SISSA - SCUOLA
INTERNAZIONALE
SUPERIORE
DI STUDI AVANZATI
TRIESTE
Strada Costiera 11

ISAS - INTERNATIONAL SCHOOL
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Quantum Groups and Conformal Field Theory

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Candidate:

Valeria Bonservizi

Supervisor:

Professor
Loriano Bonora

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Introduction

Historically, the first reason which motivated the interest of physicists in quantum group theory was the powerful tools that quantum groups provide to solve the (quantum) Yang Baxter equations [(Q)YBE's]. On the other hand, one can invert the terms of the problem and look for physical applications of quantum groups, which become the algebraic structure giving new meaning to QYBE's. A significant example of the context in which this philosophy can be verified is represented by the conformal field theory (CFT) and one of the aim of the present Thesis is to review some recent results on this subject. On the other hand, many problems remain still open, as those concerning the theory of quantum group representations, when the deformation parameter is a root of unity. From a physical point of view, the most interesting question is probably how all the different fields, in which YBE's play an important role, can be connected by means of the abstract language of quantum groups.

Indeed, during the last few decades, YBE's appear more and more frequently, in their classical and quantum version, as the condition for the solvability of physical problems in many different fields, from the classical and quantum field theory models in 1+1 dimension, to the two-dimensional lattice models of classical statistical mechanics.

In the literature, YBE's firstly manifested themselves in the works of McGuire in 1964 [24] and Yang in 1967 [35]. They considered a particular quantum mechanical many-body problem, and, exploiting a technique, known as the *Bethe's Ansatz*, to built exact wavefunctions, showed that the scattering matrix factorized to that of the two-body one. Thus, they was able to solve exactly the problem. In this context, YBE arose as the consistency condition for the factorization.

In statistical mechanics, the research of *solvable lattice models*, actively pursued since Onsager [28] proposed in 1944 his solution of Ising model and culminated in Baxter's solution of the eight vertex model in 1972 [4], find unifying criterion in the exactly solvable vertex model construction, where the Boltzman weights satisfy the QYBE's. Another line of development was the theory of *factorized S-matrix* in two-dimensional quantum field theory. Zamolodchicov [37] pointed out that the algebraic mechanism working here is the same as that in Baxter's and others' works.

A first attempt to point out a common feature was represented by the *quantum inverse scattering method*, proposed by Faddeev, Sklyanin and Takhtajan in 1978-79

[9,10] as a unification of the classical inverse scattering method (soliton theory, Toda lattices) and the quantum integrable models mentioned above. In their theory, the basic commutation relations of operators is described by a solution of YBE's. Hereafter YBE's became a mathematical tool which allowed the classification of exactly solvable models and their study as an abstract subject led to the idea of introducing certain deformations of groups or Lie algebras, called quantum groups by Drinfel'd [8].

At about the same time a new link invariant was discovered [21], and, subsequently, the aspect of YBE's as the characterizing relations of braid groups has been brought to attention. Closely related structures have also been revealed in CFT [34] [27], introducing the problem of a hidden quantum group symmetry. An alternative approach, based on the *Toda field theory* [14], led to the same results in a more direct way.

Therefore, the general tendency of recent developments seems more and more to attribute a significant physical role to quantum groups. It is the purpose of the present Thesis to provide a tentative analysis of limits and possibilities in this direction, as far as CFT is concerned.

The work is organized in three chapters. Chapter 1 is devoted to provide an introductory discussion about the axioms defining (quasi) triangular Hopf algebras and their connection with braid statistics. Furthermore, quantum group representation theory will be presented in a categorial frame, in the attempt to make the philosophy of physical applications more comprehensible. The properties of CFT and, in particular, the Coulomb gas representation of minimal models will be the arguments of Chapter 2. This will constitutes the ground on which, in Chapter 3, the connection between the non-local statistics of conformal blocks and a quantum internal symmetry will be investigated.

Chapter 1

Quantum groups

A quantum group is a Hopf algebra which is neither commutative nor cocommutative. An alternative definition, which clarifies the relation with groups, could be given taking into consideration the algebra $\mathbb{K}(G)$ of functions defined on a group G and valued in a field $\mathbb{K}$. Then a quantum group is obtained as a deformation of $\mathbb{K}(G)$, more precisely a deformation depending on a parameter $q \in \mathbb{K}$ such as to retrieve $\mathbb{K}(G)$ when $q = 1$. This means that, introducing the parameter q, a commutative algebra is turned into a noncommutative one. To make this idea more concrete, let us consider a simple Lie group $\mathcal{G}$ and denote $\mathfrak{g}$ the complexification of its Lie algebra. One defines the *universal enveloping algebra* $U(\mathfrak{g})$, as the tensor algebra on $\mathbb{C}$ generated by 1 and elements of $\mathfrak{g}$ modulo the relations

$$\xi_1 \otimes \xi_2 - \xi_2 \otimes \xi_1 = [\xi_1, \xi_2] \tag{1.1}$$

$\forall \xi_1, \xi_2 \in \mathfrak{g}$. In order to construct the quantum version of this algebra, let us fix some notations. Hence, $\{a_{ij}\}_{i,j=1,\dots,r}$, where r is the rank of $\mathfrak{g}$, will be the Cartan matrix of the Lie algebra, while $\{\alpha_i\}_{i=1,\dots,r}$ will denote a basis of simple roots. One calls *quantum universal enveloping algebra* $U_q(\mathfrak{sl}_2)$ the tensor algebra generated by 1 and the indeterminates $\{k_i^{\pm 1}, e_i, f_i\}_{i=1,\dots,r}$ modulo the relations

$$\begin{gathered}
k_i\, k_i^{-1} = 1 = k_i^{-1}\, k_i\,, \qquad k_i\, k_j = k_j\, k_i\,, \\
k_i\, e_j\, k_i^{-1} = q_i^{a_{ij}/2} e_j\,, \qquad k_i\, f_j\, k_i^{-1} = q_i^{-a_{ij}/2} f_j\,, \\
[e_i, f_j] = \delta_{ij} \frac{k_i^2 - k_i^{-2}}{q_i - q_i^{-1}}\,, \\
\sum_{n=0}^{1-a_{ij}} (-1)^n \begin{bmatrix} 1 - a_{ij} \\ n \end{bmatrix}_{q_i} e_i^{1-a_{ij}-n}\, e_j\, e_i^n = 0 \qquad \textit{for } i \neq j \\
\sum_{n=0}^{1-a_{ij}} (-1)^n \begin{bmatrix} 1 - a_{ij} \\ n \end{bmatrix}_{q_i} f_i^{1-a_{ij}-n}\, f_j\, f_i^n = 0 \qquad \textit{for } i \neq j\,.
\end{gathered} \tag{1.2}$$

Here $q_i = q^{(\alpha_i,\alpha_i)/2}$, $(\cdot,\cdot)$ being the invariant inner product on the root system $\oplus\,\mathbb{C}\alpha_i$, with $(\alpha_i,\alpha_i)\in\mathbb{Z}$, and

$$\begin{bmatrix} m \\ n \end{bmatrix}_q = \frac{(q^m - q^{-m})(q^{m-1} - q^{-(m-1)})\cdots(q^{m-n+1} - q^{-(m-n+1)})}{(q-q^{-1})(q^2-q^{-2})\cdots(q^n-q^{-n})} \qquad for\ \ m > n > 0\,,$$

$$\begin{bmatrix} m \\ n \end{bmatrix}_q = 1 \qquad for\ \ n = 0\ \ or\ \ m = n\,. \tag{1.3}$$

As the notation suggests (Rosso [31] has shown that this analogy can be rigorously justified), one can see in the e_i and f_i generators the quantum correspondents of the elements of the Weyl basis of $\mathfrak{g}$, while k_i and k_i^{-1}, due to eqs.(1.2), should be related with the Cartan subalgebra. This idea makes more intuitive how the deformation acts on the "classical" $U(\mathfrak{g})$. Indeed, let us consider the simplest case, $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$, and denote $q^{H/2}$, $E^{\pm}$, $q^{-H/2}$ the indeterminates which generate $U_q(\mathfrak{sl}_2)$. The defining relations of this quantum universal enveloping algebra

$$\begin{aligned} q^{\pm H/2} q^{\mp H/2} &= 1\,, \\ q^{H/2} E^{\pm} q^{-H/2} &= q^{\pm 1} E^{\pm}\,, \\ [E^+, E^-] &= \frac{q^H - q^{-H}}{q - q^{-1}} \end{aligned} \tag{1.4}$$

can be easily interpreted as the deformation of the usual commutation relations

$$\begin{aligned} [H, E^{\pm}] &= \pm 2E^{\pm} \\ [E^+, E^-] &= H\,. \end{aligned} \tag{1.5}$$

for the Weyl basis $\{E^{\pm}, H\}$.

Notice that in this example the beginning algebra is already noncommutative and what one loses in the deformation is the cocommutativity of the enveloping algebra $U(g)$, in a dual way with respect to the deformation of the algebra $\mathbb{K}(G)$. In fact $U(\mathfrak{sl}_2)$ is essentially dual to $\mathbb{C}(SL_2)$. But, to make this introduction more exhaustive, some further definitions are needed.

1.1 Hopf algebras

In order to introduce the concept of Hopf algebra, a detailed definition of algebra can be useful.

Definition 1 An *algebra with unit* $(A,\ +,\ \cdot,\ \eta;\ \mathbb{K})$ is a vector space $(A,\ +;\ \mathbb{K})$ together with the linear functions $\cdot : A \otimes A \to A$ and $\eta : \mathbb{K} \to C$ (η is related to the usual unit $\mathbf{1}_A$ by the equation $\eta(\lambda) = \lambda \mathbf{1}_A$). The *product* $\cdot$ must fulfill the *associativity* property

$$\cdot \circ (id \otimes \cdot) \;=\; \cdot \circ (\cdot \otimes id) \tag{1.6}$$

while for the *unit* it is required that

$$\cdot \circ (id \otimes \eta) \;=\; \cdot \circ (\eta \otimes id) \tag{1.7}$$

□

The dual concept of the algebra structure is the coalgebra structure, described by the following

Definition 2 A *coalgebra with counit* $(C, +, \Delta, \epsilon; \mathbb{K})$ is a vector space $(C, +; \mathbb{K})$ together with the linear functions $\Delta : C \to C \otimes C$ and $\epsilon : C \to \mathbb{K}$. The *coproduct* Δ must fulfill the *coassociativity* property

$$(id \otimes \Delta) \circ \Delta \;=\; (\Delta \otimes id) \circ \Delta \tag{1.8}$$

while for the *counit* it is required that

$$(id \otimes \epsilon) \circ \Delta \;=\; (\epsilon \otimes id) \circ \Delta \tag{1.9}$$

□

The duality between algebras and coalgebras are easily established. In fact, if the coalgebra C is finite dimensional as a vector space, C^* is its dual space and $\langle , \rangle$ the pairing relation defined as $\langle \phi, c \rangle = \phi(c) \, \forall c \in C, \; \forall \phi \in C^*$, then one defines an induced algebra structure in C^* with the relations

$$\begin{aligned} \langle \phi\psi, c \rangle \;&\equiv\; \langle \phi \otimes \psi, \Delta(c) \rangle \\ \langle \mathbf{1}_{C^*}, c \rangle \;&\equiv\; \epsilon(c) \, . \end{aligned} \tag{1.10}$$

The finite dimensional condition is needed because only in that case the pairing $\langle , \rangle$ induces an isomorphism between C and C^{**}, otherwise one has just an inclusion $C \hookrightarrow C^{**}$. Nevertheless, here and in the following these results can be generalized if a suitable topology is introduced.

The most natural generalization of the last definitions consists in introducing "self-dual" structures.

Definition 3 A *bialgebra* $(B, +, \cdot, \Delta, \eta, \epsilon; \mathbb{K})$ is both an algebra and a coalgebra, such that the two structures are compatible. The compatibility condition consists in imposing Δ and ϵ to be algebra maps. □

An example of bialgebra could be the *quantum matrices* ${}_qM_2(\mathbb{C})$. It is the tensor algebra generated by 1 and the indeterminates a, b, c, d modulo the relation

$$ac = q^{-1}\,ca\,, \qquad bd = q^{-1}\,db\,, \qquad ad - da = q^{-1}\,cb - q\,bc\,. \tag{1.11}$$

The coproduct and the counit are

$$\Delta(u^i{}_j) = \sum_{k=1,2} u^i{}_k \otimes u^k{}_j\,, \qquad \epsilon(u^i{}_j) = \delta^i{}_j \tag{1.12}$$

To make the meaning of this structure less obscure, let us note that ${}_qM_2(\mathbb{C})$ can be constructed as the product of the quantum plane with the quantum super plane, exactly as, for a finite dimensional vector space V, $End(V) = V \otimes V^*$. Remember that the *quantum plane* $\mathbb{C}_q^{2|0}$ is the tensor algebra on $\mathbb{C}^2$ together with the relation

$$xy = q^{-1}\,yx\,, \qquad\qquad \textit{relations}\ \rho_b \tag{1.13}$$

while the *quantum super plane* $\mathbb{C}_q^{0|2}$ is defined by

$$\xi\xi = 0\,, \qquad \eta\eta = 0\,, \qquad \xi\eta = -q\,\eta\xi\,, \qquad\qquad \textit{relations}\ \rho_f \tag{1.14}$$

where $\{x, y\}$ and $\{\xi, \eta\}$ are two bases of $\mathbb{C}^2$. Notice that $\mathbb{C}_q^{2|0}$ is, indeed, the algebra generated by the relations of quantum mechanics, $[P, Q] = -i\hbar$, in exponentiated form $x = e^{iP}$, $y = e^{iQ}$, for $q = e^{-i\hbar}$. The duality between $\mathbb{C}_q^{2|0}$ and $\mathbb{C}_q^{0|2}$ follows if one supposes that $\{\xi, \eta\}$ is the dual basis of $\{x, y\}$. This means that, due to this hypothesis, the relations ρ_f can be obtained in the form $\rho_f = \phi(\rho_b)$, where $\phi = \phi_1\xi \otimes \xi + \phi_2\xi \otimes \eta + \phi_3\eta \otimes \xi + \phi_4\eta \otimes \eta$. If, now, one identifies $\{a = x \otimes \xi, b = x \otimes \eta, c = y \otimes \xi, d = y \otimes \eta\}$, then ρ_b and ρ_f induces the relations (1.11), which define the quantum matrices ${}_qM_2(\mathbb{C})$.

A second interesting "self-dual" structure is

Definition 4 A *Hopf algebra* is a bialgebra $(H, +, \cdot, \Delta, \eta, \epsilon, S; \mathbb{K})$ equipped with a linear *antipode map* $S : H \to H$ obeying

$$\cdot \circ (id \otimes S) \circ \Delta = \eta \circ \epsilon = \cdot \circ (S \otimes id) \circ \Delta \tag{1.15}$$

□

A straightforward calculation shows that S must be a antialgebra and anticoalgebra map, while in general it is not true that $S^2 = id$. Moreover, taking into account what has been said about duality between algebras and coalgebras, one concludes that for each finite dimensional Hopf algebra there exists a dual one, with product, unit, coproduct and counit defined as eq.(1.10) suggests and with the antipode map which obeys the obvious condition

$$\langle S\phi, h\rangle = \langle \phi, Sh\rangle\,, \qquad h \in H\,, \phi \in H^* \tag{1.16}$$

As before, the dimensional limitation can be avoided.

On these grounds, one can consider again the example of (quantum) universal enveloping algebra of a Lie algebra. In fact, remember that, $U_q(\mathfrak{g})$, obtained as a deformation of $U(\mathfrak{g})$, according to the first definition of quantum group must be a neither commutative nor cocommutative Hopf algebra, where $U(\mathfrak{g})$ is a cocommutative Hopf algebra (that is, the coproduct obeys the relation $\tau \circ \Delta = \Delta$, where $\tau : H \otimes H \to H \otimes H$ such that $h_1 \otimes h_2 \mapsto h_2 \otimes h_1$). This can be shown just reporting the expressions of coproduct, counit and antipode map that make $U(\mathfrak{g})$ a cocommutative Hopf algebra

$$\Delta(\xi) = \xi \otimes 1 + 1 \otimes \xi\,, \qquad \epsilon(\xi) = 0\,, \qquad S(\xi) = -\xi\,, \qquad for\ \xi \in g \tag{1.17}$$

and their deformation in $U_q(\mathfrak{g})$

$$\begin{aligned} \Delta(k_i^{\pm 1}) &= k_i^{\pm 1} \otimes k_i^{\pm 1}\,, \\ \Delta(e_i) &= e_i \otimes k_i^{-1} + k_i \otimes e_i\,, \\ \Delta(f_i) &= f_i \otimes k_i^{-1} + k_i \otimes f_i\,, \end{aligned} \tag{1.18}$$

or, in the particular case $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$,

$$\begin{aligned} \Delta(q^{\pm H/2}) &= q^{\pm H/2} \otimes q^{\pm H/2} \\ \Delta(E^{\pm}) &= E^{\pm} \otimes q^{H/2} + q^{-H/2} \otimes E^{\pm}\,, \\ \epsilon(q^{\pm H/2}) &= 1 \qquad\qquad S(q^{\pm H/2}) = q^{\mp H/2} \\ \epsilon(E^{\pm}) &= 0\,, \qquad\qquad S(E^{\pm}) = -q^{\pm} E^{\pm}\,, \end{aligned} \tag{1.19}$$

where coproducts and counits are extended to the tensor producted elements as algebra maps, and the antipodes as antialgebra maps. It is easy to verify the axioms and the noncommutativity of $U_q(\mathfrak{sl}_2)$ is evident, being $\tau \circ \Delta(E^{\pm}) \neq \Delta(E^{\pm})$. Notice that, for the $U(\mathfrak{g})$-antipode, the usual condition $S^2 = id$ is satisfied, while this is not true in general for the quantum antipode. It is also interesting to note how the last equations reduce to $U(\mathfrak{g})$-ones when $q \to 1$.

An important class of the Hopf algebras are the quasitriangular Hopf algebras. They are in general noncommutative and noncocommutative, but the lack of cocommutation is, in some sense, under control. This restriction make possible to extend a lot of properties that are proper of the cocommutative case.

Definition 5 A *quasitriangular Hopf algebra* is a pair $(H, \mathfrak{R})$ where H is a Hopf algebra and $\mathfrak{R} \in H \otimes H$ is invertible and obeys

$$(\Delta \otimes id)\mathfrak{R} = \mathfrak{R}_{13}\mathfrak{R}_{23}\,, \qquad (id \otimes \Delta)\mathfrak{R} = \mathfrak{R}_{13}\mathfrak{R}_{12} \tag{1.20}$$

$$\tau \circ \Delta h = \mathfrak{R}(\Delta h)\mathfrak{R}^{-1}\,, \qquad \forall h \in H \tag{1.21}$$

where, writing $\mathfrak{R} \equiv \sum \mathfrak{R}^{(1)} \otimes \mathfrak{R}^{(2)}$, the notation used is

$$\mathfrak{R}_{ij} = \sum 1 \otimes 1 \otimes \cdots \otimes \mathfrak{R}^{(1)} \otimes \cdots \otimes \mathfrak{R}^{(2)} \otimes \cdots \otimes 1\,, \tag{1.22}$$

the element of $H \otimes H \otimes \cdots \otimes H$ which is $\mathfrak{R}$ in the i'th and j'th factors. In particular, $(H, \mathfrak{R})$ is called *triangular* if, in addition

$$\tau(\mathfrak{R}^{-1}) = \mathfrak{R} \tag{1.23}$$

$\mathfrak{R}$ is called *universal $\mathfrak{R}$-matrix.* □

Drinfel'd [8] has demonstrated that every quantum enveloping algebra is quasitriangular. As an example, $\mathfrak{R}$ for $U_q(\mathfrak{sl}_2)$ is

$$\mathfrak{R} = q^{H\otimes H/2} \sum_{n=0}^{\infty} \frac{(1-q^{-2})^n}{[n]_q!} (q^{H/2}E^+ \otimes q^{-H/2}E^-)^n q^{n(n-1)/2}\,. \tag{1.24}$$

Here, coherently with the definition (1.3),

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} \tag{1.25}$$

and $[n]_q! = [n]_q \cdot [n-1]_q \cdots [1]_q$. Notice that, if q is the p^{th} root of unity, this quantity vanishes for n proportional to p, and singularities might be present in the expression (1.24). On the other hand, due to the second of the defining eqs.(1.4), if p is even, then $(E^{\pm})^{p/2}$ vanish too. Furthermore, strictly speaking $\mathfrak{R}$ is not an element of $U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2)$, because this contains only finitely many strings in the generators, but the problem is easily solvable thinking, in a topological context, to some completion. However, this consideration can be avoided if finite dimensional representation of H are considered, as it is made clearer in the sequel.

The meaning of the axiom (1.20) becomes clear if $\mathfrak{R}$ is regarded as a function $\mathfrak{R}:\ H^* \to H$ such that $\phi \mapsto \mathfrak{R}^{(1)}\phi(\mathfrak{R}^{(2)})$. Indeed, almost in the finite dimensional case, this axiom is equivalent to impose that $\mathfrak{R}$ as a map is coalgebric and antialgebric.

The same defining eq.(1.20) implies also some other important relations involving $\mathfrak{R}$

$$(\epsilon \otimes id)\mathfrak{R} = 1 = (id \otimes \epsilon)\mathfrak{R} \tag{1.26}$$

$$(S \otimes id)\mathfrak{R} = \mathfrak{R}^{-1}\,, \qquad (id \otimes S)\mathfrak{R}^{-1} = \mathfrak{R} \tag{1.27}$$

Moreover, due to the first part of the eq.(1.20) and eq.(1.21), $\mathfrak{R}$ fulfills the so-called *abstract Quantum Yang Baxter Equations* (QYBE's)

$$\mathfrak{R}_{12}\mathfrak{R}_{13}\mathfrak{R}_{23} = \mathfrak{R}_{23}\mathfrak{R}_{13}\mathfrak{R}_{12} \tag{1.28}$$

If, remembering its defining equations, one interprets $\mathfrak{R}$ as the "structure constants" of the (quasi)triangular Hopf algebra, these equations become nothing else that the corresponding "Jacobi Identity". But one can go further in this analogy. In fact, it is possible to develop from each matrix solution of the QYBE's a quasitriangular Hopf algebra and its dual, just as one can characterized a Lie algebra on the ground of the structure constants ${c_{ij}}^k$ such that $[\xi_i, \xi_j] = {c_{ij}}^k \xi_k$. In the next section, matrix solutions of the QYBE's will be studied in different contexts.

1.2 R-matrix

Given a vector space V, one can construct the associated *symmetric algebra* $S(V)$ as the tensor algebra generated by 1 and the elements of V modulo the relations

$$v_1 \otimes v_2 = v_2 \otimes v_1 \tag{1.29}$$

For finite dimensional V, such an algebra is equivalent to polynomial functions on V. The definition of $S(V)$ can be given, alternatively, referring to the permutation groups S_N. In fact, notice that the twist map $\tau: V \otimes V \to V \otimes V$, $v_1 \otimes v_2 \to v_2 \otimes v_1$ generates a representation of S_N on the spaces V^N. These representations are constructed factorizing each permutation σ into transpositions, $\sigma = (i_1 j_1)(i_2 j_2)\dots(i_N j_N)$, and relating each transposition to a twist map acting on V^N. Then, the operator which represents the action of σ on the space $S_N(V)$ is $\tau_{i_1 j_1}\tau_{i_2 j_2}\dots\tau_{i_N j_N}$, where τ_{ij} is the twist map acting in the i^{th} and j^{th} terms of V^N. On these grounds, the symmetric algebra $S(V)$ can be thought as the tensor algebra constructed on V, modulo such an action of S_N. This new definition allows to generalize such a construction, when an arbitrary representation is taken into account.

Let us consider the assignment $(i_1 j_1)(i_2 j_2)\dots(i_N j_N) \xrightarrow{\rho} B_{i_1 j_1} B_{i_2 j_2}\dots B_{i_N j_N}$, where the *statistical matrix* $B \in End(V) \otimes End(V) \subseteq End(V \otimes V)$. This defines an action of the permutation group S_N on the space V^N if and only if B is an unitary solution ($B^2 = 1$) of the equations

$$B_{12} B_{23} B_{12} = B_{23} B_{12} B_{23}\,. \tag{1.30}$$

Then, symmetric algebras can be generalized substituting the quotient relations (1.29) with

$$v_1 \otimes v_2 = B\, v_2 \otimes v_1 \tag{1.31}$$

The new algebras so obtained is said B-symmetric algebra and denoted $S_B(V)$. The simplest non trivial example of B-symmetric algebra is the exterior algebra $\Lambda(V)$ defined as the tensor algebra constructed on V, with the relations $v_1 v_2 = -v_2 v_1$ (understanding the tensor product symbol). Analogously to the symmetric case, this algebra can be thought as that of functions on a super-vector space. How this considerations are connected to the argument of this section becomes evident if one puts $B = \tau \circ R$: indeed, in such a way, the eqs.(1.30) reduce to the matrix QYBE's.

The unitarity condition in terms of the R-matrix, means that $R_{21} = R_{12}^{-1}$. Removing this requirement, one finds a further generalization of B-symmetric algebras in which the role of the permutation group is played by the braid group, described in the following

Definition 6 Let M be a manifold of dimension $d \geq 2$ and $D(M^N)$ the subspace of M^N

$$D(M^N) = \{(p_1, p_2, \dots, p_N) :\ p_i \in M,\ p_i \neq p_j \ \textit{if}\ i \neq j\}\,. \tag{1.32}$$

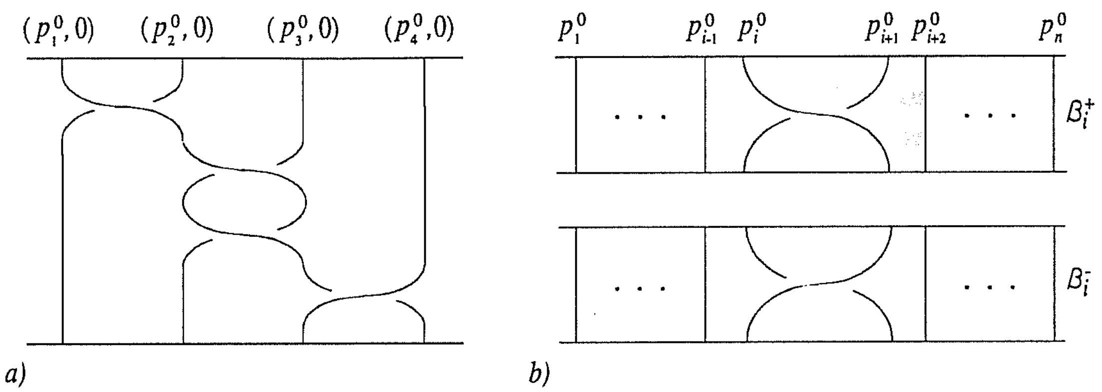


Figure 1.1: *a) A geometric braid representing an element of B_3. b) The elementary braids.*

The fundamental group $\pi_1\, D(M^{\rm N})$ of the space $D(M^{\rm N})$ is the *pure braid group* (or *monodromy group*) for N strings of the manifold M, denoted $P_{\rm N}(M)$. Let $\overline{D}(M^{\rm N})$ be the space $D(M^{\rm N})$ modulo permutations. The fundamental group $\pi_1\,\overline{D}(M^{\rm N})$ of the space $\overline{D}(M^{\rm N})$ is called the *full braid group* or, more simply, the *braid group* of M, denoted $B_{\rm N}(M)$. □

When the manifold is the euclidean plane E^2, one speaks of the *classical braid group* $B_{\rm N}$. In this case, an alternatively graphical definition is possible. In fact, any element in $B_{\rm N}(M) = \pi_1\,\overline{D}(E^2)^{\rm N})$ is represented by a loop

$$f\,:\;[0,1] \to \overline{D}(E^2)^{\rm N}\,, \qquad f(0) \;=\; f(1) \;=\; \bar{p}^0 \tag{1.33}$$

which lifts uniquely to a path

$$f\,:\;[0,1] \to D(E^2)^{\rm N}\,, \qquad f(0) \;=\; p^0\,, \tag{1.34}$$

where p^0 is an element of the permutation class $\bar{p}^0$. The graph β of this path in the $E^2 \times [0,1]$ is called a *geometric braid* or, more simply, a *braid*, while the graph of a single component f_i is called the i^{th} *braid string*. An example of braid is given in figure (1.1). In order that β is a representative of class in the fundamental group $B_{\rm N}$, one must identify two braids $\beta \sim \beta'$ if the paths f and f' which define them are homotopic relative to the base point $(p_1^0, p_2^0, \ldots, p_{\rm N}^0)$ in the space $D((E^2)^{\rm N})$. Nevertheless, this equivalence relation can be reformulated in the context of the graphical representation, regarding β and β' as subsets of $E^2 \times [0,1]$. Thus, two braids can be identified if they can be "regularly" deformed one into the other. It follows that, for example, the braids in figure (1.2) are equivalent to the trivial braid. In the sequel, this criterion will be used to prove some important properties of $B_{\rm N}$.

