heating, which is called the turbulent heating; when an initial overstability is instead only saturated (i.e., $\gamma(t) \rightarrow 0$ for $t \rightarrow \infty$) the same phenomenon occurs, but it is restricted to the particles belonging to certain regions of velocity space, due to the presence of a plateau on the distribution function. In the case of transient waves (i.e., $\gamma(t)<0$) the corresponding heating process is effective as long as the initial amplitudes are properly large. Part of the ordered motion associated with the waves is thus converted into random motion of the particles; this is just an example of anomalous-transport phenomena occurring in a plasma which, as a consequence of an overstability, passes from a laminar to a turbulent state. Other weaker heating processes due to wave-particle interactions can be described in the framework of the more general weakly non-linear theory; such non-linear wave-particle interactions are responsible for the damping phenomena occurring in the waves which lead to these heating processes, and are thus referred to as the non-linear Landau damping.

We shall now inquire into the validity of the quasi-linear theory, by explaining in what physical situations it becomes inapplicable. To the order in the amplitude of the waves to which the quasi-linear theory is valid no interaction between the waves themselves occurs, but the second order effect of the interaction between waves and particles is included. As one considers the situation in which the amplitude is larger, it is expected that even this interaction is not well represented by the theory. Since the theory basically assumes the orbits of the particles to be modified by small amounts from their unperturbed orbits, it can be guessed that a limit on the theory will occur when the amplitude of the waves is large enough to trap the particles; therefore, if the bounce period of a particle trapped in a wave is shorter than the time this particle spends in the wave packet, it could be reasonably expected the theory to be inaccurate.

So far we have considered the situation in which so many waves of finite amplitude are present that a statistical approach can be employed to formulate a quasi-linear theory of plasma waves. However, in some important physical situations only a few waves are involved, so that a different treatment is required (theory of weak coherent waves). Using the Drummond and Pines (1962) approach, which is not based on statistical assumptions, it can be shown that the time evolution of the slowly varying distribution function f_0 is indeed described by a quasi-linear diffusion equation exactly of the same form as that derived in the case of many waves (to which this approach refers). In this context (wave-particle interaction) only the calculation of the diffusion tensor cannot be carried out along the same line (an integration over the wave-vector $\underline{k}$ is replaced by a finite sum over $\underline{k}_i$). Because of this fact, bearing in mind that we are interested in the wave-particle interaction alone, we shall extend the meaning of the term "quasi-linear theory" to include also the case in which a few waves are considered.

4.4 ATTEMPTS TO ACHIEVE A SATISFACTORY FORMULATION OF A QUASI-LINEAR THEORY IN THE GRAVITATIONAL CASE

This section is mostly devoted to explain the main difficulties which are to be tackled, from a general point of view, for extending the quasi-linear theory to stellar systems as well, and to mention the suggestions of various authors to achieve a satisfactory formulation of such a theory, which is not available yet. My own contribution and proposals in the particular framework of spiral structure theory will be stressed in the second part of this thesis, as they are intended to be a contribution to the understanding of the heating of galactic disks.

In plasma physics one usually deals with systems many orders of magnitude larger than the scale of the waves contributing to collective processes, namely the Debye shielding length; this makes the analysis comparatively simple, since the standard Fourier expansion technique can be used. Stated in another way, in plasmas the requirement of performing a global modal analysis can often be by-passed because the assumption of spatial homogeneity is reasonably satisfied in many cases of physical interest, so that a local analysis can suitably be used. Unfortunately this is not the case for self-gravitating systems, whose size is not so drastically different from the wavelength scale; this makes it necessary to treat the waves as eigenmodes of the system. There is, however, even in the case of stellar systems a cunning trick for eliminating, from a formal point of view, the "unpleasant" effect of large-scale inhomogeneities; it simply consists in using the action-angle variables $\{(J_i, w_i); i = 1,2,3\}$ as the proper canonical coordinates. In this representation, in fact, the equilibrium quantities of integrable systems turn out to depend only on J_i, which in this context (and also in a more general context; cf. the Hamilton equations) play thus the role of the velocity components v_i (see, e.g., Kalnajs 1971; Galgani 1985). Taking into account the physical meaning of these variables (the actions J_i correspond to adiabatic invariants) and the "strengthened" Jeans theorem, it can be shown that even in quasi-equilibrium situations a similar result holds apart from the fact that in this case an explicit dependence on time is allowed (see, e.g., Binney and Lacey 1987); in particular, this is true for the slowly varying part of the distribution function $f_0 = f_0(J_i,t)$ and for the self-consistent Hamiltonian $H_0 = H_0(J_i,t)$, so that $[f_0,H_0] = 0$. The most direct mathematical implication of the dependence of f_0 and H_0 on the actions alone consists in the possibility of adopting the standard Fourier representation[6] in the angle space for the perturbations. While on the one hand there is the advantage of using this local treatment avoiding the difficulties connected with the solution of a global-mode equation, on the other hand this canonical representation has the drawback of being not simply related to directly observable quantities as indeed $\underline{x}$ and $\underline{v}$. It is just for this reason that basically important theories of stellar systems, such as for instance the spiral structure theory, are expressed in terms of the usual canonical coordinates $(\underline{x},\underline{v})$; the formal elegance and the compactness deriving from the use of the action-angle variables are indeed sacrificed in favour of a higher level of predictability and a simpler interpretation in terms of observable phenomena.

Because of the above-mentioned difficulties, only local formulations of the quasi-linear theory for gravitational plasmas have been given so far, and mostly in the framework of spiral structure theory (Marochnik and Suchkov 1969; see also Marochnik 1970; Dekker 1975, who extended the work of Lynden-Bell and Kalnajs 1972; see also Contopoulos 1974). These works, however, neither have shed new light on the problem of spiral structure (apart from some local results which can easily be extrapolated from the plasmistic analogue of spiral waves, i.e. the Bernstein waves), nor have tried to incorporate the fundamental role played excitation mechanisms at the corotation resonance shown in global linear treatments (for a more exhaustive discussion see section 6.2a of Part II). A local formulation of considerably different type related to the theory of violent relaxation (Lynden-Bell 1967) has been proposed by Severne and Luwel (1980) along a line similar to that followed by Kadomtsev and Pogutse (1970) in the context of weak homogeneous plasma turbulence. Extremely interesting discussions on related subjects can be found in Saslaw (1985).

FOOT-NOTES

1 The wave-particle resonance condition in the absence of an external magnetic field is expressed by the requirement that the velocity component of the particle along the direction of propagation of the wave should be equal to its phase velocity (see the formula given in the text). In the more general case in which an external magnetic field is also present two resonance conditions are possible: the Cherenkov resonance condition $\omega - l\omega_c = k_{\|}v_{\|}$, and the cyclotron resonance condition $\omega - l\omega_c = k_{\|}v_{\perp}$, $(l \in \mathbb{Z})$ where reference is made to the direction of the magnetic field and ω_c is the corresponding cyclotron frequency.

2 The wave-wave resonance condition when more than three waves are involved in the scattering process (higher-order effects) is $\Sigma_i^{i^\circ}\omega_i = 0$, $\Sigma_i^{i^\circ}\underline{k}_i = 0$ $(i^\circ \geq 3)$.

3 Bear in mind that to zeroth order of expansion f_0 satisfies the stationary Vlasov equation $[f_0, H_0] = 0$.

4 Since this interaction involves resonant particles, it cannot be considered within the framework of an equivalent fluid theory.

5 Given a perturbation whose time-dependence is of the form $f_1 \sim e^{-i\omega t}$ with $\omega = \omega_0 + i\gamma$, the following terminology is used:

- Instability: $\gamma > 0$, $\omega_0 = 0$.
- Overstability: $\gamma > 0$, $\omega_0 \neq 0$.

6 Standard Fourier expansion means that the perturbation is of the form $f_1 = \hat{f}_1 e^{ikx}$. If the system is inhomogeneous along the x-direction, the previous expression is to be replaced by the more general dependence $f_1 = \hat{f}_1(x)\exp[i\int^x k(x')dx']$.

CHAPTER 5. HEATING MECHANISMS IN GALACTIC DISKS

5.1 INTRODUCTION

In section 1.2 we have seen that the components of the velocity dispersion of disk stars in a solar neighbourhood show a tendency to increase with increasing spectral type, and this has been interpreted in terms of a corresponding increase of the components of the stellar velocity dispersion with age. Because of the observational difficulties connected with the determination of the stellar velocity dispersion, we do not know yet for certain whether such a systematic behaviour is restricted only to a small solar neighbourhood or instead is a general feature of galactic disks.[1] We shall see, however, that from a theoretical point of view there is no problem to account for the same phenomenon on larger scales or even in other similar stellar systems, provided the sun is not thought of as belonging to a privileged region of the Galaxy. Two classes of explanations have been invoked; the most diffused and currently accepted of them is based on the existence of relaxation mechanisms leading to a secular heating of galactic disks, the other regards the observed velocity dispersions as native properties. In this chapter we shall review the relaxation mechanisms which are thought to contribute more effectively to the increase of the components of the stellar velocity dispersion with age; other less important relaxation mechanisms will be mentioned for the sake of completeness in the forthcoming discussion. We shall also describe in less detail the basic ideas underlying the second class of explanations.

The velocity dispersion of disk stars can be affected by the following mechanisms[2] (see Wielen and Fuchs 1985; see also Binney and Tremaine 1987; Mihalas and Binney 1981):

- The stochastic heating (random increase of the stellar velocity dispersion) caused by local irregularities in the galactic gravitational field due to the existence of massive perturbers, as giant molecular clouds (GMC's) and hypothetical massive halo objects, or by large-scale phenomena as transient spiral waves;[3] ordinary binary encounters between stars, in fact, are known to be completely inefficient (see section 2.2).

- The deflections (random changes in the direction of the stellar velocity) caused by the same phenomena responsible for the stochastic heating. Their overall importance lies in the fact that they can transfer energy (and energy changes) between the motions of a star perpendicular and parallel to the galactic plane, so that deflections may be of primary importance for the axial ratios of the velocity ellipsoid even though the heating effect of the same irregularities is nearly negligible. This fact occurs when the velocity dispersion of the massive perturbers is much smaller than that of the test stars; in this case, in fact, the relaxation time for deflections T_D turns out to be much smaller than the relaxation time for equipartition of energy T_E (see Chandrasekhar 1960).
- The adiabatic heating or cooling (adiabatic changes in the stellar velocity) produced by slow changes in the regular gravitational field of galactic disks. Its effect is probably stronger perpendicularly to the galactic plane, because the disk is nearly self-gravitating in this direction. Two typical examples of adiabatic changes in galactic disks are the adiabatic cooling due to stochastic heating and the adiabatic heating due to infall of gas from haloes.

