Fourier Spectral Methods for Soliton Equations

Thesis submitted for the degree of

"Magister Philosophiæ"

CANDIDATE

Alfio E. Borzì

SUPERVISOR

Prof. Alfio Quarteroni

October 1990

Scuola Internazionale Superiore di Studi Avanzati

International School for Advanced Studies

Fourier Spectral Methods for Soliton Equations

Thesis submitted for the degree of

"Magister Philosophiæ"

CANDIDATE

Alfio E. Borzì

SUPERVISOR

Prof. Alfio Quarteroni

October 1990

ad Ann:

Index

Preface and Introduction

Since before Aristotele's time the concepts of the wave and the particle have been the two cardinal reference points in physics. Along the centuries a lot of scientists and philosophers debated whether the matter can be explained in terms of waves and/or particles. Among the others the last important example of this debate is that of Einstein and Bohr. But in this century a new concept that seems to be the solution (at least for some aspects) of this dialectic battle has arisen in physics: the concept of soliton. Now let us give some news about the history of the "soliton".

In 1834 a Scottish scientist, J. Scott Russel, made the first documented observation of what he called a "solitary wave". The important thing he noticed was that, once momentum has been transferred to a fluid, this does not result in a rippling motion in which this momentum would be scattering over the whole surface of the water. Instead, it remains localized in a stable propagating "thing" which passes over the fluid and leaves it as it was before the "thing" had arrived. In the following decades, the "solitary wave" was mentioned by various mathematicians, e.g. Stokes (1847) and Boussinesq (1872), but there was no attempt for a theoretical treatment of the phenomenon until 1895, when Korteweg and de Vries derived their equation (KdV) which describes the propagation of waves on the surface of a flimsy shallow canal. After a subsequent gap of 60 years, numerical experiments on wave propagation (Fermi *et al.*, 1955) showed the existence of wave-like excitations which, rather than dispersing their energy, maintained a stable shape in the course of their propagation and emerged from collisions unaltered. Zabusky and

Kruskal (1965) introduced the word "solitons" to characterize such waves, so they took their place in the modern science.

Since 1970 the literature about soliton solutions of non linear equations grew up even if up to nowadays essentially few non linear equations as the Korteweg-de Vries (KdV) equation and the non linear Schrödinger (NLSE) equation have been solved. Notice that, at this time, there is not a complete and satisfactory theory for the solution of non linear partial differential equations. Just the first important question about existence of solutions may be prohibitive to answer.

We look at the history and we recognize that the well known KdV equation was first investigated numerically obtaining in this way some useful informations for its subsequent analysis. For the same reason a numerical investigation of non linear differential equations is fruitful and should be suggested as the first step in order to solve these non linear equations.

Then it is important to develop powerful numerical methods which make us able to handle with the problem of non linearity. For these reasons the aim of this thesis is the study and the development of numerical methods for the solution of non linear partial differential equations in 1 and 2 space dimensions. We consider explicitly the Korteweg-de Vries equation and the Kadomtsev- Petviashvili (KP) equation as model equations because they present both a non linear term and terms with derivatives of high order. More exactly we will consider the modern numerical spectral methods, which are really new, in fact they do not yet cover the complete spectrum of applications. In addition, in recent years it has been done a deep analytical examination of convergence and stability of these methods so that the use of them is a mathematical science and no more an "art", how it seems to be for more "popular" methods. Due to this success the theory of spectral methods has evolved vigorously so that now is impossible to present within a thesis the entire class of spectral discretization schemes for differential equations.

This thesis will be organized as follows. Chapter 1 consists of a colloquial introduction to the Korteweg-de Vries equation. We present an easy way to derive its soliton solutions and then we show their uniqueness. The final part of the first chapter is devoted to the problem of integrability and to the original Inverse Scattering Transform (IST) method. In Chapter 2 we review briefly some properties of the Kadomtsev-Petviashvili equation. Notice that there are two different equations which have the same name, but they are different for the sign of the dispersion term. We stress that the one with negative dispersion is obtained from the KdV equation by extending it in two space dimensions. We will consider the more interesting one with positive dispersion.

In the same chapter we introduce a method complementary to IST, developed by Hirota in 1971. In our opinion the Hirota method can be considered one of the most important methods to solve non linear partial differential equations. It is a general method which can be applied to a wide class of non linear equations (we give a brief list of these equations in Appendix III). Furthermore it works directly on the studied equation without passing through the solution of any other equation. On the other hand the IST method remains a tricky way to solve just the KdV equation (and few others).

The aim of the first and the second chapter is to present the main features of the KdV and the KP equations and to set them in the class of equations which can be studied with the Hirota method. If the reader has experienced with these equations, he can skip these first two chapters and start with Chapter 3.

In Chapter 3 we give the main features of the Fourier spectral methods, in particular, the Fourier Galerkin and the Fourier collocation schemes. There we explain how to treat non linearity using a model equation. Then we will give an example of convergence and stability analysis.