Geometric intuition suggests that, as for $S_{\rm N}$, an arbitrary braid is equivalent to a product of elementary braids β_i^+ and β_i^- $(= (\beta_i^+)^{-1})$, represented graphically in figure (1.1). In the example of figure (1.1)

$$\beta^{(4)} \;=\; \beta_1^+\beta_2^+\beta_2^+\beta_3^-\,. \tag{1.35}$$

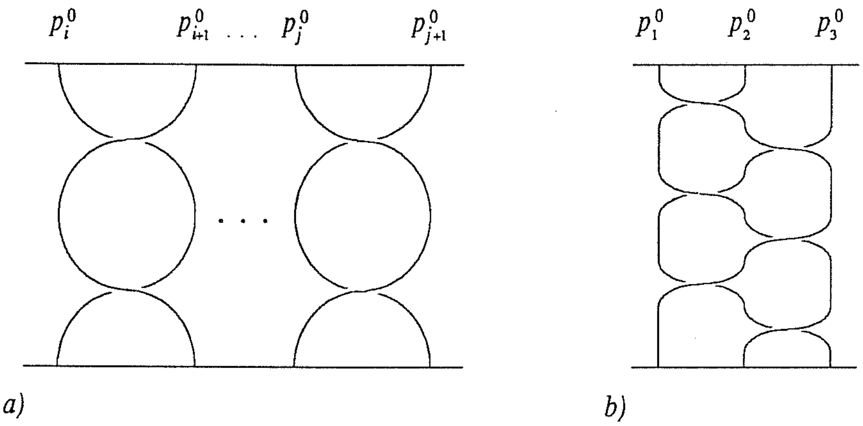


Figure 1.2: *Examples of braids equivalent to the trivial one. The braid b) provides a graphical representation of the second of the eqs.(1.37) (QYBE)*

This means that $\beta_1^+, \beta_2^+, \dots, \beta_{N-1}^+$ generate the group B_N[1]. The following relations hold

$$\beta_i^+ \beta_j^+ = \beta_j^+ \beta_i^+ \qquad \textit{for } 1 \leq i,j \leq n-1\,, \textit{ if } |i-j| \geq 2 \tag{1.36}$$

$$\beta_i^+ \beta_{i+1}^+ \beta_i^+ = \beta_{i+1}^+ \beta_i^+ \beta_{i+1}^+ \qquad \textit{for } 1 \leq i \leq n-2\,. \tag{1.37}$$

The path deformations which establish the relations $\beta_i^+ \beta_i^- = 1$ and (1.37) are called *Reidemeister moves* of type *II* and *III*. Eqs.(1.36) and (1.37) provide an alternative algebric definition of the classical braid groups B_N. If one considers different two-dimensional manifold, some additional relations arise. For example, the generator of the braid group on the sphere S^2 must satisfy the further conditions

$$\beta_1^+ \dots \beta_{N-2}^+ \beta_{N-1}^+ \beta_{N-1}^+ \beta_{N-2}^+ \dots \beta_1^+ = 0\,, \tag{1.38}$$

$$(\beta_1^+ \beta_2^+ \dots \beta_{N-1}^+)^N = 1\,. \tag{1.39}$$

As in the case of the permutation groups, if one looks at the representations of B_N as statistical matrices $B^{\pm}$ acting on V^N, then the eqs.(1.37) result equivalent to the matrix QYBE's.

1.3 Construction of quasitriangular Hopf algebras

Here we will face the question, already brought out at the end of §1.1 in order to emphasize the role of the R-matrix, of the construction of a quasitriangular Hopf algebra from the knowledge of a regular invertible solution $R \in M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$ of the matrix QYBE's

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}\,. \tag{1.40}$$

[1]For a rigorous proof of this statement, see Birman [6]

After what has been explained about R-matrices and their statistical and physical meaning, this result represents an important progress, since, consequently, each problem which involves R-matrices or, that is the same, braid group representations, should exhibit a hidden quantum group symmetry.

Let $A(R)$ be the tensor algebra generated by 1 and n^2 indeterminates $u^i{}_j$, $i,j = 1,\dots n$ modulo the relations

$$R^{i\ k}_{\ m\ n} u^m_{\ j} u^n_{\ l} = u^k_{\ n} u^i_{\ m} R^{m\ n}_{\ j\ l} \tag{1.41}$$

where $R^{i\ k}_{\ j\ l}$ are the matrix elements of $\sum R^{(1)}{}^i{}_j \otimes R^{(2)}{}^k{}_l$, being *otimes* a matrix tensor product, [see eq.(1.22)], and understanding the tensor product among the generators $u^i{}_j$.[2] The same equations can be written using a more compact notation

$$R(u \otimes 1)(1 \otimes u) - (1 \otimes u)(u \otimes 1)R = 0 \tag{1.42}$$

where $u \in M_n(A(R)) \otimes M_n(A(R))$ is the matrix of the indeterminates $u^i{}_j$. Notice that $R \in M_n(A(R)) \otimes M_n(A(R))$ because $\mathbb{C} \subset A(R)$ through the unit. It is possible to verify that the coproduct and the counit

$$\Delta(u^i{}_j) = \sum_k u^i{}_k u^k{}_j, \qquad \epsilon(u^i{}_j) = \delta^i{}_j \tag{1.43}$$

(extended as algebra maps), together with the usual algebra structure, make $A(R)$ a bialgebra. In particular, when $R = 1 \otimes 1$ one finds the bialgebra of functions on the semigroup $M_n(\mathbb{C})$. A more significant example is the bialgebra associated to the R-matrix

$$R = q^{-1/2} \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & q - q^{-1} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix} \tag{1.44}$$

where R must be viewed as an element of $M(2, M(2,\mathbb{C}))$ and $q \in \mathbb{C} - \{0\}$. It is easy to verify that, in such a case, $A(R) = {}_qM_2(\mathbb{C})$, the quantum matrices defined by the relations (1.11).

Analogously, one defines the bialgebra $\tilde{U}(R)$ as the tensor algebra generated by 1 and the $2n^2$ indeterminates l^+ and l^- modulo the relations

$$(l^\pm \otimes 1)(1 \otimes l^\pm)R - R(1 \otimes l^\pm)(l^\pm \otimes 1) = 0 \tag{1.45}$$

$$(l^- \otimes 1)(1 \otimes l^+)R - R(1 \otimes l^+)(l^- \otimes 1) = 0 \tag{1.46}$$

where $l^\pm \in M_n(\tilde{U}(R)) \otimes M_n(\tilde{U}(R))$, as explained. The coalgebra structure will be the usual one.

[2] Hereafter in this section the $\otimes$ will be explicitly indicated only when the matrix tensor product is involved, omitting it when represents multiplication among the generators of the algebra, so that for instance, the generic string $u^{i_1}_{j_1} \otimes u^{i_2}_{j_2} \otimes \cdots \otimes u^{i_p}_{j_p}$ will be denoted $u^{i_1}_{j_1} u^{i_2}_{j_2} \cdots u^{i_p}_{j_p}$.

Both in the case of $A(R)$ and in that of $\tilde{U}(R)$ the R-matrix as *fundamental representation* of these bialgebras in $M_n(\mathbb{C})$. More precisely one can show that the algebra maps

$$\begin{aligned} \rho^+ &: A(R) \to M_n(\mathbb{C}) &: \rho^+(u^i{}_j)^k{}_l &= R^i{}_j{}^k{}_l\,, \\ \rho^- &: A(R) \to M_n(\mathbb{C}) &: \rho^-(u^i{}_j)^k{}_l &= (R^{-1})^k{}_l{}^i{}_j\,, \\ \rho &: \tilde{U}(R) \to M_n(\mathbb{C}) &: &\begin{cases} \rho(l^+{}^i{}_j)^k{}_l = R^i{}_j{}^k{}_l \\ \rho(l^-{}^i{}_j)^k{}_l = (R^{-1})^k{}_l{}^i{}_j \end{cases}, \end{aligned} \tag{1.47}$$

are well-defined representations of $A(R)$ and $\tilde{U}(R)$, respectively. This means that they are compatible with the quotient relations (1.42) and (1.46), property which is strictly connected to the QYBE's.

Another important consequence of the QYBE's is that $A(R)$ and $\tilde{U}(R)$ are paired by bialgebra map

$$\left\langle u, l^+ \right\rangle = R^+\,, \qquad \left\langle u, l^- \right\rangle = R^- \tag{1.48}$$

being $R^+ = R$, $R^- = \tau(R^{-1})$. Indeed, this can be interpreted simply as a different way of casting the previously defined fundamental representations, $u \mapsto \langle u, l^\pm \rangle$, $l^\pm \mapsto \langle u, l^\pm \rangle$. Although, generally, this pairing is degenerate (for instance, if R is unitary then $R^+ = R^-$ and $l^+ - l^-$ has zero pairing with every element of $A(R)$), dividing both $A(R)$ and $\tilde{U}(R)$ by further relations, one obtains two "essentially" dual Hopf algebras (her and in the sequel the attribute "essentially" is to remember the problems which arise due to the infinite dimensional space involved). The main features which allows this construction are two:

i) the kernels of the pairing $\langle , \rangle$ (*null subspaces*) are bi-ideals, so that the bialgebra structure is preserved in the quotient,

ii) the relations

$$\left\langle S(u), l^+ \right\rangle = \left\langle u, S(l^+) \right\rangle = R^{-1}\,, \tag{1.49}$$

$$\left\langle S(u), l^- \right\rangle = \left\langle u, S(l^-) \right\rangle = \tau(R) \tag{1.50}$$

define antipodes on the quotient algebras $\check{A}(R)$ and $\check{U}(R)$, when extended as antialgebra maps.

Finally, a quasitriangular structure can be introduced in $\check{U}(R)$ defining the antialgebra and coalgebra map

$$\mathfrak{R}: \check{A}(R) \to \check{U}(R) \quad : \quad u \mapsto l^+\,. \tag{1.51}$$

In fact, as explained in §1.1 , the existence of an element in (the completion of) $\check{U}(R) \otimes \check{U}(R)$ which satisfies the property (1.20) is related to the existence of an antialgebra and coalgebra map $\breve{U}(R)^* \to \breve{U}(R)$, *i.e.*, essentially, $\breve{A}(R) \to \breve{U}(R)$. It remains to be checked that $\mathfrak{R}$ fulfills the second property (1.21) and is well-defined, *i.e.* it respects the quotient relations (null relations included). The proof is based, again, on the QYBE's.

A possible example of this prescription could regard the well-known bialgebra ${}_qM_2(\mathbb{C})$. Let us summarize briefly the results.

- **The null relations**, mimicking the classical case, are

$$det_q(u) \;=\; u^1{}_1 u^2{}_2 - q^{-1}\, u^1{}_2 u^2{}_1 \;=\; 1 \tag{1.52}$$

  or, in the notation of §1.1,

$$ad - q^{-1}\, bc \;=\; 1\,. \tag{1.53}$$

  To support this statement, one can easily verify that

$$\rho^{\pm}(det_q(u) - 1)^i{}_j \;=\; \left\langle det_q(u) - 1, l^{\pm}{}^i{}_j \right\rangle \;=\; 0\,. \tag{1.54}$$

  It is possible to show that this is also a sufficient condition. Moreover, that $ad - q^{-1}\, bc$ is central and that the coproduct

$$\Delta(ad - q^{-1}\, bc) \;=\; ad - q^{-1}\, bc \otimes ad - q^{-1}\, bc \tag{1.55}$$

  is group-like, confirms the consistency of this relation with the bialgebra structure. Hence, $\breve{A}(R) = \frac{A(R)}{det_q(u)-1} = \frac{{}_qM_2(\mathbb{C})}{det_q(u)-1}$.

- **The antipode**

$$S(u) \;=\; \begin{pmatrix} u^2{}_2 & -q\, u^1{}_2 \\ -q^{-1}\, u^2{}_1 & u^1_1 \end{pmatrix}, \tag{1.56}$$

  makes $\breve{A}(R)$ a Hopf algebra denoted ${}_qSL_2(\mathbb{C})$. It is easy to verify the required properties for S, *e.g.* $\left\langle u, S(l^+{}^2_1)\right\rangle$ $=(R^{-1})^{\cdot}{}^{2}_{\cdot}{}_1 = q^{1/2}\,(q^{-1} - q)\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} =$ $\left\langle u, -q\, l^+{}^2_1\right\rangle$ so that $S(E^+) = -q\, E^+$ *etc.*

- **The essentially dual Hopf algebra** is $\breve{U}(R) = U_q(\mathfrak{sl}_2)$. This derives from the identifications

$$l^+ \;=\; \begin{pmatrix} q^{H/2} & 0 \\ q^{-1/2}(q - q^{-1})E^+ & q^{-H/2} \end{pmatrix}, \tag{1.57}$$

$$l^- \;=\; \begin{pmatrix} q^{-H/2} & q^{1/2}(q^{-1} - q)E^- \\ 0 & q^{H/2} \end{pmatrix}, \tag{1.58}$$

incorporating the corresponding null relations

$$l^{+}{}^{1}_{2} = 0 = l^{-}{}^{2}_{1}, \quad l^{+}{}^{1}_{1} l^{-}{}^{1}_{1} = 1,$$
$$l^{+}{}^{2}_{2} l^{-}{}^{2}_{2} = 1, \quad det\, l^{\pm} = 1 \tag{1.59}$$

Indeed, the nondegeneracy of the pairing (1.48) can be proven [30]. Notice that eqs.(1.59), together with the remaining relations in $\check{U}(R)$ (coming from the quotient relations in $\tilde{U}(R)$) comprehend those of the deformed algebra of $U_q(\mathfrak{sl}_2)$. In particular, each one of the last three null relations indifferently implies the condition $q^{H/2}q^{-H/2}$, while

$$\left.\begin{array}{r} l_1^+ l_2^+ R = R\, l_2^+ l_1^+ \\ \\ l_1^- l_2^- R = R\, l_2^- l_1^- \end{array}\right\} \Rightarrow q^{H/2} E^{\pm} q^{-H/2} = q^{\pm} E^{\pm} \tag{1.60}$$

$$l_1^- l_2^+ R = R\, l_2^+ l_1^- \Rightarrow [E^+, E^-] = \frac{q^H - q^{-H}}{q - q^{-1}} \tag{1.61}$$

Finally, due to the nondegeneracy of the pairing between $\breve{A}(R)$ and $\check{U}(R)$, the quasitriangular structure induced by the map (1.51) coincides with that defined through the universal $\mathfrak{R}$-matrix (1.24).

1.4 Representation theory of quasitriangular Hopf algebras

Hopf algebras can act on other structures in a variety of way, depending on which part of H is involved, the algebra or the coalgebra one. Nevertheless, due to the duality properties of Hopf algebras, these different kinds of representations are related, for example to each left H-action corresponds a right H^*-coaction, etc. Hence, conventionally, one can choose to consider only one of these.

Definition 7 Let H be an algebra and V a vector space, one says that H *acts on* V through the linear map $\alpha :\ H \otimes V \longrightarrow V$ if

$$\alpha \circ (\cdot \otimes id) = \alpha \circ (\alpha \otimes id) \tag{1.62}$$

This condition becomes more explicit using the module notation, $h.v = \alpha(h \otimes v)$, so that eq.(1.62) can be rewritten as

$$(hg).v = h.(g.v)\,. \tag{1.63}$$

Then the pair (α, V) is said an *algebra representation* or, more simply, a *representation* of the algebra H and V a *left H-module.* □

The representations of a Hopf algebra have many properties similar to those of group representations, and this analogy is more evident in the case of quasitriangular and triangular Hopf algebras. In particular, the studies carried out independently by Lusztig [23] and Rosso [31] on the quantum universal enveloping algebra $U_q(\mathfrak{g})$, lead to the conclusion that, for a generic value of the deformation parameter q different from a root of unit and being $\mathfrak{g}$ a complex simple Lie algebra, finite dimensional $U_q(\mathfrak{g})$-modules can be obtained as deformations of those of the "classical" $U(\mathfrak{g})$. Entering into details, Rosso shows in his paper that, given a representation (α, V) of $U_q(\mathfrak{g})$ in a finite dimensional space V,

i) $\alpha(e_i)$ and $\alpha(f_i)$ are nilpotent and, if α is irreducible, then the operators $\alpha(k_i)$ are simultaneously diagonalizable and $V = \oplus V_\mu$, where, for $\mu = (\mu_1, \mu_2, \ldots, \mu_r)$,

$$V_\mu = \{v \in V / \ \forall i = 1, \ldots, r, \quad \alpha(k_i)\, v = \mu_i\, v\}\,. \tag{1.64}$$

This lemma allows to speak about weights of the representation.

ii) For each finite dimensional representation (α, V), there is at least a *highest weight vector*, *i.e.* a vector $v_\lambda \in V - \{0\}$ which fulfills the properties

$$\alpha(k_i)\, v_\lambda = \lambda_i\, v_\lambda\,, \qquad \alpha(e_i)\, v_\lambda = 0 \qquad \forall i = 1, \ldots, r\,, \tag{1.65}$$

being $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \in (\mathbb{C}^*)^r$ its *highest weight.* If (α, V) is irreducible then its module V is spanned by v_λ and the non-zero vectors of the form $\alpha(f_{i_1})\alpha(f_{i_2}) \cdots \alpha(f_{i_p})\, v_\lambda$, $(i_1, \ldots, i_p = 1, 2, \ldots r)$, of weight $\mu = (\mu_1, \mu_2, \ldots, \mu_r)$, being $\mu_k = \lambda_k\, q_k^{-\frac{1}{2}\sum_n a_{kin}}$. Such representations are said *highest weight representations.*

iii) If (α, V) is a finite dimensional irreducible representation with highest weight λ, then $\lambda = \omega \cdot q^{\tilde{\lambda}/2}$, where $\omega \in \{1, -1, i, -i\}^r$ and $\tilde{\lambda}$ is a dominant weight of $\mathfrak{g}$. On the other hand, any character of this form, defined on the subgroup of invertible elements generated by the k_i's, is the highest weight of a finite dimensional irreducible representation.

iv) Every finite dimensional representation of $U_q(\mathfrak{g})$ is completely reducible.

To clarify these points and to make more evident the similarity with groups, an example could be useful. The quantum universal enveloping algebra of $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$, beside being the simplest non trivial case in which the previous statements can be verified, is also a complete example. Indeed, as for the "classical" group, all the results concerning the classification of the quantum group irreducible representations [see point (*iii*)] can be proved by virtue of the theory of the $U_q(\mathfrak{sl}_2)$-modules. Therefore, it is easy to check that, setting

$$\alpha(E^+) = {}^t\alpha(E^-) = \begin{pmatrix} 0 & \sqrt{[2j]_q[1]_q} & 0 & \cdots & 0 \\ 0 & 0 & \sqrt{[2j-1]_q[2]_q} & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & \ddots & \sqrt{[1]_q[2j]_q} \\ 0 & 0 & \cdots & \cdots & 0 \end{pmatrix}, \tag{1.66}$$

$$\alpha(q^{\pm H/2}) = \begin{pmatrix} q^{\pm j} & 0 & \cdots & 0 \\ 0 & q^{\pm(j-1)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & q^{\mp j} \end{pmatrix}, \tag{1.67}$$

where $j \in \frac{1}{2}\mathbb{Z}_{\geq 0}$, one obtains the quantum deformation of the $(2j+1)$-dimensional irreducible representation of $\mathfrak{sl}_2(\mathbb{C})^3$. Remember that the notation $[n]_q$, already introduced in the expression of the universal $\mathfrak{R}$ matrix, denotes

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}. \tag{1.25}$$

A general important feature of the Hopf algebra representations, which makes even deeper the connection with the group case, consists in the possibility to define a tensor product between the representations. Such a product structure on ${}_H\mathfrak{M}$ can be expressed in the language of category theory as follows.

1.4.1 Hopf algebra representations as a category

A *category* $\mathfrak{C}$ is a collection of *objects* $Ob(\mathfrak{C})$ together with a set $Mor(X,Y)$ for each pair $X, Y \in Ob(\mathfrak{C})$. The elements of the latter sets are called *morphisms* and $Mor(X_1, Y_1)$, $Mor(X_1, Y_1)$ are disjoint unless $X_1 = X_2$ and $Y_1 = Y_2$. They should have properties analogous to those of maps from X to Y which respect the structure of X and Y. Thus, if $\phi_1 \in Mor(X,Z)$, $\phi_2 \in Mor(Z,Y)$ then there should be an element, denoted $\phi_2 \circ \phi_1$ in $Mor(X,Y)$ and for any three elements for which $\circ$ is defined, $\phi_1 \circ (\phi_2 \circ \phi_3 = (\phi_1 \circ \phi_2) \circ \phi_3$. Further, every set $Mor(X,X)$ should contain an identity element id_X such that $\phi_1 \circ id_X = \phi_1$, $id_X \circ \phi_2 = \phi_2$ for any morphism for which $\circ$ is defined. A morphism $\phi \in Mor(X,Y)$ is called an *isomorphism* if there exists a morphism $\phi^{-1} \in Mor(Y,X)$ such that $\phi \circ \phi^{-1} \in Mor(Y,Y)$ and $\phi^{-1} \circ \phi \in Mor(X,X)$ are identity morphisms.

[3] Hereafter the degeneracy introduced by $\omega \in \{1, -1, i, -i\}^r$ will be neglected.

A map between categories $F: \mathfrak{C}_1 \rightarrow \mathfrak{C}_2$ is called a *covariant functor* if to each object $X \in Ob(\mathfrak{C}_1)$ it assigns an object $F(X) \in Ob(\mathfrak{C}_2)$ and to each morphism $\phi \in Mor(X,Y)$ it assigns a morphism $F(\phi) \in Mor(F(X), F(Y))$ in a way compatible with $\circ$, namely such that

$$F(\phi_1 \circ \phi_2) = F(\phi_1) \circ F(\phi_2) \tag{1.68}$$

A *contravariant functor* $F: \mathfrak{C}_1 \rightarrow \mathfrak{C}_2$ also assigns an object $F(X) \in Ob(\mathfrak{C}_2)$to each object $X \in Ob(\mathfrak{C}_1)$ $X \in Ob(\mathfrak{C}_1)$, but assigns an element $F(\phi) \in Mor(F(X), F(Y))$ for each $\phi \in Mor(X,Y)$ such that $F(\phi_1 \circ \phi_2) = F(\phi_1) \circ F(\phi_2)$.

A *natural transformation* $\Phi: F_1 \rightarrow F_2$ between two covariant functor $F_1,\ F_2: \mathfrak{C}_1 \rightarrow \mathfrak{C}_2$, is a map that assigns to each object $X \in Ob(\mathfrak{C}_1)$ a morphism $\Phi_X \in Mor(F_1(X), F_2(X))$ such that for any morphism $\phi \in Mor(X,Y)$ in $\mathfrak{C}_1$

$$\Phi_Y \circ F_1(\phi) = F_2(\phi) \circ \Phi_X\,. \tag{1.69}$$

There is a similar formula if $F_1,\ F_2$ are contravariant. A natural transformation Φ is called a *natural equivalence of functors* if each map Φ_X is an isomorphism. The maps Φ_X in this case are also said to be *functorial isomorphism.*

A category $(\mathfrak{C}, \tilde{\otimes}, \underline{1})$ is called *monoidal* if it has a product functor $\tilde{\otimes}: \mathfrak{C} \times \mathfrak{C} \rightarrow \mathfrak{C}$ and a unit $\underline{1} \in Ob(\mathfrak{C})$ obeying the conditions

i) the two functors $\tilde{\otimes}(\tilde{\otimes}), (\tilde{\otimes})\tilde{\otimes}: \mathfrak{C} \times \mathfrak{C} \times \mathfrak{C} \rightarrow \mathfrak{C}$ are naturally equivalent (associativity). *I.e* there are functorial isomorphisms

$$\Phi_{X,Y,Z}: X\tilde{\otimes}(Y\tilde{\otimes}Z) \rightarrow (X\tilde{\otimes}Y)\tilde{\otimes}Z\,. \tag{1.70}$$

ii) the natural equivalence Φ obeys the *pentagon identity*

$$\begin{array}{ccccc} X\tilde{\otimes}(Y\tilde{\otimes}(Z\tilde{\otimes}W)) & \xrightarrow{\Phi} & (X\tilde{\otimes}Y)\tilde{\otimes}(Z\tilde{\otimes}W) & \xrightarrow{\Phi} & ((X\tilde{\otimes}Y)\tilde{\otimes}Z)\tilde{\otimes}W \\ \downarrow{\scriptstyle id\tilde{\otimes}\Phi} & & & & \uparrow{\scriptstyle \Phi\tilde{\otimes}id} \\ X\tilde{\otimes}((Y\tilde{\otimes}Z)\tilde{\otimes}W) & & \xrightarrow{\Phi} & & (X\tilde{\otimes}(Y\tilde{\otimes}Z))\tilde{\otimes}W \end{array} \tag{1.71}$$

iii) the functors $X \mapsto X\tilde{\otimes}\underline{1}$ and $X \mapsto \underline{1}\tilde{\otimes}X$ from $\mathfrak{C}$ to itself are naturally equivalent to the identity functor.

On these grounds, one can show that the representations of any Hopf algebra H forms a monoidal category, ${}_H\mathfrak{M}$.

