Tensor Products of Conformal Models

Thesis submitted for the degree of

"Magister Philosophiæ"

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October 1989

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Acknowledgements

I would like to thank Prof. G.Sotkov for his advice, supervision and, most of all, friendship.

I am highly greatfull also to Dr. C.Crnkovic for our scientific colaboration and for the enormous technical help, without which this thesis couldnt be realized.

A thank is due also to Dr. R.Paunov for the colaboration and friendship.

My collegues D.Tommasini and E.Nardi deserve particular thanks for introducing me in the complicated area of the computer employnment.

Introduction

The extensive study of 2D conformal QFT in the last years was motivated mainly by their close connection with (super) string theories and 2D statistical mechanics. In the former case the conformal symmetry automaticaly arises after the gauge fixing as a symmetry on the world-sheet and and this requires that every vacuum solution of the string theory is nessesserily conformal QFT. The latter observation is based on the suggestion of Polyakov [1] that the fluctuations of the order parameter fields at the second order phase transition possesses conformal as well as scale invariance. Therefore the problem of classification of all types of universal critical behaviour can be formulated as the problem of finding the conformaly invariant solutions of the QFT.

The two space-time dimensions are distinguished between others by the fact that the conformal group (= group of analytic transformations of the complex plane) is infinite dimensional. The corresponding infinite dimensional Lie algebra is generated by the zz- ($\bar{z}\bar{z}$-) component of the stress-energy tensor $T(z)$ ($\bar{T}(\bar{z})$) and is equivalent to the well known Virasoro algebra [2]

$$[L_n, L_m] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}$$

Here L_n are the coefficients in the Laurent expansion of $T(z)$ and c is the Virasoro central charge. Consequently all the fields present in the theory can be classified completely in terms of the representations of this algebra.

The simplest fields,transforming homogeneously under the action of the generators,are called "primary" ones [3]. They give rise to a representation characterized by the value of the conformal dimension Δ of the primary field (highest weight). Such a representation is not in general irreducible - at certain level it appears a field which is again primary. Setting the corresponding Hilbert space states ("null vectors") to zero assures the irredusibility of the representation and imposes nontrivial conditions on the fields of the theory. Together with the conformal Ward identities this leads to differential equations satisfied by their correlation functions which permit to construct them and hence to solve the theory. A particular consequence of the existence of null vectors is that the operator algebra closes on finite number of fields only. These models are knoun under the name "minimal models" (m.m.) [3].

All interesting and tractable models of 2D conformal QFT correspond to certain rational values of the Virasoro central charge and the conformal dimensions. They are based on the presence of null vectors and contain a finite number of primary fields (rational conformal QFT). The first and simplest of them, described by Belavin,Polyakov and Zamolodchikov, are the minimal conformal models corresponding to the Virasoro central charge $c < 1$. Of most interest, actualy,is their infinite subfamily obeying the unitarity conditions - the "unitary discrete series" [4]

$$c = 1 - \frac{6}{(M+2)(M+3)}, \qquad M = 1, 2, \ldots .$$

However the m.m do not exhaust all the solutions of the conformal QFTs in 2D.One can consider the case when the theory (at the fixed point $\beta(g) = 0$) has higher than conformal infinite symmetries. The two simplest examples are the $N = 1$ superconformal symmetry [5] and the conformal current algebra [6]. By using these and other infinite symmetries one can construct new exact solutions of the conformal field theories and thereby to describe new fixed points.

The infinite dimensional algebra of superconformal symmetry - the Neveu-Schwartz-Ramond algebra - contains the Virasoro algebra as a subalgebra.All the fields in the theory can be classified according to its representations.The unitary minimal superconformal models [5] correspond to the value

$$c = \frac{3}{2} - \frac{12}{(M+2)(M+4)}, \qquad M = 1, 2, \ldots$$

of the central charge of the Virasoro algebra.

The Wess-Zumino-Witten model possesses at the fixed point symmetry with respect to $\hat{G} \times \hat{G}$ current algebra,i.e.,with respect to the direct product of "left" and "right" Kac-Moody algebras $\hat{G}$. The energy momentum tensor of this theory is expressible quadratically in terms of the currents by the Sugawara construction [7]. The corresponding values of the central charge are given (for the case of the semisimple group G) by

$$c = \frac{d_G\, k}{k + C_v}$$

where d_G is the dimension of the group G,C_v is the quadratic Casimir operator in the adjoint representation and k is the central charge of the Kac-Moody algebra,which takes (in unitary theory) integer values $k = 1, 2, \ldots$.

The superconformal and Kac-Moody symmetries are generated by local currents of spin $\frac{3}{2}$ and 1 respectively. One can consider also the general case of local currents of higher order spins(see [8]). Another series of models are generated by the nonlocal currents with fractional spins ("parafermions").Fields with such spins arise naturaly in Z_N symmetric statistical systems [9]. A series of exactly solvable unitary models of central charges

$$c = 2 - \frac{6}{N+2}, \qquad N = 2, 3, \ldots,$$

arising as representations of the algebra of the parafermionic currents, was proposed by Zamolodchikov and Fateev [10].

One can continue this list of conformal models adding the $N = 2$ superconformal m.m.'s [11], generalized parafermions , [12], the variuos W-algebras [13] etc. The question arises: how to classify all the possible 2D symmetries (and their m.m.'s)containing Virasoro algebra as a subalgebra? Despite recent progress in the classification of the CFT's two important problems remain open:

a)the explicite construction of the solutions of all of these models, i.e.,4-point functions,structure constants,the fusion algebras etc.

b)the problem of their irreducibility.In other words: can one find a minimal set of conformal models in terms of which one can construct and explicitely solve all the other "reducible" models?

One of the powerfull approaches to these problems is the so called GKO construction [14].

Let $\hat{G}$ be a Kac-Moody algebra and $\hat{H}$ its subalgebra, T_G and T_H being the generators of the Virasoro algebra (i.e., stress-energy tensors in the Sugawara construction). Then the difference $T = T_G - T_H$ commutes with all the generators of $\hat{H}$ and represents the Virasoro algebra with

$$c(G/H) = c(G,k) - c(G,k')$$

where k and k' are the central charges of the algebras $\hat{G}$ and $\hat{H}$ respectively. A particular,but very general case is presented by the symmetric coset

$$\frac{\hat{G}_k \times \hat{G}_l}{\hat{G}_{k+l}}$$

with G being some semisimple Lie algebra. The corresponding Virasoro central charge in the case $l = 1$ is given by

$$c = \frac{rk(2h+k+1)}{(h+k)(h+k+1)},$$

r is the rank and h - the dual Coxeter number of G.

There is a big evidence that in order to solve the models so constructed it is only nesessary to solve the ones with $l = 1$.One of the main goals of this thesis is to make this sugestion more clear.Consider in particular the construction with $G = SU(N)$ and

$$c = \frac{l(N^2-1)}{N+l}\left(1 - \frac{N(N+l)}{(N+k)(N+k+l)}\right).$$

In Chapters 1 and 2 below it is shown on the example of $SU(2)$ that all these models can be constructed as specific projected tensor products of the Virasoro m.m. (with $l = 1$).

All the presently known unitary minimal models can be realized by the GKO construction.Important examples which are not included in the above ones are the parafermionic theories $SU(2)/(U(1)$ and the $N = 2$ superconformal m.m.'s wich have a GKO realization $SU(2) \times U(1)/U(1)$ and a central charge

$$c = 3 - \frac{6}{N+2}, \qquad N = 1, 2, \ldots.$$

However,with this construction of the solutions,the question of the internal symmetries of the models usualy remains open.

Having solved the various m.m.'s on the sphere the natural question to address is: how one can define and solve these models on compact higher genus Riemann surfaces? Besides the pure statistical interpretation, [15],and the analytic-geometrical bootstrap programm of Friedan and Shenker, [16],there are at least two additional motivations to study the conformal models on compact Riemann surfaces. The first one is their role in the perturbative description of the heterotic string vacua [17]. Many results on the partition functions relevant to string theory were obtained [18]. These,however,deal essentialy with the free field theories, where the partition functions are expressible as a functional determinants of the ∂ operator. The second reason for this interest is to examine the specific features of the QFT on curved spaces with nontrivial topology and to understand the role of the moduli space [19], [20]. For the conformal QFT's of nongaussian type (e.g. the conformal m.m.'s),general results are known in the case of torus ($g = 1$) where the partition function can be constructed in terms of the characters of the irreducible representations of the conformal algebra (or higher symmetry algebra) [15]. In particular,the general classification of the modular invariant toroidal partition functions relevant to unitary m.m.'s was achieved in [21].

In the present thesis we describe the conformal m.m. on a restricted class of surfaces which can be represented as a double covering of the branched sphere. Such surfaces are known as hyperelliptic surfaces.The strategy we will use is to reduce the genus g problem to the corresponding $g = 0$ problem. The central idea of this method is that the topological properties of the Z_2-surface $X_g^{(2)}$ of genus g are simulated by the specific vertices $V(a_i)$ (called branching operators) placed at the points a_i of the branched sphere. The main advantage of this approach is that the calculation of the n-point correlation functions of the conformal fields on $X_g^{(2)}$ reduces to the problem of the construction of $n+2g+2$-point function on the sphere [22]. Hence ,the problem we address is the following: for each conformal m.m. defined on $X_g^{(2)}$ find the relevant conformal model (central charge,fields and branching operators) on the (branched) sphere.

A natural extension of this procedure to the case of superconformal (Chapter 4) and higher level $SU(2)$-coset models can also be considered. Following the methods of Chapter 1 we show in Chapter 4 below that the latter ones can be constructed and solved using only the information about the level 1 m.m. on $X_g^{(2)}$.In this way one can compute the partition functions of $V(L,M)$ models on the torus and genus two Riemann surfaces as 4- and 6-point functions of certain conformal fields from the Virasoro models on the branched sphere.

The construction of the exact conformal solution corresponding to a fixed point enables us to develope a perturbation theory and to examine the behaviour of the renormalization group in a neighborhood of this point.

The study of the relevant perturbations of 2D CFTs [3] is motivated by their role in the description of the off-critical behaviour of the corresponding 2D statistical models. Zamolodchikov has shown [23] that in 2D renormalizable field theory there exists a function $c(g)$ of coupling constants $g = (g^1, g^2, \ldots, g^n)$ which decreases monotonically under renormalization group (RG) transformations, and which is equal to the central charge of the corresponding CFT in the RG-fixed points. Furthermore, in the context of the Virasoro minimal models $V(M)$ with $M >> 1$, one can construct a field theory which, in a leading approximation in $1/M$, corresponds to the RG-flow connecting the fixed points $V(M)$ and $V(M-1)$ [24]. For the superVirasoro models $V(2,M)$ with $M >> 1$, a similar calculation [25,26] shows that the corresponding RG-flow is from $V(M)$ to $V(M-2)$.

Let us make a brief review of the Zamolodchikov's construction [24]. Starting from the action $\mathcal{H}(1,M)$ corresponding to the M-th model ($M >> 1$) in the $L = 1$ series, one perturbs it with $\mathcal{H}_{int}(1,M) = g \int \phi_{13}(1,M)(x) d^2x$. Here, $\phi_{13}(1,M)$ is the primary field in the M-th model with the conformal weight $\Delta_{13}(1,M) = 1 - \epsilon(1,M)$, $\epsilon(1,M) = 2/(M+1) \ll 1$. In the perturbation theory expansion for $g \sim \epsilon \ll 1$, one then obtains for the β-function

$$\beta(g) = \epsilon g - \frac{1}{2} C g^2 + \mathcal{O}(g^3),$$

where $C = C_{(13)(13)(13)}(1, M)$ is the corresponding structure constant. Using the results of [27] for C, it is easy to show that $\beta(g)$ has another (IR) fixed point g^* in the vicinity and that $c(g^*) = c(M) - 12 \int_0^{g^*} \beta(g) dg = c(M-1)$.The same procedure can be applyed also to the case of superVirasoro models.

Within the $SU(2)$ coset models $V(L, M)$, [14], the Virasoro and superVirasoro series are just the first two members ($L = 1$ and $L = 2$) of an infinite series $SU_M(2) \times SU_L(2)/SU_{M+L}(2)$. The natural question is: can one say something about the RG-flow for a general coset model?

In generalizing this construction to an arbitrary L, the main obstacle is the lack of detailed information, such as the structure constants, about arbitrary level models. To obtain this information and extend the calculation to a general level L, we shall use the representation of the higher level coset models in terms of the (projected) tensor products of the lower level models [28], described in Chapters 1 and 2. Let $V(L, M)$ be the M-th model in the discrete series at the L-th level.In Chapter 5 below we construct a perturbation of $V(L, M)$ which ,in a leading approximation in $1/M$ for $M >> 1$,gives the RG-flow

$$V(L, M) \longrightarrow V(L, M - L).$$

This thesis is organized as follows.In Chapter 1 we show that the higher level $SU(2)$- coset models can be represented by the projected tensor products of the Virasoro models. We prove the modular covariance of the prescription and construct explicitely the higher level primary fields. Chapter 2 is devoted to the construction of the monodromy - invariant correlation functions of these fields and calculate some of the structure constants. For level $L \geq 4$, the projected tensor products span a $\frac{1}{12}L(L-1)L-2)(L-3)$- dimensional space. In Chapter 3 we describe the Virasoro m.m.'s on the hyperelliptic surfaces. They are mapped into the specific models on the branched sphere. The latter are described by a generalized Coulomb gas representation. The results of this Chapter serve as a ingredient for the construction of the higher level $SU(2)$ coset models on the hyperelliptic surfaces, considered in Chapter 4. We discuss in more detailes the superconformal m.m.'s which are mapped into the m.m.'s of the $D_4^{p=2}$ parafermionic algebra on the sphere. In Chapter 5 is investigated the the behaviour of the RG in the vicinity of the fixed point of the latter models.We show that there exists a RG- flow from the M-th to the $(M-L)$-thy model in the L-th level $SU(2)$ coset series, for M large. Finaly, in the three Appendices, we give a detailed prove of some statements about the correlation functions and the structure constants used in Chapters 1, 2 and 5. We show also that the model $V(1) \times V(2,1)$ with $c = 6/5$ coincides with the second model of the W- algebra series of models.

Chapter 1

Fusions of conformal models

1.1 Introduction

After the initial period of proliferation of new minimal models and new symmetry algebras [3, 13, 29, 30, 31, 14, 32, 33], the study of the 2D conformal field theories (CFTs) has turned to the analysis of the relations between these models and search of the classification principles [34]. To make the meaning of "irreducibility" of the minimal models (and their symmetries) clearer, we start with the description of the "quark structure" of the $\widehat{SU}(2)$ coset family of minimal models $V(L,M) \sim \widehat{SU}(2)_L \times \widehat{SU}(2)_M / \widehat{SU}(2)_{L+M}$, $L, M = 1, 2, \ldots$ [14]. The main statement in the present Chapter is that *a general L-th family of minimal models $V(L,M)$ $L > 1$, $M = 1, 2, \ldots$, can be realized as a projected tensor product of consequent Virasoro minimal models $V(1,M) \equiv V(M)$.* As we shall show, all the data for a general $V(L,M)$ model: the primary fields, the conformal blocks and the 4-point functions, the structure constants, the fusion algebra, the characters *etc.* can be expressed explicitly in terms of the corresponding data from the Virasoro minimal models only. It is in this sense that all the minimal models $V(L,M)$ with $L > 1$ are *reducible*.

The simplest example for such reducible models are the $N=1$ superconformal models [5], *i.e.* $V(2,M)$. The initial observation is that:

$$
\begin{aligned}
\frac{\widehat{SU}(2)_1\times\widehat{SU}(2)_1}{\widehat{SU}(2)_2}\times\frac{\widehat{SU}(2)_2\times\widehat{SU}(2)_M}{\widehat{SU}(2)_{M+2}}&=\\
=\frac{\widehat{SU}(2)_1\times\widehat{SU}(2)_M}{\widehat{SU}(2)_{M+1}}&\times\frac{\widehat{SU}(2)_1\times\widehat{SU}(2)_{M+1}}{\widehat{SU}(2)_{M+2}}.
\end{aligned}
\tag{1.1}
$$

A consequence of (1.1) is an identity for the central charges:

$$
\begin{gathered}
c(1,1)+c(2,M)=c(1,M)+c(1,M+1),\\
c(1,M)=1-\frac{6}{(M+2)(M+3)},\qquad c(2,M)=\frac{3}{2}-\frac{12}{(M+2)(M+4)}.
\end{gathered}
\tag{1.2}
$$

The observations (1.1) and (1.2) motivate our basic statement:

$$
V(1)\otimes V(2,M)=\mathsf{P}(V(M)\otimes V(M+1)),\qquad M=1,2,\ldots,
\tag{1.3}
$$

where P is a certain projection whose rigorous definition in terms of characters is given in the next section. In terms of the primary fields, P projects from the space of all products of the fields* $\{\phi^M_{rq}\phi^{M+1}_{ps}\}=V(M)\otimes V(M+1)$ to the subspace

$$
\mathsf{P}(V(M)\otimes V(M+1))=\{\phi^M_{rp}\phi^{M+1}_{ps}\},\qquad p=1,\ldots,M+2,
$$

which is isomorphic to the representation space $V(1)\otimes V(2,M)$. This isomorphism is based on the following simple relations between the dimensions of the primary fields from two consequent Virasoro minimal models [3], $N=1$ superconformal minimal models [5] and Ising model $V(1)$:

$$
\begin{aligned}
\Delta_{rp}(1,M)+\Delta_{ps}(1,M+1)-\Delta^{NS}_{rs}(2,M)&=\frac{1}{2}\left(p-\frac{r+s}{2}\right)^2,\\
&\qquad r-s\in 2\mathsf{Z},\\
\Delta_{rp}(1,M)+\Delta_{ps}(1,M+1)-\Delta^{R}_{rs}(2,M)&=\frac{1}{2}\left(p-\frac{r+s}{2}\right)^2-\frac{1}{16},\\
&\qquad m-n\in 2\mathsf{Z}+1,
\end{aligned}
\tag{1.4}
$$

and therefore

$$
\begin{aligned}
N_{rs}&=\phi^M_{r,\frac{1}{2}(r+s)}\phi^{M+1}_{\frac{1}{2}(r+s),s},\\
\sigma R^i_{rs}&=\phi^M_{r,\frac{1}{2}(r+s\mp1)}\phi^{M+1}_{\frac{1}{2}(r+s\mp1),s},\quad i=1,2.
\end{aligned}
\tag{1.5}
$$

Here, σ is the Ising field with $\Delta=\frac{1}{16}$, N_{rs} and R^i_{rs} are the NS- and R-fields of

---

$\star$ Throughout the thesis both ϕ^M_{rs} and $\phi_{rs}(1,M)$, and Δ^M_{rs} and $\Delta_{rs}(1,M)$ are used to denote the Virasoro fields and their dimensions.

the $N = 1$ superconformal minimal model. The rest of the products $\phi^M_{rp}\phi^{M+1}_{ps}$ for $p \neq (r+s)/2$ or $(r+s\mp 1)/2$ correspond to the descendants of the primary fields (1.5).

A remarkable property of this construction is that $N = 1$ superconformal symmetry naturally arises as a symmetry of $\mathsf{P}(V(M)\otimes V(M+1))$, and the supercurrent is simply realized in terms of the Virasoro primary fields:

$$G = i\sqrt{\frac{1}{(M+2)(M+4)}}\left[M\phi^M_{12}\partial\phi^{M+1}_{21} - (M+6)(\partial\phi^M_{12})\phi^{M+1}_{21}\right]. \qquad (1.6)$$

Although ϕ^M_{12} and ϕ^{M+1}_{21} are not free fields and have complicated 4-point functions, $G(z)$ defined by (1.6) indeed has the well-known free 4-point function (see sec.1.3 and App. A for the proof) and reproduces the standard OPEs with the primary fields (1.5).

In fact the representation space $\mathsf{P}(V(M)\otimes V(M+1))$ is full of symmetries. The simplest one is generated by the free fermion

$$\psi = \phi^M_{12}\phi^{M+1}_{21}$$

and is related to the Ising model $V(1)$. The spin-2 field $T = \phi^M_{13}\phi^{M+1}_{31}$, the spin-3 field $W = \lambda\phi^M_{13}\partial\phi^{M+1}_{31} + \mu(\partial\phi^M_{13})\phi^{M+1}_{31}$ and the stress-energy tensors $T^{(M)}$ and $T^{(M+1)}$ realize a large symmetry algebra of three spin-2 currents and one spin-3 current. As is shown in App. C, in the particular case $M = 1$, *i.e.* $c(1) + c(2) = 6/5$, this algebra has as a subalgebra the well-known W_3 algebra [29].

The above construction has a straightforward generalization for general L:

$$V(1, L-1)\otimes V(L, M) = \mathsf{P}(V(M)\otimes V(L-1, M+1)). \qquad (1.7)$$

By iterating (1.7) we arrive at

$$\begin{aligned} &V(L, M)\otimes \mathsf{P}(V(1,1)\otimes V(1,2)\otimes\cdots\otimes V(1, L-1)) = \\ &\quad = \mathsf{P}(V(1, M)\otimes V(1, M+2)\otimes\cdots\otimes V(1, M+L-1)). \end{aligned} \qquad (1.8)$$

In words, (1.8) means that *any model $V(L, M)$, $L > 1$ can be constructed and explicitly solved in terms of the Virasoro models only.* Note that we have imposed the projection P on the LHS of (1.8) too. We will take (1.7) and (1.8) to mean that for any field from $V(L,M)$, one can find fields from $V(1,L-1)$, $V(M)$ and $V(L-1,M+1)$ such that the (projected) products of the fields that are identified like in (1.5) have the same correlation functions. Furthermore, where there is no projection P (like between $V(1,L-1)$ and $V(L,M)$), the monodromy-invariant 2D correlation function of the product of the fields factorizes into the product of the correlation functions. The bulk of the proof of (1.8) appears in secs.1.2 through 2.2. The crucial role will be played by the projection P, *i.e.* the restriction to the products of the type $\phi^{M}_{rp_1}\phi^{M+1}_{p_1p_2}\ldots\phi^{M+L-1}_{p_{L-1}s}$ only. In particular, in computing the 4-point functions only the products of conformal blocks corresponding to such products of fields are allowed. Still, in secs.1.4 and 2.1 we will show that this is enough to construct monodromy-invariant correlation functions. In this way, we will obtain the corresponding structure constants as the products of the structure constants of the Virasoro models.

One could wonder how general is this procedure of reducing and solving a general coset model in terms of the lowest level coset models only. Concerning the $\widehat{SU}(2)$ coset models, we present in chap. 2 one more example proving that $N=2$ superconformal minimal models (*i.e.* $\widehat{SU}(2)_M \times U(1)/U(1)$ coset models) can be solved in terms of the $V(2,M)$ and the parafermionic models [30] $V_{pf}(M)$:

$$\mathsf{P}(V_{pf}(M+2)\otimes V(2,M+2)) = V^{N=2}(M)\otimes V(1). \tag{1.9}$$

Our conjecture for the arbitrary (symmetric) coset series of models [35] $G(k,l) = G_k \times G_l/G_{k+l}$ (G_k denotes level k of the affine algebra $\hat{G}$) is that $G(k,l)$ *are reducible to the products of the first level models only.*

In this preliminary discussion we have succeeded in avoiding the question: *what is the origin of the reducibility of the $V(L,M)$, $L>1$ coset models?* A formal answer is that it follows from the obvious coset identities:

$$\begin{aligned}\frac{\widehat{SU}(2)_1\times\widehat{SU}(2)_{L-1}}{\widehat{SU}(2)_L}\times\frac{\widehat{SU}(2)_L\times\widehat{SU}(2)_M}{\widehat{SU}(2)_{M+L}} &= \\ =\frac{\widehat{SU}(2)_1\times\widehat{SU}(2)_M}{\widehat{SU}(2)_{M+1}}&\times\frac{\widehat{SU}(2)_{L-1}\times\widehat{SU}(2)_{M+1}}{\widehat{SU}(2)_{M+L}}.\end{aligned} \tag{1.10}$$

As we shall show less formally in the next section, the basic relations (1.7), (1.4) and (1.5) responsible for the reducibility follow from the branching rules for the

$\widehat{SU}(2)$-characters and specific associativity properties of the branching functions. Finally, one can think that the reducibility of $V(L, M)$ reflects the well-known fusing property of the associated lattice spin-$l/2$ models [39], since all of them can be realized as fusions of the basic (spin-1/2) 6-vertex model.

1.2 Modular properties of the projected tensor products

As will be shown in detail in sec. 1.4, the dimensions of the primary fields of the $V(L, M)$ agree with the dimensions of the specific products of the primary fields of the lower level models, and we would like to identify them with those products. In particular, the proposed identifications are: for $n - m \in L\mathsf{Z}$,

$$\phi_{mn}(L, M) = \phi_{mp}(1, M)\phi_{pn}(L-1, M+1), \quad p = \frac{n + (L-1)m}{L}, \tag{1.11}$$

and for $n - m \in L\mathsf{Z} \mp l$, $1 \le l \le L - 1$

$$\begin{aligned} \phi_{l,l+1}(1, L-1)\phi_{mn}(L, M) &= \phi_{mp}(1, M)\phi_{pn}(L-1, M+1), \\ p &= \frac{n + (L-1)m \mp (L-l)}{L}. \end{aligned} \tag{1.12}$$

We would like to prove now that the subspace

$$\mathsf{P}(V(M) \otimes V(L-1, M+1)) = \{\phi_{mp}(1, M)\phi_{pn}(L-1, M+1)\}$$

is isomorphic to the representation space $V(1, L-1) \otimes V(L, M)$ given by the LHS of (1.11) and (1.12). To prove this we have to answer the following two questions:

a) do the fields $\phi_{mn}(L, M)$ defined by (1.11) and (1.12) span a representation of the conformal chiral algebra specific to $V(L, M)$?

b) is the projection P as defined in (1.11) and (1.12) consistent with the modular invariance, *i.e.* is the subspace $\mathsf{P}(V(M) \otimes V(L-1, M+1))$ invariant w.r.t. the modular transformations?

We will start with the simplest case ($L = 2$) of the $N = 1$ superconformal minimal models . The proof of the above statement is based on the following simple relations between the characters $\hat{\chi}_{rs}(\tau, M)$, $\tilde{\chi}_{rs}(\tau, M)$, $\hat{\chi}^R_{rs}(\tau, M)$ of $V(2, M)$ and these of $V(1)$, $V(M)$ and $V(M + 1)$. For $r - s \in 2\mathbb{Z}$ we have

$$
\begin{aligned}
H_{rs}(\tau) &= \sum_{p=1}^{M+2} \chi_{rp}(\tau, M)\chi_{ps}(\tau, M+1) = (\chi_0 + \chi_{\frac{1}{2}})\hat{\chi}_{rs}(\tau, M), \\
\tilde{H}_{rs}(\tau) &= \sum_{p=1}^{M+2} (-1)^{p+(r+s)/2}\chi_{rp}(\tau, M)\chi_{ps}(\tau, M+1) = (\chi_0 - \chi_{\frac{1}{2}})\tilde{\hat{\chi}}_{rs}(\tau, M).
\end{aligned}
\tag{1.13}
$$

If $r - s \in 2\mathbb{Z} + 1$,

$$
H^R_{rs}(\tau) = \sum_{p=1}^{M+2} \chi_{rp}(\tau, M)\chi_{ps}(\tau, M+1) = 2\chi_{\frac{1}{16}}(\tau)\hat{\chi}^R_{rs}(\tau, M). \tag{1.14}
$$

One can derive (1.13) and (1.14) in three different ways. The first observation is that $H_{rs}(\tau)$ (defined as a sum of products of the Virasoro characters) under the modular transformations behaves as a product of Ising model and $N = 1$ minimal model characters and has the same behaviour for small q. The modular properties of both sides of (1.13) and (1.14) can be easily checked using the standard formulae [21]:

$$
\begin{aligned}
T: &\quad \chi_{rp}(\tau + 1, M) = e^{2\pi i(\Delta^M_{rp} - c(M)/24)}\chi_{rp}(\tau, M), \\
S: &\quad \chi_{rp}(-1/\tau, M) = \sum_{r',p'} S^{[M,1]}_{rp;r'p'}\chi_{r'p'}(\tau, M), \\
&\quad S^{[M,1]}_{rp;r'p'} = \sqrt{\frac{2}{(M+2)(M+3)}}(-1)^{(r+p)(r'+p')}\sin\frac{\pi r r'}{M+2}\sin\frac{\pi p p'}{M+3},
\end{aligned}
$$

and a few identities like

$$
\sum_{p=1}^{M+2} (-1)^{p(p'+p'')}\sin\frac{\pi p p'}{M+3}\sin\frac{\pi p p''}{M+3} = \frac{M+3}{2}(\delta_{p',p''} - \delta_{p',M+3-p''}), \quad M \in 2\mathbb{Z}.
$$

Another way to prove (1.13) and (1.14) is to use the explicit expression for $\chi_{rp}(\tau, M)$ [21]:

$$
\begin{aligned}
\chi_{rp}(\tau, M) &= q^{-1/24}\eta^{-1}(q)\sum_{n\in\mathbb{Z}}\left(q^{\alpha^{(M)}_{r,p}(n)} - q^{\alpha^{(M)}_{r,-p}(n)}\right), \\
\alpha^{(M)}_{r,p}(n) &= \frac{[2(M+2)(M+3)n + r(M+3) - p(M+2)]^2 - 1}{4(M+2)(M+3)}, \qquad q = e^{2\pi i\tau}.
\end{aligned}
$$

By resumming the LHS of (1.13) and (1.14) one obtains the product of Ising and

$N = 1$ characters.

The equations (1.13) and (1.14) are in fact simple consequences of the associativity of the branching rules for the $SU(2)_k$-characters $\chi_{k,l}(\tau, z)$ (k is the level and $l/2$ is the spin) [40]:

$$\chi_{1,\epsilon}(\tau, z)\chi_{M,p-1}(\tau, z) = \sum_{\substack{q=1 \\ q-p=\epsilon \bmod 2}}^{M+2} \chi_{pq}(\tau, M)\chi_{M+1,q-1}(\tau, z), \qquad \epsilon = 0, 1,$$
$$(\chi_{2,0}(\tau, z) + \chi_{2,2}(\tau, z))\chi_{M,p-1}(\tau, z) = \sum_{\substack{q=1 \\ q-p=0 \bmod 2}}^{M+3} \hat{\chi}_{pq}(\tau, M)\chi_{M+2,q-1}(\tau, z). \tag{1.15}$$

Applying (1.15) several times we get:

$$(\chi_{1,0}\chi_{1,0} + \chi_{1,1}\chi_{1,1})\chi_{M,p-1} = \sum_{\substack{r=1 \\ r-p=0 \bmod 2}}^{M+3} \chi_{M+2,r-1} \sum_{q=1}^{M+2} \chi_{pq}(\tau, M)\chi_{qr}(\tau, M+1)$$

and

$$\chi_{1,0}\chi_{1,0} + \chi_{1,1}\chi_{1,1} = (\chi_0 + \chi_{\frac{1}{2}})(\chi_{2,0} + \chi_{2,2}).$$

These identities, together with the second of equations (1.15), lead to the first of equations (1.13). The remaining two equations in (1.13) can be obtained by the modular transformations of the first one or directly by following the procedure described above:

$$2\chi_{1,1}\chi_{1,0}\chi_{M,p-1} = \sum_{\substack{r=1 \\ r-p=1 \bmod 2}}^{M+3} \chi_{M+2,r-1}(\tau, z) \sum_{q=1}^{M+2} \chi_{pq}(\tau, M)\chi_{qr}(\tau, M+1),$$
$$\chi_{1,0}(\tau, z)\chi_{1,1}(\tau, z) = \chi_{\frac{1}{16}}(\tau)\chi_{2,1}(\tau, z),$$
$$\chi_{2,1}(\tau, z)\chi_{M,p-1}(\tau, z) = \sum_{\substack{r-1 \\ r-p=1 \bmod 2}}^{M+3} \chi_{M+2,r-1}(\tau, z)\hat{\chi}_{pr}(\tau, M).$$

Consider an arbitrary L-th level coset model $V(L, M)$. Our main tool in the proof of (1.7), (1.11) and (1.12) are the branching properties of the $SU(2)$-

characters [40]:

$$\chi_{L,r}(\tau,z)\chi_{M,p-1}(\tau,z)=\sum_{\substack{q=1\\q-p=r\bmod 2}}^{M+L+1} b^{(r)}_{pq}(\tau|L,M)\chi_{M+L,q-1}(\tau,z).$$

The branching functions $b^{(r)}_{pq}(\tau|L,M)$ have the properties

$$\begin{aligned} b^{(r)}_{pq} &= b^{(L-r)}_{M+2-p,M+2+L-q},\\ b^{(r)}_{pq} &= 0 \quad \text{for} \quad p-q\in 2\mathbb{Z}+r+1, \end{aligned}$$

and the following modular transformations:

$$\begin{aligned} b^{(r)}_{pq}(-1/\tau) &= \sum_{p'=1}^{M+1}\sum_{q'=1}^{M+L+1}\sum_{r'=0}^{L} A^{(M+2)}_{pp'}A^{(M+L+2)}_{qq'}A^{(L+2)}_{r+1,r'+1}b^{(r')}_{p'q'}(\tau),\\ A^{(M+2)}_{pp'} &= \sqrt{\frac{2}{M+2}}\sin\frac{\pi pp'}{M+2},\\ b^{(r)}_{pq}(\tau+1) &= e^{2\pi i(\Delta^{(r)}_{pq}-\frac{c}{24})}b^{(r)}_{pq}(\tau+1). \end{aligned}$$

The corresponding characters of irreducible representations of $V(L,M)$ are linear combinations of $b^{(r)}_{pq}$ with integer coefficients n_r:

$$\chi_{pq}(\tau|L,M)=\sum_{r=0}^{L} n_r b^{(r)}_{pq},\qquad \chi_{pq}(\tau|1,M)=b_{pq}.$$

Applying the branching rules to the triple product $\chi_{M-2,p-1}\chi_{1,0}\chi_{L,s}$ and using their associativity, we obtain the following identity:

$$\sum_{t=1}^{M+2}\chi_{pt}(M)b^{(s)}_{tr}(L,M+1)=\sum_{q=1}^{M+2}\chi_{s+1,q}(L+2)b^{(q-1)}_{pr}(L+1,M), \tag{1.16}$$

which appears as a generalization of the equations (1.13) and (1.14). One can easily check the modular covariance of (1.16). This establishes the consistence of the projection defined by (1.7), (1.11) and (1.12)with the modular transformations. Note that the sum on the RHS of (1.16) reflects a certain projection in the product $V(1,L-1)\otimes V(L,M)$ as well. In the simplest case of $L=2$ this property is expressed by the fact that only the states with zero total $\mathbb{Z}_2$ charge (*i.e.* $\chi_{\frac{1}{16}}\hat{\chi}^{(R)}_{pq}$ and $(\chi_0+\chi_{\frac{1}{2}})\hat{\chi}^{(NS)}_{pq}$) are consistent with the projection P on the LHS of (1.13) and (1.14).