While the adiabatic heating and cooling discussed above primarily affect the perpendicular motion of stars, deflections can partially transfer this energy change to their parallel motions. Only the first two relaxation mechanisms will be considered in this chapter, the third one acting on much longer timescales.

5.2 THE ROLE OF MASSIVE PERTURBERS

5.2a GIANT MOLECULAR CLOUDS AND COMPLEXES

The importance that massive perturbers might have in the dynamical evolution of disk stars has first been stressed by Spitzer and Schwarzschild (1951, 1953), who hypothesized the existence of low-velocity dispersion ($c_g \sim 10$ km sec^{-1}) massive ($M_g \sim \sim 10^5 \div 10^6$ $M_\odot$) gas clouds and complexes to account for the observed velocity dispersion of stars of different spectral class. In the first paper they performed a numerical integration of the Fokker-Planck equation, taking the small gas velocity dispersion into account but disregarding the effects of galactic rotation and the vertical motion of stars. They noted that the so-obtained age-dependence of the stellar velocity dispersion, $c(\hat{t}) \simeq c_g(1+\hat{t})^{1/5}$ with time expressed in dimensionless units, could more simply be derived assuming that the stellar distribution function remains Maxwellian at all times (a Maxwellian distribution function was taken as the initial condition). Taking this fact into account, in the second paper they performed analytical calculations of the kind used for studying the secular effects of binary encounters (Chandrasekhar 1960; see also Hénon 1973), considering the epicyclic motion of stars in the galactic plane but still disregarding their vertical motion and also the small turbulent velocities of the interstellar clouds. They obtained a different power-law for the evolution of the stellar velocity dispersion: $c(\hat{t}) \simeq c_0(1+\hat{t})^{1/3}$, where the same notations as before have been used.

The existence of giant molecular clouds and complexes has definitely been shown in the 1970's. Since then a lot of observational and theoretical works have been devoted to investigate their physical and kinematical properties, as well as to study their role in the context of galactic disk stability and evolution (for a more detailed discussion see Romeo 1985; Bertin and Romeo 1987; and references therein cited). It is quite surprising that the mass of GMC's has been estimated to be of the same order as that theoretically predicted by Spitzer and Schwarzschild (1951, 1953), even though their turbulent velocities are thought to be smaller ($\sim 4 \div 8$ km sec^{-1}).

As regards the problem of stochastic heating of galactic disks, in which we are mostly interested, the contribution of Lacey (1984a) (see also Lacey 1984b, 1985) deserves particular attention, and thus will be discussed in detail. It can be considered a generalization of Spitzer-Schwarzschild's (1953) work, since it relies on the same assumptions and employs similar methods but takes the vertical motion of stars into account. This extension is expected to be important both because of the intrinsic interest in predicting the evolution of the vertical velocity dispersion of disk stars, and because their vertical epicyclic oscillations can take stars out of the layer of perturbing GMC's so as to reduce the scattering rate. The assumptions made by Lacey (1984) are those commonly used to treat star-GMC encounters in such a degree of approximation as to make analytical calculations not particularly stiff, but at the same time to give a correct physical description retaining the essential features of the relaxation mechanism. They are the following:

- The orbits of disk stars in the background galactic potential (assumed to be axi symmetric and plane-symmetric) are described by the first-order epicyclic theory.

- GMC's are long-lived, much more massive than disk stars and move in circular orbits.
- GMC's are randomly distributed and act independently; thus the possibility of the GMC distribution being organized on a large scale (i.e., into spiral arms) is neglected.
- For a typical star-GMC encounter the effective interaction time is short compared to the epicyclic time, and the velocity difference between the Local Standard of Rest (LSR) at the stars and at the GMC positions is negligible with respect to the peculiar velocity of the star.
- The change of the stellar peculiar velocities is dominated by the effect of many distant weak encounters.

The first, the second and the fifth assumption are reasonably satisfied, whereas the third and the fourth assumption may be criticized, the former being the most drastic one; these working assumptions, however, allow us to make use of standard methods for deriving the diffusion and the dynamical-friction coefficients in binary-encounter processes (Chandrasekhar 1960; see also Hénon 1973).

The evolution of the stellar velocity dispersion can be divided into two phases:

- A transient relaxation in which the shape of the velocity ellipsoid relaxes to a final steady state with $c_r : c_\vartheta : c_z = 1 : 1/\beta : \alpha_s(\beta)$, where $\beta \equiv 2\Omega/\varkappa$ and $\alpha \equiv c_z/c_r$, $\Omega = \Omega(r)$ and $\varkappa = \varkappa(r)$ being the angular velocity and the epicyclic frequency respectively. The existence of this phase depends on the non-vanishing of the dynamical-friction coefficient.
- A steady heating (absent for a solid-body rotation curve) in which the velocity dispersion increases steadily on a longer timescale, while its components maintain constant ratios depending only on the value of the local parameter β: $c(\hat{t}) \simeq c_0(1+\hat{t})^{1/4}$. It is worth noting that Spitzer and Schwarzschild (1953) 1/3 -power-law is recovered in the unphysical limit in which the scale-height of disk stars is much smaller than the scale-height of GMC's.

The apparent discrepancy between these theoretical predictions and observational results (up to the year of publication of this paper) seemed to rule GMC's out of the role of most promising heating mechanism in galactic disks. It should be born in mind, however, as stressed in section 1.2 and less explicitly pointed out by Lacey (1984), that observations do not put yet any stringent constraint on the age-dependence of the components of the stellar velocity dispersion and on the shape of the velocity ellipsoid because strong selection and contamination effects, inherent in the choice of otherwise claimed to be reliable samples, tend to bias such samples in a not simply estimable manner; a proof of this lies in the fact that even some recent observational surveys are inconsistent with each other.

The contribution of GMC's to the stochastic heating of galactic disks has been investigated also by several other authors both analytically (e.g., Fujimoto 1980: 1/2; Kamahori and Fujimoto 1986a: 1/3 for vanishing dynamical friction, otherwise saturation; Semenzato 1987: 1/3) and numerically (e.g., Icke 1982: 1/3 $\div$ 1/2; Villumsen 1983, 1985a,b: 1/2: Kamahori and Fujimoto 1987: 1/3). Differences in the results can be ascribed to the different approaches and approximations employed in the various cases. However, they all predict observationally consistent power-laws for the age-dependence of the stellar velocity dispersion (as indicated at the side of each reference) except Kamahori and Fujimoto 1986a, where the observed saturation of its components is clearly due to a wrong treatment of dynamical friction in the framework of the Langevin approach.

5.2b HYPOTHETICAL MASSIVE HALO OBJECTS

Stimulated by the ever more growing interest in the problem of dark matter in the universe, a number of authors have recently speculated upon the existence of massive ($\sim 10^6$ $M_\odot$) halo objects, as massive black holes and dark clusters, as possible candidades for the heating mechanism invoked in galactic disks (e.g., Lacey 1984b; Lacey and Ostriker 1985; Ipser and Semenzato 1985; see also Ipser and Semenzato 1983; Kamahori and Fujimoto 1986b, 1987; Carr and Lacey 1987). Dark clusters produce similar heating effects as massive black holes, but have the advantage of circumventing some of the problems inherent in the black hole model; in particular, dynamical friction is prevented from building up too much mass at the galactic centre if the clusters are disrupted by mutual collisions or tidal effects before dynamical friction can become operative, and the problem that these halo objects may generate too much luminosity through accretion is avoided. The methods employed in this case are substantially the same as those used in the case of GMC's; the calculations are anyway more complicated because the velocity dispersion of these hypothetical halo objects cannot of course be neglected as well as their vertical distribution. For suitable values of some free parameters the results are in agreement with observations, but the large number of these free parameters makes indeed the theory not highly predictive.

5.3 THE ROLE OF TRANSIENT SPIRAL WAVES

A different point of view was introduced by Barbanis and Woltjer (1967), who stressed the importance that large-scale phenomena as spiral waves[4] might have in the secular evolution of the components of the stellar velocity dispersion parallel to the galactic disk; a heuristic argument was also presented to show that the same heating mechanism might account for the increase of the vertical component of the velocity dispersion with age as well.[5] Their analysis does not make reference to any specific formulation of spiral structure theory. They only investigated, in fact, the effect of an imposed spiral potential of particular form on the epicyclic motion of disk stars; in this sense the spiral waves they considered are not self-sustained. Their suggestion that recurrent transient spiral waves might naturally heat galactic disks, even though no explicit time-dependence was derived, lies at the basis of further analytical (e.g., Byl 1974; Carlberg 1984; Carlberg and Sellwood 1985) and numerical investigations (e.g., Carlberg and Sellwood 1983; Sellwood and Carlberg 1984; Carlberg and Freedman 1985). In this context it should be noted that the restriction to transient non-self-sustained spiral waves, while on the one hand it avoids the difficulties arising from the consideration of the Poisson equation and from all its physical implications (i.e., excitation mechanisms in self-sustained spiral waves, etc.), on the other hand at the same time lowers the level of predictability of the theory (which is, in fact, less constrained).

The theoretical framework of these recent investigations (except the confused attempt made by Byl 1974) is a simplified formulation of the quasi-linear theory in the action-angle canonical representation; the simplification corresponds indeed not to care about the self-consistency of the theory (hence, only the quasi-linear diffusion equation is considered), which instead is expected to have a crucial role (see section <u>6.2a</u> of Part <u>II</u>). Only spiral waves varying on a timescale comparable to the basic periods of oscillation in the disk are considered, because otherwise counterreacting relaxation mechanisms would take adiabatically the system back to its initial unperturbed state without any appreciable dynamical effect. If transient spiral waves of a particularly simple form recur at a constant rate in time, then the increase of the planar velocity dispersion with age for a coeval population of disk stars follows a 1/2 -power-law; however, as the velocity dispersion becomes as large as to make the size of the epicycles comparable to the wavelength of the spiral wave, the horizontal heating rate follows a 1/5 -power-law. On the other hand, the vertical heating associated with such transient spiral waves is completely inefficient, just because they propagate along the galactic plane. Carlberg (1984) suggested that a way of overcoming this difficulty in the context of a similar scenario is to invoke the existence of bending waves, whose observational counterparts are the well-known warps in spiral galaxies.