An object in the category ${}_H\mathfrak{M}$ is a representation (α, V), while the morphisms in $Mor((\alpha_1, V_1), (\alpha_2, V_2))$ are the intertwiners, *i.e.* the maps $\phi \in Lin_{\mathbb{K}}(V_1, V_2)$ such that

$$\alpha_2(\phi(v)) = \phi(\alpha_1(v)) \tag{1.72}$$

or, using the module notation,

$$h.\phi(v) = \phi(h.v) \tag{1.73}$$

The product functor $\tilde{\otimes}$ defined by $V_1\tilde{\otimes}V_2 = V_1 \otimes V_2$ as vector spaces,

$$h.(v_1\tilde{\otimes}v_2) = h_{(1)}.v_1\tilde{\otimes}h_{(2)}v_2\,, \qquad (\Delta h = h_{(1)} \otimes h_{(2)})\,, \qquad \forall v_1\tilde{\otimes}v_2 \in V_1\tilde{\otimes}V_2 \tag{1.74}$$

as regards the category objects and $\phi_1\tilde{\otimes}\phi_2 = \phi_1 \otimes \phi_2$, and the unit defined by the trivial representation on the field $\mathbb{K}$, make ${}_H\mathfrak{M}$ a monoidal category. The role of the Hopf algebra structure of H, already evident in the defining eq.(1.74), becomes determinant to show the consistency of the above definitions. For example, one should prove that $\tilde{\otimes}$ is a functor. To this end, the first step is to check that $\alpha_1\tilde{\otimes}\alpha_2$ is a well defined representation, that is, it verifies the condition (1.62) or, equivalently, (1.63). But, by the fact that Δ is an algebra map,

$$\begin{aligned}
(hg).v_1\tilde{\otimes}v_2 &= (hg)_{(1)}.v_1\tilde{\otimes}(hg)_{(2)}.v_2 && \textit{product definition} \\
&= (h_{(1)}g_{(1)}).v_1\tilde{\otimes}(h_{(2)}g_{(2)}).v_2 && \Delta \textit{ is an algebra map} \\
&= h_{(1)}.(g_{(1)}.v_1)\tilde{\otimes}h_{(2)}.(g_{(2)}.v_2) && V_1, V_2 \textit{ are } H\textit{-module} \\
&= h.(g.(v_1\tilde{\otimes}v_2))
\end{aligned} \tag{1.75}$$

Moreover, it is easy to see that $\phi_1 \otimes \phi_2$ is an intertwiner and that this assignment is compatible with the composition $\circ$. Then the product functor is well defined. In a similar way, due to the Hopf algebra properties, one can show that $\tilde{\otimes}$ functor obeys the associativity condition and the pentagon identity and that also the requirements for the unit object $\underline{1}$ are fulfilled. In particular, due to the coassociativity of Δ, one can identify the functorial isomorphisms of eq.(1.70) with the standard vector space isomorphisms $V_1\tilde{\otimes}(V_2\tilde{\otimes}V_3) \cong (V_1\tilde{\otimes}V_2)\tilde{\otimes}V_3$.

1.4.2 Tensor and quasitensor categories

If $(H, \mathfrak{R})$ is a quasitriangular or triangular Hopf algebra, then the "algebra-like" structure in the category ${}_H\mathfrak{M}$ acquires many further analogies with the case of group representations. The first one concerns the order in the product of representations, irrelevant for group representations. The situation is exactly the same when the Hopf algebra is triangular and one speaks of *tensor category* $({}_H\mathfrak{M}, \tilde{\otimes}, \Phi, \Psi, \underline{1})$. This is a monoidal category such that

i) the two functors $(X,Y) \mapsto X\tilde{\otimes}Y$ and $(X,Y) \mapsto Y\tilde{\otimes}X$ from $\mathfrak{C} \times \mathfrak{C}$ to $\mathfrak{C}$ are naturally equivalent, *i.e.* there exists functorial isomorphisms $\Psi_{X,Y} : X\tilde{\otimes}Y \to Y\tilde{\otimes}X$.

ii) it is $\Psi_{Y,X} \circ \Psi_{X,Y} = id$.

iii) the natural equivalences Φ and Ψ obey the *hexagon identity.*

$$\begin{array}{ccccc} X\tilde{\otimes}(Y\tilde{\otimes}Z) & \xrightarrow{\Phi} & (X\tilde{\otimes}Y)\tilde{\otimes}Z & \xrightarrow{\Psi} & Z\tilde{\otimes}(X\tilde{\otimes}Y) \\ \Big\downarrow {\scriptstyle id\tilde{\otimes}\Psi} & & & & \Big\downarrow {\scriptstyle \Phi} \\ X\tilde{\otimes}(Z\tilde{\otimes}Y) & \xrightarrow{\Phi} & (X\tilde{\otimes}Z)\tilde{\otimes}Y) & \xrightarrow{id\tilde{\otimes}\Psi} & (Z\tilde{\otimes}X)\tilde{\otimes}Y \end{array} \tag{1.76}$$

One can show that once the pentagon and the hexagon axioms have been satisfied, then all the other obvious compatibility checks between Φ and Ψ corresponding to different bracketting and ordering also hold (MacLane's theorem). To be more precise, this means that, constructing from Φ and Ψ any map of the form $X_1\tilde{\otimes}X_2\cdots\tilde{\otimes}X_{\mathrm{N}}\xrightarrow{\beta} X_{\beta(1)}\tilde{\otimes}X_{\beta(2)}\cdots\tilde{\otimes}X_{\beta(N)}$, one finds a representation of the permutation group S_{N}.

For the triangular Hopf algebra representation category ${}_H\mathfrak{M}$ the natural equivalence Ψ is generated by the functorial isomorphisms

$$\Psi_{V_1,V_2} : V_1\tilde{\otimes}V_2 \to V_2\tilde{\otimes}V_1 \tag{1.77}$$

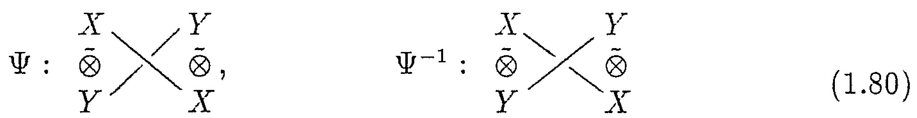
$$\Psi_{V_1,V_2}(v_1\tilde{\otimes}v_2) = \mathfrak{R}^{(2)}.v_2\tilde{\otimes}\mathfrak{R}^{(1)}.v_2 \tag{1.78}$$

($\mathfrak{R} = \mathfrak{R}^{(1)}\otimes\mathfrak{R}^{(2)}$, understanding the sum indicated explicitly in eq.(1.22)) makes ${}_H\mathfrak{M}$ into a tensor category.

Notice that the twist map $\tau : V_1\otimes V_2 \to V_2\otimes V_1$ is not an intertwiner unless H is cocommutative. In particular, the property $\Psi\circ\Psi = id$ is related to the requirement that R is unitary ($\tau(\mathfrak{R}^{-1}) = \mathfrak{R}$). Hence, in the quasitriangular case this requirement cannot be fulfilled, but the natural equivalence Ψ obeys a *second hexagon identity*

$$\begin{array}{ccccc} (X\tilde{\otimes}Y)\tilde{\otimes}Z & \xrightarrow{\Phi^{-1}} & X\tilde{\otimes}(Y\tilde{\otimes}Z) & \xrightarrow{\Psi} & (Y\tilde{\otimes}Z)\tilde{\otimes}X \\ \Big\downarrow {\scriptstyle \Psi\tilde{\otimes}id} & & & & \Big\downarrow {\scriptstyle \Phi^{-1}} \\ (Y\tilde{\otimes}X)\tilde{\otimes}Z & \xrightarrow{\Phi^{-1}} & Y\tilde{\otimes}(X\tilde{\otimes}Z) & \xrightarrow{id\tilde{\otimes}\Psi} & Y\tilde{\otimes}(Z\tilde{\otimes}X) \end{array} \tag{1.79}$$

and $({}_H\mathfrak{M},\tilde{\otimes},\Phi,\Psi,\underline{1})$ becomes a so-called *quasitensor category.* As in the tensor category case, due to the pentagon and the two hexagon identities, there are no further natural constrains between Φ and Ψ arising from brackets and ordering, but now the maps of the form $X_1\tilde{\otimes}X_2\cdots\tilde{\otimes}X_{\mathrm{N}} \xrightarrow{\beta} X_{\beta(1)}\tilde{\otimes}X_{\beta(2)}\cdots\tilde{\otimes}X_{\beta(\mathrm{N})}$, constructed from Φ, Φ^{-1} and Ψ, Ψ^{-1}, constitute a representation of the braid group B_{N}. The coherence for quasitensor categories is then that any diagram corresponding to a closed path in any B_{N} commutes. The functorial isomorphisms



are, hence, the braiding operator and its inverse, respectively.

1.4.3 Rigid monoidal categories

Another property that one would also like to generalize is duality. This is expressed in the axioms of rigidity.

A *rigid monoidal category* is essentially a monoidal category for which there exists an "internal hom" object which fulfills particular finiteness or reflexivity conditions. Before describing them, note that, if $\mathfrak{C}$ is any category, the sets of morphisms $Mor(X,Y)$, $X,Y \in \mathfrak{C}$ can be regarded as the objects of another category (the category Set of sets, for which morphisms are maps and $\circ$ is the composition of maps). Thus one can construct the respectively covariant and contravariant functors $Y \mapsto Mor(X,Y)$ and $X \mapsto Mor(X,Y)$, from $\mathfrak{C}$ to Set. Then one says that a functor to Set is representable if it is naturally equivalent to a functor of this form, and X is the *representing object*. Hence, a monoidal category has an *internal hom* if the contravariant functors $F_{X,Y} = Mor(\tilde{\otimes}X, Y)$ are each representable. In this case the representing object, the *internal hom*, is denoted $\underline{Hom}(X,Y)$.

Thus, for each $X, Y \in Ob(\mathfrak{C})$ one needs a third object $\underline{Hom}(X,Y) \in \mathfrak{C}$ such that

$$Mor(Z\tilde{\otimes}X, Y) \cong Mor(Z, \underline{Hom}(X,Y)) \tag{1.81}$$

by functorial isomorphisms. Setting $Z = \underline{Hom}(X,Y)$, the identity morphism on the right correspond to a morphism $ev_{X,Y} : \underline{Hom}(X,Y)\tilde{\otimes}X \to Y$, called the *evaluation map*. One can image that $\underline{Hom}(X,Y)$ is like "linear maps from X to Y" and $ev_{X,Y}$ "applies" this to an element of X to obtain an element of Y. Moreover, given an internal hom one also define a *duality functor* $* : \mathfrak{C} \to \mathfrak{C}$ by $X^* = \underline{Hom}(X,\underline{1})$. One can check that the properties of such functor justify this definition.

For a monoidal category with internal hom to be *rigid* the functorial morphisms

$$\underline{Hom}(X_1,Y_1)\tilde{\otimes}\underline{Hom}(X_2,Y_2) \to \underline{Hom}(X_1\tilde{\otimes}X_2, Y_1\tilde{\otimes}Y_2) \tag{1.82}$$

$$X \to X^{**}, \tag{1.83}$$

induced by the properties of the internal hom, must be isomorphisms.

If $\mathfrak{C}$ is the category of the finite dimensional algebra representations of a Hopf algebra H, denoted ${}_H\mathfrak{M}^{f.d.}$, the internal hom is defined by

$$\underline{Hom}(V_1,V_2) = Lin_{\mathbb{K}}(V_1,V_2) \tag{1.84}$$

and the functorial isomorphism (1.81) relates a morphism $\psi \in Mor(V_3\tilde{\otimes}V_1, V_2)$ to $\hat{\psi} \in Mor(V_3, \underline{Hom}(V_1,V_2))$ given by

$$(\hat{\psi}(v_3))(v_1) = \psi(v_3\tilde{\otimes}v_1) \tag{1.85}$$

Finally, the module structure in $\underline{Hom}(V_1,V_2)$ (the internal hom is an object of the category) is

$$(h.f)(v_1) = h_{(1)}.(f(Sh_{(2)}.v_1)), \tag{1.86}$$

$\forall h \in H$, $f \in \underline{Hom}(V_1, V_2)$, $v_1 \in V_1$. This is just the action on $Lin_{\mathbb{K}}(V_1, V_2) = V_2 \tilde{\otimes} V_1^*$, that is the tensor product of the action on V_2 with the adjoint of the action on V_1.

The ${}_H\mathfrak{M}^{f.d.}$ internal hom, for H a (quasi)triangular Hopf algebra with bijective antipode, satisfies the axioms of rigidity. In particular, the functorial isomorphism $V \cong V^{**}$ is obtained constructing the morphism corresponding to $\psi = ev_{V,\underline{1}} \circ \Psi_{V,V^*}$: $V \tilde{\otimes} V^* \to \underline{1}$ through the functorial isomorphism (1.81). In fact, this is exactly the desired map $\hat{\psi}: \ V \to \underline{Hom}(V^*, \underline{1}) = V^{**}$.

1.4.4 Tensor product "algebra" in ${}_H\mathfrak{M}$

The rich structure found, as shown in the previous sections, in the category ${}_H\mathfrak{M}$, particularly if H is a triangular or quasitriangular Hopf algebra, can be made more interesting if one introduces the possibility to decompose the algebra representations of H into the "direct sum" of any basis of irreducible representations. To translate this in the category language, one needs to define in this context an addition structure $\tilde{\oplus}: \ \mathfrak{C} \times \mathfrak{C} \to \mathfrak{C}$, with distributive properties with respect to $\tilde{\otimes}$ and which admits a zero object $\underline{0} \in Ob(\mathfrak{C})$ (some further technical assumptions are required). In ${}_H\mathfrak{M}$ this structure is realized by the usual direct sum of modules and by the zero dimensional representation $\{0\}$. If now one supposes that every object V in ${}_H\mathfrak{M}$ can be decomposed into a direct sum of irreducible objects V_j, for example restricting oneself to consider only the finite dimensional representations, then the tensor algebra of this category is characterized by the relations

$$V_{j_1} \tilde{\otimes} V_{j_2} \cong \tilde{\bigoplus_j} N_{j_1 j_2}{}^j V_j \, . \tag{1.87}$$

The factors $N_{j_1 j_2}{}^j$ are the multiplicities of V_j in the decomposition of the tensor product of V_{j_1} and V_{j_2}. In an analogous way one can decompose the functorial isomorphisms corresponding to Φ and Ψ. To make this evident, it is convenient to rewrite eq.(1.87) abstractly as

$$V_{j_1} \tilde{\otimes} V_{j_2} \cong \tilde{\bigoplus_j} W_{j_1 j_2}{}^j \tilde{\otimes} V_j \, , \tag{1.88}$$

where $W_{j_1 j_2}{}^j$ is a module constitutes by $N_{j_1 j_2}{}^j$ copies of the trivial representation $\underline{1}$. Moreover, thus functorial isomorphisms are intertwiners, *i.e.* commute with the action of H, they are block diagonal in the direct sum decomposition. This means that, referring to the eq.(1.88), $\Phi_{j_1 j_2 j_3}: \ V_{j_1} \tilde{\otimes} (V_{j_2} \tilde{\otimes} V_{j_3}) \to (V_{j_1} \tilde{\otimes} V_{j_2}) \tilde{\otimes} V_{j_3}$ and $\Psi_{j_1 j_2}: \ V_{j_1} V_{j_2} \to V_{j_2} \tilde{\otimes} V_{j_1}$ can be written as

$$\begin{aligned} \Phi_{j_1 j_2 j_3} &= \tilde{\bigoplus_j} \Phi_{j_1 j_2 j_3}{}^j \tilde{\otimes} id \\ \Psi_{j_1 j_2} &= \tilde{\bigoplus_j} \Psi_{j_1 j_2}{}^j \tilde{\otimes} id \, . \end{aligned} \tag{1.89}$$

This is called the *spectral decomposition.* Using eq.(1.88) for the tensor producted modules on which these functorial isomorphism are applied, one finds

$$\begin{aligned} \Phi_{j_1 j_2 j_3} &: \bigoplus_{j_{23}}^{\sim} W_{j_1 j_{23}}{}^{j} \tilde{\otimes} W_{j_2 j_3}{}^{j_{23}} \longrightarrow \bigoplus_{j_{12}}^{\sim} W_{j_1 j_2}{}^{j_{12}} \tilde{\otimes} W_{j_{12} j_3}{}^{j} \\ \Psi_{j_1 j_2} &: W_{j_1 j_2}{}^{j} \longrightarrow W_{j_2 j_1}{}^{j} . \end{aligned} \tag{1.90}$$

To establish the first of these last equations one has used the fact that, due to unit axioms, $W\tilde{\otimes}V \cong V\tilde{\otimes}W$ by functorial isomorphisms whenever H acts trivially on W.

Obviously, the pentagon and the two hexagon identities impose some conditions on the spectral decomposition components of the functorial isomorphisms corresponding to Φ and Ψ. These appear more explicitly if one refers to bases of the irreducible modules V_j. Thus, in terms of these bases $\{e^j_m\}$, the general decomposition (1.88) takes the form

$$e_{m_1} \tilde{\otimes} e_{m_2} = \sum_{j,m} \begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix} \tilde{\otimes} e_m^{j(j_1 j_2)} \tag{1.91}$$

where $\begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix} \in W_{j_1 j_2}{}^{j}$ are called the *generalized Clebsh Gordan Coefficients* (CGC's) and the superfix $^{(j_1 j_2)}$ on the vectors e^j_m is to remind that they are being viewed in $V_{j_1}\tilde{\otimes}V_{j_2}$ according to the isomorphism implicit in the decomposition (1.88). Thus, the generalized CGC's transform the basis $\{e_m^{j(j_1 j_2)}\}$ into the standard one $\{e^{j_1}_{m_1}\tilde{\otimes}e^{j_2}_{m_2}\}$ of $V_{j_1}\tilde{\otimes}V_{j_2}$.

The two hexagon identities are reflected, in this context, by equations involving the generalized CGC's, which characterize the representations of a given quasitriangular Hopf algebra. To show how these relations arise, considered, first, the action of $\Psi_{j_1 j_2}$ on an element of the standard basis of $V_{j_1}\tilde{\otimes}V_{j_2}$

$$\Psi_{j_1 j_2}(e^{j_1}_{m_1} \tilde{\otimes} e^{j_2}_{m_2}) = \sum_{m_1' m_2'} (B^{j_1 j_2})^{m_1' m_2'}_{m_1 m_2} (e^{j_2}_{m_2'} \tilde{\otimes} e^{j_1}_{m_1'}) \tag{1.92}$$

where $(B^{j_1 j_2})^{m_1' m_2'}_{m_1 m_2}$ are the matrix elements of $\Psi_{j_1 j_2}$. From the definition of the natural equivalence Ψ in the Hopf algebra case (§1.2.2), $B^{j_1 j_2}$ has the form

$$B^{j_1 j_2} = \tau \circ (\alpha_1 \tilde{\otimes} \alpha_2)(\mathfrak{R}) \equiv \tau R^{j_1 j_2} , \tag{1.93}$$

where α_{j_1}, α_{j_2} are the actions of H on V_{j_1} and V_{j_2}, respectively. Decomposing both sides according to the eqs.(1.88) and (1.89), one obtains

$$\Psi_{j_1 j_2}{}^{j}\left(\begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix}\right) = \sum_{m_1' m_2'} (B^{j_1 j_2})^{m_1' m_2'}_{m_1 m_2} \begin{bmatrix} j_2 & j_1 & j \\ m_2' & m_1' & m \end{bmatrix} . \tag{1.94}$$

On the other hand, as the maps $\Phi_{j_1 j_2 j_3}$ are just the usual vector space isomorphisms $V_{j_1}\tilde{\otimes}(V_{j_2}\tilde{\otimes}V_{j_3}) \cong (V_{j_1}\tilde{\otimes}V_{j_2})\tilde{\otimes}V_{j_3}$, one can identify the bases $\{e^{j_1}_{m_1}\tilde{\otimes}(e^{j_2}_{m_2}\tilde{\otimes}e^{j_3}_{m_3})\}$,

$\{(e^{j_1}_{m_1}\tilde{\otimes}e^{j_2}_{m_2})\tilde{\otimes}e^{j_3}_{m_3}\}$ so that $\Phi_{j_1j_2j_3}$ become the identity matrices, and hence the first hexagon identity takes the form

$$\Psi_{j_{12}j_3}((e^{j_1}_{m_1}\tilde{\otimes}e^{j_2}_{m_2})\tilde{\otimes}e^{j_3}_{m_3}) = (\Psi_{j_1j_{32}}\tilde{\otimes}id)\circ(id\tilde{\otimes}\Psi_{j_2j_3})(e^{j_1}_{m_1}\tilde{\otimes}(e^{j_2}_{m_2}\tilde{\otimes}e^{j_3}_{m_3})) \quad (1.95)$$

Thus, applying eq.(1.94), this relation becomes

$$\sum_{m}\begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m\end{bmatrix}(R^{jj_3})^{m'm_3'}_{mm_3} = \sum_{m_1',m_2'}\begin{bmatrix} j_1 & j_2 & j \\ m_1' & m_2' & m'\end{bmatrix}(R^{j_1j_3}R^{j_2j_3})^{m_1'm_2'm_3'}_{m_1m_2m_3} \quad (1.96)$$

which is exactly the matrix version of the equation $(\Delta\otimes id)\mathfrak{R} = \mathfrak{R}_{13}\mathfrak{R}_{23}$. In an analogous way, the second hexagon identity, needed when H is only quasitriangular, is the matrix corresponding of the equation $(id\otimes\Delta)\mathfrak{R} = \mathfrak{R}_{13}\mathfrak{R}_{12}$, and therefore is given by a similar formula,

$$\sum_{m}\begin{bmatrix} j_2 & j_3 & j \\ m_2 & m_3 & m\end{bmatrix}(R^{j_1j})^{m_1'm'}_{m_1m} = \sum_{m_2',m_3'}\begin{bmatrix} j_2 & j_3 & j \\ m_2' & m_3' & m'\end{bmatrix}(R^{j_1j_3}R^{j_1j_2})^{m_1'm_2'm_3'}_{m_1m_2m_3} \quad (1.97)$$

As the axioms (1.20) these last two equations imply that R obeys the QYBE's on a basis $e^{j_1}_{m_1}\tilde{\otimes}e^{j_2}_{m_2}\tilde{\otimes}e^{j_3}_{m_3}$,

$$R^{j_1j_2}_{12}R^{j_1j_3}_{13}R^{j_2j_3}_{23} = R^{j_2j_3}_{23}R^{j_1j_3}_{13}R^{j_1j_2}_{12}\,. \quad (1.98)$$

Analogously to the hexagon identities, also the pentagon can be cast in a matricial form and in such a form it becomes a condition on the 6-j symbols. To make this result explicit, consider, firstly, the decompositions for $V_{j_1|j_2j_3}\equiv V_{j_1}\tilde{\otimes}(V_{j_2}\tilde{\otimes}V_{j_3})$ and $V_{j_1j_2|j_3}\equiv(V_{j_1}\tilde{\otimes}V_{j_2})\tilde{\otimes}V_{j_3}$, which respectively are

$$e^{j_1}_{m_1}\tilde{\otimes}(e^{j_2}_{m_2}\tilde{\otimes}e^{j_3}_{m_3}) = \sum_{\substack{j,j_{23},\\ m,m_{23}}}\begin{bmatrix} j_1 & j_{23} & j \\ m_1 & m_{23} & m\end{bmatrix}\begin{bmatrix} j_2 & j_3 & j_{23} \\ m_2 & m_3 & m_{23}\end{bmatrix}e^{j_{23}j(j_1|j_2j_3)}_{m}$$

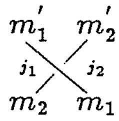
$$(e^{j_1}_{m_1}\tilde{\otimes}e^{j_2}_{m_2})\tilde{\otimes}e^{j_3}_{m_3} = \sum_{\substack{j,j_{12},\\ m,m_{12}}}\begin{bmatrix} j_{12} & j_3 & j \\ m_{12} & m_3 & m\end{bmatrix}\begin{bmatrix} j_1 & j_2 & j_{12} \\ m_1 & m_2 & m_{12}\end{bmatrix}e^{j_{12}j(j_1j_2|j_3)}_{m}, \quad (1.99)$$

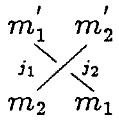
where the superfixes $^{(j_1|j_2j_3)}$ and $^{(j_1j_2|j_3)}$ indicate in which of the two spaces $V_{j_1|j_2j_3}$ or $V_{j_1j_2|j_3}$ the vectors $e^{j_{23}j}_{m}$ and $e^{j_{12}j}_{m}$ are being viewed via the isomorphisms in the decomposition (1.88). This to emphasize that these vectors must be regarded as new bases of their respective spaces. Therefore, due to the isomorphism between the two modules $V_{j_1|j_2j_3}$ and $V_{j_1j_2|j_3}$ induced by Φ, $\{e^{j_{12}j(j_1j_2|j_3)}_{m}\}$ and $\Phi\{e^{j_{23}j(j_1|j_2j_3)}_{m}\}$ must be related. The pentagon identity will be a constraint on the matrix of such a transformation. In the case of $\mathfrak{sl}_2(\mathbb{C})$ representations the elements of the matrix connecting these bases are called the *Racah-Wigner 6-j symbols*. Referring to the canonical basis in the tensor products $V_{j_1|j_2j_3}$ and $V_{j_1j_2|j_3}$,

$$e^{j_{12}j(j_1j_2|j_3)}_{m} = \sum_{j_{23}}\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23}\end{Bmatrix}e^{j_{23}j(j_1|j_2j_3)}_{m}\,. \quad (1.100)$$

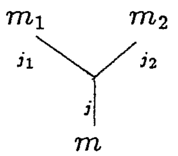
Hence, the pentagon identity for Φ will coincides with the Biedenharn- Elliot identity for the 6-j symbols.