The goal of the above discussion was the proof (at the level of characters and branching functions) that the generic $V(L,M)$ model can be realized in terms of $V(M)$, $V(L-1,M+1)$ and $V(1,L-1)$ models. Obviously, this is not the only way to construct $V(L,M)$. For example, the 4/3-parafermionic models $V(4,M)$ [33] have an alternative construction in terms of $N=1$ superconformal models only:

$$V(2,2)\otimes V(4,M)=\mathrm{P}(V(2,M)\otimes V(2,M+2)).$$

Applying again the branching rules method and using the fact that [41]

$$b_{pq}^{(1)}(4,M)+b_{pq}^{(3)}(4,M)=\chi_{pq}^{R}(4,M),\qquad p-q\in 2\mathrm{Z}+1,$$

we find that the following identity holds:

$$\sum_{r=1}^{M+3}\hat{\chi}_{pr}(\tau,M)\hat{\chi}_{rq}(\tau,M+2)=2\hat{\chi}_{\frac{1}{16}}^{(NS)}(\tau,2)\chi_{pq}^{R}(\tau|4,M)$$

for $p-q\in 2\mathrm{Z}+1$, where $\hat{\chi}_{\frac{1}{16}}$ is the character of the representation with $\Delta=\frac{1}{16}$ in the NS-sector of $V(2,2)$.

The realization (1.9) of $N=2$ superconformal models $V^{N=2}(M)$ in terms of $V(2,M+2)$, $V^{pf}(M+2)$ and $V(1)$ announced in the introduction is based on the following relations between the dimensions of the fields in these models:

$$\begin{aligned}\Delta_{n}^{k(pf)}(M+2)+\Delta_{l+1,k+1}^{NS}(M+2)&=\Delta_{n,0}^{l}(M)+\frac{1}{8}(k-l)^{2},\\ \Delta_{n\pm1}^{k+1(pf)}(M+2)+\Delta_{l+1,k+2}^{R}(M+2)&=\Delta_{n,\pm1}^{l}(M)+\frac{1}{8}(k-l+1)^{2}-\frac{1}{16},\end{aligned}$$

where $k-l$ is even, which imply the following identifications:

$$\begin{aligned}N_{n,0}^{l}(M)&=\phi_{n}^{l}(M+2)N_{l+1,l+1}(M+2),\\ R_{n,\pm1}^{l}(M)\sigma_{\frac{1}{16}}&=\phi_{n\pm1}^{l+1}(M+2)R_{l+1,l+2}(M+2).\end{aligned}$$

Denote by $\chi_{q,s}^{l}(\tau,z|M)$ ($|q|\le l$, $0\le l\le M$) with $s=0,2 \bmod 4$ the characters for the NS sector ($\chi_{q}^{l(NS)}=\chi_{q,0}^{l}+\chi_{q,2}^{l}$) and with $s=\pm1$ the ones for the R sector. They are related to the $SU(2)$-characters $\chi_{m,l}$ by the following identity [42]:

$$\sum_{q \bmod 2(m+2)} \chi_q^l \Theta_{q,m+2} = \chi_{m,l}(\Theta_{0,2} + \Theta_{2,2}), \qquad l - q \in 2\mathbb{Z}.$$

Remembering that

$$\chi_{2,0} + \chi_{2,2} = (\chi_0 + \chi_{\frac{1}{2}})(\Theta_{0,2} + \Theta_{2,2})$$

we obtain

$$\begin{aligned}(\chi_0 + \chi_{\frac{1}{2}}) \sum_q \chi_q^l \Theta_{q,m+2} &= \chi_{m,l}(\chi_{2,0} + \chi_{2,2}) \\ &= \sum_{\substack{s=0 \\ s-l \in 2\mathbb{Z}}}^{m+2} \hat{\chi}^{(NS)}_{l+1,s+1}(\tau, m+2) \chi_{m+2,s}.\end{aligned}$$

Since the parafermionic characters $\eta(\tau) C_m^l(\tau)$ are simply related with $\chi_{m+2,s}$ as in

$$\chi_{m+2,s} = \sum_q C_q^s(m+2) \Theta_{s,m+2},$$

the above identities take the desired form:

$$(\chi_0 + \chi_{\frac{1}{2}}) \chi_q^l(\tau, m) = \sum_{\substack{s=0 \\ s-l \in 2\mathbb{Z}}}^{m+2} \hat{\chi}^{(NS)}_{l+1,s+1}(\tau, m+2) C_q^s(\tau, m+2). \tag{1.17}$$

In closing, we would like to point out that the relations (1.13), (1.14), (1.16)and (1.17)provide us with a simple method for the construction of nontrivial nondiagonal modular invariant partition functions for the projected tensor product models $\mathsf{P}(\otimes_{k=0}^{L-1} V(M+k))$. For example, the partition functions for the models $\mathsf{P}(V(M) \otimes V(M+1))$ with $c = 2 - 12/(M+2)(M+4)$ have the form:

$$\begin{aligned}Z(\tau, \bar{\tau}) = &\sum_{r,s,r',s'} N^{(NS)}_{rs,r's'} H_{rs}(\tau) H^*_{r's'}(\bar{\tau}) + \\ &+ \sum_{r,s,r',s'} N^{(\widetilde{NS})}_{rs,r's'} \tilde{H}_{rs}(\tau) \tilde{H}^*_{r's'}(\bar{\tau}) + \frac{1}{2} \sum_{r,s,r',s'} N^{(R)}_{rs,r's'} H^R_{rs}(\tau) H^{R*}_{r's'}(\bar{\tau}).\end{aligned} \tag{1.18}$$

The matrices $N^{(NS)}$, $N^{(\widetilde{NS})}$ and $N^{(R)}$ are the same as in the $V(2, M)$ partition functions and are classified in [43]. Similarly, using the corresponding N-matrices for the $N = 2$ minimal models [44], one can construct the partition functions for the $\mathsf{P}(V^{pf}(M+2) \otimes V(2, M+2))$ models.

The two simplest cases illustrating the above discussion are the models with $c = 3/2$ and $c = 6/5$. Taking the matrices $N^{(NS,\widetilde{NS},R)}_{rs,r's'}$ for the nondiagonal invariant for the superVirasoro model $c(2,2) = 1$, we obtain the following modular invariant for the model $c = 3/2 = 1 + 1/2 = 7/10 + 4/5$:

$$Z(\tau,\bar{\tau}) = \frac{1}{2}|H^{NS}_{11} + H^{NS}_{31}|^2 + \frac{1}{2}|\tilde{H}^{NS}_{11} + \tilde{H}^{NS}_{31}|^2 + |H^{NS}_{13}|^2 + \\ + |\tilde{H}^{NS}_{13}|^2 + |H^{R}_{21}|^2 + \frac{1}{2}|H^{R}_{23}|^2,$$

where H_{rs} are realized in terms of the products of the characters of $c(1,2) = 7/10$ and $c(1,3) = 4/5$ models. For example:

$$H_{21} = \chi_{\frac{7}{16}}(2)(\chi_0(3) + \chi_3(3)) + \chi_{\frac{3}{80}}(2)(\chi_{\frac{2}{5}}(3) + \chi_{\frac{7}{5}}(3)),$$
$$H_{11} + \tilde{H}_{11} = 2(\chi_0(2)\chi_0(3) + \chi_{\frac{3}{5}}(2)\chi_{\frac{7}{5}}(3)).$$

This partition function coincides with the $Z_{s-a}(\sqrt{3})$ for the $c = 3/2$ model found by Dixon, Ginsparg and Harvey in [45]. Using the corresponding matrices for the diagonal invariant of $c = 1$ superVirasoro model one can write one more invariant for the projected tensor product model $\mathsf{P}(V(2) \otimes V(3))$, $c = 7/10 + 4/5 = 3/2$. As it is shown in App. C, the tensor product model $\mathsf{P}(V(1) \otimes V(2))$ with $c = 1/2 + 7/10 = 6/5$ can be reinterpreted as the second model in the series of minimal models of the W_3-algebra [29]. The corresponding modular invariant obtained by using the matrices $N^{(NS,\widetilde{NS},R)}_{rs,r's'}$ of $c = 7/10$ model,

$$Z = \frac{1}{2}\left(|H^{NS}_{11}|^2 + |\tilde{H}^{NS}_{11}|^2 + |H^{NS}_{13}|^2 + |\tilde{H}^{NS}_{13}|^2\right) + |H^{R}_{12}|^2 + |H^{R}_{21}|^2,$$

is a nondiagonal invariant of $c = 6/5$ model of the W-algebra.

1.3 Operator constructions

Consider now the problem of the realization of the $V(L,M)$ chiral algebra and its field representations in the space $\mathsf{P}(V(M) \otimes V(L-1, M+1))$. We will begin with the simplest case $L = 2$. The natural candidates for the generators of the $N = 1$ superconformal algebra in $\mathsf{P}(V(M) \otimes V(M+1))$ are the fields $\psi_p = \phi^M_{1p}\phi^{M+1}_{p1}$, $p = 2,3$ with dimensions $\Delta_p = \frac{1}{2}(p-1)^2$, their derivatives, and the stress-energy tensors $T^{(M)}$, $T^{(M+1)}$ of $V(M)$ and $V(M+1)$. Define the following field combinations:

$$\begin{aligned}\psi &= \phi^M_{12}\phi^{M+1}_{21},\\ T^I &= \frac{M+2}{4(M+5)}T^{(M)} + \frac{M+4}{4(M+1)}T^{(M+1)} + \frac{1}{2}\sqrt{\frac{3M(M+6)}{4(M+1)(M+5)}}\phi^M_{13}\phi^{M+1}_{31},\end{aligned} \tag{1.19}$$

and

$$\begin{aligned}G &= i\sqrt{\frac{1}{(M+2)(M+4)}}\Big[M\phi^M_{12}\partial\phi^{M+1}_{21} - (M+6)(\partial\phi^M_{12})\phi^{M+1}_{21}\Big],\\ T^{SUSY} &= \frac{3(M+6)}{4(M+5)}T^{(M)} + \frac{3M}{4(M+1)}T^{(M+1)} - \frac{1}{2}\sqrt{\frac{3M(M+6)}{4(M+1)(M+5)}}\phi^M_{13}\phi^{M+1}_{31}.\end{aligned} \tag{1.20}$$

The statements we are going to prove in the following are:

(a) T^I and ψ generate the usual Ising model algebra of central charge $c = 1/2$;

(b) T^{SUSY} and G are the generators of the $N = 1$ superconformal algebra with $c = 3/2 - 12/(M+2)(M+4)$;

(c) the (T^I, ψ) and (T^{SUSY}, G) algebras are in direct product;

(d) the fields N_{rs} and R^i_{rs} defined in (1.5) are the primary fields of the $N = 1$ algebra generated by G and T^{SUSY} of (1.20).

Let us start with (a). Using the OPEs of $V(M)$ and $V(M+1)$ models we have to prove that ψ and T^I given by (1.19) satisfy the well-known OPEs [3]:

$$\begin{aligned}T^I(1)T^I(2) &= \frac{1}{4z_{12}^4} + \frac{2}{z_{12}^2}T^I(2) + \frac{1}{z_{12}}\partial T^I(2) + \cdots,\\ T^I(1)\psi(2) &= \frac{1}{2z_{12}^2}\psi(2) + \frac{1}{z_{12}}\partial\psi(2) + \cdots,\\ \psi(1)\psi(2) &= \frac{1}{z_{12}} + 2z_{12}T^I(2) + \cdots.\end{aligned}$$

To do this we have to implement the projection P in the OPEs and in the construction of the conformal blocks of ψ, G, T and of the primary fields in terms of $V(M) \otimes V(M+1)$ blocks. Postponing the general discussion to secs.1.4 and

2.1, we consider here the specific problem of constructing the 4-point functions and the OPEs of the currents ψ, G and ψ_p, using the conformal blocks of the ingredients ϕ_{1p}^M and ϕ_{p1}^{M+1}, $p = 1, 2, 3$. According to the construction (1.19), the 4-point function of $\psi(z)$ can be written as a sum of the products of the conformal blocks I_i^M of ϕ_{12}^M and I_j^{M+1} of ϕ_{21}^{M+1}:

$$F_\psi(z) = \langle\psi(0)\psi(z)\psi(1)\psi(\infty)\rangle = z(1-z)\sum_{i,j=1}^{2} Y_{ij} I_i^M(z) I_j^{M+1}(z).$$

Note that at $z = 0$ I_1^M and I_1^{M+1} are analytic and $I_2^M \sim z^{1-2(M+2)/(M+3)}$, $I_2^{M+1} \sim z^{1-2(M+4)/(M+3)}$ [27]. Then the condition for the monodromy invariance of $F_\psi(z)$ at $z = 0$ implies $Y_{12} = 0 = Y_{21}$, *i.e.*

$$\begin{aligned} F_\psi(z) &= (z(1-z))^{-1} \\ &\times \left\{ F\left(-\frac{M}{M+3}, \frac{1}{M+3}; \frac{2}{M+3}; z\right) F\left(-\frac{M+6}{M+3}, -\frac{1}{M+3}; -\frac{2}{M+3}; z\right) \right. \\ &\left. + Y_{22} z^2 F\left(\frac{M+2}{M+3}, \frac{1}{M+3}; \frac{2M+4}{M+3}; z\right) F\left(\frac{M+4}{M+3}, -\frac{1}{M+3}; \frac{2M+8}{M+3}; z\right) \right\}, \end{aligned} \tag{1.21}$$

where $Y_{11} = 1$ is a convenient normalization. Considering the small distance behaviour $z \to 0$ of (1.21), we conclude that the first term gives rise to $\phi_{11}^M \phi_{11}^{M+1}(0)$ in the OPE $\psi(z)\psi(0)$ and the second one to $\phi_{13}^M \phi_{31}^{M+1}(0)$, *i.e.* the terms $\phi_{11}^M \phi_{31}^{M+1}$ and $\phi_{13}^M \phi_{11}^{M+1}$ are projected out. What we have learned is that in this case *applying the projection* P *is the same as requiring monodromy invariance around* $z = 0$ *for the 4-point functions.*

As it is shown in Apps. A and B, the monodromy invariance around $z = 1$ fixes

$$Y_{22} = C^M_{(12)(12)(13)} C^{M+1}_{(21)(21)(31)} = \frac{3M(M+6)}{4(M+1)(M+5)}, \tag{1.22}$$

where $C^M_{(12)(12)(13)}$ and $C^{M+1}_{(21)(21)(31)}$ are the structure constants of the scalar fields $\phi_{12}^M(z)\phi_{12}^M(\bar z)$ and $\phi_{21}^{M+1}(z)\phi_{21}^{M+1}(\bar z)$ [27]:

$$C^M_{(12)(12)(13)} = \frac{M}{M+1} A, \qquad C^{M+1}_{(21)(21)(31)} = \frac{3(M+6)}{4(M+5)} A^{-1},$$

$$A = \left(\frac{\Gamma\left(\frac{M+2}{M+3}\right)\Gamma\left(\frac{2}{M+3}\right)^2\Gamma\left(\frac{M}{M+3}\right)}{\Gamma\left(\frac{1}{M+3}\right)\Gamma\left(\frac{M+1}{M+3}\right)^2\Gamma\left(\frac{3}{M+3}\right)} \right)^{\frac{1}{2}}.$$

This determines completely the 4-point function $F_\psi(z)$. Since the crossing-symmetric, monodromy-invariant 4-point function of the free Majorana field $\psi(z)$ is given by

$$\langle\psi(0)\psi(z)\psi(1)\psi(\infty)\rangle = (z(1-z))^{-1}(1-z+z^2),$$

we have obtained as a bonus a proof of the nontrivial identity

$$\begin{aligned}
&F\left(-\frac{M}{M+3},\frac{1}{M+3};\frac{2}{M+3};z\right)F\left(-\frac{M+6}{M+3},-\frac{1}{M+3};-\frac{2}{M+3};z\right)\\
&\quad+\frac{3M(M+6)}{4(M+1)(M+5)}z^2F\left(\frac{M+2}{M+3},\frac{1}{M+3};\frac{2M+4}{M+3};z\right)\\
&\quad\times F\left(\frac{M+4}{M+3},-\frac{1}{M+3};\frac{2M+8}{M+3};z\right)=1-z+z^2.
\end{aligned}\tag{1.23}$$

We provide an independent proof of (1.23) in App. B. Clearly, the present formalism can be used as a machine for generating similar identities among the sums of products of (generalized) hypergeometric functions.

Since the structure constants appearing in the OPEs are square roots of the corresponding coefficients in the 4-point functions [27], we obtain the following OPE:

$$\begin{aligned}
\psi(z)\psi(0)=\frac{1}{z}+2z\Bigg[&\frac{\Delta_{12}^{M}}{c(M)}T^{(M)}(0)+\frac{\Delta_{21}^{M+1}}{c(M+1)}T^{(M+1)}(0)+\\
&+\frac{1}{2}\sqrt{Y_{22}}\phi_{13}^{M}\phi_{31}^{M+1}(0)\Bigg]+\cdots.
\end{aligned}\tag{1.24}$$

From (1.21), (1.22) and (1.23) we see that the structure constant in front of the $\phi_{13}^{M}\phi_{31}^{M+1}(0)$ term in (1.24) is $\sqrt{C^{M}_{(12)(12)(13)}C^{M+1}_{(21)(21)(31)}}$, a square root of what one would naively expect. To understand this, remember that the OPEs should always be thought of as operations performed within well-defined correlation functions. Since the currents are distinguished from all the other conformal fields by having well-defined 1D (*i.e.* dependent only on z, *i.e.*with only left-moving fields) correlation functions, their 1D OPEs are well-defined. In the present context, the currents are realized as sums of products of ordinary conformal fields, whose only well-defined correlation functions (and therefore OPEs and structure constants) are usually 2D. Still, as is shown in detail in App. A, the particular combinations used to construct currents *can* have well-defined 1D correlation functions like

$$\left\langle \prod_{i=1}^{4} \phi^M_{1p_i} \phi^{M+1}_{p_i 1}(z_i) \right\rangle, \qquad 1 \le p_i \le M+2,$$

and

$$\left\langle \phi^M_{1p} \phi^{M+1}_{p1}(0) \phi_{rs}(2,M)(z_2,\bar{z}_2) \phi^M_{1p} \phi^{M+1}_{p1}(z_3) \phi_{rs}(2,M)(z_4,\bar{z}_4) \right\rangle, \quad r-s \in 2\mathrm{Z}.$$

Thus, as we have seen in (1.21), (1.22) and (1.23), and as is shown for all the other relevant cases in App. A, *the monodromy invariance around $z=1$ of the 1D 4-point functions of the currents results in the structure constants of 1D OPEs being constructed from the square roots of the standard 2D structure constants.* Heuristically, one could think of the square root appearing since only the left-moving fields contribute to the OPE.

After this digression we return to the proof of (a) and consider the OPE $T^I\psi$. Keeping in mind what was just said and using the Ward identity

$$T^{(M)}(z)\phi^M_{12}(0) = \frac{\Delta^M_{12}}{z^2}\phi^M_{12}(0) + \frac{1}{z}\partial\phi^M_{12}(0) + \cdots,$$

we find that

$$\begin{aligned} T^I(z)\psi(0) &= \left[\frac{(\Delta^M_{12})^2}{c(M)} + \frac{(\Delta^{M+1}_{21})^2}{c(M+1)} + \frac{3M(M+6)}{8(M+1)(M+5)}\right]\frac{1}{z^2}\psi(0) + \cdots, \\ &= \frac{1}{2z^2}\psi(0) + \cdots. \end{aligned}$$

This demonstrates that there are infinitely many ways (indexed by M), to realize the free spin-1/2 Majorana field ψ in terms of its "asymmetric square roots" ϕ^M_{12} and ϕ^{M+1}_{21}, which in their turn are non-free fields with fractional spins $\frac{M}{4(M+3)}$ and $\frac{M+6}{4(M+3)}$, respectively.

In proving (b), (c) and (d) we follow the same procedure, *i.e.* start with the construction (1.19) and (1.20), and do the OPEs. In the OPEs, keep only the terms consistent with P and use the square roots of the 2D structure constants in

1D OPEs. For example, applying this recipe to the product of two supercurrents $G(z)G(0)$, we obtain the well-known OPE [5]

$$G(z)G(0)=\frac{M(M+6)}{(M+2)(M+4)}\frac{1}{z^3}+\frac{2}{z}\left[\frac{3}{4}\frac{M+6}{M+5}T^{(M)}(0)+\right.$$
$$\left.\frac{3}{4}\frac{M}{M+1}T^{(M+1)}(0)-\frac{1}{2}\sqrt{Y_{22}}\phi^M_{13}\phi^{M+1}_{31}(0)\right]+\cdots$$

implying $c(2,M)=3/2-12/(M+2)(M+4)$. Analogous calculations for $\psi(z)G(0)$ and $T^I(z)T^{SUSY}(0)$ show that no singular terms appear in these OPEs. For example,

$$\begin{aligned}\psi(z)G(0)=i\frac{1}{\sqrt{(M+2)(M+4)}}\bigg\{&\frac{(M+2)(M+6)}{M+5}T^{(M)}\\&-\frac{M(M+4)}{M+1}T^{(M+1)}+2\sqrt{Y_{22}}\phi^M_{13}\phi^{M+1}_{31}\bigg\},\end{aligned}\tag{1.25}$$

i.e. the Ising model algebra (T^I, ψ) and the $N=1$ superconformal algebra are in fact in direct product.

It remains to consider the supercurrent Ward identities and the properties of the primary fields. Starting with the NS sector, we have to find a realization of the second component N^{II}_{rs}, with the dimension $\Delta_{rs}(2,M)+1/2$, consistent with the constructions for G and N_{rs}, *i.e.* satisfying the OPEs:

$$\begin{aligned}G(z)N_{rs}(0)&=\frac{1}{z}iN^{II}_{rs}(0)+\cdots,\qquad r-s\in 2\mathbb{Z},\\G(z)N^{II}_{rs}(0)&=\frac{2\Delta_{rs}(2,M)}{z^2}iN_{rs}(0)+\frac{1}{z}i\partial N_{rs}(0)+\cdots.\end{aligned}\tag{1.26}$$

The result is

$$\begin{aligned}N^{II}_{rs}(M)&=a_-(M)\phi^M_{r,\frac12(r+s)-1}\phi^{M+1}_{\frac12(r+s)-1,s}+a_+(M)\phi^M_{r,\frac12(r+s)+1}\phi^{M+1}_{\frac12(r+s)+1,s},\\a_\mp(M)&=\frac{1}{\sqrt{(M+2)(M+4)}}\bigg[M\left(\Delta^{M+1}_{\frac12(r+s)\mp1,s}-\Delta^{M+1}_{21}-\Delta^{M+1}_{\frac12(r+s),s}\right)\\&\qquad-(M+6)\left(\Delta^M_{r,\frac12(r+s)\mp1}-\Delta^M_{21}-\Delta^M_{r,\frac12(r+s)}\right)\bigg]\\&\qquad\times\sqrt{C^M_{(12)(r,\frac12(r+s))(r,\frac12(r+s)\mp1)}C^{M+1}_{(21)(\frac12(r+s),s)(\frac12(r+s)\mp1,s)}}.\end{aligned}\tag{1.27}$$

For example, for $r=1$, $s=3$ we obtain the field driving the RG-flow $V(2,M)\rightarrow$

$V(2, M-2)$ [49]:*

$$N_{13}^{II} = \sqrt{\frac{M}{M+4}} \left[\sqrt{\frac{M}{2(M+1)}} \phi_{11}^{M} \phi_{13}^{M+1} + \sqrt{\frac{M+2}{2(M+1)}} \phi_{13}^{M} \phi_{33}^{M+1} \right]. \tag{1.28}$$

Note that there is one more field with the same dimension:

$$\psi N_{13} = \frac{1}{2(M+1)} \left[(M+2) \phi_{11}^{M} \phi_{13}^{M+1} + M \phi_{13}^{M} \phi_{33}^{M+1} \right],$$

but only N_{13}^{II} given by (1.28) has all the properties of the second component of N_{13}.

To conclude the discussion of the supercurrent Ward identities, we turn to the Ramond sector. Using (1.5), one can show that

$$G(z) R_{rs}^{1(2)}(0) = \sqrt{\Delta_{rs} - \frac{c}{24}} \frac{1}{z^{3/2}} R^{2(1)}(0) + \cdots .$$

To prove that N_{rs} and R_{rs} constructed as above obey all the required null-vector properties [5, 46], what we have done so far is still not enough. We have to show that their fusion rules, structure constants, and 4-point functions coincide with the ones for the $N=1$ minimal models [46, 47, 26]. We postpone this discussion to the following sections.

In extending the discussion of the current algebra and the Ward identities to the higher level coset models, one encounters new difficulties. We will outline them here briefly, motivating the change of strategy we will use for $L>2$, which will entail abandoning of the study of the current algebra and focusing on the direct construction of monodromy invariants. The first difficulty is the simple fact that even the dimensions of all the currents are not known.$^\diamond$ The other added difficulty has to do with the fact that the dimension of the $A_{\frac{L+4}{L+2}}$ current stops being a multiple of $1/2$ for $L>2$. Therefore, we cannot expect that such a curent has a monodromy-invariant 1D correlation function. As explained in

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$\star$ Do not forget that these are 1D second components. The full 2D second component is $N_{13}^{II}(z,\bar{z}) = N_{13}^{II}(z) N_{13}^{II}(\bar{z})$.

$\diamond$ For $L \geq 5$ there seem to exist additional currents over and above the well-known $\Delta = 2$ (T) and $\Delta = \frac{L+4}{L+2}$ ones.

App. A, one can construct monodromy-invariant 1D correlation functions of the following kind:

$$G = \left\langle \prod_{i=1}^{4} \phi_{1p_1}^{M} \phi_{p_1p_2}^{M+1} \dots \phi_{p_{L-1}1}^{M+L-1}(z_i) \right\rangle . \tag{1.29}$$

The dimension of $\phi_{1p_1}^{M} \phi_{p_1p_2}^{M+1} \dots \phi_{p_{L-1}1}^{M+L-1}$ is always a multiple of $1/2$. Therefore, it can be identified with a product of $A_{\frac{L+4}{L+2}}$ with some non-trivial field from $V(1, L-1)$ on the LHS of (1.7). For example, for $L = 3$, among the projected products that could be used for the representation of $A_{7/5}$ the lowest dimension ones have dimension equal to $3/2$, as for example $\phi_{13}^{M} \phi_{32}^{M+1} \phi_{21}^{M+2}$. Since $\frac{3}{2} - \frac{7}{5} = \frac{1}{10}$, a look at (1.8) tells us that $A_{7/5}$ appears multiplied by the field $\phi_{13}(2,1)$. A similar structure persists for higher L; in constructing the current A out of projected products of dimension $3/2$ one has to multiply it with the field $\phi_{13}(2, L-2)$ of dimension $\frac{L-2}{2(L+2)}$. Therefore, the correlation function (1.29) has to be interpreted as

$$G_{A\phi} = \left\langle \prod_{1}^{4} A\phi_{13}(2, L-2)(z_i) \right\rangle .$$

It seems impossible to factorize $G_{A\phi}$ into some 1D correlation functions $\langle \prod A(z) \rangle$ and $\langle \prod \phi_{13}(2, L-2)(z) \rangle$.

Obviously, at this stage we cannot be very precise about the current algebras for $L = 3, 5, 6, \dots$. Therefore, for the study of these higher levels we adopt a different strategy. In the next section, we will start by constructing all the primary fields for any L in terms of the projected products of the Virasoro fields. Then, we will construct the corresponding conformal blocks *and their monodromy-invariant combinations*. That will allow us to obtain the fusion rules (which we work out explicitly for the fusions of two vacuum sector fields) and the structure constants. In the cases where there are any previous results to compare with ($L = 2$ and $L = 4$), our results will be confirmed.

1.4 Construction of the higher level primary fields

In this section we present an explicit construction of the fields belonging to the higher level models. Our goal is to reduce everything to the products of the

Virasoro fields since their conformal blocks, correlation functions *etc.* are fully understood and explicitly calculated. At the outset we will guide ourselves solely by comparing the dimensions of the fields. Having established a tentative identification, we will show in the sections that follow that such an identification leads to monodromy-invariant correlation functions and to the structure constants that agree with the previous results.

Here we limit ourselves to the fields which are primary relative to the stress-energy tensor and all the additional currents present in the higher level models. We will have more to say about the descendants later on. The primary fields of the model M at level L are $\phi_{mn}(L,M)$, with the conformal dimension given by [32]

$$\Delta_{mn}(L,M) = \frac{[(L+M+2)m-(M+2)n]^2 - L^2}{4L(M+2)(L+M+2)} + \frac{\mathcal{L}(L-2\mathcal{L})}{L(L+2)},$$
$$\mathcal{L} = \frac{1}{2}|m-n \bmod 2L|, \qquad 0 \leq \mathcal{L} \leq \frac{L}{2},$$
$$1 \leq m \leq M+1, \quad 1 \leq n \leq L+M+1, \quad M,L = 1,2,\ldots.$$

If $n-m \in LZ$, the expression for Δ_{mn} simplifies since $\mathcal{L}(L-2\mathcal{L}) = 0$. For $L=2$ such fields belong to the Neveu-Schwarz sector, for general L we will call such sector the "vacuum sector." Since it is significantly simpler, we present the construction first for such fields.

It is easy to check that

$$\Delta_{mn}(L,M) = \Delta_{mx}(1,M) + \Delta_{xn}(L-1,M+1)$$

if $x = \frac{1}{L}(n+(L-1)m)$. This identity leads us to write

$$\phi_{mn}(M,L) = \phi_{m,\frac{1}{L}(n+(L-1)m)}(1,M)\phi_{\frac{1}{L}(n+(L-1)m),n}(L-1,M+1). \qquad (1.30)$$

Two remarks are in order: first, note that $V(1,L-1)$ from the LHS of (1.7) contributes the identity field to the LHS of (1.30); second, as will become clear later on, the products $\phi_{mt}(1,M)\phi_{tn}(L-1,M+1)$ with $t \neq x$ represent (parts of) descendants of $\phi_{mn}(L,M)$. Since

$$n - \frac{n+(L-1)m}{L} \in (L-1)\mathrm{Z},$$

we can immediately iterate (1.30) and finally obtain $\phi_{mn}(L,M)$ written in terms of the Virasoro fields:

$$\begin{aligned}\phi_{mn}(L,M) &= \prod_{i=0}^{L-1} \phi_{k_i k_{i+1}}(1, M+i), \\ k_i &= \frac{in+(L-i)m}{L}, \qquad n-m \in L\mathrm{Z}.\end{aligned} \tag{1.31}$$

Let us pause for a moment to study (1.31). Firstly, note that we can treat any two consequtive Virasoro fields $\phi_{k_i k_{i+1}}(1, M+i)\phi_{k_{i+1}k_{i+2}}(1, M+i+1)$, $i = 0, \ldots, L-2$ as representing a superVirasoro NS field $\phi_{k_i k_{i+2}}(2, M+i)$. Namely, since $k_i = \frac{1}{L}(in+(L-i)m) = m+iK$, $K = \frac{1}{L}(n-m)$, it is clear that $k_{i+2} - k_i = 2K \in 2\mathrm{Z}$. Since $k_{i+1} = \frac{1}{2}(k_i + k_{i+2})$, all of these NS fields are primaries. Furthermore, starting from (1.31) one can reach any other projected product

$$\phi_{m\hat{k}_1}(1,M)\phi_{\hat{k}_1\hat{k}_2}(1,M+1)\cdots\phi_{\hat{k}_{L-1}n}(1,M+L-1), \tag{1.32}$$

by changing k_1 into $\hat{k}_1$, k_2 into $\hat{k}_2$ *etc.*. By (1.4), the dimension of (1.32) is higher than the one of (1.31) by a multiple of 1/2. We interpret products as (1.32) as (parts of) descendants of $\phi_{mn}(L,M)$ w.r.t. T, G ($L=2$), $\phi_{13}(2,L-2)A_{\frac{L+4}{L+2}}$, any other additional currents that appear for $L > 4$, or a product of these currents.*

To summarize, among all the products (1.32) we search for the one with the lowest dimension to identify it with the primary field $\phi_{mn}(L,M)$. Minimizing the dimension is equivalent to minimizing $S \equiv \sum_{i=0}^{L-1}(k_i - k_{i+1})^2$. If $m-n \in L\mathrm{Z}$ there is a unique solution (with $k_0 = m$, $k_L = n$) that gives $S = LK^2$, namely equidistant k_i's , $k_i = m+iK$, $K = \frac{1}{L}(n-m)$ that we already wrote down in (1.31).