5.4 MORE GENERAL APPROACHES

Since the basic physical mechanism responsible for the heating of galactic disks is not well-known at present, a phenomenological description of the heating process by the theory of orbital diffusion seems to be rather adequate; this line was first pursued by Wielen (1977), Wielen and Fuchs (1983, 1985). More precisely, in this approach the heating of galactic disks is basically described by a diffusion process in velocity space, in which dynamical friction is not taken into account and the diffusion coefficient is empirically determined from the observed age-dependence of the components of the velocity dispersion of nearby stars. The advantage of such an empiric procedure lies in the fact that it avoids as far as possible uncertain assumptions on the basic physical source of the irregular part of the galactic gravitational field, apart from those inherent in the choice of a diffusion process among all the possible stochastic processes. Wielen (1977) showed that a constant (time and velocity independent) diffusion coefficient, despite its extremely simple form, can explain fairly well both the age-dependence of the components of the stellar velocity dispersion (a 1/2 -power-law is obtained) and the axial ratios of the velocity ellipsoid. It turns out also that for a constant diffusion coefficient the Fokker-Planck equation admits self-similar solutions of Schwarzschild type (i.e., gaussian distribu tion functions with an anisotropic time-dependent velocity dispersion), provided the radial gradient of the (axisymmetric) distribution function is neglected with respect to its vertical variation and the epicyclic approximation is used (Wielen and Fuchs 1983); this result is extremely important in view of the relevance that the Schwarzschild distribution function has on observational grounds. Although it is appealing due to its simplicity, a constant diffusion coefficient is not the only physically relevant one; other more physically meaningful choices of isotropic (in velocity space) time-dependent diffusion coefficients can satisfactorily mimic the observed behaviour of the components of the stellar velocity dispersion as well (Wielen 1977). Amore detailed analysis (Wielen and Fuchs 1985) suggests that the stochastic heating is the main relaxation process in galactic disks, while other processes as adiabatic cooling and infall of gas from haloes are only of secondary importance from a dynamical point of view.

A similar approach has recently been undertaken by Binney and Lacey (1987), who transposed Wielen's results in the action-angle canonical representation, more elegant from a formal point of view but also less predictive when it is applied to real physical situations (see section 4.4). As particular cases of heating mechanisms they considered the effects of GMC's and transient spiral waves, for which they derived the quasi-linear diffusion equation within the framework of the Fokker-Planck approach calculating the diffusion tensor by means of the Hamilton perturbation theory. They showed that both these heating mechanisms are inconsistent with the "observed" 1/2 -power-law for the age-dependence of the components of the stellar velocity dispersion (bear in mind, however, the low confidence level of such observations; see section 1.2 and cf. section 5.2a). This fact led them to the conclusion that other relaxation mechanisms, as the scattering of disk stars by massive halo objects, might play a major role in the stochastic heating of galactic disks.

5.5 ANOTHER CLASS OF EXPLANATIONS

As mentioned in the introductory section 5.1, there is another class of explanations which interpret the observed increase of the stellar velocity dispersion with age in terms of native properties of disk stars. Tinsley and Larson (1978) suggested that the kinematics of stars older than 10^9 yr can be explained by a gradual decay of turbulent motions, as is predicted by certain extremely slow collapse models, and showed that the correlation between velocity dispersion and metallicity predicted by such models is in agreement with observations. This effect cannot directly account for the rapid variation of the velocity dispersion with age observed even for stars younger than 10^9 yr, but they suggested that this could be explained if the velocity dispersion of the youngest stars reflects only the local turbulent motions in the gas, while the velocity dispersion of older stars reflects in addition larger-scale non-circular motions in the galactic gas layer. If interstellar medium possesses a hierarchy of motions whose velocity dispersion increases with the size of the region considered older stars, which have travelled farther since their formation, will experience gas motions over a larger space volume and thus will acquire larger velocity dispersions than younger stars.

This possibility was further investigated by Larson (1979). The relation between the gaseous velocity dispersion and the region size that is required if such interstellar motions are to explain the dependence of the stellar velocity dispersion c on age t can be estimated from the empiric relation $c \sim t^{1/2}$. If c is equal to the velocity dispersion of the gas in a region of size L in which the stars of age t have originated, then we obtain $c \sim L^{1/3}$ since $L \sim ct$. The agreement between this power-law and the <u>Kolmogorov spectrum</u> for incompressible turbulence is suggestive, if perhaps only accidental. The Kolmogorov law depends on the assumption that energy is successively transferred into motions on ever smaller scales until it is entirely dissipated by viscosity. In general this is not expected if the motions are supersonic, as in interstellar medium, since energy can then directly be dissipated on large scales by shock fronts; this leaves less energy for small-scale motions, and produces a steeper dependence of c on L. Data assembled by Larson (1979) from a variety of sources show indeed that the velocity dispersion of young stars and cold interstellar gas increases systematically with the size of the region considered over a wide range of scale-lengths, and this effect is sufficient to account for the observed age-dependence of the velocity dispersion of disk stars for ages up to about 10^9 yr. The observed dependence of the gas velocity dispersion on region size suggests the existence of a hierarchy of turbulent motions in which smaller-scale motions are produced by the turbulent decay of larger-scale motions.

FOOT-NOTES

1 Recall that the spheroidal component has a different cosmological origin than the disk component, so that the same age-velocity dispersion relation is not expected to hold.

2 Recall that the restriction to nearly integrable situations is assumed. For strong departures from the integrability condition other relaxation mechanisms become operative due to the effects of Lyapunov instability (see section 2.3).

3 Transient spiral waves are not self-sustained. In the second part of this thesis we shall propose the interaction of disk stars with self-sustained spiral waves as the dominant heating mechanism in galactic disks.

4 Small-scale irregular spiral features can be induced as a wake by massive perturbers in the galactic plane (Julian and Toomre 1966), and thus can be studied in that context by taking collective effects into account.

5 The suspicion that spiral-arm formation might be the dominant relaxation mechanism in galactic disks had already been expressed by Goldreich and Lynden-Bell (1965b).

PART II

MY OWN CONTRIBUTION: PROPOSALS AND PRELIMINARY RESULTS

CHAPTER 6. SELF-REGULATION MECHANISMS IN GALACTIC DISKS

6.1 INTRODUCTION

A careful inspection of the heating mechanisms of galactic disks described in chapter 5. shows that none of them takes collective effects properly into account; this is indeed a very severe restriction, because collective effects are known to play a crucial role in systems whose dynamics is governed by long-range interactions. It may be objected that the heating mechanisms by recurrent transient large-scale spiral waves proposed by Barbanis and Woltjer (1967) and further investigated by Carlberg and Sellwood (1985) represent an attempt to estimate the role that such effects may have in driving the secular evolution of galactic disks. This is in part true, because the large-scale spiral structure observed in disk galaxies is a visible manifestation of them. The problem, however, lies in the fact that these simplified models do not even retain the most essential physics of the phenomenon, which derives from the self-sustenance property of spiral waves. In fact, when a global linear analysis is performed taking the self-gravity of the perturbations into account, these modes are found to be maintained by internal excitation and feedback mechanisms, which a local treatment cannot predict; in this context the role played by the resonances turns out to be crucial.

While on the one hand stellar dynamics tends to give too much emphasis to binary relaxation processes, on the other hand the spiral structure theory suggests the existence of global self-regulation mechanisms, due to collective effects, which lead to an increase of the stellar velocity dispersion in such a way as to saturate otherwise exponentially growing overstabilities. In simpler local self-regulation processes, which do not take internal excitation mechanisms into account, the stellar velocity dispersion is expected to settle at a critical local value which ensures a situation of marginal stability of the system. These self-regulation mechanisms, together with the crucial role that the cold interstellar gas plays in them, will be discussed in more detail in this chapter.

6.2 PROPOSED GLOBAL COLLECTIVE HEATING MECHANISM

6.2a BASIC IDEAS FOR THE FORMULATION OF A GLOBAL QUASI-LINEAR THEORY OF SPIRAL STRUCTURE

As we have discussed in section 4.4, a number of attempts have been made to extend the quasi-linear theory of plasma waves to the gravitational case as well. The major difficulty which is to be tackled for achieving a satisfactory formulation of this theory consists in the necessity of using a global approach. The following discussion is just devoted to explain it in more detail.

Let us first examine the case of self-gravitating purely stellar disks in differential rotation $\Omega(r)$. As in the linear theory of spiral structure, perturbations of the form

$$f_1 = f_1(r,\theta,z;v_r,v_\theta,v_z;t) = \hat{f}_1(r,z;v_r,v_\theta,v_z)\exp\left[i(m\theta-\omega t)\right] \qquad (6.1)^1$$

are self-consistently imposed on an axisymmetric and plane-symmetric state described by a distribution function f_0, which in the quasi-linear treatment is allowed to vary slowly with time:

$$f_0 = f_0(r,z;v_r,v_\theta,v_z;t)\,, \qquad \left|\frac{\partial \ln f_0}{\partial t}\right| \ll \left|\frac{\partial \ln f_1}{\partial t}\right| . \qquad (6.2)$$

In writing these expressions a system of cylindrical coordinates has been used to take into account the approximate symmetry properties of galactic disks; furthermore, $k = k(r) \equiv -i(\partial \ln f_1/\partial r)$ is the complex radial wavenumber of the perturbation, whose radial dependence takes the inhomogeneity of the system into account, m is the number of spiral arms, $\Omega_p \equiv \mathrm{Re}(\omega)/m$ is the pattern frequency, and $\gamma \equiv \mathrm{Im}(\omega)$ is the growth (or damping) rate of the spiral wave. The standard linear theory relies on a number of assumptions which allow to make the system of coupled Vlasov and Poisson equations more tractable; the same assumptions are taken to hold in the quasi-linear theory as well:

- Only infinitesimally thin disk systems are considered; as regards the perturbations this assumption implies $|k\langle z\rangle| \ll 1$, where $\langle z\rangle$ is the thickness-scale of the system.

- Only tightly wound spiral waves are considered: $|k|r \gg m$; consistently with this assumption the radial gradient of f_0 is neglected with respect to that of f_1, and WKBJ asymptotic expansion techniques are employed.
- Only small departures from circular orbits are considered: $c_r/r\varkappa \ll 1$ (epicyclic approximation), where c_r is the radial velocity dispersion and $\varkappa$ the epicyclic frequency.
- The winding and the epicyclic parameters are formally taken to be of the same order: $m/|k|r \sim c_r/r\varkappa \ll 1$.

In a local approach, when finite-thickness corrections are taken into account, the form of the final dispersion relation remains the same provided the unperturbed surface density is multiplied by a suitable reduction factor; furthermore, another reduction factor lowers the response of high-velocity dispersion stars (see section 7.3a). From a formal point of view, a quasi-linear diffusion equation for the evolution of f_0 can be derived along a line similar to that followed in the case of plasma waves, and the corresponding diffusion tensor turns out to be of second order in the perturbations. The wave-particle resonances which play the dominant role are the following:

$$m\left[\Omega_p - \Omega(r_{co})\right] \equiv 0 \qquad \text{(Corotation resonance)}, \tag{6.3}$$

$$\frac{m\left[\Omega_p - \Omega(r_{ILR})\right]}{K(r_{ILR})} \equiv -1 \qquad \text{(Inner Lindblad resonance)}, \tag{6.4}$$

$$\frac{m\left[\Omega_p - \Omega(r_{OLR})\right]}{K(r_{OLR})} \equiv +1 \qquad \text{(Outer Lindblad resonance)}. \tag{6.5}$$

At this stage the global approach has been required just because the diffusion tensor is dominated by the effects of these resonances, which cannot properly be described in the framework of a local approach (breakdown of the concept of dispersion relation). But what makes the use of a global approach really essential is the fact that in a local treatment propagating spiral waves ($m[\Omega_p - \Omega] \neq 0$) turn out to be neutral ($\gamma = 0$), while the actual situation is not indeed so simple.

