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SUPERIORE
DI STUDI AVANZATI
TRIESTE
Strada Costiera 11

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THE OPERATOR FORMALISM OF CONFORMAL FIELD THEORIES ON GENUS g RIEMANN SURFACES

Thesis submitted for the degree of
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Candidate:
Jorge Guillermo Russo

Supervisor:
Loriano Bonora

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Unless otherwise stated, the work is original and has not been submitted before for a degree of this or any other University.

ABSTRACT

The notion of SuperVirasoro algebras are extended to genus g Riemann surfaces. These superalgebras and their central extensions are realized in the framework of superstring theory by providing a genus g operator formalism. The corresponding BRST operators are constructed, and they turn out to be nilpotent in ten space-time dimensions.

The Sugawara contruction for a generalized Kac-Moody algebra is carried out on a generic Riemann surface.

The operator formalism is applied to compute the propagators for b-c systems with arbitrary integer or half integer weight λ (in the Ramond and Neveu-Schwarz sectors). Explicit expressions for the zero modes and for the Teichmuller deformations are given.

The Hamiltonian formulation for string theory on genus g Riemann surface is provided. Scattering amplitudes are defined in this operator context. These turn out to be the natural extension of the g=0 scattering amplitudes. Correlation functions involving the matter field are computed, reobtaining the well known results.

C O N T E N T S

Abstract..3

Contents..4

Preface...7

Introduction....................................9

CHAPTER I - BACKGROUND

I.1.Definition of "time"..........................14

I.2.The Krichever-Novikov bases...................17

I.3.The Krichever-Novikov algebra.................21

I.4.Operator formalism in string theory...........25

I.5.Fock and Physical spaces......................29

CHAPTER II - GENERALIZING SUPERVIRASORO ALGEBRAS TO GENUS g RIEMANN SURFACES

II.1.Generalized superVirasoro algebra............33

II.2.Operator formalism in superstring theory at genus g.36

II.3.Construction of a BRST operator..............42

CHAPTER III - THE SUGAWARA CONSTRUCTION

ON GENUS g RIEMANN SURFACES

III.1.The Kac-Moody algebras over a genus g Riemann surfaces..46

III.2.The Sugawara construction on genus g Riemann surfaces..49

III.3.Proof of the identities..........................52

CHAPTER IV - b-c SYSTEMS

IV.1.Explicit construction of the KN bases..............56

IV.2.The operator formalism............................59

IV.3.Propagators for b-c systems........................61

IV.4.Propagators computed by starting from other vacuum states...66

IV.5.Some remarks concerning zero modes.................69

CHAPTER V - HAMILTONIAN FORMULATION AND

SCATTERING AMPLITUDES

V.1.Hamiltonian and equations of motion.................73

V.2.Vertex operators and scattering amplitudes..........79

V.3.Computation of correlation functions................83

Conclusions and discussion....86

Acknowledgements....88

Appendix A....89

Appendix B....92

References....94

Preface

Nobody, absolutely nobody, will be entitled to say in the far future that the "string Age" of physics was just a regretful waste of time. Even though string theories did not satisfy the expectatives of many people as a candidate for unifying all the interaccions or failed in providing a consistent theory of quantum gravity, even though they were, rather than a realistic theory, the biggest lie in the human history , the progress reached in these last 15 years in the field of theoretical physics would always be relevant, and then the efforts of so many physicists and mathematicians would not have been completely useless.

One will be perhaps entitled to say that string theories did not deserve so much atention, and that the number of people working on this theory was unjustified and excessive, but he will also have to accept that otherwise the rate at which the developement of string theory had taken place would have been hopelessly much slower.

The fact is that nowadays string theory is being investigated by thousands of scientists; and it is anyway possible (in spite of the skeptics) that the universe, and therefore the world and everything, be made of strings.

I have tried, without much success, to be as clear as possible. I have wanted to avoid speaking by "enigmas", as some physicists do for not assuming responsibilities, so I have explained everything in as much detail as time and the volume of this thesis have allowed me.

These pages try to summarize the results of a long year of work. As every summary, this work has the drawback of having lost some information and clarity. As every summary, this thesis emphasizes results but hides their meaning.

I hope the reader knows how to forgive me for the many mistakes which I could not find but are surely present in this thesis. A scientific work is destinated to provide new discoveries. Often, the only news of a scientific work are errors. The number of errors of a work is quite larger than the number of new true statements. The latter is a very small number, in most of the cases between zero and one.

J.G.R.

Trieste, September 1988.

INTRODUCTION

The fundamental principles behind string theory [1] are so far unknown, and there is a spread belief that they might arise from a deeper understanding of string field theory [2]. Most of the progress has centered around studying perturbation theory on string backgrounds which are classical solutions of string theory, namely the conformal theories [3].

Conformal theories on Riemann surfaces have been essentially developed along two lines: By using path integral techniques [4], and by means of operator methods. In the former, the g-loop contribution to a vacuum expectation value (VEV) is given by the functional integral over all geometries of a two-dimensional surface of genus g and over quantum fields living on this surface. This approach necessarily leads to complicated problems of the algebraic geometry of Riemann surfaces. The second line of work has been extensively studied at g=0, but the operator formalism at higher genus is thus far incomplete. Some computations have been performed by using the genus zero formalism and extending it to higher genus by unitarity [5]. Other very ingenious approach is the one by L.Alvarez-Gaume *et al.*[6]. It has the virtue of keeping the simplicity of the operator formalism on the plane and simultaneously incorporates all the geometrical features characteristic of surfaces with non-trivial topology; but it has the drawback that it is not manifestly global. Results are obtained in a patch of local coordinates and extended unambiguously to the rest of the surface. Recently Krichever and Novikov [7,8] introduced a new formalism wich may prove to be a very important tool in the study of conformal theories on Riemann surfaces. This formalism provides the necessary elements to construct a natural operatorial formulation which allows one to easily reobtain the results of the path integral approach [9,10,11]. The purpose of this thesis is to apply the Krichever-Novikov (KN) formalism to the study of the conformal field theories, and to construct an operator formalism at arbitrary genus.

In ref.[7] Krichever and Novikov showed that it is possible to explicitly provide bases for the spaces of meromorphic tensors of weight λ which are holomorphic outside two distinguished points P_+ and P_- of the Riemann surface Σ. The construction of these kind of bases follows as a simple application of the Riemann-Roch theorem [12]. In particular, for λ=-1 one gets a set of vector fields

$\{e_n\}$ which can be used to generate either reparametrizations or Teichmuller deformations of the Riemann surface. These vector fields obey an algebra (called KN algebra) which is a generalization to higher genus of the usual Virasoro algebra. Similarly, one can extend the notion of Kac-Moody algebra. If $\mathcal{A}^{\Sigma}$ denotes the space of meromorphic functions on Σ having poles only at the points $P_{\pm}$, by the multiplicative structure of $\mathcal{A}^{\Sigma}$, one defines the algebra $\mathcal{G}^{\Sigma} = \mathcal{G} \times \mathcal{A}^{\Sigma}$, for any semisimple algebra $\mathcal{G}$, which is the generalization to arbitrary Riemann surfaces of the Kac-Moody algebra. In ref.[8] they apply these results for constructing an operator formalism for the bosonic string theory.

Chapter I is devoted to set up the notation and to provide the necessary background that will be useful in the reading of the next chapters. In chapter II we extend the KN formalism to supersymmetric theories [13]. We construct, in addition, a BRST operator and demonstrate that it is nilpotent in d=10 (d≡space-time dimension). In chapter III we carry out the Sugawara construction [14] for a generic Riemann surface [15]. In chapter IV we compute the correlation functions for b-c systems with arbitrary integer or half-integer weight (in the Ramond and Neveu-Schwarz sectors). We also give explicit expressions for the zero modes and for the Teichmuller deformations for a generic Riemann surface [9]. In chapter V we provide a Hamiltonian formulation for string theories and define multiloop amplitudes in this operator context. These matters are still under study [10,11,16]. Finally, in chapter VI we draw some conclusions and analyse the prospects of future work along these lines. In appendices A and B we recall some basic facts about theta functions, theta divisors and spin structures.

C H A P T E R I

B A C K G R O U N D

I.1 Definition of "time"

A closed string is conventionally parametrized by an angular coordinate σ and a time evolution parameter τ, which is a sort of time coordinate for an observer sitting at the position σ along the string. When it propagates in space-time (fig.I.1a), it sweeps out a world sheet described by specifying $X^{\mu}(\sigma,\tau)$, the position of the string at given values of σ and τ. By going to euclidean time, this world sheet can be conformally mapped to a Riemann surface without two points (fig.I.1b). At g=0 this is conformal to

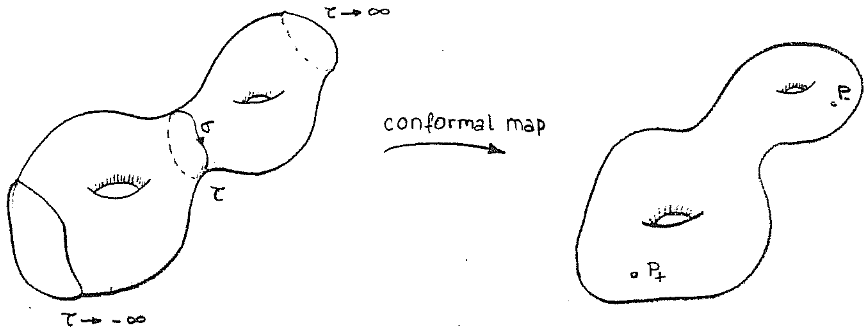

Fig. I.1a

Fig I.1b

Fig.I.1a-When a string propagates in space-time it sweeps out a two-dimensional manifold with boundaries.
Fig.I.1b-By going to euclidean time, this is conformal to a Riemann surface without two points.

the complex plane without the points z=0 and z=∞. In this case the parameters are usually defined by $z=e^{\tau+i\sigma}$, or equivalently

$$\tau(z) = \operatorname{Re}\int_{z_0}^{z} \frac{dz}{z} \tag{I.1}$$

and

$$\sigma(z) = \mathrm{Im} \int_{z_0}^{z} \frac{dz}{z} \tag{I.2}$$

where $z \neq 0, \infty$ is an arbitrary point of the complex plane.

The integral in (I.2) depends on the path from z_0 to z. If we choose two different paths differing by a number n of cycles around $z=0$, then the corresponding σ are related by $\sigma' = \sigma + 2\pi n$, which of course represent the same point z of the complex plane.

The natural generalization of eqs.(I.1,I.2) to higher genus is to define

$$\tau(P) = \mathrm{Re} \int_{P_0}^{P} dk \tag{I.3}$$

$$\sigma(P) = \mathrm{Im} \int_{P_0}^{P} dk \tag{I.4}$$

where dk is the differential of the third kind on Σ with simple poles at the points P_+ and P_- with residues +1 and -1 respectively. By the Riemann-Roch theorem this fixes dk univoquely up to the adition of (holomorphic) abelian differentials. A further requirement, namely, that the integral in (I.3) do not depend on the path, determines unambiguously

$$dk(P) = d\left[\log\left(E(P,P_+)/E(P,P_-)\right)\right] - 2\pi i \sum_{i,j=1}^{g} \mathrm{Im}\left(\int_{P_-}^{P_+} \eta_i\right)(\mathrm{Im}\,\Omega)^{-1}_{ij}\,\eta_j(P) \tag{I.5}$$

where η_j are the g abelian differentials with the standard normalization (see appendix A).

The integral (I.4), however, does depend on the path, so in order to give sense to this definition of σ one should specify the path from P_0 to P .

By inserting (I.5) into (I.3), one obtains an explicit expression for the time parameter τ :

$$\tau(P) = \mathrm{Re}\left[\log\left(\frac{E(P,P_+)\,E(P_0,P_-)}{E(P,P_-)\,E(P_0,P_+)}\right) - 2\pi i \sum_{i,j=1}^{g}\left(\mathrm{Im}\int_{P_-}^{P_+}\eta_i\right)(\mathrm{Im}\,\Omega)^{-1}_{ij}\int_{P_0}^{P}\eta_j\right] \qquad (\text{I}.6)$$

One can define a one-parameter family C_τ of contours as follows

$$C_\tau = \{Q \in \Sigma : \tau(Q)=\tau,\ \ \tau \in \mathbf{R}\ \}$$

For $\tau \longrightarrow \pm\infty$ the contours C_τ are small circles around the points $P_\mp$. These contours can be thought of as the string propagating along the Riemann surface with splittings and joinings if g>0 (fig.I.2).

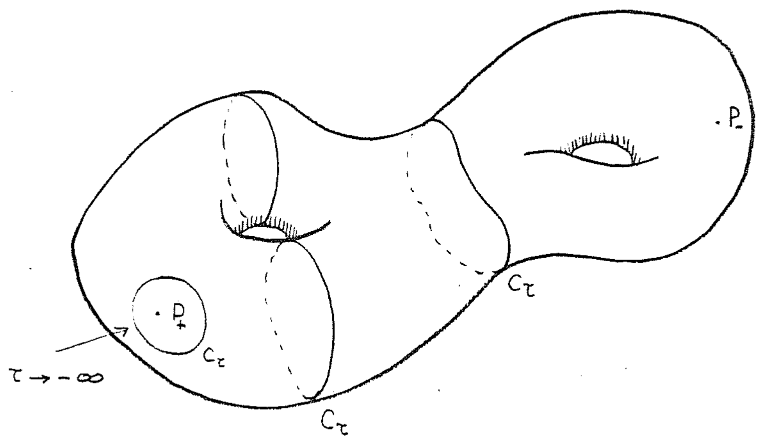

Fig I.2- It is possible to define a one-parameter family C of contours so that for each τ the corresponding contour represents the string at time τ. In this way one gets a of the string propagating along the Riemann surface.

I.2 The Krichever-Novikov bases

In conformal field theories on Riemann surfaces one deals with fields which are sections of a certain line bundle. An intrinsic feature of these theories is that the equations of motion leads to a decomposition of the fields in holomorphic and antiholomorphic parts. On-shell fields must obey their corresponding equation of motion for any point of the Riemann surface different from $P_{\pm}$. This fact already happens at g=0, and the reason is that these points correspond to times $\mp\infty$ and must be excluded (if we required the fields to be everywhere holomorphic, then the space of solutions to the equations of motion would have finite dimension (in some cases even zero or one). This space would be generated by the zero modes of the corresponding operator).

We are thus interested in finding a basis for the generic space $\mathcal{F}^{\Sigma}_{(\lambda)}$ meromorphic tensors of weight λ which are holomorphic outside the points $P_{\pm}$. The Riemann-Roch theorem guarantees for g>1 the existence and uniqueness of tensors of this type which in a neighborhood of $P_{\pm}$ have the following form

$$f_j^{(\lambda)} = \varphi_j^{\pm(\lambda)}\, z_{\pm}^{\pm j - s(\lambda)}\,\left(1 + O(z_{\pm})\right)(dz_{\pm})^{\lambda} \quad ; \qquad (\mathrm{I}.8)$$

$$\varphi_j^{+(\lambda)} \equiv 1 \quad , \quad \lambda \text{ integer} \neq 0$$

where $S(\lambda)=g/2-\lambda(g-1)$

The index j runs through integral or half-integral values, for g even and odd respectively.

The case of half-integer λ will be treated separately below.

In the case of functions ($\lambda=0$) we know from the Noether "gap" theorem [12] that eq.(I.8) can not hold for $|j|\leq g/2$ *. In fact, in the case of the funcions the basis is constructed as follows. Let A $|j|\geq g/2+1$ be the unique functions $A_j \in \mathcal{A}^{\Sigma}$, which in neighborhoods of $P_\pm$ have the form

$$A_j(z_\pm) = a_\pm^{(j)}\, z_\pm^{\pm j-g/2}\,(1+O(z_\pm)) \;; \quad a_+^j \equiv 1 \;, |j|>g/2 \qquad (I.9)$$

(as before, j is integral or half integral, depending on the parity of g). For $j=-g/2,\ldots,g/2-1$ we denote by $A_j \in \mathcal{A}^{\Sigma}$ a function which, in neighborhoods of $P_\pm$, has the form

$$A_j(z_\pm) = a_\pm^{(j)}\, z^{\pm j-g/2 \,{}^{+0}_{-1}}\,(1+O(z_\pm)) \;; \quad a_+^j \equiv 1 \,, \; j=-\frac{g}{2},\ldots,\frac{g}{2}-1 \qquad (I.10)$$

This behaviour actually determines the A_j up to the addition of a constant, which is generated by the remaining element of the basis $A_{g/2}\equiv 1$.