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* The dimension of $\phi_{13}(2,L-2)$ is $\frac{L-2}{2(L+2)}$. The current $A(z) = \sum_r A_r z^{-r-\frac{L+4}{L+2}}$ is not branched around the vacuum sector fields, and therefore is moded by $r \in \mathrm{Z} - \frac{2}{L+2}$. Putting this together, we obtain $\frac{L-2}{2(L+2)} + \frac{2}{L+2} = \frac{1}{2}$.

Turning to the non-vacuum sector (Ramond and the analogues), namely fields $\phi_{mn}(L,M)$ with $m-n \notin L\mathsf{Z}$, we will see that things stop being quite so simple. We will begin by deriving the expression for one product of the Virasoro fields having the required (minimal) dimension. Unfortunately, it will become obvious that for non-vacuum sectors that expression is not unique. We will study the field $\phi_{mn}(L,M)$ with $n-m \in L\mathsf{Z} \mp l$, $1 \leq l \leq L-1$. (A part of the reason for non-uniqueness should be already obvious, namely if $n-m \in L\mathsf{Z} \mp l$ then also $n-m \in L\mathsf{Z} \pm (L-l)$.) Some straightforward algebra will show that

$$\Delta_{l,l+1}(1,L-1) + \Delta_{mn}(L,M) = \Delta_{my}(1,M) + \Delta_{yn}(L-1,M+1),$$

where $y = \frac{1}{L}((L-1)m + n \mp (L-l))$. Therefore, we write

$$\begin{aligned} \phi_{l,l+1}(1,L-1)\phi_{mn}(L,M) = \phi_{m,\frac{1}{L}((L-1)m+n\mp(L-l))}&(1,M) \\ \times\, \phi_{\frac{1}{L}((L-1)m+n\mp(L-l)),n}&(L-1,M+1). \end{aligned} \tag{1.33}$$

This time $V(1,L-1)$ contributes a nontrivial field° $\phi_{l,l+1}(1,L-1)$. Analogously to the discussion for the vacuum sector, $n-m \in L\mathsf{Z} \mp l$ implies

$$n - \frac{(L-1)m + n \mp (L-l)}{L} \in (L-1)\mathsf{Z} \mp (l-1).$$

The field from $V(L-1,M+1)$ is again from a non-vacuum sector, even though one step closer to the vacuum sector. We again iterate the process and after l steps obtain the following identification among the products of the fields:

$$\begin{aligned} &\phi_{12}(1,L-l)\phi_{23}(1,L-l+1)\cdots\phi_{l,l+1}(1,L-1)\phi_{mn}(L,M) = \\ &= \phi_{m,\frac{1}{L}((L-1)m+n\mp(L-l))}(1,M)\phi_{\frac{1}{L}((L-1)m+n\mp(L-l)),\frac{1}{L}((L-2)m+2n\mp2(L-l))}(1,M+1) \\ &\quad \times \phi_{\frac{1}{L}((L-2)m+2n\mp2(L-l)),\frac{1}{L}((L-3)m+3n\mp3(L-l))}(1,M+2) \times \cdots \\ &\quad \cdots \times \phi_{\frac{1}{L}((L-l+1)m+(l-1)n\mp(l-1)(L-l)),\frac{1}{L}((L-l)m+ln\mp l(L-l))}(1,M+l-1) \\ &\quad \times \phi_{\frac{1}{L}((L-l)m+ln\mp l(L-l)),n}(L-l,M+l). \end{aligned}$$

This time

$$n - \frac{(L-l)m + ln \mp l(L-l)}{L} \in (L-l)\mathsf{Z},$$

and the field $\phi_{\frac{1}{L}((L-l)m+ln\mp l(L-l)),n}(L-l,M+l)$ is in the vacuum sector. Using

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° Note that (1.33) includes also the vacuum sector expression (1.30). For $l=L$, $\phi_{L,L+1}(1,L-1) = \phi_{11}(1,L-1)$, using the identification $\phi_{mn}(L,M) = \phi_{M+2-m,M+L+2-n}(L,M)$.

(1.31) we can write

$$\begin{aligned}
&\phi_{\frac{1}{L}((L-l)m+ln\mp l(L-l)),n}(L-l,M+l)= \\
&=\phi_{\frac{1}{L}((L-l)m+ln\mp l(L-l)),\frac{1}{L}((L-l-1)m+(l+1)n\mp l(L-l-1))}(1,M+l) \\
&\quad\times\phi_{\frac{1}{L}((L-l-1)m+(l+1)n\mp l(L-l-1)),\frac{1}{L}((L-l-2)m+(l+2)n\mp l(L-l-2))}(1,M+l+1)\times\cdots \\
&\quad\cdots\times\phi_{\frac{1}{L}(m+(L-1)n\mp l),n}(1,M+L-1).
\end{aligned}$$

We arrive at the general formula for the non-vacuum sector fields:

$$\begin{aligned}
&\phi_{12}(1,L-l)\phi_{23}(1,L-l+1)\cdots\phi_{l,l+1}(1,L-1)\phi_{mn}(L,M)= \\
&\quad=\phi_{1,l+1}(l,L-l)\phi_{mn}(L,M)=\prod_{i=0}^{L-1}\phi_{k_ik_{i+1}}(1,M+i), \\
&\quad n-m\in L\mathbb{Z}\mp l,\quad 1\le l\le L-1, \\
&\quad k_i=\frac{(L-i)m+in+d_i^l}{L}, \\
&\quad d_i^l=\begin{cases}\mp i(L-l), & \text{if } i\le l,\\ \mp l(L-i), & \text{if } i>l.\end{cases}
\end{aligned}\tag{1.34}$$

The minimum of S would again be obtained for $k_i-k_{i+1}=\frac{1}{L}(n-m)$, if $\frac{1}{L}(n-m)$ were an integer. Since it is not, we have to content ourselves with the next best thing, provided by (1.34). According to (1.34), $k_{i+1}-k_i=\frac{1}{L}(n-m)\mp\frac{1}{L}(L-l)$ or $\frac{1}{L}(n-m)\pm\frac{l}{L}$, so it is always one of the two integers closest to $\frac{1}{L}(n-m)$. In other words, since $n-m=LK\mp l$, $L-l$ of the $(k_{i+1}-k_i)$'s can remain equal to K, but the remaining l have to be equal to $K\mp 1$. Obviously, it makes no difference *which* $(k_{i+1}-k_i)$'s will be equal to $K\mp 1$, and we arrive thus at $\binom{L}{l}$ different products of the Virasoro fields that have the same dimension, equal to the dimension of $\phi_{1,l+1}(l,L-l)\phi_{mn}(L,M)$.

How should we interpret this embarassment of riches, namely, having too many ways to represent a given field as a product of the Virasoro fields? One could try to explain it away claiming that it is only a certain linear combination of these $\binom{L}{l}$ products that is well-defined. Since we will show in the next chapter that each of these products separately has a perfectly well-defined monodromy-invariant 4-point function that cannot be true. To understand better the origin of this degeneracy, note that $\phi_{1,l+1}(l,L-l)$ in (1.34) represents actually (remember

(1.8)) the product

$$\phi_{11}(1,1)\cdots\phi_{11}(1,L-l-1)\phi_{12}(1,L-l)\phi_{23}(1,L-l+1)\cdots\phi_{l,l+1}(1,L-1).$$

From the discussion bellow (1.34) follows that there are exactly $\binom{L-1}{l}$ such projected products of the fields from $\mathsf{P}(V(1,1)\otimes\cdots\otimes V(1,L-1))$ that have the same dimension. What is more, assuming that $L-l\neq l$ (*i.e.* $l\neq L/2$), we could view the original $\phi_{mn}(L,M)$ field as belonging to the $(L-l)$-sector. In that case in (1.34) would appear the field $\phi_{1,L-l+1}(L-l,l)$ ($\Delta_{1,l+1}(l,L-l)=\Delta_{1,L-l+1}(L-l,l)=\frac{l(L-l)}{4(L+2)}$) which in its turn would represent $\binom{L-1}{L-l}$ products like

$$\phi_{11}(1,1)\cdots\phi_{11}(1,l-1)\phi_{12}(1,l)\phi_{23}(1,l+1)\cdots\phi_{L-l,L-l+1}(1,L-1).$$

Finally, since

$$\binom{L}{l}=\binom{L-1}{l}+\binom{L-1}{L-l},$$

we conclude that for $l\neq L/2$ all of the degeneracy on the RHS of (1.34) is accounted for by the degeneracy on the LHS. As a simple example, consider the field $\phi_{12}(4,M)$. It can be viewed as belonging to the $l=1$ or $l=3$ sector. In the projected tensor product it appears as

$$\left.\begin{array}{l}\phi_{11}^{M}\phi_{11}^{M+1}\phi_{11}^{M+2}\phi_{12}^{M+3}\\ \phi_{11}^{M}\phi_{11}^{M+1}\phi_{12}^{M+2}\phi_{22}^{M+3}\\ \phi_{11}^{M}\phi_{12}^{M+1}\phi_{22}^{M+2}\phi_{22}^{M+3}\\ \phi_{12}^{M}\phi_{22}^{M+1}\phi_{22}^{M+2}\phi_{22}^{M+3}\end{array}\right\}=\phi_{12}(4,M)\times\left\{\begin{array}{l}\phi_{12}^{1}\phi_{23}^{2}\phi_{34}^{3}\\ \phi_{11}^{1}\phi_{11}^{2}\phi_{12}^{3}\\ \phi_{11}^{1}\phi_{12}^{2}\phi_{22}^{3}\\ \phi_{12}^{1}\phi_{22}^{2}\phi_{22}^{3}\end{array}\right..$$

That leaves us with the case $l=L/2$, for which replacing l by $L-l$ obviously does not produce any new products of fields with equal dimension. In that case

$$\binom{L}{l}=2\binom{L-l}{l},$$

and we conclude that *for every even L, the fields from the $l=L/2$ sector are doubly degenerate*, *i.e.* for every m and n such that $m-n\in L\mathbb{Z}+L/2$ there are two distinct fields $\phi^{i}_{mn}(L,M)$, $i=1,2$, with the same dimension but with, in principle, different correlation functions and structure constants. The simplest example for this is (1.5), the Ramond fields in the superVirasoro models. A more non-trivial example is provided by $\phi_{13}(4, M)$ field, belonging to the sector $l = 2$ for $L = 4$. The products with the same dimension are:

$$\left.\begin{array}{l} \phi_{11}^{M}\phi_{11}^{M+1}\phi_{12}^{M+2}\phi_{23}^{M+3} \\ \phi_{11}^{M}\phi_{12}^{M+1}\phi_{22}^{M+2}\phi_{23}^{M+3} \\ \phi_{12}^{M}\phi_{22}^{M+1}\phi_{22}^{M+2}\phi_{23}^{M+3} \\ \phi_{11}^{M}\phi_{12}^{M+1}\phi_{23}^{M+2}\phi_{33}^{M+3} \\ \phi_{12}^{M}\phi_{22}^{M+1}\phi_{23}^{M+2}\phi_{33}^{M+3} \\ \phi_{12}^{M}\phi_{23}^{M+1}\phi_{33}^{M+2}\phi_{33}^{M+3} \end{array}\right\} = \phi_{13}^{i}(4, M) \times \left\{\begin{array}{l} \phi_{11}^{1}\phi_{12}^{2}\phi_{23}^{3} \\ \phi_{12}^{1}\phi_{22}^{2}\phi_{23}^{3} \\ \phi_{12}^{1}\phi_{23}^{2}\phi_{33}^{3}. \end{array}\right. \tag{1.35}$$

On the LHS of (1.35) we could form six linearly independent combinations with the same dimension, whereas on the RHS there are $3n$ linearly independent combinations, where n is the degree of degeneracy of $\phi_{13}(4, M)$. Obviously, $n = 2$, which we denote by $\phi_{13}^{i}(4, M)$, $i = 1, 2$.

In [33], the two $\phi_{13}(4, M)$ fields are denoted $\mathcal{D}_{13}$ and $\mathcal{D}_{13}^{\dagger}$. They form a two-dimensional representation of S_3, generated by the charge conjugation C and Z_3 generator Ω and defined by the relations

$$C^2 = 1, \qquad \Omega^3 = 1, \qquad C\Omega = \Omega^2 C.$$

Since

$$\mathcal{D}_{13}(1)\mathcal{D}_{13}^{\dagger}(2) \sim \frac{1}{z_{12}^{2\Delta}}, \tag{1.36}$$

$\mathcal{D}_{13}$ and $\mathcal{D}_{13}^{\dagger}$ are conjugate to each other, with Z_3 charges equal to $+1$ and -1, respectively. We see that in the basis spanned by $\mathcal{D}_{13}$ and $\mathcal{D}_{13}^{\dagger}$ C is realized off-diagonaly and Z_3 is diagonal. In contrast, the six products on the LHS of (1.35) are self-conjugate. Therefore, in this basis C is diagonal and Ω has to be off-diagonal to preserve (1.36).

Chapter 2

Correlation Functions and Structure Constants

2.1 Correlation functions

In this section we will present the explicit construction of the 4-point correlation functions for arbitrary fields from a higher level model. We start with the simplest example of

$$G(z,\bar{z}) = \langle \phi_{mn}(L,M)(0)\phi_{mn}(L,M)(z,\bar{z})\phi_{mn}(L,M)(1)\phi_{mn}(L,M)(\infty)\rangle \ , \quad (2.1)$$

where $n-m \in L\mathbb{Z}$. There are two basic steps in the calculation of G, according to [27]. First, one obtains the conformal blocks, *i.e.*the linearly independent solutions of the differential equation obeyed by the correlation function, and second, one combines them in a monodromy-invariant expression which is the correlation function. By (1.31) $\phi_{mn}(L,M)$ is a product of the Virasoro fields and therefore the conformal blocks for (2.1) will be the products of the Virasoro conformal blocks. Of course, only certain products of conformal blocks will survive the projection P.

To proceed we need to introduce some notation. For instance, the conformal blocks of the correlation function of the Virasoro fields

$$G_V(z,\bar{z}) = \langle \phi_{kl}(1,M)(0)\phi_{kl}(1,M)(z,\bar{z})\phi_{kl}(1,M)(1)\phi_{kl}(1,M)(\infty)\rangle\ ,$$

can be obtained with the Coulomb gas technology as certain multi-contour integrals [27], denoted as $I^M_{ij}(a,a';z)$, where $i=1,\ldots,k$, $j=1,\ldots,l$, and

$$\begin{aligned}
a &= 2\alpha_-\alpha_{kl}, \qquad a' = 2\alpha_+\alpha_{kl},\\
\alpha_{kl} &= \frac{1}{2}[(1-k)\alpha_+ + (1-l)\alpha_-], \qquad \alpha_+\alpha_- = -1,\\
\alpha_+^2 &= \frac{M+3}{M+2} \equiv \rho', \qquad \alpha_-^2 = \frac{M+2}{M+3} \equiv \rho.
\end{aligned}$$

By studying the leading behaviour of $I^M_{ij}(z)$ as $z \to 0$, it is easy to see that it corresponds to the field $\phi_{2(k-i)+1,2(l-j)+1}(1,M)$ in the intermediate channel.* Therefore, in order to preserve the projection P in the intermediate channel, we allow only products of the conformal blocks of the form

$$I^M_{i_0i_1} I^{M+1}_{i_1i_2} \cdots I^{M+L-1}_{i_{L-1}i_L}. \tag{2.2}$$

Having obtained the conformal blocks, we want to construct their monodromy-invariant combinations. We start with the simple example of $L=2$, *i.e.* 4-point functions of NS fields $\phi_{mn}(2,M) = \phi_{mx}(1,M)\phi_{xn}(1,M+1)$ and $x = \frac{1}{2}(m+n)$. The task is to find the coefficients $X_{i_0i_1i_2,j_0j_1j_2}$ such that

$$G(z,\bar{z}) = \sum_{\substack{i_0,j_0=1,\ldots,m\\ i_1,j_1=1,\ldots,x\\ i_2,j_2=1,\ldots,n}} X_{i_0i_1i_2,j_0j_1j_2} I^M_{i_0i_1} I^{M+1}_{i_1i_2}(z) \overline{I^M_{j_0j_1} I^{M+1}_{j_1j_2}(z)}$$

is monodromy-invariant. In other words, we want $G(z,\bar{z})$ to be well-defined, that is single-valued in the complex plane. Since the conformal blocks only have poles at $z=0,1$ and ∞, $G(z,\bar{z})$ will be single-valued everywhere if it is invariant under analytic continuation in z along a contour surrounding $z=0$ and along a contour surrounding $z=1$.

* In the correlation function I^M_{ij} is multiplied by $[z(1-z)]^{2\alpha^2_{kl}}$, which will be tacitly assumed in all our expressions for the correlation functions, even though we will omit to write it explicitly.

We start with the behaviour around $z = 0$. The conformal block $I^M_{i_0 i_1}$ (together with the factor $[z(1-z)]^{2\alpha^2}$) has the leading behaviour for $z \to 0$ given by

$$z^{\Delta_{2(m-i_0)+1,2(x-i_1)+1}(1,M) - 2\Delta_{mx}(1,M)}.$$

Therefore, under the monodromy transformation $z \to ze^{2\pi i}$, $\bar{z} \to \bar{z}e^{-2\pi i}$, the term $I^M_{i_0 i_1} I^{M+1}_{i_1 i_2}(z) \overline{I^M_{j_0 j_1} I^{M+1}_{j_1 j_2}(z)}$ will get a phase $e^{2\pi i \varphi}$, where

$$\begin{aligned}\varphi = \Delta^M_{2(m-i_0)+1,2(x-i_1)+1} + \Delta^{M+1}_{2(x-i_1)+1,2(n-i_2)+1} \\ - \Delta^M_{2(m-j_0)+1,2(x-j_1)+1} - \Delta^{M+1}_{2(x-j_1)+1,2(n-j_2)+1}.\end{aligned}$$

Again, some straightforward algebra will show that in order for φ to be an integer, we have to demand

$$i_0 = j_0, \qquad i_2 = j_2.$$

Note that there are no new restrictions on i_1 and j_1. We conclude that

$$G(z, \bar{z}) = \sum_{\substack{i_0=1,\ldots,m \\ i_1,j_1=1,\ldots,x \\ i_2=1,\ldots,n}} X_{i_0 i_1 j_1 i_2} I^M_{i_0 i_1} I^{M+1}_{i_1 i_2}(z) \overline{I^M_{i_0 j_1} I^{M+1}_{j_1 i_2}(z)} \tag{2.3}$$

is invariant under $z \to ze^{2\pi i}$.

We turn to the study of the analytic continuation around $z = 1$. Let us return to the 4-point function of the Virasoro field $\phi_{kl}(1, M)$, and study its conformal blocks $I^{(kl)}_{ij}(z)$. Unfortunately, they do not have as simple transformation properties under monodromy around $z = 1$ as they do around $z = 0$. Following [27], we will rewrite them in terms of $\tilde{I}_{ij}(z) = I_{ij}(1-z)$, and then repeat the same argument as for $z = 0$. Define α-matrices by

$$I^{(kl)}_{ij}(a, a'; z) = \sum_{\substack{p=1,\ldots,k \\ q=1,\ldots,l}} \alpha^{(kl)}_{ij,pq}(a, a') I^{(kl)}_{pq}(a, a'; 1-z).$$

The main ingredient in obtaining the fully monodromy-invariant $G(z, \bar{z})$ is a rather special and surprising identity that the α-matrices satisfy, which we now proceed to discuss.

Among the Virasoro fields ϕ_{kl}^{M}, there is a particularly simple subclass containing fields like ϕ_{1l}^{M} and ϕ_{k1}^{M}. Since the 4-point functions of such fields require only a single kind of screening charges, the corresponding conformal blocks I_i and the matrix α_{ij} depend on only one of the two parameters a and a'. A very important, simple fact proved in [27] is that

$$\alpha_{ij,pq}^{(kl)}(a,a') = \alpha_{ip}^{(kl)}(a')\alpha_{jq}^{(kl)}(a). \tag{2.4}$$

In the following, we will sometimes write this as $\alpha_{ij,pq} = \alpha'_{ip}\alpha_{jq}$. We focus onto the two-index matrix α_{ij}. The explicit expressions for the α-matrices have been obtained in [27, 48]. In particular, they state that a generic matrix element α_{ij} is a sum of the products of even number of factors of the kind

$$f = \sin[\pi(Aa + B\rho)],$$

where A and B are some integers. In determining the monodromy properties of the 4-point function of $\phi_{mn}(2,M) = \phi_{mx}(1,M)\phi_{xn}(1,M+1)$, we will encounter

$$\alpha_{ij,rs}^{(mx)}(M) = \alpha_{ir}^{\prime(mx)}(M)\alpha_{js}^{(mx)}(M) \text{ and } \alpha_{kl,uv}^{(xn)}(M+1) = \alpha_{ku}^{\prime(xn)}(M+1)\alpha_{lv}^{(xn)}(M+1).$$

The identity we mentioned and are now going to prove is

$$\alpha_{ij}^{(mx)}(M) = \alpha_{ij}^{\prime(xn)}(M+1). \tag{2.5}$$

The proof is immediate. Since

$$a(M) = 2\alpha_-\alpha_{mx} = m - x + \frac{x-1}{M+3}, \quad \rho(M) = \frac{M+2}{M+3} = 1 - \frac{1}{M+3},$$
$$a'(M+1) = 2\alpha_+\alpha_{xn} = n - x + \frac{1-x}{M+3}, \quad \rho'(M+1) = \frac{M+4}{M+3} = 1 + \frac{1}{M+3},$$

$f(M)$ is up to a sign equal to $f'(M+1)$.

Now we are fully equiped to conclude our construction of the monodromy-invariant 4-point functions of the NS fields in the superVirasoro models. To study (2.3) under $(1-z) \to (1-z)e^{2\pi i}$ we use α-matrices to transform $I(z)$'s into $I(1-z)$'s. The result is

$$G(z,\bar{z}) = \sum X_{i_0 i_1 j_1 i_2} \alpha^{M}_{i_0 i_1, ef} \alpha^{M+1}_{i_1 i_2, gh} \alpha^{M}_{i_o j_1, rs} \alpha^{M+1}_{j_1 i_2, tu} I^{M}_{ef} I^{M+1}_{gh}(1-z) \overline{I^{M}_{rs} I^{M+1}_{tu}(1-z)}. \tag{2.6}$$

There are two requirements that have to be satisfied. First, α-transformation should not take us outside of the subspace defined by P. Second, $G(z,\bar{z})$ should be of the same form w.r.t. $I(1-z)$ as (2.3) is w.r.t. $I(z)$ to insure invariance under the monodromy transformation around $z=1$. Both of these requirements will be satisfied if

$$\sum_{i_0,i_1,j_1,i_2} X_{i_0 i_1 j_1 i_2} \alpha^{M}_{i_0 i_1, ef} \alpha^{M+1}_{i_1 i_2, gh} \alpha^{M}_{i_o j_1, rs} \alpha^{M+1}_{j_1 i_2, tu} \propto \delta_{fg} \delta_{st} \delta_{er} \delta_{hu}. \tag{2.7}$$

By (2.4) and (2.5) the LHS of (2.7) is equal to

$$\begin{aligned} &\sum_{i_0,i_1,j_1,i_2} X_{i_0 i_1 j_1 i_2} \alpha'^{M}_{i_0 e} \alpha^{M}_{i_1 f} \alpha'^{M+1}_{i_1 g} \alpha^{M+1}_{i_2 h} \alpha'^{M}_{i_0 r} \alpha^{M}_{j_1 s} \alpha'^{M+1}_{j_1 t} \alpha^{M+1}_{i_2 u} = \\ &= \sum_{i_0,i_1,j_1,i_2} X_{i_0 i_1 j_1 i_2} \alpha'^{M}_{i_0 e} \alpha^{M}_{i_1 f} \alpha^{M}_{i_1 g} \alpha^{M+1}_{i_2 h} \alpha'^{M}_{i_0 r} \alpha'^{M+1}_{j_1 s} \alpha'^{M+1}_{j_1 t} \alpha^{M+1}_{i_2 u}. \end{aligned}$$

It is clear that (2.7) will be satisfied if we choose

$$X_{i_0 i_1 j_1 i_2} = X'^{M}_{i_0} X^{M}_{i_1} X'^{M+1}_{j_1} X^{M+1}_{i_2},$$

where each of the X_i's on the RHS is a solution of the equation

$$\sum_i X_i \alpha_{ir} \alpha_{is} \propto \delta_{rs}. \tag{2.8}$$

The solution to (2.8) can be written as [27]

$$\frac{X_k}{X_N} = \frac{\alpha_{Nk}}{\alpha_{kN}},$$

where N is the maximum value of k. Since α's factorize as in (2.4), Xs do too. We write

$$X_{kl} = X_{kl}(a, a') = X_k(a') X_l(a) = X'_k X_l.$$

Thus we arrive at the final form of our monodromy-invariant correlation function

$$G(z,\bar{z}) = \sum_{i_0,i_1,j_1,i_2} X^{M}_{i_0 i_1} X^{M+1}_{j_1 i_2} I^{M}_{i_0 i_1} I^{M+1}_{i_1 i_2}(z) \overline{I^{M}_{i_0 j_1} I^{M+1}_{j_1 i_2}(z)}. \tag{2.9}$$

Since Xs (up to certain normalization constants) give the structure constants, we can already see that the *Neveu-Schwarz structure constants for* $(2, M)$ *will* *be given by certain products of the Virasoro structure constants for* $(1, M)$ *and* $(1, M+1)$. But, before turning to the structure constants, we would like to generalize the simple example that resulted in (2.9).

As a first step in the generalization, let us consider the correlation function (2.1). The relevant conformal blocks are of the form (2.2). An analysis similar to the one for $L = 2$ shows that only the terms in the correlation function of the form

$$I^{M}_{i_0 i_1} I^{M+1}_{i_1 i_2} \cdots I^{M+L-1}_{i_{L-1} i_L}(z) \overline{I^{M}_{i_0 j_1} I^{M+1}_{j_1 j_2} \cdots I^{M+L-1}_{j_{L-1} i_L}(z)}$$

are invariant under $z \to z e^{2\pi i}$. Since it is obvious that (2.5) does not depend on the particular values of m, x, n or M, the monodromy-invariant correlation function will be

$$G(z,\bar{z}) = \sum X^{M}_{i_0 i_1} X^{M+1}_{j_1 i_2} \cdots X^{M+L-1}_{j_{L-1} i_L} I^{M}_{i_0 i_1} \cdots I^{M+L-1}_{i_{L-1} i_L}(z) \overline{I^{M}_{i_0 j_1} \cdots I^{M+L-1}_{j_{L-1} i_L}(z)}. \tag{2.10}$$

We can generalize further and discuss asymmetrical correlation functions

$$G_a(z,\bar{z}) = \left\langle \prod_{A=1}^{4} \phi_{m_A n_A}(L, M)(z_A, \bar{z}_A) \right\rangle, \qquad m_A - n_A \in LZ.$$

Now Is, α's and Xs depend on 3 sets of parameters:

$$\begin{aligned} a &= 2\alpha_- \alpha_{m_1 n_1}, & b &= 2\alpha_- \alpha_{m_3 n_3}, & c &= 2\alpha_- \alpha_{m_2 n_2}, \\ a' &= 2\alpha_+ \alpha_{m_1 n_1}, & b' &= 2\alpha_+ \alpha_{m_3 n_3}, & c' &= 2\alpha_+ \alpha_{m_2 n_2}. \end{aligned}$$

It is straightforward to go over the arguments and convince oneself that there are no significant changes. For instance, α-matrices will now be the sums of an even number of the factors of the form

$$\sin[\pi(Aa + Bb + Cc + D\rho)],$$

where A, B, C and D are some integers. Obviously, (2.5) remains to hold.

Turning to the non-vacuum sectors, we want to calculate the 4-point function of $\phi_{mn}(L, M)$, where $n - m \in LZ \mp l$, $1 \le l \le L-1$. By (1.34) we know that the product $\phi_{1,l+1}(l, L-l)\phi_{mn}(L, M)$ can be expressed as various products of the

Virasoro fields. The construction of the 4-point functions of these products of the Virasoro fields proceeds as above and we conclude that the 4-point function of $\phi_{1,l+1}(l, L-l)\phi_{mn}(L, M)$ has the form (2.10). Furthermore, since there is no projection between $\phi_{1,l+1}(l, L-l)$ and $\phi_{mn}(L, M)$, the 4-point function of the product factorizes into the product of the 4-point function of $\phi_{1,l+1}(l, L-l)$ and 4-point function of $\phi_{mn}(L, M)$.

2.2 Fusion algebras and structure constants

In this section we will use the construction of the monodromy-invariant 4-point functions performed in the previous section to study the fusion algebras and the structure constants for higher level models. The discussion is going to be illustrative rather than exhaustive, since we discuss explicitly only the fusions of two vacuum sector fields.

L=2

The NS fields in question are constructed as follows:

$$\phi_{mn}(2, M) = \phi^{M}_{m,\frac{1}{2}(m+n)}\phi^{M+1}_{\frac{1}{2}(m+n),n}, \qquad n-m \in 2\mathbb{Z} \tag{2.11}$$

All the other combinations $\phi^{M}_{mp}\phi^{M+1}_{pn}$ belong to the descendants of $\phi_{mn}(2, M)$, with the dimension $\Delta_{mn}(2, M) + \frac{1}{2}(p - \frac{1}{2}(m+n))^2$. Since we have seen that the conformal blocks used in the previous section are projected products of $(1, M)$ and $(1, M+1)$ conformal blocks, the fusion algebra will follow the same recipe.

For the study of the fusion rules, it suffices to consider only the diagonal terms $(i_1 = j_1)$ in (2.9). To see this, remember that the non-diagonal terms can have left dimension equal or different by an integer from the right dimension. In the later case, the intermediate field is a descendant w.r.t. $T(z)$ or $T(\bar{z})$ of a field corresponding to a diagonal term. In the former case, the left and the right field are the two parts of a descendant w.r.t. $G(z)G(\bar{z})$, a field already represented by its diagonal terms. Such non-diagonal scalar fields appearing in the supersymmetric fusions are discussed in all the detail in App. A.

Taking the diagonal terms in (2.9) is equivalent to using the Virasoro fusion rules [3]:

$$\phi^M_{m_1n_1} \times \phi^M_{m_2n_2} = \sum_{k=|m_1-m_2|+1}^{\min(m_1+m_2-1,2(M+2)-m_1-m_2-1)} \times \sum_{l=|n_1-n_2|+1}^{\min(n_1+n_2-1,2(M+3)-n_1-n_2-1)} \phi^M_{kl}$$

(k and l advance in steps of two), for each of the fields in (2.11) and then imposing the projection by identifying the middle indices. The result is

$$\phi_{m_1n_1}(2,M) \times \phi_{m_2n_2}(2,M) = \sum_{p=|m_1-m_2|+1}^{\min(m_1+m_2-1,2(M+2)-m_1-m_2-1)} \times \sum_{q=|\frac{1}{2}(m_1+n_1)-\frac{1}{2}(m_2+n_2)|+1}^{\min(\frac{1}{2}(m_1+n_1+m_2+n_2)-1,2(M+3)-\frac{1}{2}(m_1+n_1+m_2+n_2)-1)} \times \sum_{r=|n_1-n_2|+1}^{\min(n_1+n_2-1,2(M+4)-n_1-n_2-1)} \phi^M_{pq}\phi^{M+1}_{qr}.$$

The remaining problem is identifying the products $\phi^M_{pq}\phi^{M+1}_{qr}$ with the superVirasoro fields. Firstly, note that

$$\begin{aligned} r-p &= |n_1-n_2| - |m_1-m_2| \bmod 2, \\ &= n_1 - m_1 - (n_2-m_2) \bmod 2 \in 2Z. \end{aligned}$$

Therefore, $\phi^M_{pq}\phi^{M+1}_{qr}$ stands for a NS field $\phi_{pr}(2,M)$ or its descendant. A little thought shows that the range of $\frac{1}{2}(p+r)$ is *within* the range of q, *i.e.* for every choice of p and r, there is a $q = q_{\min}$ that minimizes $\frac{1}{2}(q-\frac{1}{2}(p+r))^2$, and such a $q_{\min}$ differs from $\frac{1}{2}(p+r)$ by at most 1. If it does differ by 1, the dimension of $\phi^M_{pq_{\min}}\phi^{M+1}_{q_{\min}r}$ will be $\Delta_{pr}(2,M)+1/2$ and we interpret it as (a part of) the descendant $\phi^{II}_{pr}(2,M)$ of $\phi_{pr}(2,M)$ w.r.t. G. All the other $\phi^M_{pq}\phi^{M+1}_{qr}$ (with q differing by an even integer from $q_{\min}$) are parts of the descendants (relative to T) of $\phi^{II}_{pr}(2,M)$. On the other hand, if p and r are such that there exists a $q_{\min} = \frac{1}{2}(p+r)$, then $\phi^M_{p,\frac{1}{2}(p+r)}\phi^{M+1}_{\frac{1}{2}(p+r),r} = \phi_{pr}(2,M)$. Again, the remaining $\phi^M_{pq}\phi^{M+1}_{qr}$ are descendants relative to T of $\phi_{pr}(2,M)$. We conclude that the following NS fusion rules hold;

$$\begin{aligned}
\phi_{m_1n_1}(2,M)\times\phi_{m_2n_2}(2,M) = & \sum_{k=|m_1-m_2|+1}^{\min(m_1+m_2-1,2(M+2)-m_1-m_2-1)} \\
& \times \sum_{l=|n_1-n_2|+1}^{\min(n_1+n_2-1,2(M+4)-n_1-n_2-1)} \phi_{kl}^{(II)}(2,M), \\
m_i - n_i \in 2\mathrm{Z}, & \\
\phi_{kl}^{(II)} = & \begin{cases} \phi_{kl} & \text{if } k+l-|m_1+n_1-m_2-n_2| \in 4\mathrm{Z}+2, \\ \phi_{kl}^{II} & \text{if } k+l-|m_1+n_1-m_2-n_2| \in 4\mathrm{Z}, \end{cases}
\end{aligned} \tag{2.12}$$

in agreement with [46].