In the study of these identities, diagrammatic methods can be applied, disclosing the topological context of such relations. This graphical representation is constructed on the fundamental diagrams associated with the R-matrix elements and the generalized CGC's,



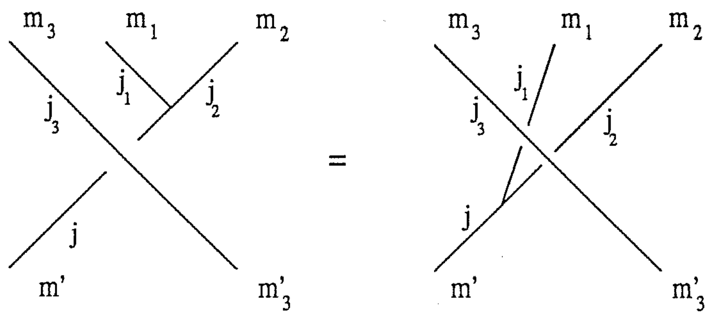
$= (R^{j_1 j_2})^{m_1' m_2'}_{m_1 m_2}$,


$= (R^{-1\ j_1 j_2})^{m_1' m_2'}_{m_1 m_2}$, (1.101)



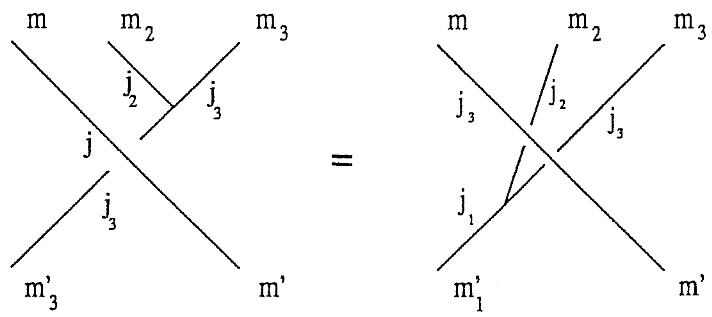
$$= \begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix} \tag{1.102}$$

so that the hexagon identities can be rewritten in the form



1st *hexagon identity*

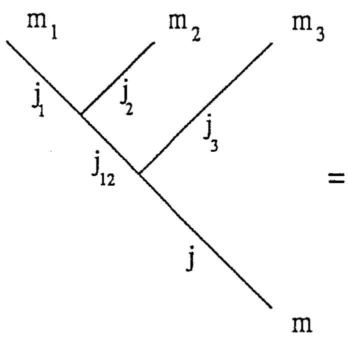
(1.103)



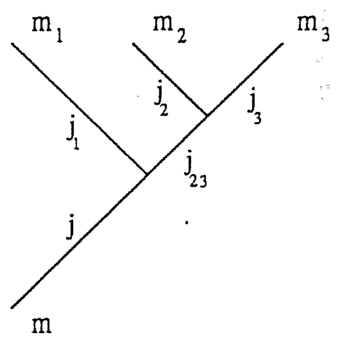
2nd *hexagon identity*

(1.104)

and the 6-j symbols can be defined by the diagrammatic equation



$$= \sum_{j_{23}} \left\{ \begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{array} \right\}$$



(1.105)

1.4.5 $U_q(\mathfrak{sl}_2)$-modules

Here the results obtained in the previous sections will be illustrated with the help of an example, the quantum enveloping algebra $U_q(\mathfrak{sl}_2)$. In addition to the opportunity of an exemplification of the general theory, another motivation for this choice lies in the applications of the theory of quantum group which will be studied in the sequel. The main features of the representation theory of $U_q(\mathfrak{sl}_2)$ have been already discussed, but let us remember again some essential points. It has been already stressed that the $U_q(\mathfrak{sl}_2)$-modules can be viewed as a deformation of the classical counterpart. It follows that the finite dimensional irreducible representations are labeled by the integer or half integer number j and have dimension $2j+1$. Choosing in the module V_j the basis $\{e^j_m\}_{\substack{m\in\mathbb{Z} \\ |m|\leq j}}$ defined by the relations

$$\alpha_{j_1}(E^{\pm}).e^j_m = ([j \mp m]_q [j \pm m + 1]_q)^{1/2} e^j_{m\pm 1}\,, \qquad \alpha_{j_1}(H).e^j_m = 2m\, e^j_m\,, \tag{1.106}$$

the non zero entries of the matrix $\alpha_{j_1} \otimes \alpha_{j_2}(\mathfrak{R})$ are

$$\begin{aligned} (R^{j_1 j_2})^{m_1' m_2'}_{m_1 m_2} &= \delta^{m_1'+m_2'}_{m_1+m_2} \frac{(1-q^2)^{m_1'-m_1}}{[m_1'-m_1]_q!} \times \\ &\times \left(\frac{[j_1+m_1']_q! [j_1-m_1]_q! [j_2-m_2']_q! [j_2+m_2]_q!}{[j_1-m_1']_q! [j_1+m_1]_q! [j_2+m_2']_q! [j_2-m_2]_q!} \right)^{1/2} q^{m_1 m_2' + m_2 m_1'} \end{aligned} \tag{1.107}$$

for $m_1' \geq m_1$. As regards the q-analog of the CGC's, they can be cast in the form

$$\left[\begin{array}{ccc} j_1 & j_2 & j \\ m_1 & m_2 & m \end{array} \right]_q = \left[\begin{array}{ccc} j_1 & j_2 & j_{12} \\ m_1 & m_2 & m \end{array} \right]^{\sim}_q \delta(j_1 j_2 j)\, \delta_{m_1+m_2,m} \tag{1.108}$$

where

$$\delta(j_1 j_2 j) = \begin{cases} 1 & |j_1 - j_2| \leq j \leq j_1 + j_2 \\ 0 & otherwise \end{cases} . \tag{1.109}$$

Eq.(1.108) shows that the tensor product "algebra" exhibits $\mathfrak{sl}_2$-like selection rules. The analogy with the "classical" case is, so, extended to the properties of the CGC's. Indeed, one can also generalize to $U_q(\mathfrak{sl}_2)$ the orthogonality conditions

$$\sum_{\substack{m_1 m_2 \\ |m_i| \leq j_i}} \begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix}_q^{\sim} \begin{bmatrix} j_1 & j_2 & j' \\ m_1 & m_2 & m' \end{bmatrix}_q^{\sim} = \delta_{jj'}\,\delta_{mm'} , \tag{1.110}$$

$$\sum_{\substack{jm \\ |m| \leq j}} \begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix}_q^{\sim} \begin{bmatrix} j_1 & j_2 & j \\ m_1' & m_2' & m \end{bmatrix}_q^{\sim} = \delta_{m_1 m_1'}\,\delta_{m_2 m_2'} . \tag{1.111}$$

Using these relations,one obtains an expression of the q-analog of the $6-j$ symbols in terms of the CGC's,

$$\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix}_q = \begin{bmatrix} j_1 & j_{23} & j \\ m_1 & m_{23} & m \end{bmatrix}_q^{-1} \sum_{\substack{m_1 m_2 \\ m_2+m_3=m-m_1}} \begin{bmatrix} j_1 & j_2 & j_{23} \\ m_1 & m_2 & m_{23} \end{bmatrix}_q \times$$
$$\times \begin{bmatrix} j_{12} & j_3 & j \\ m_{12} & m_3 & m \end{bmatrix}_q \begin{bmatrix} j_1 & j_2 & j_{12} \\ m_1 & m_2 & m_{12} \end{bmatrix}_q \tag{1.112}$$

from which follows their explicit dependence on their 6 arguments

$$\begin{aligned} \begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix}_q^{(RW)} &= \Delta(abe)\,\Delta(acf)\,\Delta(cef)\,\Delta(dbf) \times \\ &\times \sum_{\substack{i \\ i\in I}} (-1)^i [i+1]_q! \,\{[i-a-b-e]_q!\,[i-a-c-f]_q! \times \\ &\times [i-b-d-f]_q![i-d-c-e]_q![a+b+c+d-i]_q! \times \\ &\times [a+d+e+f-i]_q![b+c+e+f+1-i]_q!\}^{-1} , \end{aligned} \tag{1.113}$$

where I is the set of those integer values of i which give rise to non negative arguments in the q-deformed factorial $[\cdot]$. The superfix $^{(RW)}$ denotes the Racah-Wigner normalization

$$\begin{aligned} \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix}_q^{(RW)} &= (-1)^{j_1+j_2-j-j_3-2j_{12}} \left([2j_{12}+1]\,[2j_{23}+1]\right)^{1/2} \times \\ &\times \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix}_q , \end{aligned} \tag{1.114}$$

while Δ is the function

$$\Delta(abc) = \left(\frac{[-a+b+c]_q![a-b+c]_q![a+b-c]_q!}{[a+b+c+1]_q!}\right)^{1/2}. \qquad (1.115)$$

Chapter 2

Conformal field theory

Already in the first years of the special theory of relativity physicist knew the possibility to extend to the conformal one the group of space-time transformations which leave the Maxwell equations (in vacuum) invariant. But a revival of interest in the subject arose with the work of Wilson [36] on renormalization group approach to critical systems, which are, as well-known, conformally invariant.

Even if the peculiarity of the two-dimensional case was known (in particular, the intuition that the stress-energy tensor of conformal invariant two-dimensional quantum field theory (QFT) models generates the Virasoro algebra came about in the early 1970's), a deeper understanding of the representation theory of Virasoro algebra was needed before further developments. In fact, these mathematical progresses constituted the essential basis for the paper by Belavin, Polyakov and Zamolodchikov (BPZ) [5] in which the authors, working out the bootstrap program proposed in the previous work of Polyakov [29], use the Kac determinant for the highest weight representations of the Virasoro algebra in their studies of massless, two-dimensional, interacting field theories. In the sequel, many attempts have been devoted to bring to an end Polyakov's idea and to formulate the theory in some more general frame, which might have suggest a criterion for the classification of two-dimension conformal field theory (CFT), at least in the rational case. The purpose of this chapter is, more simply, to summarize the main features of the BPZ's approach and remember briefly the properties of the Dotzenko-Fateev (DF) models, which, in the physical limit, reduces to the minimal ones.

2.1 Conformal invariance

The conformal (*i.e.* angle-preserving) group C on a manifold M of dimension d is the group of the coordinate transformations

$$\xi^a \to \eta^a(\xi), \qquad a = 0, \ldots, d-1 \tag{2.1}$$

which leave invariant the ratio

$$\frac{d\xi^a(P)d\xi_a(Q)}{|d\xi(P)||d\xi(Q)|}\,. \tag{2.2}$$

This corresponds to the metric tensor transformation law

$$g_{ab} \;\rightarrow\; \nabla_a\eta^{a'}\nabla_b\eta^{b'}g_{a'b'} \;=\; f(\xi)g_{ab}\,, \tag{2.3}$$

where f is a function of coordinates. For an infinitesimal transformation

$$\eta^a(\xi) \;=\; \xi^a + \epsilon^a(\xi)\,, \tag{2.4}$$

one therefore obtains the condition

$$\nabla_a\epsilon_b + \nabla_b\epsilon_a \;=\; h(\xi)g_{ab}\,, \tag{2.5}$$

where $h(\xi) = f(\xi) - 1$. Considering the case of a flat metric and taking the derivative of both sides of eqs.(2.5), one finds

$$\begin{aligned}(d-2)\,\partial_a\partial_b h(\xi) \;&=\; 0\\ \partial^c\partial_c\epsilon_a \;&=\; \tfrac{1}{2}(d-2)\,\partial_a h(\xi)\,.\end{aligned} \tag{2.6}$$

Thus, the conclusion is that, if $d > 2$, then $h(\xi)$ must be at most linear in ξ and, consequently, $\epsilon^a(\xi)$ must be at most quadratic in ξ. The general solution that satisfies these constraints is

$$\epsilon^a \;=\; \alpha^a + \lambda\xi^a + \omega^{ab}\xi_b + (2\beta\cdot\xi\,\xi^a - \beta^a\xi^2)\,, \qquad \omega^{ab} \;=\; -\omega^{ba}\,. \tag{2.7}$$

Consequently, for $d > 2$, the conformal group has finite dimension (dimension 15 for $d = 4$) and includes the following finite transformations:

$$\begin{array}{llll} \textit{translations} & \eta^a \;=\; \xi + \alpha^a\,, & h(\xi) \;=\; 0 \\ \textit{Lorentz transformations} & \eta^a \;=\; \Lambda^{ab}(\omega)\xi_b\,, & h(\xi) \;=\; 0 \\ \textit{dilatations} & \eta^a \;=\; e^\lambda\xi^a\,, & h(\xi) \;=\; 2\lambda \\ \textit{special conformal transformations} & \eta^a \;=\; \frac{\xi^a - \beta^a\xi^2}{1-2\beta\cdot\xi+\beta^2\xi^2}\,, & h(\xi) \;=\; 4\beta\cdot\xi\,. \end{array} \tag{2.8}$$

The situation is completely different for $d = 2$. In this case the condition (2.5) becomes simply an analyticity requirement on the infinitesimal variations ϵ^a: deriving the terms of these equations no more constraints are obtained. This means that the conformal group C in a two-dimensional flat space is the infinite-dimensional group of the analytic reparametrization $\xi^a \rightarrow \eta^a = \eta^a(\xi)$, $a = 0, 1$. Moreover, if one consider the space-time (ξ_0, ξ_1) as a proper section of the complex space $\mathbb{C}^2$ with coordinates

$$z \;=\; \xi_0 + \xi_1 \qquad \bar{z} \;=\; \xi_0 - \xi_1 \tag{2.9}$$

and having the metric

$$ds^2 = dz d\bar{z}\,, \tag{2.10}$$

then, due to the condition (2.5), C factorizes into the product

$$\mathbf{C} = \Gamma \otimes \bar{\Gamma}\,, \tag{2.11}$$

where the groups $\Gamma = \{z \to w = w(z)\}$ and $\bar{\Gamma} = \{\bar{z} \to \bar{w} = \bar{w}(\bar{z})\}$ are the independent, respectively, holomorphic and antiholomorphic component. Consequently, an identical factorization holds for the Lie algebra of C, that split in the direct sum of a holomorphic and an antiholomorphic component, and for its representations. Being the space $\mathfrak{F}$ of state of a conformal invariant two-dimensional QFT model a suitable representation space of this algebra, it will have the form $\mathfrak{F} = \mathfrak{F} \otimes \bar{\mathfrak{F}}$, and an analogous decomposition holds (up to completion) also for the operators acting on $\mathfrak{F}$. This property permits, in the sequel, to forget about the antiholomorphic part $\bar{\Gamma}$.

The generators of the group Γ,

$$l_n = z^{n+1}\frac{d}{dz}\,, \qquad n \in \mathbb{Z}\,, \tag{2.12}$$

satisfy the Witt or classical Virasoro algebra V_0

$$[l_n, l_m] = (n-m)\, l_{n+m}\,. \tag{2.13}$$

It is important now to emphasize that only a particular class of representations of this algebra will be taken into account. In order to characterize such a class, notice firstly, that, generally, they shall be projective representations or, that is the same, representations of the central extension of the Witt algebra, the Virasoro algebra $Vir = V_0 \oplus \mathbb{C}\hat{c}$,

$$\begin{aligned} [\hat{l}_n, \hat{l}_m] &= (n-m)\, \hat{l}_{n+m} + \tfrac{1}{12}\hat{c}\,(n^3 - n)\, \delta_{n+m} \\ [\hat{l}_n, \hat{c}] &= 0\,. \end{aligned} \tag{2.14}$$

Indeed, this is a typical physical requirement: properly, the states of a QFT model are not vectors in a certain space, but rays, *i.e.* vectors defined up to a phases. To support this statement, a more cogent argument comes from the study of the commutation rules fulfilled by the stress-energy tensor T_{ab} of a quantum conformal field theories (QCFT)[1]. At the same time, this will clarify the physical meaning of the central element $\hat{c}$. The crucial point is that, as in the case of the Poincaré group, the generators of the extended algebra are represented by the components of the stress-energy tensor. A model independent proof of this statement, based on Wightman-type general axioms, was originally given by Lüscher and Mack [22]. They showed that, in a dilatation invariant local relativistic two-dimensional QFT, the holomorphic component of the stress-energy tensor $T(z)$ obeys the commutation relation

$$[T(z), T(w)] = \delta(z,w)\, T'(w) - 2\delta'(z,w)\, T(w) - \tfrac{1}{12} c\, \delta'''(z,w)\,, \tag{2.15}$$

[1]The theory is assumed to be free of Weyl anomalies, *i.e.* $T^a_{\ a} = 0$.

being c is a positive real constant. Here the function δ is defined by the relation

$$\frac{1}{2\pi i}\oint_{C_z} dw\, f(w)\,\delta(z,w) \;=\; f(z) \tag{2.16}$$

for any analytic function f. Here, the integration contour C_z surround the point z. It is straightforward to verify that a possible representation of δ is

$$\delta(z,w) = \sum_{n\in\mathbb{Z}} z^n w^{-n-1}\,. \tag{2.17}$$

By expanding, now, the holomorphic component of the stress-energy tensor in the two-form basis $\{z^{-n-2}(dz)^2\}_{n\in\mathbb{Z}}$,

$$T(z) \;=\; \sum_{n\in\mathbb{Z}} L_n\, z^{-n-2}(dz)^2, \qquad L_n \;=\; \frac{1}{2\pi i}\oint_{C_0} dz\, z^{n+1}T(z)\,, \tag{2.18}$$

one finds that the components L_n, together with the positive real constant c, constitute a representation of the Virasoro algebra (2.14). In addition, notice that equation (2.15) correspond to the finite transformation law

$$T(z) \;\rightarrow\; (w')^2\,T(z) + \tfrac{1}{12}c\{w,z\} \tag{2.19}$$

for a finite change of coordinates $z \rightarrow w = w(z)$. Here

$$\{w,z\} \;=\; \frac{w'''}{w'} - \frac{3}{2}\left(\frac{w''}{w'}\right)^2 \tag{2.20}$$

is known as the Schwarzian derivative. Hence, a non-zero central charge c implies that the stress-energy tensor $T(z)$ is not properly a quadratic differential, *i.e.* a precise geometrical object.

Another restriction on the class of physical representations arises imposing the positive energy condition. But to understand which is the constrain and how it is originated, one must come back to a real space. In particular, one must consider the real section of the complex space $\mathbb{C}^2$, with coordinates τ and σ, related to z and $\bar{z}$ through the formulae

$$z \;=\; e^{\tau+i\sigma} \qquad \bar{z} \;=\; e^{\tau-i\sigma} \qquad\qquad 0<\sigma\leq\pi\,. \tag{2.21}$$

Notice that this space can be regarded just as the euclidean real section $z=\xi_0+i\xi_1$, $\bar{z}=\xi_0-i\xi_1$, with ξ_0 and ξ_1 real numbers, with compactified space coordinate. Since, with respect to τ, the time translation generator is

$$H \;=\; L_0+\bar{L}_0 \;=\; \frac{d}{d\tau}\,, \tag{2.22}$$

in order to find a positive defined Hamiltonian H, the only acceptable representations are those in which L_0 and $\bar{L}_0$ are bounded from below. This correspond to say that they must be decomposable into representations such that there exists a vector $|\phi_\Delta\rangle$ with the properties

$$\begin{aligned} L_0\,|\phi_\Delta\rangle &= \Delta\,|\phi_\Delta\rangle \\ L_n\,|\phi_\Delta\rangle &= 0 \qquad\qquad for\quad n>0 \\ \hat{c}\,|\phi_\Delta\rangle &= c\,|\phi_\Delta\rangle\,, \end{aligned} \tag{2.23}$$

and each vector has the form

$$(L_{-k_n})^{j_n}(L_{-k_{n-1}})^{j_{n-1}}\dots(L_{-k_1})^{j_1}\;|\phi_\Delta\rangle\,, \qquad 0<k_1<k_2<\dots<k_n\,. \tag{2.24}$$

Here Δ is a positive real number and only the holomorphic part of the algebra $Vir \oplus \overline{Vir}$ has been taken into account. In fact, it is clear from their commutation relations that the generators L_n for $n>0$ lower the eigenvalue of L_0 by n, and, on the contrary, increase it by the same amount when $n<0$, so that the spectrum of L_0 has a minimum, Δ. Relations (2.23) define a so-called *highest weight representations* (hwr) of the Virasoro algebra, corresponding to *highest weight* Δ and central charge c, while the space of the vectors (2.24) is the *Verma module* V_Δ generated by the vector $|\phi_\Delta\rangle$.

Once the properties that characterize a physical representation of the Virasoro (or of the Witt) algebra are established , the identification of the vectors in the module V with the states of a conformal invariant model can be made possible by introducing the notion of scalar product and a proper defined vacuum state. To this end, let us consider a "dual" (right) module $V^\dagger$ associated to the states at infinity, in the same way as V is associated to the states generated at the origin[2], and, hence, a bilinear form $\langle\ |\ \rangle:\ V^\dagger\times V\to\mathbb{C}$. Moreover, let $|0\rangle$ ($\langle 0|$) be the *in (out)* vacuum state of the model, which generates V ($V^\dagger$) when the algebra of the model at $z=0$ ($z=\infty$) is applied on it. Some constraints arise when one requires that the stress-energy tensor is a regular, hermitean operator. In particular, in order that the vacuum expectation value of the stress-energy tensor is regular function of z in the origin, one must impose the condition

$$L_n\,|0\rangle = 0 \qquad for \qquad n\geq -1 \quad (-n-2<0)\,, \tag{2.25}$$

while, for the regularity at infinity, it is needed that

$$\langle 0|\,L_n = 0 \qquad for \qquad n\leq 1 \qquad (n-2<0)\,. \tag{2.26}$$

This can be easily derived when one refers to the expansion (2.18). Furthermore, equation (2.18) allows to find the *unitarity condition*

$$L_n^\dagger = L_{-n}\,, \tag{2.27}$$

that guarantees the hermiticity of $T(z)$. Among the consequences of equation (2.27), it can be stressed that each one of the eqs.(2.25) and (2.26) implies the other. Another

[2]origin and infinity in the (τ,σ) plane correspond to $z=0$ and $z=\infty$, respectively.

possible observation is that, since L_0 is self-adjoint, its eigenspaces (*levels of the representation*) are orthogonal. More generally, the unitarity condition, by defining the coaction of the Virasoro algebra on the dual space of V,

$$\left\langle L_n^\dagger \phi^\dagger \middle| \phi \right\rangle = \left\langle \phi^\dagger \left| L_n \phi \right\rangle \right. , \tag{2.28}$$

determines completely the bilinear form $\langle \mid \rangle$ (up to a normalization constant). To obtain a unitary representation one still needs to require the positivity of the norm induced by $\langle \mid \rangle$. About it, the above observations allow to anticipate two results:

1. one will need to impose the norm positivity condition at each level, and
2. it should generate constrains on the highest weight and on the central charge of the Verma module.

As will be pointed out in the sequel, for a particular range of value of the central charge, the conditions imposed by unitarity give rise to very peculiar properties of the corresponding Virasoro algebra representations and, consequently, sharply characterize a physical theory whose space of states coincides with one of such modules.

2.2 Correlation functions

Till now, we have faced the problem to pick out the properties fulfilled by the space of states when the physical model exhibits conformal invariance. The conclusion drawn in the previous section has been that the states of such a model should be vectors in a module of a unitary representation of the Virasoro algebra, but, likely, many further informations can be achieved. The next question might regard how conformal invariance contributes to the solution of such a model. Since this latter is characterized by the values of the correlation functions

$$\langle \phi_1 \phi_2 \ldots \phi_n \rangle = \langle 0 | \, \phi_1(z_1, \bar{z}_1) \phi_2(z_2, \bar{z}_2) \ldots \phi_n(z_n, \bar{z}_n) \, | 0 \rangle , \tag{2.29}$$

it is on these quantities that one should concentrate his attention. Therefore, a more detailed description of the algebra of a QCFT becomes an even more necessary introduction.

Among the elements of a conformal invariant operator algebra, one can distinguish those fields which transform as a tensor

$$\phi(z, \bar{z})(dz)^\Delta (d\bar{z})^{\bar{\Delta}} = \phi(w, \bar{w})(dw)^\Delta (d\bar{w})^{\bar{\Delta}} , \tag{2.30}$$

where $w = w(z)$ and $\bar{w} = \bar{w}(\bar{z})$ are arbitrary holomorphic and antiholomorphic reparametrization, while $\Delta + \bar{\Delta}$ and $\Delta - \bar{\Delta}$ are said, respectively, the conformal weight and the spin of ϕ. In connection with conformal invariance, the peculiarity of these fields, that are called *primary field*, becomes evident when their infinitesimal (holomorphic) variation

$$\delta_\epsilon \phi(z,\bar{z}) = \epsilon(z) \frac{\partial}{\partial z}\phi(z,\bar{z}) + \Delta \epsilon'(z)\, \phi(z,\bar{z}) \qquad \textit{for } z \longrightarrow w(z) = z + \epsilon(z) \quad (2.31)$$

is considered. In fact, choosing $\epsilon(z) = \epsilon z^{n+1}$, the last equation can be cast in the form

$$[L_n, \phi(z,\bar{z})] = z^{n+1}\frac{\partial}{\partial z}\phi(z,\bar{z}) + \Delta\,(n+1)\, z^n \phi(z,\bar{z})\,. \qquad (2.32)$$

Since these commutation relations determine completely the properties of the state created by the primary field, one can study its transformation under the action of the generators of the Virasoro algebra. Omitting the $\bar{z}$-dependence, the result for $n = 0$ is

$$L_0\,|\phi\rangle = \lim_{z\to 0} L_0\,\phi(z)\,|0\rangle = \lim_{z\to 0}\,[L_0, \phi(z)]\,|0\rangle = \Delta\,|0\rangle\,, \qquad (2.33)$$

while, for $n > 0$,

$$L_n\,|\phi\rangle = \lim_{z\to 0} L_n\,\phi(z)\,|0\rangle = \lim_{z\to 0}\,[L_n, \phi(z)]\,|0\rangle = 0\,. \qquad (2.34)$$

Such equations express precisely the defining properties of a highest weight vector of weight Δ (the central charge c is here just a number and, hence, acts multiplicatively). Accordingly, the algebra associated to the Verma module V_Δ contains, besides the stress-energy tensor, at least the primary field of weight[3] Δ and those fields which appear in their operator product expansion (OPE). For example

$$T(w)\phi(z,\bar{z}) = \sum_{k=0}^{\infty} (w-z)^{-2+k}\phi^{(-k)}(z,\bar{z})\,, \qquad (2.35)$$

$$\begin{aligned} T(w)\phi^{(-k_1)}(z,\bar{z}) &= \tfrac{1}{12}c\,(w-z)^{-k_1-2}(k_1^3 - k_1)\phi(z,\bar{z}) \\ &+ \sum_{l=1}^{k_1} (w-z)^{-l-2}(l+k_1)\phi^{(l-k_1)}(z,\bar{z}) \\ &+ \sum_{k_2=0}^{\infty} (w-z)^{-2+k_2}\phi^{(-k_1,-k_2)}(z,\bar{z})\,, \end{aligned} \qquad (2.36)$$

(one does not find any new contribution to the field algebra in the expansion of the product of the type $\phi\phi^{(-k)}$).

Analogous considerations hold for the antiholomorphic part of the conformal algebra, that is, the primary field ϕ, with the transformation law (2.30), generates the highest weight vector of the Verma module $\bar{V}_\Delta$ and expansions similar to (2.35) and 2.36), but involving the antiholomorphic component $\bar{T}(\bar{z})$ of the stress-energy tensor, can be performed. In this way, one finds an infinite set of fields $\phi^{\{k\}\{\bar{k}\}}$,

[3]More exactly this is the holomorphic contribution to the conformal weight of the primary field, which coincides with the highest weight of the state associated to the primary field. Throughout we will use the word weight or dimension, understanding this ambiguity.

where $\{k\} = (-k_1, \ldots, -k_N)$ and $\{\bar{k}\} = (-\bar{k}_1, \ldots, -\bar{k}_N)$, which constitute the *family* of *secondary field* or *descendants* generated by $\phi(z, \bar{z})$, denoted usually $[\phi]$. As particular cases

$$\phi^{(0)}(z, \bar{z}) = \Delta\, \phi(z, \bar{z}), \qquad \phi^{(-1)}(z, \bar{z}) = \frac{\partial}{\partial z}\phi(z, \bar{z}). \tag{2.37}$$

from which one deduces that $T(z)$ is a secondary field belonging to the family $[\mathbf{1}]$, being $\mathbf{1}$ the unit of the operator algebra and $T(z) = \mathbf{1}^{(-1)}(z)$.

A deeper insight shows that the family $[\phi]$ constitutes the whole contribution to field content of the theory associated to the Verma module $V_\Delta \otimes \bar{V}_{\bar{\Delta}}$. Indeed, by substituting in eqs. (2.35) and (2.36) the expansion (2.18) of the stress-energy tensor, one realizes that

$$\phi^{\{k\}\{\bar{k}\}}(z, \bar{z}) = L_{-k_n}(z) \ldots L_{-k_1}(z) \bar{L}_{-\bar{k}_n}(z) \ldots \bar{L}_{-\bar{k}_1}(z, \bar{z})\, \phi(z, \bar{z}). \tag{2.38}$$

Here, the operator introduced in the previous equation are

$$L_{-k}(z) = \frac{1}{2\pi i} \oint_{C_z} dw\, \frac{T(w)}{(w - z)^{k-1}}, \tag{2.39}$$

being C_z is a contour of integration surrounding the point z. (The same expression holds for the antiholomorphic $\bar{L}(\bar{z})$). Thus, remembering eq.(2.24), one, thus, concludes that the descendants create all and only the states of the Verma module associated to the corresponding primary field. On the other hand, the initial values of primary and secondary fields determines them uniquely. This permits, finally, to establish a one to one correspondence between primary fields and Verma modules, so that it is exactly equivalent to formulate the theory in terms of highest weight representations or in those of conformal families. In particular, as the first ones are characterized by their highest weight vectors, in the same way, one can show that all the informations about the QCFT are contained in the correlation functions of primary fields. From another point of view, the operator product algebra of the model is generated by the primary fields.