Also in the case of elliptic curves (g=1), eq.(I.8)

*

This is because P and P are supposed to be in general position, i.e. they are not Weierstrass points.

needs to be slightly modified due to the Weierstrass "gap" theorem [12]. The change occurs for j=-1/2,1/2, for which we define

$$f^{(\lambda)}_{1/2} = \varphi^{\pm(\lambda)}_{1/2} \quad (1 + O(z_\pm))(dz_\pm)^\lambda \qquad , \varphi^{+}_{1/2} \equiv 1 \qquad (I.11)$$

$$f^{(\lambda)}_{-1/2} = \varphi^{\pm(\lambda)}_{-1/2} \; z_\pm^{-1} \; (1 + O(z_\pm))(dz_\pm)^\lambda \quad , \varphi^{+}_{-1/2} \equiv 1 \qquad (I.12)$$

The dual bases are defined through the following duality relation

$$\frac{1}{2\pi i} \oint_{C_\tau} f^{(\lambda)}_j \; f^{+(1-\lambda)}_i \; = \; \delta_{ij} \qquad (I.13)$$

Now let us consider the case of half-integer λ. We shall look for sections of K^λ with a given spin structure $[\alpha,\beta]$. We are interested in two kind of bases:

i) Basis for the space of meromorphic tensors of weight λ with the spin structure $[\alpha,\beta]$ which are holomorphic outside P_+ and P_- ("Ramond(R)-type" basis);

ii) Basis for the space of meromorphic tensors of weight λ with the spin structure $[\alpha,\beta]$ which are holomorphic outside P_+ and P_- and a slit from P_+ to P_- ("Neveu-Schwarz(NS)-type" bases).

By Riemann-Roch theorem, there exists a unique section $f^{(\lambda)}_n$ which in neighborhoods of $P_\pm$ have the form (if the spin structure is odd, the following expression needs to be modified for $|n|=1/2$ in the NS sector, in the cases $\lambda=1/2$ or g=1, see below)

$$f_n^{(\lambda)}(z_\pm) = \varphi_n^{\pm(\lambda)} \, z_\pm^{\pm n - S(\lambda)} \left(1 + O(z_\pm)\right) \quad , \quad \varphi_n^{+(\lambda)} = 1 \qquad (I.14)$$

where n takes integer values in the Ramond case i), and half-integer values in the NS case ii).

Even though (I.14) looks like (I.8), there is a difference due to the fact that the indices j or n run in general over distinct values. Throughout this section indices i,j will be used to label the elements of the bases for integer λ, and m, n to label the elements of the bases for half-integer λ.

Let us now consider the NS sector with odd spin structure. If $n \neq \pm 1/2$ the generic formula (I.14) still holds. When $n = \pm 1/2$ a modification is needed for the cases either $\lambda = 1/2$ or $g = 1$. In these cases these sections are given by eqs. (I.11) and (I.12).

This completes the presentation of the KN bases.

I.3 Krichever-Novikov algebra

Denote by L^{Σ} the algebra of the vector fields generated by the basis $\{e_i\}$. From eq. (I.8), one finds that the vector fields e_i have the following behaviour in neighborhoods of $P_{\pm}$

$$e_i = \varepsilon_i^{\pm} \; z_{\pm}^{\pm i - g_0 + 1} \left(1 + O(z_{\pm}) \right) \quad , \quad \varepsilon_i^{+} \equiv 1 \quad , \quad g_0 \equiv 3g/2$$

By using the above equation, one easily finds that the elements of the basis satisfy the following commutation relation ($g_0 \equiv 3g/2$)

$$[e_i, e_j] = \sum_{s=-g_0}^{g_0} C_{ij}^{s} \; e_{i+j-s} \tag{I.15}$$

where the structure constants can be found by integrating eq.(I.15) with the dual (λ=2) basis $\{\Omega_i\}$ to the basis $\{e_i\}$

$$C_{ij}^{s} = \frac{1}{2\pi i} \oint_{C_\tau} [e_i, e_j] \; \Omega_{i+j-s} \tag{I.16}$$

Remark i). Insertion of the expansions of the e_i around $P_{\pm}$ into eq.(I.16) gives

$$C_{ij}^{g_0} = (j-i) \qquad ; \quad C_{ij}^{-g_0} = (i-j) \; \frac{\varepsilon_i^{-} \, \varepsilon_j^{-}}{\varepsilon_{i+j+g_0}^{-}} \tag{I.17}$$

Thus one sees that at g=0 this algebra becomes the usual Virasoro algebra.

Remark ii). We denote by $L_{\pm}^{(s)}$ the subspaces of L^{Σ}, generated by the vector fields e_i with indices $\pm i \geq g_0 + s$, $s \in Z$. It follows from (I.15) that the subspaces $L_\pm^{(s)}$ with $s \geq -1$ are subalgebras of L^Σ. In particular, $L_\pm^{(-1)}$ are the subalgebras of vector fields from L^Σ which, at the points $P_\pm$ respectively, are holomorphic.

Remark iii). The vector fields $e_i \in L_\pm^{(-1)}$ can be used to generate reparametrizations of the Riemann surface, while the $e_i \in L^\Sigma$ but $\notin L_\pm^{(-1)}$ can be used to generate Teichmuller deformations. The subspace generated for the latter has dimension 3g-3, and can be naturally identified with the Teichmuller space (the tangent space to the manifold of moduli of curves of genus g).

There exists a unique "local" central extension of L^Σ and is given by the cocycle [7]

$$\chi(e_i, e_j) = \frac{1}{24\pi i} \oint_{C_\tau} \tilde{\chi}(e_i, e_j) \qquad (I.18)$$

where $$\tilde{\chi}\left(f(z)\frac{\partial}{\partial z}, g(z)\frac{\partial}{\partial z} \right) = \frac{1}{2}\left(f'''g - g'''f \right) dz$$

(by "local" we mean $\chi(e_i, e_j) = 0$ if $|i+j| > 3g$) (I.19)

Central extensions $\hat{L}^\Sigma$ of L^Σ, defined by the cocycle (I.18), are the algebras generated by the elements e_i and a central element t with the following commutation relations:

$$[e_i, e_j] = \sum_{s=-g_0}^{g_0} C_{ij}^s \, e_{i+j-s} + t\, \chi(e_i, e_j)$$
$$[e_i, t] = 0 \qquad (I.20)$$

Remarks:

$\check{\chi}(e_i, e_j)$ is not a one form, so (I.18) depend on the choice of coordinate system . Under a reparametrization $z \to w(z)$, it transforms as

$$\check{\chi}(f,g) \to \left[\frac{1}{2}\left(\frac{d^3\hat{f}}{dw^3}\hat{g} - \frac{d^3\hat{g}}{dw^3}\hat{f}\right) + 2\left(\hat{f}\frac{d\hat{g}}{dw} - \frac{d\hat{f}}{dw}\hat{g}\right)\{z,w\}\right] dw \quad (I.21)$$

Here $S(z)=\{z,w\}$ is the Schwarzian derivative,

$$S(z) = \{z,w\} = \frac{d^3z/dw^3}{dz/dw} - \frac{3}{2}\frac{(d^2z/dw^2)^2}{(dz/dw)^2} \quad (I.22)$$

There are two ways to solve this problem:

a) Define a projective complex structure on Σ [12], which implies that the only admissible local coordinate systems are the ones with homographic transition function (i.e., of the form $w = \frac{az+b}{cz+d}$; $a,b,c,d \in \mathbb{C}$, $ad-bc=1$

For this kind of transition functions the Schwarzian derivative is zero, as it can easily be seen from the definition, and therefore the cocycle (I.18) is well defined.

b) The cocycle may also be defined by providing on Σ a projective connection: It is said that on Σ is given a holomorphic projective connection R if for an arbitrary local system of coordinates $z_\alpha(Q)$, defined in $U_\alpha \subset \Sigma$ is given a holomorphic function $R_\alpha(z_\alpha)$, such that in the intersection of charts $U_\alpha \cap U_\beta$ the corresponding functions are related by

$$R_\beta(z_\beta)\left(\frac{\partial z_\beta}{\partial z_\alpha}\right)^2 = R_\alpha(z_\alpha) + S(f_{\alpha\beta}) \quad (I.23)$$

It follows that the difference between two holomorphic projective connections is a quadratic differential. Now we can redefine $\tilde{\chi}(e_i, e_j)$ as follows

$$\tilde{\chi}\left(f \frac{\partial}{\partial z}, g \frac{\partial}{\partial z}\right) = \left[\frac{1}{2}\left(f'''g - fg'''\right) - R\left(f'g - fg'\right)\right] dz \qquad (\mathrm{I}.24)$$

As a result, $\tilde{\chi}(e_i, e_j)$ is a well defined one-form and therefore (I.18) does not depend on the parametrization chosen.

I.4 Operator formalism in string theory

The phase space for the classical closed bosonic string theory at fixed τ is defined as the space of functions $X^\mu(Q)$ and 1-differentials $P^\mu(Q)$, with the Poisson bracket

$$\left\{ P^\mu(Q), X^\nu(Q') \right\} = \eta^{\mu\nu} \Delta_\tau(Q,Q') \qquad ; \quad Q, Q' \in C_\tau \tag{I.25}$$

where $\Delta_\tau(Q,Q')$ is the "delta"-function over the contour C_τ , i.e. for any continuously differentiable funcion f over C_τ one has

$$f(Q) = \oint_{C_\tau} f(Q') \;\; \Delta_\tau(Q',Q) \tag{I.26}$$

Since the restriction of the KN basis over C_τ is dense in the space of smooth tensors of the corresponding weight, $\Delta_\tau(Q,Q')$ can be written as follows

$$\Delta_\tau(Q,Q') = \frac{1}{2\pi i} \sum_i \omega_i(Q) \; A_i(Q') \tag{I.27}$$

where $\{\omega_i\}$ stands for the dual basis of $\{A_i\}$. The function $X^\mu(Q)$ and the one-form $P^\mu(Q)$ can be thus respectively expanded on a given C_τ in terms of the bases $\{A_i\}$ and $\{\omega_i\}$,

$$X^\mu(Q) = \sum_i x_i^\mu \;\; A_i(Q) \tag{I.28}$$

$$P^\mu(Q) = \sum_i p_i^\mu \;\; \omega_i(Q) \tag{I.29}$$

The coefficients of these expansions become, after quantization, operators acting on a Fock space. So, let $X^{\mu}(Q)$ and $P^{\mu}(Q')$ the operator -valued functions and 1-differentials , commuting if Q and Q′ belong to two different C_{τ} (i.e., at different times), and

$$[\, X^{\mu}(Q)\, ,\, P^{\nu}(Q')\,] = -i\eta^{\mu\nu}\Delta_{\tau}(Q,Q') \quad ;\; Q, Q' \in C_{\tau} \qquad (I.30)$$

It follows for x^{μ}_{i} ,p^{μ}_{i}

$$[\, x^{\mu}_{i}\, ,\, p^{\nu}_{j}\,] = \frac{1}{2\pi}\,\eta^{\mu\nu}\,\delta_{ij} \qquad (I.31)$$

It is convenient to introduce the generalization of the familiar operators α^{μ}_{i} by

$$\pi P^{\mu} + dX^{\mu} = \sum_{j} \alpha^{\mu}_{j}\; \omega_{j}(Q) \qquad (I.32)$$

By integrating (I.32) with $A_i(Q)$, one obtains

$$\alpha^{\mu}_{i} = \pi\, p^{\mu}_{i} + \sum_{j} \gamma_{ij}\, x^{\mu}_{j} \quad ; \quad \gamma_{ij} \equiv \frac{1}{2\pi i}\oint_{C_{\tau}} A_i\, dA_j \qquad (I.33)$$

Therefore, the α^{μ}_{i} satisfy the following commutation relation

$$[\, \alpha^{\mu}_{i}\, ,\, \alpha^{\nu}_{j}\,] \equiv \gamma_{ij}\,\eta^{\mu\nu} \qquad (I.34)$$

which at g=0 reduces to the expected result: $[\alpha^{\mu}_{i}\, ,\alpha^{\nu}_{j}\,]=n\delta_{i+j}\eta^{\mu\nu}$

Similarly, one introduces $\bar{\alpha}^{\mu}_{i}$ by

$$-\pi P^{\mu} + dX^{\mu} = \sum_{j} \bar{\alpha}^{\mu}_{j}\; \bar{\omega}_{j}(Q) \qquad (I.35)$$

It is straightfoward to verify the rules

$$[\bar{\alpha}_i^\mu, \bar{\alpha}_j^\nu] = \eta^{\mu\nu}\, \bar{\gamma}_{ij} \qquad (I.36)$$

$$[\alpha_i^\mu, \bar{\alpha}_j^\nu] = 0 \qquad (I.37)$$

The components of the energy-momentum tensor in string theory are given by

$$T = \frac{1}{2}\,(dx + \pi P)^2 \qquad (I.38a)$$

$$\bar{T} = \frac{1}{2}\,(dx - \pi P)^2 \qquad (I.38b)$$

They are quadratic differentials, so they can be expanded as

$$T(Q) = \sum_i L_i\, \Omega_i(Q) \qquad (I.39a)$$

$$\bar{T}(Q) = \sum_i \bar{L}_i\, \bar{\Omega}_i(Q) \qquad (I.39b)$$

In order to give a sense to eqs.(I.38), they must be normal ordered. Due to the non-commutativity of α_i^μ and α_j^ν with $|i|, |j| < g/2$, there is a large number of non-equivalent notions of possible choices of normal ordering.

In ref.[8] Krichever and Novikov introduced a normal order product defined as

$$:\alpha_i \alpha_j: \;=\; \alpha_i \alpha_j + \tilde{\gamma}_{ij} \qquad (I.40)$$

where $\tilde{\gamma}_{ij}$ are arbitrary constants equal to zero for all except a finite number of points of the half-plane i≤j and equal to γ_{ij} for all except a finite number of points of the half-plane i>j (a more general notion of normal product is introduced in [11])

Using equations (I.38) and (I.39) one obtains a formula for the L_i, $\bar{L}_i$ expressed in terms of the operators α_i^μ, $\bar{\alpha}_i^\mu$

$$L_n = \frac{1}{2} \sum_{i,j} \ell^n_{ij} \quad :\alpha_i \cdot \alpha_j: \qquad (I.41a)$$

$$\bar{L}_n = \frac{1}{2} \sum_{i,j} \bar{\ell}^n_{i,j} \quad :\bar{\alpha}_i \cdot \bar{\alpha}_j: \qquad (I.41b)$$

where

$$\ell^n_{ij} = \frac{1}{2\pi i} \oint_{C_\tau} e_n \, \omega_i \, \omega_j \qquad (I.42)$$

The explicit calculation of the commutator $[L_i, L_j]$ gives

$$[L_i, L_j] \equiv L_i L_j - L_j L_i = \sum_{s=-g_0}^{g_0} C^s_{ji} \, L_{i+j-s} \quad + D \cdot \chi^\Lambda_{ij} \qquad (I.43)$$

where χ^Λ_{ij} depends on the normal ordering (simbolically denoted by Λ) through trivial cocycles. This is the operatorial realization of the KN algebra (I.20).

I.5 Fock and physical spaces

The operators α_i^μ and $\bar{\alpha}_i^\mu$, with i>g/2 will be called annihilation operators; and the ones with i<g/2, creation operator (the operator $\alpha_{g/2}^\mu = p^\mu$ is the center of mass momentum. It commutes with every α_i^μ , so the representation for this operator and $x_{g/2}^\mu$ can be treated separately). As standard, one can introduces a Fock space generated by the states constructed from the action of creation operators on a "vacuum" state, defined by the conditions

$$\alpha_i^\mu |0\rangle = \bar{\alpha}_i^\mu |0\rangle = 0 \qquad , \quad i > g/2 \qquad (I.44)$$

Similarly, the dual vacuum state is defined by

$$\langle 0| \alpha_i^\mu = \langle 0| \bar{\alpha}_i^\mu = 0 \qquad i < -g/2 \qquad (I.45)$$

The subspaces $A_\pm$, generated by the A_i with ±i>g/2 are dense in the spaces of holomorphic functions in the neighborhoods of $P_\pm$, respectively. In consequence, the Fock space that we have defined is the same as the usual Fock space, constructed according to the Fourier expansions over a small contour around $P_\pm$.