<u>L=3</u>

We study the fusions of two vacuum sector fields like

$$\phi_{mn}(3,M) = \phi^{M}_{m,\frac{1}{3}(n+2m)}\phi^{M+1}_{\frac{1}{3}(n+2m),\frac{1}{3}(2n+m)}\phi^{M+2}_{\frac{1}{3}(2n+m),n}, \quad n-m\in 3\mathrm{Z}.$$

By the same sort of arguments as for $L=2$, we can immediately write

$$\begin{aligned}
\phi_{m_1n_1}(3,M)\times\phi_{m_2n_2}(3,M) = & \sum_{k_0=|m_1-m_2|+1}^{\min(m_1+m_2-1,2(M+2)-m_1-m_2-1)} \\
& \times \sum_{k_1=|\frac{1}{3}(n_1+2m_1)-\frac{1}{3}(n_2+2m_2)|+1}^{\min(\frac{1}{3}(n_1+2m_1+n_2+2m_2)-1,2(M+3)-\frac{1}{3}(n_1+2m_1+n_2+2m_2)-1)} \\
& \times \sum_{k_2=|\frac{1}{3}(2n_1+m_1)-\frac{1}{3}(2n_2+m_2)|+1}^{\min(\frac{1}{3}(2n_1+m_1+2n_2+m_2)-1,2(M+4)-\frac{1}{3}(2n_1+m_1+2n_2+m_2)-1)} \\
& \times \sum_{k_3=|n_1-n_2|+1}^{\min(n_1+n_2-1,2(M+5)-n_1-n_2-1)} \phi^{M}_{k_0k_1}\phi^{M+1}_{k_1k_2}\phi^{M+2}_{k_2k_3}.
\end{aligned} \tag{2.13}$$

In contrast to the $L=2$ case, here

$$k_3 - k_0 = n_1 - m_1 - (n_2 - m_2) \bmod 2 \tag{2.14}$$

is *not* any more in 3Z in general. We encounter a new feature; in the fusion of two vacuum sector fields (1.31) appears a non-vacuum sector field (1.34), or a descendant of such a field.

The field with $k_3 - k_0 \notin 3Z$ cannot be a primary field. To see this think of the $V(1,2)$ (cf. LHS of (1.7)) part of the fusion rules. If the field was primary those fusion rules would imply that $\phi_{12}(1,2)$ can arise in the fusion of the identity fields, an impossibility. We will show that all such fields are descendants of non-vacuum sector fields w.r.t. the current $A_{7/5}$.

We want to establish a precise correspondence between the products of the Virasoro fields appearing in (2.13) and the $L = 3$ fields. Start by checking again that the range of k_1 includes the range of $\frac{1}{3}(k_3 + 2k_0)$ and that the range of k_2 includes the range of $\frac{1}{3}(2k_3 + k_0)$. Then determine k_1, k_2 which minimize the dimension of $\Phi \equiv \phi^M_{k_0 k_1} \phi^{M+1}_{k_1 k_2} \phi^{M+2}_{k_2 k_3}$ for a given pair of k_0, k_3. The only obstacle to minimizing $\Delta(\Phi)$ might come from the parities of k_1 and k_2.

Since $k_i = |a + ix| + 1 \bmod 2$, where $a = m_1 - m_2$, $x = \frac{1}{3}(n_1 - m_1 - (n_2 - m_2))$, k_0 and k_1 have the same parity as k_2 and k_3, respectively. This fact is general; for arbitrary L, $k_0, \ldots, k_L$ split into two equivalence classes relative to parity; k_i, $i = 0 \bmod 2$ and k_i, $i = 1 \bmod 2$. This results in two kinds of behaviour of the fusion rules, relative to L. For L even k_L and k_0 are in the same class, $k_L - k_0 \in 2Z$, and the knowledge of k_0 and k_L is not enough to determine all the relative parities of the k_i's, whereas for L odd the opposite is true. In other words, for L odd all sectors can in general appear in the fusion of two vacuum sector fields, whereas for L even only the sectors with even l appear.

Going back to $L = 3$, assume $k_3 - k_0 = 3K$. Then, setting $k_i = k_0 + iK$ minimizes $\Delta(\Phi)$. We obtain the primary field

$$\phi_{k_0 k_3}(3, M) = \phi^M_{k_0, \frac{1}{3}(k_3 + 2k_0)} \phi^{M+1}_{\frac{1}{3}(k_3 + 2k_0), \frac{1}{3}(2k_3 + k_0)} \phi^{M+2}_{\frac{1}{3}(2k_3 + k_0), k_3}.$$

Again, the other choices of k_1 and k_2 give the descendants of $\phi_{k_0 k_3}(3, M)$. We turn now to the case $k_3 - k_0 = 3K \pm 1$. Note that if K is odd all k_i's have the same parity, whereas if K is even $k_0 = k_2 \bmod 2 \neq k_1 = k_3 \bmod 2$. In both cases $\Delta^c_{\min}$, the minimum dimension consistent with the parities, is obtained with the following three sets of choices for $k_{i+1} - k_i$, $i = 0, 1, 2$:

$k_1 - k_0$	$k_2 - k_1$	$k_3 - k_2$
$K \pm 1$	$K \pm 1$	$K \mp 1$
$K \pm 1$	$K \mp 1$	$K \pm 1$
$K \mp 1$	$K \pm 1$	$K \pm 1$

If there was no restriction on k_1 and k_2, the minimum dimension $\Delta_{\min}$ with given k_0 and k_3, $k_3 - k_0 = 3K \pm 1$ is obtained with

$$\begin{array}{ccc} k_1 - k_0 & k_2 - k_1 & k_3 - k_2 \\ K & K & K \pm 1 \\ K & K \pm 1 & K \\ K \pm 1 & K & K \end{array}$$

As discussed in sec. 1.4,

$$\Delta_{\min} = \Delta_{k_0 k_3}(3, M) + \Delta_{12}(1,2) = \Delta_{k_0 k_3}(3, M) + \frac{1}{10}.$$

Since

$$\Delta^c_{\min} = \Delta_{\min} + \frac{1}{2} = \Delta_{k_0 k_3}(3, M) + \frac{3}{5},$$

we see that the field belonging to the class $[\phi_{k_0 k_3}(3, M)]$, with $k_3 - k_0 \in 3\mathrm{Z} \pm 1$, appearing in the fusion of two vacuum sector fields, is a descendant of the primary field $\phi_{k_0 k_3}(3, M)$ w.r.t. $A_{7/5}$. In conclusion, we have shown that the following $L = 3$ fusion rules hold:

$$\phi_{m_1 n_1}(3, M) \times \phi_{m_2 n_2}(3, M) = \sum_{k=|m_1 - m_2|+1}^{\min(m_1+m_2-1, 2(M+2)-m_1-m_2-1)} \times \sum_{l=|n_1-n_2|+1}^{\min(n_1+n_2-1, 2(M+5)-n_1-n_2-1)} \phi^{(d)}_{kl}(3, M),$$

where k and l advance in steps of 2 and $\phi^{(d)}_{kl}(3, M)$ is the primary field $\phi_{kl}(3, M)$ if $k - l \in 3\mathrm{Z}$ and its descendant w.r.t. $A_{7/5}$ otherwise.

As an illustration of the foregoing somewhat dry discussion, let us look at the fusion of $\phi_{14}(3, M) = \phi^M_{12} \phi^{M+1}_{23} \phi^{M+2}_{34}$ with itself. The terms with $k_0 = 1$ and $k_3 = 3$ are

$$\begin{array}{lll} \phi^M_{11} \phi^{M+1}_{11} \phi^{M+2}_{13}, & \phi^M_{11} \phi^{M+1}_{13} \phi^{M+2}_{33}, & \phi^M_{11} \phi^{M+1}_{15} \phi^{M+2}_{53}, \\ \phi^M_{13} \phi^{M+1}_{31} \phi^{M+2}_{13}, & \phi^M_{13} \phi^{M+1}_{33} \phi^{M+2}_{33}, & \phi^M_{13} \phi^{M+1}_{35} \phi^{M+2}_{53}, \end{array}$$

Here, $k_3 - k_0 = 3K - 1$, $K = 1$. Therefore, the dimension would be minimized by $\phi^M_{11} \phi^{M+1}_{12} \phi^{M+2}_{23}$, $\phi^M_{12} \phi^{M+1}_{22} \phi^{M+2}_{23}$ and $\phi^M_{12} \phi^{M+1}_{23} \phi^{M+2}_{33}$. Since these terms cannot appear due to the wrong k_1 and/or k_2 parity, the terms with the lowest dimension are $\phi_{11}^{M}\phi_{11}^{M+1}\phi_{13}^{M+2}$, $\phi_{11}^{M}\phi_{13}^{M+1}\phi_{33}^{M+2}$ and $\phi_{13}^{M}\phi_{33}^{M+1}\phi_{33}^{M+2}$, all with dimension higher by 1/2. These three terms form the descendant $A_{7/5}\phi_{13}(3,M)$. The remaining terms have dimensions higher than $A_{7/5}\phi_{13}(3,M)$ by a multiple of 2 and can be identified with its descendants relative to the energy-momentum tensor.

L=4

We can immediately write down the analogue of (2.13):

$$\begin{aligned}\phi_{m_1n_1}(4,M)\times\phi_{m_2n_2}(4,M) = & \sum_{k_0=|m_1-m_2|+1}^{\min(m_1+m_2-1,2(M+2)-m_1-m_2-1)} \sum_{k_1}\sum_{k_2}\sum_{k_3} \\ \times & \sum_{k_4=|n_1-n_2|+1}^{\min(n_1+n_2-1,2(M+6)-n_1-n_2-1)} \phi_{k_0k_1}^{M}\phi_{k_1k_2}^{M+1}\phi_{k_2k_3}^{M+2}\phi_{k_3k_4}^{M+3}.\end{aligned} \tag{2.15}$$

One can distinguish the following cases:

a) $k_4-k_0=4K$ and K and k_1-k_0 have the same parity; $k_i=k_0+iK$ and the field from $[\phi_{k_0k_4}(4,M)]$ appearing in the fusion is the primary field $\phi_{k_0k_4}(4,M)$.

b) $k_4-k_0=4K+2$; $\Delta_{\min}$ is obtained, for example, by

$$\begin{array}{cccc} k_1-k_0 & k_2-k_1 & k_3-k_2 & k_4-k_3 \\ K & K & K+1 & K+1 \end{array}$$

whereas $\Delta^c_{\min}$ is obtained, for example, by

$$\begin{array}{cccc} k_1-k_0 & k_2-k_1 & k_3-k_2 & k_4-k_3 \\ K & K & K & K+2 \end{array}$$

or

$$\begin{array}{cccc} k_1-k_0 & k_2-k_1 & k_3-k_2 & k_4-k_3 \\ K+1 & K+1 & K+1 & K-1 \end{array}$$

depending on the relative parity of K and k_1-k_0. Again,

$$\Delta^c_{\min}=\Delta_{\min}+\frac{1}{2}=\Delta_{k_0k_4}(4,M)+\Delta_{13}(2,2)+\frac{1}{2}=\Delta_{k_0k_4}(4,M)+\frac{2}{3},$$

and we conclude that the field from $[\phi_{k_0k_4}(4,M)]$ appearing in the fusion is the descendant $A_{4/3}\phi_{k_0k_4}(4,M)$.

c) $k_4 - k_0 = 4K$, but K and $k_1 - k_0$ do *not* have the same parity; $\Delta^c_{\rm min}$ is obtained from, for example,

$$\begin{array}{cccc} k_1 - k_0 & k_2 - k_1 & k_3 - k_2 & k_4 - k_3 \\ K+1 & K-1 & K+1 & K-1 \end{array}$$

Since $\Delta^c_{\rm min} = \Delta_{k_0 k_4}(4, M) + 1$, we conclude that the field from $[\phi_{k_0 k_4}(4, M)]$ appearing in the fusion is the "double" descendant $A^\dagger_{4/3}[A_{4/3}(\phi_{k_0 k_4}(4, M))]$.

In summary, the fusion rules of two $L = 4$ vacuum sector fields are

$$\phi_{m_1 n_1}(4, M) \times \phi_{m_2 n_2}(4, M) = \sum_{k=|m_1-m_2|+1}^{\min(m_1+m_2-1, 2(M+2)-m_1-m_2-1)} \times \sum_{l=|n_1-n_2|+1}^{\min(n_1+n_2-1, 2(M+6)-n_1-n_2-1)} \phi^{(d)}_{kl}(4, M),$$

where

$$\phi^{(d)}_{kl}(4, M) = \begin{cases} \phi_{kl}(4, M) & \text{if } l - k = 4K \text{ and } \mathcal{K} \text{ is even,} \\ A^{(\dagger)}_{4/3}\phi_{kl}(4, M) & \text{if } l - k = 4K + 2, \\ A_{4/3}A^\dagger_{4/3}\phi_{kl}(4, M) & \text{if } l - k = 4K \text{ and } \mathcal{K} \text{ is odd,} \end{cases}$$

where $K \in \mathsf{Z}$ and $\mathcal{K} \equiv K - \frac{1}{4}[|n_1 + 3m_1 - (n_2 + 3m_2)| - |m_1 - m_2|]$.

L=5

This is where the lack of more detailed knowledge of the extended current algebra for $L \geq 5$ catches up with us. Let us just briefly illustrate what is happening. As a result of the fusion we obtain a product of 5 Virasoro fields

$$\phi^M_{k_0 k_1}\phi^{M+1}_{k_1 k_2}\phi^{M+2}_{k_2 k_3}\phi^{M+3}_{k_3 k_4}\phi^{M+4}_{k_4 k_5}. \tag{2.16}$$

As long as $k_0 - k_5 = 5K$, we obtain the primary field $\phi_{k_0 k_5}(5, M)$ by setting $k_i = k_0 + iK$. What happens if, for example, $k_0 = 1$ and $k_5 = 3$? Such fields should belong to the class $[\phi_{13}(5, M)]$. By (1.34), we know that

$$\Delta(\phi_{13}(2,3)\phi_{13}(5, M)) = \Delta(\phi^M_{12}\phi^{M+1}_{23}\phi^{M+2}_{33}\phi^{M+3}_{33}\phi^{M+4}_{33}).$$

Since a product like (2.16), obtained from the fusion of two vacuum sector fields, has $k_0 = k_2 = k_4 \bmod 2$ and $k_1 = k_3 = k_5 \bmod 2$, the lowest dimension descendant of $\phi_{13}(4,M)$ that can appear is

$$\phi_{13}^{M}\phi_{33}^{M+1}\phi_{33}^{M+2}\phi_{33}^{M+3}\phi_{33}^{M+4}, \tag{2.17}$$

with the dimension

$$\Delta_{13}(5,M)+\Delta_{13}(2,3)+\frac{1}{2}=\Delta_{13}(5,M)+\frac{5}{7}.$$

We interpret (2.17) as a (part of the) descendant of $\phi_{13}(5,M)$ w.r.t. $A_{\frac{L+4}{L+2}=\frac{9}{7}}$ with dimension $\frac{L}{L+2}=\frac{5}{7}$ greater than $\Delta_{13}(5,M)$. This is the field used in [49] to perturb the conformal Lagrangian. Finally, the problems begin if, for example, $k_0=1$ but $k_5=2$. Such a field should belong to $[\phi_{12}(5,M)]$. Again, by (1.34) we know that

$$\Delta(\phi_{12}(1,4)\phi_{12}(5,M))=\Delta(\phi_{12}^{M}\phi_{22}^{M+1}\phi_{22}^{M+2}\phi_{22}^{M+3}\phi_{22}^{M+4}).$$

Again, because of parities of k_i's coming from the fusion rules, the lowest dimension descendant of $\phi_{12}(5,M)$ that can appear is

$$\phi_{12}^{M}\phi_{21}^{M+1}\phi_{12}^{M+2}\phi_{21}^{M+3}\phi_{12}^{M+4}, \tag{2.18}$$

with the dimension

$$\Delta_{12}(5,M)+\Delta_{12}(1,4)+\frac{1}{2}+\frac{1}{2}=\Delta_{12}(5,M)+\frac{8}{7}.$$

One should probably interpret (2.18) as a descendant of $\phi_{12}(5,M)$ w.r.t. some new current B. One could probably also extend the present line of argument and obtain the dimension and the analytic properties of B, but we will refrain from trying to do it here and leave it for future work.

Arbitrary L

In conclusion, we can write the general vacuum-sector fusion rules, for general L, in the following form:

$$\phi_{m_1n_1}(L,M)\times\phi_{m_2n_2}(L,M)=\sum_{k=|m_1-m_2|+1}^{\min(m_1+m_2-1,2(M+2)-m_1-m_2-1)}\times\sum_{l=|n_1-n_2|+1}^{\min(n_1+n_2-1,2(M+L+2)-n_1-n_2-1)}\phi_{kl}^{(d)}(L,M),$$

where $m_i-n_i\in LZ$, k and l advance in steps of 2, and $\phi_{kl}^{(d)}(L,M)$ is the primary field $\phi_{kl}(L,M)$ (if $k-l=LK$, $K\in Z$ and $K-\frac{1}{L}[|n_1+(L-1)m_1-(n_2+(L-$

$1)m_2)| - |m_1 - m_2|] \in 2Z)$, and its descendant w.r.t. $A_{\frac{L+4}{L+2}}$, one of the additional currents appearing for $L \geq 5$, or some product of those currents otherwise.

Structure constants

In the last section we have exhaustively demonstrated that, as long as we stay within the part of the fusion rules that maps the primary vacuum sector fields into the primary vacuum sector fields, we have full control, for any L. We will use that control now to obtain explicit expressions for the structure constants connecting 3 vacuum sector primary fields, for arbitrary L.

The structure constants appear as a limit of monodromy-invariant 4-point functions. The conformal blocks whose limits one is taking are nothing but the products of the Virasoro conformal blocks. Since for the primary fields from the vacuum sector the mapping from the L-level fields into the products of the Virasoro fields remains strictly one-to-one and does not involve any non-trivial fields from $V(1, L-1)$ (cf. (1.7)) and/or additional currents, it should be obvious that the L-level vacuum sector structure constants are given by the products of the Virasoro structure constants.

Explicitly, for the fields

$$
\begin{gathered}
\phi_{m_a n_a}(L, M) = \prod_{i=0}^{L-1} \phi_{k_i^a k_{i+1}^a}(1, M+i), \\
n_a - m_a \in LZ, \quad a = 1, 2, 3, \qquad k_i^a = \frac{i n_a + (L-i) m_a}{L}, \\
(n_3 - m_3) - [|n_1 + (L-1)m_1 - (n_2 + (L-1)m_2)| - |m_1 - m_2|] \in 2LZ,
\end{gathered}
$$

the structure constants are given by

$$
C_{(m_1 n_1)(m_2 n_2)(m_3 n_3)}(L, M) = \prod_{i=0}^{L-1} C_{(k_i^1 k_{i+1}^1)(k_i^2 k_{i+1}^2)(k_i^3 k_{i+1}^3)}(1, M+i). \tag{2.19}
$$

One could consider (2.19) as one of the main results obtained in the present thesis by an application of the techniques developed here.

As a simple check and illustration of (2.19), let us set $L = 2$ and reexpress a (known) superVirasoro structure constant as a product of Virasoro ones. For example, take

$$\begin{aligned}\phi_{33}(2,M) &= \phi_{33}^{M}\phi_{33}^{M+1},\\ \phi_{22}(2,M) &= \phi_{22}^{M}\phi_{22}^{M+1}.\end{aligned} \tag{2.20}$$

As it is well-known [46, 47, 26], and as it also follows from (2.12), the following fusion rules hold:

$$\begin{aligned}\phi_{22}(2,M)\phi_{22}(2,M) = \phi_{11}(2,M) &+ C_{(22)(22)(13)'}(2,M)\phi_{13}^{II}(2,M)\\ &+ C_{(22)(22)(31)'}(2,M)\phi_{31}^{II}(2,M) + C_{(22)(22)(33)}(2,M)\phi_{33}(2,M),\end{aligned}$$

where* [47, 26]

$$\begin{aligned}C_{(22)(22)(33)}(2,M) = \Big\{\gamma\left(\frac{M}{M+2}\right)^2\gamma\left(\frac{1}{M+2}\right)\gamma\left(\frac{3}{M+2}\right)\\ \gamma\left(\frac{2}{M+4}\right)^2\gamma\left(\frac{M+3}{M+4}\right)\gamma\left(\frac{M+1}{M+4}\right)\Big\}^{\frac{1}{2}}.\end{aligned}$$

The construction (2.20) gives

$$\phi_{22}^{M}\phi_{22}^{M+1}\times\phi_{22}^{M}\phi_{22}^{M+1} = \phi_{11}^{M}\phi_{11}^{M+1} + C_{(22)(22)(33)}^{M}C_{(22)(22)(33)}^{M+1}\phi_{33}^{M}\phi_{33}^{M+1} + \cdots.$$

Using the expression [27]

$$\begin{aligned}C_{(22)(22)(33)}^{M} = \Big\{\gamma\left(\frac{M}{M+2}\right)^2\gamma\left(\frac{1}{M+2}\right)\gamma\left(\frac{3}{M+2}\right)\\ \times\gamma\left(\frac{2}{M+3}\right)^2\gamma\left(\frac{M+2}{M+3}\right)\gamma\left(\frac{M}{M+3}\right)\Big\}^{\frac{1}{2}},\end{aligned}$$

it is easy to show that

$$C_{(22)(22)(33)}^{M}C_{(22)(22)(33)}^{M+1} = C_{(22)(22)(33)}(2,M),$$

as predicted by (2.19).

* We remind the reader again that in our notation $M = 1,2,\ldots$.

2.3 Moduli of the projected products

There are two remarkable properties of the projected tensor products, related to their possible interpretation in terms of some target geometry. The two properties are:

a) each model $V^P_{L,M} \equiv \mathsf{P}\Big(\otimes_{i=1}^{L-1} V(M+i)\Big)$ contains $h(1) = \frac{1}{12}L(L-1)(L-2)(L-3)$ *integrable marginal operators* $\equiv$ *moduli* $\equiv$ *"massless states"*. This provides $V^P_{L,M}$ with the structure of *an h-dimensional space of conformal models* with $c^P(L,M) = L - \frac{6L}{(M+2)(M+L+2)}$, in analogy with the line ($h(1)=1$) of $c=1$ models.

b) the models $V^P_{L,M}$ obey higher supersymmetries, in particular $N=1$ and $N=2$.

Start by counting the fields ψ^L_k of dimension $\Delta = 1/2$ in $V^P_{L,M}$. They can be written in terms of the Virasoro fields $\phi_{12}(M+i) \equiv 12$, $\phi_{22}(M+k) \equiv 22$ and $\phi_{21}(M+s) \equiv 21$, for some $0 \le i < k < s \le L-1$, using the basic elements 12 21 and 12 22 ... 22 21, for example:

$$
\begin{array}{cccccccc}
M & M+1 & M+2 & \dots & M+i & M+i+1 & \dots & M+L-1 \\
12 & 21 & 11 & \dots & 11 & 11 & \dots & 11 \\
\vdots & & & & & & & \\
11 & 11 & 11 & \dots & 12 & 21 & \dots & 11 \\
12 & 22 & 21 & \dots & 11 & 11 & \dots & 11 \\
\vdots & & & & & & & \\
12 & 22 & 22 & \dots & 22 & 22 & \dots & 21
\end{array}
$$

Their total number is $h(1/2) = \frac{1}{2}L(L-1)$. The fields J^L_k of dimension 1 can be obtained from the ψ^L_k's applying the following rules:

1) joining two ψ^L_k's of appropriate lenghts : $J^L_k = \psi^{L-p}_{k_1}\psi^p_{k_2}$,

2) replacing a pair 22 22 in ψ^L_k with the pair 23 32:

$$12 \;\dots\; 22 \;\; 22 \;\dots\; 21 \to 12 \;\dots\; 23 \;\; 32 \;\dots\; 21$$

3) inserting 33 in a J^L_k from 2) according to the rule

$$\dots \;\; 23 \;\; 32 \;\; \dots \to \dots \;\; 23 \;\; 33 \;\; 32 \;\; \dots$$

The result of the counting is

$$h(1) = \sum_{l=1}^{L-3}(L-l-2)(l+1)l = \frac{1}{12}L(L-1)(L-2)(L-3).$$

The candidates for the supercurrents G_k^L, *i.e.* fields with $\Delta = 3/2$ can be constructed by the following rules:

i) as appropriate combinations of derivatives of ψ_k^L's, see for example (1.20),

ii) all the products $G_k^L = \psi_{k_1}^{L-p} J_{k_2}^p$,

iii) from J_k^L by replacing a pair like 22 22 by 23 32 or a pair like 33 33 by 32 23 or 34 43,

iv) inserting the field kk, $k = 1,2,3,4$:

$$\ldots \quad lk \quad ks \quad \ldots \rightarrow \ldots \quad lk \quad kk \quad ks \quad \ldots$$

in the G_k^L's from ii) and iii) or in the new basic fields appearing for $L = 3$: $G_1^3 = 13\ 32\ 21$ and $G_2^3 = 12\ 23\ 31$.

One may continue the above constructions thus enumerating the fields of dimensions 2, 5/2, 3, *etc.*.

Consider now the OPEs of J_k^L's and G_m^L's, crucial for establishing the announced geometrical properties of $V_{L,M}^P$. Applying the general fusion rules of sec.2.2 to J_k^L's we obtain the following two kinds of OPEs:

$$\begin{aligned}
&(\mathrm{a}) \qquad J_i^L(1)J_j^L(2) = \frac{\delta_{ij}}{z_{12}^2} + \text{reg.terms}, \\
&(\mathrm{b}) \qquad J_k^L(1)J_l^L(2) = \frac{\delta_{kl}}{z_{12}^2} + \frac{A_{klm}}{z_{12}^{3/2}}\psi_m^L(2) + \frac{B_{klm}}{z_{12}^{1/2}}G_m^L(2) + \text{reg.terms}, \\
&\qquad\qquad A_{kkm} = 0 = B_{kkm}.
\end{aligned}$$

For example, the only two marginal operators for $L = 4$, $J_1^4 = 12\ 21\ 12\ 21$ and $J_2^4 = 12\ 23\ 32\ 21$ close an algebra of type (a). For $L = 5$, $h(1) = 10$, and one can find 5 pairs of J_i^5's with the algebra of type (a). The OPEs among the J_i^5's from different pairs are of type (b). For example:

$$\begin{array}{lllllll} J_1^5 & = & 12 & 21 & 12 & 22 & 21 \\ J_2^5 & = & 12 & 23 & 32 & 22 & 21 \end{array} \qquad \begin{array}{lllllll} J_3^5 & = & 12 & 21 & 11 & 12 & 21 \\ J_4^5 & = & 12 & 23 & 33 & 32 & 21 \end{array}$$

then (J_1^5, J_2^5) and (J_3^5, J_4^5) combine in pairs with the discussed properties.

According to the general criteria [50, 51], all the operators $J_k^L = J_k^L(z)J_k^L(\bar{z})$ of dimension $(\Delta, \bar{\Delta}) = (1,1)$ are marginal operators to the first order in the perturbation theory. This means that while deforming the $V_{L,M}^P$ model by adding the marginal perturbations

$$\sum_{k=1}^{h(1)} g_k \int d^2 z J_k^L(z)\bar{J}_k^L(\bar{z}),$$

the central charge $c^P(L,M)$ does not change whereas the dimensions of the fields $\phi_\alpha \in V_{L,M}^P$ change continuously by

$$\delta\Delta_\alpha = \sum_k C_{k\alpha\alpha}\delta g_k = \delta\bar{\Delta}_\alpha,$$

where $C_{k\alpha\alpha}$ is the structure constant $\langle J_k^L \phi_\alpha \phi_\alpha \rangle$ (see chap. 6). Thus we obtain a $h(1)$-dimensional space of conformal models with, in general, fractional central charge $c^P(L,M)$. The rational CFTs correspond to the specific points $\{g_k^2\} = \{n_k/m_k, \quad n_k, m_k \in Z_+\}$. One can consider this construction as a generalization of the well-known lines ($h(1) = 1$) of CFTs with $c = 1$ and $c = 3/2$ [50, 52, 45].

Among all marginal operators $J_k^L = J_k^L(z)J_k^L(\bar{z})$, the ones whose OPEs with all the other J_i^L's are of the type (a) remain integrable to all orders of the perturbation theory. The question of the integrability to higher orders of the remaining marginal operators (or of their specific linear combinations) requires further analysis of certain integrals of their 4-, 5- and higher point functions appearing in the higher order contributions to the central charge and the anomalous dimensions [51].

In the case of integer* central charge c (for example, $L = 25$, $5M = L$, $c^P(L,M) = 24$), the natural attempt is to find a c-dimensional$^\diamond$ space-time interpretation of $V_{L,M}^P$ with $h(1)$-dimensional moduli space. A necessary condition for $V_{L,M}^P$ to represent an acceptable superstring vacuum is for it to have $N = 2$ world-sheet supersymmetry (*i.e.* $N = 1$ space-time supersymmetry). With a large number of fields of dimensions 3/2 and 1 (G_k^L and J_k^L) in hand, it is not

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$\star$ The non-integer c (say $10\frac{10}{11}$) can be interesting for the construction of (dilaton) time-dependent string vacua in the spirit of [53].

$\diamond$ For $N = 1$ space-time supersymmetry, the space-time is $2c/3$-dimensional.

difficult to find combinations that close the $N = 2$ algebra. In the simplest case ($L = 4$) the fields

$$\begin{aligned} G_1 &= 13\ 32\ 22\ 21 \qquad G_2 = 12\ 22\ 23\ 31, \\ J &= J_1 + J_2, \\ J_1 &= 12\ 21\ 12\ 21, \qquad J_2 = 12\ 23\ 32\ 21, \end{aligned}$$

and the total stress-energy tensor $T = \sum_{i=1}^{4} T_i$ indeed close the $N = 2$ OPE algebra:

$$\begin{aligned} G_1(1)G_1(2) = G_2(1)G_2(2) &= \frac{c_0}{z_{12}^3} + \frac{1}{z_{12}} T(2) + \cdots, \\ G_1(1)G_2(2) &= \frac{1}{z_{12}} J(2) + \cdots, \\ J(1)J(2) &= \frac{1}{z_{12}^2} + \cdots, \\ J(1)G_1(2) &= \frac{1}{z_{12}} G_2(2) + \cdots. \end{aligned}$$

One can easily generalize this construction to arbitrary L by considering

$$\begin{aligned} G_1 &= \underbrace{11\ \ldots\ 11}_{L-4}\ 13\ 32\ 22\ 21, \\ G_2 &= 11\ \ldots\ 11\ 12\ 22\ 23\ 31, \\ J_1 &= 11\ \ldots\ 11\ 12\ 21\ 12\ 21, \\ J_2 &= 11\ \ldots\ 11\ 12\ 23\ 32\ 21. \end{aligned}$$

However, since it is possible that the $N = 2$ algebra with $c_0 = c$ includes many more G_k^L's, the establishment of the $N = 2$ structure of $V_{L,M}^P$ requires further analysis. An important open problem in this case is how the ψ_k^L's and part of the J_k^L's can be organized in the $N = 2$ multiplets of the massless states.

In closing, let us point out that the purpose of the present far-from-conclusive discussion of the geometry of $V_{L,M}^P$ was to demonstrate that the 2D conformal data of the projected tensor product models provide interesting possibilities for the various geometrical constructions useful in the description of the multicritical behaviour of the statistical systems or for the constructions of the superstring vacua.

Chapter 3

Minimal Models on Hyperelliptic Surfaces

3.1 Construction of the models

In this Chapter, we describe the conformal minimal models on a restricted class of surfaces which can be represented as a double (in general n-) covering of the branched sphere. Such surfaces are known as hyperelliptic (in general Z_n-) surfaces. The strategy we will use is to reduce the genus-g problem to the corresponding $g = 0$ problem.

For $g = 1$ and $g = 2$ the Z_2-surfaces exhaust the entire moduli space. Therefore, the hyperelliptic construction of the minimal models that we present in this Chapter (Coulomb gas, partition functions *etc.*), possesses all the specific features of the models on arbitrary Riemann surfaces. As such it can be used as a starting point for the further generalizations.

Recently, the hyperelliptic formalism was widely explored in the two-loop (super)string calculations, [22, 55], and in the description of the orbifolds ,[22,56], and the critical Ashkin-Teller model [57]. Our discussion of the conformal minimal models is based on an appropriate generalization of the methods of [22] and [57].