In order to verify these statements in details, notice that, from the expansion (2.35) and the expression (2.37) of the coefficients of its singular part, one achieves the relation

$$\begin{aligned} &\langle T(z)\phi_1(z_1, \bar{z}_1) \ldots \phi_n(z_n, \bar{z}_n)\rangle = \\ &= \sum_{i=1}^{n} \left(\frac{\Delta_i}{(z - z_i)^2} + \frac{1}{z - z_i} \frac{\partial}{\partial z_i} \right) \langle \phi_1(z_1, \bar{z}_1) \ldots \phi_n(z_n, \bar{z}_n)\rangle, \end{aligned} \tag{2.40}$$

being ϕ_{l_i}, $(i = 1, \ldots, n)$, primary fields. Due to this equation, the previous statement is then essentially proved. Indeed, secondary fields are defined precisely as the coefficients of OPE's like (2.35) and (2.36). In the simplest case,

$$\phi^{(-k)}(z, \bar{z}) = L_{-k}(z)\, \phi(z, \bar{z}) = \frac{1}{2\pi i} \oint_{C_z} dw\, \frac{T(w)\, \phi(z, \bar{z})}{(w - z)^{k-1}}, \tag{2.41}$$

where the operators $L_{-k}(z)$ are defined in eq.(2.39). Therefore, deforming C_z into a set of contours C_i around the points z_i, the correlation function reads

$$\begin{aligned}
&\left\langle \phi^{(-k)}(z,\bar{z})\phi_1(z_1,\bar{z}_1)\dots\phi_n(z_n,\bar{z}_n)\right\rangle = \\
&= -\frac{1}{2\pi i}\sum_{i=1}^{n}\oint_{C_i} dw\,\frac{1}{(w-z)^{k-1}}\left(\frac{\Delta_i}{(w-z_i)^2}+\frac{1}{w-z_i}\frac{\partial}{\partial z_i}\right)\times \\
&\qquad \times\langle\phi(z,\bar{z})\phi_1(z_1,\bar{z}_1)\dots\phi(z_i,\bar{z}_i)\dots\phi_1(z_n,\bar{z}_n)\rangle = \\
&= \sum_{i=1}^{n}\left(\frac{(k-1)\,\Delta_i}{(z_i-z)^k}-\frac{1}{(z_i-z)^{k-1}}\frac{\partial}{\partial z_i}\right)\times \\
&\qquad \times\langle\phi(z,\bar{z})\phi_1(z_1,\bar{z}_1)\dots\phi(z_i,\bar{z}_i)\dots\phi_1(z_n,\bar{z}_n)\rangle \equiv \\
&\equiv \mathcal{L}_{-k}(z,z_1,\dots,z_n)\,\langle\phi(z,\bar{z})\phi_1(z_1,\bar{z}_1)\dots\phi(z_n,\bar{z}_n)\rangle\;. \qquad (2.42)
\end{aligned}$$

With successive applications of the differential operators $\mathcal{L}_{-k}$, one can computes the correlation functions for the descendant fields,

$$\left\langle\phi_1^{\{k_1\}}(z_1,\bar{z}_1)\phi_2^{\{k_2\}}(z_2,\bar{z}_2)\dots\phi_n^{\{k_n\}}(z_n,\bar{z}_n)\right\rangle\;, \qquad (2.43)$$

once those relevant to primary fields are known.

As it has been suggested at the beginning of this section, conformal symmetry imposes severe restrictions on the correlators $\langle\phi_1,\phi_2\dots\phi_n\rangle$. The first conditions derive from the *projective invariance* of the vacuum

$$\begin{aligned}
L_s\,|0\rangle &= 0 \\
&\qquad\qquad \textit{for}\;\; s = -1,0,1 \qquad (2.44)\\
\langle 0|\,L_s &= 0\,.
\end{aligned}$$

Indeed, since

$$\oint dz\, z^{s+1}\,\langle T(z)\phi_1(z_1,\bar{z}_1)\dots\phi_n(z_n,\bar{z}_n)\rangle = 0 \qquad \textit{for}\;\; s = -1,0,1\,, \qquad (2.45)$$

due to eqs. (2.40), the correlation functions among primary fields should fulfills the conditions

$$\begin{aligned}
&\sum_{i=1}^{n}\frac{\partial}{\partial z_i}\,\langle\phi_1(z_1,\bar{z}_1)\dots\phi_i(z_i,\bar{z}_i)\dots\phi_n(z_n,\bar{z}_n)\rangle = 0 \\
&\sum_{i=1}^{n}\left(z_i\frac{\partial}{\partial z_i}+\Delta_i\right)\langle\phi_1(z_1,\bar{z}_1)\dots\phi_i(z_i,\bar{z}_i)\dots\phi_n(z_n,\bar{z}_n)\rangle = 0 \\
&\sum_{i=1}^{n}\left(z_i^2\frac{\partial}{\partial z_I}+2z_i\,\Delta_i\right)\langle\phi_1(z_1,\bar{z}_1)\dots\phi_i(z_i,\bar{z}_i)\dots\phi_n(z_n,\bar{z}_n)\rangle = 0\,, \qquad (2.46)
\end{aligned}$$

whose more general solution has the form

$$\langle\phi_1(z_1,\bar{z}_1)\phi_2(z_2,\bar{z}_2)\dots\phi_n(z_n,\bar{z}_n)\rangle = \\ = \prod_{\substack{ij\\ i<j}} (z_i - z_j)^{\gamma_{ij}}\,(\bar{z}_i - \bar{z}_j)^{\bar{\gamma}_{ij}}\, f(z_{ij}^{kl}, \bar{z}_{ij}^{kl}) . \tag{2.47}$$

Here the notation is

i) γ_{ij} and $\bar{\gamma}_{ij}$ are the solutions of the systems of equations

$$\begin{aligned} \sum_{\substack{j\\ j\neq i}} \gamma_{ij} &= 2\Delta_i \qquad & i &= 1, 2, \dots, n \\ \sum_{\substack{j\\ j\neq i}} \bar{\gamma}_{ij} &= 2\bar{\Delta}_i \qquad & i &= 1, 2, \dots, n \end{aligned} \tag{2.48}$$

where $\gamma_{ij} = \gamma_{ji}$ and $\bar{\gamma}_{ij} = \bar{\gamma}_{ji}$,

ii) f is an arbitrary function depending on the $2(n-3)$ harmonic quotient

$$z_{ij}^{kl} = \frac{(z_i - z_j)(z_k - z_l)}{(z_i - z_l)(z_k - z_j)} \qquad \bar{z}_{ij}^{kl} = \frac{(\bar{z}_i - \bar{z}_j)(\bar{z}_k - \bar{z}_l)}{(\bar{z}_i - \bar{z}_l)(\bar{z}_k - \bar{z}_j)} . \tag{2.49}$$

In the particular cases $n = 2, 3$ the correlation functions of primary fields results, thus, completely determined and one achieves

$$\langle\phi_n(z_1,\bar{z}_1)\phi_m(z_2,\bar{z}_2)\rangle = \delta_{nm}\, D_n\, (z_1 - z_2)^{-2\Delta_n}(\bar{z}_1 - \bar{z}_2)^{-2\bar{\Delta}_n} \tag{2.50}$$

$$\langle\phi_n(z_1,\bar{z}_1)\phi_m(z_2,\bar{z}_2)\phi_p(z_3,\bar{z}_3)\rangle = \\ = C_{nmp}\, (z_1 - z_2)^{-2\Delta_{mp,n}}(z_2 - z_3)^{-2\Delta_{mp,n}}(\bar{z}_1 - \bar{z}_2)^{-2\bar{\Delta}_{nm,p}}(\bar{z}_2 - \bar{z}_3)^{-2\bar{\Delta}_{mp,n}} \tag{2.51}$$

where $\Delta_{nm,p} = \Delta_n + \Delta_m - \Delta_p$ and $\bar{\Delta}_{nm,p} = \bar{\Delta}_n + \bar{\Delta}_m - \bar{\Delta}_p$. Notice that the highest weight vectors $|n\rangle$ and $|m\rangle$, associated with the fields ϕ_n and ϕ_m, can be always chosen orthonormal, as they are L_0 eigenvectors, so that it is $D_n = 1$. With this normalization, the coefficients C_{nmp} becomes symmetric in the indices n, m, p. Furthermore, from eq.(2.50) one derives the definition of the *out* state created by a primary field as

$$\langle\Delta| = \lim_{z\to\infty} \langle 0|\, \phi_\Delta(z)\, z^{2L_0} , \tag{2.52}$$

having understood the $\bar{z}$-dependence.

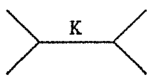
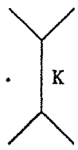
Another important constrain (see appendix C of ref. [5]), arises as the consequence of the associativity assumption for the operator product algebra

$$O_I O_J = \sum_K C_{IJ}^K O_K , \tag{2.53}$$

where O_I is a generic field, the index I combining the space-time dependence and the label of the operator. Consequently, the associativity condition is expressed by the equation

$$\sum_K C_{IJ}^K C_{KL}^M = \sum_K C_{IK}^M C_{JL}^K \tag{2.54}$$

which can be graphically represented as

Σ_K  $= \Sigma_K \cdot$ 

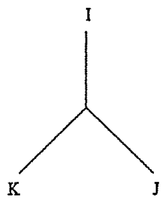
(2.55)

Here the "propagator" and the "vertex" coincide with the two-point correlation function,

$= D_{IJ} = \langle O_I O_J \rangle \ ,$

(2.56)

and with the three-point one,



$= C_{IJK} = \Sigma'_K \ D_{KK'} C_{IJ}^{K'} = \langle O_I O_J O_K \rangle \ ,$

(2.57)

respectively. Therefore, eq.(2.54) acquires the meaning of a crossing symmetry condition for the four-point correlation function $\langle O_I O_J O_K O_L \rangle$. A more explicit notation is recovered denoting

$$G^{lk}_{nm}(z,\bar{z}) = \langle k | \, \phi_l(1,1) \phi_n(z,\bar{z}) \, | m \rangle \ , \tag{2.58}$$

which represents the most general form of the four-point correlation function including the constraints imposed by modular invariance. Therefore, the *crossing symmetry condition* reads

$$G^{lk}_{nm}(z,\bar{z}) = G^{mk}_{nl}(1-z, 1-\bar{z}) = G^{lm}_{nk}\left(\frac{1}{z}, \frac{1}{\bar{z}}\right) . \tag{2.59}$$

This equation can be cast in a more useful way if one performs the OPE

$$\phi_n(z,\bar{z})\phi_m(0,0) = \sum_p \sum_{\{k\}\{\bar{k}\}} C_{nm}^{p;\{k\}\{\bar{k}\}} z^{-\Delta_{nm,p}+\Sigma_i k_i} \bar{z}^{-\bar{\Delta}_{nm,p}+\Sigma_i \bar{k}_i} \phi_p^{\{k\}\{\bar{k}\}}(0,0) \quad (2.60)$$

in the correlation functions G_{nm}^{lk}. Indeed, due to conformal invariance, the numerical coefficients $C_{nm}^{p;\{k\}\{\bar{k}\}}$ split into $C_{nm}^{p}\,\beta_{nm}^{\{k\}}\,\bar{\beta}_{nm}^{\{\bar{k}\}}$, where $\beta_{nm}^{\{k\}}$ and $\bar{\beta}_{nm}^{\{\bar{k}\}}$ are expressed unambiguously in terms of the weights Δ_n, Δ_m, Δ_p and $\bar{\Delta}_n$, $\bar{\Delta}_m$, $\bar{\Delta}_p$, respectively, and $\beta_{nm}^{\{0\}} = 1 = \bar{\beta}_{nm}^{\{\bar{0}\}}$ so that the coefficients C_{nm}^{p} coincide with those of the three-point correlation function (2.51). Therefore, by inserting this result in the definition (2.58), one obtains

$$G_{nm}^{lk}(z,\bar{z}) = \sum_p C_{nm}^{p}\, C_{klp}\, \mathcal{F}_{nm}^{lk}(p\,|z)\, \bar{\mathcal{F}}_{nm}^{lk}(p\,|\bar{z}) . \quad (2.61)$$

In the above equation the generally many-valued functions $\mathcal{F}_{nm}^{lk}$ $(\bar{\mathcal{F}}_{nm}^{lk})$ are called *conformal block functions* and depend on Δ_n, Δ_m, Δ_k, Δ_l, Δ_p ($\bar{\Delta}_n$, $\bar{\Delta}_m$, $\bar{\Delta}_k$, $\bar{\Delta}_l$, $\bar{\Delta}_p$), as it appears evident from their expression

$$\mathcal{F}_{nm}^{lk}(p\,|z) = z^{-\Delta_{nm,p}} \sum_{\{k\}} \beta_{nm}^{p\,\{k\}} z^{\Sigma_i k_i} \frac{\langle k|\,\phi_l(1,1)L_{k_1}\ldots L_{k_n}\,|p\rangle}{\langle k|\,\phi_l(1,1)\,|p\rangle} , \quad (2.62)$$

[the same expression, with the substitution $z \to \bar{z}$, $\Delta_{nm,p} \to \bar{\Delta}_{nm,p}$, $\beta_{nm}^{p\,\{k\}} \to \bar{\beta}_{nm}^{p\,\{k\}}$, holds for $\bar{\mathcal{F}}_{nm}^{lk}(p\,|\bar{z})$]. Notice that (2.62) are generally many-valued, even if the correlators $G_{nm}^{lk}(z,\bar{z})$ are single-valued meromorphic functions. The crossing symmetry condition in terms of such conformal block functions reads

$$\sum_p C_{nm}^{p}\, C_{lkp}\, \mathcal{F}_{nm}^{lk}(p|z)\, \bar{\mathcal{F}}_{nm}^{lk}(p|\bar{z}) = \sum_q C_{nl}^{q}\, C_{mkq}\, \mathcal{F}_{nl}^{mk}(p|1-z)\, \bar{\mathcal{F}}_{nl}^{mk}(p|1-\bar{z}) . \quad (2.63)$$

These are the fundamental equations of the Polyakov's bootstrap program, whose aim is the computation of the constants C_{nm}^{l} and the dimensions Δ_n, $\bar{\Delta}_n$, the most important dynamical characteristics of the QCFT. Indeed, once the conformal block functions are known, eq. (2.63) yields a system of equations which are sufficient to determine these numerical parameters. Notice that the problem can be formulated in a slight different way, pointing out the role of local assumption. Indeed, the constants C_{mn}^{p}, constrained by the requirement of locality, *i.e.* of single-valuedness of the physical correlation functions, shall be such that the combination (2.61) will be monodromy invariant.

The expression (2.61) is an explicit example of the factorization of the operator algebra into a holomorphic and an antiholomorphic component and can be obviously generalized to the n-point correlation functions, by the introduction of the n-point conformal block functions $\mathcal{F}^{l_1 l_2 \ldots l_n}(\vec{p}\,|z_1, z_2, \ldots, z_n)$, where $\vec{p} = (p_1, p_2, \ldots, p_{n-1})$. Such quantities, in fact, allow to write the physical correlators as the monodromy invariant combination

$$G^{l_1 l_2 \ldots l_n}(z_1,\bar{z}_1,\ldots z_n,\bar{z}_n) = \sum_{\vec{p}\vec{p}'} h_{\vec{p}\vec{p}'}\, \mathcal{F}^{l_1 l_2 \ldots l_n}(\vec{p}\,|z_1,z_2,\ldots,z_n)\bar{\mathcal{F}}^{l_1 l_2 \ldots l_n}(\vec{p}'\,|\bar{z}_1,\bar{z}_2,\ldots,\bar{z}_n) , \quad (2.64)$$

with the invariant matrix $h_{\vec{p}\vec{p}'}$ built out of the constants C^p_{nm}. This enables to state that the same decomposition holds at the operator level

$$\phi_n(z,\bar{z}) = \sum_{m\,p} D^p_{nm} \varphi^p_{nm}(z) \otimes \bar{\varphi}^p_{nm}(\bar{z}), \tag{2.65}$$

where φ^p_{nm} are many valued conformal fields of weight Δ_n whose action and coaction are non-zero only on the Verma modules V_m and V_p, respectively. In addition, if one imposes the normalization condition

$$\langle p|\varphi^p_{nm}(1)|m\rangle = 1 \tag{2.66}$$

then the conformal block functions $\mathcal{F}^{l_1 l_2 \ldots l_n}$ are the vacuum expectation values of the product of such conformal fields

$$\mathcal{F}^{l_1 l_2 \ldots l_n}(\vec{p}\,|z_1, z_2, \ldots, z_n) = \left\langle \varphi^0_{l_1 p_1}(z_1) \varphi^{p_1}_{l_2 p_2}(z_2) \ldots \varphi^{p_{n-1}}_{l_n 0}(z_n) \right\rangle . \tag{2.67}$$

Here, 0 denote the vacuum representation. Therefore, φ^p_{nm} are said *conformal block operators* or, more simply, *conformal blocks*. The possibility to describe QCFT by focusing the attention on conformal blocks, rather than directly on local primary fields, will be investigated in the next chapter, where locality arguments will become the main ingredients of the discussion.

Although Polyakov's idea can be applied to any conformal invariant model, it has to be stressed that the explicit expression of conformal block functions, in principle completely determined by conformal symmetry, is not known for a general Virasoro algebra representation. Nevertheless, in the next section it will be shown that there exists a particular class of such representations, the so-called degenerate representations, in which conformal block functions are the solutions of partial differential equations.

2.3 Unitarity condition and minimal models

Let us remember that a representation of the Virasoro algebra is said to be unitary when a scalar product structure has been introduced in it, coherently with the unitarity condition

$$L_n^\dagger = L_{-n} . \tag{2.27}$$

On the other hand, in order for the bilinear form $\langle\ |\ \rangle$, introduced at the end of §2.1, to fulfill the positivity condition, $\left\langle \psi^\dagger | \psi \right\rangle \in \mathbb{R}_{\geq 0}$, $= 0 \Leftrightarrow |\psi\rangle = 0$, some restrictions are needed on the possible values of the central charge c and the weight Δ. As an example, consider the states $L_{-n}\,|\phi_\Delta\rangle$, with $n > 0$. In this case, if one imposes the positivity of the quantity

$$\begin{aligned} \|\, L_{-n}\,|\phi_\Delta\rangle\,\|^2 &= \langle\phi_\Delta|\, L_n L_{-n}\,|\phi_\Delta\rangle \\ &= \langle\phi_\Delta|\,[L_n, L_{-n}]\,|\phi_\Delta\rangle \\ &= \left[2n\,\Delta + \frac{c}{12} n(n^2-1)\right] , \end{aligned} \tag{2.68}$$

(with the choice of the normalization $\| \, |\phi_\Delta\rangle \, \| = 1$), for $n = 1$ and then $n \to \infty$, one finds that

$$\Delta \geq 0 \qquad and \qquad c \geq 0 \tag{2.69}$$

represents a necessary requirement for an irreducible highest weight representation to be unitary.

The general analysis is based on the explicit expression, due to Kac [19,20], of the determinant of the matrix associated to the quadratic form $\| \, |\psi\rangle \, \|^2 = \langle\psi|\psi\rangle$. As already hinted, since any representation is L_0-graded, that is, can be decomposed in the direct sum of orthogonal finite-dimensional L_0-eigenspaces (levels of the representation), Kac's formula can be given at each level N [*i.e.*, the level of those vectors such that $L_0 \, |\psi\rangle = (\Delta + N) \, |\psi\rangle$] and reads

$$det\, M_N(c,\Delta) = \prod_{k=1}^{N} \prod_{\substack{n\,n' \\ n'n = k}} (\Delta - \Delta_{n',n}(c))^{\pi(N-k)} . \tag{2.70}$$

In eq.(2.70) $\pi(N)$ is the dimension of the level N, whose value can be derived from the formula

$$\prod_{N=1}^{\infty} (1 - q^N)^{-1} = \sum_{N=0}^{\infty} \pi(N)\, q^N . \tag{2.71}$$

Moreover, the functions

$$\Delta_{n',n}(c) = \left(\tfrac{1}{2}\alpha_+ n + \tfrac{1}{2}\alpha_- n'\right)^2 - \alpha_0^2 \qquad\qquad n, n' \in \mathbb{Z}_{>0} , \tag{2.72}$$

are expressed in terms of the quantities

$$\begin{aligned} \alpha_0^2 &= \tfrac{1}{24}(1-c) \\ \alpha_\pm &= \frac{\sqrt{1-c} \pm \sqrt{25-c}}{\sqrt{24}} \end{aligned} \tag{2.73}$$

and give, for each value of the central charge c, the highest weight of the reducible Verma module. Indeed, since $det\, M_N(c, \Delta_{n',n}(c)) = 0$ for $N \geq n'n$, the respective Verma module $V_{n',n} = V_{\Delta_{n',n}}$ should contains at the level $n'n$ a so-called *null* or *singular vector* $|\chi\rangle$, with zero norm and orthogonal to any state in $V_{n',n}$,

$$\begin{aligned} \langle\chi|\chi\rangle &= 0 \\ \langle\psi|\chi\rangle &= 0 \qquad\qquad \forall \, |\psi\rangle \in V_{n',n} . \end{aligned} \tag{2.74}$$

From the above relations, it follows that $|\chi\rangle$ satisfies the equations

$$\begin{aligned} L_0 \, |\chi\rangle &= (\Delta_{n',n} + nn') \, |\chi\rangle \\ L_m \, |\chi\rangle &= 0 \qquad\qquad for \;\; m > 0 , \end{aligned} \tag{2.75}$$

that are characteristic of the highest weight vectors. Therefore, to achieve an irreducible representation from $V_{n',n}$, it is sufficient to factor out the submodule generated by $|\chi\rangle$,

$$\tilde{V}_{n',n} = V_{n',n} \Big/ V_{n',-n} \tag{2.76}$$

(notice that $\Delta_{n',n} + n'n = \Delta_{n',-n}$). Indeed, due to eqs. (2.74) $\tilde{V}_{n',n}$ turns out to be a well-defined module, corresponding to the so-called *degenerate representations.*

As for the unitarity property, Kac's formula allows to state that the quadratic form $\| \, |\psi\rangle \, \|^2$ or, equivalently, the matrix $M(c,\Delta) = \bigoplus_{\mathrm{N}} M_{\mathrm{N}}(c,\Delta)$, is positive semi-definite if $c \geq 1$ (strictly positive definite if $c > 1$), with no further assumption for the weight Δ, besides $\Delta \geq 0$. In this range of values, in fact, either the functions $\Delta_{n',n}(c)$ are complex ($1 < c < 25$) or they become negative for sufficiently large n and n' ($c \geq 25$)[4]. In the remaining region, $0 \leq c < 1$, the matrices $M_{\mathrm{N}}(c,\Delta)$ are always negative semi-definite and one can establish that

$$\begin{aligned} c &= 1 - \frac{6}{p(p+1)} && \textit{with } \; p = 2, 3, 4, \ldots \\ \Delta &= \Delta_{n',n}(p) = \frac{[(p+1)n - pn']^2 - 1}{4p(p+1)} && \textit{with } \; \begin{matrix} r = 1, 2 \ldots, p-1 \\ s = 1, 2, \ldots, r \end{matrix} \end{aligned} \tag{2.77}$$

are the only acceptable values (see ref.'s [15] and [16]). Notice that the cases $n' > n$ have been excluded only because $\Delta_{n',n} = \Delta_{p+1-n',p-n}$. Moreover, for $n > p$ and $n' > p+1$ the series repeats itself exactly, in accordance with the rule

$$\Delta_{n',n} = \Delta_{p+1+n',p+n} \, . \tag{2.78}$$

In particular, for $p = 2$, one retrieves the trivial representation $c = 0$, $L_n = 0 \; \forall n \in \mathbb{Z}$. The conformal invariant models associated to the irreducible Virasoro algebra representations $\tilde{V}_{n',n}$ are called *unitary minimal models* and belong to a wider class, the *rational conformal field theory* (RCFT), characterized by a space of states consisting in a finite direct sum of irreducible highest weight representations of $Vir \oplus \overline{Vir}$, or, equivalently, by an operator algebra which involves a finite number of conformal families. A common feature of these models is the possibility to compute the conformal block functions as the solutions of some partial differential equations. In the case of the minimal models they arise because of the reducibility of the Verma modules into consideration. Indeed, let $\chi(z)$ be the primary field corresponding to the Verma module $V_{n',-n} = V_{\Delta_{n',n}+n'n}$, submodule of $V_{n',n}$. Coherently with the reduction (2.76), any correlation function of the form $\langle \chi\phi_1 \ldots \phi_{n-1} \rangle$ vanishes. But, since this quantity can be expressed through a differential operator in terms of the correlators of the corresponding primary field $\psi_{n',n}$, [see eq.(2.42)], the presence in

[4]One can show that, for $c > 1$, a complete operator algebra containing one of the primary field $\psi_{n',n}$ of weight $\Delta_{n',n}$, must comprehend all of them.

$V_{n',n}$ of a null vector and hence the reducibility of such a representation, is reflected into the partial differential equations

$$\mathcal{L}_\chi(z, z_1, \ldots, z_{n-1}) \left\langle \psi_{n',n}(z)\phi_1(z_1)\ldots\phi_n(z_{n-1})\right\rangle = 0 . \tag{2.79}$$

Here the omission of the $\bar{z}$-dependence is justified, only the holomorphic component of the correlation functions being involved in the condition. Therefore, more precisely, conformal block functions are the arguments of eqs.(2.79),

$$\mathcal{L}_\chi(z, z_1, z_2, \ldots, z_n)\, \mathcal{F}_{n',n}(\vec{p}\,|z, z_1, \ldots, z_{n-1}) = 0 . \tag{2.80}$$

The differential operator $\mathcal{L}_\chi$ has maximal order of derivatives $n'n$, and can be obtained as a combination of the differential operators $\mathcal{L}_{-m}$ defined in equation (2.42), exactly in the same way as the primary field $\chi(z)$ is a combination of descendants belonging to the family $[\psi_{n',n}]$.

The simplest non trivial example of unitary degenerate representation is that associated to $\Delta = \Delta_{1,2}$ (the value of the central charge is left arbitrary). The null state of the Verma module $V_{1,2}$ is

$$|\chi\rangle = \left(L_{-2} - \frac{3}{2(2\Delta_{1,2}+1)}L_{-1}^2\right)|\chi\rangle , \tag{2.81}$$

as one can easily show, equating to zero the norm of the generic second level vector $(a\,L_{-2} + b\,L_{-1}^2)\,|\Delta_{1,2}\rangle$. Due to the relation (2.42), the desired partial differential eqs.(2.79) is achieved simply by substituting to the Virasoro generators in the expression (2.81) the corresponding differential operators $\mathcal{L}_{-k}$. The result is

$$\left(\frac{3}{2(2\Delta_{1,2}+1)}\frac{\partial^2}{\partial z^2} - \sum_{i=1}^{n}\frac{\Delta_i}{(z-z_i)^2} + \right.$$
$$\left. - \sum_{i=1}^{n}\frac{1}{z-z_i}\frac{\partial}{\partial z_i}\right)\mathcal{F}_{1,2}(\vec{p}\,|z, z_1, \ldots, z_{n-1}) = 0 , \tag{2.82}$$

where $\Delta_1, \Delta_2, \ldots, \Delta_n$ are the dimensions of the primary fields $\phi_1, \phi_2, \ldots, \phi_n$. If $0 < c < 1$ and the theory is unitary, *i.e.* in the case of unitary minimal models, every field belongs to a degenerate family (the conformal families corresponding to degenerate representations) and, consequently, every conformal block function must satisfy similar equation. Notice that if one consider, in particular, the four-point correlation functions

$$\mathcal{F}_{1,2}(\vec{p}\,|z, z_1, z_2, z_3) , \tag{2.83}$$

due to projective invariance of the vacuum, the partial differential eqs.(2.79) reduce to ordinary ones and, more precisely, to the Riemann ordinary differential equation

$$\left\{\frac{3}{2(2\Delta_{1,2}+1)}\frac{d^2}{dz^2} + \sum_{i=1}^{3}\left(\frac{1}{z-z_i}\frac{d}{dz} - \frac{\Delta_i}{(z-z_i)^2}\right) + \right.$$
$$\left. + \sum_{\substack{i\,j \\ j<i}}\frac{\Delta_{1,2}+\Delta_{ij,k}}{(z-z_i)(z-z_j)}\right\}\mathcal{F}_{1,2}(\vec{p}\,|z, z_1, z_2, z_3) = 0 . \tag{2.84}$$

Therefore, for the cases $(n',n) = (1,2)$ or $(n',n) = (2,1)$, the conformal block functions $\mathcal{F}^{l_1 l_2 l_3}_{1,2}(\vec{p}\,|z, z_1, z_2, z_3)$ can be expressed in terms of the hypergeometric functions.