The physical space is the space generated by the states belonging to the Fock space satisfying

$$\begin{aligned} L_i |Phys\rangle &= \bar{L}_i |Phys\rangle = 0 \qquad , \quad i > g_0 \\ L_{g_0} |Phys\rangle &= \bar{L}_{g_0} |Phys\rangle = |Phys\rangle \end{aligned} \qquad (I.46)$$

Similarly, the dual physical space is made of states satisfying

$$\langle Phys| L_i = \langle Phys| \bar{L}_i = 0 \quad , \quad i < -g_0$$

$$\langle Phys| L_{-g_0} = \epsilon^-_{-g_0} \langle Phys| \qquad (I.47)$$

$$\langle Phys| \bar{L}_{-g_0} = \bar{\epsilon}^-_{-g_0} \langle Phys|$$

(We recall that in a neighborhood of P_- the field e_{-g_0} behaves as $\epsilon^-_{-g_0} \cdot z\partial/\partial z$).

The correspondence $e_i \to L_i$ allows one to construct a representation for L_i operators. The representation for the e_i is constructed by means of semiinfinite forms of the type

$$f^{(\lambda)}_{i_1} \wedge f^{(\lambda)}_{i_2} \wedge \ldots \wedge f^{(\lambda)}_{i_{m-1}} \wedge f^{(\lambda)}_{m} \wedge f^{(\lambda)}_{m+1} \wedge \ldots \qquad (I.48)$$
$$i_1 < i_2 < \ldots < i_{m-1} < m$$

The e_i acts on this semiinfinite wedge product by the Leibnitz rule, and on each tensor as Lie derivative. One can verify that this provides a representation for the central extended algebra [7].

Similarly the dual space can be represented by the space generated by the forms

$$f^{+}_{j_1} \wedge f^{+}_{j_2} \wedge \ldots \wedge f^{+}_{j_{n-1}} \wedge f^{+}_{n} \wedge f^{+}_{n+1} \wedge \ldots \qquad (I.49)$$

$$j_1 < j_2 < \ldots < j_{n-1} < n$$

We define the scalar product between elements of these spaces, by defining it on the basis elements

$$\langle j_1 \cdots j_{m-1} | i_1 \cdots i_{m-1} \rangle = \delta_{i_1 j_1} \delta_{i_2 j_2} \cdots \delta_{i_{m-1} j_{m-1}} \qquad (I.50)$$

and extending it to the rest by linearity.

The scalar product introduced above lest us define, for any operator from the associated ring generated by the operators L_i , the concept of its mean

$$\langle L_{i_1} \cdots L_{i_k} \rangle \equiv \langle \text{phys} | L_{i_1} \cdots L_{i_k} | \text{phys} \rangle \qquad (I.51)$$

A representation for the algebra of the α_i^μ operators will be given in chapter III. This will allow us to compute correlation functions involving the fields $X^\mu(Q)$.

CHAPTER II

GENERALIZING SUPER VIRASORO ALGEBRAS TO GENUS g RIEMANN SURFACES

II.1 Generalized SuperVirasoro Algebra [13]

In the particular case of $\lambda=-1/2$, equation (I.14) takes the form

$$g_n(z_\pm) = a_n^\pm \; z_\pm^{\pm n - g + \frac{1}{2}} \; (1 + O(z_\pm))\,(dz_\pm)^{-1/2} \qquad (\text{II}.1)$$

$$a_n^+ \equiv 1$$

Denote by LL^Σ the algebra of the vector fields generated by the basis $\{e_i\}$ and the -1/2 differentials generated by the basis $\{g_n\}$, with the following binary operations:

$[e_i, e_j] \equiv$ Lie bracket

$\{g_n, g_m\} \equiv g_n g_m + g_m g_n$, tensor product of sections

$[e_i, g_n] \equiv L_{e_i} g_n$.

By using the expansions around $P_\pm$, one finds that the elements of the bases satisfy the following relations

$$[e_i, e_j] = \sum_{s=-g_0}^{g_0} C_{ij}^s \; e_{i+j-s}$$

$$[e_i, g_n] = \sum_{s=-g_0}^{g_0} H_{in}^s \; g_{i+n-s} \qquad (\text{II}.2)$$

$$\{g_n, g_m\} = \sum_{p=-g}^{g} B_{nm}^p \; e_{n+m-p/2}$$

The algebra LL^Σ will be called the generalized superVirasoro algebra.

Remark. The coefficients C_{ij}^s , H_{in}^s , B_{nm}^p can be calculated from the constants appearing in the expansions of e_i and g_n near $P_\pm$. For example, in the simplest cases, we have

$$C_{ij}^{g_0} = j-i \quad , \quad H_{in}^{g_0} = n - \frac{i}{2} - g + \frac{g_0}{2} \quad , \quad B_{nm}^{P} = 2$$

Let us now move on the central extension of this algebra. Beside the cocycle $\kappa(e_i, e_j)$ introduced in section I.3 of the last chapter, one introduces

$$\varphi(g_n, g_m) = \frac{1}{6\pi i} \oint_{C_\tau} \tilde{\varphi}(g_n, g_m) \tag{II.3}$$

where $\quad \tilde{\varphi}(\rho, \sigma) = \rho' \sigma' dz \quad$ (II.4)

and $\quad \rho = \rho(z)\, dz^{-1/2} \, , \; \sigma = \sigma(z)\, dz^{-1/2} \quad$ are two arbitrary -1/2 differentials belonging to the space generated by $\{g_n\}$.

It is immediate to see that they verify the following properties:

(i) $\kappa(e_i, e_j) = -\kappa(e_j, e_i)$; $\varphi(g_n, g_m) = \varphi(g_m, g_n)$

(ii) They are independent of the coordinate system

(iii) They satisfy the following cocycle conditions

$$\chi(f, [g,h]) + \chi(g, [h,f]) + \chi(h, [f,g]) = 0 \tag{II.5}$$

$$\varphi(\rho, [\sigma, f]) - \varphi(\sigma, [f, \rho]) + \chi(f, \{\rho, \sigma\}) = 0 \tag{II.6}$$

where f, g, h are arbitrary vector fields belonging to the space generated by the $\{e_i\}$.

(iv) They are "local", in the sense that

$$\chi(e_i, e_j) = 0 \qquad \text{for } |i+j| > 3g \tag{II.7}$$

$$\varphi(g_n, g_m) = 0 \qquad \text{for } |n+m| > 2g \tag{II.8}$$

as follows from an elementary computation of the zeroes and poles in $P_\pm$.

Central extensions of LL^Σ , defined by the cocycles

(I.18) and (II.3), and denoted by $\widehat{LL}^{\Sigma}$, are the algebras generated by the elements e_i , g_n , and a central element t with the following relations

$$
\begin{aligned}
[e_i, e_j] &= \sum_{s=-g_0}^{g_0} C_{ij}^{s}\, e_{i+j-s} + t\, \chi(e_i, e_j) \\
[e_i, g_n] &= \sum_{s=-g_0}^{g_0} H_{in}^{s}\, g_{i+n-s} \\
\{ g_n, g_m \} &= \sum_{p=-g}^{g} B_{nm}^{p}\, e_{n+m-p/2} + t\, \varphi(g_n, g_m) \\
[e_i, t] &= [g_n, t] = 0
\end{aligned}
\qquad (\mathrm{II}.9)
$$

Remarks

- The cocycles κ and φ are easily calculated in a few cases. For example, for R=0

$$\chi(e_i, e_{3g-i}) = \frac{1}{12} \left((i-g_0)^3 - (i-g_0) \right) \qquad (\mathrm{II}.10a)$$

$$\varphi(g_n, g_{2g-n}) = -\frac{1}{3} (n-g)^2 + \frac{1}{12} \qquad (\mathrm{II}.10b)$$

-When g=0, the algebras $\widehat{LL}^{\Sigma}$ reduce to the usual superVirasoro algebras, in either the R or NS sectors

II.2 Operator formalism in superstring theory at genus g

The energy-momentum tensor t of superstring theory can be decomposed in a (right) holomorphic and (left) antiholomorphic (outside $P_\pm$) parts T and $\overline{T}$ respectively. Throughout this chapter, we will only consider the right part of the theory. All the results similarly apply for the left part.

In terms of the basic fields, the energy-momentum tensor takes the form

$$T = T^{X\psi} + T^{gh} \qquad (\text{II}.11)$$

where

$$T^{X\psi} = -\frac{1}{2}\partial X^\mu \partial X_\mu - \frac{1}{2}\partial\psi^\mu \psi_\mu \qquad (\text{II}.12)$$

$$T^{gh} = c\,\partial b + 2\partial c\, b - \frac{1}{2}\gamma\partial\beta - \frac{3}{2}\partial\gamma\beta \qquad (\text{II}.13)$$

The supersymmetric current (right part) is

$$J = J^{X\psi} + J^{gh} \qquad (\text{II}.14)$$

where

$$J^{X\psi} = \psi_\mu \partial X^\mu \qquad (\text{II}.15)$$

$$J^{gh} = 2c\partial\beta + 3\partial c\,\beta - \gamma b \qquad (\text{II}.16)$$

The field ψ^μ (Q) has weight λ=1/2. The (commuting) reparametrization ghosts b(Q) and c(Q) have weight 2 and -1 respectively; and the superghosts β(Q) and γ(Q), weight equal to 3/2 and -1/2 respectively.

The restriction of a KN basis of weight λ to a contour

C_τ generates a space which is dense in the space of (piece-wise) smooth tensors of weight λ on the contour C . This allows one to expand the fields on any contour C by using the corresponding KN basis.

Denoting by $\{h_n\}$ the KN basis with $\lambda=1/2$, and by $\{k_n\}$ the dual ($\lambda=3/2$) basis to $\{g_n\}$, the fields of superstring theory can be expanded as follows

$$\lambda=-1 \qquad C(Q) = \sum_i c_i \, e_i(Q) \tag{II.17a}$$

$$\lambda=0 \qquad X^\mu(Q) = \sum_i x_i^\mu \, A_i(Q) \tag{II.17b}$$

$$\lambda=2 \qquad b(Q) = \sum_i b_i \, \Omega_i(Q) \tag{II.17c}$$

$$\lambda=-1/2 \qquad \gamma(Q) = \sum_n \gamma_n \, g_n(Q) \tag{II.17d}$$

$$\lambda=1/2 \qquad \psi^\mu(Q) = \sum_n d_n^\mu \, h_n(Q) \; , \quad h_n^+ = h_{-n} \tag{II.17e}$$

$$\lambda=3/2 \qquad \beta(Q) = \sum_n \beta_n \, k_n(Q) \tag{II.17f}$$

$$\lambda=1 \qquad P^\mu(Q) = \sum_i p_i^\mu \, \omega_i(Q) \tag{II.17g}$$

$$dX^\mu + \pi P^\mu = \sum_i \alpha_i^\mu \, \omega_i(Q) \tag{II.17h}$$

Now let us introduce the Poisson brackets

$$[X^\mu(Q), P^\nu(Q')]_{P.B.} = -i\eta^{\mu\nu} \Delta_\tau(Q,Q') \; , \quad Q,Q' \in C_\tau \tag{II.18a}$$

$$\{\psi^\mu(Q), \psi^\nu(Q')\}_{P.B.} = 2\pi \eta^{\mu\nu} \delta_\tau(Q,Q') \tag{II.18b}$$

$$\{c(Q), b(Q')\}_{P.B.} = 2\pi \, D_\tau(Q,Q') \tag{II.18c}$$

$$[\gamma(Q), \beta(Q')]_{P.B.} = 2\pi \, d_\tau(Q,Q') \tag{II.18d}$$

where

$$\Delta_\tau (Q, Q') = \frac{1}{2\pi i} \sum_i A_i(Q)\, \omega_i(Q') \tag{II.19a}$$

$$\delta_\tau (Q, Q') = \frac{1}{2\pi i} \sum_n h_n(Q)\, h_n^+(Q') \tag{II.19b}$$

$$D_\tau (Q, Q') = \frac{1}{2\pi i} \sum_i e_i(Q)\, \Omega_i(Q') \tag{II.19c}$$

$$d_\tau (Q, Q') = \frac{1}{2\pi i} \sum_n g_n(Q)\, k_n(Q') \tag{II.19d}$$

These are the "delta"-functions over C_τ for smooth tensors of weights 0,1/2,-1,-1/2, respectively (i.e. for any smooth tensor $F^{(\lambda)}$ on the C_τ, one has

$$F^{(\lambda)}(Q) = \oint_{C_\tau} \delta^{(\lambda)}(Q, Q')\; F^{(\lambda)}(Q') \quad ; \quad Q \in C_\tau \tag{II.20}$$

where $\delta^{(\lambda)}(Q,Q')$ denotes the corresponding delta-function).

As a consequence of eqs.(II.17,18) we have the following Poisson brackets for the coefficients of the expansions

$$[\, x_i^\mu ,\, p_j^\nu \,] = -i\, \eta^{\mu\nu}\, \delta_{ij} \tag{II.21a}$$

$$[\, \alpha_i^\mu ,\, \alpha_j^\nu \,] = -i\, \delta_{ij}\, \eta^{\mu\nu} \tag{II.21b}$$

$$\{ d_n^\mu ,\, d_m^\nu \} = -i\, \eta^{\mu\nu}\, \delta_{n+m} \tag{II.21c}$$

$$\{ b_i ,\, c_j \} = -i\, \delta_{ij} \tag{II.21d}$$

$$[\, \gamma_n ,\, \beta_m \,] = -i\, \delta_{nm} \tag{II.21e}$$

Now let us consider $L_i = L_i^{X\psi} + L_i^{gh}$ and $G_n = G_n^{X\psi} + G_n^{gh}$ defined by

$$T(Q) = \sum_i L_i \Omega_i(Q) \tag{II.22}$$

$$J(Q) = \sum_n G_n k_n(Q) \tag{II.23}$$

It follows

$$L_i^{X\psi} = -\frac{1}{2}\sum_{jk} \ell^i_{jk}\, \alpha_j \cdot \alpha_k + \frac{1}{4}\sum_{nm} d_n \cdot d_m\, F^i_{nm} \tag{II.24}$$

$$L_i^{gh} = \sum_j \sum_{s=-g_0}^{g_0} C^s_{ij}\, c_j\, b_{i+j-s} - \sum_n \sum_{s=-g_0}^{g_0} H^s_{in}\, \gamma_n\, \beta_{i+n-s} \tag{II.25}$$

and

$$G_n^{X\psi} = \sum_{mj} d_m \cdot \alpha_j\, D^n_{mj} \tag{II.26}$$

$$G_n^{gh} = -2\sum_j \sum_{s=-g_0}^{g_0} c_j\, \beta_{j+n-s}\, H^s_{jn} - \frac{1}{2}\sum_m \sum_{p=-g}^{g} B^p_{nm}\, \gamma_m\, b_{n+m-p/2} \tag{II.27}$$

where

$$F^i_{nm} = \frac{1}{2\pi i}\oint_{C_\tau} (h_m \partial h_n - h_n \partial h_m)\, e_i$$

$$D^m_{nj} = \frac{1}{2\pi i}\oint_{C_\tau} h_m\, \omega_j\, g_n \tag{II.28}$$

Then the Poisson brackets for L_i and G_n are:

$$[L_i, L_j] = -i\sum_{s=-g_0}^{g_0} C^s_{ji}\, L_{i+j-s}$$

$$[L_i, G_n] = -i\sum_{s=-g_0}^{g_0} H^s_{in}\, G_{i+n-s} \tag{II.29}$$

$$\{G_n, G_m\} = -i\sum_{p=-g}^{g} B^p_{nm}\, L_{n+m-p/2}$$

which is the operatorial realization of eq.(II.2) (apart from the opposite sign in the first equation and the factor -i).

The next step is quantization. All the classical quantities considered so far are promoted to operators acting on a Fock space. The Poisson brackets are replaced by quantum commutators according to the recipe:

$$[\ ,\ ]_{P.B.} \longrightarrow -i\,[\ ,\ ]_{quantum}$$

The normal ordering for the α_i^μ operators has been defined in the previous chapter. For the other operators it is defined by considering as annihilation operators b_i for $i>0$ and c_i for $i\leq 0$, d_n^μ and γ_n for $n\leq 0$ and β_n for $n>0$, and as creation operators the complementary ones†.