Unfortunately an exhaustive discussion cannot be given straightforwardly in this context, because the subject is extremely specific; we shall only try to express the basic ideas underlying the internal excitation and feedback mechanisms which make the maintenance of global spiral modes possible. For a more detailed description of these mechanisms in the framework of the linear theory reference is made to the original papers by Mark (1974a,b, 1976a,b,c, 1977) and to the review papers by Bertin (1980, 1987), Lin and Bertin (1985), Lin and Lau (1979). The starting point is the global-mode equation deduced by combining the linearized Vlasov and Poisson equations.[2] This equation, which is Schrödinger type (but not identical to the quantum-mechanical wave equation), exhibits two turning points: a first-order turning point, the bulge radius r_{ce}, and a second-order turning point, the corotation radius r_{co}. Far away from them and from the outer Lindblad resonance (m = 2 models do not generally exhibit the inner Lindblad resonance) the global-mode equation approximately reduces to a local dispersion relation. The solution of this wave equation can be found by methods similar to those employed in quantum mechanics when a WKBJ approach is used: the global solution is obtained by performing an asymptotic matching of the local solutions[3] at the turning points, and by imposing a radiation condition at infinity (which in our case is represented by the outer Lindblad resonance). In particular, a quantum condition for the wave-vector similar to the Bohr-Sommerfeld quantum condition is found. This analysis shows also that linear wave-wave interactions occur at the two turning points; these interactions are at the basis of the above-mentioned internal excitation and feedback processes for the maintenance of global spiral modes. The overstabilities, therefore, are not produced by non-monotonic features in the velocity distribution, as instead occurs in plasmas (in the cases generally considered by the quasi-linear theory). The simplest mechanism (only lowest-order terms in the WKBJ expansion are retained) involves only trailing spiral waves.[3] When a long trailing wave propagating away from the bulge enters the corotation region, it undergoes an over-reflection process (WASER) in which two short trailing waves are produced:

- The reflected wave is always amplified, because the energy flux associated with the wave changes sign when it passes through the corotation resonance. This wave propagates back towards the bulge, where it is turned into a long trailing wave by a feedback mechanism. During this cycle non-linear effects act in such a way as to damp the wave, whereas the amplification process occurs only inside the corotation region.

- The transmitted wave propagates out towards the outer Lindblad resonance, where it is absorbed due to Landau damping mechanisms.

Another possible over-reflection process (swing amplification), involving both trailing and leading open waves[3], is shown to occur when second-order terms in the WKBJ expansion are retained.

We expect the same kind of internal excitation and feedback mechanisms to occur also in the framework of the quasi-linear theory; the only complication which might invalidate the linear results at a quantitative level lies in the fact that in the linear approach a time-independent properly modified Schwarzschild distribution function has been assumed, while it is not known a priori whether the form of the diffusion tensor allows self-similar solutions of this type for the quasi-linear diffusion equation (cf. section 5.4). We recall that the use of local Schwarzschild distribution functions has mainly been invoked on observational grounds; the effect that a different choice of the stellar distribution function may have in the global stability properties of galactic disks is not known yet (see Lin and Bertin 1985; Romeo 1985).

Although this physical picture is already extremely complicated, it is not yet satisfactorily complete; the dual dynamical role that the cold interstellar gas has in the stability of galactic disks cannot in fact be disregarded (Romeo 1985, 1987; Bertin and Romeo 1987). On the one hand, the presence of such a cold component can significantly destabilize the sistem in the linear regime, and in some pathological situations might even excite more complicated wave channels and wave cycles. On the other hand, the cold interstellar gas can be shocked and thus contributes, together with non-linear effects, to saturate otherwise exponentially growing spiral overstabilities; in the context of these self-regulation mechanisms, it thus tends to inhibit an excessive heating of the stellar component which would be produced by the above-mentioned non-linear effects (see also section 6.3).

6.2b GLOBAL SELF-REGULATION MECHANISMS: ASTROPHYSICAL IMPLICATIONS AND OBSERVATIONAL EVIDENCES

We shall now discuss the expected astrophysical implications of the heating pro cess associated with such global self-regulation mechanisms. First of all it is apparent that the proposed global heating mechanism can be effective in the galactic plane alone, just because spiral waves propagate in it. The corresponding vertical heating is thus expected to be almost vanishing, and the consideration of finite-thickness effects cannot appreciably change the situation at all; other global or local relaxation mechanisms, such as those associated with bending wave-star interactions or GMC-star encounters respectively, are surely more effective. From an observational point of view, this fact implies that the age-dependence of the horizontal and vertical components of the stellar velocity dispersion is not constrained to obey the same power-law.

The proposed global collective heating mechanism is expected to be dominant, or at least competitive, with respect to the other local non-collective heating mechanisms so far invoked (see chapter 5.). There are indeed some observational evidences which seem to support and suggest this fact. Some normal spiral galaxies, whose most representative case is that of NGC 488, are characterized by high planar stellar velocity dispersions, whereas the vertical stellar velocity dispersions are expected to be very low, because the extremely tightly wound (nearly circular) spiral structure of these galaxies suggests that their disks should be very thin (see Romeo 1985, 1987 also for other implications in connection with their global stability properties). It seems reasonable to interpret this strong "temperature" anisotropy as produced by radically different heating mechanisms, and to conclude that at least in such disk galaxies the horizontal heating mechanism is much more effective than the vertical one; this is indeed in agreement with our theoretical predictions.

Finally, we want to mention the fact that this description has a defect which might make the comparison with observations not straightforward: as in the linear spiral structure theory, a single equivalent stellar component is taken to be representative of the whole active stellar disk (see section 7.1). The consideration of more stellar components would give rise to several complications due to their gravitational coupling via the Poisson equation, as required by the self-consistency condition. On the other hand, it should be noted that observations tend to overestimate the effect of high-velocity dispersion stars, while only low-velocity dispersion stars are dynamically relevant (see Lin and Bertin 1985; Romeo 1985).

6.3 LOCAL SELF-REGULATION MECHANISMS

While on the one hand the formulation of a global quasi-linear theory of spiral structure is required for making these theoretical predictions quantitative, on the other hand some physically relevant qualitative conclusions can be drawn even by making use of simple local models of self-regulation in which the effects of the cold interstellar gas are taken into account, as shown by Bertin and Romeo (1987). The two-component local self-regulation processes underlying these models are based on the following facts:

- In the disk the stellar component tends to heat up via gravitational instabilities and possibly because of interactions with GMC's; the former, being a collective heating mechanism, is very sensitive to the local level of stability. No obvious cooling mechanisms are at hand.
- The cold gaseous component tends to cool off on a short timescale due to turbulent dissipation (inelastic cloud-cloud collisions); this cooling mechanism is not sensitive to the local level of stability. It also suffers heating via the same gravitational instabilities, but at a faster rate because of the stronger reaction of the (thinner) gaseous component.
- Cooling is a source of dynamical instabilities, so that it generates heating, ensuring self-regulation.

Such self-regulation mechanisms produce a rapid increase of the planar stellar velocity dispersion up to a quasi-stationary (secularly increasing) critical value which ensures a situation of marginal stability of the system[4]. Note that if the cold interstellar gas is not taken into account the critical value of the planar stellar velocity dispersion remains fixed in time, so that no secular evolution occurs[5]; this fact shows how crucial is the role of the cold interstellar gas in the physical picture of spiral galaxies.

The rapid phase of these local self-regulation mechanisms may be expected to dominate the initial evolution of the planar stellar velocity dispersion up to its critical value for marginal stability; the subsequent secular evolution is likely to be governed both by local and global self-regulation mechanisms[6].

FOOT-NOTES

1 In the local approach the radial dependence of f_1 is partially explicited:

$$f_1 = f_1(r,\theta,z;v_r,v_\theta,v_z;t) = \tilde{f}_1(r,z;v_r,v_\theta,v_z)\exp\left[i\left(\int^r k(r')dr' + m\theta - \omega t\right)\right], \quad (6.1)'$$

where it is assumed that $|\partial \ln \tilde{f}_1/\partial r| \ll |k|$ and $|\partial k/\partial r| \ll k^2$, so that the final global-mode equation reduces to a local dispersion relation (in the case of infinitesimally thin disk systems).

2 The same global-mode equation holds in the quasi-linear theory as well, because only the evolution of the slowly varying part of the distribution function, f_0, is determined by second-order terms in the perturbations.

3 Three kinds of spiral waves have been studied so far: short waves, long waves and finally open waves, which have intermediate properties with respect to the others (they are more open than long waves, but they propagate similarly to short waves). In each case the trailing and leading configurations are possible; only trailing spiral waves can propagate between the corotation and the outer Lindblad resonance in such a way as to satisfy the radiation condition.

4 When such marginal stability condition is imposed throughout the disk and reasonable equilibrium models are considered, the profiles of the radial stellar velocity dispersion resemble those inferred from observations of external galaxies (see Romeo 1985; Bertin and Romeo 1987).

5 Recall that in local self-regulation processes internal excitation mechanisms contributing to the secular evolution of the stellar velocity dispersion are not taken into account.

6 Recall that the global modal analysis considers propagating spiral waves, and the propagation condition (see section 6.2a) is just the stability condition given in the local analysis. This explains why local and global self-regulation mechanisms are expected to operate simultaneously in the secular phase.

CHAPTER 7. FINITE-THICKNESS EFFECTS IN TWO-COMPONENT GALACTIC DISKS: PRELIMINARY RESULTS

7.1 INTRODUCTION

As we have mentioned in section 6.2a, the spiral structure theory relies on a number of work assumptions which allow to make the linearized system of coupled Vlasov and Poisson equations more tractable; however, in some situations of astrophysical interest the validity of such assumptions may be questioned. For instance, we know that real galactic disks have finite, although small, thickness and the possibility of regarding them as infinitesimally thin depends on the relevant wavelengths of the perturbations excited; in some cases, when the underlying spiral structure has a high winding degree, finite-thickness effects should be taken into account in the stability analysis. In view of the importance that such effects might have in the self-regulation mechanisms which are expected to operate in galactic disks and to be at the basis of their secular heating, we have tried to evaluate them. This can be done only after that their perpendicular structure at equilibrium has carefully been investigated.