The central idea of this method is that the topological properties of the Z_2-surface $X_g^{(2)}$ of genus g:

$$y^2(z) = \prod_{i=1}^{2g+2} (z - a_i), \qquad y^{(k)} = e^{i\pi k} \sqrt{\prod_{i=1}^{2g+2} (z - a_i)}, \quad k = 0, 1 \qquad (3.1)$$

are simulated by the specific vertices $V(a_i)$ (called branching operators) placed at the points a_i of the branched sphere. Three of these points can be fixed by an $SL(2, \mathrm{R})$ transformation, and the remaining $2g - 1$ points a_i play the role of the moduli of the corresponding surface. The main advantage of this approach is that the calculation of the n-point correlation functions of the conformal fields $\Psi_i(y)$ on $X_g^{(2)}$ reduces to the problem of the construction of the $n + 2g + 2$-point function

$$\langle \prod_{i=1}^{n} \Psi_i^{(k)}(z_i) \prod_{j=1}^{2g+2} V(a_j) \rangle \qquad k = 0, 1$$

on the sphere [22]. Under the hyperelliptic map (3.1) the field $\Psi(y)$ on $X_g^{(2)}$ maps into two fields

$$\Psi^{(k)}(z) = \left(\frac{dy^{(k)}}{dz} \right)^{\Delta} \Psi(y^{(k)}),$$

living on the corresponding Riemann sheets ($\equiv$branched spheres). These fields have to satisfy, on top of the usual conformal properties (encoded in their OPE's with the stress-energy tensors $T^{(k)}(z)$) an additional monodromy property [22]

$$\Pi_a \Psi^{(0)}(z) = e^{-2\pi i \Delta} \Psi^{(1)}(z), \qquad \Pi_a \Psi^{(1)}(z) = \Psi^{(0)}, \qquad \Pi_a^2 = e^{-2\pi i \Delta}, \qquad (3.2)$$

around each branch point a_i. (The general form of (3.2) is, say, $\Pi_a \Psi^{(0)}(z, \bar{z}) = e^{-2\pi i (\Delta - \bar{\Delta})} \Psi^{(1)}(z, \bar{z})$, but we shall systematically omit the $\bar{z}$ dependence.) These boundary conditions are obtained by the analytic continuation of $y^{(k)}$ through the cut to the next sheet $y^{(k+1)}$:

$$\Pi_a y^{(k)} = y^{(k+1)}, \qquad \Pi_a z = z e^{2\pi i} + a.$$

Similarly to the Ramond (spin) fields, the geometrical vertices $V(a_i)$ appear in the branching points a_i in order to generate (by their OPE's with the Π_a-diagonal combinations of $\Psi^{(k)}$'s) the monodromies (3.2).

The problem we address is the following: for each conformal minimal model,[3], given by the central charge $\hat{c}_p = 1 - 6/p(p+1)$, $p = 3, 4, \ldots$ and finite set of primary fields $\{\Psi_i(y)\}$ defined on $X_g^{(2)}$, find the relevant conformal model $\left\{\Psi_i^{(k)}(z), V(a_i)\right\}$ on the sphere. The main idea of this paper is to consider the algebra $\mathcal{V}_{br}$ on the branched sphere, obtained by the hyperelliptic map (3.1) from the stress-energy tensor $T(y)$ ("Virasoro") algebra on $X_g^{(2)}$ [19]. As we shall see, the $\{\Psi, V\}$ models that we are looking for are the minimal models of $\mathcal{V}_{br}$.

To obtain $\mathcal{V}_{br}$, note that (3.1) maps $T(y)$ into two fields $T^{(k)}(z)$, where $T^{(k)}(z_1)T^{(k)}(z_2)$ is the usual (compactified plane) OPE [3], and $T^{(k)}(z_1)T^{(k')}(z_2) \sim$ finite for $k \neq k'$. In addition, $T^{(k)}(z)$ obey the monodromy condition

$$\Pi_a T^{(k)} = T^{(k+1)}, \qquad \Pi_a^2 = 1. \tag{3.3}$$

We see that $\mathcal{V}_{br}$ is the doubled Virasoro algebra with specific Z_2 boundary conditions (3.3) for the spin 2 currents $T^{(k)}(z)$.

It is convenient to analyze the properties of this algebra and its representations in the Z_2-diagonal basis

$$\begin{aligned} T &= T^{(0)} + T^{(1)}, \qquad & \Pi_a T &= T, \\ T^\dagger &= T^{(0)} - T^{(1)}, \qquad & \Pi_a T^\dagger &= -T^\dagger. \end{aligned} \tag{3.4}$$

i.e. T and $T^\dagger$ carry Z_2-charges 0 and 1. The corresponding OPE algebra describing the symmetries of the diagonalised sheets is a simple consequence of the $T^{(k)}$ OPE's:

$$\begin{aligned} T(1)T(2) &= \frac{c}{2z_{12}^4} + \frac{2}{z_{12}^2}T(2) + \frac{1}{z_{12}}\partial T(2) + \cdots, \\ T(1)T^\dagger(2) &= \frac{2}{z_{12}^2}T^\dagger(2) + \frac{1}{z_{12}}\partial T^\dagger(2) + \cdots, \\ T^\dagger(1)T^\dagger(2) &= \frac{c}{2z_{12}^4} + \frac{2}{z_{12}^2}T(2) + \frac{1}{z_{12}}\partial T(2) + \cdots, \qquad (c = 2\hat{c}). \end{aligned} \tag{3.5}$$

The primary states and fields of this algebra can be organized in two sectors according to its Z_2-structure. The states on which the boundary conditions (3.4) are realized have Z_2-charge $l = 1$ and represent the "branching sector" ($\equiv$ twisted

$T^\dagger$). The nontwisted Z_2-sector ($l = 0$) is represented by the states corresponding to the following ("nonbranching points") boundary conditions:

$$\Pi_b T = T, \qquad \Pi_b T^\dagger = T^\dagger.$$

Following the analogy with the generalized parafermionic algebras [30] $Z_N^p(\Delta_k = pk(N-k)/N + M_k)$ ($N = 2$, $p = 4$ in our case), we define the mode expansions of T and $T^\dagger$:

$$\begin{aligned} T(z)V_{[l]}(0) &= \sum_{n \in Z} z^{n-2} L_{-n} V_{[l]}(0), \\ T^\dagger(z)V_{[l]}(0) &= \sum_{n \in Z} z^{n-2-\frac{l}{2}} M_{-n+\frac{l}{2}} V_{[l]}(0). \end{aligned} \tag{3.6}$$

In terms of the Laurent modes L_n, $M_{n+\frac{l}{2}}$ ($l = 0, 1$) the OPE algebra (3.5) takes the form:

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12} m(m^2-1)\delta_{m+n,0}, \\ \left[M_{m+\frac{l}{2}}, M_{n+\frac{l}{2}}\right] &= (m-n)L_{m+n+l} + \frac{c}{12}\left[\left(m+\frac{l}{2}\right)^2 - 1\right]\left(m+\frac{l}{2}\right)\delta_{m+n+l,0}, \\ \left[L_m, M_{n+\frac{l}{2}}\right] &= \left(m-n-\frac{l}{2}\right) M_{m+n+\frac{l}{2}}. \end{aligned} \tag{3.7}$$

According to the standard BPZ procedure [3] the primary states of (3.7) are given by:

$l = 0$ (untwisted sector):

$$\begin{aligned} &L_0|\Delta, \Delta^\dagger\rangle = \Delta|\Delta, \Delta^\dagger\rangle, \qquad L_n|\Delta, \Delta^\dagger\rangle = 0 = M_n|\Delta, \Delta^\dagger\rangle \\ &M_0|\Delta, \Delta^\dagger\rangle = \Delta^\dagger|\Delta, \Delta^\dagger\rangle \qquad\qquad n \geq 1 \\ &V_{[0]}(0)|0,0\rangle = |\Delta, \Delta^\dagger\rangle \end{aligned}$$

$l = 1$ (branched sector):

$$\begin{aligned} &L_0|\Delta\rangle = \Delta|\Delta\rangle, \qquad L_n|\Delta\rangle = 0 = M_{n-\frac{1}{2}}|\Delta\rangle, \quad n \geq 1, \\ &V_{[1]}(0)|0,0\rangle = |\Delta\rangle. \end{aligned}$$

Since we are interested in the degenerate unitary representations of (3.7), we shall start with the analysis of the null-vectors:

$$l = 1 \text{ sector:} \quad \text{level } \frac{1}{2} \qquad M_{-\frac{1}{2}}|\Delta\rangle = 0 \quad \text{iff } \Delta = \frac{c}{32} \tag{8}$$

$$\text{level 1} \qquad \left\{M^2_{-\frac{1}{2}} + \frac{c-8(4\Delta+1)}{12c}L_{-1}\right\}|\Delta\rangle = 0 \quad \text{iff}$$

$$\Delta = \frac{c+10 \pm \sqrt{(2-c)(50-c)}}{64} \tag{9}$$

$$l=0 \text{ sector:} \quad \text{level 1} \qquad \left\{L_{-1} - \frac{\Delta}{\Delta^\dagger}M_{-1}\right\}|\Delta, \Delta^\dagger\rangle = 0 \text{ iff } \Delta = \pm\Delta^\dagger \quad etc. \tag{10}$$

The degenerate field $V_{c/32}(z)$ corresponding to the level 1/2 null state (3.8) will play the central role in our construction of the minimal models on $X_g^{(2)}$. It has all the properties of the Knizhnik's branch point operators introduced in [22] for the ghost and matter string systems on $X_g^{(2)}$.

Applying the GKO (coset constructions) method, [14], for the group $SU(2) \times SU(2) \approx O(4)$, *i.e.*considering $O(4) \times O(4)/O(4)$, we conclude that the central charge of the degenerate unitary representations of (3.7) is quantized:

$$c = 2 - \frac{12}{p(p+1)} = 2\hat{c}, \qquad p = 3, 4, \ldots. \tag{3.11}$$

In order to obtain the Kac-spectrum, correlation functions, fusion rules, *etc.* for the minimal models of (3.7) one can continue with the analysis of the null vectors, Ward identities and corresponding differential equations. The other, more powerfull approach to the minimal models is the Dotsenko-Fateev Coulomb gas representation[27]. Our description of the minimal models of (3.7) is based on the following Coulomb gas representation

$$T^{(k)} = \frac{1}{2}(\partial\varphi^{(k)})^2 + \alpha_0\partial^2\varphi^{(k)},$$

$$\Pi_a\partial\varphi^{(k)} = \partial\varphi^{(k+1)}, \qquad \langle\varphi^{(i)}(1)\varphi^{(j)}(2)\rangle = \delta^{ij}\ln z_{12}$$

($\alpha_0^2 = 1/2p(p+1)$ for the discrete series (3.11)). That is, we start with the usual (sphere) Coulomb gas for each sheet separately. Passing to the Z_2-diagonal basis (3.4)

$$\begin{aligned} \phi &= \varphi^{(0)} + \varphi^{(1)}, \qquad \Pi_a\partial\phi = \partial\phi, \quad \Pi_a\partial\phi^\dagger = -\partial\phi^\dagger, \\ \phi^\dagger &= \varphi^{(0)} - \varphi^{(1)}, \qquad \Pi_b\partial\phi^{(\dagger)} = \partial\phi^{(\dagger)}, \\ \langle\phi(1)\phi(2)\rangle &= 2\ln z_{12} = \langle\phi^\dagger(1)\phi^\dagger(2)\rangle, \qquad \langle\phi(1)\phi^\dagger(2)\rangle = 0. \end{aligned} \tag{3.12}$$

we can write T and $T^\dagger$ in the form:

$$T = \frac{1}{4}(\partial\phi)^2 + \alpha_0\partial^2\phi + \frac{1}{4}(\partial\phi^\dagger)^2, \qquad T^\dagger = \frac{1}{2}\partial\phi\partial\phi^\dagger + \alpha_0\partial^2\phi^\dagger, \qquad c = 2 - 24\alpha_0^2, \tag{3.13}$$

which makes transparent the splitting of the Coulomb gas system with respect to the Virasoro subalgebra of (3.7) into the sphere minimal models with $c_{sp} = 1 - 12/p(p+1)$ and the Z_2-orbifold S^1/Z_2-model with $c_{orb} = 1$. In this basis the vertex operator construction of the primary fields is straightforward:

$l = 0$ sector:

$$V_{a,b}(z) = e^{a\phi + b\phi^\dagger}, \tag{14}$$

$$\Delta_{a,b} = a^2 - 2\alpha_0 a + b^2,$$

$$\Delta^\dagger_{a,b} = 2ab - 2\alpha_0 b, \tag{15}$$

$l = 1$ sector:

$$V_{\alpha,\epsilon}(z) = e^{\alpha\phi}\sigma_\epsilon(z), \quad \epsilon = 0, 1 \tag{16}$$

$$\Delta_\alpha = \alpha^2 - 2\alpha_0\alpha + \frac{1}{16}. \tag{17}$$

Since $\phi^\dagger(z)$ lives on the orbifold S^1/Z_2 having two fixed points $\phi_0^\dagger = 0$ and $\phi_1^\dagger = \frac{1}{2}(2\pi R)$, the lowest energy states of $T_{orb} = \frac{1}{4}(\partial\phi^\dagger)^2$ are represented by the well-known twist fields $\sigma_\epsilon(z)$ (see refs. [57] and [56]) of dimension $\Delta_\epsilon = \frac{1}{16}$. They satisfy the following basic OPE's:

$$\begin{aligned} \partial\phi^\dagger(z)\sigma_\epsilon(0) &= \frac{1}{2\sqrt{z}}\hat{\sigma}_\epsilon(0) + \cdots, \\ \partial\phi^\dagger(z)\hat{\sigma}_\epsilon(0) &= \frac{1}{2}\frac{1}{z^{3/2}}\sigma_\epsilon(0) + \frac{2}{\sqrt{z}}(\partial_z\sigma_\epsilon)(0) + \cdots, \end{aligned} \tag{3.18}$$

where $\hat{\sigma}_\epsilon$ is an excited twist field of dimension $\hat{\Delta}_\epsilon = 9/16$. Direct inspection, using (3.16) and (3.18), shows that the branching operators $V_{\alpha,\epsilon}(z)$ reproduce the boundary conditions (3.4) and the OPE's (3.6), *i.e.*

$$\begin{aligned} T^\dagger(z)V_{\alpha,\epsilon}(0) &= \frac{2\alpha - \alpha_0}{2z^{3/2}}\hat{V}_{\alpha,\epsilon}(0) + O\left(\frac{1}{\sqrt{z}}\right) \equiv \frac{1}{z^{3/2}}\left(M_{-\frac{1}{2}}V_{\alpha,\epsilon}\right)(0) + \cdots, \\ \hat{V}_{\alpha,\epsilon} &= e^{\alpha\phi}\hat{\sigma}_\epsilon. \end{aligned} \tag{3.19}$$

A simple consequence of (3.19) is the explicit construction of the lowest-dimensional branching operators representing the degenerate state at level 1/2 (3.8):

$$\alpha = \frac{\alpha_0}{2}, \ \textit{i.e.} \quad V_{\alpha,\epsilon} = e^{\frac{\alpha_0}{2}\phi}\sigma_\epsilon \tag{3.20}$$

with dimension $\Delta_\epsilon = (1 - 12\alpha_0^2)/16 = c/32$.

To find the screening operators we take a special $l = 0$ vertex $S_{a,b}$ and require that $Q_{a,b} = \oint dz S_{a,b}(z)$ be invariant under the action of T and $T^\dagger$, *i.e.*

$$T(z_1)S_{a,b}(z_2) = \partial_2 \left(\frac{S_{a,b}(z_2)}{z_{12}} \right) + \text{ reg. terms,}$$
$$T^\dagger(z_1)S_{a,b}(z_2) = a\partial_2 \left(\frac{S_{a,b}(z_2)}{z_{12}} \right) + \text{ reg. terms.}$$

These conditions are satisfied iff $a = \pm b$, $a^2 - \alpha_0 a = 1/2$, which gives

$$a_\pm = \frac{\alpha_0}{2} \pm \sqrt{\left(\frac{\alpha_0}{2}\right)^2 + \frac{1}{2}}, \quad a_+ + a_- = \alpha_0, \quad a_+ a_- = -\frac{1}{2}. \tag{3.21}$$

Therefore, the screening operators we are looking for have the form:

$$S_+^\pm = e^{a_\pm(\phi+\phi^\dagger)}, \qquad S_-^\pm = e^{a_\pm(\phi-\phi^\dagger)}. \tag{3.22}$$

In the derivation of the dimensions of the primary fields of the minimal models of (3.7) we closely follow the methods developed in [29], addapting them to the case of the $O(4) \approx SU(2) \times SU(2)$ underlying algebra. Representing the completely degenerate stetes of (3.7) as contour integrals of products of arbitrary number of screening operators (3.22) and one vertex (3.14) or (3.16), we get the following Kac-spectrum for the degenerate primary fields from the discrete series (3.11) ([58] for details):

$$l = 0 \text{ sector:} \quad a[{}^{n}_{n'}{}^{m}_{m'}] = \frac{2-n-n'}{2}a_+ + \frac{2-m-m'}{2}a_-$$
$$b[{}^{n}_{n'}{}^{m}_{m'}] = \frac{n-n'}{2}a_+ + \frac{m-m'}{2}a_-$$
$$1 \le n, n' \le p, \qquad 1 \le m, m' \le p-1 \tag{23}$$
$$\Delta[{}^{n}_{n'}{}^{m}_{m'}] = \frac{[(n+n')(p+1)-(m+m')p]^2 + [(n-n')(p+1)-(m-m')p]^2 - 4}{8p(p+1)}$$
$$\Delta^\dagger[{}^{n}_{n'}{}^{m}_{m'}] = -\frac{[(n+n')(p+1)-(m+m')p]\,[(n-n')(p+1)-(m-m')p]}{4p(p+1)}$$
$$l = 1 \text{ sector:} \quad \alpha^{(p)}_{n,m} = \frac{2-n}{2}a_+ + \frac{2-m}{2}a_-$$
$$\Delta^{(p)}_{n,m} = \frac{[(p+1)n-mp]^2 - 4}{8p(p+1)} + \frac{1}{16} = \Delta^{(p)}_{p-n,p+1-m} \tag{24}$$
$$1 \le n \le p, \qquad 1 \le m \le p-1$$

The branching operator (3.20) appears as the lowest dimension field in the branched sector ($n = 1 = m$), $\Delta_{1,1} = \frac{1}{16}(1 - \frac{6}{p(p+1)})$. As usual, the corresponding lowest energy field in the untwisted sector ($n = 1 = n'$, $m = 1 = m'$) has quantum numbers $\Delta = 0 = \Delta^\dagger$ and is represented by the unity operator. We shall give as an example the constructions of some of the primary fields in the simplest model $c = 1$ ($\hat{c} = 1/2$, corresponding to the Ising model on $X_g^{(2)}$):

$$
\begin{array}{ll}
l = 1 & V^\epsilon_{1,2} = e^{\frac{a_+}{2}\phi}\sigma_\epsilon; \quad V^\epsilon_{2,2} \equiv \sigma_\epsilon \qquad \Delta_{1,2} = \Delta_{2,2} = \frac{1}{16} \\
 & V^\epsilon_{2,1} = e^{\frac{a_-}{2}\phi}\sigma_\epsilon, \qquad \Delta_{2,1} = \frac{9}{32} \\
l = 0 & \Delta^\dagger = 0: \quad \Psi\left[\begin{smallmatrix} 2 & 1 \\ 2 & 1 \end{smallmatrix}\right] = e^{-a_-\phi}, \quad \Delta\left[\begin{smallmatrix} 2 & 1 \\ 2 & 1 \end{smallmatrix}\right] = \frac{1}{8}; \quad \Psi\left[\begin{smallmatrix} 1 & 2 \\ 1 & 2 \end{smallmatrix}\right] = e^{-a_+\phi}, \quad \Delta\left[\begin{smallmatrix} 1 & 2 \\ 1 & 2 \end{smallmatrix}\right] = 1 \\
 & \Delta^\dagger = -\Delta: \quad \Psi\left[\begin{smallmatrix} 1 & 1 \\ 2 & 1 \end{smallmatrix}\right] = e^{-\frac{a_-}{2}(\phi - \phi^\dagger)}, \quad \Delta\left[\begin{smallmatrix} 1 & 1 \\ 2 & 1 \end{smallmatrix}\right] = \frac{1}{16}; \\
 & \qquad \Psi\left[\begin{smallmatrix} 1 & 1 \\ 1 & 2 \end{smallmatrix}\right] = e^{-\frac{a_+}{2}(\phi - \phi^\dagger)}, \quad \Delta\left[\begin{smallmatrix} 1 & 1 \\ 1 & 2 \end{smallmatrix}\right] = \frac{1}{2} \\
 & \Delta^\dagger = \Delta: \quad \Psi\left[\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right] = e^{-\frac{a_-}{2}(\phi + \phi^\dagger)}, \quad \Delta\left[\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right] = \frac{1}{16} \quad etc. \\
 & \Delta^\dagger = \frac{7}{16}: \quad \Psi\left[\begin{smallmatrix} 1 & 2 \\ 2 & 1 \end{smallmatrix}\right] = e^{-\frac{a_0}{2}\phi + \frac{a_+ - a_-}{2}\phi^\dagger}, \quad \Delta\left[\begin{smallmatrix} 1 & 2 \\ 2 & 1 \end{smallmatrix}\right] = \frac{9}{16} \quad etc.
\end{array}
\tag{3.25}
$$

3.2 Partition- and correlation functions on hyperelliptic surfaces

The next important issue in the description of the minimal models of (3.7) is the explicit structure of the corresponding OPE-algebras, *i.e.* the fusion rules and structure constants. Omitting the details (see [58]), we present here only a part of the fusion rules we need for the construction of the partition functions of the Virasoro minimal models on $X_g^{(2)}$. As it is explained in ref. [59], the fusion rules can be obtained by examining all the possible ways to screen the 3-point functions:

$$
\langle V^\epsilon_{n_1,m_1}(1) V^\epsilon_{n_2,m_2}(2) \Psi\left[\begin{smallmatrix} n & m \\ n' & m' \end{smallmatrix}\right](3) \prod_i^a Q^\pm_\pm(i) \rangle, \quad \langle \Psi\left[\begin{smallmatrix} n_1 & m_1 \\ n_1' & m_1' \end{smallmatrix}\right] \Psi\left[\begin{smallmatrix} n_2 & m_2 \\ n_2' & m_2' \end{smallmatrix}\right] \Psi\left[\begin{smallmatrix} n & m \\ n' & m' \end{smallmatrix}\right] \prod_i Q^\pm_\pm(i) \rangle
$$

satisfying each time the neutrality condition:

$$\sum \vec{\alpha}_i + n\vec{a}_+ + n'\vec{a}'_+ + m\vec{a}_- + m'\vec{a}'_- = 2\vec{\alpha}_0 \tag{3.26}$$

where

$$\vec{\alpha}_i = (a_i, b_i), \qquad \vec{a}_\pm = a_\pm(1,1), \qquad \vec{a}'_\pm = a_\pm(1,-1), \qquad 2\vec{\alpha}_0 = (2\alpha_0, 0).$$

The screening procedure together with the symmetry conditions

$$\Delta\left[\begin{smallmatrix} n & m \\ n' & m' \end{smallmatrix}\right] = \Delta\left[\begin{smallmatrix} p-n & p+1-m \\ n' & m' \end{smallmatrix}\right] = \Delta\left[\begin{smallmatrix} n & m \\ p-n' & p+1-m' \end{smallmatrix}\right] = \Delta\left[\begin{smallmatrix} p-n & p+1-m \\ p-n' & p+1-m' \end{smallmatrix}\right]$$

(and the same for $\Delta^\dagger$) and (3.24) leads to the following fusion rules ($\vec{n} = (n,m)$):

$$\Psi\begin{pmatrix} \vec{n}_1 \\ \vec{n}'_1 \end{pmatrix} \Psi\begin{pmatrix} \vec{n}_2 \\ \vec{n}'_2 \end{pmatrix} = \sum_{\vec{k}=|\vec{n}_1-\vec{n}_2|+1}^{\vec{n}_1+\vec{n}_2-1} \sum_{\vec{l}=|\vec{n}'_1-\vec{n}'_2|+1}^{\vec{n}'_1+\vec{n}'_2-1} \left[\Psi\begin{pmatrix} \vec{k} \\ \vec{l} \end{pmatrix}\right] \tag{3.27}$$

$$V^\epsilon_{1,1} V^\epsilon_{1,1} = \sum_{n=1}^{p} \sum_{m=1}^{p+1} \left[\Psi\left(\begin{smallmatrix} n & m \\ n & m \end{smallmatrix}\right)\right] \tag{3.28}$$

$$\Psi\left[\begin{smallmatrix} n & m \\ 1 & 1 \end{smallmatrix}\right] V^\epsilon_{1,1} = \left[V^\epsilon_{n,m}\right] \tag{3.29}$$

The OPE's corresponding to the fusion rules (3.29)

$$\begin{aligned} &\Psi\left(\begin{smallmatrix} n & m \\ 1 & 1 \end{smallmatrix}\right)(z) V^{(0)}_{1,1}(0) = C^{(0)}_{n,m} z^{-\frac{1}{2}\Delta\left(\begin{smallmatrix} n & m \\ 1 & 1 \end{smallmatrix}\right)} V^{(0)}_{n,m}(0) + \cdots \\ &\Psi\left(\begin{smallmatrix} n & m \\ 1 & 1 \end{smallmatrix}\right)(z) V^{(1)}_{1,1}(0) = C^{(1)}_{n,m} z^{-\frac{1}{2}\Delta\left[\begin{smallmatrix} n & m \\ 1 & 1 \end{smallmatrix}\right]+\frac{1}{2}} \hat{V}^{(1)}_{n,m}(0) + \cdots \\ &\hat{V}^\epsilon_{n,m} = e^{\alpha_{nm}\phi} \hat{\sigma}_\epsilon \end{aligned} \tag{3.30}$$

play a crucial role in the description of the possible boundary conditions of the fields $\Psi\left[\begin{smallmatrix} n & m \\ 1 & 1 \end{smallmatrix}\right]$ around the basic cycles A_a, B_a ($a = 1, \ldots, g$) of $X^{(2)}_g$. Another important consequence of the analysis of the 3-point functions and the fusion rules is that the radius of the orbifold S^1/Z_2 on which $\phi^\dagger$ lives is fixed by the screening procedure to $R = -2a_-$ (up to the symmetry ,[60], $R \to 2/R$). In fact, the possible screenings of the 3-point functions $\langle V^{\epsilon}_{1,1} V^{\epsilon}_{1,1} \Psi[\begin{smallmatrix} n & m \\ n' & m' \end{smallmatrix}]\rangle$ reflected in the fusion rules (3.28) imply that the function:

$$\langle \sigma_\epsilon \sigma_\epsilon \prod_i e^{\pm a_+ \phi^\dagger(i)} \prod_j e^{\pm a_- \phi^\dagger(j)} \rangle$$

is not vanishing, *i.e.*in the OPE [57,60]

$$\begin{aligned} \sigma_{\epsilon_1}(1)\sigma_{\epsilon_2}(2) &= \sum_{n,m} z_{12}^{(\beta^{\epsilon_1\epsilon_2}_{n,m})^2 - \frac{1}{8}} e^{\beta^{\epsilon_1\epsilon_2}_{n,m} \phi^\dagger(2)} \\ \beta^{\epsilon_1\epsilon_2}_{n,m} &= \frac{n}{R} + \frac{1}{2}\left(m + \frac{\epsilon_1+\epsilon_2}{2}\right) R \end{aligned} \tag{3.31}$$

only the fields with $R = -2a_-$ contribute. The fusion rules (3.28) describe qualitatively the handle degeneration of $X_g^{(2)}$ to $X_{g-1}^{(2)}$ with marked points and the projection of the null-states in the corresponding channel.

In the construction of the correlation functions of the primary fields of (3.7) we restrict ourselves to the functions

$$\langle \prod_{i=1}^{2g+2} V^{\epsilon_i}_{1,1}(a_i) \prod_{k=1}^{N} \Psi\left[\begin{smallmatrix} n_k & m_k \\ n'_k & m'_k \end{smallmatrix}\right](z_k) (Q^+_\pm)^s (Q^-_\pm)^t \rangle, \quad \sum \epsilon_i = 0 \bmod 2. \tag{3.32}$$

As we shall see, they are simply related to the correlation functions of the Virasoro minimal models on $X_g^{(2)}$. Using the vertex constructions (3.20) and (3.22) and the neutrality condition (3.26), we can write an integral representation for the "partition functions" on $X_g^{(2)}$:

$$\begin{aligned} Y^g_{C^\pm}(a_i) &= \langle \prod_{i=1}^{2g+2} V^{\epsilon_i}_{1,1}(a_i) (\tilde{Q}^-_\pm)^r (Q^-_\pm Q^+_\pm)^{rp+1-g} \rangle \\ &= \prod_{s=0}^{r} \prod_{l,m=0}^{rp+1-g} \oint_{C_s} dx_s \oint_{C^-_l} du_l \oint_{C^+_m} dv_m \langle \prod_{i=1}^{2g+2} e^{\frac{\alpha_0}{2}\phi(a_i)} e^{-p\alpha_0 \phi(x_s)} e^{a_- \phi(u_l)} e^{a_+ \phi(v_m)} \rangle \\ &\quad \times \langle \prod_{i=1}^{2g+2} \sigma_{\epsilon_i}(a_i) e^{\mp p\alpha_0 \phi^\dagger(x_s)} e^{\pm a_- \phi^\dagger(u_l)} e^{\pm a_+ \phi^\dagger(v_m)} \rangle \end{aligned} \tag{3.33}$$

where

$$\tilde{Q}^-_\pm = \oint_C dx e^{-p\alpha_0(\phi(x) \pm \phi^\dagger(x))}, \qquad a_- = -p\alpha_0.$$

The contours C_s, $C^\pm_m$ can be fixed by the branch cuts of the integrand as in ref. [27]. It remains to construct explicitly the correlation function of $2g+2$-twist fields σ_ϵ and an arbitrary number of untwisted vertices $e^{q\phi^\dagger}$:

$$
\begin{aligned}
G(\{a_i\},\{x_k\}) &= \langle \prod_{i=1}^{2g+2} \sigma_{\epsilon_i}(a_i) \prod_{k=1}^{n} e^{q_k\phi^\dagger(x_k)}\rangle = \sum_{p_a} \tilde{G}(p_a; a_i, x_k) \\
&\sum_{a=1}^{g} p_a + p_{g+1} + \sum_{k=1}^{n} q_k = 0, \qquad \sum \epsilon_i = 0 \bmod 2 \\
p_a^{\epsilon_i+\epsilon_{i+1}} &= \frac{n_a}{R} + \frac{1}{2}\left(m_a + \frac{\epsilon_i+\epsilon_{i+1}}{2}\right) R, \qquad R = -2a_-.
\end{aligned}
\tag{3.34}
$$

The direct generalization, [58], of the method of ref. [57] to the case of the function (3.34) allows us to derive a system of differential equations for the multipoint blocks $\tilde{G}(p_a; a_i, x_k)$. The starting point is the construction of the auxiliary functions

$$
\begin{aligned}
\Gamma(p_a; a_i, x_k, z) &= \langle \partial\phi^\dagger(z) \prod_{i=1}^{2g+2} \sigma_{\epsilon_i}(a_i) \prod_{k=1}^{n} e^{q_k\phi^\dagger(x_k)}\rangle \\
\hat{\Gamma}(p_a; a_i, x_k, z) &= \langle \partial\phi^\dagger(z)\hat{\sigma}_\epsilon(a_i) \prod_{i=2}^{2g+2} \sigma_{\epsilon_i}(a_i) \prod_{k=1}^{n} e^{q_k\phi^\dagger(x_k)}\rangle
\end{aligned}
$$

using the OPE's (3.18), the Ward identity

$$
\partial\phi^\dagger(z) e^{q\phi^\dagger(x)} = \frac{2q}{z-x} e^{q\phi^\dagger(x)} + \frac{1}{q}\partial_x e^{q\phi^\dagger(x)} + \cdots
$$

and the block condition coded in the OPE (3.31):

$$
\oint_{A_a} \Gamma(z, a_i, x_k; p_a) dz = 2p_a \tilde{G}(p_a; a_i, x_k).
$$

(The basic cycles A_a, B_a are chosen as in ref. [57], fig. 2.) The differential equations we look for have the following form [58]:

$$
\begin{aligned}
\frac{\partial \tilde{G}}{\partial a_i} &= \Bigg\{ \prod_{\substack{j=1\\ j\neq i}}^{2g+2} a_{ij}^{-1} \left[\sum_{a=1}^{g} p_a R_a(a_i) + \sum_{k=1}^{n} q_k y(x_k) \left(\frac{1}{a_i - x_k} - \sum_{a=1}^{g} M_a(x_k) R_a(a_i) \right) \right]^2 \\
&\quad - \frac{1}{8} \sum_{\substack{j=1\\ j\neq i}}^{2g+2} \frac{1}{a_{ij}} - \frac{1}{2}\frac{\partial}{\partial a_i} \ln\det \hat{K}^{(g)} \Bigg\} \tilde{G} \\
\frac{\partial \tilde{G}}{\partial x_k} &= \Bigg\{ \frac{q_k}{y(x_k)} \Bigg[\sum_{a=1}^{g} p_a R_a(x_k) - q_k y(x_k) \sum_{a=1}^{g} M_a(x_k) R_a(x_k) + \\
&\quad + \sum_{\substack{i=1\\ i\neq k}}^{n} q_i y(x_i) \left(\frac{1}{x_{ik}} - \sum_{a=1}^{g} M_a(x_i) R_a(x_k) \right) \Bigg] - \frac{1}{2} q_k^2 \sum_{l=1}^{2g+2} \frac{1}{x_k - a_l} \Bigg\} \tilde{G}
\end{aligned}
\tag{3.35}
$$

where $R_a(z) = \sum_{b=1}^{g} z^{b-1}(\hat{K}^{-1})_{ba}$, $M_a(x_i)$ and $K_{ab}(a_i)$ denote the complete hyperelliptic integrals of third and first kind respectively:

$$M_a(x_i) = \oint\limits_{A_a} \frac{dz}{(z-x_i)y(z)}, \qquad K_{ab}(a_i) = \oint\limits_{A_a} \frac{z^{b-1}}{y(z)} dz.$$

The solutions of this system can be easily found taking into account the simple fact, [58], that the equations (3.35) are the handle degeneration limits $X_{g'}^{(2)} \to X_g^{(2)}$, $g'-g=n$, of the corresponding equations[57] for the proper twist function $\langle \prod_{i=1}^{2g'+2} \sigma_{\epsilon_i}(a_i) \rangle$, *i.e.*:

$$\begin{aligned}\bar{G}(p_a; a_i, x_k) &= \prod_{k=1}^{g'-g} \lim_{a_{2k-1,2k}\to 0} (a_{2k-1,2k})^{\frac{1}{8}-q_k^2} \langle \prod_{l=1}^{2g'+2} \sigma_{\epsilon_l}(a_l) \rangle \\ &= \prod_{i<j}^{2g+2} a_{ij}^{-\frac{1}{8}} (\det \hat{K}^{(g)})^{-\frac{1}{2}} \exp\left[\pi i W^{(n,g)}(p_a, q_k; a_i, x_k)\right]\end{aligned}$$

$$W^{(n,g)} = \sum_{a,b=1}^{g} (p_a+\delta)\tau_{ab}(p_b+\delta) + \sum_{i,j=1}^{n} \sum_{s=1}^{2g+2} \frac{q_i}{g+1} \int_{a_s}^{x_i} [(2p_a+\delta)\hat{v}^{(a)} + q_j \omega^{(j)}],$$
$$\delta = \frac{\sum q_i}{g+1}. \tag{3.36}$$

The abelian differentials $\hat{v}^{(a)}(z) = R_a(z)/y(z)$ have the standard normalization

$$\oint\limits_{A_b} \hat{v}^{(a)} = \delta_{ab}, \qquad \oint\limits_{B_a} \hat{v}^{(b)} = \tau_{ab},$$

and the third kind abelian differentials

$$\omega^{(k)} = y(x_i) \left[\frac{1}{(z-x_k)y(z)} - \sum_{a=1}^{g} M_a(x_k)\hat{v}^{(a)}(z) \right]$$

have zero A_a-periods $\oint_{A_a} \omega^{(k)} = 0$ and residua normalized to ± 1.