With this prescription one can calculate the algebra of $:L_i:$ and $:G_n:$ and obtain

$$\begin{aligned} [\,:L_i:\,,\,:L_j:\,] &= \sum_{s=-g_0}^{g_0} C_{ji}^{s}\,:L_{i+j-s}: + \hat{\chi}_{ij} \\ [\,:G_n:\,,\,:L_i:\,] &= \sum_{s=-g_0}^{g_0} H_{in}^{s}\,:G_{i+n-s}: \\ \{\,:G_n:\,,\,:G_m:\,\} &= \sum_{p=-g}^{g} B_{nm}^{p}\,:L_{n+m-p/2}: + \hat{\varphi}_{\alpha\beta} \end{aligned} \qquad (\text{II}.30)$$

This is the operator realization of eq.(II.9). For the central charges $\hat{\kappa}_{ij}$ and $\hat{\varphi}_{nm}$ one has the following formulas:

$$\hat{\chi}_{ij} = D\,\chi_{ij}^{\Lambda} + \chi_{ij}^{bc} + D\,\chi_{ij}^{\Psi} + \chi_{ij}^{\gamma\beta} \qquad (\text{II}.31)$$

where χ_{ij}^{Λ} was given in the previous chapter and

$$\chi_{ij}^{bc} = \sum_{\substack{r,s=-g_0 \\ i+j=r+s}}^{g_0} \Big(\sum_{\ell\leq 0} C_{i\ell}^{s}\, C_{j,i+\ell-s}^{r}\, \theta(i+\ell-s) - \sum_{\ell>0} C_{i\ell}^{s}\, C_{j,i+\ell-s}^{r}\, \theta(-i-\ell+s) \Big) \qquad (\text{II}.32\text{a})$$

† In the definition of normal ordering the discriminating value could be chosen, for example, to be a constant "a" instead of 0. This would amount to modifying the central charges by trivial cocycles.

$$\chi^{\psi}_{ij} = \frac{1}{8} \sum_{n\le 0,\, m>0} \left(F^{i}_{m,-n} F^{j}_{n,-m} - F^{j}_{m,-n} F^{i}_{n,-m} \right) \tag{II.32b}$$

$$\chi^{\gamma\beta}_{ij} = \sum_{\substack{r,s=-g_0 \\ r+s=i+j}}^{g_0} \sum_{n\le 0} \left(-H^{r}_{in} H^{s}_{j,i+n-r}\, \theta(i+n-r) + H^{s}_{jn} H^{r}_{i\, n+r-i}\, \theta(-i+n+r) \right) \tag{II.32c}$$

while

$$\hat{\varphi}_{nm} = D\, \varphi^{\chi\psi}_{nm} + \varphi^{gh}_{nm} \tag{II.33}$$

with

$$\varphi^{\chi\psi}_{nm} = \sum_{\bar{n}} \sum_{ij} D^{n}_{\bar{n}j} D^{m}_{-\bar{n}j} \left(\gamma_{ij}\, \Theta(\Lambda^{-}) - \gamma_{ij}\, \theta(n+\epsilon) \right) \tag{II.34a}$$

$$\varphi^{gh}_{nm} = \left(\sum_{\substack{i>0 \\ \bar{n}\le 0}} - \sum_{\substack{\bar{n}>0 \\ i\le 0}} \right) \sum_{s=-g_0}^{g_0} \sum_{p=-g}^{g} \left(H^{s}_{in} B^{p}_{\bar{n}m}\, \delta_{\pi, i+n-s}\, \delta_{i, m+\bar{n}-p/2} + \{ n \leftrightarrow m \} \right. \tag{II.34b}$$

where ϵ=1 (0) in the R (NS) sector.

$\Theta(\Lambda^{-})$ is 1 if $(i,j) \in \Lambda^{-}$ and is 0 otherwise (see next section).

II.3 Construction of a BRST operator

The central charges $\hat{\chi}_{ij}$ and $\hat{\varphi}_{nm}$ are antisymmetric and symmetric respectively, and they satisfy the locality conditions (II.7,8). So, by uniqueness of the cocycles, they must be proportional to the cocycles $\kappa(e_i, e_j)$ and $\varphi(g_n, g_m)$ defined above. Therefore it is enough to calculate them for a particular value of the indices in order to know the proportionality constant. We have calculated $\hat{\chi}_{ij}$ for i+j=3g and $\hat{\varphi}_{nm}$ for n+m=2g and found

$$\chi^{\Lambda}_{i,3g-i} = \frac{1}{12}(i-g_0)^3 + (i-g_0)A(\Lambda) \qquad (\mathrm{II}.35a)$$

$$\chi^{bc}_{i,3g-i} = -\frac{13}{6}(i-g_0)^3 + (i-g_0)\left(\frac{1}{6} + g_0^2 - g_0\right) \qquad (\mathrm{II}.35b)$$

$$\chi^{\psi}_{i,3g-i} = \frac{1}{24}(i-g_0)^3 + \frac{1}{12}(i-g_0) \qquad (\mathrm{II}.35c)$$

$$\chi^{\gamma\beta}_{i,3g-i} = \frac{11}{12}(i-g_0)^3 - (i-g_0)\left(\frac{1}{6} + g^2 - g\right) \qquad (\mathrm{II}.35d)$$

$$\varphi^{gh}_{\alpha,2g-\alpha} = -5(\alpha-g)^2 + \left(\frac{5}{4}g^2 - \frac{g}{2}\right) \; ; \; \varphi^{x\psi}_{\alpha,2g-\alpha} = \frac{1}{2}(\alpha-g)^2 + B(\Lambda) \qquad (\mathrm{II}.36)$$

$A(\Lambda)$ and $B(\Lambda)$ are coefficients which depend on the choice of normal ordering for the α_i^{μ}. For instance, if we choose

$$:\alpha_i \alpha_j: = \begin{cases} \alpha_i \alpha_j & (i,j) \in \Lambda^+ \\ \alpha_j \alpha_i & (i,j) \in \Lambda^- \end{cases}$$

where $\Lambda^{\pm}$ are defined as follows : $(i,j) \in \Lambda^+$ if $i+j=s$, $i \le \sigma_s$, $s=-g,\dots,g$, where $\sigma_{-g},\dots,\sigma_g$ are real numbers, and (i,j)

$\in \Lambda^-$ otherwise, we obtain

$$A(\Lambda) = -\frac{1}{12}\left(1 + 3(2\sigma_g - g) + \frac{3}{2}(2\sigma_g - g)^2\right) \qquad (\text{II}.37a)$$

$$B(\Lambda) = -\frac{1}{2}\left(\sigma_g - \frac{g}{2}\right)\left(1 + \left(\sigma_g - \frac{g}{2}\right)\right) \qquad (\text{II}.37b)$$

We observe that at g=0 the usual expressions are obtained.

The anomaly cancels out , up to trivial cocycles, in D=10. The trivial cocycles can be eliminated by making a redefinition (analogous to $L'_0 = L_0 + 1$ corresponding to g=0, in the bosonic string)

$$L'_N = L_N + S_N\, t \quad ; \; N = -g_0, \ldots, g_0$$

Thus, up to trivial cocycles, one finds

$$\hat{\chi}_{ij} = \left(\frac{3}{2}D - 15\right)\chi(e_i, e_j) \quad ; \quad \hat{\varphi}_{nm} = -\left(\frac{3}{2}D - 15\right)\varphi(g_n, g_m) \qquad (\text{II}.38)$$

The BRST operator on Σ is defined as

$$Q = \frac{1}{2\pi i}\oint_{C_\tau}\left\{\frac{1}{2}\left(c\,\partial X^\mu \partial X_\mu + c\,\partial\psi^\mu\psi_\mu - \gamma\psi_\mu\partial X^\mu\right) + \partial c\, c\, b - \frac{3}{2}\partial c\,\gamma\beta \right.$$

$$\left. - \gamma\, c\,\partial\beta + \gamma^2 b\right\} \qquad (\text{II}.39)$$

or equivalently

$$Q = \frac{1}{2\pi i}\oint_{C_\tau}\left(T^{X\psi}(Q)c(Q) + J^{X\psi}(Q)\,\gamma(Q) + \frac{1}{2}b(Q)\left[c(Q), c(Q)\right]\right.$$

$$\left. - \beta(Q)\left[c(Q), \gamma(Q)\right] - \frac{1}{2}\left\{\gamma(Q), \gamma(Q)\right\} b(Q)\right) \qquad (\text{II}.40)$$

In terms of the expansion coefficients

$$Q = \sum_i L_i^{X\psi} c_i + \sum_n \gamma_n G_n^{X\psi} + \frac{1}{2}\sum_{i,j}\sum_{s=-g_0}^{g_0} C_{ij}^s\, c_i c_j\, b_{i+j-s} +$$

$$- \sum_{i,n}\sum_{s=-g_0}^{g_0} H_{in}^s\, c_i \gamma_n \beta_{i+n-s} - \frac{1}{2}\sum_{n,m}\sum_{p=-g}^{g} B_{nm}^p\, \gamma_n\gamma_m\, b_{n+m-p/2} \qquad (\text{II}.41)$$

After quantization we have to consider $\hat{Q}$ = :Q:. One obtains

$$\hat{Q}^2 = \{\hat{Q}, \hat{Q}\} = \sum_{i,j} \hat{\chi}_{ij} :c_i c_j: + \sum_{n,m} \hat{\varphi}_{nm} :\gamma_n \gamma_m:$$

So, it follows from eq.(II.38) that it is nilpotent in D=10.

C H A P T E R III

THE SUGAWARA CONSTRUCION

ON GENUS g RIEMANN SURFACES

III.1 The Kac-Moody algebras over a genus g Riemann surface [15]

As outlined in [7], a generalization of a Kac-Moody algebra for a genus g Riemann surface is obtained by considering the tensor product of a semisimple Lie algebra $\mathcal{G}$ and the algebra $\mathcal{A}^{\Sigma}$ of meromorphic functions over Σ, given by the multiplicative structure of this space. Let us denote by T^a a basis of $\mathcal{G}$ and set $J_i^a \equiv A_i \otimes T^a$. We have immediately the KN-Kac-Moody (KNKM) algebra $\mathcal{K}^{\Sigma}$

$$
\begin{aligned}
[J_i^a, J_j^b] &= f^{abc}\, \alpha_{ij}^{s}\, J_s^c + t\, \gamma_{ij}\, \delta^{ab} \\
[J_i^a, t] &= 0
\end{aligned}
\qquad (\mathrm{III}.1)
$$

where

$$
\alpha_{ij}^{s} = \frac{1}{2\pi i} \oint_{C_\tau} A_i A_j \omega_s \qquad (\mathrm{III}.2)
$$

In eq.(III.1) we understand the summation over s which is limited to a finite range (of width g+1) as it is easy to verify from the definition (III.2). From now on two repeated lower and upper indices are understood to be summed from $-\infty$ to $+\infty$.

Let us now construct the 1-differentials "fields"

$$
J^a(Q) = J_i^a\, \omega_i(Q) \quad , \quad Q \in \Sigma \qquad (\mathrm{III}.3)
$$

where now J_i^a are to be considered as expansion coefficients satisfying (III.1). Then

$$[J^a(Q), J^b(Q')] = f^{abc}\,\Delta(Q,Q')\,J^c(Q) + t\, d\Delta(Q,Q') \qquad (\mathrm{III}.4)$$

We would like to give an example of a realization of the algebra (III.1) or (III.4) in terms of fermionic currents. Let ψ_i be a multiplet of spin 1/2-fields holomorphic outside $P_\pm$, transforming according to a real representation of $\mathcal{G}$ and let T^a denote the antisymmetric matrices representing the generators of $\mathcal{G}$. We start from the anticommutation relations

$$\{\psi_i(Q), \psi_j(Q')\} = \delta_{ij}\ \delta(Q,Q') \qquad ; \ Q,Q' \in C_\tau \qquad (\mathrm{III}.5)$$

The field ψ_i is expanded, as in chapter II, in terms of the basis $\{h_i\}$. Now we construct

$$\begin{aligned} J^a(Q) &= \frac{1}{2}\ :\psi(Q)\, T^a\, \psi(Q): \\ &= \frac{1}{2}\ :\psi_i(Q)\, (T^a)^{ij}\ \psi_j(Q): \end{aligned} \qquad (\mathrm{III}.6)$$

The normal ordering for the d^i_n operators is chosen as in the previous chapter. A simple calculation gives

$$\begin{aligned} [J^a(Q), J^b(Q')] &= :\psi(Q)\,(T^a T^b)\,\psi(Q'):\ \delta(Q,Q') + \\ &\quad - \frac{1}{2}\ \mathrm{tr}\,(T^a T^b)\ \mathfrak{D}(Q,Q') \end{aligned} \qquad (\mathrm{III}.7)$$

where

$$\mathfrak{D}(Q,Q') = \Big(\sum_{\substack{n<0\\ m\geq 0}} - \sum_{\substack{n\geq 0\\ m<0}}\Big)\ h_n(Q)\, h^+_n(Q')\, h_m(Q')\, h^+_n(Q) \qquad (\mathrm{III}.8)$$

We will prove in section III.3 the remarkable identity

$$\mathfrak{D}(Q,Q') = d'\Delta(Q',Q) \qquad (\mathrm{III}.9)$$

With this and the conventions

$$[T^a, T^b] = f^{abc}\, T^c \qquad , \qquad \mathrm{tr}\,(T^a T^b) = -k\, \delta^{ab} \qquad (\mathrm{III}.10)$$

where k is the index of the given representation, we can write eq.(III.7) as

$$[J^a(Q), J^b(Q')] = f^{abc}\, J^c(Q)\, \Delta(Q,Q') - \frac{1}{2} k\, \delta^{ab}\, d\Delta(Q,Q') \qquad (\mathrm{III}.\ 11)$$

III.2 The Sugawara construction on genus g Riemann surfaces

Lets us start from equation (III.4) or, equivalently, from equation (III.1) with central charge k/2. We consider the Sugawara energy-momentum tensor [14]

$$T(Q) = \frac{-1}{c_v + k} : J^a(Q) J^a(Q) : \qquad (\text{III}.12)$$

and its "momenta"

$$L_i = \frac{1}{2\pi i} \oint_{C_\tau} T(Q)\, e_i(Q) = \frac{-1}{c_v+k} \sum_{pq} \ell^i_{pq} : J^a_p\, J^a_q : \qquad (\text{III}.13)$$

Here c_v is the second Casimir of the adjoint representation

$$c_v\, \delta_{ab} = f^{acd} f^{bcd}$$

whereas the normal ordering is defined by

$$: J^a_p\, J^b_q : = \begin{cases} J^a_p\, J^b_q & p < N \\ J^b_q\, J^a_p & p \geq N \end{cases} \qquad (\text{III}.14)$$

where N is a fixed integer. Let us first compute $[L_i, J^b_k]$:

$$[L_i, J^b_k] = \frac{c_v}{c_v+k} \Theta^\ell_{ik}\, J^b_\ell - \frac{k}{c_v+k} S^\ell_{ik}\, J^b_\ell \qquad (\text{III}.15)$$

where

$$S^\ell_{ik} = \frac{1}{2\pi i} \oint \omega_\ell\, e_i\, dA_k$$

$$\Theta^\ell_{ik} = \Big(\sum_{\substack{p \geq N \\ q < N}} - \sum_{\substack{p < N \\ q \geq N}} \Big) \ell^i_{pr}\, \alpha^q_{pk}\, \alpha^\ell_{rq} \qquad (\text{III}.16)$$

In the next section we will prove the identity

$$\Big(\sum_{\substack{p \geq N \\ q < N}} - \sum_{\substack{p < N \\ q \geq N}} \Big) \omega_p(Q)\, \omega_q(Q')\, A_p(Q')\, A_q(Q) = d'\Delta(Q', Q) \qquad (\text{III}.17)$$

As it will turn out, eq.(III.17) is independent of N. It follows

$$\theta^{l}_{ik} = - S^{l}_{ik} \tag{III.18}$$

and

$$[L_i, J^b_k] = - S^l_{ik} J^b_l \tag{III.19}$$

In the g=0 case this equation reduces to the well known one

$$[L_i, J^k_b] = -k \, J^b_{i+k} \tag{III.20}$$

Using eq.(III.19) it is not difficult to find

$$[L_i, L_j] = \frac{1}{c_v + k} \left[\left(l^i_{pq} S^k_{jq} - l^j_{pq} S^k_{iq} \right) : J^a_p J^a_k : + k \dim g \, \chi_{ij} \right] \tag{III.21}$$

where

$$\chi_{ij} = \frac{1}{2} \left(\sum_{\substack{p \geq N \\ q < N}} - \sum_{\substack{p < N \\ q \geq N}} \right) S^q_{ip} S^p_{jq} \tag{III.22}$$

But using the definitions (I.42) and (III.16) one easily verifies that

$$l^i_{pq} S^k_{jq} - l^j_{pq} S^k_{iq} = - C^s_{ij} \, l^{i+j-s}_{pk}$$

so equation (III.21) becomes

$$[L_i, L_j] = C^s_{ij} L_{i+j-s} + k \frac{\dim g}{c_v + k} \chi_{ij} \tag{III.23}$$

It will be proven in the next section that κ_{ij} coincides with $\kappa(e_i , e_j)$ defined in chapter I. Equation (III.23) completes our construction of a KN algebra over a Riemann surface of genus g by means of the Sugawara Ansatz.