In performing this analysis, we have made use of simplified models of galactic disks in which stars and the cold interstellar gas are treated as two different components. Although such models might be thought of as being inaccurate to describe actual galactic disks, which are known to consist of different populations of stars and gas components, they incorporate indeed the most essential features as regards their stability properties. In this context it should be noted that such a single stellar component is taken to be representative of the whole active stellar disk consisting of low-velocity dispersion stars (high-velocity dispersion stars do not participate appreciably in spiral structure), whereas the single gaseous component is taken to simulate H I regions of neutral atomic hydrogen and giant molecular clouds. Generally even more drastically simplified galactic models are used, in which the cold interstellar gas is not taken into account; in some situations of astrophysical interest this further simplification may not be justified, because the cold interstellar gas is expected to play an important or even crucial role in the stability of galactic disks due to its low turbulent velocities (Romeo 1985, 1987; Bertin and Romeo 1987).

Several attempts have already been made to estimate finite-thickness corrections to the local dispersion relation in one-component galactic disks (e.g., Toomre 1964, 1974; Goldreich and Lynden-Bell 1965a; Vandervoort 1970b; Genkin and Safronov 1975; see also Fridman and Polyachenko 1984 for a review). While it is generally agreed that the form of the dispersion relation remains the same provided the unperturbed surface density is multiplied by a suitable reduction factor, different estimates of such reduction factor have been given by the various authors. The most reliable and complete analysis is that of Vandervoort (1970b), which is local in the galactic plane and global perpendicularly to it; the reduction factor, found by solving an eigenvalue problem, has been shown to be very well approximated by the simple expression

$$T = \frac{1}{1 + |k\langle z\rangle|}\,, \quad k\langle z\rangle = O(1)\,, \tag{7.1}$$

where k is the local radial wavenumber of the perturbation and $\langle z\rangle$ is the thickness-scale of the galactic disk. This estimate should be compared to that naively obtained by Toomre (1964):

$$T = \frac{1 - \exp(-|k\langle z\rangle|)}{|k\langle z\rangle|}\,, \quad |k\langle z\rangle| \ll 1\,. \tag{7.2}$$

As regards more realistic models of galactic disks in which more (than one) components are present, no so rigorous partially global analysis has been performed (e.g., Shu 1968; Vandervoort 1970c; Nakamura 1978; Jog and Solomon 1984); among these attempts Shu's (1968) contribution is surely the most important one, while Vandervoort (1970c) refers only to a particular continuous model of stellar populations without performing a proper stability analysis. The aim of our calculations is just to extend the rigorous partially global analysis of Vandevoort (1970b) in such a way as to include the cold interstellar gas as well.

7.2 PERPENDICULAR STRUCTURE OF GALACTIC DISKS

The perpendicular structure of galactic disks has recently been investigated in detail by a number of authors, who made use of multi-component locally isothermal models (e.g., Bahcall 1984; Bahcall and Casertano 1984). These analyses take into account the fact that galactic disks are nearly self-gravitating perpendicularly to their symmetry plane, so that standard asymptotic expansion techniques can be employed. While in the case of one-component stellar disks this is all that is needed to make the problem analytically tractable (see Vandervoort 1967, 1970a for the most rigorous analysis in this context), when multi-component models are considered further assumptions are to be made to this end: generally the component with the largest scale-height is taken to have the largest mass density, so that a perturbative approach can be employed. In our two-component model this assumption is certainly satisfied, but a perturbative approach of this kind may not always be suitable because in the outermost regions of galactic disks the mass density of the cold interstellar gas becomes comparable to that of disk stars. For this reason we have tried to relax this assumption, and this has allowed us to perform a detailed analysis only at small and large distances from the galactic plane; anyway it is shown that these are the distances most relevant to the stability analysis (see section 7.3b), and also from an observational point of view. A number of asymptotic regimes of astrophysical interest have finally been investigated. Our analysis on the perpendicular structure of galactic disks should further be refined in view of its relevance to the stability analysis concerning finite-thickness effects.

7.2a ONE-COMPONENT CASE

We shall now briefly discuss the one-component case because the investigation of two-component galactic disks, although it is much more complicated, employs similar methods. The system, which is assumed to be in an axisymmetric and plane-symmetric equilibrium state and to be locally isothermal perpendicularly to the galactic plane, is described by the Poisson equation

$$\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{\partial^2 \Phi}{\partial z^2} = 4\pi G \rho \qquad (7.3)$$

supplemented by the "locally isothermal" condition

$$\rho(r,z) = \rho_o(r)\,\exp\left[-\frac{1}{c_z^2(r)}\left\{\Phi(r,z)-\Phi(r,0)\right\}\right]. \tag{7.4}$$

If $\langle r\rangle$ and $\langle z\rangle$ are respectively the characteristic radius and the thickness-scale (defined to be positive) of the system, then $(\partial^2\Phi/\partial r^2) + r^{-1}(\partial\Phi/\partial r) = O(\Phi/\langle r\rangle^2)$ and $(\partial^2\Phi/\partial z^2) = O(\Phi/\langle z\rangle^2)$. Therefore, taking into account that galactic disks are highly flattened, we can perform an asymptotic expansion in powers of the small parameter $\varepsilon \equiv \langle z\rangle/\langle r\rangle \ll 1$; we obtain the following hierarchy of equations derived from the Poisson equation (Vandervoort 1967, 1970a):

$$\frac{\partial^2\Phi^{(0)}}{\partial z^2} = 0\,, \tag{7.3)$^{(0)}$$$

$$\frac{\partial^2\Phi^{(1)}}{\partial z^2} = 4\pi G\rho^{(0)}\,, \tag{7.3)$^{(1)}$$$

$$\frac{\partial^2\Phi^{(n-2)}}{\partial r^2} + \frac{1}{r}\,\frac{\partial\Phi^{(n-2)}}{\partial r} + \frac{\partial^2\Phi^{(n)}}{\partial z^2} = 4\pi G\rho^{(n-1)} \qquad (n \geq 2)\,, \tag{7.3)$^{(n)}$$$

where in this notation $\Phi^{(n)}$, $\rho^{(n)} = O(\varepsilon^n)$. In the following we shall consider only lowest-order non-trivial contributions, which are represented by equation $(7.3)^{(1)}$. Combining this equation with the "locally isothermal" condition and suppressing the order-indices for simplicity of notations, we find in dimensionless form:

$$\frac{d^2\hat\Phi}{d\hat z^2} = 2\,e^{-\hat\Phi}\,, \tag{7.5}$$

where we have adopted the following scaling:

$$\hat z \equiv \frac{z}{\Delta}\,, \quad \Delta \equiv \left(\frac{c_z^2}{2\pi G\rho_o}\right)^{1/2}; \tag{7.6}$$

$$\hat{\Phi} \equiv \left(\frac{\Phi - \Phi_0}{c_z^2}\right) , \quad \Phi_0 \equiv \Phi(r,0) \cdot \tag{7.7}$$

This non-linear second-order differential equation can easily be integrated by quadratures (see, e.g., Bahcall 1984); imposing the boundary conditions

$$\hat{\Phi}(0) = 0 , \quad \left.\frac{d\hat{\Phi}}{d\hat{z}}\right|_{\hat{z}=0} = 0 , \tag{7.8}$$

we find:

$$\int_0^{\hat{\Phi}} \frac{d\hat{\Phi}}{(1 - e^{-\hat{\Phi}})^{1/2}} = 2|\hat{z}| , \tag{7.9}$$

whose solution is

$$\hat{\Phi}(\hat{z}) = \ln\left(\cosh^2 \hat{z}\right) \cdot \tag{7.10}$$

The corresponding volume density is

$$\rho(r,z) = \rho_0(r) \operatorname{sech}^2\left(\frac{z}{\Delta(r)}\right) \cdot \tag{7.11}$$

The following two asymptotic limits are of interest:

$$\rho(r,z) \underset{|z| \ll \Delta}{\simeq} \rho_0 \exp\left(-\frac{z^2}{z_G^2}\right) , \quad z_G(\text{Gaussian thickness-scale}) \equiv \Delta; \tag{7.12}$$

$$\rho(r,z) \underset{|z| \gg \Delta}{\simeq} 4\rho_0 \exp\left(-\frac{|z|}{z_E}\right) , \quad z_E(\text{Exponential thickness-scale}) \equiv \Delta/2. \tag{7.13}$$

Integrating the volume density given in (7.11) over z, we find the following expression for the surface density:

$$\sigma(r) = \rho_0(r)\left[2\Delta(r)\right] , \tag{7.14}$$

so that in this one-component case the expansion thickness scale is

$$\langle z \rangle = \Delta \ ; \tag{7.15}$$

expressing it in terms of the surface density, we find:

$$\Delta = \left(\frac{c_z^2}{\pi G \sigma} \right). \tag{7.6)'$$

7.2b TWO-COMPONENT CASE

While the investigation of the one-component case is more or less trivial at this order of approximation, the same is not true for two-component galactic disks: several complications arise, some of which were already hidden in the one-component case. The two components will be denoted by different labels, H(=Hot) and C(=Cold), in order to recall that they are characterized by different vertical velocity dispersions. Having in mind cases of astrophysical interest, we shall refer to them as the stars (H) and the cold interstellar gas (C); however, we could also consider the case where gas is absent but two stellar populations with different scale-heights can be identified. It is assumed that the system possesses the same symmetry properties as in the one-component case, and that each component is locally isothermal in the z-direction; moreover, the only interaction between the two components is taken to occur via the gravitational field (Poisson equation):

$$\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r} \frac{\partial \Phi}{\partial r} + \frac{\partial^2 \Phi}{\partial z^2} = 4 \pi G \left(\varrho_H + \varrho_C \right) , \tag{7.16}$$

$$\varrho_i (r, z) = \varrho_{oi} (r) \exp \left[- \frac{1}{c_{zi}^2 (r)} \left\{ \Phi (r, z) - \Phi (r, 0) \right\} \right] \quad (i = H, C) \ . \tag{7.17}$$

Proceeding along the same line as in the one-component case, we find that equation (7.5) is replaced by

(7.18)

where we have adopted the following scaling and parametrization:

$$\hat{r} \equiv \frac{r}{\Delta_H} \quad , \quad \Delta_i \equiv \left(\frac{c_{ri}^2}{2\pi G \rho_{oi}}\right)^{1/2} \quad (i = H, C) \quad ; \tag{7.19}$$

$$\hat{\Phi} \equiv \left(\frac{\Phi - \Phi_o}{c_{rH}^2}\right) \quad , \quad \Phi_o \equiv \Phi(r, 0) \quad ; \tag{7.20}$$

$$\beta_r \equiv \frac{c_{rC}^2}{c_{rH}^2} \quad , \tag{7.21}$$

$$\gamma \equiv \frac{\rho_{oC}}{\rho_{oH}} \quad . \tag{7.22}$$

This equation can formally be integrated by quadratures with the same boundary conditions as in the previous case:

$$\int_0^{\hat{\Phi}} \frac{d\hat{\Phi}}{\left\{(1 - e^{-\hat{\Phi}}) + \gamma\beta_r\left(1 - e^{-\beta_r^{-1}\hat{\Phi}}\right)\right\}^{1/2}} \equiv F[\hat{\Phi}] = 2|\hat{r}| \quad ; \tag{7.23}$$

however, in contrast to the one-component case, the left-hand side of this equation cannot explicitly be expressed in terms of elementary or special functions for arbitrary values of the local parameter β_z. In order to overcome this difficulty two strategies can be employed:

- One consists in investigating the asymptotic behaviour of $F[\hat{\Phi}]$ as $\hat{\Phi} \to 0$ and $\hat{\Phi} \to +\infty$ so as to estimate $\hat{\Phi}(\hat{z})$ at small and large $\hat{z}$ (non-perturbative method).
- The other consists in assuming $\gamma\beta_z \ll 1$ and in carrying out an asymptotic expansion in this small quantity (perturbative method).