Now we are able to write explicitly the correlation function (3.33):

$$\begin{aligned}Y_{C^\pm}^g(a_i) = &\prod_{l=0}^{r(p+1)+1-g} \prod_{m=0}^{rp+1-g} \oint\limits_{C_l^-} du_l \oint\limits_{C_m^+} dv_m (v_m-u_l)^{-1} \prod_{i<j}^{2g+2} (a_{ij})^{-\frac{c}{24}-\frac{1}{12}} (\det \hat{K}^g)^{-\frac{1}{2}} \\ &\times \prod_{s=1}^{2g+2} (a_s-u_l)^{\alpha_0 a_-} (a_s - v_m)^{\alpha_0 a_+} \prod_{k>l}^{r(p+1)+1-g} (u_{lk})^{2a_-^2} \prod_{n>m}^{rp+1-g} (v_{mn})^{2a_+^2} \\ &\times \sum_{(n_a,m_a)\in\Gamma_R^{\epsilon_a}} \exp\left[\pi i W^{(n,g)}(p_a; q_k; a_i, u_l, v_m)\right]\end{aligned} \tag{3.37}$$

where

$$\Gamma_R^{\epsilon_a} = \left(Z^g, \left(Z + \frac{\epsilon_i + \epsilon_{i+1}}{2}\right)^g\right)$$

Extending the arguments of ref. [57] (Sect. 5) to the case of $Y^g_{C\pm}(a_i)$, we can construct the corresponding crossing-symmetric 2-D correlation function. Since the contour integrals can be rewritten as integrals on the whole complex plane, [61], we get the following monodromy- (and modular-) invariant function:

$$\begin{aligned}Z_g^{(2)}(a_i, \bar{a}_i) = |a_{ij}|^{-\frac{c}{24}-\frac{1}{6}} |\det \hat{K}^g|^{-1} &\prod_{l=0}^{r(p+1)+1-g} \prod_{m=0}^{rp+1-g} \int d^2u_l d^2v_m |v_m - u_l|^{-2} \\ \times \prod_{s=1}^{2g+2} |a_s - u_l|^{2\alpha_0 a_-} |a_s - v_m|^{2\alpha_0 a_-} &\prod_{k>l}^{r(p+1)+1-g} |u_{lk}|^{4a_-^2} \\ \times \prod_{n>m}^{rp+1-g} |v_{mn}|^{4a_+^2} \prod_{(p,\bar{p})\in\Gamma_R^{\epsilon_a}} &e^{\pi i (W - \bar{W})}\end{aligned} \tag{3.38}$$

where ϵ_i's and $\bar{\epsilon}_i$'s are either all equal to 0 or all equal to 1 and

$$\begin{aligned}\bar{W} = \sum_{a,b=1}^{g} (\bar{p}_a + \delta)\bar{\tau}_{ab}(\bar{p}_b + \delta) + \sum_{i,j=1}^{n} \sum_{s=1}^{2g+2} \frac{q_i}{g+1} \int_{a_s}^{\bar{x}_i} [(2\bar{p}_a + \delta)\bar{\bar{v}}^{(a)} + q_j \bar{\omega}^{(j)}] \\ \bar{p}_a = \frac{n^a}{R} - \frac{1}{2} m^a R\end{aligned} \tag{3.39}$$

(*i.e.* the classical solutions contributing only to the untwisted part of the Z_2-orbifold's correlation functions (3.34) are considered.) Another solution is to take the twisted part of the Z_2-orbifold function (3.34), *i.e.* to change the momenta

$$p_a^t = \frac{n_a + \delta_1^a}{R} + \frac{1}{2} R(m^a + \delta_2^a), \qquad \bar{p}_a^t = \frac{n_a + \delta_1^a}{R} - \frac{1}{2} R(m^a + \delta_2^a), \tag{3.40}$$

to redefine

$$W^t - \bar{W}^t = W(p_a^t) - \bar{W}(\bar{p}_a^t) + 2\epsilon_2^a(m^a + \delta_2^a) + 2\epsilon_1^a(n^a + \delta_1^a) \tag{3.41}$$

and to sum in a modular-invariant way over all possible choices of the g-dimensional vectors δ_i^a, ϵ_i^a (with components 0 or 1/2), provided that $(-1)^{4\vec{\epsilon}_1\cdot\vec{\delta}_1} = 1 = (-1)^{4\vec{\epsilon}_2\cdot\vec{\delta}_2}$. For $R^2 = p/q$ the infinite sum over $(n^a, m^a) \in Z^g$ reduces to a finite sum, [57, 60], which is important for the interpretation of $Z_g^{(2)}$ as a "partition function" of the Virasoro minimal models on $X_g^{(2)}$. The choice $\epsilon_1^a = 0 = \delta_2^a$ corresponds to the following crossing-symmetric function:

$$\sum \langle \prod_{i=1}^{g+1} V^{(0)}(a_i, \bar{a}_i) \prod_{j=1}^{g+1} V^{(1)}(b_i, \bar{b}_i) \prod Q_{\pm}^{\pm} \rangle,$$

where the sum is over all independent permutations of a_i and b_i.

So far we have constructed the most important ingredients of the minimal models of (3.7) on the branched sphere and now we have to answer the question how can one construct the partition and the correlation functions of the Virasoro minimal models on $X_g^{(2)}$ in terms of the crossing-symmetric functions (3.32). Start with the path-integral definition of the partition function on $X_g^{(2)}$ in the spirit of the ghost system constructions [22,55,63,62]

$$Z_g(m_i, \bar{m}_i) = \sum_{\substack{winding\\numbers}} \int \mathcal{D}\varphi e^{\frac{1}{2}\int \partial\varphi\bar{\partial}\varphi + 2\alpha_0 \int R^{(g)}\varphi} (\tilde{Q}_g^-)^r (Q_g^- Q_g^+)^{rp+1-g} \tag{3.42}$$

where the screening charges

$$Q_g^{\pm} = \int_{X_g^{(2)}} d^2x e^{\alpha_{\pm}\varphi(x) + \alpha_{\pm}\bar{\varphi}(\bar{x})}, \qquad \tilde{Q}_g^- = \int d^2x e^{-2p\alpha_0(\varphi(x)+\bar{\varphi}(\bar{x}))}, \quad \alpha_{\pm} = 2a_{\pm}$$

make neutral the measure of integration:

$$-2pr\alpha_0 + (rp+1-g)\alpha_+ + (rp+1-g)\alpha_- \equiv -2\alpha_0(g-1)$$

(The anomaly for $g = 0$ is $2\alpha_0$.) The hyperelliptic map (3.1) (see also (3.12)) reduces (3.42) to the specific partition function on the branched sphere, introducing the branching operators $V_{1,1}^{(\epsilon)}$ to simulate the boundary conditions of the fields around the basic cycles A_a, B_a of $X_g^{(2)}$:

$$\begin{aligned} Z_g(a_i, \bar{a}_i) = & \sum_{\substack{winding\\numbers}} \prod_{l=0}^{r(p+1)+1-g} \prod_{m=0}^{rp+1-g} \int d^2u_l d^2v_m \int \mathcal{D}\phi\mathcal{D}\phi^\dagger \exp\Big[\frac{1}{4}\int \partial\phi\bar{\partial}\phi \\ & + 2\alpha_0 \int R^{g=0}\phi + \frac{1}{4}\int \partial\phi^\dagger\bar{\partial}\phi^\dagger\Big] \prod_{i=1}^{2g+2} V^{(\epsilon_i)}((a_i, \bar{a}_i) e^{a_-(\phi\pm\phi^\dagger)(u_l)} e^{a_+(\phi\pm\phi^\dagger)(v_l)} \\ = & \sum_{\substack{winding\\numbers}} \prod_l \prod_m \int d^2u_l d^2v_m \langle \prod_{i=1}^{2g+2} e^{\frac{\alpha_0}{2}\phi(a_i,\bar{a}_i)} e^{a_-\phi(u_l)} e^{a_+\phi(v_l)} \rangle_{g=0} \\ & \langle \prod_{i=1}^{2g+2} \sigma_\epsilon(a_i, \bar{a}_i) e^{\pm a_-\phi^\dagger(u_l)} e^{\pm a_+\phi^\dagger(v_l)} \rangle_{g=0}. \end{aligned} \tag{3.43}$$

Note that the branched sphere correlation functions in (3.43) satisfy the neutrality condition with the sphere anomaly:

$$(g+1)\alpha_0 - rp\alpha_0 + (rp+1-g)\alpha_0 = 2\alpha_0$$

for the first one and (3.34) for the second one. The above arguments strongly suggest that the crossing-symmetric (*e.g.* modular invariant) correlation functions (3.38), (3.40) and (3.41) or their linear combinations (even with different but conjugate radii R of the Z_2-orbifold) can be identified with the partition function Z_g of the Virasoro minimal models on $X_g^{(2)}$. For $g \geq 2$ the problem of the classification of all modular invariants is still open. For the torus, the corresponding partition function does not contain any screening operators. As it can be seen from (3.38), (3.40), (3.41) and (3.43), $Z^2_{g=1}$ is proportional to the twisted part of the Z_2-orbifold partition function and it reproduces, [58], the well-known modular invariants [21]. Remember that for $\hat{c} \neq 0$ partition functions transform covariantly under the local rescaling of the surface metric, [16],:

$$Z(e^f \hat{g}) = Z(\hat{g}) e^{\hat{c} S_L(f,\hat{g})}$$

where $S_L(f,\hat{g})$ is the Liouville action of the function f in the metric $\hat{g}$. Our metric on $X_g^{(2)}$ is the singular one induced from the metric $\hat{g} = dzd\bar{z}$ on the sphere by the hyperelliptic map $z \rightarrow y(z)$. So, for instance, to obtain the partition function for the flat metric $g = dyd\bar{y}$ on the torus, one has to multiply (3.38), (3.43) by the factor $\prod_{i<j} |a_{ij}|^{\hat{c}/12}$.

Our quite incomplete description of the minimal models on $X_g^{(2)}$ was concentrated on the construction of $g > 1$ partition functions and left unanswered a lot of important questions: 1) the factorization properties of these partition functions; 2) the explicit construction of the correlation functions; 3) the classification (according to some physical principle) of the different modular invariant partition functions for $g \geq 2$ *etc.*. Nevertheless, we hope that our description of minimal models on some Riemann surfaces with $g > 1$ contains certain prompts on how to treat these models even on arbitrary compact surfaces.

Chapter 4

Higher Level Models on Hyperelliptic Surfaces

4.1 Superconformal models

We continue our study of the minimal models generalizing the method we have used in Chap. 3 (see also [36]) to the case of the $N = 1$ superconformal minimal models [5] on Z_2-hyperelliptic supersurfaces ,[64], $SX_g^{(2)}$. The problem is to find the appropriate m.m.'s on the branched supersphere which describe the $N = 1$ minimal models on $SX_g^{(2)}$.

Analogously to $X_g^{(2)}$, the $SX_g^{(2)}$ can be defined as a double cover of the supersphere $\mathrm{CP}^{1,1}$, branched over $2g+2$ points, by defining all of the coordinate patches and the transition functions among them. The supersphere has two coordinate patches: (z, θ) and $(w, \chi) = (-1/z, \theta/z)$. To define the transition functions around the superbranch points we recall the general solution of the superconformal constraint , [65], $Dw = \chi D\chi$:

$$w(z,\theta) = u(z) - u'(z)\epsilon(z)\theta,$$
$$\chi(z,\theta) = \sqrt{u'(z)}\left(\theta + \epsilon(z) + \frac{1}{2}\epsilon\partial\epsilon(z)\theta\right),$$

where $u(z)$ is a function and $\epsilon(z)$ is a $(-1/2)$-differential defined on the intersection of charts. Most of the u's and ϵ's can be removed by the superconformal transformations in various charts, the remaining ones contain information on the supermoduli of the SRS. For $SX_g^{(2)}$ $u_i = \sqrt{z-a_i}$, and $\epsilon(z) = \alpha_i(z-a_i)^{-1/4}$, where α_i is an odd parameter, is a $(-1/2)$-differential with a first order pole at a_i. In such a way, we define a super hyperelliptic map

$$\begin{aligned} w_i(z,\theta) &= \sqrt{z-a_i-\alpha_i\theta(z-a_i)^{-\frac{1}{4}}}, \\ \chi_i(z,\theta) &= \frac{\theta+\alpha_i(z-a_i)^{-\frac{1}{4}}}{\sqrt{2}(z-a_i)^{\frac{1}{4}}}. \end{aligned} \tag{4.1}$$

As shown in [64], accepting this definition of the hyperelliptic supersurface $SX_g^{(2)}$ one can derive the dimensions d_g of the corresponding supermoduli space $s\mathcal{M}_g^{(2)}$. For $g=1$ one obtains $d_1^{even} = (1|0)$ for the even spin structures and $d_1^{odd} = (1|1)$ for the odd one. For $g=2$ $d_2 = (3|2)$ for all spin structures.

The most attractive feature of the $N=1$ superconformal algebra on the $SX_g^{(2)}$, generated by the super stress-energy tensor $W(w,\chi) \equiv \{T(w), G(w)\}$, is that it maps into $Z_4^{p=2}$ parafermionic algebra, [30], on the branched supersphere [64]. Under (4.1) $T(w)$ and $G(w)$ get mapped onto $T^{(k)}(z)$ and $G^{(k)}(z)$ $(k=0,1)$, defined on the corresponding sheets. For a given k they satisfy the usual OPE's [5] and for $k \neq m$ the OPE's contain only regular terms:

$$T^{(k)}(z_1)T^{(m)}(z_2) \sim T^{(k)}(z_1)G^{(m)}(z_2) \sim G^{(k)}(z_1)G^{(m)}(z_2) \sim \text{ reg. for } k \neq m.$$

The analytic continuation of $T^{(k)}(z)$ and $G^{(m)}(z)$ around the branch point leads to the following monodromy conditions:

$$\Pi_a T^{(k)} = T^{(k+1)}, \qquad \Pi_a G^{(0)} = -G^{(1)}, \qquad \Pi_a G^{(1)} = G^{(0)}.$$

We can diagonalize Π_a by changing the basis as follows:

$$\begin{aligned} T &\equiv T^{(0)} + T^{(1)}, & \Pi_a T &= T, \\ T^\dagger &\equiv T^{(0)} - T^{(1)}, & \Pi_a T^\dagger &= -T^\dagger, \\ G &\equiv G^{(0)} - iG^{(1)}, & \Pi_a G &= -iG, \\ G^\dagger &\equiv G^{(0)} + iG^{(1)}, & \Pi_a G^\dagger &= iG^\dagger, \end{aligned} \qquad \Pi_a^4 = 1. \tag{4.2}$$

The Z_4 charge of T, $T^\dagger$, G and $G^\dagger$ is 0, 1, 2 and 3, respectively. Around the other arbitrary "nonbranching" points T and $T^\dagger$ are periodic (see Chapter 3) and G

and $G^\dagger$ have Z_2-boundary conditions corresponding to the NS (Z_4-charge $l=0$) and Ramond ($l=2$) marked points on $CP^{1,1}$ [64]. Using the sheetwise OPE's it is easy to derive the OPE algebra of T, $T^\dagger$, G and $G^\dagger$ [64]:

$$\begin{aligned}
&T(1)T(2) \sim T^\dagger(1)T^\dagger(2) \sim \frac{\tilde{c}/2}{z_{12}^4} + \frac{2}{z_{12}^2}T(2) + \frac{1}{z_{12}}\partial_2 T(2),\\
&T(1)T^\dagger(2) \sim \frac{2}{z_{12}^2}T^\dagger(2) + \frac{1}{z_{12}}\partial_2 T^\dagger(2),\\
&T(1)G(2) \sim T^\dagger(1)G^\dagger(2) \sim \frac{3/2}{z_{12}^2}G(2) + \frac{1}{z_{12}}\partial_2 G(2),\\
&T(1)G^\dagger(2) \sim T^\dagger(1)G(2) \sim \frac{3/2}{z_{12}^2}G^\dagger(2) + \frac{1}{z_{12}}\partial_2 G^\dagger(2),\\
&G(1)G(2) \sim G^\dagger(1)G^\dagger(2) \sim \frac{1/2}{z_{12}}T^\dagger(2),\\
&G(1)G^\dagger(2) \sim \frac{\tilde{c}/6}{z_{12}^3} + \frac{1/2}{z_{12}}T(2), \qquad\qquad \tilde{c} = 2c.
\end{aligned} \tag{4.3}$$

The primary states and fields of (4.3) are divided into four sectors $V_{[l]}(z)$ ($l = 0,1,2,3$) according to the four possible sets of branching conditions for the generators T, $T^\dagger$, G, $G^\dagger$. The primary fields $V_{[l]}(z)$ realize these boundary conditions through their OPE's with the generators:

$$\begin{aligned}
T(z)V_{[l]}(0) &= \sum_{n\in Z} z^{-n-2}L_n V_{[l]}(0), & L_n V_{[l]}(0) &= \oint \frac{dz}{2\pi i} z^{n+1}T(z)V_{[l]}(0),\\
T^\dagger(z)V_{[l]}(0) &= \sum_{n\in Z} z^{-n-\frac{l}{2}-2}M_{n+\frac{l}{2}} V_{[l]}(0), & M_{n+\frac{l}{2}} V_{[l]}(0) &= \oint \frac{dz}{2\pi i} z^{n+\frac{l}{2}+1}T^\dagger(z)V_{[l]}(0),\\
G(z)V_{[l]}(0) &= \sum_{n\in Z} z^{-n-\frac{l}{4}}G_{n+\frac{l}{4}-\frac{1}{2}} V_{[l]}(0), & G_{n+\frac{l}{4}-\frac{1}{2}} V_{[l]}(0) &= \oint \frac{dz}{2\pi i} z^{n+\frac{l}{4}}G(z)V_{[l]}(0),\\
G^\dagger(z)V_{[l]}(0) &= \sum_{n\in Z} z^{-n+\frac{l}{4}}G^\dagger_{n-\frac{l}{4}-\frac{3}{2}} V_{[l]}(0), & G^\dagger_{n-\frac{l}{4}-\frac{1}{2}} V_{[l]}(0) &= \oint \frac{dz}{2\pi i} z^{n-\frac{l}{4}}G^\dagger(z)V_{[l]}(0).
\end{aligned} \tag{4.4}$$

A simple consequence of (4.3) and (4.4) is the following Laurent modes algebra:

$$
\begin{aligned}
&[L_m, L_n] = \frac{\bar{c}}{12} m(m^2-1)\delta_{n+m,0} + (m-n)L_{n+m}, \\
&\left[M_{m+\frac{l}{2}}, M_{n+\frac{l}{2}}\right] = \frac{\bar{c}}{12}\left(\left(m+\frac{l}{2}\right)^2 - 1\right)\left(m+\frac{l}{2}\right)\delta_{n+m+l,0} + (m-n)L_{n+m+l}, \\
&\left[L_m, M_{n+\frac{l}{2}}\right] = \left(m-n-\frac{l}{2}\right) M_{m+n+\frac{l}{2}}, \\
&\left[L_m, G_{n+\frac{l}{4}-\frac{1}{2}}\right] = \left[\frac{1}{2}(m+1)-n-\frac{l}{4}\right] G_{n+m+\frac{l}{4}-\frac{1}{2}}, \\
&\left[L_m, G^\dagger_{n-\frac{l}{4}-\frac{1}{2}}\right] = \left[\frac{1}{2}(m+1)-n+\frac{l}{4}\right] G^\dagger_{n+m-\frac{l}{4}-\frac{1}{2}}, \\
&\left[M_{m+\frac{l}{2}}, G_{n+\frac{l}{4}-\frac{1}{2}}\right] = \left[\frac{1}{2}(m+1)-n\right] G^\dagger_{n+m+\frac{3l}{4}-\frac{1}{2}}, \\
&\left[M_{m+\frac{l}{2}}, G^\dagger_{n-\frac{l}{4}-\frac{1}{2}}\right] = \left[\frac{1}{2}(m+l+1)-n\right] G_{n+m+\frac{l}{4}-\frac{1}{2}}, \\
&\left\{G_{m+\frac{l}{4}+\frac{1}{2}}, G_{n+\frac{l}{4}-\frac{1}{2}}\right\} = \frac{1}{2} M_{n+m+\frac{l}{2}}, \\
&\left\{G^\dagger_{m-\frac{l}{4}+\frac{1}{2}}, G^\dagger_{n-\frac{l}{4}-\frac{1}{2}}\right\} = \frac{1}{2} M_{n+m-\frac{l}{2}}, \\
&\left\{G_{m+\frac{l}{4}+\frac{1}{2}}, G^\dagger_{n-\frac{l}{4}-\frac{1}{2}}\right\} = \frac{\bar{c}}{12}\left(m+\frac{l}{4}+1\right)\left(m+\frac{l}{4}\right)\delta_{n+m,0} + \frac{1}{2}L_{n+m}.
\end{aligned}
\tag{4.5}
$$

The direct comparison with the parafermionic algebras of [30] shows that (4.5) represents the generalized parafermionic $Z_4^{p=2}$ algebra. However, for $SX_g^{(2)}$ (and in general in the presence of fields of half-integer spin on $X_g^{(2)}$ the monodromies (4.2) do not exhaust all the discrete symmetries, *i.e.* all the boundary conditions for $T, T^\dagger, G, G^\dagger$. In fact, the sheet-interchange Z_2 symmetry, together with (4.2) leads to D_4 as a group of monodromies [38]. It turns out, [38], that only the vertices from the Z_4-sectors (the usual order-disorder fields) play an important role in the construction of the $N=1$ superconformal partition functions. Therefore, in what follows, we shall consider the even (order) sectors of $D_4^{p=2}$ parafermionic algebra only.

The unitary degenerate representations of this algebra can be obtained by the GKO method, [14], for the coset $\hat{D}_2(p-2) \times \hat{D}_2(2)/\hat{D}_2(p)$, which results in the central charge $\bar{c}$ of (4.5) being quantized as follows:

$$
\bar{c} = 3 - \frac{24}{p(p+2)} = 2c, \qquad p = 3, 4, \ldots
$$

We start the construction of these representations by taking the usual (supersymmetric) Coulomb gas realization, [5,59], of $T^{(k)}$ and $G^{(k)}$ ($k = 0, 1$):

$$
\begin{aligned}
T^{(k)} &= \frac{1}{2}(\partial\varphi^{(k)})^2 + \frac{1}{2}(\partial\psi^{(k)})\psi^{(k)} + \alpha_0\partial^2\varphi^{(k)}, \\
G^{(k)} &= \psi^{(k)}\partial\varphi^{(k)} + 2\alpha_0\partial\psi^{(k)}, \qquad \alpha_0^2 = \frac{1}{p(p+2)},
\end{aligned}
\tag{4.6}
$$

where $\psi^{(k)}$ are free Majorana fermions

$$
\begin{aligned}
\langle\psi^{(k)}(1)\psi^{(l)}(2)\rangle &= \delta^{kl}\frac{1}{z_{12}}, \\
\Pi_a\psi^{(0)} = -\psi^{(1)}, &\qquad \Pi_a\psi^{(1)} = \psi^{(0)},
\end{aligned}
$$

and the free scalar fields $\varphi^{(k)}$ are defined as in Chapter 3 (see also [38] eq.12). In the Z_4-diagonal basis (4.2) eqs. (4.6) become

$$
\begin{aligned}
T &= \frac{1}{4}(\partial\phi)^2 + \frac{1}{4}(\partial\phi^\dagger)^2 + \frac{1}{4}(\partial\psi)\psi^\dagger + \frac{1}{4}(\partial\psi^\dagger)\psi + \alpha_0\partial^2\phi, \\
T^\dagger &= \frac{1}{2}\partial\phi\partial\phi^\dagger + \alpha_0\partial^2\phi^\dagger + \frac{1}{4}(\partial\psi)\psi + \frac{1}{4}(\partial\psi^\dagger)\psi^\dagger, \\
G &= \frac{1}{2}\psi^\dagger\partial\phi + \frac{1}{2}\psi\partial\phi^\dagger + 2\alpha_0\partial\psi^\dagger, \\
G^\dagger &= \frac{1}{2}\psi\partial\phi + \frac{1}{2}\psi^\dagger\partial\phi^\dagger + 2\alpha_0\partial\psi.
\end{aligned}
\tag{4.7}
$$

where the 1/2-spin fields $\psi = \psi^{(0)} + i\psi^{(1)}$ and $\psi^\dagger = \psi^{(0)} - i\psi^{(1)}$ obey the following monodromy conditions around the branch points:

$$
\Pi_a\psi = i\psi, \qquad \Pi_a\psi^\dagger = -i\psi^\dagger.
$$

Note that with respect to the Virasoro subalgebra of (4.7) the Coulomb gas system splits into the sphere minimal model with $c_{sp} = 1 - \frac{24}{p(p+2)}$, the Z_2-orbifold with $c_{orb} = 1$ and $R = -2\hat{a}_-$, and the $X_g^{(2)}$ Ising model with $c_{Is} = 1$.

The lowest energy vertex operators in the different sectors, satisfying the OPE's (4.4), can be constructed in terms of ϕ, $\phi^\dagger$, the Z_2-twist fields σ_ϵ and the Ising model ($\tilde{c} = 1$, $c = 1/2$) fields $V_{1,1}^{(0)} \equiv V_{[1]}^{\psi}$, $V_{1,1}^{(1)} \equiv V_{[3]}^{\psi}$ with $\Delta = 1/32$ and $\psi[\begin{smallmatrix}1&1\\2&1\end{smallmatrix}] \equiv V_{[2]}^{\psi}$ with $\Delta = 1/16$ (see (3.30)):

$$
\begin{aligned}
V_{[0]} &= e^{a_0\phi + b_0\phi^\dagger} & V_{[2]} &= V_{[2]}^{\psi} e^{\frac{a_0}{2}(\phi+\phi^\dagger)} \\
V_{[1]} &= V_{[1]}^{\psi}\sigma_{(0)} e^{\frac{a_0}{2}\phi} & V_{[3]} &= V_{[3]}^{\psi}\sigma_{(1)} e^{\frac{a_0}{2}\phi}
\end{aligned}
\tag{4.8}
$$

The two $\Delta = 1/2$ Ising model fields $\psi[\begin{smallmatrix}1&1\\2&1\end{smallmatrix}] = \exp(-\frac{a_+}{2}(\phi - \phi^\dagger))$ with $\Delta^\dagger = -1/2$ and $\psi[\begin{smallmatrix}2&1\\1&1\end{smallmatrix}] = \exp(-\frac{a_-}{2}(\phi + \phi^\dagger))$ with $\Delta^\dagger = 1/2$ are related to $\psi^{(k)}$ through

$\psi[{}^{2\,1}_{1\,1}] = \psi^{(0)} = \frac{1}{2}(\psi + \psi^\dagger)$ and $\psi[{}^{1\,1}_{2\,1}] = \psi^{(1)} = \frac{1}{2i}(\psi - \psi^\dagger)$. We have to mention that the fields in the $l = 0$ (NS) sector can be realized as $N = 1$ superfields. In fact the vertex $V_{[0]}$ represents the first component of the NS superfield, the second component is given by $\hat{V}_{[0]} = (a_0\psi + b_0\psi^\dagger)\,\exp(a_0\phi + b_0\phi^\dagger)$.