Finally we would like to remind that in the case of the bosonic string compactified over a group manifold the target space is $M^d \times G$, where M^d is a d-dimensional Minkowski space. Then the generators of the KN algebra relevant to this theory are the sum of the KN generators for the usual bosonic string theory introduced in chapter I and the L introduced in eq.(III.13). The central charge cocycles corresponding to these two types of generators are proportional since they are both proportional to $\kappa(e_i, e_j)$, and the proportionality constants are known from the result of chapter II and the present chapter. Since the ghost contribution to the central charge is unchanged it is easy to deduce the equation

$$26 = d + k \frac{\dim \mathfrak{g}}{c_v + k} \tag{III.24}$$

which characterizes the critical dimension.

III.3 Proof of the identities

Let us first consider the equation (III.17) with N=0. In order to demonstrate it we construct a representation of the central extended algebra $\hat{\mathcal{A}}^{\Sigma}$ of meromorphic functions generated by the $\{A_i\}$, where the basis elements satisfy

$$[A_i, A_j] = \gamma_{ij}\, t \tag{III.25}$$

This representation can be constructed with the use of semi-infinite forms. We start from the highest weight vector

$$\begin{aligned} \varphi_0 &= \omega_0 \wedge \omega_1 \wedge \dots \qquad , \quad g \text{ even} \\ \varphi_0 &= \omega_{1/2} \wedge \omega_{3/2} \wedge \dots \qquad , \quad g \text{ odd} \end{aligned} \tag{III.26}$$

The action of the A_i on this form is defined by the Leibnitz rule, where A_i acts on each ω_j by multiplication

$$A_i\, \omega_j = \alpha^j_{ik}\, \omega_k \tag{III.27}$$

We obtain

$$[A_i, A_j]\, \varphi_0 = \Big(\sum_{\substack{l \geq 0 \\ k<0}} - \sum_{\substack{l<0 \\ k \geq 0}} \Big)\, \alpha^k_{il}\, \alpha^l_{jk}\, \varphi_0 \tag{III.28}$$

By computing eq. (III.28) in the particular case i+j=g, one finds that t=1. This prove eq.(III.17).

Eq.(III.9) can be proven along similar lines. We define the highest weight vector

$$\psi_0 = h_0 \wedge h_1 \wedge \ldots \qquad , \text{R-sector}$$
$$= h_{1/2} \wedge h_{3/2} \wedge \ldots \qquad , \text{NS- sector} \qquad (\text{III}.29)$$

We then calculate $[A_i , A_j]\psi$, the action of A over h_n being

$$A_i h_n = R^m_{in} h_m \qquad (\text{III}.30)$$

$$R^m_{in} = \frac{1}{2\pi i} \oint_{C_\tau} A_i h_n h^+_m \qquad (\text{III}.31)$$

Following the same procedure as before we find

$$\gamma_{ij} = \Big(\sum_{\substack{m \geq 0 \\ n < 0}} - \sum_{\substack{m < 0 \\ n \geq 0}} \Big) R^n_{im} R^m_{jn} \qquad (\text{III}.32)$$

From this result, eq.(III.9) follows immediately.

The last part of this section will be devoted to justify the statement made after eq.(III.23) that κ_{ij} and $\kappa(e_i , e_j)$ coincide up to trivial cocycles. To this end let us calculate $[e_i , e_j]$ on the highest weight semi-infinite form

$$\varphi_N = A_N \wedge A_{N+1} \wedge \ldots \qquad (\text{III}.33)$$

Using the uniqueness of the cocycles satisfying the "locality" condition mentioned in chapter I, one knows that $\kappa(e_i , e_j)$ should be proportional up to trivial cocycles to κ_{ij} given by eq.(III.22). The proportionality constant can be found by computing them for any particular values of i and j. We have compute them for i+j=3g, and N=g/2 finding

$$\chi_{i,2q_0-i} = \frac{1}{12}\left[(i-q_0)^3 - (i-q_0) \right] = \chi(e_i, e_{2q_0-i}) \qquad (\text{III}.34)$$

So the proportionality constant is 1. This completes the proof of eq. (III.24).

CHAPTER IV

b - c SYSTEMS

IV.1 Explicit construction of the KN bases [9]

In this chapter, we will make extensive use of the definitions and properties given in appendices A and B.

Looking at (I.8), we observe that this behaviour is correctly reproduced by using prime forms as follows

$$f_j^{(\lambda)} \approx \frac{E(P,P_+)^{j-s(\lambda)}}{E(P,P_-)^{j+s(\lambda)}} \tag{IV.1}$$

The correct weight in the P-variable is obtained by mean of the σ-differential

$$f_j^{(\lambda)} \approx \frac{E(P,P_+)^{j-s(\lambda)}}{E(P,P_-)^{j+s(\lambda)}} \cdot \sigma(P)^{2\lambda-1} \tag{IV.2}$$

Finally, we require $f_j^{(\lambda)}$ to be single-valued. To this purpose we introduce a θ-function

$$f_j^{(\lambda)}(P) = N_j^{(\lambda)}(P_+,P_-)\,\frac{E(P,P_+)^{j-s(\lambda)}}{E(P,P_-)^{j+s(\lambda)}}\,\sigma(P)^{2\lambda-1}\,\Theta(P+e(\lambda,j)) \tag{IV.3}$$

where

$$e(\lambda,j) = (j-s(\lambda))P_+ - (j+s(\lambda))P_- + (1-2\lambda)\Delta \quad ,$$

and the normalization constant is given by

$$N_j^{(\lambda)}(P_+,P_-) = \frac{E(P_+,P_-)^{j+s(\lambda)}\,\sigma(P_+)^{1-2\lambda}\,h(P_+)^{2(j-s(\lambda)+\lambda)}}{\Theta(P_+ + e(\lambda,j))}$$

Note that the θ-function gives the zeroes of $f_j^{(\lambda)}$ outside $P_\pm$.

As explained in chapter I, for g=1 and λ=0 the expressions are slightly modified. From the expansions given in section I.2, one can, similarly as above, derive the following formulas

$\lambda = 0$: if $|j| > g/_2$ $\quad A_j =$ as (IV.3) with $\lambda = 0$

if $j = -g/_2, \dots, g/_2 - 1$:

$$A_j(P) = N_j^{(0)}(P_\pm, P_{g+1}) \frac{E(P,P_+)^{j-g/2}\; E(P,P_{g+1})}{E(P,P_-)^{j+g/2+1}} \; \sigma(P)^{2\lambda-1} \; \theta(P + e(j)) \qquad (\text{IV}.4)$$

where $\quad e(j) = (j - g/_2)P_+ - (j + g/_2 + 1)P_- + P_{g+1} + \Delta$,

P_{g+1} is an arbitrary point $\neq P_\pm$,

and $\quad N_j^{(0)}(P_\pm, P_{g+1}) = \dfrac{E(P_+,P_-)^{j+g/2+1}\; \sigma(P_+)\; h(P_+)^{2(j-g/2)}}{\theta(P_\pm + e(j)) E(P_+, P_{g+1})}$

$g = 1$

$$f_j^{(\lambda)}(P) = N_j^{(\lambda)} \; A_j(P) \; \sigma(P)^{2\lambda} \quad , \forall j \qquad (\text{IV}.5)$$

where $A_j(P)$ is given by (IV.4) with $g = 1$.

Case of half-integer λ

The explicit construction of these bases is made in the same way as for integer λ, but now we have to take into account the spin structure. This is accomplished by introducing θ-functions with characteristics $\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]$. We quote the results below

$$f_n^{(\lambda)}(P) = N_n^{(\lambda)}(P_+,P_-)\,\frac{E(P,P_+)^{n-S(\lambda)}}{E(P,P_-)^{n+S(\lambda)}}\,\sigma(P)^{2\lambda-1}\,\theta\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](P+e(\lambda,n)) \tag{IV.6}$$

where

$$e(\lambda,n) = (n-S(\lambda))P_+ - (n+S(\lambda))P_- + (1-2\lambda)\Delta\ ,$$

$$N_n^{(\lambda)}(P_+,P_-) = \frac{E(P_+,P_-)^{n+S(\lambda)}\,\sigma(P_+)^{2n}}{\theta\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](P_+ + e(\lambda,n))}$$

In the NS sector, odd spin structure and $\lambda = 1/2$, according to chapter I, $f_n^{(\lambda)}(P)$ must be modified for $n = \pm\frac{1}{2}$:

$$f_{1/2}^{(1/2)}(P) = N_{1/2}^{(1/2)}(P_+,P_-)\,\frac{\theta\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](P-P_-)}{E(P,P_-)} \tag{IV.7}$$

$$f_{-1/2}^{(1/2)}(P) = N_{-1/2}^{(1/2)}(P_\pm,Q)\,\frac{E(P,Q)}{E(P,P_+)\,E(P,P_-)}\,\theta\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](P+e) \tag{IV.8}$$

where $\quad e = Q - P_+ - P_-\ ,\quad Q$: arbitrary point $\neq P_\pm$

$$N_{1/2}^{(1/2)}(P_+,P_-) = \frac{E(P_+,P_-)\,\sigma(P_+)}{\theta\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](P_+-P_-)}\ ;\quad N_{-1/2}^{(\lambda)}(P_\pm,Q) = \frac{E(P_+,P_-)\,\theta\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](P_++e)}{E(P_+,Q)\,\sigma(P_+)}$$

IV.2 The operator formalism

Let b and c be tensors of weight λ and $1-\lambda$ respectively. As is well known from conformal field theories, they satisfy the equations of motion

$$\bar{\partial}\, b(P) = 0 \quad ; \quad \bar{\partial}\, c(P) = 0 \qquad (\text{IV}.9)$$

As claimed in chapter I, these must hold everywhere except possibly at the points $P_\pm$. If the equation still holds even at these points, then we are dealing with the zero modes. A discussion about zero modes will be given in the last section of this chapter. Thus, the equation of motion implies that the tensors b and c belong to the space generated by the KN bases $\{f_j^{(\lambda)\dagger}\}$ and $\{f_j^{(1-\lambda)}\}$ respectively, and therefore they can be written as

$$b(P) = \sum_i b_i \; f_i^{\dagger(\lambda)}(P) \qquad (\text{IV}.10)$$

$$c(P) = \sum_i c_i \; f_i^{(1-\lambda)}(P) \qquad (\text{IV}.11)$$

The summations run over the integers or half integers depending on the case we are considering.

Upon quantization, b(P) and c(P) become operators and satisfy canonical (anti-) commutation relations. From these, it follows that the coefficients obey

$$\begin{aligned} \{b_i, c_j\} &= \delta_{ij} \\ \{c_i, c_j\} &= 0 = \{b_i, b_j\} \end{aligned} \qquad (\text{IV}.12)$$

The Fock space is defined in the standard way. It is convenient to choose the vacuum state as the one obeying the conditions

$$
\begin{aligned}
c_i|0\rangle &= 0 \qquad , \quad i < S(1-\lambda) \\
b_i|0\rangle &= 0 \qquad , \quad i \geqslant S(1-\lambda)
\end{aligned}
\tag{IV.13}
$$

Similarly, the dual vacuum $\langle 0|$ is defined by means of

$$
\begin{aligned}
\langle 0|\, c_i &= 0 \qquad , \quad i \geqslant S(1-\lambda) \\
\langle 0|\, b_i &= 0 \qquad , \quad i < S(1-\lambda)
\end{aligned}
\tag{IV.14}
$$

(an example of a different choice for $\langle 0|$ will be given later).

We normalize the vacuum states by requiring

$$\langle 0|0\rangle \equiv 1 \tag{IV.15}$$

IV.3 Propagators for b-c systems

Let us first consider the $g>1$, $\lambda>1$ case. The propagator $S(P,Q)$ is defined as

$$S(P,Q) = \langle 0 | T\{ b(P)\, c(Q)\} | 0 \rangle =$$

$$= \begin{cases} \langle 0 | b(P)\, c(Q) | 0 \rangle & , \tau_P > \tau_Q \\ -\langle 0 | c(Q)\, b(P) | 0 \rangle & , \tau_Q > \tau_P \end{cases} \qquad \text{(IV.16)}$$

where τ_P means the value of τ at the point P. By inserting (IV.10,11) into (IV.16) and using the definition of the vacuum state, one obtains

$$S(P,Q) = \begin{cases} \displaystyle\sum_{k=S(1-\lambda)}^{\infty} f_k^{+(\lambda)}(P)\, f_k^{(1-\lambda)}(Q) & , \tau_P > \tau_Q \\ \displaystyle -\sum_{k=-\infty}^{S(1-\lambda)-1} f_k^{+(\lambda)}(P)\, f_k^{(1-\lambda)}(Q) & , \tau_Q > \tau_P \end{cases} \qquad \text{(IV.17)}$$

We would like to compare these expressions with the well known Szego kernel of the literature [6,17,18], namely

$$S(P,Q) = \frac{1}{E(P,Q)} \left(\frac{E(P,P_-)}{E(Q,P_-)} \right)^{(2\lambda-1)(g-1)} \left(\frac{\sigma(P)}{\sigma(Q)} \right)^{(2\lambda-1)} \cdot \frac{\theta\left(Q - P + u(\lambda)\right)}{\theta\left(u(\lambda)\right)} \qquad \text{(IV.18)}$$

where

$$u(\lambda) = (2\lambda-1)(g-1)\,P_- + (2\lambda-1)\Delta$$

A check that eq. (IV.18) and (IV.17) coincide is the following. Consider the propagator to be a tensor $F^{(1-\lambda)}(Q)$ of weight 1-λ depending on Q, at fixed P. Then it can be expanded in terms of the basis $\{f_k^{(1-\lambda)}\}$

$$F^{(1-\lambda)}(Q) = \sum_k a_k \, f_k^{(1-\lambda)}(Q) \qquad (\mathrm{IV}.19)$$

Multiplying by $f_k^{+(\lambda)}$ and integrating over C_τ we obtain

$$a_k = \frac{1}{2\pi i} \oint_{C_\tau} F^{(1-\lambda)} \, f_k^{+(\lambda)} \qquad (\mathrm{IV}.20)$$

Now we can use eq. (IV.18) and the explicit expressions for $f_k^{+(\lambda)}$ in order to arrive to eq.(IV.17). In fact, solving the integral (IV.20) we obtain

$$a_k = \begin{cases} f_k^{+(\lambda)} & , \; k \geq S(1-\lambda) \\ 0 & , \; k < S(1-\lambda) \end{cases} \quad , \quad \tau_Q < \tau_P$$

$$a_k = \begin{cases} 0 & , \; k \geq S(1-\lambda) \\ -f_k^{+(\lambda)} & , \; k < S(1-\lambda) \end{cases} \quad , \quad \tau_P < \tau_Q \qquad (\mathrm{IV}.21)$$

in agreement with eq.(IV.17).

Another way to arrive to the same result consists in looking at the behaviour of the sums (IV.17) in neighborhoods of $P_\pm$, and P=Q. After that, one uses the Riemann-Roch theorem to prove the existence and uniqueness of sections with such behaviours. The explicit expression follows from a similar construction as in section IV.1.