Both methods will be considered and the corresponding results will be compared in such a way as to draw as much information as possible.

Let us first consider the non-perturbative approach. Expanding the integrand function in powers of $\hat{\Phi}$ as $\hat{\Phi} \to 0$ and retaining only first-order terms, we obtain:

$$F[\hat{\Phi}] \underset{\hat{\Phi}\to 0}{\simeq} \frac{2\,\hat{\Phi}^{1/2}}{(1+\gamma)^{1/2}} \quad\Longrightarrow\quad \hat{\Phi}(\hat{r}) \underset{|\hat{r}|\to 0}{\simeq} (1+\gamma)\,\hat{r}^{2} \;. \tag{7.24}$$

Studying the behaviour of the integrand function in the range $0\leq\hat{\Phi}<+\infty$, we realize that the following reasonable lower estimate can be given for $F[\hat{\Phi}]$ as $\hat{\Phi}\to+\infty$:

$$F[\hat{\Phi}] \underset{\hat{\Phi}\to +\infty}{\simeq} \frac{\hat{\Phi}}{(1+\gamma\beta_{r})^{1/2}} \quad\Longrightarrow\quad \hat{\Phi}(\hat{r}) \underset{|\hat{r}|\to +\infty}{\simeq} 2\,(1+\gamma\beta_{r})^{1/2}\,|\hat{r}| \;. \tag{7.25}$$

Later on we shall see that a more precise estimate is indeed required for the stability analysis; unfortunately this cannot easily be obtained. The correct lower and upper asymptotic scales are respectively:

$$r_0 = r_{GC} \;, \tag{7.26}$$

$$r_\infty = r_{EH}\, \max\left[1, \ln a^{-1}\right] \tag{7.27}$$

(see equations (7.28), (7.29), (7.35) below); these scales have not properly been estimated by Bahcall (1984). The volume densities of the two components have the following asymptotic behaviours:

$$\rho_i(r,z) \underset{|z|\ll r_0}{\simeq} \rho_{oi}\, \exp\left(-\frac{z^2}{z_{Gi}^2}\right) \;, \quad z_{Gi} \equiv \left(\frac{c_{zi}^2}{2\pi G\rho_o}\right)^{1/2} \quad (i=H,C) \;; \tag{7.28}$$

$$\rho_i(r,z) \underset{|z|\gg r_\infty}{\simeq} 4^{K_i}\rho_{oi}\, \exp\left(-\frac{|z|}{z_{Ei}}\right) \;, \quad z_{Ei} \equiv \left[\frac{c_{zi}^4}{8\pi G(\rho_{oH}c_{zH}^2+\rho_{oC}c_{zC}^2)}\right]^{1/2} (i=H,C) \,. \tag{7.29}$$

Observe that

$$\frac{z_{GC}}{z_{GH}} = \beta_{z}^{1/2} \;, \tag{7.30}$$

$$\frac{r_{EC}}{r_{EH}} = \beta_r \; . \tag{7.31}$$

The z-independent exponents K_i in equation (7.29) depend on the higher-order corrections which we have not been able to estimate properly in equation (7.25). Although the volume densities of the two components are not known in the whole range $0 \leq |z| < +\infty$ but only asymptotically, their surface densities can all the same be determined exactly without performing any integration. This is indeed a delicate point; some authors (Talbot and Arnett 1975), in fact, generalized superficially equation (7.14) found in the one-component case obtaining the wrong relation $\sigma_i = \varrho_{oi}\,(2z_{Gi})$. It is not the gaussian thickness-scales of the two components that determine their surface densities but their exponential thickness-scales, as can be deduced from the work of Bahcall and Casertano (1984) when equation (7.29) is taken into account; this fact is not apparent in the one-component case, because there the gaussian and the exponential thickness-scales are identical apart from numerical factors (see equations (7.12) and (7.13)). Therefore, the surface densities of the two components are:

$$\sigma_i(r) = \rho_{oi}(r)\left[4\,r_{Ei}(r)\right] \qquad (i = H, C) \; ; \tag{7.32}$$

this result is not intuitive at all. Expressing the gaussian and the exponential thickness-scales of the two components in terms of their surface densities, we find:

$$r_{Gi} = 2\,r_{Ei}\left[\frac{(\sigma / c_{ri}^2)}{\{(\sigma_H / c_{rH}^2) + (\sigma_C / c_{rC}^2)\}}\right]^{1/2} \qquad (i = H, C) \; , \tag{7.28)'$$

$$r_{Ei} = \left(\frac{c_{ri}^2}{2\pi G \sigma}\right) \qquad (i = H, C) \; ; \tag{7.29)'$$

in the one-component case $[\,....\,]^{1/2}$ in equation (7.28)' reduces to unity. The meaning of equations (7.28) and (7.29)' is clear: the total volume density in the plane determines the gaussian thickness-scales of the two components, while the total surface density determines their exponential thickness-scales. A global thickness-scale of the system, which can be identified with the expansion thickness-scale $\langle z \rangle$, may be defined as

$$\langle z \rangle \equiv \frac{\sigma}{2\rho_0} = \frac{z_{GH}\, z_{GC}}{(4 z_{EH}\, z_{EC})^{1/2}} \tag{7.33}$$

in analogy with the one-component case; observe that the following inequalities hold:

$$2 z_{EC} < z_{GC} < \langle z \rangle < z_{GH} < 2 z_{EH} \,. \tag{7.34}$$

Let us now introduce the dimensionless local parameter

$$\alpha \equiv \frac{\sigma_C}{\sigma_H} = \gamma \beta_z \,, \tag{7.35}$$

which in a flat galactic disk is more relevant than γ, and investigate some asymptotic regimes of astrophysical interest (see Table 7.1). The interpretation of these results, which are very sensitive to the ordering between α and β_z, is straightforward once equations (7.30), (7.31), (7.35) are taken into account. Substantial differences with respect to the one-component case (e.g., note the frequent presence of the geometric mean of the exponential thickness-scales) can be ascribed to the strong gravitational coupling of the two components.

Let us now consider the perturbative approach (Bahcall 1984). Performing an asymptotic expansion in powers of the small local parameter $\alpha \ll 1$ and retaining only first-order terms, we obtain:

$$\hat{\Phi} \simeq \hat{\Phi}_0 + \alpha \hat{\Phi}_1 \,, \quad \alpha \hat{\Phi}_1 << \hat{\Phi}_0 \qquad \text{(Self-consistency condition)}; \tag{7.36}$$

$$\hat{\Phi}_0(\hat{z}) = \ln\left(\cosh^2 \hat{z}\right) ; \tag{7.37}$$

$$\hat{\Phi}_1(\hat{z}) = I(\hat{z}, \beta_z) \tanh \hat{z} \,, \quad I(\hat{z}, \beta_z) \equiv \int_0^{\hat{z}} \frac{1 - \operatorname{sech}^{2\beta_z^{-1}} \hat{z}}{\tanh^2 \hat{z}} \, d\hat{z} \,. \tag{7.38}$$

$I(\hat{z},\beta_z)$ cannot explicitly be expressed in terms of elementary or special functions for arbitrary values of the local parameter β_z, so that only its asymptotic behaviour can be investigated analytically. We shall not pursue this line, because this has been done even without the restriction $\alpha \ll 1$ in the non-perturbative approach discussed previously; only a further, still not satisfactory (for the stability analysis), information on the z-independent exponents K_i in equation (7.29) can be drawn.[1] Rather, we shall consider the case in which first-order contributions can be neglected, so that only $\hat{\Phi}_0$ is relevant to the following analysis. The volume densities of the two components are:

$$\rho_H(r,z) \underset{\substack{\alpha\ll 1\\(\alpha\ll\beta_z)}}{\simeq} \rho_{0H}(r)\,\mathrm{sech}^2\left(\frac{z}{\Delta_H(r)}\right), \tag{7.39}$$

$$\rho_c(r,z) \underset{\substack{\alpha\ll 1\\(\alpha\ll\beta_z)}}{\simeq} \rho_{0c}(r)\,\mathrm{sech}^{2\beta_z^{-1}}\left(\frac{z}{\Delta_H(r)}\right). \tag{7.40}$$

The further restriction $\alpha \ll \beta_z$ has been derived by comparing the gaussian and the exponential thickness-scales of each component found in this lowest-order approximation to those found in the non-perturbative approach (see Table 7.1); it can be identified with the asymptotic self-consistency condition of this perturbative approach. For an interesting discussion on the observational implications of this analysis see Bahcall (1984), Bahcall and Casertano (1984).

From the results obtained in this section it follows that our non-perturbative approach is richer in information than the perturbative approach employed by Bahcall (1984) except in the asymptotic regime $\gamma \ll 1$ ($\alpha \ll \beta_z$). Nevertheless, this is not yet all that is required for performing a detailed stability analysis concerning finite-thickness effects; in particular, a more accurate estimate of the z-independent exponents K_i would be necessary.

7.3 STABILITY ANALYSIS: FINITE-THICKNESS CORRECTIONS TO THE LOCAL DISPERSION RELATION

The basis of our investigation on finite-thickness effects in two-component galac tic disks is the stability analysis performed by Vandervoort (1970b) in the case of one-component purely stellar disks. The method we have employed is in fact a trivial extension of that of Vandervoort (1970b) to the case in which two stellar populations are present (the consideration of a fluid component might give rise to some difficulties), but the resulting analysis is indeed much more complicated than in the one-component case. For this reason we shall briefly discuss the one-component case before.