As usual, the screening operators are particular $l = 0$ vertices, whose contour integrals are invariant under the action of the generators. In our case these requirements are satisfied by:

$$\hat{Q}^{\pm}_{+} = \oint dz(\psi + \psi^\dagger)e^{\hat{a}_\pm(\phi+\phi^\dagger)}, \qquad \hat{Q}^{\pm}_{-} = \oint dz(\psi - \psi^\dagger)e^{\hat{a}_\pm(\phi-\phi^\dagger)}$$

$$\hat{a}_\pm^2 - \alpha_0\hat{a}_\pm = \frac{1}{4}, \quad \hat{a}_\pm = \frac{\alpha_0 \pm \sqrt{\alpha_0^2+1}}{2}, \quad \hat{a}_+ + \hat{a}_- = \alpha_0, \quad \hat{a}_+\hat{a}_- = -\frac{1}{4}.$$

To find the vertices representing the primary fields of the l-sector of the discrete series of the unitary representations of (4.5) we simply repeat the procedure described in Chapter 3. The primary fields we are looking for have the same form as (4.8), but with the following quantized charges:

$$\begin{array}{lll}
\begin{array}{l} l=1 \\ l=3 \end{array} : & V^{[l]}_{n,m} = V^{\psi}_{[l]}\sigma_\epsilon e^{\alpha_{n,m}\phi}, & \begin{array}{l} \alpha_{n,m} = \frac{2-n}{2}\hat{a}_+ + \frac{2-m}{2}\hat{a}_- \\ \Delta^{[l]}_{n,m} = \alpha^2_{n,m} - 2\alpha_0\alpha_{n,m} + \frac{3}{32} \end{array} \\
l=0 & n^{(l)} - m^{(l)} \in 2Z & V_{[0]}[{}^{n\ \ m}_{n'\ m'}] = \exp\left(a[{}^{n\ \ m}_{n'\ m'}]\phi + b[{}^{n\ \ m}_{n'\ m'}]\phi^\dagger\right) \\
l=2 & n^{(l)} - m^{(l)} \in 2Z+1 & V_{[2]}[{}^{n\ \ m}_{n'\ m'}] = V^{\psi}_{[2]}\exp\left(a[{}^{n\ \ m}_{n'\ m'}]\phi + b[{}^{n\ \ m}_{n'\ m'}]\phi^\dagger\right)
\end{array}$$

where $a[\ ]$ and $b[\ ]$ are of the form (3.23) (with $\hat{a}_\pm$ instead of $a_\pm$) and

$$\Delta[{}^{n\ \ m}_{n'\ m'}] = \vec{\alpha}\cdot(\vec{\alpha} - 2\vec{\alpha}_0) + \frac{2 - (-1)^{m-n} - (-1)^{m'-n'}}{32}$$

$$\vec{\alpha} = (a, b), \qquad \vec{\alpha}_0 = (\alpha_0, 0).$$

An important peculiarity of the supersymmetric partition functions is their dependence on the odd moduli [66]. According to the discussion for the superstring case in [64,67], the relevant branching operators for the non-split $SX_g^{(2)}$ are $\tilde{V}_{[l]} = V_{[l]} + 2\alpha G_{-3/4}V_{[l]}$, $l = 1, 3$ and α is the odd modulus (cf. (4.1)). In the case of the $N = 1$ superconformal minimal models these operators are realized as:

$$\tilde{V}_{[1]} = (V^{(0)}_{\Delta=\frac{1}{32}}\sigma^{(0)}_{\frac{1}{16}} + 2\alpha V^{(0)}_{\Delta=\frac{9}{32}}\hat{\sigma}^{(0)}_{\frac{9}{16}})e^{\frac{\alpha_0}{2}\phi},$$
$$\tilde{V}_{[3]} = (V^{(1)}_{\Delta=\frac{1}{32}} + 2\alpha V^{(1)}_{\Delta=\frac{25}{32}})\sigma^{(1)}_{\frac{1}{16}}e^{\frac{\alpha_0}{2}\phi}.$$

Leaving the general case for [58], we restrict ourselves to the construction of the "partition function" for the split $(\alpha_1 = \alpha_2 = 0)$ $SX^{(2)}_{g=2}$ surface and for the models with even $p = 2k$:

$$\begin{aligned}
Y^{g=2}_{C^\pm,p}(a_i) &= \left\langle \prod_{i=1}^{6} V_{\epsilon_i}(a_i)\tilde{Q}^-_\pm(\hat{Q}^-_\pm\hat{Q}^+_\pm)^{k-1}\right\rangle \\
&= \prod_{l,m}^{k-1}\oint_C dx \oint_{C_l^-} du_l \oint_{C_m^+} dv_m \left\langle \prod_{i=1}^{6} e^{\frac{\alpha_0}{2}\phi(a_i)}e^{-k\alpha_0\phi(x)}e^{\hat{a}_-\phi(u_l)}e^{\hat{a}_+\phi(v_m)}\right\rangle \\
&\qquad \times \left\langle \prod_{i=1}^{6}\sigma_{\epsilon_i}(a_i)e^{\mp k\alpha_0\phi^\dagger(x)}e^{\pm\hat{a}_-\phi^\dagger(u_l)}e^{\pm\hat{a}_+\phi^\dagger(v_m)}\right\rangle \\
&\qquad \times \left\langle \prod_{i=1}^{6} V^\psi_{\epsilon_i}(a_i)\psi^{(k)}(x)\psi^{(k_l)}(u_l)\psi^{(k_m)}(v_m)\right\rangle,
\end{aligned} \tag{4.9}$$

where $\sum \epsilon_i = 0 \bmod 2$ and

$$\tilde{Q}^-_\pm = \oint_C dx\left(\psi(x) \pm \psi^\dagger(x)\right)\exp\left(-\frac{p}{2}\alpha_0(\phi(x) \pm \phi^\dagger(x))\right),$$
$$2\hat{a}_- = -p\alpha_0, \qquad k_i = 0,1, \qquad \psi^{(0)} \sim \psi[{}^{2\,1}_{1\,1}], \qquad \psi^{(1)} \sim \psi[{}^{1\,1}_{2\,1}],$$

and

$$V_{\epsilon_i} = V^\psi_{\epsilon_i}\sigma_{\epsilon_i}e^{\frac{\alpha_0}{2}\phi}, \qquad V^\psi_0 \equiv V^\psi_{[1]}, \qquad V^\psi_1 \equiv V^\psi_{[3]}.$$

The correlation function $\tilde{G}(p_a, a_i, x_k)$ of $2g+2 = 6$ twist fields σ_{ϵ_i} and an arbitrary number of the untwisted vertices $e^{q\phi^\dagger}$ has the form (3.36)It remains to construct the multipoint Ising fermion correlation function on $X^{(2)}_{g=2}$ [58]:

$$\begin{aligned}
S^{g=2}_{\tilde{p}=3}(a_i;u_l,v_m) &= \left\langle \prod_{i=1}^{6} V^{\psi}_{\epsilon_i}(a_i) \prod_{l=1}^{L} \psi[{}^{2\ 1}_{1\ 1}](u_l) \prod_{m=1}^{M} \psi[{}^{1\ 1}_{2\ 1}](v_m) \tilde{Q}^{-}_{\pm} Q^{-}_{\pm} (Q^{+}_{\pm})^{s} \right\rangle, \\
&= \prod_{j=1}^{s} \oint_{C_j^+} dz_j \oint_{C^-} dx \oint_{C^-} dy \left\langle \prod_{i=1}^{6} e^{\frac{\alpha_0}{2}\phi(a_i)} \prod_{l,m} e^{-\frac{a_+}{2}\phi(u_l)} \right. \\
&\qquad \left. \times\, e^{-\frac{a_+}{2}\phi(v_m)} e^{-3\alpha_0\phi(x)} e^{a_-\phi(y)} e^{a_+\phi(z_j)} \right\rangle \\
&\quad \times \left\langle \prod_{i=1}^{6} \sigma_{\epsilon_i}(a_i) \prod_{l,m} e^{-\frac{a_+}{2}\phi^\dagger(u_l)} e^{\frac{a_+}{2}\phi^\dagger(v_m)} e^{\pm 3\alpha_0\phi^\dagger(x)} e^{\mp a_-\phi^\dagger(y)} e^{\mp a_+\phi(z_j)} \right\rangle \\
&= \sum_{p_1,p_2} \prod_{j=1}^{s} \oint dz_j \oint dx \oint dy \tilde{G}_3(a_i,p_1,p_2,\{q_i\},R^2=\frac{3}{2})(POWERS)
\end{aligned} \tag{4.10}$$

where

$$L+M=2s, \qquad \alpha_0^2=\frac{1}{24}, \qquad a_+=\frac{2}{\sqrt{6}}, \qquad a_-=-\frac{3}{2\sqrt{6}},$$

and $\tilde{G}_{p=3}$ is a special case of eq. (I.36). Putting all the ingredients, (3.36)and (4.10), into (4.9) we get the integral representation for the holomorphic ("left") part of the partition function. The modular invariant construction of the full $g=2$ partition function for the $N=1$ superconformal models will be given in [58].

For the non-split case we have to construct, in addition to (4.9), the following 6-point functions:

$$\left\langle \prod_{i=1}^{6} \tilde{V}_{\epsilon_i}(a_i,\alpha_i) \tilde{Q}^{-}_{\pm} (\hat{Q}^{-}_{\pm} \hat{Q}^{+}_{\pm})^{k-1} \right\rangle,$$

which requires evaluation of the Ising correlators of the type

$$\left\langle V_{\frac{9}{32}}(a_1) \prod_{i=2}^{5} V_{\frac{1}{32}}(a_i) V_{\frac{9}{32}}(a_6) \prod^{L} \psi[{}^{2\ 1}_{1\ 1}](u_l) \prod_{m=1}^{M} \psi[{}^{1\ 1}_{2\ 1}](v_m) \prod Q \right\rangle,$$

and of the modified Ashkin-Teller correlators

$$\left\langle \hat{\sigma}_{\frac{9}{16}}(a_1) \prod_{i=2}^{5} \sigma_{\frac{1}{16}}(a_i) \hat{\sigma}_{\frac{9}{16}}(a_6) \prod_{1=1}^{N} e^{q_i\phi^\dagger(z_i)} \right\rangle.$$

In conclusion we shall briefly describe another generalization of the method of Chapter 3, the construction of the Virasoro minimal models on the Z_N-

hyperelliptic surfaces $X_g^{(N)}$. Representing the stress-energy tensor $T(y)$ by N fields $T^{(k)}(z)$, $k = 0, \ldots, N-1$ (one for each sheet $y^{(k)} = e^{2\pi i \frac{k}{N}}(z-a)^{\frac{1}{N}}$) in the Z_N-diagonal basis

$$\begin{aligned} T_{(\mu)}(z) &= \sum_{k=0}^{N-1} e^{-2\pi i \frac{k\mu}{N}} T^{(k)}(z), \\ \Pi_a T_{(\mu)} &= e^{2\pi i \frac{\mu}{N}} T_{(\mu)}, \qquad \Pi_a^N = 1 \end{aligned} \tag{4.11}$$

we obtain the following OPE algebra:

$$T_{(\mu)}(1)T_{(\nu)}(2) = \frac{N\hat{c}}{2z_{12}^4}\delta^{Z_N}_{\mu+\nu,0} + \frac{2}{z_{12}^2}T_{(\mu+\nu)}(2) + \frac{1}{z_{12}}\partial T_{(\mu+\nu)}(2) + \cdots$$

In the discrete basis

$$T_{(\mu)}(z)V_{[l]}(0) = \sum_n z^{-n+\frac{\mu l}{N}-2} L^{(\mu)}_{n-\frac{\mu l}{N}} V_{[l]}(0)$$

the above OPE algebra takes the form:

$$\begin{aligned} \left[L^{(\mu)}_{m-\frac{\mu l}{N}}, L^{(\nu)}_{n-\frac{\nu l}{N}}\right] &= \left(m - n - \frac{(\mu-\nu)l}{N}\right) L^{(\mu+\nu)}_{n+m-\frac{(\mu+\nu)l}{N}} + \\ &+ \frac{N\hat{c}}{12}\left(m - \frac{\mu l}{N}\right)\left[\left(m - \frac{\mu l}{N}\right)^2 - 1\right] \delta^{Z_N}_{\mu+\nu,0}\delta_{n+m-\frac{(\mu+\nu)l}{N},0} \end{aligned} \tag{4.12}$$

The simplest null-vector in the branched sector $l = 1$ is at level $1/N$:

$$L^{(1)}_{-\frac{1}{N}} V_{[1]}(0)|0\rangle = 0 \text{ iff } \Delta_{[1]} = \frac{N\hat{c}}{24}\left(1 - \frac{1}{N^2}\right).$$

A natural generalization of the Z_2 Coulomb gas representation is the following Z_N Coulomb gas:

$$\begin{aligned} T_{(\mu)} &= \frac{1}{2N}\sum_{\nu=0}^{N-1} \partial\varphi_{(\nu)}\partial\varphi_{(\mu-\nu)} + \alpha_0 \partial^2 \varphi_{(\mu)} \\ \Pi_a \partial\varphi_{(\mu)} &= e^{2\pi i \frac{\mu}{N}} \partial\varphi_{(\mu)}, \qquad \varphi_{(0)} \equiv \phi \\ V_{[1]} = e^{\alpha_0(1-\frac{1}{N})\phi}\sigma, \quad \sigma &= \prod_{k=1}^{N-1} \sigma_k, \quad \Delta(\sigma_k) = \frac{1}{4}\frac{k}{N}\left(1 - \frac{k}{N}\right), \quad \Delta(\sigma) = \frac{N^2-1}{24N} \end{aligned} \tag{4.13}$$

where σ_k are the corresponding Z_N-orbifold twisted fields [56]. The Kac-spectrum for the branched sectors $l = 1, \ldots, N-1$ of the minimal models of (4.12) is given by the following finite set of primary fields:

$$\begin{aligned} V_{[l]}^{n,m} &= e^{\alpha_{n,m}(l)\phi}\sigma, \qquad \alpha_{n,m}(l) = \frac{N-ln}{N}a_{+} + \frac{N-lm}{N}a_{-} \\ \Delta_{n,m}^{[l]} &= \alpha_{n,m}(l)N\left(\frac{\alpha_{n,m}(l)}{2} - \alpha_0\right) + \frac{N^2-1}{24N} \end{aligned} \tag{4.14}$$

As in the case of the Z_2-surfaces, the partition function of the minimal models on $X_g^{(N)}$ can be written in terms of the lowest energy branching operators $V_{[1]}^{1,1} \equiv V_{[N-1]}^{1,1}$ with $\Delta = \frac{N\hat{c}}{24}(1-\frac{1}{N^2})$ and the screening operators

$$S_{(k)}^{\pm} = \exp\left[\frac{\alpha_{\pm}}{N}\sum e^{2\pi i\frac{k\mu}{N}}\varphi_{(\mu)}\right]$$

Again, the nontrivial ingredients of these constructions are the mixed correlation functions of twisted fields $\sigma_{(\mu)}$ and untwisted vertices $\exp\left[\frac{\alpha_{\pm}}{N}(-1)^{\frac{2k\mu}{N}\varphi_{(\mu)}}\right]$ (see [58]).

4.2 Generalization for the higher levels

The natural question that poses itself is does the strategy of solving general $V(L,M)$ models by using the Virasoro models as the building blocks extend from the sphere to higher genus Riemann surfaces. In the following we consider hyperelliptic surfaces X_g.

As it was shown in Chapter 3 the Virasoro models $V(M)$ on X_g can be described in terms of the minimal models of $WD_2(M) \equiv O(4)_k \times O(4)_1/O(4)_{k+1}$ on the branched sphere [36, 57]. The supersymmetric minimal models $V(2,M)$ on splited hyperelliptic supersurfaces SX_g are mapped into the $WD_2(2,M)$ minimal models ($\sim D_4^{p=2}$ parafermionic minimal models) on the branched sphere [38]. Then the hyperelliptic version of (1.3) reads as:

$$WD_2(1) \otimes WD_2(2,M) = \mathsf{P}(WD_2(M) \otimes WD_2(M+1)). \tag{4.15}$$

One can easily repeat all the constructions of Chapter 1. For example, the currents of $WD_2(2,M)$: T, $T^{\dagger}$, G, $G^{\dagger}$ have a form similar to (1.19) and (1.20), where the basic ingredients now are the fields $\psi[{}^{1\,1}_{1\,2}] \equiv \psi_{12}(M)$, $\psi[{}^{1\,2}_{1\,1}] \equiv \psi_{12}^{\dagger}(M)$, $\psi_{21}(M+1)$, $\psi_{21}^{\dagger}(M+1)$ *etc.* (see [36, 38] for the notation).

Following our method we will indicate that the hyperelliptic counterpart of (1.7) is

$$WD_2(L-1)\otimes WD_2^{(L)}(M)=\mathsf{P}(WD_2(M)\otimes WD_2^{(L-1)}(M+1)). \qquad (4.16)$$

In this way one could compute the partition functions of $V(L,M)$ models on the torus and genus two Riemann surfaces as 4- and 6- point functions of certain conformal fields from $WD_2(M)$ models on the branched sphere.

Let us consider in more detail the relations between the fields $V_{rs}(L,M)$ from the branching sectors. They are the images on the sphere of the fields $\phi_{rs}(L,M)(a)$ on X_g under the hyperelliptic map $y^2=z-a$, with dimensions [36]:

$$\Delta_{rs}^{br}(L,M)=\frac{1}{2}\Delta_{rs}(L,M)+\frac{1}{16}c(L,M).$$

According to (1.2) and (1.4), we find the following identities:

$$\begin{aligned}\Delta_{rp}^{br}(1,M)+\Delta_{ps}^{br}(1,M+1)&=\Delta_{rs}^{br}(2,M)+\frac{1}{4}\left(p-\frac{r+s}{2}\right)^2+\frac{\epsilon_{rs}}{32},\\ \epsilon_{rs}&=\begin{cases}1 & \text{if } r-s\in 2\mathsf{Z},\\ 0 & \text{if } r-s\in 2\mathsf{Z}+1.\end{cases}\end{aligned} \qquad (4.17)$$

In terms of the fields we get:

$$\begin{aligned}V_{rs}^{br}(2,M)V_{11}^{br}(1,1)&=V_{r,\frac{1}{2}(r+s)}^{br}(1,M)V_{\frac{1}{2}(r+s),s}^{br}(1,M+1), \quad r-s\in 2\mathsf{Z},\\ V_{rs}^{br}(2,M)V_{12}^{br}(1,1)&=V_{r,\frac{1}{2}(r+s\mp1)}^{br}(1,M)V_{\frac{1}{2}(r+s\mp1),s}^{br}(1,M+1), \quad r-s\in 2\mathsf{Z}+1.\end{aligned}$$

These constructions have a simple generalization for arbitrary level L coset model:

$$\begin{aligned}&V_{rs}^{br}(L,M)V_{l,l+1}^{br}(1,L-1)=\\ &=V_{r,\frac{1}{L}((L-1)r+s\pm(L-l))}^{br}(1,M)V_{\frac{1}{L}((L-1)r+s\pm(L-l)),s}^{br}(L-1,M+1),\end{aligned}$$

where $r-s\in L\mathsf{Z}\pm l$. An important open problem is to construct the multipoint correlation funtions of $V_{11}^{br}(L,M)$ in terms of the conformal blocks of the Virasoro branching fields $V_{rs}^{br}(1,M)$ only. One can hope that there exists an appropriate generalization of our recipe for the construction of the 4-point functions of the $V(L,M)$-models to the 4- and 6-point functions of the fields from the branching sector of the $WD_2(L,M)$ models. In that case following the methods of [36, 38] one could find higher genus partition functions of the $SU(2)$-coset models.

Chapter 5

RG- Flow in General SU(2) Coset Models

With the solutions of the conformal coset models $V(L,M)$ in hand, one can study their off-critical behaviour perturbing the conformal action by adding certain fields. In the general case the most relevant operator is not unique, and one can consider various (one- and multi-parameter) perturbations. As a simple example, consider the product of two Virasoro models. There are two most relevant fields with equal dimension $\Delta = 1 - \frac{2}{M+4} \equiv 1 - \epsilon$:

$$\Phi_1 = \phi_{11}^M \phi_{13}^{M+1}, \qquad \Phi_2 = \phi_{13}^M \phi_{33}^{M+1}.$$

The perturbed theory

$$S = S_0 + g_1 \int \Phi_1 d^2 z + g_2 \int \Phi_2 d^2 z \tag{5.1}$$

will have a UV fixed point described by the conformal theory S_0. The conformal

OPEs of Φ_1 and Φ_2 have the form

$$\begin{aligned}\Phi_1\Phi_1 &= C_{111}\Phi_1 + \cdots,\\ \Phi_1\Phi_2 &= C_{122}\Phi_2 + \cdots,\\ \Phi_2\Phi_2 &= C_{221}\Phi_1 + C_{222}\Phi_2 + \cdots.\end{aligned} \tag{5.2}$$

The corresponding β-functions can be computed in the perturbation theory:

$$\begin{aligned}\beta_1 &= \epsilon g_1 - \frac{1}{2}C_{111}g_1^2 - \frac{1}{2}C_{122}g_2^2 + \mathcal{O}(g^3),\\ \beta_2 &= \epsilon g_2 - C_{212}g_1g_2 - \frac{1}{2}C_{222}g_2^2 + \mathcal{O}(g^3).\end{aligned} \tag{5.3}$$

It is obvious that (β_1, β_2) have *two* nontrivial fixed points in the vicinity of $g_1 = 0$, $g_2 = 0$. The first of them is given by $g_2 = 0$ and the solution of the equation

$$\beta_1 = \epsilon g_1 + C_{111}g_1^2 = 0.$$

Since $C_{111} \equiv C^{M+1}_{(13)(13)(13)}$ it is obvious that this solution corresponds to the RG-flow

$$V(M) \otimes V(M+1) \to V(M) \otimes V(M), \tag{5.4}$$

where $V(M) \otimes V(M)$ model is equivalent to $V(M)$ on the hyperelliptic surface. The second nontrivial fixed point is described below and reduces effectively to the fixed point of the one-parameter flow $V(2, M) \to V(2, M-2)$ generated by

$$S^{int} = g\int(a\phi^M_{11}\phi^{M+1}_{13} + b\phi_{13}\phi_{33})d^2z, \tag{5.5}$$

The integrand in (5.5) is in fact the second component of the NS field N_{13} in $V(2, M)$, which we will denote by $\varphi_{13}(2, M)$, and the coefficients a and b are such that

$$\varphi_{13} \times \varphi_{13} = 1 + C_{(13)'(13)'(13)'}\varphi_{13} + \cdots.$$

Therefore (5.5) leads to the RG flow*

$$V(M) \otimes V(M+1) \to V(M-2) \otimes V(M-1). \tag{5.6}$$

Maybe the most interesting aspect of the two flows (5.4) and (5.6) is that the latter one preserves the supersymmetry in the projected part of the tensor product,

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* Note that an RG-flow of a projected product of models implies such a flow for the product of models itself. The reason is that an RG-flow to one loop is caused by the existence (and precise value) of certain 3-point functions, and 3-point functions that exist in a projected product certainly exist and have the same value also in the product itself.

while the former one breaks the supersymmetry.

The general case can be studied inductively [49]. Let

$$\varphi_{13}(L,M) = a(L,M)\phi_{11}^M\varphi_{13}(L-1,M+1) + b(L,M)\phi_{13}^M\phi_{33}(L-1,M+1).$$

In a similar way as above, the perturbation

$$S = S_0 + g_1\int\phi_{11}^M\varphi_{13}(L-1,M+1) + g_2\int\phi_{13}^M\phi_{33}(L-1,M+1)$$

gives the following scheme of RG-flows:

$$\begin{tikzcd}
 & V(1,M)\otimes V(L-1,M+1) \arrow[dl, "\phi_{11}^M\varphi_{13}(L-1,M+1)"'] \arrow[dr, "\varphi_{13}(L,M)"] & \\
V(M)\otimes V(L-1,M+2-L) & & V(1,M-L)\otimes V(L-1,M-L+1).
\end{tikzcd}$$

Now we want to prove the statement annonced above.Using our construction one can obtain the fusion rules and the 4-point functions in the L-th series of models.In doing this one has to follow the projection described above - only the products of the type $\phi_{mk}\phi_{kn}$ are permitted. This means that in computing the four-point functions only the products of conformal blocks corresponding to such products of fields have to be taken into account. In this way the structure constants needed for our purposes here can be obtained inductively as products of the structure constants of the Virasoro models. In the case of the $M(2,p)$ models the structure constants obtained in this way exactly coincide with the known ones [47,26].

We will perturb $M(l,p)$ with the field $\phi_{13}(l,p)$ with the dimension $\Delta(l,p) = 1-\epsilon(l,p)$, $\epsilon(l,p) = 2/(p+l) \ll 1$. This field is the first descendant, with respect to the additional currents of the theory, of the field $\Phi_{13}(l,p)$. We construct it in terms of the fields from the lower levels:

$$\phi_{13}(l,p) = a(l,p)\phi_{11}(1,p)\phi_{13}(l-1,p+1) + b(l,p)\phi_{13}(1,p)\phi_{33}(l-1,p+1), \quad (5.7)$$

where $a(l,p)$ and $b(l,p)$ are coefficients to be determined. We will also need

$$\phi_{33}(l,p) = \phi_{33}(1,p)\phi_{33}(l-1,p+1). \quad (5.8)$$

Before we proceed with the general argument, let us look at the concrete examples of $l = 2$ and $l = 4$. For the superconformal series $M(2,p)$, the field $\phi_{13}(2,p)$ has all the properties of the second component of the NS field $N_{13}(2,p)$. Indeed, acting with the supercurrent G given by ((1.6)) on the first component $N_{13}(2,p) = \phi_{12}(1,p)\phi_{23}(1,p+1)$ we obtain

$$\begin{aligned} G(z)N_{13}(2,p)(0) &= \frac{1}{z}\sqrt{\frac{p-2}{p+2}}\left(\sqrt{\frac{p-2}{2(p-1)}}\phi_{11}(1,p)\phi_{13}(1,p+1)\right. \\ &\qquad \left. + \sqrt{\frac{p}{2(p-1)}}\phi_{13}(1,p)\phi_{33}(1,p+1)\right) + \mathcal{O}(1) \qquad (5.9) \\ &= \frac{1}{z}\sqrt{2\Delta_{13}(2,p)}N_{13}^{II}(2,p)(0) + \cdots . \end{aligned}$$

Using ((5.9)) we reproduce the well-known result

$$N_{13}^{II}(2,p)N_{13}^{II}(2,p) = 1 + (p-4)\sqrt{\frac{2}{p(p-2)}}\mathcal{G}(p+1)N_{13}^{II}(2,p) + \cdots, \qquad (5.10)$$

where

$$\mathcal{G}(p) = \left\{\gamma^3\left(\frac{p}{p+1}\right)\gamma^4\left(\frac{2}{p+1}\right)\gamma^2\left(\frac{p-3}{p+1}\right)\gamma\left(\frac{3}{p+1}\right)\right\}^{\frac{1}{4}},$$

and $\gamma(p) = \Gamma(p)/\Gamma(1-p)$. The structure constants that appear in (5.9)and (5.10)are square roots of the standard ones. The square root is due to the fact that we are using the "asymmetric" second component $G_{-1/2}N_{13}(2,p)$ instead of the "symmetric" one $\bar{G}_{-1/2}G_{-1/2}N_{13}(2,p)$ (see App. A for detailes). Similarly, the other fields $\phi_{13}(l,p)$ in (5.7)are descendants with respect to the additional currents of the theory. For example, in the spin 4/3 theory $M(4,p)$, [33], the field $\phi_{13}(4,p)$ is obtained by acting with the spin 4/3 current Ψ on the field $\mathcal{D}_{13}(4,p)$

$$\phi_{13}(4,p) = \Psi^{\dagger}_{-2/3}(z)\mathcal{D}_{13}(4,p)(0) + \Psi_{-2/3}\mathcal{D}_{13}^{\dagger}(4,p)(0). \qquad (5.11)$$

Therefore, the structure constants C appearing in our construction below (for example in (14)) are square roots of the structure constants $\mathcal{C}$ corresponding to the "symmetric" fields used to perturb the theory.

Using what we have learned about the superconformal series, we can rewrite $\phi_{13}(4,p)$ in terms of the $M(2,p)$ fields:

$$\phi_{13}(4,p) = \sqrt{\frac{p-2}{2p}}N_{11}(2,p)N_{13}^{II}(2,p+2) + \sqrt{\frac{p+2}{2p}}N_{13}^{II}(2,p)N_{33}(2,p+2). \qquad (5.12)$$

Since the $M(2,p)$ structure constants are known [26,47], for the OPE of the field

(5.10)with itself we obtain

$$\phi_{13}(4,p)\phi_{13}(4,p) = \frac{p-6}{\sqrt{(p-2)(p+2)}}\mathcal{G}(p+3)\phi_{13}(4,p) + \cdots . \tag{5.13}$$

In closing with this example, note that there is one more slightly relevant field in this theory; $S_{15}(4,p)$ with dimension $\Delta_{15}(4,p) = 1 - 6/(p+4)$. It is constructed as $S_{15}(4,p) = N_{13}(2,p)N_{35}(2,p+2)$ and has an OPE

$$S_{15}(4,p)S_{15}(4,p) = 2\frac{p-2}{p}\mathcal{G}'(p)\phi_{13}(4,p) + \cdots ,$$

$$\mathcal{G}'(p) = \left[\gamma^3\left(\frac{p+3}{p+4}\right)\gamma^2\left(\frac{2}{p+4}\right)\gamma^2\left(\frac{4}{p+4}\right)\gamma^2\left(\frac{p-2}{p+4}\right)\gamma\left(\frac{3}{p+4}\right)\right]^{\frac{1}{2}} . \tag{5.14}$$

With two slightly relevant fields, we can use a more general perturbation of the CFT given by $\mathcal{H}_{int} = \int d^2z(g_1\phi_{13} + g_2 S_{15})$. Now we have two β-functions β_i and a system of equations for their zeros:

$$\beta_1(g_1^*, g_2^*) \equiv \epsilon_1 g_1^* - \frac{1}{2}(C_{111}g_1^{*2} + C_{122}g_2^{*2}) = 0, \tag{(5.15)a}$$

$$\beta_2(g_1^*, g_2^*) \equiv \epsilon_2 g_2^* - \frac{1}{2}(2C_{212}g_1^* g_2^*) = 0, \tag{(5.15)b}$$

$$\epsilon_1 = \frac{2}{p+4}, \quad \epsilon_2 = \frac{6}{p+4}, \quad C_{111} = \frac{1}{\sqrt{3}} + \mathcal{O}\left(\frac{1}{p}\right), \quad C_{122} \equiv C_{212} = \frac{\sqrt{3}}{2} + \mathcal{O}\left(\frac{1}{p}\right).$$

The obvious solution $g_2^* = 0$, $g_1^* = 2\sqrt{3}\epsilon_1$ coincides with the original case in consideration and leads to the change in the central charge

$$\Delta c = -\frac{192}{p^3} = c_{p-4} - c_p.$$

Surprisingly, it turns out that this is the only nontrivial solution. Starting with ((5.15)b) we obtain $g_1^* = 2\sqrt{3}\epsilon_1$, which by ((5.15)a) gives $g_2^* = 0$.

Returning to the discussion for a general l, we demand that the following OPEs be satisfied (supressing the obvious factors of z):

$$\phi_{13}(l,p)(z)\phi_{13}(l,p)(0) = 1 + C_{(13)(13)(13)}(l,p)\phi_{13}(l,p)(0) + \cdots , \tag{(5.16)a}$$

$$\begin{aligned}\phi_{33}(l,p)(z)\phi_{33}(l,p)(0) = 1 &+ C_{(33)(33)(13)}(l,p)\phi_{13}(l,p)(0) \\ &+ C_{(33)(33)(33)}(l,p)\phi_{33}(l,p)(0) + \cdots \end{aligned} \tag{(5.16)b}$$

By using (5.7)in ((5.16)a) we obtain the equations for the structure constants at a level l in terms of the structure constants at the lower levels:

$$a^2(l,p)C_{(13)(13)(13)}(l-1,p+1)$$
$$+\,b^2(l,p)C_{(33)(33)(13)}(l-1,p+1) = a(l,p)C_{(13)(13)(13)}(l,p), \quad (5.17)a$$
$$a^2(l,p)C_{(33)(33)(13)}(l-1,p+1)$$
$$+b(l,p)C_{(13)(13)(13)}(1,p)C_{(33)(33)(33)}(l-1,p+1) = C_{(13)(13)(13)}(l,p). \quad (5.17)b$$

Similarly, using (5.8)in ((5.16)b) results in

$$C_{(33)(33)(13)}(l-1,p+1) = a(l,p)C_{(33)(33)(13)}(l,p), \quad (5.18)a$$
$$C_{(33)(33)(13)}(1,p)C_{(33)(33)(33)}(l-1,p+1) = b(l,p)C_{(33)(33)(13)}(l,p), \quad (5.18)b$$
$$C_{(33)(33)(33)}(1,p)C_{(33)(33)(33)}(l-1,p+1) = C_{(33)(33)(33)}(l,p). \quad (5.18)c$$

Finally, requiring that the two-point function of $\phi_{13}(l,p)$ be normalized to 1 gives the final constraint

$$a^2(l,p) + b^2(l,p) = 1, \qquad \forall l,p. \quad (5.19)$$

Prompted by the results for $l = 1, 2$ and 4, we claim that

$$C_{(13)(13)(13)}(l,p) = \frac{2(p-l-2)}{\sqrt{l(p+l-2)(p-2)}}\mathcal{G}(p+l-1), \quad (5.20)a$$
$$C_{(33)(33)(13)}(l,p) = -2\sqrt{\frac{l}{(p+l-2)(p-2)}}\mathcal{G}(p+l-1), \quad (5.20)b$$
$$C_{(33)(33)(33)}(l,p) = \frac{\mathcal{G}(p+l-1)}{\mathcal{G}(p-1)}, \quad (5.20)c$$
$$a(l,p) = \sqrt{\frac{(l-1)(p-2)}{l(p-1)}}, \qquad b(l,p) = \sqrt{\frac{l+p-2}{l(p-1)}}. \quad (5.20)d$$

Note that (5.20)satisfies (5.18)and (5.19).

The proof proceeds inductively: assume (5.20)for $l-1$, and then, using (5.17), (5.18)and (5.19), derive (5.20)for l. From (5.17)and (5.19)follows a quadratic equation for $a^2(l,p)$:

$$a^4(l,p)[(p+2l-3)^2(p-2) + (p-3)^2(l-1)(p+l-2)]$$
$$+\,a^2(l,p)[-2(l-1)(p+2l-3)(p-2) - (p-3)^2(l-1)(p+l-2)]$$
$$+\,(l-1)^2(p-2) = 0.$$

The discriminant is $[(p-3)(l-1)(p+l-2)(p-1)]^2$ and one of the solutions is

$a^2(l,p) = \frac{(l-1)(p-2)}{l(p-1)}$, in agreement with ((5.20)d). Using that result in ((5.17)b), we verify ((5.20)a). Finally, ((5.20)b) and ((5.20)c) follow from (5.18)and the knowledge of $a(l,p)$ and $b(l,p)$.

For our present purposes of studying the RG-flow generated by $\phi_{13}(l,p)$, the crucial identity is ((5.20)a), from which we read off the structure constant at an arbitrary level l:

$$\begin{aligned} \mathcal{C}_{(13)(13)(13)}(l,p) = [C_{(13)(13)(13)}(l,p)]^2 &= \frac{4(p-l-2)^2}{l(p+l-2)(p-2)}[\mathcal{G}(p+l-1)]^2, \\ &= \frac{4}{l\sqrt{3}} + \mathcal{O}\left(\frac{1}{p}\right). \end{aligned} \tag{5.21}$$

From (5.3)and (5.21), the fixed point is at $g^* = \sqrt{3}l/p$, and the corresponding $\Delta c = -12l^2/p^3$. Therefore, to the required accuracy,

$$\Delta c = c(l, p-l) - c(l,p),$$

thus proving the announced result.