Consider now the case λ=1. We have already seen that the bases $\{A_i\}$, $\{\omega_i\}$ are slightly modified with respect to the generic $f_i^{(\lambda)}$. It is convenient to define the vacuum state in this case by the conditions

$$\begin{aligned} c_i|0\rangle &= 0 \quad , \quad i \leq g/2 \\ b_i|0\rangle &= 0 \quad , \quad i > g/2 \end{aligned} \tag{IV.22}$$

So, the propagator is

$$S(P,Q) = \begin{cases} \displaystyle\sum_{k=g/2+1}^{\infty} \omega_k(P)\, A_k(Q) & , \quad \tau_P > \tau_Q \\ \displaystyle -\sum_{k=-\infty}^{g/2} \omega_k(P)\, A_k(Q) & , \quad \tau_Q > \tau_P \end{cases} \tag{IV.23}$$

The summations in (IV.23) can be performed by the two methods explained above. We will not repeat the computation, since it follows the same lines as before. We just quote here the result which agrees with the well-known Szego kernel for λ=1 [17]

$$S(P,Q) = \frac{E(Q,P_+)\,\theta(Q-P-u)\,\theta(P-P_+-u)}{E(P,Q)\;E(P,P_+)\,\theta(u)\,\theta(Q-P_+-u)} \tag{IV.24}$$

where $u = g\,P_- - P_+ - \Delta$

Finally, let us consider the genus one case. The vacuum state is defined by the conditions

$$\begin{aligned} c_i|0\rangle &= 0 \quad , \quad i \leq 1/2 \\ b_i|0\rangle &= 0 \quad , \quad i > 1/2 \end{aligned} \tag{IV.25}$$

The propagator can be computed in much the same way as for the previous cases, and we obtain

$$S(P,Q) = \frac{1}{E(P,Q)}\,\frac{E(P,P_-)E(Q,P_+)}{E(P,P_+)E(Q,P_-)}\left[\frac{\sigma(P)}{\sigma(Q)}\right]^{(2\lambda-1)}\frac{\theta(Q-P+u)}{\theta(u)} \tag{IV.26}$$

where $u = P_- - P_+ - \Delta\,(2\lambda-1)$

Propagators for half-integer λ

For half-integer case the only subtleties that arise in the computation of $S(P,Q)$ come from the presence of branch points in the NS sector. These branches points are absent in integrals of the type (IV.20), due to the fact that in the NS sector both $F^{(1-\lambda)}$ and $f_k^{+(\lambda)}$ in eq.(IV.20) have branch points in $P_\pm$. Following the same lines as earlier, we obtained the results given below:

$g \geq 2$, NS sector, $\lambda \neq 1/2$

$$S(P,Q) = \frac{1}{E(P,Q)} \left(\frac{E(P,P_-)}{E(Q,P_-)}\right)^{(2\lambda-1)(g-1)} \left[\frac{\sigma(P)}{\sigma(Q)}\right]^{(2\lambda-1)} \frac{\theta\left[{\alpha \atop \beta}\right](Q-P+u(\lambda))}{\theta\left[{\alpha \atop \beta}\right](u(\lambda))} \tag{IV.27}$$

where $u(\lambda) = -(2\lambda-1)(g-1)P_- + (2\lambda-1)\Delta$

$g \geq 2$, R sector

$$S(P,Q) = \frac{1}{E(P,Q)} \left[\frac{E(P,P_-)}{E(Q,P_-)}\right]^{(2\lambda-1)(g-1)+1/2} \left[\frac{E(P,P_+)}{E(Q,P_+)}\right]^{-1/2} \left[\frac{\sigma(P)}{\sigma(Q)}\right]^{(2\lambda-1)} \frac{\theta\left[{\alpha \atop \beta}\right](Q-P+u(\lambda))}{\theta\left[{\alpha \atop \beta}\right](u(\lambda))} \tag{IV.28}$$

where $u(\lambda) = 1/2(P_+ - P_-) - (2\lambda-1)(g-1)P_- + (2\lambda-1)\Delta$

In the $\lambda=1/2$ case, formulas (IV.27,28) still hold, except when the spin structure is odd and the sector is NS, for which we have

$$S(P,Q) = \frac{E(P,P_-)\,E(Q,P_+)\,\theta\left[{\alpha \atop \beta}\right](Q-P+P_+-P_-)}{E(P,Q)\,E(P,P_+)\,E(Q,P_-)\,\theta\left[{\alpha \atop \beta}\right](P_+-P_-)} \tag{IV.29}$$

If g=1, and the spin structure is odd, the propagator in the NS sector for any half-integer λ is given by

$$S(P,Q) = \frac{1}{E(P,Q)}\,\frac{E(P,P_-)}{E(P,P_+)}\,\frac{E(Q,P_+)}{E(Q,P_-)}\left(\frac{\sigma(P)}{\sigma(Q)}\right)^{(2\lambda-1)}\frac{\theta\left[{\alpha \atop \beta}\right](Q-P+u(\lambda))}{\theta\left[{\alpha \atop \beta}\right](u(\lambda))} \tag{IV.30}$$

where $u(\lambda) = P_+ - P_- + (2\lambda-1)\Delta$

For any other case with g=1 the propagator is given by eq.(IV.27,28).

N-points correlation function can be calculated by using Wick's theorem. The only non-vanishing correlation functions are of the form $\langle 0|T\{\prod_{i=1}^{N}(b(P_i)\,c(Q_i))\}|0\rangle$

The rule to calculate them is

$$\langle 0|T\{\prod_{i=1}^{N}(b(P_i)\,c(Q_i))\}|0\rangle = \begin{cases} \sum_{\sigma}(-)^{\sigma}\prod_{i=1}^{N}(\langle 0|T\{b(P_i)\,c(Q_i)\}|0\rangle) \\ \sum_{\sigma}\prod_{i=1}^{N}(\langle 0|T\{b(P_i)\,c(Q_i)\}|0\rangle) \end{cases} \tag{IV.31}$$

where σ runs over all permutations.

IV.4 Propagators computed by starting from other vacuum states

In section IV.2 we have arbitrarily introduced vacuum states defined by conditions (IV.13,14). One may wonder whether this is the only possible choice, or what happens if we define it by means of other conditions. One can see that the only changes which will arise in the propagators are poles in P_- rather than P_+ (or viceversa) and so on. This is clearer in the path integral approach, where the propagator is defined through

$$\langle b(P)\,c(Q)\rangle = \frac{1}{Z}\int [db\,dc]\; b(z_1)\ldots b(z_N)\,c(w_1)\ldots c(w_M)\,b(P)\,c(Q)\,\exp - S[b,c] \tag{IV.32}$$

where $$Z = \int [db\,dc]\; b(z_1)\ldots b(z_N)\,c(w_1)\ldots c(w_M)\,\exp - S[b,c]$$

N: number of zero modes of λ-differentials

M: number of zero modes of $1-\lambda$-differentials

From this, one immediately see that poles will arise whenever P be equal to w_i, or Q equal to z_i. In the KN formalism, all the poles are either in P_+ or P_-. There are no more arbitrary points.

As an example, consider the vacuum state defined from the reqirement $b(P)|0\rangle$, $c(P)|0\rangle$ finite in P_+, and $\langle 0|b(P)$, $\langle 0|c(P)$ finite in P_-. This leads for $\lambda>1$ and $g>1$ to

$$c_i|0\rangle = 0 \qquad , \quad i < s(1-\lambda)$$

$$b_i|0\rangle = 0 \qquad , \quad i \geq s(1-\lambda) \tag{IV.33}$$

and

$$\langle 0| c_i = 0 \qquad , \quad i > -s(1-\lambda)$$

$$\langle 0| b_i = 0 \qquad , \quad i \leq -s(1-\lambda) \tag{IV.34}$$

If we defined the propagator by

$$\langle b(P)c(Q)\rangle = \langle 0|T\{b(P)c(Q)\}|0\rangle$$

then we would obtain a vanishing result. This is because the vacuum defined above gives $\langle 0|0\rangle = 0$, as it can be seen by using the algebra. This is analogous to what happens in the path integral formalism.

We define a correlation function for this case as follows

$$\langle b(P_1)\ldots b(P_r)\, c(Q_1)\ldots c(Q_s)\rangle =$$

$$= \frac{\langle 0|\,T\{b(z_1)\ldots c(w_1)\ldots b(P_1)\ldots c(Q_1)\ldots\}|0\rangle}{\langle 0|\,T\{b(z_1)\ldots c(w_1)\ldots\}|0\rangle} \tag{IV.35}$$

We will recover the results of the preceding section when we identify the points z_i with P_- and w_i with P_+ .

Let us first consider the case $\lambda>1$ for $g\geq 2$. The propagator is

$$\langle b(P)\,c(Q)\rangle = \frac{\langle 0|\,T\{b(z_1)\ldots b(z_N)\, b(P)\, c(Q)\}|0\rangle}{\langle 0|\,T\{b(z_1)\ldots b(z_N)\}|0\rangle} \tag{IV.36}$$

where $N = -2S(\lambda)+1$.

By using the algebra one arrives to the following result

$$\langle b(P)c(Q)\rangle = \frac{1}{\det \| g_i(z_j)\|} \cdot \det \begin{vmatrix} S(P,Q) & S(z_1,Q) & \dots & S(z_N,Q) \\ g_1(P) & g_1(z_1) & \dots & g_1(z_N) \\ \dots & \dots & \dots & \dots \\ g_N(P) & g_N(z_1) & \dots & g_N(z_N) \end{vmatrix} \tag{IV.37}$$

where g_i stands for $f^{+S(\lambda)}, \dots, f^{+-S(\lambda)}$

By taking the limit $z_i \to P_-$, for which $S(z_i, Q)=0$ we get

$$\lim_{z_i \to P_-} \langle b(P)c(Q)\rangle = S(P,Q) \tag{IV.38}$$

as asserted above.

Let us consider as another example the case $\lambda=1$. We obtain

$$\langle b(P)\, c(Q)\rangle = \frac{1}{\det \|\omega_i(z_j)\|} \times \tag{IV.39}$$

$$\times \det \begin{vmatrix} S(P,Q)-S(P,w) & S(z_1,Q)-S(z_1,w) & \dots & S(z_g,Q)-S(z_g,w) \\ \omega_1(P) & \omega_1(z_1) & \dots & \omega_1(z_g) \\ \dots & \dots & \dots & \dots \\ \omega_g(P) & \omega_g(z_1) & & \omega_g(z_g) \end{vmatrix}$$

in the limit $z_i \to P_-$, $w \to P_+$, we recover the expected result

$$\lim_{\substack{z_i \to P_- \\ w \to P_+}} \langle b(P)c(Q)\rangle = S(P,Q) \tag{IV.40}$$

IV.5 Some remarks concerning zero modes

It is interesting to note that the KN bases have among their elements the zero modes for λ-differentials, which are by definition the holomorphic sections of K^{λ}.

For example, we observe that from the explicit expressions given in section IV.1, the basis of meromorphic vector fields has three zero modes for $i=\pm1,0$ when $g=0$, one zero mode when $g=1$ (corresponding to $i=1/2$), and no zero mode if $g\geq2$.

It is well known that the number of zero modes of quadratic differentials coincides with the dimension of the moduli space. In fact, for $\lambda=2$ eq.(I.8) becomes

$$\Omega_j = \varphi_j^{(2)\pm} z_\pm^{\pm j-2+g_0} \left(1+O(z_\pm)\right)(dz_\pm)^2 \qquad (IV.41)$$

This is a zero mode provided that $|j|\leq g_0-2$; therefore there are $3g-3$ quadratic differentials for $g\geq2$ and no zero modes for $g=0$.

If $g=1$ there is only one holomorphic section of K for any $\lambda \in Z$ (the one labeled by $i=1/2$ in eq.(IV.5).

On the other hand, we know that the number of zero modes of the 3/2 differentials plus the number of quadratic differentials gives the dimension of the supermoduli space. In fact, for $\lambda=3/2$ we obtain

$$f_n^{(3/2)}(z_\pm) = \varphi_n^{(3/2)\pm} z_\pm^{\pm n+g-3/2} \left(1+O(z_\pm)\right)(dz_\pm)^{3/2} \qquad (IV.42)$$

from where we observe the existence of 2g-2 zero modes.

The explicit global expressions of the zero modes can be obtained from the formulas of section (IV.I). As an example, let us write the basis of holomorphic quadratic differentials

$$\Omega_j(P) = N_j^{(2)}(P_+,P_-)\,\frac{E(P,P_+)^{j+g_0-2}}{E(P,P_-)^{j-g_0+2}}\;\sigma(P)^3\;\Theta(P+e(2,j)) \qquad (IV.43)$$

where

$$e(2,j) = (j+g_0-2)P_+ - (j-g_0+2)P_- + 3\lambda\Delta$$

The dual to the Ω_j form a basis for the Beltrami differentials μ^i. They obey the duality relation

$$\frac{1}{2\pi i}\int_\Sigma \mu_i\,\Omega_j = \delta_{ij} \qquad (IV.44)$$

The vector fields e_i with $|i|\leq g_0-2$ can be used to generate Teichmuller deformations of the Riemann surface in the following way. Divide the Riemann surface in two parts D^+ and D^- containing P_+ and P_- respectively such that D^+ be a small disk whose center is P_+ and $D^+\cap D^-=A$, where A is an annulus. Take a local coordinate z on the disk. We can use the vector field e to obtain a new Riemann surface as follows. We deform $A\to A'$ by

$$z \to z + \epsilon\,\tilde{e}_i \;\; ; \; z\in A,\; \epsilon\in C \qquad (IV.45)$$

where $e = \tilde{e}(z)\partial/\partial z$. Now D^- is glued to the disk D^+ by identifying the new annulus with the previous collar on D^+. This new Riemann surface is not analytically equivalent to the old one when e_i has poles both in P_+ and P_- which corresponds to $|i|\leq g_0-2$.

Under the infinitesimal deformation (IV.45) the metric transforms to

$$\gamma(\mu) \propto \left| dz + \epsilon \mu_i \, d\bar{z} \right|^2 \tag{IV.46}$$

where

$$\mu_i(P) = \begin{cases} \bar{\partial} \tilde{e}_i & \text{if } P \in \Sigma^- \\ 0 & \text{if } P \in \Sigma^- - A \end{cases} \tag{IV.47}$$

Now we are ready to give the explicit expression for the variation of the period matrix Ω under Teichmuller deformations. Under the deformation of the complex structure given by (IV.47) we have

$$\delta_k \Omega_{ij} = \int_\Sigma \eta_i \eta_j \mu_k = - \oint \eta_i \eta_j e_k \quad , \tag{IV.48}$$

where the integration contour separates P_+ and P_-. Then it is easy to see that the variation $\delta_k \Omega_{ij}$ vanishes if $|k| \geq g_o - 1$. Now suppose v is a linear combination of meromorphic vector fields e_i of the KN basis. Then the most general infinitesimal variation of the period matrix is given by eq.(IV.48) with e_i replaced by ($g>1$)

$$V = \sum_{k=-g_o+1}^{g_o-1} y_k e_k \tag{IV.49}$$

where the explicit expression of the e_k can be extracted from the general formulas of section IV.1

CHAPTER V

HAMILTONIAN FORMULATION OF STRING THEORY AND SCATTERING AMPLITUDES

V.1 HAMILTONIAN AND EQUATIONS OF MOTION [10]

The energy-momentum tensor is by definition

$$t = \frac{2\pi}{|h|^{1/2}} \frac{\delta S[X,h]}{\delta h^{ab}} d\sigma^a \otimes d\sigma^b \tag{V.1}$$

where $h = h_{ab}\, d\sigma^a \otimes d\sigma^b$ is a metric on Σ and $|h|$ is its determinant.

Let us take as parameters for any chart $\sigma^1 = \tau$, which plays the role of (euclidean) time, and $\sigma^2 = \sigma$. We define the Hamiltonian in the standard way as the integral of t_{11} over C_τ:

$$H(\tau) = \frac{1}{2\pi} \oint_{C_\tau} d\sigma\, t_{11} \quad ; \; (d\sigma = \mathrm{Im}\, dk) \tag{V.2}$$

The momentum is similarly defined as

$$P(\tau) = \frac{1}{2\pi} \oint_{C_\tau} d\sigma\, t_{12} \tag{V.3}$$

H and P are the generators of translation in τ and σ respectively.

Taking into account the tracelessness and symmetry of the energy-momentum tensor, eqs.(V.2,3) can also be written as follows

$$H(\tau) = -\frac{1}{2\pi} \oint_{C_\tau} (t|e_\sigma) \tag{V.4}$$

$$P(\tau) = \frac{1}{2\pi} \oint_{C_\tau} (t|e_\tau) \tag{V.5}$$

where e_σ , e_τ are the following vector fields

$$e_\tau = e_k + e_{\bar{k}} \quad , \quad e_\sigma = i(e_k - e_{\bar{k}}) \tag{V.6}$$

with e_k, $e_{\bar{k}}$, the dual meromorphic vector fields to dk and $d\bar{k}$ respectively‡ ,i.e.,

$$\begin{aligned} dk(e_k) &= (e_k|dk) = 1 = (e_{\bar{k}}|d\bar{k}) \\ d\bar{k}(e_k) &= (e_k|d\bar{k}) = 0 = (e_{\bar{k}}|dk) \end{aligned} \tag{V.7}$$

Whenever dk has a zero (pole), e_k must have a pole (zero) for (V.7) to hold. Since by Riemann-Roch theorem dk has 2g zeroes out of $P_\pm$ we conclude that e_k has simple zeroes at $P_\pm$ and 2g poles outside them. Of course, the same occurs for its complex conjugate $e_{\bar{k}}$. At these 2g points one has $d\tau=0=d\sigma$. These points correspond to the critical points where the C_τ split or join.