Since correlation functions and OPE of primary fields are strictly related, important constrains on the latter arise naturally in the degenerate case, due to differential equations of the type (2.80). In the simplest case, when OPE involves $\psi_{1,2}$ or $\psi_{2,1}$, one can easily verify that (2.82) implies the symbolic formulae

$$\begin{aligned} \psi_{1,2}\psi_{n',n} &= \left[\psi_{n',n-1}\right] + \left[\psi_{n',n+1}\right] \\ \psi_{2,1}\psi_{n',n} &= \left[\psi_{n'-1,n}\right] + \left[\psi_{n'+1,n}\right] , \end{aligned} \qquad (2.85)$$

where the square brackets denote here the contribution of the corresponding conformal families to the OPE. Therefore, the degenerate conformal fields $\psi_{1,2}$ and $\psi_{2,1}$ acts as a sort of shift operator with respect to the OPE algebra. The relations expressed by eq.(2.85), which represent the simplest example of *fusion rules* in minimal models, must be understood with the limitation that no negative labels n', n arise in the expansion, otherwise the series results truncated. The general fusion rules for degenerate fields has the form

$$\psi_{n',n}\psi_{m',m} = \sum_{l'=|n'-m'|+1}^{k'} \; \sum_{l=|n-m|+1}^{k} \psi_{l',l} , \qquad (2.86)$$

where $k^{(')} = min(n^{(')}+m^{(')}-1,\, 2p^{(')}-n^{(')}-m^{(')}-1)$ and $n^{(')}+m^{(')}-l^{(')}-1 = 0 \; mod\, 2$.

2.4 Coulomb gas representation

Although the question of the computation of conformal block functions would seem to be solved, as explained in the previous section, the problem is only shifted, since the expression of such differential equations is not known in general. A different approach was proposed by Dotzenko and Fateev [7] and consists in a representation of the conformal blocks of minimal models in terms of the correlators of Coulomb gas vertex operators, the so-called Feigin-Fuks integral representation [11]. Furthermore, this interpretation allows to formalize the construction of the space of physical states in the case of degenerate representations, with the introduction of a BRST charge. Henceforth, only the holomorphic component of the operator algebra will be considered and the terminology proper of QCFT (primary fields, correlation functions, etc.) shall be understood by taking into account this restriction.

The DF model is constructed on a direct sum of "charged" bosonic Fock spaces $\mathfrak{F}_{\alpha,\alpha_0}$ of charge α, which a Verma module structure is given to, with central charge $c = 1 - 24\alpha_0^2$. To enter in more details, $\mathfrak{F}_{\alpha,\alpha_0}$ is the module of the Heisenberg algebra

$$[a_n, a_m] = 2n\,\delta_{n,-m} \qquad\qquad n, m \in \mathbb{Z} , \qquad (2.87)$$

built upon a vector $|\alpha\rangle_{\alpha_0}$ (ground state of the Fock space $\mathfrak{F}_{\alpha,\alpha_0}$) with the properties $a_n\,|\alpha\rangle_{\alpha_0} = 0$ for $n > 0$ and $a_0\,|\alpha\rangle_{\alpha_0} = 2\alpha\,|\alpha\rangle_{\alpha_0}$, by acting with the creation operators,

$$\mathfrak{F}_{\alpha,\alpha_0} = \bigoplus_{k=0}^{\infty} \bigoplus_{\substack{n_1 n_2 \ldots n_k \\ 1\leq n_1 \leq \ldots \leq n_k}} \mathbb{C} a_{-n_1}\ldots a_{-n_k}\,|\alpha\rangle_{\alpha_0}\,. \tag{2.88}$$

In the spaces $\mathfrak{F}_{\alpha,\alpha_0}$ one can introduce a Virasoro structure whose generators are represented by

$$L_n = \tfrac{1}{4}\sum_{k=-\infty}^{\infty} a_{n-k}a_k - \alpha_0(n+1)a_n\,, \qquad n \neq 0$$

$$L_0 = \tfrac{1}{2}\sum_{k=1}^{\infty} a_{-k}a_k - \tfrac{1}{4}\,a_0^2 - \alpha_0\,a_0\,. \tag{2.89}$$

One can easily verify that these operators obey the Virasoro commutation relations with central charge $1-24\alpha_0^2$. Moreover, $|\alpha\rangle_{\alpha_0}$ is the highest weight state of the Verma module $\mathfrak{F}_{\alpha,\alpha_0}$, with highest weight $\alpha^2 - 2\alpha\alpha_0$. Actually, to show that $\mathfrak{F}_{\alpha,\alpha_0}$ is isomorphic to such a Verma module, a further necessary condition is that subspaces of Verma modules and of charged Fock spaces with fixed degree in the L_0-gradation have the same dimension, as can be easily checked. The dual space $\mathfrak{F}^\dagger_{\alpha,\alpha_0}$, defined by the coaction $L_n^\dagger = L_{-n}$ on the space

$$\mathfrak{F}^\dagger_{\alpha,\alpha_0} = \bigoplus_{k=0}^{\infty} \left(\mathfrak{F}_{\alpha,\alpha_0}\right)^*_k\,, \tag{2.90}$$

coherently with the unitarity condition, has a Fock space structure whose Heisenberg generators are $a_n^\dagger = 4\alpha_0\,\delta_{n,0} - a_{-n}$. Therefore, $\mathfrak{F}^\dagger_{\alpha,\alpha_0}$ is isomorphic as a Verma module to $\mathfrak{F}_{2\alpha_0-\alpha,\alpha_0}$, a fact that suggest a physical interpretation of the parameter α_0 as the "charge at infinity". This can be made clearer if one consider the example of the vacuum module $\mathfrak{F}_{0,\alpha_0}$ and the respective dual $\mathfrak{F}^\dagger_{0,\alpha_0} \cong \mathfrak{F}_{2\alpha_0,\alpha_0}$ (see ref.[7]).

For the purpose of this section, the interesting values of the charges α and α_0 are

$$\alpha_0^2 = \frac{(p'-p)^2}{4p'p}$$

$$\alpha = \alpha_{n',n} = \tfrac{1}{2}(1-n')\alpha_- + \tfrac{1}{2}(1-n)\alpha_+\,. \tag{2.91}$$

Here p' and p are positive coprime integer with the restriction $p' > p$, while $n',\, n \in \mathbb{Z}$ have to fulfill the condition $n' \neq 0\ mod\, p'$, $n' \neq 0\ mod\, p'$, and $\alpha_\pm = \alpha_0 \pm (1+\alpha_0^2)^{1/2}$ are the same function of the central charge $c = 1-24\alpha_0^2$ introduced in equation (2.72). Thus, the weights of the Verma modules are

$$\begin{aligned}\Delta_{n',n} &= \alpha^2_{n',n} - 2\alpha_{n',n}\alpha_0 = \\ &= \left[\tfrac{1}{2}n'\alpha_- + \tfrac{1}{2}n\alpha_+\right]^2 - \alpha_0^2 = \\ &= \tfrac{1}{2}(n'^2-1)\,\alpha_- \tfrac{1}{2}(n'n-1) + \tfrac{1}{2}(n^2-1)\,\alpha_+^2\,,\end{aligned} \tag{2.92}$$

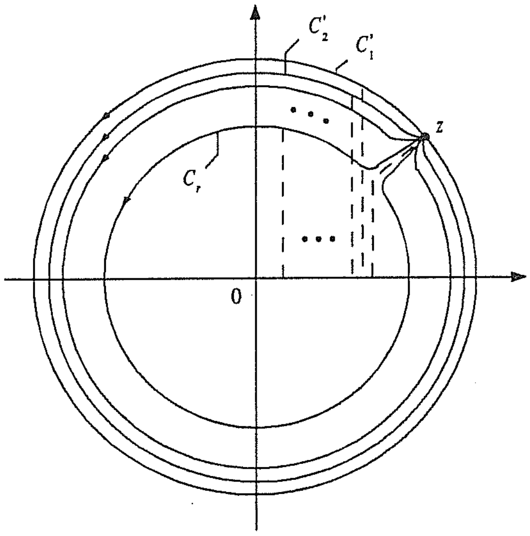


Figure 2.1: *Contours of integration in the definition of screened vertex operators (2.94).*

so that, when $p' = p + 1$ and $1 \leq n^{(')} \leq p^{(')} - 1$, one recovers, after the needed factorization, the unitary degenerate representations listed in the previous section. However, since also when such a unitary case is not considered the recursive formula

$$\alpha_{n',n} = \alpha_{p'+n',p+n} \tag{2.93}$$

holds, one obtains, for each value of p and p' (*i.e.* of the corresponding central charge), a finite complete set of Virasoro algebra representations. For this reason, the corresponding physical models (comprehending unitary and non-unitary minimal models) can still be count among the RCFT ones.

In this frame, the primary fields associated to the Verma modules $\mathfrak{F}_{n',n}$ are represented, in their most general form, by the so-called *screened vertex operators* (SVO's)

$$V_{n',n}^{r',r}(z) = \oint V_{\alpha_{n',n}}(z) V_{\alpha_-}(u_1) \dots V_{\alpha_-}(u_{r'}) V_{\alpha_+}(v_1) \dots V_{\alpha_+}(v_r) \prod_{i=1}^{r'} du_i \prod_{j=1}^{r} dv_j\,, \tag{2.94}$$

where the contours of integration, C_i' of u_i and C_j of v_j, are depicted in figure (2.1). The vertex operators $V_{\alpha_{n',n}}$ and $V_{\alpha_\pm}$ introduced in eq.(2.94) are Wick-ordered exponential of the free fields of the type

$$V_\alpha = T_\alpha\, z^{\alpha a_0} exp\left(\alpha \sum_{n=1}^{\infty} \frac{a_{-n}}{n} z^n\right) exp\left(-\alpha \sum_{n=1}^{\infty} \frac{a_n}{n} z^{-n}\right). \tag{2.95}$$

Here T_α the $U(1)$-charge generator, $[a_n, T_\alpha] = 2\,\delta_{n,0}\alpha T_\alpha$. In the sequel, SVO $V_{n',n}^{r',r}$ will be recognized as the conformal blocks of the primary field of weight $\Delta_{n',n}$ [see eq.(2.65)]. To support this statement, let us remember the expression of the vertex operator correlation functions

$$\langle \Omega^* | V_{\alpha_1}(z_1) V_{\alpha_2}(z_2) \ldots V_{\alpha_n}(z_n) \Omega \rangle = \begin{cases} \prod_{\substack{ij \\ i<j}} (z_i - z_j)^{2\alpha_i \alpha_j} & \textit{if } \sum_i \alpha_i = 0 \\ 0 & \textit{otherwise} \end{cases} \tag{2.96}$$

where the vacuum state $|\Omega\rangle$ and its dual are the highest weight vectors of the Verma modules $\mathfrak{F}_{0,\alpha_0}$ and $\mathfrak{F}^{\dagger}_{0,al_0} \cong \mathfrak{F}_{2\alpha_0,\alpha_0}$, respectively. Moreover, being the correlators (2.96) generally many-valued, one chooses $|z_1| > |z_2| > \ldots > |z_n|$. Therefore, the matrix element of the primary field of weight $\Delta_{n',n}$ between the ground states $\left|\alpha_{l',l}\right\rangle$ and $\left|\alpha_{m',m}\right\rangle$ is

$$\left\langle \alpha_{l',l} \right| V_{n',n}^{\frac{1}{2}(m'+n'-l'-1),\, \frac{1}{2}(m+n-l-1)} \left. \alpha_{m',m} \right\rangle , \tag{2.97}$$

the only possible choice to fulfill the neutrality condition $\sum_i \alpha_i = 0$. Indeed, $V_{\alpha_\pm}$, which do not contribute to the weight of the screen vertex operator, as it has dimension one, nevertheless contribute to its charge. Therefore, one can easily verify that $V^{r',r}_{n',n}$ maps the states in the Fock space $\mathfrak{F}_{m',m}$ to those in $\mathfrak{F}_{m'+n'-2r'-1,\, m+n-2r-1}$. The analogy with conformal blocks becomes, so, evident, if one interprets φ^l_{nm} as a homomorphism between V_{Δ_m} and V_{Δ_l}.

It remains to clarify the motivation of that particular choice of integration contours in the definition (2.94). Notice, firstly, that, although the integrand is many-valued, the contour C_i' and C_j are chosen so to be contained in a region where such a product of vertex operators is well-defined, namely $|z| > |u_1| > \ldots > |v_r|$, and one can fix conventionally its matrix elements between two highest weight states to be real when the variable are ordered on the positive real axis $(z > u_1 > \ldots > v_r > 0)$. The value of (2.94) is then unambiguously defined if one gives a path of analytic continuation in $\mathbb{C}^{r+r'+1} - \bigcup_{\alpha<\beta} \{z_\alpha = z_\beta\}$, as shown in figure (2.1) (dashed lines). Of course, due to the divergence of the product of vertex operators as their arguments approach, the integrals in eq.(2.94) have to be regularized somehow. A possibility is to evaluate the matrix elements of a SVO as analytic continuation from a region of the complex α_-^2 plane (with $\alpha_+^2 = 1/\alpha_-^2$) where the integral converge. Such region is given by

$$-\frac{1}{2} < \mathrm{Re}\, \alpha_\pm^2 < 0 \tag{2.98}$$

and, hence, does not contain any real point. However, one can observe that every other choice of the integration contours is allowable, provided that they surround the point z. Indeed, they become completely equivalent when expectation values are considered. Nevertheless, the present definition has the advantage to enable a direct identification of conformal blocks with screened vertex operators, already at operator level, instead that only at the level of correlators.

The analysis brought out in the previous section led to the conclusion that the Verma modules $V_{n',n}$ contain irreducible maximal proper submodules of weight

$\Delta_{n',n} + n'n$. Since the Fock spaces $\mathfrak{F}_{n',n}$ are isomorphic to $V_{n',n}$, rather than to the irreducible quotient modules $\tilde{V}_{n',n}$, they cannot be identified with minimal models before decoupling physical from unphysical states. A BRST cohomology of Fock spaces can be the tool to realize this program. In order to find a good candidate as BRST charge, let us consider the operators

$$Q_m = \frac{1}{m} \oint V_{\alpha_+}(v_0) \dots V_{\alpha_+}(v_{m-1}) \prod_{i=0}^{m-1} dv_i \tag{2.99}$$

mapping the states in $\mathfrak{F}_{m',m}$ to those in $\mathfrak{F}_{m',-m}$. Here the integration contours for $v_1, \dots, v_{m-1}$ are those chosen for the SVO's, while v_0 is integrate over the unit circle. This operator has some crucial properties.

i) The operators (2.99) can be viewed as the charge corresponding to the currents

$$J_m(z) = \frac{1}{m} V_{1,-1}^{0,m-1}(z). \tag{2.100}$$

Since $J_m(z)$ are single-valued weight-one operators, the respective charges Q_m preserve the Virasoro algebra structure, *i.e.*

$$[Q_m, L_k] = 0 \qquad \forall k \in \mathbb{Z}, \tag{2.101}$$

which allows to state that Q_m are injective homomorphisms of $\mathfrak{F}_{m',m}$ into $\mathfrak{F}_{m',-m}$, provided that one can prove the sufficient condition $Q_m \left|\alpha_{m',m}\right\rangle \neq 0$. It has been shown (see appendix in ref.[12]) that this requirement is fulfilled when $m' < 0$ and $1 \leq m \leq p-1$. Since $\Delta_{m',m} = \Delta_{m',-m} - m'm$, for these values of m' and m, the state $Q_m \left|\alpha_{m',m}\right\rangle$ has to be a non-zero singular vector of $\mathfrak{F}_{m',-m}$.

ii) Let us consider the charges

$$\mathfrak{F}_{m',2p-m} \xrightarrow{Q_{p-m}} \mathfrak{F}_{m',m} \xrightarrow{Q_m} \mathfrak{F}_{m',m} \tag{2.102}$$

with $1 \leq m^{(\prime)} \leq p^{(\prime)} - 1$. In this sequence Q_{p-m} maps $\mathfrak{F}_{m',2p-m} = \mathfrak{F}_{m'-p',p-m}$ into $\mathfrak{F}_{m'-p',m-p} = \mathfrak{F}_{m',m}$. Due to what has been said in the previous point about the condition under which the charges Q_m are injection of Verma modules, one can verify the *BRST property*

$$Q_m Q_{p-m} = 0. \tag{2.103}$$

iii) The spaces of BRST states (cohomology group)

$$B_{m',m} = Ker\, Q_m / Im\, Q_{p-m} \tag{2.104}$$

are isomorphic, as a Virasoro modules, to the irreducible highest weight modules $\tilde{V}_{m',m}$. Notice that a complete analysis would require computing the BRST cohomology in the case where $m^{(\prime)} \neq 0 \, mod \, p^{(\prime)}$. For this it would be necessary to consider the more complicated BRST complex $C_{m',m} = \bigoplus_{j\in\mathbb{Z}} \mathfrak{F}_{m',n-2j} \oplus \bigoplus_{j\in\mathbb{Z}} \mathfrak{F}_{-m',m-2j}$,

$$\ldots \xrightarrow{Q_m} \mathfrak{F}_{m',-m+2p} \xrightarrow{Q_{p-m}} \mathfrak{F}_{m',m} \xrightarrow{Q_m} \mathfrak{F}_{m',-m} \xrightarrow{Q_{p-m}} \mathfrak{F}_{m',m-2p} \xrightarrow{Q_m} \ldots \tag{2.105}$$

However, the conclusions would not change. Indeed, if one denotes $Q^{(2j)}$ the BRST charge $Q_m : \mathfrak{F}_{m',m-2jp} \to \mathfrak{F}_{m',-m-2jp}$ and $Q^{(2j)}$ the BRST charge $Q_{p-m} : \mathfrak{F}_{m',-m-2jp} \to \mathfrak{F}_{m',m-2(j+1)p}$, the BRST property $Q^{(2j)}Q^{(2j-1)} = 0$ is again fulfilled. Furthermore, the only non-vanishing cohomology group among those defined as

$$H^j(C_{m',m}) = Ker\, Q^{(2j)} \big/ Im\, Q^{(2j-1)} \tag{2.106}$$

is H^0, the space of BRST states already introduced.

The last important result is the BRST-invariance of SVO's, *i.e.*, the statement that, on $\mathfrak{F}_{m',m}$,

$$Q_{m+n-2r-1} V^{r',r}_{n',n}(z) = e^{2\pi i \alpha_{n',n}\alpha_+(m+n-2r-1)} V^{r',n-r-1}_{n',n}(z) Q_m \,, \tag{2.107}$$

where the phase factor is generated by the "braiding" of the vertex operators V_{α_+} with $V_{\alpha_{n',n}}$. Eq.(2.107) is illustrated by the (up to a phase commutative[5]) diagram

$$\begin{array}{ccc} \mathfrak{F}_{m',m} & \xrightarrow{V^{r',r}_{n',n}(z)} & \mathfrak{F}_{m'+n'-2r'-1,\, m+n-2r-1} \\ \Big\downarrow Q_m & & \Big\downarrow Q_{m+n-2r-1} \\ \mathfrak{F}_{m',-m} & \xrightarrow{V^{r',n-r-1}_{n',n}(z)} & \mathfrak{F}_{m'+n'-2r'-1,\, -m-n+2r+1} \end{array} \tag{2.108}$$

The physical meaning of the commutation (2.107) is that SVO's are well-defined on the BRST states and maps BRST states to BRST states. The interpretation of these properties in the context of Virasoro modules leads to identify, up to a constant, the SVO's $V^{r',r}_{n',n}$ and the conformal blocks $\varphi^{l',l}_{n',n\; m',m}$, with $l^{(\prime)} = n^{(\prime)} + m^{(\prime)} - 2r^{(\prime)} - 1$. An exact coincidence arises if SVO's are normalized by imposing

$$N^{l',l}_{n',n\; m',m} = \left\langle \alpha_{l',l} \middle| V^{r',r}_{n',n}\, \alpha_{m',m} \right\rangle = 1\,. \tag{2.109}$$

Moreover, $N^{l',l}_{n',n\; m',m}$ can be computed explicitly (see [12] and reference therein) and from their expression one achieves the conditions under which $V^{r',r}_{n',n} \neq 0$. Indeed

[5] The phase could be eliminated by rescaling SVO's.

the vanishing of normalization constants implies the vanishing of the corresponding SVO's themselves on the BRST states, since all their matrix elements can be expressed in terms of $N^{l',l}_{n',n\ m',m}$ using the conformal properties of SVO's. Therefore, requiring $N^{l',l}_{n',n\ m',m} \neq 0$, one gets the constraints

$$n^{(')} - m^{(')} + 1 \leq l^{(')} \leq min(n^{(')} + m^{(')} - 1,\, 2p^{(')} - n^{(')} - m^{(')} - 1)\,, \tag{2.110}$$

while further conditions[6] arise from the global projective invariance of the three-point correlators. For this reason the allowable values restrict to

$$|n^{(')} - m^{(')}| + 1 \leq l^{(')} \leq min(n^{(')} + m^{(')} - 1,\, 2p^{(')} - n^{(')} - m^{(')} - 1)\,. \tag{2.111}$$

This, together with the neutrality condition

$$n^{(')} + m^{(')} - l^{(')} - 1 = 0\; mod\, 2\,, \tag{2.112}$$

is in agreement with the fusion rules of conformal families reported in equation (2.86).

[6]One has to require that $N^{l',l}_{n',n\ m',m}$ do not vanish for every permutation of the indices (n', n), (m', m) and (l', l)

Chapter 3

Quantum group structure in minimal models

Although till now quantum group theory and QCFT are described separately, some similarity can be already stressed. In sections §1.2 and §1.3 QYBE's established a deep connection between quantum groups and braid group representations. On the other hand, the decomposition (2.64) of any correlators into many-valued factors [the conformal block functions $\mathcal{F}^{l_1 l_2 \ldots l_n}(\vec{p}\,|z_1, z_2, \ldots, z_n)$] leads straightforwardly to the introduction of a braid statistics. Indeed, to define block functions one has to choose a particular ordering of their arguments (for example $|z_1| > |z_2| > \ldots > |z_n|$) and then, via analytic continuation, extend their definition to different domains. Hence, the symmetry of single-valued correlation functions with respect to the permutation of their arguments, is replaced by the invariance under braiding of paths of analytic continuation. Evidently, the importance of the role played by braid groups in both these fields is not accidental. In a local QFT on a space-time of dimension $d \geq 3$, a symmetry is described by a group of unitary transformations which act on the field algebra (or, equivalently, on the tensor product algebra of observable algebra representations) by automorphisms commuting with the permutation of the factors in a product of space-like separated fields. However, if $d = 2$, *i.e.* on such space-times where the space-like complement of a point is not connected, the possibility to find fields with statistics determined by a representation of a braid group cannot be excluded. Therefore, in order to describe their symmetry a more general algebraic structure is needed. The idea that this structure can be identified with quantum groups is supported by the properties of the category of quasitriangular Hopf algebra representations, whose main characteristics were summarized in section §1.4. The crucial point is that such a deformation of a "classical" group transforms a cocommutative structure, linked to "local" permutation statistics, into a non-cocommutative one, with some "structure constants" (the universal $\mathfrak{R}$-matrix) which constrain the cocommutation relations.

In the sequel, minimal models will provide an example of such physical application of quantum groups. After having pointed out some features which seem to suggest a $U_q(\mathfrak{sl}_2)$ hidden symmetry, the attempts till now produced to construct an explicit quantum group action on the chiral field algebra of the theory will be briefly described. The limits of the argumentations given in the sequel are due expecially to the inadequate comprehension of quantum group representation theory when the parameter of deformation is a root of unity[1], $q = e^{2\pi i n}$. Indeed, this is precisely the case of those $U_q(\mathfrak{sl}_2)$ representations related to minimal models and, more generally, to RCFT, as one can argue from the explicit expression of the monodromy matrices. The basic problem arising when $q = e^{2\pi i/p}$ is that $(E_\pm)^p = 0$ and this generates null vectors in some representations. Therefore, the results achieved in Chapter 1, according to which the representation theory of $U_q(\mathfrak{sl}_2)$ for q generic is obtained as a deformation of the Borel-Weyl construction for "classical" groups, can be extended straightforwardly to the present case only in some favorable circumstances. Generally, the tensor product of irreducible representations might not be completely reducible, which means that some of the representation appearing in this decomposition might be reducible, but not fully reducible. The problem can be eluded if one imposes particular conditions on what representation can be considered "integrable" and modifies the definition of the tensor product of representations. More precisely, the requirements are that in the tensor product of fundamental representations [spin $\frac{1}{2}$ representations of $U_q(\mathfrak{sl}_2)$] one only keeps those highest weight vectors annihilated by E_+ and, at the same time, not in the image of $(E_+)^{p-1}$. This restricts the possible representations to those with spin smaller or equal to $(p-2)/2$. As a consequence, in all the relations reported in §1.4.5 involving summations on the allowable representations, the series will result truncated at this value of j. It must be again stressed that this is an *ad hoc* construction and a conclusive theory on quantum group representations when q is a root of unity has not yet been formulated.

3.1 Chiral vertex operators

On the ground of what already said in the previous chapter, minimal models are characterized by a local chiral algebra $\mathcal{A}$, the enveloping algebra of the Virasoro algebra Vir, and by a space of states

$$\mathfrak{F} = \bigoplus_{\mathrm{N},\bar{\mathrm{N}} \in I_c} \mathfrak{F}_{\mathrm{N}} \otimes \overline{\mathfrak{F}_{\bar{\mathrm{N}}}}\,, \tag{3.1}$$

being $\mathfrak{F}_{\mathrm{N}}$ an irreducible degenerate highest weight module of $\mathcal{A}$, acting on it as $L_n \otimes 1$. Analogously, $\overline{\mathfrak{F}_{\bar{\mathrm{N}}}}$ is an irreducible degenerate highest weight module of $\bar{\mathcal{A}}$, the antiholomorphic copy of the chiral algebra $\mathcal{A}$, acting on it as $1 \otimes \bar{L}_n$. The index N is an abbreviate notation for (n', n) and I_c is a finite set of allowable values related to the central charge $c < 1$. The algebra $\mathcal{A}$ can be constructed by imposing some locality assumptions as the holomorphic operator algebra which generates the vacuum module $\mathfrak{F}_0 = \mathfrak{F}_{(1,1)}$ (remind that $\Delta_{1,1} = 0$).

[1]Throughout, the deformation parameter will be the square of that considered in Chapter 1.