7.3a ONE-COMPONENT CASE

The work assumptions made by Vandervoort (1970b) are the same as those used in the local spiral structure theory (see section 6.2a), except that concerning the infinitesimal thickness of the models. The method employed to solve the Vlasov equation when z-motions are taken into account is based on the existence of an adiabatic invariant J_z, whose approximate constancy characterizes the perpendicular motion of disk stars. This is a characteristic of highly flattened galactic disks, where the frequency of the oscillation in the z-direction is large compared to frequency of the epicyclic motion in the symmetry plane, which in turn is generally of the same order as the pattern frequency of spiral waves. To lowest order, assuming an unperturbed distribution function of Schwarzschild type, it is found that the perturbed volume density is

$$\begin{aligned} \rho_1 = & -\frac{\rho}{2\pi G\rho_0\Delta^2}\,\frac{1}{(2\pi c_r^2)^{1/2}}\int\left(\Phi_1-\langle\Phi_1\rangle\right)\exp\left(-\frac{v_r^2}{2c_r^2}\right)dv_r - \\ & -\frac{\rho|k|}{4\pi G\rho_0\Delta}\,\frac{1}{(2\pi c_r^2)^{1/2}}\int D\langle\Phi_1\rangle\exp\left(-\frac{v_r^2}{2c_r^2}\right)dv_r\,, \end{aligned} \tag{7.41}$$

where we have adopted the same notations as in sections 6.2a and 7.2a; $\langle\ldots\rangle$ denotes the average over the angles conjugated to the action J_z,

$$D\equiv\frac{4\pi G\rho_0\Delta|k|}{\kappa^2-(\omega-m\Omega)^2}\,F_\nu(x)=\frac{2\pi G\sigma|k|}{\kappa^2-(\omega-m\Omega)^2}\,F_\nu(x)\,, \tag{7.42}$$ [2]

$$F_\nu(x)\equiv\frac{1-\nu^2}{x}\left[1-\frac{\nu e^{-x}}{2\sin(\nu\pi)}\int_{-\pi}^{+\pi}e^{-x\cos p}\cos(\nu\pi)\,dp\right], \tag{7.43}$$

$$\nu \equiv \left(\frac{\omega - m\Omega}{\kappa}\right), \quad x \equiv \frac{c_r^2 k^2}{\kappa^2}, \tag{7.44}$$

where $F_\nu(x)$ is a reduction factor which lowers the response of high-radial velocity dispersion stars and ν is the dimensionless Doppler-shifted frequency of the spiral wave. In considering the Poisson equation, we shall assume that the following ordering holds:

$$\frac{\Delta}{\langle r \rangle} \ll 1, \quad |\langle k \rangle| \langle r \rangle \gg m, \quad \langle k \rangle \Delta = O(1), \tag{7.45}$$

where $\langle k \rangle$ is the characteristic radial wavenumber of the perturbation. Self-consistency requires:

$$\frac{\partial^2 \Phi_1}{\partial r^2} - k^2 \Phi_1 = \frac{2\rho}{\rho_0 \Delta^2}\left[\frac{1}{(2\pi c_r^2)^{1/2}} \int \langle \Phi_1 \rangle \exp\left(-\frac{v_r^2}{2c_r^2}\right) dv_r - \Phi_1\right] - \\ - \frac{\Lambda \rho}{\rho_0 \Delta^2} \frac{1}{(2\pi c_r^2)^{1/2}} \int \langle \Phi_1 \rangle \exp\left(-\frac{v_r^2}{2c_r^2}\right) dv_r \tag{7.46}$$

to the same order of approximation, where we have defined

$$\Lambda \equiv |k| \Delta D; \tag{7.47}$$

this wave equation is to be solved with the boundary conditions

$$\Phi_1(r) \xrightarrow[|r| \to +\infty]{} 0. \tag{7.48}$$

This complicated eigenvalue problem for Λ can be reduced to a simpler eigenvalue problem, by observing that the perturbed volume density

$$\rho_1^* = -\frac{\rho |k|}{4\pi G \rho_0 \Delta} D \Phi_1 \tag{7.49}$$

gives rise to the same surface density as ρ_1 to the order of approximation to which we are working. Thus a reasonable approximation to the eigenvalue problem for Λ can be obtained by replacing ϱ_1 with ϱ_1^*:

$$\frac{\partial^2 \Phi_1}{\partial z^2} - k^2 \Phi_1 = -\frac{\Lambda}{\Delta^2} \operatorname{sech}^2\left(\frac{z}{\Delta}\right)\Phi_1 \ , \quad \Phi_1(z) \xrightarrow[|z|\to+\infty]{} 0 \ , \tag{7.50}$$

which is a Schrödinger-type wave equation. Note, however, that this eigenvalue problem differs from the usual quantum-mechanical problem, for it consists in fixing the energy of the particle (in a bound state) and in seeking the depths of the potential well for which that energy is allowed. Nevertheless, the solution can be obtained along a line similar to the quantum-mechanical case (see, e.g., Landau and Lifshitz 1977); the corresponding quantum condition is

$$\Lambda_m = (m + |k|\Delta)(m + |k|\Delta + 1) \qquad (m \in \mathbb{N}) \ . \tag{7.51}$$ [3]

Note that only the lowest eigenvalue is physically relevant, because all the others do not vanish as $|k|\Delta \rightarrow 0$ in such a way as to recover the equation valid for infinitesimally thin disks (this is just equation (7.50) with the right-hand side replaced by the term $4\pi G\, \sigma_1\, \delta(z)$, where the self-consistency condition has not been imposed). Therefore, imposing the restricted quantum condition

$$\Lambda_0 = |k|\Delta\,(1 + |k|\Delta) \implies D_0 = 1 + |k|\Delta \ , \tag{7.52}$$

and taking into account equation (7.42) and foot-note [2], we find that finite-thickness corrections to the local dispersion relation correspond to a lowering in the response of disk stars, which can equivalently be thought of as a lowering in their equilibrium surface density by the reduction factor

$$T = \frac{1}{1 + |k|\Delta} = \frac{1}{1 + |k\langle z\rangle|} \ , \quad \sigma_{eff} = T\sigma \ . \tag{7.1}$$

Therefore, the effective surface density σ_{eff} and not σ is relevant to the local stability of one-component purely stellar disks. From the form of the reduction factor it is apparent that such a simple result does no longer hold when a global (in the galactic plane) analysis is performed.

7.3b TWO-COMPONENT CASE

When two stellar populations are considered instead of one, the method employed by Vandervoort (1970b) to solve the Vlasov equation when z-motions are taken into account applies separately to each component. Therefore, assuming that the ordering

$$\frac{\langle z \rangle}{\langle r \rangle} \ll 1 \,, \quad |\langle k \rangle| \langle r \rangle \gg m \,, \quad \langle k \rangle \langle z \rangle = O(1) \tag{7.53}$$

holds, we find that to lowest non-trivial order the wave equation is now written as

$$\frac{\partial^2 \Phi_1}{\partial z^2} - k^2 \Phi_1 = - \left[\frac{\Lambda_H}{(2 z_{EH})^2} \frac{\rho_H}{\rho_{oH}} + \frac{\Lambda_C}{(2 z_{EC})^2} \frac{\rho_C}{\rho_{oC}} \right] \Phi_1 \,, \quad \Phi_1(z) \xrightarrow[|z| \to +\infty]{} 0 \,, \tag{7.54}$$

where we have defined

$$\Lambda_i \equiv |k| (2 z_{Ei}) D_i \qquad (i = H, C) \,, \tag{7.55}$$

$$D_i \equiv \frac{4\pi G \rho_{oi} (2 z_{Ei}) |k|}{k^2 - (\omega - m\Omega)^2} F_\nu(x_i) = \frac{2\pi G \sigma_i |k|}{k^2 - (\omega - m\Omega)^2} F_\nu(x_i) \qquad (i = H, C) \,. \tag{7.56}$$ [4]

When one of the two components is fluid, such as in the case in which the cold interstellar gas is considered, the same wave equation with the corresponding gaseous reduction factor

$$F_\nu(x_c) \equiv \frac{1}{1 + \{x_c/(1-\nu^2)\}} \tag{7.57}$$

is not expected to hold, because for such a component the Vlasov equation is replaced by the standard hydrodynamical equations and the method developed by Vandervoort (1970b) cannot be applied. A more careful analysis is thus required in this case (finite-thickness effects in one-component fluid models have been estimated by Goldreich and Lynden-Bell 1965a).

Let us now make some general remarks on the wave equation obtained when two stellar components are present. We observe that, since the quantum condition is determined by imposing the boundary conditions $\Phi_1(z) \to 0$ as $|z| \to +\infty$, only the asymptotic behaviour of the solutions as $|z| \to +\infty$ is indeed required. For this reason we can approximate $\varrho_i(r,z)/\varrho_{oi}(r,z)$ in equation (7.54) with their asymptotic limits given in (7.29) (this explains why the z-independent exponents K_i should be estimated accurately):

$$\frac{\partial^2 \Phi_1}{\partial \pi^2} - k^2 \Phi_1 = -\left[\frac{\Lambda_H}{(2\pi_{EH})^2} 4^{K_H} \exp\left(-\frac{|\pi|}{\pi_{EH}}\right) + \frac{\Lambda_C}{(2\pi_{EC})^2} 4^{K_C} \exp\left(-\frac{|\pi|}{\pi_{EC}}\right)\right]\Phi_1 , \quad \Phi_1(\pi) \xrightarrow[|\pi|\to+\infty]{} 0 \tag{7.58}$$

Moreover, the quantum condition of this double-eigenvalue problem for Λ_H and Λ_C, which determines the correct local dispersion relation, is expected to be of the form

$$f(\Lambda_H, \Lambda_C; 2|k|\pi_{EH}, 2|k|\pi_{EC}; m) = 0 \tag{7.59}$$

apart from the dependence on the local parameters α and β_z contained in K_i. While it seems reasonable that even in this two-component case only the lowest eigenvalues (n==0) are physically relevant, it cannot be expected a priori that such relation could be reduced to the particular form

$$D_H T_H(2|k|\pi_{EH}, 2|k|\pi_{EC}) + D_C T_C(2|k|\pi_{EH}, 2|k|\pi_{EC}) = 1 \tag{7.60}$$

in analogy with the one-component case, where T_H and T_C are the reduction factors of the two components. If this were not the case, finite-thickness corrections to the local dispersion relation could not simply be expressed in terms of two reduction factors, one for each component; in other words, the local dispersion relation could not be reduced to the form obtained in the case of infinitesimally thin disks by a suitable scaling of the surface densities of the two components.