Inserting eqs.(I.38) for the energy momentum tensor into (V.2) ($t=T+\bar{T}$) we obtain the Hamiltonian which takes the form

$$\begin{aligned} H(\tau) &= \frac{-1}{4\pi} \oint_{C_\tau} \left((dX+2\pi P)^2 + (dX-2\pi P)^2 | e_\sigma \right) \\ &= \frac{-1}{2\pi} \oint_{C_\tau} \left(dX^2 + 4\pi^2 P^2 | e_\sigma \right) \end{aligned} \tag{V.8}$$

Now we impose the equations of motion which follow from this Hamiltonian. These are easily obtained by using the Poisson bracket (I.25)

‡We hope these vector fields not be confused with anyone of the $\{e_i\}$ of the KN basis.

$$L_{e_\tau} X^\mu \equiv i\{H, X^\mu\} = -2\pi i\,(P^\mu | e_\sigma) \qquad (V.9a)$$

$$L_{e_\tau} P^\mu \equiv i\{H, P^\mu\} = \frac{1}{2\pi i}\, d(dX^\mu \backslash e_\sigma) \qquad (V.9b)$$

where L_e denotes the Lie derivative with respect to the vector field e.

The first one tells us that $P^\mu = 1/2\pi(\partial-\bar{\partial})X^\mu$, so eqs.(I.38) become $T=-1/2\partial X.\partial X$, $\bar{T}=-1/2\bar{\partial}X.\bar{\partial}X$.

By contracting with e the second equation in (V.9) becomes

$$L^2_{e_\tau} X^\mu = -L^2_{e_\sigma} X^\mu \qquad (V.10)$$

which is the analogous to the g=0 equation of motion $\ddot{X}+\overset{\shortparallel}{X}=0$. Eq.(V.10) implies

$$\partial\bar{\partial} X = 0 \qquad (V.11)$$

The most general solution to this equation is

$$X^\mu(Q) = x^\mu + p^\mu \tau(Q) + \sum_{n \neq g/2} \left(X^\mu_n A_n(Q) + \bar{X}^\mu_n \bar{A}_n(Q) \right) \qquad (V.12)$$

The familiar coefficients α^μ_n are introduced by the following definition

$$\partial X^\mu = dX^\mu + 2\pi P^\mu \equiv \sum_n \alpha^\mu_n \omega_n \qquad (V.13a)$$

$$\bar{\partial} X^\mu = dX^\mu - 2\pi P^\mu \equiv \sum_n \bar{\alpha}^\mu_n \bar{\omega}_n \qquad (V.13b)$$

The next step is quantization of the theory: the coefficients of these expansions become second quantized operators acting on a Fock space, whose commutation rules are to be derived from the canonical commutation relation

$$[P^\mu(Q), X^\nu(Q')] = i\eta^{\mu\nu}\Delta_\tau(Q,Q') \quad ; Q, Q' \in C_\tau \qquad (V.14)$$

This leads to the following commutation rules for the $\alpha_n^\mu, \bar{\alpha}_n^\mu$, p^μ, x^μ :

$$[\alpha_n^\mu, \alpha_m^\nu] = \delta_{nm}\eta^{\mu\nu} \quad , \quad [\bar{\alpha}_n^\mu, \bar{\alpha}_m^\nu] = \bar{\delta}_{nm}\eta^{\mu\nu}$$
$$[\alpha_n^\mu, \bar{\alpha}_m^\nu] = 0 \quad , \quad [p^\mu, x^\nu] = i\eta^{\mu\nu} \qquad (V.15)$$

In terms of α_n^μ, $\bar{\alpha}_n^\mu$ the Hamiltonian and momentum operators take the following form

$$H(\tau) = \frac{1}{2}\sum_{n,m}\left(l_{nm}(\tau) :\alpha_n\cdot\alpha_m: + \bar{l}_{nm}(\tau) :\bar{\alpha}_n\cdot\bar{\alpha}_m:\right) \qquad (V.16a)$$

$$P(\tau) = \frac{i}{2}\sum_{nm}\left(l_{nm}(\tau) :\alpha_n\cdot\alpha_m: - \bar{l}_{nm}(\tau) :\bar{\alpha}_n\cdot\bar{\alpha}_m:\right) \qquad (V.16b)$$

where

$$l_{nm}(\tau) = \frac{1}{2\pi i}\oint_{C_\tau}(e_k|\omega_n)\omega_m = l_{mn}(\tau) \qquad (V.17)$$

<u>Remarks</u>

i) At g=0 H and P reduce to the well known expressions

$$H = L_0 + \bar{L}_0 \quad , \quad P = i(L_0 - \bar{L}_0)$$

ii) $H(\tau)$ and $P(\tau)$ depend on time. This is due to the 2g poles of the vector fields e_k, $e_{\bar{k}}$. The variation of $l_{nm}(\tau)$ is however very simple. It is like a step funcion in the sense that it remains constant until it reaches a splitting or a joining of the C_τ (because the integrand picks a pole from e_k), where it changes value by a discrete quantity.

iii) If τ is small enough ($\tau<\tau_1$, where τ_1 is the time at which the first bifurcation of C_τ takes place), one can verify that

$$H(\tau)\,|0\rangle = 0 \tag{V.18}$$

$$H(\tau)\,|\mathrm{phys}\rangle = 2\,|\mathrm{phys}\rangle \tag{V.19}$$

(the definition of the vacuum state and Physical states is as in chapter I). Similar considerations apply for $\tau>\tau$ and the dual states.

iv) Note that our definition of the coefficients X^μ_n and P^μ_n is not the same as the used by KN, given in chapter I. There, $X^\mu(Q)$ and $P^\mu(Q)$ are expanded at fixed τ in terms of the bases $\{A_i\}$ and $\{\omega_i\}$ respectively. In consequence, the coefficients of these expansions depend on time. It is easy to obtain the relation between both sets of coefficients,

$$x^\mu_n(\tau) = X^\mu_n + \sum_{m\neq g/2} f_{nm}(\tau)\,\bar{X}^\mu_m \qquad ; \; n\neq g/2 \tag{V.20a}$$

$$P^\mu_n(\tau) = P^\mu_n - \sum_{m\neq g/2} \bar{f}_{mn}(\tau)\,\bar{P}^\mu_m \tag{V.20b}$$

where

$$f_{nm}(\tau) = \frac{1}{2\pi i}\oint_{C_\tau} \omega_n \bar{A}_m$$

The time dependence of these variables follows from the equation of motion derived from the Hamiltonian

$$\dot{x}^\mu_n(\tau) = \left[H(\tau)\,,\, x^\mu_n(\tau)\right] \qquad ; \quad \dot{P}^\mu_n(\tau) = \left[H(\tau),\, P^\mu_n(\tau)\right] \tag{V.21}$$

v) Ghost system

It is straightforward to repeat all these steps for the ghost system. We are now particularly interested in the anticommuting reparametrization ghosts (bc) of the bosonic string theory, with weights $\lambda=2$ and $\lambda=-1$ respectively. Their contribution to the energy-momentum tensor is

$$t^{gh} = c\,\partial b + 2\,\partial c\, b + \bar c\,\bar\partial \bar b + 2\,\bar\partial \bar c\, \bar b \qquad \text{(V.22)}$$

So, the ghost Hamiltonian is given by

$$H^{gh}(\tau) = \frac{-1}{2\pi} \oint_{C_\tau} \left(e_\sigma \mid c\,\partial b + 2\,\partial c\, b + \bar c\,\bar\partial \bar b + 2\,\bar\partial \bar c\, \bar b \right) \qquad \text{(V.23)}$$

which allows, by using the canonical anticommutation relations between b and c, to obtain the equations of motion

$$L_{e_\tau} c = [H, c] = -i\, L_{e_\sigma} c \qquad \text{(V.24a)}$$

$$L_{e_\tau} b = [H, b] = -i\, L_{e_\sigma} b \qquad \text{(V.24b)}$$

These equations imply $\bar\partial c=0=\bar\partial b$, whose most general solutions are

$$c(Q) = \sum_i c_i\; e_i(Q) \qquad \text{(V.25a)}$$

$$b(Q) = \sum_i b_i\; \Omega_i(Q) \qquad \text{(V.25b)}$$

and similarly for the complex conjugates $\bar b$, $\bar c$.

V.2 Vertex operators and scattering amplitudes

Let $A^M[\Sigma]$ denote the M-particle scattering amplitude of a process performed on an equivalence class of Riemann surfaces $[\Sigma]$. In order to obtain the total scattering amplitude, one has to sum over all equivalence classes and over all topologies. This is done in [11]. Here we shall be mainly concerned with the definition of $A^M[\Sigma]$.

In string theory, a scattering amplitude for M particles is given by the vacuum expactation value of a product of suitable vertex operators $\{V_i\ ,\ i=1,\ldots,M\}$ representing the external particles

$$A^M[\Sigma] = \langle 0 | T\{V_1 \ldots V_M\} | 0 \rangle \tag{V.26}$$

Now we would like to find the right definition of the vertex operators within the present formalism . A vertex operator for an on-shell string state of momentum p^μ is some local field $W(Q;p^\mu)$ with the correct Lorentz number of the represented particle. Moreover, it must change the momentum of any state by p^μ , so it must carry a factor $\exp(ip.X(Q))$. Finally, because the vertex must be invariant under reparametrization, one requires $W(Q,p^\mu)$ to have dimension (1,1), so that

$$V = \int_\Sigma W(Q;p^\mu) \tag{V.27}$$

be invariant. The tachyon vertex at g=0 has the following form

$$V_{tach} = \int_{\Sigma} \frac{dz \wedge d\bar{z}}{z\bar{z}} \; e^{ip.X(z,\bar{z})} \qquad (V.28)$$

Up to the addition of holomorphic terms, Riemann-Roch theorem guarantees that the only one differential which has this behaviour in a neighborhood of $P_{\pm}$, and which has no other poles anywhere on the Riemann surface is dk; and similarly for $d\bar{k}$. Thus we are led to define the tachyon vertex operator as

$$V_{tach} = \frac{1}{4\pi i} \int_{\Sigma} dk \wedge d\bar{k} \; e^{ip.X} \qquad (V.29)$$

Note from the definitions that

$$dk \wedge d\bar{k} = d\sigma \wedge d\tau \; (2i)$$

Therefore the tachyon can also be written in the form

$$V_{tach} = \int_{-\infty}^{\infty} d\tau \; \frac{1}{2\pi} \oint_{C_\tau} d\sigma \; e^{ip.X} \qquad (V.30)$$

Similarly, one defines the graviton vertex as

$$V_{grav} = \xi_{\mu\nu}(p) \int_{\Sigma} \partial X^{\mu} \wedge \bar{\partial} X^{\nu} \; e^{ip.X} \qquad (V.31)$$

where $\xi_{\mu\nu}(p)$ is a symmetric tensor obeying the conditions $\xi_{\mu\nu}\, p^{\mu} = 0$, $\xi_{\mu}{}^{\mu} = 0$, and $p^2 = 0$, in such a way there is no anomaly in the products $(\partial X^{\mu} \bar{\partial} X^{\nu}).\, e^{ip.X}$, $\partial X^{\mu}.\bar{\partial} X^{\nu}$ and $e^{ip.X}$ respectively. When $\xi_{\mu\nu}$ is an antisymmetric tensor, this vertex represents the antisymmetric particle, present in the massless level of the spectrum of the string.

Finally we find the "dilaton", represented by the vertex operator

$$V_D = \int_\Sigma \partial X_\mu \wedge \bar{\partial} X^\mu \; e^{ip.X} \tag{V.32}$$

A generic vertex vertex representing a nth level state will have the form

$$V_{\gamma^n} = \xi_{\mu_1 \cdots \mu_{2n}}(p) \int_\Sigma \partial X^{\mu_1} \wedge \bar{\partial} X^{\mu_2} \, (e_k | \partial X^{\mu_3})(e_{\bar{k}} | \bar{\partial} X^{\mu_4}) \ldots (e_k | \partial X^{\mu_{2n-1}})(e_{\bar{k}} | \bar{\partial} X^{\mu_{2n}}) \, . \, e^{ip.X} \tag{V.33}$$

where the polarization tensor has to satisfy the constraints that follow from the requirement that W_{γ^n} have the right conformal dimension. Eq.(V.33) can be written simbolically as

$$V_{\gamma^n} = \xi^{2n} \int_\Sigma (e_k \bar{e}_{\bar{k}})^n (\partial X \bar{\partial} X)^{n+1} \, e^{ip.X} \tag{V.34}$$

This is not however the most general vertex operator yet. It is also possible to include X^μ-factors with higher derivatives. General expressions for vertex operators, with the proper trace and transversality conditions for the polarization tensors, are given in [19].

In ref.[11], vertex operators have been defined in this context with the use of an arbitrary metric. The choice made here for the measure has the virtue of making apparent the factorization property in holomorphic and antiholomorphic parts, and arise as the natural generalization of the g=0 expressions.

Computation of $[L_i, V]$

We require the vertex operators W to have conformal dimension (1,1) in the following sense

$$[L_n, W] = L_{e_n} W \quad ; \quad [\bar{L}_n, W] = L_{\bar{e}_n} W \tag{V.35}$$

These are the generalizations of the requirements at g=0. The physical reason of conditions (V.35) is unitarity: unitarity requires that only physical states, and not negative norm states, contribute as poles in the amplitude (V.26). This amount to say that the state

$$V_i(\tau_i) \ldots V_M(\tau_M) |0\rangle$$

when it is on-shell, be annihilated by the L_n, for $n>g$. This is indeed the case provided W has dimension (1,1). In fact; by acting with L_n on this state and commuting it with the vertices on the left one arrives to the following state

$$V_i(\tau_i) \ldots L_n V_M(\tau_M) |0\rangle$$

plus states containing a commutator placed between two vertex operators. It turns out that these terms vanishes because of conditions (V.35), and what remains gives zero if $n>g_0$ and 1 if $n=g_0$ (for details, see ref.[10]).

V.3 Computation of correlation functions

Consider first the correlation function $<\partial X^\mu(P)\partial X^\nu(Q)>$. By definition it is given by

$$\langle 0 | T\{\partial X^\mu(P)\partial X^\nu(Q)\} | 0\rangle = A^{\mu\nu}(P,Q)\,\theta(\tau_P-\tau_Q) + \\ + A^{\nu\mu}(Q,P)\,\theta(\tau_Q-\tau_P) \qquad (V.36)$$

where

$$A^{\mu\nu}(P,Q) \equiv \langle 0 | \partial X^\mu(P)\partial X^\nu(Q) | 0\rangle$$

By using the expansions, we have

$$A^{\mu\nu}(P,Q) = \sum_{nm} \langle 0 | \alpha_n^\mu \alpha_m^\nu | 0\rangle\, \omega_n(P)\,\omega_m(Q) \qquad (V.37)$$

Now, from the vacuum definition given in chapter I, and the commutation rules of the α one finds

$$A^{\mu\nu}(P,Q) = A_1^{\mu\nu}(P,Q) + A_2^{\mu\nu}(P,Q) + A_3^{\mu\nu}(P,Q) \qquad (V.38)$$

where

$$A_1^{\mu\nu}(P,Q) = \eta^{\mu\nu} \sum_{\substack{i \geq g/2 \\ j < g/2}} \gamma_{ij}\,\omega_i(P)\,\omega_j(Q)$$

$$A_2^{\mu\nu}(P,Q) = \eta^{\mu\nu} \sum_{\substack{i \in I \\ j < -g/2}} \delta_{ij}\,\omega_i(P)\,\omega_j(Q) \qquad ; \quad I \equiv [-g/2, g/2)$$

$$A_3^{\mu\nu}(P,Q) = \sum_{ij \in I} C_{ij}^{\mu\nu}\,\omega_i(P)\,\omega_j(Q)$$

As we will see later there is no need to calculate the constants $C_{ij}^{\mu\nu} = <0|\alpha_i^\mu \alpha_j^\nu|0>$ because $A_3^{\mu\nu}(P,Q)$ is holomorphic in P and Q (they are computed anyway in ref.[10]). $A_1^{\mu\nu}(P,Q)$ is nothing but

$$A_1^{\mu\nu}(P,Q) = \partial_Q S(P,Q) \qquad (V.39)$$

where S(P,Q) is the propagator for a b-c system of $\lambda=0$. Now we would like to compare these results with the propagator that one finds in the literature [18]. The latter is the unique exact(*) meromorphic one-form G(z,w) in both z and w variables which satisfies

$$\partial_{\bar{z}} G(z,w) = \pi \partial_w \delta^2(z-w) \qquad (V.40a)$$

$$\partial_{\bar{w}} G(z,w) = \pi \partial_z \delta^2(z-w) \qquad (V.40b)$$

Uniqueness is seen as follows: if G(z,w) obeys the above equations, then it is determined up to the addition of a holomorphic tensor in z and w, i.e.

$$G'(z,w) = G(z,w) + \sum_{i,j=1}^{g} a_{ij} \eta_i(z) \eta_j(w) \qquad (V.41)$$

Now the requirement of "exactness" completely determines it. In fact, if G and G′ are both exact, then for the α cycles we get $0=\sum_{ij} a_{ij}\eta_j$ which implies a=0.