In this context, in the attempt to give a precise meaning to conformal blocks and conformal block functions, the concept of chiral vertex operators must be introduced. With this purpose, let us consider three representations of the chiral algebra $\mathcal{A}$, labelled by j_1, j_2 and j. Furthermore, suppose that the respective modules are localized at the points z_1, z_2 and ∞. Then, a chiral vertex operator (CVO) of type $\begin{pmatrix} j \\ j_1\, j_2 \end{pmatrix}$ is defined as an intertwiner operator

$$\begin{pmatrix} j \\ j_1\; j_2 \end{pmatrix}_{z_1,z_2} : \mathfrak{F}^{(z_1)}_{j_1} \otimes \mathfrak{F}^{(z_2)}_{j_2} \to \mathfrak{F}^{(\infty)}_{j} . \tag{3.2}$$

The intertwining property is expressed by the relation

$$\begin{aligned} \rho^j(O_n) \begin{pmatrix} j \\ j_1\; j_2 \end{pmatrix}_{z_1,z_2} &(v^{j_1}(z_1) \otimes v^{j_2}(z_2)) = \\ = \begin{pmatrix} j \\ j_1\; j_2 \end{pmatrix}_{z_1,z_2} &\rho^{j_1} \otimes \rho^{j_2} \left(\Delta_{z_1 z_2}(O_n)\right)(v^{j_1}(z_1) \otimes v^{j_2}(z_2)) , \end{aligned} \tag{3.3}$$

where $v^{j_1}(z_1) \in \mathfrak{F}^{(z_1)}_{j_1}$, $v^{j_2}(z_2) \in \mathfrak{F}^{(z_2)}_{j_2}$ and O_n is the n^{th} mode of an operator $O(z)$ belonging to the chiral algebra $\mathcal{A}$ of dimension Δ, $O(z) = \sum_n O_n z^{-n-\Delta}$. Moreover, ρ^{j_1}, ρ^{j_2} and ρ^j represent the actions on the modules $\mathfrak{F}^{(z_1)}_{j_1}$, $\mathfrak{F}^{(z_2)}_{j_2}$ and $\mathfrak{F}^{(\infty)}_{j}$, respectively. In addition, CVO's must satisfy the equation of motion

$$\frac{d}{dz} \begin{pmatrix} j \\ j_1\; j_2 \end{pmatrix}_{z,0} (v^{j_1}(z) \otimes v^{j_2}(0)) = \begin{pmatrix} j \\ j_1\; j_2 \end{pmatrix}_{z,0} (L_{-1} v^{j_1}(z) \otimes v^{j_2}(0)) . \tag{3.4}$$

In defining the tensor product action $\rho^{j_1} \tilde{\otimes} \rho^{j_2}$, the coproduct Δ_{z_1,z_2} has been introduced. With z_2 identified to the origin, contour deformation allows to write

$$\Delta_{z,0}(O_n) = \sum_{k=0}^{\infty} \begin{pmatrix} n+\Delta-1 \\ k \end{pmatrix} z^{n+\Delta-1+k}\, O_{1+k-\Delta} \otimes 1 + 1 \otimes O_n , \tag{3.5}$$

being $O(z) \in \mathcal{A}$ an operator of given dimension Δ. Following Moore and Seiberg [27], one can observe that eq.(3.5) does not define a coassociative coproduct, while the role of this property in the case of Hopf algebras had been stressed in Chapter 1. However, admitting the possibility for the functorial isomorphisms $\Phi_{X,Y,Z} : X\tilde{\otimes}(Y\tilde{\otimes}Z) \to (X\tilde{\otimes}Y)\tilde{\otimes}Z$ to be no longer trivial, the coproduct (3.5) still allows to introduce a tensor product algebra in the category of representations of the chiral algebra $\mathcal{A}$. As a particular case of the definition (3.5),

$$\Delta_{z,0}(L_n) = \begin{cases} (z^{n+1}L_{-1} + z^n(n+1)L_0 + \cdots + L_n) \otimes 1 + 1 \otimes L_n & \textit{for } n \geq -1 \\ (z^{n+1}L_{-1} + z^n(n+1)L_0 + \cdots) \otimes 1 + 1 \otimes L_n & \textit{for } n < -1 . \end{cases} \tag{3.6}$$

The requirement that CVO's obey eq.(3.3), together with the intertwining condition (3.4), implies that the map ${}_j\phi^{j_2}_{v_0^{j_1}}(z) : \mathfrak{F}^{(0)}_{j_2} \to \mathfrak{F}^{(\infty)}_j$, defined as

$$ {}_j\phi^{j_2}_{v_0^{j_1}}(z) = \begin{pmatrix} & j & \\ j_1 & & j_2 \end{pmatrix}_{z,0} (v_0^{j_1}(z) \otimes \cdot\,), \tag{3.7} $$

is a primary field of weight Δ_i, whenever $v_0^{j_1}$ is the highest weight vector of $\mathfrak{F}_{j_1}$. Indeed, by definition, for $n \geq 1$,

$$ L_n \circ {}_j\phi^{j_2}_{v_0^{j_1}}(z)(v^{j_2}) = {}_j\phi^{j_2}_{v_0^{j_1}}(z)(L_n\, v^{j_2}) + {}_j\phi^{j_2}_{(z^{n+1}L_{-1} + (n+1)z^n L_0 + \cdots + L_n)v_0^{j_1}}(z)(v^{j_2}), \tag{3.8} $$

so that, if $L_0 v_0^{j_1} = \Delta_{j_1} v_0^{j_1}$ and $L_n v_0^{j_1} = 0$ for $n > 0$, then

$$ \left[L_n, {}_j\phi^{j_2}_{v_0^{j_1}}(z)\right] = \left(z^{n+1}\frac{d}{dz} + (n+1)z^n\Delta_{j_1}\right) {}_j\phi^{j_2}_{v_0^{j_1}}(z). \tag{3.9} $$

Consequently, conformal blocks can be identified with the operators in eq.(3.7) and conformal block functions with the vacuum expectation values of products of such operators,

$$ \left\langle {}_0\phi^{p_1}_{v_0^{j_1}}(z_1)\, {}_{p_1}\phi^{p_2}_{v_0^{j_2}}(z_2) \cdots {}_{p_{n-1}}\phi^{0}_{v_0^{j_n}}(z_n) \right\rangle . \tag{3.10} $$

Furthermore, since a module $\mathfrak{F}_j$ is completely characterized by the corresponding primary field, every conformal block determines its associated CVO.

The introduction of CVO's allows to formulate the bootstrap program in a different language. Indeed, one can show that the vector space spanned by the CVO's of type $\begin{pmatrix} & j & \\ j_1 & & j_2 \end{pmatrix}$ and denoted $V^j_{j_1 j_2}$, provides a representation of the centralizer of the chiral algebra $\mathcal{A}$ acting on the tensor product $\mathfrak{F}_{j_1} \tilde{\otimes} \mathfrak{F}_{j_2}$. In order to better explain this statement, consider the centralizer $C(A, V^{\otimes N})$ of an algebra A in $End(V^{\otimes N})$, *i.e.* those linear maps $V^{\otimes N} \to V^{\otimes N}$ that commute with the action of A[2]. It is easy to see that, then, $C(A, V^{\otimes N})$ coincides with $Mor(V^{\otimes N}, V^{\otimes N})$, the set of intertwiners from $V^{\otimes N}$ to $V^{\otimes N}$. In addition, note that using the decomposition of $V \tilde{\otimes} V$ [see eq.(1.88)] repeatedly, one obtains $V^{\otimes N} \cong \bigoplus_j W^j \tilde{\otimes} V_j$, where W^j is a vector space on which A acts trivially. Consequently, every intertwiner $\phi \in Mor(V^{\otimes N}, V^{\otimes N})$, admitting a spectral decomposition $\phi = \tilde{\bigoplus}_j \phi^j \tilde{\otimes} id$ (it commutes with the action of A), can be uniquely associated to the endomorphism $\phi^j \in End(W^j)$. This defines an action of the centralizer on W^j. On the other hand it is evident the relation between W^j and the space of the intetwiners $Mor(V^{\otimes N}, V_j)$, thought as an A-module[3]. Hence, once the space $V^j_{j_1 j_2}$ is identified with the set of intertwiners $Mor(\mathfrak{F}_{j_1} \tilde{\otimes} \mathfrak{F}_{j_2}, \mathfrak{F}_j)$, the previous statment results clarified. On these grounds BPZ axioms can be expressed in terms

[2]Such a definition understand that the A-modules can be tensor producted, or in a categorial language, that ${}_A\mathfrak{M}$ is a monoidal category.

[3]Actually, what can be generally pointed out is the existence of an homomorphism between $Mor(V^{\otimes N}, V_j)^*$ and W^j. In the context of the following discussion these two space are assumed to be identified.

of operations on the spaces $V^j_{j_1 j_2}$ and their content amount to the existence of maps B and F (the braiding and fusion moves) between the $V^j_{j_1 j_2}$'s which obey identities such that the representations of the chiral algebra $\mathcal{A}$ form some kind of rigid Abelian tensor category. Hence, the bootstrap program can be viewed as a reconstruction problem. This idea was developed by More and Seiberg [26,27] and by Alvarez-Gaumé, Gómez and Sierra [1,2,3]. They consider the basic transformations

$$\begin{pmatrix} j_1 \\ j_2 \; p \end{pmatrix}_{z,0} \begin{pmatrix} p \\ j_3 \; j_4 \end{pmatrix}_{w,0} \xrightarrow{B} \begin{pmatrix} j_1 \\ j_2 \; q \end{pmatrix}_{w,0} \begin{pmatrix} q \\ j_3 \; j_4 \end{pmatrix}_{z,0} \tag{3.11}$$

$$\begin{pmatrix} j_1 \\ j_2 \; p \end{pmatrix}_{z,0} \begin{pmatrix} p \\ j_3 \; j_4 \end{pmatrix}_{w,0} \xrightarrow{F} \begin{pmatrix} j_1 \\ q \; j_4 \end{pmatrix}_{w,0} \begin{pmatrix} q \\ j_2 \; j_3 \end{pmatrix}_{z,0} \tag{3.12}$$

The equations fulfilled by such maps follow as a consequence of associativity of the OPE algebra discussed in §2.2. Indeed, the role of the conformal block functions $\mathcal{F}^{lk}_{nm}(p\,|z)$ is here played by the corresponding CVO's.

A further possibility disclosed by this kind of approach consists in interpreting the equation satisfied by the maps B and F as a consequence of a hidden quantum group symmetry. Indeed, as it will be shown in the next section, the explicit expression of braiding and fusing matrices, representing the moves B and F, seems to indicate a connection with the quantum group $U_q(\mathfrak{sl}_2)$ where q is a root of unity. The natural procedure to make such a symmetry manifest involves the introduction of an internal degree of freedom, characterized by a new quantum number, m, on which the quantum group generators will act. This can be realized by associating to each conformal block operator $\phi_{v^{j_1}_0}(z)$ a suitable U_q-*vertex operator* (U_q-covariant field). Then the aim of this construction is to find that more information about the two-dimensional CFT can be coded in a chiral (say right movers) model when the quantum group symmetry is required. In particular, pairing conformal block operators with U_q-vertex operators should the action of the braid group on the correlation functions of U_q-invariant fields (see ref.[33] [25] for more details). A different approach will be explain in §3.3, where, following Gómez and Sierra [17,18], the quantum number m arises in a more natural way, rather than from an *ad hoc* prescription.

3.2 Monodromy matrices

The aim of this section is to outline the computation of the monodromy matrices, describing the analytic continuation properties of conformal blocks in minimal models. The knowledge of the explicit expression of such monodromy matrices will provide significant indication on the connection with the quantum group $U_q(\mathfrak{sl}_2)$. This is only an example among many other "phenomenological" observations about the close relation between RCFT and quantum groups. The possibility to construct knot invariants and their role in both these fields, the coincidence between the monodromy matrices of level k $SU(2)$-WZW model and the Racah 6-j symbol of the quantum group $U_q(\mathfrak{sl}_2)$ for $q = e^{\frac{2\pi i}{k+n}}$, might be two further examples. Even if in the next section an attempt to give a theoretical explanation on the subject (limited to the case of minimal models) will be described, and in the previous section different attempt has been mentioned, a satisfactory solution of the problem has not been yet found.

Following Felder, Frölich and Keller [13], we will use the Coulomb gas representation of minimal models, whose main feature are discussed in §2.4. Although here such a representation will be exploited only as a computational method, the next section will show its peculiarity in making possible the definition of a quantum group action on the operator algebra. In this context, conformal block operators are identified with the SVO's of eq.(2.94). In order to simplify the expression of the final result, it is convenient to introduce a phase in the definition given in eq.(2.94). Accordingly,

$$V^{(l',l)}_{(n',n)(m',m)}(z) = e^{i\theta^{(l',l)}_{(n',n)(m',m)}} V^{r',r}_{n',n}(z) \tag{3.13}$$

denotes an operator which maps BRST states in $\mathfrak{F}_{m',m}$ to BRST states in $\mathfrak{F}_{l',l}$, being $l^{(')} = n^{(')} + m^{(')} - 2r^{(')} - 1$, due to neutrality condition (see §2.4 for a detailed discussion). Here the phase $\theta^{(l',l)}_{(n',n)(m',m)}$ is

$$\theta^{(l',l)}_{(n',n)(m',m)} = \\ = \frac{\pi}{2}\{(2n'+1)r + (2n+1)r'\} - \pi\alpha_-^2 r'(r'-m') - \pi\alpha_+^2 r(r-m)\,. \tag{3.14}$$

The monodromy matrices are defined by the equation

$$V^A_{MB}(z)V^B_{NC}(w) = \sum_D \mathcal{B}(A,M,N,C)_{BD}\, V^A_{ND}(w)V^D_{MC}(z)\,, \tag{3.15}$$

where the abbreviate notation $M = (m', m)$, $N = (n', n)$, etc. has been introduced. The l.h.s. of eq.(3.15), valid for $|w| > |z|$ and $0 \leq arg\ z,\ arg\ w < 2\pi$ along a path such that z circumvents w counterclockwise. From the OPE of primary fields and, in particular, of degenerate primary fields, one obtains the basic results which enable to perform the calculation. They are the braiding-commuting relations for vertex operators,

$$V_\alpha(z)V_\beta(w) = e^{2\pi i\alpha\beta}V_\beta(w)V_\alpha(z)\,, \tag{3.16}$$

with the same convention as in eq.(3.15), and the fusion relations

$$\begin{aligned} V^{(l',l)}_{(n'+1,n)(m',m)}(z) &= \\ &= (-1)^{\frac{1}{2}(n+m-l-1)} \lim_{w\to z}(w-z)^{-2\alpha_{21}\alpha_{n',n}} V^{(l',l)}_{(2,1)(l'\pm1,l)}(w) V^{(l'\pm1,l)}_{(n',n)(m',m)}(z)\,, \\ V^{(l',l)}_{(n',n+1)(m',m)}(z) &= \\ &= (-1)^{\frac{1}{2}(n'+m'-l'-1)} \lim_{w\to z}(w-z)^{-2\alpha_{12}\alpha_{n',n}} V^{(l',l)}_{(1,2)(l',l\pm1)}(w) V^{(l',l\pm1)}_{(n',n)(m',m)}(z)\,. \end{aligned} \tag{3.17}$$

Here the value of α_-^2 $(= \alpha_+^{-2})$ is such to make the limit for $w \to z$ convergent [see eq.(2.98)]. The crucial point in the derivation of eq.(3.17) is that the contours over which the screening operators $J_\pm(z)$ (*i.e.*, the vertex operators with charge α_- or α_+) are integrated can be interchanged without affecting the value of the SVO (see [13]).

Consequently, monodromy matrices can be computed recursively in terms of the "elementary" monodromy matrices, related to the vertex operators $V_{2,1}(z)$ and $V_{1,2}(z)$. Indeed, it easy to deduce from eq.(3.15) the relations

$$\begin{aligned}
&\mathcal{B}(A,\, M+(1,0),\, N,\, C)_{BD} = \\
&\qquad = (-1)^{\frac{1}{2}(a-b+c-d)} \sum_{D_1} \mathcal{B}(A,\, (2,1),\, N,\, D_1)_{A_1\,D} \mathcal{B}(A_1,\, M,\, N,\, C)_{B\,D_1} \\
&\mathcal{B}(A,\, M+(0,1),\, N,\, C)_{BD} = \\
&\qquad = (-1)^{\frac{1}{2}(a'-b'+c'-d')} \sum_{D_1} \mathcal{B}(A,\, (1,2),\, N,\, D_1)_{A_1\,D} \mathcal{B}(A_1,\, M,\, N,\, C)_{B\,D_1} \\
&\mathcal{B}(A,\, M,\, N+(1,0),\, C)_{BD} = \\
&\qquad = (-1)^{\frac{1}{2}(a-b+c-d)} \sum_{A_1} \mathcal{B}(A,\, M,\, (2,1),\, D_1)_{A_1\,D} \mathcal{B}(A_1,\, M,\, N,\, C)_{B\,D_1} \\
&\mathcal{B}(A,\, M,\, N+(0,1),\, C)_{BD} = \\
&\qquad = (-1)^{\frac{1}{2}(a'-b'+c'-d')} \sum_{A_1} \mathcal{B}(A,\, M,\, (1,2),\, D_1)_{A_1\,D} \mathcal{B}(A_1,\, M,\, N,\, C)_{B\,D_1}\,.
\end{aligned} \tag{3.18}$$

Hence, the problem reduces to the computation of the matrices $\mathcal{B}(L',\,(2,1),\,(2,1),\,L)$, $\mathcal{B}(L',\,(2,1),\,(1,2),\,L)$, $\mathcal{B}(L',\,(1,2),\,(2,1),\,L)$ and $\mathcal{B}(L',\,(1,2),\,(1,2),\,L)$ that will be outlined in the following, taking α_-^2 as an arbitrary complex parameter (the more interesting case of rational α_-^2, will be discuss in the sequel).

1. $\mathcal{B}(L',\,(2,1),\,(2,1),\,L)$. Consider the product of two SVO's of weight $\Delta_{2,1}$ mapping $\mathfrak{F}_L$ to $\mathfrak{F}_{L'}$. Due to the fusion rules expressed in eq.(2.86), the possible values for L' are $L+(2,0)$, L, and $L-(2,0)$ (provided that none of these labels becomes zero). In the first case these SVO's coincide with the vertex operators themselves, with no screening charge, and the monodromy matrix is just a phase,
$$\mathcal{B}(L',\,(2,1),\,(2,1),\,L)_{L+(1,0),\,L+(1,0)} = e^{\pi i\alpha_-^2/2}\,. \tag{3.19}$$
In the second case there is a screening operator, that may be attached to either of the two fields. By deforming the contour integration one can write both sides of eq.(3.15) in terms of the integrals
$$I_{1,2}(z,w) = \int_{\gamma_{1,2}} V_{\alpha_{2,1}}(w) J_-(u) V_{\alpha_{2,1}}(z)\, du\,, \tag{3.20}$$

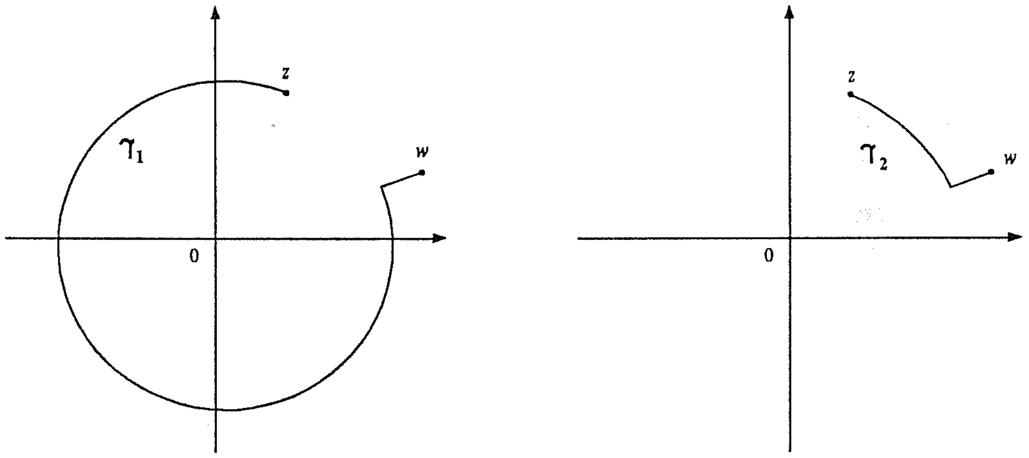


Figure 3.1: *Integration contours in eq.(3.20).*

where $\gamma_{1,2}$ are the contours drawn in figure (1). Then one obtains

$$
\begin{aligned}
V^L_{(2,1),L+(1,0)}(z)V^{L+(1,0)}_{(2,1)L}(w) &= -i\,q_-^{-\frac{1}{4}-\frac{l'}{2}}\left(q_-^{\frac{1}{2}}I_1 + q_-^{-\frac{1}{2}}I_2\right)\\
V^L_{(2,1),L-(1,0)}(z)V^{L-(1,0)}_{(2,1)L}(w) &= -i\,q_-^{-\frac{1}{4}}\left(q_-^{\frac{1-l'}{2}}I_1 + q_-^{-\frac{1-l'}{2}}I_2\right)\\
V^L_{(2,1),L+(1,0)}(w)V^{L+(1,0)}_{(2,1)L}(z) &= -i\left(q_-^{\frac{-l'}{2}}I_1 + q_-^{\frac{l'}{2}}I_2\right)\\
V^L_{(2,1),L-(1,0)}(w)V^{L-(1,0)}_{(2,1)L}(z) &= -i\,q_-^{-\frac{l'}{2}}\left(I_1 + I_2\right)\,,
\end{aligned}
\tag{3.21}
$$

where $q_- = e^{2\pi i\alpha_-^2}$. The elimination of I_1 and I_2 from the above equations gives the braid matrix

$$
\begin{aligned}
\mathcal{B}(L,\,(2,1),\,(2,1),\,L)_{L\pm(1,0),L\pm(1,0)} &= \mp q_-^{-\frac{1}{4}\mp\frac{l'}{2}}\,\frac{q_-^{\frac{1}{2}} - q_-^{-\frac{1}{2}}}{q_-^{\frac{l'}{2}} - q_-^{-\frac{l'}{2}}}\\
\mathcal{B}(L,\,(2,1),\,(2,1),\,L)_{L\pm(1,0),L\mp(1,0)} &= \mp q_-^{-\frac{1}{4}}\,\frac{q_-^{\frac{l'\pm1}{2}} - q_-^{-\frac{l'\pm1}{2}}}{q_-^{\frac{l'}{2}} - q_-^{-\frac{l'}{2}}}\,.
\end{aligned}
\tag{3.22}
$$

The same analysis may be carried out in third case, $L' = L-(2,0)$, where one has two contours of integration, to be split into parts as above. The simple result is

$$
\mathcal{B}(L-(2,0),\,(2,1),\,(2,1),\,L)_{L-(1,0),L-(1,0)} = e^{\pi i\alpha_-^2/2}\,. \tag{3.23}
$$

2. $\mathcal{B}(L',\,(2,1),\,(1,2),\,L)$. Because of the interchangeability of the contours on

which the screening operators $J_-(z)$ and $J_+(z)$ are integrated and

$$\oint_{C_w} J_+(z)V_{\alpha_{2,1}}dz = 0 = \oint_{C_w} J_-(z)V_{\alpha_{1,2}}dz\,, \tag{3.24}$$

due to the regularity of the operator product expansion, one gets no contributions from screening operators and one is left with the phase $e^{2\pi i\alpha_{2,1}\alpha_{1,2}} = i^{-1}$. Thus,

$$\mathcal{B}(L', (2,1), (1,2), L)_{AB} = i^{-1}. \tag{3.25}$$

3. $\mathcal{B}(L', (1,2), (2,1), L)$. From an analogous discussion as at point 2, one finds

$$\mathcal{B}(L', (2,1), (1,2), L)_{AB} = i^{-1}. \tag{3.26}$$

4. $\mathcal{B}(L', (1,2), (1,2), L)$. The computation repeats that at point 1 and, therefore, the result has the same expression as in eq.(3.22), without primes and substituting q_- with $q_+ = e^{2\pi i\alpha_+^2}$.

All the above results can be cast in a more concise form if one observes that the monodromy matrices of SVO's have the almost factorized expression

$$\begin{aligned}
&\mathcal{B}(A,\, M,\, N,\, C)_{BD} = \\
&\quad = i^{-(m'-1)(n-1)-(n'-1)(m-1)}(-1)^{\frac{1}{2}(a-b+c-d)(n'+m')+\frac{1}{2}(a'-b'+c'-d')(n+m)} \times \\
&\times b_-(a',\, m',\, n',\, c')_{b'\,d'} b_+(a,\, m,\, n,\, c)_{b\,d}\,.
\end{aligned} \tag{3.27}$$

This is the consequence of the existence in the $c < 1$ theory of two closed subalgebras, the so-called thermal subalgebras, corresponding to the weights $\Delta_{1,n}$ or $\Delta_{n',1}$, respectively. From the above calculation one finds that the non-vanishing matrix elements of the "elementary" b_+-matrix are

$$\begin{aligned}
b_+(a,\, 1,\, n,\, c)_{a\,c} &= b_+(a,\, m,\, 1,\, c)_{c\,a} = 1 \\
b_+(l \pm 2,\, 2,\, 2,\, l)_{l\pm1\, l\pm1} &= q_+^{\frac{1}{4}} \\
b_+(l,\, 2,\, 2,\, l)_{l\pm1\, l\pm1} &= \mp q_+^{-\frac{1}{4}\mp\frac{l}{2}} \frac{[1]_+}{[l]_+} \\
b_+(l,\, 2,\, 2,\, l)_{l\pm1\, l\mp1} &= q_+^{-\frac{1}{4}} \frac{[l\pm1]_+}{[l]_+}\,,
\end{aligned} \tag{3.28}$$

where $[l]_+ = q_+^{\frac{l}{2}} - q_+^{-\frac{l}{2}}$. The other b_+-matrices can be determined by the recursive relations

$$\begin{aligned}
b_+(a,\, m+1,\, n,\, c)_{b\,d} &= \sum_{d_1 \geq 1} b_+(a,\, 2,\, n,\, d_1)_{a_1\, d} b_+(a,\, m,\, n,\, c)_{b\, d_1} \\
b_+(a,\, m,\, n+1,\, c)_{b\,d} &= \sum_{d_1 \geq 1} b_+(a,\, m,\, 2,\, c_1)_{b\, d_1} b_+(d_1,\, m,\, n,\, c)_{c_1\, d}\,,
\end{aligned} \tag{3.29}$$

for any choice of a_1 and c_1 compatible with the fusion rules. The b_--matrices are given by the same formulas with primes and the replacement $q_+ \rightarrow q_-$, $[\ ]_+ \rightarrow [\ ]_-$. considering now the rational case $\alpha_-^2 = p/p'$, one can objects that, when α_-^2 takes rational values, divergent matrix elements might appear, as it evident from eq.(3.28). For instance, even if in the discrete series of minimal models the labels (n', n) belong to the family $K = \left\{1 \leq n' \leq p' - 1, 1 \leq n \leq p - 1\right\}$, the braiding of two SVO's with weight $\Delta_{2,1}$ or $\Delta_{1,2}$ generates SVO's $V^L_{(2,1),M}$ and $V^L_{(1,2),M}$ with $M \in K$ and $L \in \bar{K} = \left\{0 \leq n' \leq p', 0 \leq n \leq p\right\}$ or $M \in \bar{K}$ and $L \in K$. On the other hand, consistency requires that the braiding of primary fields of the family K does not generates different fields. However, as it is pointed out in ref.[13], the BRST cohomology of Fock spaces $\mathfrak{F}_L$ vanishes for $L \in \bar{K} - K$. Therefore, the same happens for BRST invariant operators mapping from or to $\mathfrak{F}_L$, as $V^L_{(2,1),M}$ and $V^M_{(2,1),L}$, etc. are, when such operators are applied on BRST states. This is sufficient to conclude that no divergences arise for the "elementary" monodromy matrices and, due to the recursive formulas (3.18), such an argument answers to the objection also in the general case.