Appendix A

This appendix should clarify further the origin of the square roots of the structure constants in 1D OPEs. As pointed out already in the discussion in sec.1.3, they appear due to our representation of the currents like ψ and G in terms of the non-free fields like ϕ_{12}^M, ϕ_{13}^M (cf. (1.19)and (1.20)). We begin by constructing explicitly the monodromy-invariant correlation functions of 1D fields like $\psi_p(z) = \phi_{1p}^M \phi_{p1}^{M+1}(z)$. In particular, we will demonstrate that the structure constants appearing in the limits of such 1D correlation functions are given by the square roots of the usual 2D structure constants. The 1D correlation function we will study first is

$$G = \left\langle \prod_{i=1}^{4} \phi_{1p_i}^M \phi_{p_i 1}^{M+1}(z_i) \right\rangle, \qquad 1 \leq p_i \leq M+2,$$

where we will set $p_1 = p_3$, $p_2 = p_4$ for simplicity. Using the notation and arguments explained in sec.2.1, we write

$$G = \sum_i x_i I_{1i}^M I_{i1}^{M+1}(z). \tag{A1}$$

We have to fix x_i such that G is well-defined on the whole complex plane. The intermediate fields that appear are

$$\phi_{1,p_1+p_2-2i+1}^M \phi_{p_1+p_2-2i+1,1}^{M+1},$$

with the dimension equal to

$$\Delta_{1,p_1+p_2-2i+1}^M + \Delta_{p_1+p_2-2i+1,1}^{M+1} = \frac{1}{2}(p_1+p_2-2i)^2.$$

Since $\frac{1}{2}(p_1+p_2-2i)^2 - \frac{1}{2}(p_1-1)^2 - \frac{1}{2}(p_2-1)^2 \in \mathbb{Z}$ each term in (A1) by itself is already invariant under $z \to ze^{2\pi i}$. To study the monodromy transformation

$(z-1) \to (z-1)e^{2\pi i}$, we use the α-matrices, introduced and discussed in chap. 5. After the α-transformation, we have

$$G = \sum_i x_i \alpha_{ij}^{(1p)}(M) \alpha_{ik}^{\prime(p1)}(M+1) I_{1j}^M (1-z) I_{k1}^{M+1}(1-z).$$

But, due to (2.5) $\sum_i x_i \alpha_{ij}^{(1p)}(M)\alpha_{ik}^{\prime(p1)}(M+1) \propto \delta_{jk}$ if x_i is proportional to $X_i^{(1p)}(M) = X_i^{\prime(p1)}(M+1)$ [27]. We conclude that $G = \sum_i X_i^M I_{1i}^M I_{i1}^{M+1}$ is a monodromy-invariant 1D correlation function. In a similar way one can show that

$$G = \left\langle \prod_{i=1}^{4} \phi_{1p_1}^M \phi_{p_1 p_2}^{M+1} \cdots \phi_{p_{L-1}1}^{M+L-1}(z_i) \right\rangle$$

is a monodromy-invariant 1D correlation function.

Now it is easy to determine the structure constants. In the i-th channel we have

$$\begin{aligned} \sqrt{X_i^M N_{1i}^M N_{i1}^{M+1}} &= X_i^M \sqrt{\frac{C^M_{(1p_1)(1p_2)(1,p_1+p_2-2i+1)}}{X_i^M}} \sqrt{\frac{C^{M+1}_{(p_1 1)(p_2 1)(p_1+p_2-2i+1,1)}}{X_i^{\prime M+1}}}, \\ &= \sqrt{C^M_{(1p_1)(1p_2)(1,p_1+p_2-2i+1)} C^{M+1}_{(p_1 1)(p_2 1)(p_1+p_2-2i+1,1)}}, \end{aligned} \tag{A2}$$

where Ns are defined by

$$I_{ij}^M(z) \to N_{ij}^M z^x (1 + O(z)) \quad \text{as} \quad z \to 0.$$

Note that the square roots in (A2) are due to the fact that the constants appearing in the limits of the correlation functions are *squares* of the structure constants appearing in the OPEs (*i.e.* $XN^2 = C_{ABD}$ for the correlation function $\langle ABAB \rangle$), and are therefore standard, both in 1D and 2D correlation functions. What is specific to the 1D correlation functions is that under those square roots one obtains C as opposed to C^2 in the 2D case.

The next on our list are the "mixed" correlation functions of the currents and fields like

$$\begin{aligned} H &= \left\langle \phi_{1p}^M \phi_{p1}^{M+1}(z_1) \phi_{mn}(2,M)(z_2,\bar{z}_2) \phi_{1p}^M \phi_{p1}^{M+1}(z_3) \phi_{mn}(2,M)(z_4,\bar{z}_4) \right\rangle, \\ &= \left\langle \phi_{1p}^M \phi_{p1}^{M+1}(z_1) \phi_{mn}(2,M)(z_2) \phi_{1p}^M \phi_{p1}^{M+1}(z_3) \phi_{mn}(2,M)(z_4) \right\rangle \frac{1}{(\bar{z}_2 - \bar{z}_4)^{2\Delta}}. \end{aligned}$$

Similarly to the preceeding case, we have

$$H = \sum_i x_i I_{1i}^M I_{i1}^{M+1}(z).$$

This time, the intermediary fields are

$$\phi^{M}_{m,p+\frac{1}{2}(m+n)-2i+1}\phi^{M+1}_{p+\frac{1}{2}(m+n)-2i+1,n}$$

with the dimensions equal to

$$\Delta_I = \Delta^{M}_{m,p+\frac{1}{2}(m+n)-2i+1} + \Delta^{M+1}_{p+\frac{1}{2}(m+n)-2i+1,n} = \Delta_{mn}(2,M) + \frac{1}{2}(p-2i+1)^2.$$

The leading z-behaviour of the terms in H is given by z^b, where

$$b = \frac{1}{2}(p-2i+1)^2 - \frac{1}{2}(p-1)^2 \in \mathsf{Z}.$$

Again, each of the terms separately is invariant under $z \to ze^{2\pi i}$. The discussion of the $(1-z) \to (1-z)e^{2\pi i}$ transformation mirrors the one for the pure current-current correlator, and we again conclude that the structure constants appearing in the 1D OPEs are the square roots of the usual ones.

One could push a present discussion a step further, and discuss a (formal) definition of 1D OPEs even for some of the fields that have only 2D correlation functions. The benefit from such a definition would be apparent in a calculation where 1D OPEs would be significantly simpler. Given a well-defined one-to-one relation between 2D OPEs and (formal) 1D ones, one could use the 1D ones and go back to the 2D ones only at the end of the calculation.

The definition of the "square root" of a 2D OPE that we propose is the following: The square root of a 2D OPE is a 1D OPE such that by multiplying such a 1D OPE in z with another such 1D OPE in $\bar{z}$ one recovers the original 2D OPE. While multiplying the 1D OPEs one multiplies all z-terms and $\bar{z}$-terms whose dimensions agree up to an integer. Note that for a 2D OPE to have a square root, it is necessary that such terms appear as "complete squares." For example, a 2D OPE

$$\begin{aligned}\phi_a(1,\bar{1}) \times \phi_b(2,\bar{2}) = C_{abc}\frac{\phi_c(2,\bar{2})}{|z_{12}|^{2(\Delta_a+\Delta_b-\Delta_c)}} &+ \frac{1}{|z_{12}|^{2(\Delta_a+\Delta_b-\Delta_d)}}\Big[C_{abd}\phi_d(2,\bar{2}) \\ &+ C_{abe}\phi_e(2,\bar{2}) + \sqrt{C_{abd}C_{abe}}(\phi_d(2)\phi_e(\bar{2}) + \phi_e(2)\phi_d(\bar{2}))\Big], \\ \Delta_c \neq \Delta_d = \Delta_e,\end{aligned}$$

has as the square root the 1D OPE

$$\phi_a(1)\times\phi_b(2)=\sqrt{C_{abc}}\frac{\phi_c(2)}{z_{12}^{\Delta_a+\Delta_b-\Delta_c}}+\sqrt{C_{abd}}\frac{\phi_d(2)}{z_{12}^{\Delta_a+\Delta_b-\Delta_d}}+\sqrt{C_{abe}}\frac{\phi_e(2)}{z_{12}^{\Delta_a+\Delta_b-\Delta_e}}.$$

Such formal 1D OPEs were used in Chapter 5 (see also [49], eq. (14)). Since the lack of space prevented us from discussing that point at any lenght there, we are going to give here some of the details of the arguments justifying their use. If for simplicity we take $L=2$, the eq. (14a) of [49] is

$$\begin{aligned}\Big(a\phi_{11}^M\phi_{13}^{M+1}+b\phi_{13}^M\phi_{33}^{M+1}\Big)(z)\Big(a\phi_{11}^M\phi_{13}^{M+1}+b\phi_{13}^M\phi_{33}^{M+1}\Big)(0)&=\\ =1+\sqrt{C}\Big(a\phi_{11}^M\phi_{13}^{M+1}+b\phi_{13}^M\phi_{33}^{M+1}\Big)(0)+\cdots,\end{aligned}$$

where $a^2+b^2=1$ and the factors of z are surpressed. This statement makes sense if one can prove that the 2D OPE is

$$\phi_{13}^{II}(z,\bar z)\times\phi_{13}^{II}(0,0)=1+C\phi_{13}^{II}(0,0)+\cdots,\tag{A3}$$

where $\phi_{13}^{II}(z,\bar z)=\Big(a\phi_{11}^M\phi_{13}^{M+1}+b\phi_{13}^M\phi_{33}^{M+1}\Big)(z)\Big(a\phi_{11}^M\phi_{13}^{M+1}+b\phi_{13}^M\phi_{33}^{M+1}\Big)(\bar z)$.

The eq. (A3) can be proved, and in the following we give some of the details of the proof. One of the non-standard OPEs encountered while proving (A3) is

$$\phi_{11}^M\phi_{13}^{M+1}(1)\phi_{13}^M\phi_{33}^{M+1}(\bar 1)\times\phi_{11}^M\phi_{13}^{M+1}(2)\phi_{11}^M\phi_{13}^{M+1}(\bar 2),$$

with one of the fields left-right asymmetric, even though scalar. This OPE can be obtained as a limit of the correlation function

$$\begin{aligned}G=\Big\langle&\phi_{11}^M\phi_{13}^{M+1}(1)\phi_{13}^M\phi_{33}^{M+1}(\bar 1)\phi_{11}^M\phi_{13}^{M+1}(2,\bar 2)\\ &\times\phi_{11}^M\phi_{13}^{M+1}(3)\phi_{13}^M\phi_{33}^{M+1}(\bar 3)\phi_{11}^M\phi_{13}^{M+1}(4,\bar 4)\Big\rangle.\end{aligned}$$

Using our expertise from sec.2.1, we write

$$G=\sum_{i=1,2,3}x_iI_i^{M+1}(z)\overline{\bar I_i^{M+1}(z)}.$$

One can check that each of the terms is invariant under $z\to ze^{2\pi i}$. Performing an α-transformation, we get

$$G=\sum x_i\alpha_{ij}^{M+1}\bar\alpha_{ik}^{M+1}I_j^{M+1}(1-z)\overline{\bar I_k^{M+1}(1-z)}.$$

Similarly to the discussion in sec.2.1, since $a^{M+1}=2\alpha_-\alpha_{13}$ differs from $\bar a^{M+1}=$

$2\alpha_- \alpha_{33}$ at most by an integer (and similarly for b and c), $\alpha^{M+1} = \tilde{\alpha}^{M+1}$, and we conclude that G is monodromy invariant if $x_i = X_i^{M+1} = \tilde{X}_i^{M+1}$. The structure constants are

$$\sqrt{X_i^{M+1} N_i^{M+1} \tilde{N}_i^{M+1}} = \sqrt{C^{M+1}_{(13)(13)(1,7-2i)} C^{M+1}_{(13)(33)(3,7-2i)}}.$$

This 2D OPE is consistent with the 1D square root OPEs

$$\begin{aligned} \phi_{11}^M \phi_{13}^{M+1}(1) \times \phi_{11}^M \phi_{13}^{M+1}(2) &= \sum_{i=1}^{3} \sqrt{C^{M+1}_{(13)(13)(1,7-2i)}}\, \phi_{11}^M \phi_{1,7-2i}^{M+1}(2), \\ \phi_{11}^M \phi_{13}^{M+1}(1) \times \phi_{13}^M \phi_{33}^{M+1}(2) &= \sum_{i=1}^{3} \sqrt{C^{M+1}_{(13)(33)(3,7-2i)}}\, \phi_{13}^M \phi_{3,7-2i}^{M+1}(2). \end{aligned} \tag{A4}$$

(We keep writing the identity operator ϕ_{11}^M in (A4) to stress that it would not be true with a different $V(1,M)$ field in its place.)

In analogous fashion one can, starting from the correlation function

$$G = \left\langle \prod_{i=1}^{4} \phi_{11}^M \phi_{13}^{M+1}(z_i) \phi_{13}^M \phi_{33}^{M+1}(\bar{z}_i) \right\rangle = \sum \tilde{X}_{ji}^{M+1} I_i^{M+1}(z) \overline{I_j^M I_{ji}^{M+1}(z)},$$

obtain the following 2D OPE

$$\begin{aligned} &\phi_{11}^M \phi_{13}^{M+1}(1) \phi_{13}^M \phi_{33}^{M+1}(\bar{1}) \times \phi_{11}^M \phi_{13}^{M+1}(2) \phi_{13}^M \phi_{33}^{M+1}(\bar{2}) = \\ &\qquad = \sum_{i,j=1}^{3} \sqrt{C^{M+1}_{(13)(13)(1,7-2i)} C^{M}_{(13)(13)(1,7-2j)} C^{M+1}_{(33)(33)(7-2j,7-2i)}} \\ &\qquad\quad \times \phi_{11}^M \phi_{1,7-2i}^{M+1}(2) \phi_{1,7-2j}^M \phi_{7-2j,7-2i}^{M+1}(\bar{2}), \end{aligned}$$

whose "square roots" are the first eq. in (A4) and

$$\begin{aligned} &\phi_{13}^M \phi_{33}^{M+1}(1) \times \phi_{13}^M \phi_{33}^{M+1}(2) = \\ &\qquad = \sum_{i,j=1}^{3} \sqrt{C^{M}_{(13)(13)(1,7-2j)} C^{M+1}_{(33)(33)(7-2j,7-2i)}}\, \phi_{1,7-2j}^M \phi_{7-2j,7-2i}^{M+1}(2). \end{aligned}$$

Finally, the correlation function

$$\begin{aligned} \Big\langle &\phi_{11}^M \phi_{13}^{M+1}(1) \phi_{13}^M \phi_{33}^{M+1}(\bar{1}) \phi_{13}^M \phi_{33}^{M+1}(2) \phi_{11}^M \phi_{13}^{M+1}(\bar{2}) \\ &\times \phi_{11}^M \phi_{13}^{M+1}(3) \phi_{13}^M \phi_{33}^{M+1}(\bar{3}) \phi_{13}^M \phi_{33}^{M+1}(4) \phi_{11}^M \phi_{13}^{M+1}(\bar{4}) \Big\rangle \end{aligned}$$

gives the 2D OPE

$$\phi_{11}^{M}\phi_{13}^{M+1}(1)\phi_{13}^{M}\phi_{33}^{M+1}(\bar{1}) \times \phi_{13}^{M}\phi_{33}^{M+1}(2)\phi_{11}^{M}\phi_{13}^{M+1}(\bar{2}) = \\ = \sum_i C^{M+1}_{(13)(33)(3,7-2i)}\phi_{13}^{M}\phi_{3,7-2i}^{M+1}(2,\bar{2}),$$

whose "square roots" are obviously

$$\phi_{11}^{M}\phi_{13}^{M+1}(1) \times \phi_{13}^{M}\phi_{33}^{M+1}(2) = \\ = \phi_{13}^{M}\phi_{33}^{M+1}(1) \times \phi_{11}^{M}\phi_{13}^{M+1}(2) = \sum_i \sqrt{C^{M+1}_{(13)(33)(3,7-2i)}}\phi_{13}^{M}\phi_{3,7-2i}^{M+1}(2).$$

With these ingredients in hand, it is now straightforward to complete the proof that the 1D OPEs (with the structure constants equal to the square roots of the standard, 2D, ones), used in sec.2.1, are well-defined.

Appendix B

In this appendix we want to prove eq. (1.23) explicitly. The hypergeometric function F is defined as

$$\begin{aligned} F(a,b;c;z) &= 1 + \frac{ab}{c}z + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!} + \cdots, \\ &= \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}, \end{aligned} \tag{B1}$$

where $(a)_n \equiv a(a+1)\cdots(a+n-1)$, $(a)_0 \equiv 1$. One can check (1.23) for z^0, z and z^2 directly using (B1). In the following we will show that the coefficient next to z^n, $n \geq 3$ on the LHS of (1.23) is zero.

It is straightforward to prove the following identities for $A \in Z$:

$$\begin{aligned} (A+1-a)_q &= (-1)^q(a-A-q)_q, \\ (a)_{n-q} &= p_{aA}(a-A)_{n-q+A}, \\ (A+1-a)_q(a)_{n-q} &= (-1)^q(a-A-q)_{n+A}\, p_{aA}, \end{aligned} \tag{B2}$$

where

$$p_{aA} = \begin{cases} \frac{1}{(a-1)(a-2)\cdots(a-A)}, & \text{if } A > 0, \\ a(a+1)\cdots(a-A-1), & \text{if } A < 0. \end{cases}$$

Using (B2) one can write for the coefficient next to the z^n in

$$F\left(-\frac{M}{M+3},\frac{1}{M+3};\frac{2}{M+3};z\right)F\left(-\frac{M+6}{M+3},-\frac{1}{M+3};-\frac{2}{M+3};z\right)\equiv$$
$$\equiv F(a,b;c;z)F(-2-a,-b;-c;z)$$

the following expression

$$\sum_{q=0}^{n}\frac{1}{(n-q)!q!}\frac{(a)_{n-q}(b)_{n-q}}{(c)_{n-q}}\frac{(-2-a)_q(-b)_q}{(-c)_q}$$
$$=\frac{a(a+1)(a+2)b}{c}\sum_{q=0}^{n}\frac{(-1)^q(a+3-q)_{n-3}(b+1-q)_{n-1}}{(n-q)!q!(c+1-q)_{n-1}}.$$

Similarly, the coefficient next to the z^m in

$$F\left(\frac{M+2}{M+3},\frac{1}{M+3};\frac{2M+4}{M+3};z\right)F\left(\frac{M+4}{M+3},-\frac{1}{M+3};\frac{2M+8}{M+3};z\right)\equiv$$
$$\equiv F(u,v;w;z)F(2-u,-v;4-w;z)$$

is

$$\frac{v(w-1)(w-2)(w-3)}{u-1}\sum_{q=0}^{m}\frac{(-1)^q(u-1-q)_{m+1}(v+1-q)_{m-1}}{(m-q)!q!(w-3-q)_{m+3}}.$$

Since

$$\frac{a(a+1)(a+2)b}{c}=\frac{3M(M+6)}{4(M+1)(M+5)}\frac{v(w-1)(w-2)(w-3)}{u-1}$$

what remains to be proved is that

$$\sum_{q=0}^{n}\frac{(-1)^q(a+3-q)_{n-3}(b+1-q)_{n-1}}{(n-q)!q!(c+1-q)_{n-1}}$$
$$+\sum_{q=0}^{n-2}\frac{(-1)^q(u-1-q)_{n-1}(v+1-q)_{n-3}}{(n-2-q)!q!(w-3-q)_{n+1}}=0\quad\text{for } n\geq 3. \tag{B3}$$

We will prove (B3) by showing that the LHS is equal to the sum of the residues of a vanishing integral [54]. The integral is

$$I_n=\int_C\frac{(a+3-s)_{n-3}(b+1-s)_{n-1}}{(-s)_{n+1}(c+1-s)_{n-1}}ds,$$

where C is the circle of radius R around the origin. Clearly, I_n is defined for

$n \geq 3$ and, as $R \to \infty$, $I_n \to 0$. It has simple poles at $s = 0, 1, \ldots, n$ and at $s = 1+c, 1+c+1, \ldots, 1+c+n-2$. Therefore, the sum of residues is given by

$$\sum_{q=0}^{n} \frac{(-1)^q (a+3-q)_{n-3}(b+1-q)_{n-1}}{(n-q)!q!(c+1-q)_{n-1}} + \sum_{q=0}^{n-2} \frac{(-1)^q (b-c-q)_{n-1}(a+2-c-q)_{n-3}}{(n-2-q)!q!(-1-c-q)_{n+1}} = 0.$$

Since $a-c+2 = v+1$, $b-c = u-1$ and $-1-c = w-3$, we have proved (B3)and thus (1.23).

Appendix C

Given the projected tensor product model

$$V(1) \otimes V(2,M) = \mathsf{P}(V(M) \otimes V(M+1))$$

and the OPE algebra of the total stress-energy tensor $T = T^I + T^{SUSY}$ and the spin-3 current

$$W(z) = i\sqrt{\frac{3}{4}}\left(\partial\psi(z)G(z) - \frac{1}{3}\psi(z)\partial G(z)\right), \tag{C1}$$

where ψ is the Ising model free fermion and G the supercurrent of $V(2,M)$:

$$\begin{aligned}
\psi(1)\psi(2) &= \frac{1}{z_{12}} + 2z_{12}T^I(2) + z_{12}^2\partial T^I(2) + z_{12}^3\left(\frac{3}{10}\partial^2 T^I(2) + \frac{2}{7}\Lambda_I(2)\right) + \cdots, \\
G(1)G(2) &= \frac{1}{z_{12}^3} + \frac{3}{c}\frac{1}{z_{12}}T^S(2) + \frac{3}{2c}\partial T^S(2) + \\
&\quad + z_{12}\left(\frac{9}{20c}\partial^2 T^S(2) + \frac{51}{2c(5c+22)}\Lambda_S(2)\right) + \cdots, \\
\Lambda_i &=: (T^i)^2 : -\frac{3}{10}\partial^2 T^i, \qquad i = I, S, \\
c &= c(2,M).
\end{aligned} \tag{C2}$$

For $c = 7/10 \equiv c(2,1)$, T and W close the Zamolodchikov W-algebra [29], and the model $V(1) \otimes V(2,1)$ with $c = 6/5$ coincides with the second model of the W-algebra minimal series of models. To see this, use the OPEs (C2) and derive the corresponding OPEs of W:

$$W(1)W(2) = \frac{1}{z_{12}^6}\Big\{1 + \frac{12}{2c+1}z_{12}^2 T(2) + \frac{10c-7}{1+2c}z_{12}t_2(2) +$$
$$+ z_{12}^3\left[\frac{6}{1+2c}\partial T(2) + \frac{10c-7}{2(2c+1)}\partial t_2(2)\right] +$$
$$+ z_{12}^4\left[\frac{36}{20(1+2c)}\partial^2 T(2) + \frac{3(10c-7)}{20(1+2c)}\partial^2 t_2(2) + \right.$$
$$\left. + \frac{384}{(1+2c)(10c+49)}\Lambda(2) + (10c-7)t_4(2)\right] + \mathcal{O}(z_{12}^5)\Big\}, \tag{C3}$$

where

$$t_2 = T^I - \frac{1}{2c}T^S, \qquad \Lambda =: T^2 : -\frac{3}{10}\partial^2 T$$

and t_4 is a certain spin-4 field primary w.r.t. T. For $c = 7/10$ the fields t_2 and t_4 disappear from the OPE (C3) and the OPE algebra of W and T coincides with the Zamolodchikov W-algebra.

References

1. A.Polyakov, *JETP Lett.* **12** (1970) 381.
2. M.Virasoro, *Phys.Rev.* **D1** (1970) 2933.
3. A. Belavin, A. Polyakov, A. Zamolodchikov, *Nucl. Phys.* **B241** (1984) 333.
4. D.Friedan,Z.Qiu, S.Shenker, *Phys.Rev.Lett.* **52** (1984) 1575.
5. D. Friedan, Z. Qiu and S. Shenker, *Phys. Lett.* **151B** (1985) 37
   M. A. Bershadsky, V. G. Knizhnik and M. G. Teitelman, *Phys. Lett.* **151B** (1985) 31.
6. V.Knizhnik,A.Zamolodchikov, *Nucl.Phys.* **B247** (1984) 83
   E.Witten, *Comm.Math.Phys.* **92** (1984) 455
   D.Gepner,E.Witten, *Nucl.Phys.* **B278** (1986) 493
   A.Polyakov,P.Wiegmann, *Phys.Lett.* **131B** (1983) 121.
7. H.Sugawara, *Phys.Rev.* **170** (1968) 1659,
   C.Sommerfeld, *Phys.Rev.* **176** (1968) 2019.
8. A.Zamolodchikov, *Theor.Math.Fiz.* **65** (1985) 347.

9. E.Fradkin,L.Kadanoff, *Nucl.Phys.* **B170[FS1]** (1980) 1.

10. V.Fateev,A.Zamolodchikov, *ZhETP* **89** (1985) 380.

11. W.Boucher,D.Friedan, A.Kent, *Phys.Lett.* **172B** (1986) 316 ,
P.Di Vecchia,J.Petersen,M.Yu,H.Zheng, *Phys.Lett.* **174B** (1986) 280,
A.Zamolodchikov,V.Fateev, *ZhETP* **90** (1986) 1553.

12. A.Zamolodchikov,V.Fateev, *Landau Institute preprint* (1986).

13. V. A. Fateev and S. L. Lukyanov, *Int. Jour. Mod. Phys.* **A3** (1988) 507.
F. A. Bais, P. Bouwknegt, K. Schoutens and M. Surridge, *Nucl. Phys.* **B304** (1988) 371.

14. P. Goddard, A. Kent, D. Olive, *Phys. Lett.* **152B** (1985) 88, *Comm. Math. Phys.* **103** (1986) 105.

15. M.Bershadski,L.Vaisburg,A.Migdal, *JETP Lett.* **43** (1986) 143,
J.Cardy, *Nucl.Phys.* **B270[FS16]** (1986) 746,, *Nucl.Phys.* **B275[FS17]** (1986) 200,
J.Cardy,I.Peschel, *Nucl.Phys.* **B300[FS22]** (1988) 377.

16. D.Friedan,S.Shenker, *Nucl.Phys.* **B281** (1987) 509.

17. D.Gepner, *Nucl.Phys.* **B296** (1988) 757,, *Phys.Lett.* **199B** (1987) 380.

18. A.Belavin,V.Knizhnik, *Phys.Lett.* **168B** (1986) 201,
L.Alvarez-Gaume,G.Moore,C.Vafa, *Comm.Math.Phys.* **106** (1986) 1

19. T.Eguchi,H.Ooguri, *Nucl.Phys.* **B282** (1987) 308.

20. S.Mathur,S.Mukhi,A.Sen,Differential equations for correlators and characters in arbitrary rational conformal field theories,preprintTIFR/TH/88-32.

21. C.Itzykson,J.B.Zuber, *Nucl.Phys.* **B275[FS17]** (1986) 580,
P.Di Francesco,H.Saleur,J.B.Zuber, *Jour.Stat.Phys.* **49** (1987) 57.

22. V.Knizhnik, *Comm.Math.Phys.* **112** (1987) 567.

23. A. B. Zamolodchikov, *JETP Lett.* **43** (1986) 731.

24. A. B. Zamolodchikov, *Sov. J. Nucl. Phys.* **46** (1987) 1090,
A. Ludwig, J. Cardy, *Nucl Phys.* **B285[FS19]** (1987) 687.

25. R. G. Pogossyan, Yerevan preprint EFI-I003(53)-87,
D. Kastor, E. Martinec, S. Shenker, EFI preprint 88-31.

26. Y. Kitazawa, N. Ishibashi, A. Kato, K. Kobayashi, Y. Matsuo, S. Odake, *Nucl. Phys.* **B306** (1988) 425.

27. Vl. S. Dotsenko and V. A. Fateev, *Nucl. Phys.* **B240[FS12]** (1984) 312, **B251[FS13]** (1985) 691, *Phys. Lett.* **154B** (1985) 291.

28. Č. Crnković, R. Paunov, G. Sotkov, M. Stanishkov, preprint ISAS - 93/89/EP

29. V. Fateev, A. B. Zamolodchikov, *Nucl. Phys.* **B251** (1985) 691.

30. A. B. Zamolodchikov, V. Fateev, *Sov. Phys. JETP* **62** (1985) 215.

31. P. Di Francesco, H. Saleur and J.-B. Zuber, *Nucl. Phys.* **B300[FS22]** (1988) 393.

32. D. Kastor, E. Martinec, Z. Qiu, *Phys. Lett.* **200B** (1988) 434,
J.Bagger,D.Nemeshansky,S.Yankielowicz, *Phys.Rev.Lett.* **60** (1988) 389,
F. Ravanini, *Mod. Phys. Lett.* **A3** (1988) 397.

33. A. B. Zamolodchikov, V. A. Fateev, *Theor. and Math. Phys.* **71** (1987) 451.

34. E. Verlinde, *Nucl. Phys.* **B300[FS22]** (1988) 360,
E. Witten, *Comm. Math. Phys.* **121** (1989) 351,
G. Moore and N. Seiberg, *Comm. Math. Phys.* **123** (1989) 177,
K. Narain and M. Caselle, A new approach to the construction of conformal field theories, Univ. of Torino preprint 11031/89.

35. P. Christe and F. Ravanini, $G_N \otimes G_L/G_{N+L}$ conformal field theories and their modular invariant partition functions, Nordita 88/17 P,
M Douglas, CALT-68-1453.

36. Č. Crnković, G. Sotkov and M. Stanishkov, *Phys. Lett.* **220B** (1989) 397.

37. Al. Zamolodchikov, *Nucl. Phys.* **B316** (1989) 573.

38. Č. Crnković, G. Sotkov and M. Stanishkov, *Phys. Lett.* **222B** (1989) 217.

39. E. Date, M. Jimbo, A. Kuniba, T. Miwa and M. Okado, *Nucl. Phys.* **B290[FS20]** (1987) 231.

40. V. G. Kac, *Infinite dimensional Lie algebras*, 2nd ed., Cambridge University Press, Cambridge, 1985.

41. F. Ravanini, *Mod. Phys. Lett.* **A3** (1988) 271.

42. D. Gepner, *Nucl. Phys.* **B287** (1987) 493.

43. Y. Matsuo and S. Yahikozawa, *Phys. Lett.* **178B** (1986) 211,
A. Capelli, *Phys. Lett.* **185B** (1987) 82,
D. Kastor, *Nucl. Phys.* **B280[FS18]** (1987) 304.

44. D. Gepner and Z. Qiu, *Nucl. Phys.* **B285[FS19]** (1987) 423,
Z. Qiu, *Phys. Lett.* **198B** (1987) 497.

45. L. Dixon, P. Ginsparg and J. Harvey, *Nucl. Phys.* **B306** (1988) 470.

46. G. Mussardo, G. Sotkov and M. Stanishkov, *Nucl. Phys.* **B305[FS23]** (1988) 69.

47. A. B. Zamolodchikov and R. G. Pogossyan, *Sov. J. Nuc. R.* **47** (1988) 929.

48. J. Fuchs, *Phys. Rev. Lett.* **62** (1989) 1705.

49. Č. Crnković, G. Sotkov and M. Stanishkov, *Phys. Lett.* **226B** (1989) 297.

50. L. P. Kadanoff, *Ann. Phys.* **120** (1979) 39,
L. P. Kadanoff and A. C. Brown, *Ann. Phys.* **121** (1979) 318.

51. S. Chauduri and J. A. Schwartz, *Phys. Lett.* **219B** (1989) 291,
J. L. Cardy, *J. Phys.* **A20** (1987) L891.

52. S.-K. Yang, *Nucl. Phys.* **B285[FS19]** (1987) 183,
P. Ginsparg, *Nucl. Phys.* **B295[FS21]** (1988) 153,
R. Dijkgraaf, E. Verlinde and H. Verlinde, *Comm. Math. Phys.* **115** (1988) 649.

53. I. Antoniadis and C. Bachas, An Expanding Universe in String Theory, CERN-TH. 5231/89,
S. P. de Alwis, J. Polchinski and R. Schimmrigk, Heterotic Strings with Tree Level Cosmological Constant, UTTG-23-88.

54. L. J. Slater, *Generalized Hypergeometric Functions*, Cambridge University Press, Cambridge, 1966.

55. V. Knizhnik, *Phys. Lett.* **B196** (1987) 173,
D. Lebedev, A. Morozov, preprint ITEP-16 (1987),
D. Montano, *Nucl. Phys.* **B297** (1988) 125,
E. Gava, R. Iengo, G. Sotkov, *Phys. Lett.* **B207** (1988) 283,
R. Iengo, C. J. Zhu, Two-loop computations of the four-particle amplitude in heterotic string theory, ISAS preprint
D. J. Gross, P. F. Mende, *Phys. Lett.* **197B** (1987) 129, *Nucl. Phys.* **B303** (1988) 407.

56. L. Dixon, D. Friedan, E. Martinec, S. Shenker, *Nucl. Phys.* **B282** (1987) 13,
M. Bershadsky, A. Radul, *Int. J. Mod. Phys.* **A2** (1987) 165.

57. Al. B. Zamolodchikov, *Nucl. Phys.* **B285[FS19]** (1987) 481, *Sov. Phys. JETP* **63(5)** (1986) 1061.

58. Č. Crnković, G. Sotkov, M. Stanishkov, "Correlation functions for minimal (super) conformal models on hyperelliptic surfaces", unpublished

59. G. Mussardo, G. Sotkov, M. Stanishkov, *Phys. Lett.* **195B** (1987) 397.

60. R. Dijkgraf, E. Verlinde, H. Verlinde, *Comm. Math. Phys.* **115** (1988) 649.

61. V. Dotsenko, Lectures on conformal field theory, RISM-559, Kyoto 1986.

62. J. Alvarez-Gaumé, B. Bost, G. Moore, P. Nelson, C. Vafa, *Phys. Lett.* **B178** (1986) 41.

63. D. Friedan, E. Martinec, S. Shenker, *Nucl. Phys.* **B271** (1986) 93.

64. Č. Crnković, G. Sotkov, "Strings on hyperelliptic supersurfaces," IC/88/356.

65. D. Friedan, in the proceedings of the Workshop on Unified String Theories, eds. D. Gross and M. Green, World Scientific, Singapore, 1986.

66. D. Cohn, D. Friedan, *Nucl. Phys.* **B296** (1988) 779.

67. M. Bershadsky, *Modern Phys. Lett.* **A3** (1988) 91.