One easily verifies that our propagator satisfies the above differential equations. Indeed

$$\partial_{\bar{z}} A_1^{\mu\nu}(z,w) = \partial_{\bar{z}} \partial_w S(z,w) = \pi \partial_w \delta^2(z-w) \qquad (V.42)$$

and similarly for the other differential equation. Since by construction our propagator is exact, from uniqueness it follows that it is equal to the well-known propagator quoted in the literature, namely

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(*) By "exact" we mean $\oint_\gamma G(z,w)=0$, with γ any cycle.

Integration of this correlation function allows one to calculate <X(P)X(Q)> up to the addition of functions depending only on P or on Q . These terms are however irrelevant in the computation of a scattering amplitude. They can be, anyway, determined from the differential equation which the propagator <XX> must satisfy. The latter involves the metric. This establishes, I guess, a connection between the metric and the representation of the algebra of the operators α_i^μ.

CONCLUSIONS AND DISCUSSION

We have introduced an elegant operator formalism for theories defined on arbitrary Riemann surfaces. We have seen many applications of this formalism. We have recovered standard results, as propagators, and found new results, as the Sugawara construction at arbitrary genus g or the Hamiltonian, and scattering amplitudes in this context.

Unlike other attempts [6] to develop operator techniques on Riemann surfaces, this formalism has the virtue of being manifestly global. And it is certainly natural, because is constructed in the same way as the standard operator formalism of field theories. As a drawback , one could mention that much of the information of the original theory has been lost. Here one starts with the gauge already fixed. As a result, everything looks "too much" holomorphic and antiholomorphic, and factors, as determinants, must be included *ad hoc* in the total scattering amplitude.

Chapter II is an open door to many problems; e.g. the classification of the unitary representations of the supersymmetric KN algebra, or the application to the heterotic string theory, the study of Ward identities by using the g-loop BRST operator defined there, etc.

The Sugawara construction performed on chapter III may be the basic element to begin with a perturbative study of string theories on group manifolds. It remains to make the super Sugawara construction in this framework.

In chapter IV there is no new result (with the dubious exception of the propagators in the R-sector) which have not been already found by path integral methods. Nevertheless, this chapter constitutes the first sign that by this formalism one can get the same results as the Polyakov approach.

In chapter V the necessary tools for computing any correlation function (in particular, scattering amplitudes) in string theories are provided. Although one (formally) already knows the results by the path integral approach, it is of interest to reobtain them in this context, not with the hope of avoiding difficulties (because this is quite improbable), but rather with the certainty of getting a closer approach to conformal field theories.

Drawing conclusions from something which is not already concluded, is not an easy task at all. Guessing the result of computations which have not been performed yet, is a job worthy of a prophet, rather than a physicist.

I hope that these last two pages have succeed in their modest purpose. Namely, transmitting an idea on what might be waiting in the future for this interesting formalism.

A C K N O W L E D G M E N T S

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Appendix A

Here we summarize the notation about theta functions [20]. Given the g holomorphic differentials η_i with the standard normalization around the basis of homology of the Riemann surface Σ

$$\oint_{a_j} \eta_i = \delta_{ij} \quad , \quad \oint_{b_j} \eta_i = \Omega_{ij} \quad , \ (\mathrm{Im}\,\Omega) > 0 \qquad (A.1)$$

the θ-function with characteristics $\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$ associated to Σ is

$$\begin{aligned}\theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(z) &= \sum_{N \in Z^g} \exp\left(i\pi (N+\alpha)^t\,\Omega\,(N+\alpha) + i2\pi (N+\alpha)^t (z+\beta)\right) \\ &= \exp\left(i\pi\alpha^t\Omega\alpha + i2\pi\alpha^t (z+\beta)\right)\theta(z+\beta+\Omega\alpha) \qquad (A.2)\end{aligned}$$

$$\theta(z) = \theta\begin{bmatrix}0\\ 0\end{bmatrix}(z) \quad , \ z \in \mathbb{C}^g \quad , \alpha, \beta \in \mathbb{R}^g$$

The Jacobian torus is defined by $J(\Sigma)=C^g/\Gamma(\Sigma)$ where the period lattice $\Gamma(\Sigma)$ is

$$\Gamma(\Sigma) = \left\{ v \in C^g : v = n + \Omega m \ , \ (n, m) \in Z^{2g} \right\} ,$$

The set $\Theta = \{z : \theta(z)=0\}$ is a variety of complex codimension one in $J(\Sigma)$ called Θ-divisor.

When the characteristics α_i, β_i are 0 or 1/2, the corresponding θ-function (A.2), known as first order theta function, is even or odd depending on the parity of $4\alpha^t\beta$. The theta function is the unique holomorphic section of a holomorphic line bundle on $J(\Sigma)$, called θ-line bundle, defined by the transition functions

$$\theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(z+n+\Omega m) = \exp\left(-i\pi m^t\Omega m - i2\pi m^t z + i2\pi(\alpha^t n - \beta^t m)\right)\theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(z) \ , \qquad (A.3)$$

The Jacobi map $I:\Sigma \to C^g$ is defined by

$$I(P) = \int_{P_0}^{P} \eta \quad , \quad P_0 , P \in \Sigma \tag{A.4}$$

where P_0 is an arbitrary reference point on Σ.

An important property of the Jacobi map is that it maps divisor classes to $J(\Sigma)$ (Abel's theorem), that is, given a divisor $D = \sum_{i=1}^{m} P_i - \sum_{i=1}^{n} Q_i$, $I(D) = \sum_{i=1}^{m} I(P_i) - \sum_{i=1}^{n} I(Q_i)$, then $I(D) = I([D]) \mathrm{mod}.\Gamma(\Sigma)$ where $[D]$ is the divisor class defined by the equivalence relation: $D_1 \approx D_2$ if $(D_1 - D_2)$ is the divisor of a meromorphic function. In the text we denote for compactness $I(D)$ by D itself.

A fundamental theorem in θ-function theory is the Riemann vanishing theorem. It states that the function

$$F(P) = \theta\left(I(P) - \sum_{i=1}^{g} I(P_i) + I(\Delta) \right) \tag{A.5}$$

either vanishes identically or it has exactly g-simple zeros in $P = P_i$, $i = 1, \ldots, g$. $\Delta = (\Delta_k)$ is the Riemann divisor class defined by

$$I(\Delta_k) = i\pi - i\pi\Omega_{kk} + \sum_{l \neq k} \int_{a_l} \eta_l(P) \int_{P_0}^{P} \eta_k \tag{A.6}$$

As a useful corollary of this theorem, we have

$$\theta\left(- \sum_{i=1}^{g-1} I(P_i) + I(\Delta) \right) = 0 \quad , \quad \forall P_i \in \Sigma \tag{A.7}$$

The prime form $E(P,Q)$ is a multivalued -1/2-differential without poles in both variables P and Q with a unique simple zero for Q=P

$$E(P,Q) = \frac{\theta\left[\begin{smallmatrix}\alpha \\ \beta\end{smallmatrix}\right]\left(I(P) - I(Q) \right)}{h(P)\, h(Q)} = - E(Q,P)$$

$$E(P,Q) \overset{\sim}{=} P - Q \qquad \text{as} \quad Q \approx P \tag{A.8}$$

where $h^2(P) = \sum_{i=1}^{g} \partial_{z_i} \vartheta \left[{\alpha \atop \beta} \right](0)\, \eta_i(P)$ is the holomorphic section of the spin bundle corresponding to a nonsingular $\left(\partial_{z_i} \vartheta \left[{\alpha \atop \beta} \right](0) \neq 0 \right)$ odd spin structure $\left[{\alpha \atop \beta} \right]$. E(P,Q) is independent of the particular choice of $\left[{\alpha \atop \beta} \right]$. For cycles winding around P, it transforms as

$$E(P,Q) \xrightarrow{P+na+mb} \exp\left(-i\pi m^t \Omega m - i2\pi m^t (I(P) - I(Q)) \right) E(P,Q) \qquad (A.9)$$

Finally, we introduce the σ-differential

$$\sigma(P) = \exp\left(- \sum_{j=1}^{g} \oint_{a_j} \eta_j(Q) \ln E(Q,P) \right) \qquad (A.10)$$

It is a g/2-differential defined on a covering of Σ without zeroes and poles. Its transformation property is

$$\sigma(P) \xrightarrow{P+na+mb} \exp\left(i\pi(g-1)\, m^t \Omega m - i2\pi m^t\, I(\Delta) - (g-1)\, I(P) \right) \sigma(P) \qquad (A.11)$$

Appendix B

In this appendix we illustrate some useful properties of the Θ-divisor. Let κ be a degree (g-1)-line bundle. It has a holomorphic section if in its corresponding divisor class $[\kappa]$ there is a positive divisor $\sum_{i=1}^{g-1} P_i$. By the Riemann-Roch theorem, such a divisor exists if and only if $K \otimes \kappa^{-1}$ has a divisor of the form $\sum_{i=1}^{g-1} Q_i$. Let $D_{(\alpha\beta)}$ be a spin bundle (i.e., $2[D_{(\alpha\beta)}]=[K]$), then the set

$$S_{(\alpha\beta)} = \left\{ I\left(\sum_{i=1}^{g-1} P_i - D_{(\alpha\beta)}\right) \Big| P_1,\ldots,P_{g-1} \in \Sigma \right\} \subset J(\Sigma)$$

is a symmetric subset with respect to the origin of $J(\Sigma)$, that is

$$-I\left(\sum_{i=1}^{g-1} P_i - D_{(\alpha\beta)}\right) = I\left(\sum_{i=1}^{g-1} Q_i - D_{(\alpha\beta)}\right) \in S_{(\alpha\beta)}$$

From the vanishing theorem, as the points P_i sweep out Σ we recover Θ

$$I(\Delta) - \left\{ I\left(\sum_{i=1}^{g-1} P_i\right) \Big| P_1,\ldots,P_{g-1} \in \Sigma \right\} = \Theta$$

Therefore

$$S_{(\alpha\beta)} = \gamma_{(\alpha\beta)} - \Theta \qquad , \quad \gamma_{(\alpha\beta)} = I(\Delta - D_{(\alpha\beta)}) \qquad (B1)$$

Since Θ and $S_{(\alpha\beta)}$ are both symmetric subsets with respect to the origin of $J(\Sigma)$, we have

$$\Theta + 2\gamma_{(\alpha\beta)} = \Theta \qquad (B2)$$

This means that $\theta(z+2\gamma_{(\alpha\beta)})/\theta(z)$ is a constant on the compact space $J(\Sigma)$, therefore $2\gamma_{(\alpha\beta)} \in \Gamma(\Sigma)$, that is, each $\gamma_{(\alpha\beta)}$ is one of the 2^{2g} points of order two. Being 2^{2g} also the number of spin structures, it follows from Abel's theorem that for each half-points of $J(\Sigma)$ there is a different $D_{(\alpha\beta)}$ and viceversa. Since we can write $\gamma_{(\alpha\beta)} = \beta + \Omega\alpha$ then

$$\Theta_{(\alpha\beta)} = \{ z : \vartheta\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](z) = 0 \} = \Theta + \gamma_{(\alpha\beta)} \; ; \quad \alpha_i, \beta_i \in \{0, 1/2\}$$

$$I(D_{(\alpha\beta)}) = I(D) - \beta - \Omega\alpha \tag{B.3}$$

Noting that

$$\{ I\left(\sum_{i=1}^{g-1} P_i - D_{(00)} \right) \mid P_1, \ldots, P_{g-1} \in \Sigma \} = \Theta \tag{B.4}$$

and that θ is independent of P_0 we see that $D_{(00)}$ depends only on the homology basis chosen. From (B.4) it follows that there is a one-to one correspondence between degree (g-1) line bundles for which $\bar{\partial}_{\kappa}$ has a zero mode and points in Θ. Moreover, it turns out that $h^0(\Sigma,\kappa)$ equals the multiplicity of the zero of $\theta(z)$ at $z = I(\Delta) - I(\kappa)$.

For odd characteristics $\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]$, $0 \in \Theta_{(\alpha\beta)}$ so that there is at least a set of points $P_1, \ldots, P_{g-1}$ such that

$$I\left(\sum_{i=1}^{g-1} P_i - D_{(\alpha\beta)} \right) = 0 \tag{B.5}$$

therefore $D_{(\alpha\beta)}$ has at least a holomorphic section with zeroes at $z=P_i$, $i=1,\ldots,g-1$. In the case of even theta functions there are certain values of Ω for which $0 \in \Theta_{(\alpha\beta)}$; for example, for g=2 this happens when the period matrix is diagonal.

REFERENCES

[1] J.Schwarz, Phys. Rep. 89 (1982) 223;
M.Green, J.Scwarz and E. Witten:"Superstring theory", vol I, II, Cambridge University Press(1987).

[2] E. Witten, Nucl.Phys.B268 (1986) 253; B276 (1986) 291.

[3] A.Belavin, A. Polyakov and A. Zamolodchikov, Nucl. Phys.B241 (1984), 333;
D.Friedan, E. Martinec, and S. Shenker, Nucl. Phys.B271 (1986) 93;
T. Eguchi and H. Ooguri, Nucl.Phys.B282 (1987) 308.

[4] A. Polyakov, Phys. Lett. 103B (1981) 207 and 211;
O.Alvarez, Nucl.Phys.B216 (1983) 125.

[5] V.Alessandrini, Nuovo Cim.2A,321 (1971);
V.Alessandrini and D. Amati, Nuovo Cim.4A, 793 (1971);
M.Kaku and L. Yu, Phys. Lett.33B, 166 (1970); Phys.Rev.D3 (1971)2992;3007;3020.

[6] L.Alvarez-Gaume, C.Gomez and C.Reina, in Proc. of the Trieste Spring School on Superstrings, CERN-TH 4775/87;
L.Alvarez-Gaume,C.Gomez, G.Moore and C.Vafa:"Strings in the operator formalism",BUHEP-87/51.
L.Alvarez-Gaume, C.Gomez, P. Nelson, G. Sierra, and C.

Vafa, preprint BUHEP 88-11.

[7] I.M.Krichever and S.P.Novikov, Funk.Anal.i. Pril,21 No.2(1987), 46.

[8] I.M.Krichever and S.P.Novikov, Funk.Anal.i. Pril,21 No.4(1987), 47.

[9] L.Bonora, A.Lugo, M.Matone and J.Russo: "A global operator formalism on higher genus Riemann surfaces: b-c systems", preprint SISSA 67/88/EP.

[10] A. Lugo and J. Russo: "Hamiltonian formulation and scattering amplitudes in string theory at genus g", preprint SISSA 83/88/EP.

[11] J. Russo: "Multiloop amplitudes for the bosonic string theories in the operator formalism", preprint SISSA, October (1988).

[12] H. Farkas and I. Kra, "Riemann surfaces" Springer,1980);
R. Gunning, "Lectures on Riemann surfaces" (Princeton Universty Press, 1966).

[13] L.Bonora, M.Martellini, M.Rinaldi and J.Russo, Phys.Lett.206B (1988), 444.

[14] E. Witten, Comm.Math.Phys.92 (1984) 451.

[15] L.Bonora, M.Rinaldi, J.Russo and K.Wu, Phys.Lett.B208 (1988) 440.

[16] J.Russo, "Hamiltonian formulation and scattering amplitudes in superstring theory at genus g", preprint SISSA 105/88/EP.

[17] T.Eguchi and H. Ooguri, Phys.Lett.B187 (1987),127;
H.Sonoda, Phys.Lett.178B (1986),390;
M.Bonini and R. Iengo, Int.J.Mod.Phys.A3 (1988),841.

[18] E. Verlinde and H. Verlinde, Nucl. Phys.B288 (1987), 357.

[19] S. Weinberg, Phys. Lett. 156B (1985) 309.

[20] J. Fay, "Theta functions on Riemann surfaces", Lectures Notes in Mathematics 356, Springer-Verlag (1973);
D.Mumford, "Tata Lectures on Theta" vol. I and II (Birkhauser, 1983).