We shall now try to be more specific. In contrast to the one-component case which is exactly soluble, the wave equation (7.58) cannot be reduced to a Fuchsian differential equation or to other well-known classes of differential equations whose solutions are expressible in terms of elementary or special functions, for arbitrary values of the local parameter $\beta_z = z_{EC}/z_{EH}$ (see, e.g., Abramowitz and Stegun 1970; Bender and Orszag 1978; Erdelyi 1956, 1981; Gradshteyn and Ryzhik 1980; Morse and Feshbach 1953; Smirnov 1964). For this reason we have developed a perturbative method[5] similar but not identical to that employed in quantum mechanics to solve the Schrödin ger equation when the potential is of the form $U(z) = V(z) + W(z)$, $W(z)$ representing a small perturbation of $V(z)$ (see, e.g., Bender and Orszag 1978).[6] Our perturbative method is based on the following two facts:

- The contribution of the cold component becomes rapidly negligible with respect to the contribution of the hot component as $|z|\to+\infty$ (recall that only this asymptotic regime is relevant to the quantum condition):

$$\exp\left(-\frac{|\lambda|}{\lambda_{EC}}\right)\Big/\exp\left(-\frac{|\lambda|}{\lambda_{EH}}\right) = \exp\left[-\frac{(\lambda_{EH}-\lambda_{EC})}{\lambda_{EH}\lambda_{EC}}|\lambda|\right] \xrightarrow[|\lambda|\to+\infty]{} 0 \,. \tag{7.61}$$

- The case in which only the hot component is present is exactly soluble.

A drawback inherent in perturbative methods of this type is that they require the "potential" to be known in the whole interval $0\leq|z|<+\infty$; for this reason we are forced to consider the general form of the wave equation (7.54), which for simplifying the notations we rewrite in the form

$$y''(\lambda) + \left[\Lambda_H V_H(\lambda) + \Lambda_C V_C(\lambda) - k^2\right] y(\lambda) = 0 \,,\quad y(\lambda) \xrightarrow[|\lambda|\to+\infty]{} 0 \,. \tag{7.54$'$}$$

Introducing the <u>local expansion parameter</u>

$$\varepsilon(\lambda) \equiv \frac{V_C(\lambda)}{V_H(\lambda)} \,, \tag{7.62}$$

we perform an asymptotic expansion of y, Λ_H and Λ_C:

$$y_\lambda = \sum_{n=0}^{\infty} y_n(\lambda) \,,\quad \Lambda_H = \sum_{n=0}^{\infty} \Lambda_{Hn} \,,\quad \Lambda_C = \sum_{n=0}^{\infty} \Lambda_{Cn} \,, \tag{7.63}$$

where in this notation $y_n, \Lambda_{Hn}, \Lambda_{Cn} = O(\varepsilon^n)$. To lowest (n=0) order we recover the limit of one-component systems:

$$y_0'' + \left[\Lambda_{H0} V_H - k^2\right] y_0 = 0 \,. \tag{7.54$'_0$}$$

To n($\geq$1)th order the wave equation becomes:

$$y_n'' + \left[\Lambda_{H0} V_H - k^2\right] y_n = - V_H \sum_{J=1}^{n} \Lambda_{HJ}\, y_{n-J} - V_C \sum_{J=1}^{n} \Lambda_{CJ-1}\, y_{n-J} \,. \tag{7.54$'_{n\geq1}$}$$

Writing y_n in the form

$$y_n \equiv y_0 F_n \tag*{(7.64)$_{n\geq 1}$}$$

and multiplying equation $(7.54)'_{n\geq 1}$ by y_0, we find:

$$\frac{d}{dz}\left(y_0^2 F_n'\right) = -y_0^2\left[V_H \sum_{J=1}^{n} \Lambda_{HJ} F_{n-J} + V_C \sum_{J=1}^{n} \Lambda_{CJ-1} F_{n-J}\right] , \tag*{(7.65)$_{n\geq 1}$}$$

where the lowest-order equation $(7.54)'_0$ has been taken into account. Integrating over z and taking the boundary conditions into account, we find:

$$\sum_{J=1}^{n} \Lambda_{HJ} \int y_0 y_{n-J} V_H \, dz + \sum_{J=1}^{n} \Lambda_{CJ-1} \int y_0 y_{n-J} V_C \, dz = 0 \; . \tag*{(7.66)$_{n\geq 1}$}$$

Let us now examine the zeroth and the first order of approximation to understand how this perturbative method works:

- n=0 →

$$\Lambda_H = \Lambda_{H0} = 2|k| z_{EH}\left(1 + 2|k| z_{EH}\right) , \tag*{(7.67)$_0$}$$

as derived in the one-component case (cf. equation (7.52)).

- n=1 →

$$\Lambda_{H1} \int y_0^2 V_H \, dz + \Lambda_{C0} \int y_0^2 V_C \, dz = 0 \; , \tag*{(7.66)$_1$}$$

$$\Lambda_H = \Lambda_{H0} + \Lambda_{H1} \; , \tag*{(7.68)$_1$}$$

$$\Lambda_C = \Lambda_{C0} \; . \tag*{(7.69)$_1$}$$

Combining these equations, we obtain:

$$\Lambda_{H0} = \Lambda_H + \frac{\int y_0^2 V_c \, d\eta}{\int y_0^2 V_H \, d\eta} \Lambda_c \,, \tag{7.67$_1$}$$

where Λ_{H0} is given in $(7.67)_0$.

In general the same procedure applies to nth order as well, where equation $(7.66)_{n\geq 1}$ is to be considered together with equations

$$\Lambda_H = \sum_{J=0}^{n} \Lambda_{HJ} \,, \tag{7.68$_{n\geq 1}$}$$

$$\Lambda_c = \sum_{J=0}^{n-1} \Lambda_{cJ} \,. \tag{7.69$_{n\geq 1}$}$$

We can eliminate Λ_{Hn} from equation $(7.66)_{n\geq 1}$, Λ_{Cn-1} from equation $(7.69)_{n\geq 1}$ and all the remaining Λ_{Hj}, Λ_{Cj-1} $(1 \leq j \leq n-1)$ from the equations obtained up to (n-1)th order; finally, equation $(7.68)_{n\geq 1}$ can be used to relate Λ_H and Λ_C to Λ_{Ho}, which is given in $(7.67)_0$, obtaining the restricted quantum condition

$$\Lambda_{H0} = f_{Hn} \Lambda_H + f_{cn} \Lambda_c \tag{7.67$_{n\geq 1}$}$$

where $f_{in} = f_{in}(2|k|z_{EH}, 2|k|z_{EC})$ $(i = H,C)$ are determined by the iterative method just discussed above.

We plan to inquire into the validity of this perturbative approach (convergence of the asymptotic series) soon; for the moment, however, it is interesting to note that, when such an asymptotic expansion is justified, finite-thickness corrections to the local dispersion relation reduce to a simple scaling of the surface densities of the two components by two corresponding reduction factors:

$$T_{in} = \frac{\eta_{Ei}}{\eta_{EH}} \frac{f_{in}}{1 + 2|k|\eta_{EH}} \,, \quad D_H T_{Hn} + D_c T_{cn} = 1 \,, \tag{7.60$_{n\geq 1}$}$$

where equations (7.55) and $(7.67)_0$ have been used to re-express equation $(7.67)_{n\geq 1}$. In order to make this perturbative method operative, we should try to give reasonable estimates of the integrals involved in this analysis, such as those contained in equa tion $(7.67)_1$. We expect the dominant contribution to be given by the gaussian asymptotic limit of the volume densities of the two components; if this is true, we can avoid the difficulties connected with the exponential asymptotic limit (proper evaluation of the z-independent exponents K_i), as well as those even more drastic deriving from our ignorance of the general z-behaviour of their volume densities (except in the asymptotic regime $\gamma \ll 1$).

FOOT-NOTES

1 The best estimate of the z-independent exponents K_i we have succeeded to give is

$$\left(\frac{1+\gamma\beta_n}{1+\gamma}\right)\log_4 e \lesssim K_H \lesssim 1-\frac{2\sqrt{2}}{3}\gamma(1-\beta_n)\log_4 e \;,\quad K_C \equiv \beta_n^{-1} K_H \;, \tag{7.70}$$

where the upper bound has been derived in the framework of the perturbative approach and thus holds provided 1.

2 D = 1 is just the local dispersion relation without finite-thickness corrections.

3 The quantum condition derived by Vandervoort (1970b) (which is just equation (7.51) with n replaced by 2n) is probably wrong.

4 $D_H + D_C = 1$ is just the local dispersion relation without finite-thickness corrections.

5 A WKBJ approach leading to a quantum condition of Bohr-Sommerfeld type cannot be employed here because it fails for low values of the quantum number n (recall that we are interested in n=0).

6 Differences between our perturbative method and that used in quantum mechanics arise because, in quantum-mechanical language, the energy of the particle is fixed and two eigenvalues are associated with a double-well potential.

FIGURE AND TABLE CAPTIONS

Table 7.1 Relation among the relevant thickness-scales in asymptotic regimes of astrophysical interest.

Table 7.1

ORDERING		RELATION AMONG THE RELEVANT THICKNESS-SCALES		
$\alpha << 1$, $\beta_z << 1$	$\alpha << \beta_z$	$z_{GH} \simeq 2z_{EH}$	$z_{GC} \simeq (4z_{EH}z_{EC})^{1/2}$	$< z > \simeq 2z_{EH}$
	$\alpha = 0(\beta_z)$	$z_{GH} = 0(2z_{EH})$	$z_{GC} = 0(4z_{EH}z_{EC})^{1/2}$	$< z > = 0(2z_{EH})$
	$\alpha >> \beta_z$	$z_{GH} \simeq (4z_{EH}z_{EC})^{1/2}\ \alpha^{-1/2}$	$z_{GC} \simeq 2z_{EC}\ \alpha^{-1/2}$	$< z > \simeq 2z_{EC}\ \alpha^{-1}$
$\alpha << 1$, $\beta_z = 0(1)$		$z_{GH} \simeq 2z_{EH}$	$z_{GC} \simeq (4z_{EH}z_{EC})^{1/2}$	$< z > \simeq 2z_{EH}$
$\alpha = 0(1)$, $\beta_z << 1$		$z_{GH} = 0(4z_{EH}z_{EC})^{1/2}$	$z_{GC} = 0(2z_{EC})$	$< z > = 0(2z_{EC})$
$\alpha = 0(1)$, $\beta_z = 0(1)$		$z_{GH} = 0(2z_{EH}) = 0(2z_{EC})$	$z_{GC} = 0(2z_{EH}) = 0(z_{EC})$	$< z > = 0(2z_{EH}) = 0(2z_{EC})$

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