Such explicit expressions of the b_+ and b_--matrices allow Felder, Frölich and Keller to point out their coincidences, up to irrelevant constants, with the Boltzmann weights of critical SOS models in a special limit of the spectral parameter. These weights are proportional to the generalized 6-j symbols of the quantum group $U_q(\mathfrak{sl}_2)$. This result can be justified reminding the pentagon identity of eq.(1.105), where the role of the generalized CG coefficients is played by SVO's.

3.3 A quantum group symmetry in minimal models

As already hinted, in principle one can try to reconstruct the quantum group acting on CVO's from the knowledge of the monodromy matrices. However, the solution of this problem may not be so simple and, moreover, such a construction does not clarify the relation of the quantum group generators with the chiral algebra. A different approach is that proposed by Gómez and Sierra [18], who developed an ideas already expressed in previous works [1,2,3]. They based their derivation on the Coulomb gas version of minimal models. Indeed, such a representation of the chiral operator algebra allows them to define a suitable space of SVO's associated to a given primary field, on which the action of a quantum group Q can be naturally built. Due to a different choice of the integration paths, the physical meaning of this new kind of SVO's does not coincides with that of the operators introduced in §2.4, where their identification with conformal blocks was shown. The main characteristic of the SVO's considered by Gómez and Sierra consists in their non-locality (as the authors observe, they remind in some formal sense Mandelstam's operators of gauge theories). The importance of this properties becomes evident when one ponders on

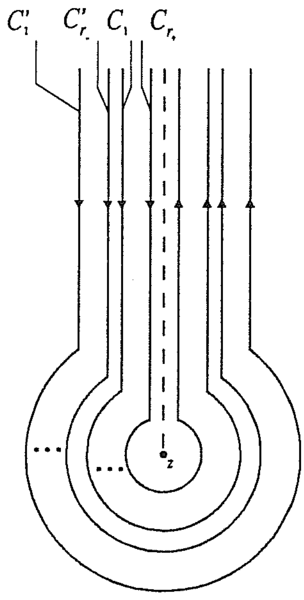


Figure 3.2: *Integration contours in the definition of the non-local screened vertex operators (3.30).*

the non-locality of the braid statistics. The consequent conclusion is that quantum group generators should be non-local functionals in the chiral algebra of the theory. In the sequel, entering in more details about the idea of Gómez and Sierra, this supposition will be confirmed by the identification of the e and f generators of the quantum group with contour creation and annihilation operators, respectively.

In defining SVO's [see eq.(2.94)], the integration contours were chosen so to make possible, already at the operator level, their interpretation as conformal blocks. However, the monodromy properties of the integrand seem to suggest the choice depicted in figure (3.2), where the contours surround the cut (dashed line) going from z to $i\infty$. Therefore, associated to these paths of integration, one define the *non-local screened vertex operators* (SVO$^\infty$'s)

$$e^{\alpha}_{r_-,r_+} = \int J_-(u_1)\dots J_-(u_{r_-})J_+(v_1)\dots J_+(v_{r_+})V_\alpha(z)\prod_i^{r_-} du_i \prod_j^{r_+} dv_j\,. \tag{3.30}$$

Referring to the discussion made about SVO's, note that the integrals in eq.(3.30) must be viewed as an analytic continuation from that region in the α_-^2 plane where they converge. If one neglects in eq.(3.30) the contributions of the integrals around the point z, SVO$^\infty$'s can be cast in the form

$$e^{\alpha}_{r_-,r_+}(z) = [r_-]_{q_-}![r_+]_{q_+}!\prod_{r'=0}^{r_--1}(1-e^{4\pi i\alpha\alpha_-}q_-^{r'})\prod_{r=0}^{r_+-1}(1-e^{4\pi i\alpha\alpha_+}q_+^{r})\,\tilde{e}^{\alpha}_{r_-,r_+}(z)\,, \tag{3.31}$$

where $q_\pm = e^{2\pi i\alpha_\pm^2}$, $[\ ]_{q\pm}$ are the q-integer introduced in eq.(1.25) and

$$\tilde{e}^\alpha_{r_-,r_+}(z) = \mathcal{P}\int_\infty^z J_-(u_1)\dots J_-(u_{r_-})J_+(v_1)\dots J_+(v_{r_+})V_\alpha(z)\prod_i^{r_-} du_i \prod_j^{r_+} dv_j\,. \qquad (3.32)$$

Here, $\mathcal{P}$ denotes the ordering $|u_1| > \dots > |u_{r_-}| > |v_1| > \dots > |v_{r_+}| > |z|$. The q-factorials in eq.(3.31) are due to the the $\mathcal{P}$-ordering, while the other factors come from the braiding between the screening operators $J_\pm(z)$ among themselves, as well as with the vertex V_α. From the expression (3.31), it is easy to conclude that when α belongs to the minimal series, $\alpha = \alpha_{n',n}$, the corresponding SVO$^\infty$'s vanish if $r_- = n'$ or $r_+ = n$. Furthermore, for $\alpha_-^2 = p/p'$, that is in the rational theory, the q-factorials $[r_\pm]_{q\pm}!$ are equal to zero if $r_- = p'$ or $r_+ = p$. Notice that this last result was to be expected on the ground of the condition for the vanishing of those SVO's considered in §2.4. Indeed, the vacuum expectation value of any kind of screened vertex operators is not influenced by the choice of integration contours. As regards the conformal properties of the operators in eq.(3.30), they can be easily determined reminding that the screening vertex operators $J_\pm(z)$ fulfill the commutation relations

$$[L_n, J_\pm(z)] = \frac{d}{dz}\left(z^{n+1}J_\pm(z)\right)\,. \qquad (3.33)$$

Hence, one obtains

$$\begin{aligned} L_n\, e^\alpha_{r_-,r_+}(z) &= e^\alpha_{r_-,r_+}(L_nV_\alpha(z)) + \\ &- \lim_{t\to\infty} t^{n+1}J_+(t)[r_+]_{q_+}(1-e^{4\pi i\alpha\alpha_+}q_+^{r_+-1})e^\alpha_{r_-,r_+-1}(z) + \\ &- \lim_{t\to\infty} t^{n+1}J_-(t)[r_-]_{q_-}(1-e^{4\pi i\alpha\alpha_-}q_-^{r_--1})e^\alpha_{r_--1,r_+}(z)\,, \qquad (3.34)\end{aligned}$$

where $e^\alpha_{r_-,r_+}(L_nV_\alpha(z))$ denotes the SVO$^\infty$ associated to the vertex $L_nV_\alpha(z)$. Therefore, albeit the operators $e^\alpha_{r_-,r_+}(z)$ seem to be fields of well defined conformal dimension, due to the choice of integration contours, boundary terms erase, altering their conformal behaviour. Gómez and Sierra interpret this phenomenon as a "covariantization" in the presence of internal quantum numbers. Indeed, if one believes that quantum group generators act as contour creation and annihilation operators, it is evident that the boundary terms must be related to the e or f generators of the quantum group.

On the ground of this definition, the idea of Gómez and Sierra consists in associating to each vertex operator $V_\alpha(z)$ a space $\mathcal{V}^\alpha$ spanned by the SVO$^\infty$'s $e^\alpha_{r_-,r_+}$, which should be the finite dimensional representation space of a quantum group to be identified. In order to realize such a program, it is necessary to show that:

i) the "representation spaces" generated by the $e^\alpha_{r_-,r_+}$'s for a given α are finite dimensional;

ii) the braiding matrices $\mathcal{B}^{\alpha\beta}$ for the tensor product $\mathcal{V}^{\alpha}\tilde{\otimes}\mathcal{V}^{\beta}$ are related to the universal $\mathfrak{R}$-matrix of a quantum group, according to eq.(1.93);

iii) the representations of e, f and k generators satisfy the right commutation relations;

iv) the quantum group action implements a symmetry of the theory, that is, it commutes with the chiral algebra.

From the previous description of SVO$^{\infty}$'s, one obtains that the requirement (*i*) is fulfilled if

$$\alpha = \alpha_{n',n} = \tfrac{1}{2}(1-n')\alpha_{-} + \tfrac{1}{2}(1-n)\alpha_{+} \tag{3.35}$$

and, in such a case, $e^{\alpha_{n',n}}_{r_-,r_+}$ is different form zero provided that $1 \le r_- \le n' - 1$ and $1 \le r_+ \le n$. Therefore, the dimension[4] of the space $\mathcal{V}^{\alpha}_{n',n}$ is $n'n$. Moreover, a further constraint on the dimension $\mathcal{V}^{\alpha}$'s comes when the rational case is considered and, hence, $1 \le n^{(')} \le p^{(')}$. Notice that the "representations" (p', n) and (n', p) can be ruled out on the basis of unitarity considerations. Indeed, in the Coulomb gas representation, the scalar product is determined by the pairing between the Fock space $\mathfrak{F}_{\alpha}$ and its dual $\mathfrak{F}_{2\alpha_0-\alpha}$. In the case $n = p$, one has $2\alpha_0 - \alpha_{1,p} = \alpha_{p',0}$. Consequently, the space $\mathcal{V}^{\alpha_{p',0}}$, having dimension $n'n = 0$, can be consistently identified with $\{0\}$. Thus, the requirement of a positive well-defined scalar product imposes to truncate the minimal series in the usual way, *i.e.* $1 \le n^{(')} \le p^{(')} - 1$. Assuming the analysis summarized in the introduction of this chapter about the representations of $U_q(\mathfrak{sl}_2)$ for q a root of unity, one can already note some connection with quantum groups. This becomes even more clear if one focus on the thermal operators $V_{1,n}$ (or $V_{n',1}$) which constitute a closed subalgebra of the $c < 1$ theory. Indeed, in the rational case, one has $q_+^p = 1$, a feature which, together with the unitarity condition, produces the constraint $1 \le n \le p-1$. This result seems to establish a one to one correspondence with the modified tensor product of quantum group representations, defined when the deformation parameter is the p^{th} root of unity. In this frame, the operator $e^{\alpha}_{0,p}$ can be interpreted as the null vector arising in $U_q(\mathfrak{sl}_2)$-modules, when its dimension is greater then $p - 1$.

A more cogent argument in favour of a quantum group symmetry is related to the capability to verify point (*ii*). In order to make more immediate the connection with the quantum $\mathfrak{R}$-matrix, consider again the thermal operators $e^{\alpha_{1,n}}_{0,r_+}$, where $\alpha_{1,n} = \frac{1-n}{2}\alpha_+$. Defining the "spin" j as $j = \frac{n-1}{2}$ and the "momenta" $m = j - r_+$, one notes that the dimension of the space $\mathcal{V}^{\alpha_{1,n}}$ is $2j - 1$ and $m = j, \ldots, -j$. The braiding-commutation relation for the SVO$^{\infty}$'s $e^{j}_{m}(z) = e^{\alpha_{1,n}}_{0,r_+}$ can be written as

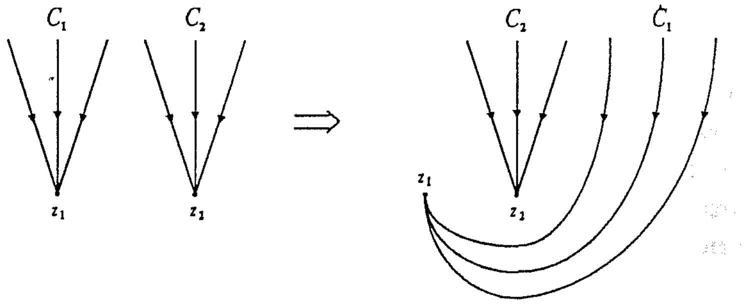
$$e^{j_1}\tilde{\otimes}e^{j_2} = \sum_{m_1',m_2'} e^{j_2}\tilde{\otimes}e^{j_1}\left(R^{j_1 j_2}\right)^{m_1' m_2'}_{m_1 m_2}, \tag{3.36}$$

[4]Notice that the order of the screening operators in the expression of the SVO$^{\infty}$'s is uninfluent.

where the tensor product $\tilde{\otimes}$ is just the product of SVO$^\infty$. Gómez and Sierra compute the matrix $R^{j_1 j_2}$ for arbitrary values of j_1 and j_2, and show that it coincides with the $U_q(\mathfrak{sl}_2)$ universal $\mathfrak{R}$-matrix in the representation $\alpha_{j_1}\tilde{\otimes}\alpha_{j_2}$. An analogous result holds for the thermal operators $e_{r_-,0}^{\alpha_{n',1}}$. Therefore, the quantum group Q acting on the representation spaces $\mathcal{V}^{\alpha_{n',n}}$ seems to consist in the "combination" of two copies of $U_q(\mathfrak{sl}_2)$. To understand how the computation can be worked out, it is convenient to consider the operators $\tilde{e}^\alpha_{\mathbf{r}}$ $[\mathbf{r} = (r_-, r_+)]$ introduced in eq.(3.31), rather than the corresponding SVO$^\infty$'s. This allows to avoid many technical complications, simplifying the needed contour manipulations. On the other hand, their braiding matrix $\tilde{R}^{\alpha\beta}$ can be easily related to the desired result. Then, the general relation

$$R^{\alpha_1\alpha_2} : \mathcal{V}^{\alpha_1}\tilde{\otimes}\mathcal{V}^{\alpha_2} \rightarrow \mathcal{V}^{\alpha_2}\tilde{\otimes}\mathcal{V}^{\alpha_1} \tag{3.37}$$

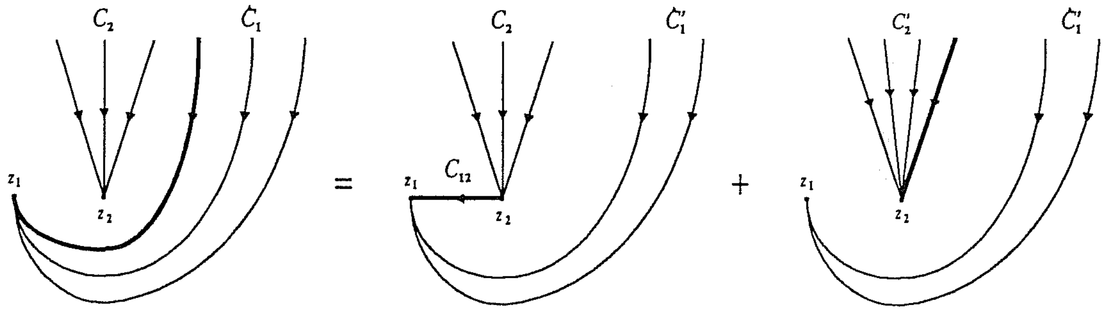
can be graphycally represented as



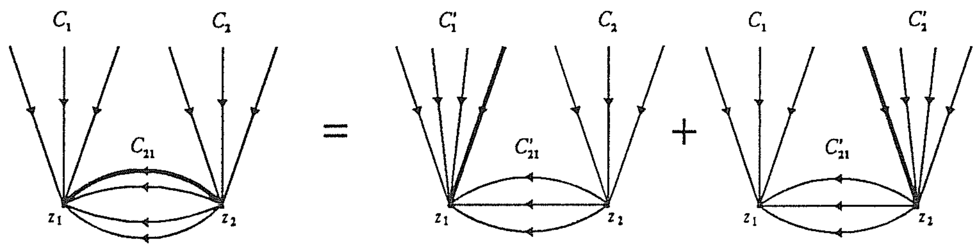
(3.38)

Here, C_1 and C_2 denote the family of contours involved in the definition of $e^{\alpha_1}_{\mathbf{r}_1}\tilde{\otimes}e^{\alpha_2}_{\mathbf{r}_2}$, which becomes $\hat{C}_1$ and C_2 for $R^{\alpha_1\alpha_2}(e^{\alpha_1}_{\mathbf{r}_1}\tilde{\otimes}e^{\alpha_2}_{\mathbf{r}_2})$. Expressing the mapping (3.37) in component is equivalent to decompose the r.h.s. of eq.(3.38) in a summation of terms analogous to the l.h.s. This can be realized through two basic deformation of the contours:

D1. splitting the contours $\hat{C}_1$ into paths joining the family C_2 and paths going from z_2 to z_1, that will be denoted C_{21},



D2. opening the contours C_{21} into paths joining the family C_1 or the family C_2,



Motivated by such a result, Gómez and Sierra develop their program with the explicit construction of the quantum group generators. To this end, they define the contour creation operators F_+ and F_-, acting on the space $\mathcal{V}^\alpha$ according to the equation

$$
\begin{aligned}
F_+\, e^\alpha_{r_-,r_+}(z) &= \int_C J_+(t) e^\alpha_{r_-,r_++1} \\
F_-\, e^\alpha_{r_-,r_+}(z) &= \int_C J_-(t) e^\alpha_{r_-+1,r_+}\,,
\end{aligned}
\tag{3.39}
$$

where the contour C surrounds the whole SVO$^\infty$. Such a definition can be naturally extended to the action of $F_\pm$ on the tensor product representation $\mathcal{V}^{\alpha_1}\tilde{\otimes}\mathcal{V}^{\alpha_2}$, if one replace C with a suitable contour ΔC surrounding the operator $e^{\alpha_1}_{\mathbf{r}_1}\tilde{\otimes}e^{\alpha_2}_{\mathbf{r}_2}$. Thus, with contour deformations not different from those described about the computation of the $\mathfrak{R}$-matrix, one achieves the expression of the coproduct of the "generators" $F_\pm$,

$$
\begin{aligned}
&\Delta F_\pm(e^{\alpha_1}_{\mathbf{r}_1}(z_1)\tilde{\otimes}e^{\alpha_2}_{\mathbf{r}_2}(z_2)) = \\
&\qquad F_\pm(e^{\alpha_1}_{\mathbf{r}_1}(z_1))e^{\alpha_2}_{\mathbf{r}_2}(z_2) + e^{2\pi i\alpha_\pm(\alpha_1+(r_-)_1\alpha_-+(r_+)_1\alpha_+)}e^{\alpha_1}_{\mathbf{r}_1}(z_1)\,F_\pm(e^{\alpha_2}_{\mathbf{r}_2}(z_2))
\end{aligned}
\tag{3.40}
$$

or

$$
\Delta F_\pm = f_\pm \otimes 1 + k^2_\pm \otimes F_\pm\,. \tag{3.41}
$$

The phase factor of the second term in eq.(3.40) arises from the braiding between the screening operators $J_\pm$ and $e^{\alpha_1}_{\mathbf{r}_1}$, necessary for $F_\pm$ to be applied on $e^{\alpha_2}_{\mathbf{r}_2}$, and has been identified in eq.(3.41) with the action of the operators

$$
k^2_\pm(e^\alpha_{\mathbf{r}_1}(z)) = exp(2\pi i\alpha_\pm \oint_C \partial\phi)\, e^\alpha_{\mathbf{r}_1}(z) = e^{2\pi i\alpha_\pm(\alpha+r_-\alpha_-+r_+\alpha_+)}e^\alpha_{\mathbf{r}_1}(z)\,. \tag{3.42}
$$

Analogously one finds the coproduct of $k_\pm$,

$$
\Delta k_\pm = k_\pm \otimes k_\pm\,. \tag{3.43}
$$

It easy to verify that $F_\pm$ and $k_\pm$ fulfill the relations

$$
\begin{aligned}
[F_+, F_-] &= 0 = [k_+, k_-] \\
k_\pm F_\pm &= q^{1/2}_\pm\, F_\pm k_\pm \\
k_\pm F_\mp &= -F_\mp k_\pm\,.
\end{aligned}
\tag{3.44}
$$

However, denoting B_- the algebra generated by $\{F_\pm,\, k_\pm\}$, the coproduct just defined together with the counit

$$\epsilon(k_\pm) = 1 \qquad \epsilon(F_\pm) = 0 \tag{3.45}$$

and the antipode map

$$S(k_\pm) = k_\pm^{-1} \qquad S(F_\pm) = -k_\pm^{-2} F_\pm\,, \tag{3.46}$$

induce an Hopf algebra structure in B_-. Finally, if one labels the SVO$^\infty$'s with the spin $j^\pm$ and the momenta $m^\pm$ given by $j^+ = \frac{n-1}{2}$, $j^- = \frac{n'-1}{2}$ and $m^\pm = j^\pm - r_\pm$, then the action of B_- on the space $\mathcal{V}^{\alpha_{n',n}}$ reads

$$\begin{aligned} k_\pm e^{j^- j^+}_{m^-, m^+}(z) &= q_\pm^{-\frac{1}{2}m^\pm} e^{\pi i m^\mp} e^{j^- j^+}_{m^-, m^+}(z) \\ F_- e^{j^- j^+}_{m^-, m^+}(z) = e^{j^- j^+}_{m^- -1, m^+}(z) \qquad & F_+ e^{j^- j^+}_{m^-, m^+}(z) = e^{j^- j^+}_{m^-, m^+ -1}(z)\,. \end{aligned} \tag{3.47}$$

Therefore, the conclusion of this analysis is that B_- represents the Borel subalgebra b_- of a quantum group Q which reduces to $U_{q_+}(\mathfrak{sl}_2)$ $(U_{q_-}(\mathfrak{sl}_2))$, when only the thermal representations $\mathcal{V}^{\alpha_{1,n}}$ $(\mathcal{V}^{\alpha_{n',1}})$ are considered. More precisely, one identifies

$$\begin{aligned} k_\pm &= q_\pm^{-\frac{1}{4}H_\pm} e^{\pi i H_\mp/2} \\ F_\pm &= f_\pm q_\pm^{-H_\pm/4}\,. \end{aligned} \tag{3.48}$$

Indeed, one recovers, in this way, the usual coproduct

$$\Delta f_\pm = f_\pm \otimes k_\pm^{-1} + k_\pm \otimes f_\pm \tag{3.49}$$

and antipode

$$S(f_\pm) = -q_\pm^{-1/2} f_\pm\,. \tag{3.50}$$

As regards the contour annihilation operators, it was already stressed that they have to appear in what Gómez and Sierra call the "covariantization" of the conformal properties of SVO$^\infty$'s. Hence, these authors write eq.(3.34) in the form

$$\begin{aligned} \delta_\xi e^\alpha_{r, r_+}(z) &= e^\alpha_{r_-, r_+}(\delta_\xi V_\alpha(z)) \\ &-(1 - q_+^{-1})\xi(\infty)\, J_+(\infty)\, E_+ e^\alpha_{r_-, r_+}(z) \\ &-(1 - q_-^{-1})\xi(\infty)\, J_-(\infty)\, E_- e^\alpha_{r_-, r_+}(z)\,, \end{aligned} \tag{3.51}$$

where $\xi(z)$ is the vector field that generates the conformal transformation and $E_\pm$ are two contours destroying operators, defined as

$$\begin{aligned} E_+ e^\alpha_{r_-, r_+}(z) &= [r_+]_{q_+} \frac{1 - e^{4\pi i \alpha \alpha_+} q_+^{r_+ -1}}{1 - q_+^{-1}}\, e^\alpha_{r_-, r_+ -1}(z) \\ E_- e^\alpha_{r_-, r_+}(z) &= [r_-]_{q_+} \frac{1 - e^{4\pi i \alpha \alpha_-} q_-^{r_- -1}}{1 - q_-^{-1}}\, e^\alpha_{r_- -1, r_+}(z)\,. \end{aligned} \tag{3.52}$$

Studying the conformal transformation of the (tensor) product $e^{\alpha_1}_{\mathbf{r}_1}(z_1)e^{\alpha_2}_{\mathbf{r}_2}(z_2)$, one easily find the coproduct of the operators $E_\pm$, while the definition of counit and antipode can be extended to the contour annihilation, so that the algebra generated by $k_\pm$ and $E_\pm$, denoted B_+, becomes an Hopf algebra. Hence, it is straightforward to recognize in B_+ the Borel subalgebra b_+ of the quantum group Q, that has been already described. Therefore, if one restricts to the thermal subalgebra $\alpha_{1,n}$ $(\alpha_{n',1})$, $\{K_+, E_+, F_+\}$ generate the quantum group $U_{q_+}(\mathfrak{sl}_2)$ $(U_{q_-}(\mathfrak{sl}_2))$. This is in agreement with the conclusion achieved about the braiding matrices of tensor product representations.

Finally, one can observe that, due to the definition of the generators $E_\pm$, their commutation relations with $F_\pm$ can be interpreted as commutation relations between $F_\pm$ and the virasoro algebra. Therefore one obtains

$$[\delta_\xi,\, F_\pm]e^{\alpha}_{\mathbf{r}} \;=\; -\xi(\infty)\, J_+(\infty)\,(1-k_\pm^4)\, e^{\alpha}_{\mathbf{r}}\,, \tag{3.53}$$

a result which seems to differ from the requirement (iv). Gómez and Sierra solve this apparent contradiction showing that conformal block functions can be identified with invariant tensors of the corresponding quantum groups Q. In the language of §3.1, this means that CVO's are strictly related to the q-CG coefficients of Q. More precisely, they demonstrate that

$$\begin{pmatrix} & \alpha_3 & \\ \alpha_1 & & \alpha_2 \end{pmatrix}_{z_1,z_2} (V_{\alpha_1}\tilde{\otimes} V_{\alpha_2}) \;=\; \sum_{\mathbf{r}_1\mathbf{r}_2} K^{\alpha_1\alpha_2\,0}_{\mathbf{r}_1\mathbf{r}\alpha_3}\, e^{\alpha_1}_{\mathbf{r}_1}(V_{\alpha_1}(z_1))e^{\alpha_2}_{\mathbf{r}_2}(V_{\alpha_2}(z_2))\,, \tag{3.54}$$

where with $K^{\alpha_1\alpha_2\,0}_{\mathbf{r}_1\mathbf{r}\alpha_3}$ the denote the (inverse of) q-CG coefficients. This allows then to prove that the actions of the quantum group Q and of the chiral algebra $\mathcal{A}$ commute when applied on the space of invariants,

$$[\Delta^{(\mathrm{N})}(Q),\, \Delta^{(\mathrm{N})}(\mathcal{A})]\, Inv\left(\mathcal{V}^{\alpha_1}_{z_1}\tilde{\otimes}\ldots\tilde{\otimes}\mathcal{V}^{\alpha_{\mathrm{N}}}_{z_{\mathrm{N}}}\right) \;=\; 0\,. \tag{3.55}$$

Hence, in this sense, also the condition (iv) seems to be fulfilled.

Conclusions

The present Thesis would represent an attempt to point out the importance of the role that quantum groups can play in many physical context. Pursuing this aim, in Chapter 1 we tried to present the axioms of (quasi) triangular Hopf algebras in a form such to emphasize their physical meaning and their analogy with the well-known "classical" group theory. Hence, the purpose of Chapter 3 is to develop the suggestions of Chapter 1 in the specific case of the CFT. In particular, In §3.3 we have shown how the Coulomb gas representation of minimal models provides a natural framework to define an explicit realization of the underlying quantum group structure. If this construction clarifies the interplay between the chiral algebra and the quantum group, however, a better understanding of the quantum group symmetry in the CFT probably requires a totally different approach. Indeed, one can face the problem starting from a conformal invariant context in which the quantum group structure is already evident and trying to deduce the correlation functions. This is precisely the philosophy of the Toda field theory. In same sense, it can be considered a generalization of the Coulomb gas representation. It is our belief that this program will contribute significantly to introduce the quantum groups and the CFT in a more general frame.

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