In Chapter 4 we report the most interesting results of the numerical analysis of the Fourier spectral methods applied to the KdV equation. Then we repeat the numerical experiment of Zabusky and Kruskal[1]. The last section of Chapter 4 is devoted to a numerical approach to the KdV equation written in the Hirota form. We do not emphasize the particular scheme used but the use of the Hirota version of the KdV equation in order to handle easily (in principle) some possible solutions of the KdV equation which have points of singularity. Furthermore the method results to be more general because it can be used, for the same purposes, for all those equations which can be written in the Hirota form.

Finally, in Chapter 5 we investigate numerically the KP equation using, for the first time, a spectral method.

Chapter 1

The Korteweg-de Vries (KdV) equation

1.1 Solitons and solitary waves

Solitons and solitary waves are certain special solutions of non linear wave equations. In order to understand why they appear and how to distinct them, let us first consider the wave equation

$$\left(\frac{1}{v^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right) u(x,t) = 0. \tag{1.1}$$

The properties of this equation are well known. For our discussion, two features of its solutions will be relevant. Since the equation is dispersionless

i) any well-behaved function of the form $f(x \pm vt)$ is a solution of eq.(1.1). If we take f as a localized function, we can construct a localized wave packet that will travel with uniform velocity $\pm v$, and without any distortion of its shape. For eq. (1.1), this is better understood saying that the plane waves $\cos(kx \pm \omega t)$, $\sin(kx \pm \omega t)$ with $\omega = kv$, constitute a complete set of solutions, and a well behaved function $f(x \pm vt)$ can be written as

$$f(x \pm vt) = \int dk\, [a_1(k)\cos(kx \pm \omega t) + a_2(k)\sin(kx \pm \omega t)]\,. \tag{2.1}$$

It is then clear that all the components travel with the same velocity $v = \frac{\omega}{k}$. Since the wave equation is linear

ii) given two localized wave packet solutions $f_1(x - vt)$, $f_2(x + vt)$, their sum f_3 is also a solution. At large negative times t, $f_3(x,t)$ consists of the two widely separated packets (suppose that they collide at $t = 0$). As t increases, they approach each other colliding at a finite time t. Then, for large positive times they separate, retaining their original shapes.

Notice that the linearity is *not* a sufficient condition to ensure i) and ii). In fact, let us suppose to add a simple constant to eq.(1.1) so that it becomes

$$\left(\frac{1}{v^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + c^2\right) u(x,t) = 0,$$

which is still linear, and still admits plane waves as a complete set of solutions. In this case nevertheless, different wavelengths travel at different velocities $v(k) = \frac{d\omega}{dk}$, $(\omega^2(k) = k^2v^2 + c^2v^2)$. Therefore, any localized wave packet spreads out, as time goes on. Furthermore, for some equations which have both non-linear and dispersive terms, it can happen that these terms balance each other in such a way that there exist solutions satisfying property i) for some special f. Strictly speaking, such solutions are called solitary waves. If feature ii) is also exhibited for $t \to \pm\infty$, these solutions are called "solitons". As the sole residual effect of collisions, the solitons may suffer a bodily displacement compared with their pre-collision trajectories.

In this thesis, we will consider only soliton solutions of two non-linear equations: the Korteweg-de Vries and the Kadomtsev-Petviashvili equation. Why will we consider them? First, because these equations appear in many fields of applied mathematics and physics such as fluid dynamics, elementary particle physics, meteorology, plasma and laser physics. Second, because for them exact (multi) soliton solutions are known, and they can be used in order to test our numerical methods. And third, because they furnish a simple and elegant example (in 2, and respectively 3 dimensions) where non linear and dispersion terms are present.

1.2 The KdV equation: a review

The KdV equation was introduced in 1895 by Korteweg and de Vries to represent dispersive waves in one direction on the surface of a shallow canal. If the canal has normal depth l, and $l + v$ represents the elevation of the surface above the bottom, the partial differential equation which governs the wave motion is:

$$\frac{\partial v}{\partial \tau} = \frac{3}{2}\sqrt{\frac{g}{l}}\frac{\partial}{\partial z}\left[\frac{2}{3}\alpha v + \frac{1}{2}v^2 + \frac{1}{3}\sigma\frac{\partial^2 v}{\partial z^2}\right],$$

where z, τ are the space and time variable and $\sigma = l^3/3 - Tl/\rho g$ is a constant, T is the surface tension and ρ the density of the fluid [2]. A scaling is usually done in order to obtain a more elegant form. Let $u = 8\alpha v$, $x' = \sqrt{2\alpha/\sigma}z$, and $t' = \sqrt{\frac{2\alpha^3 g}{\sigma l}}\tau$. Then equation (1.1) becomes

$$u_{t'} + u_{x'} + 12uu_{x'} + u_{x'x'x'} = 0$$

The term $u_{x'}$ can be eliminated by the transformation $x = x' - t'$, $\quad t = t'$. So, from now on, we will refer to the KdV equation in the form

$$u_t + \alpha uu_x + \beta u_{xxx} = 0 \tag{3.1}$$

Notice that we can manipulate the coefficients α and β as we want by rescaling the variables. For example a scaling $x \to \beta^{1/3}x$ takes the equation into the form

$$u_t + \alpha\beta^{-1/3}uu_x + u_{xxx} = 0;$$

next a scaling $u \to \alpha^{-1}\beta^{1/3}u$ gives

$$u_t + uu_x + u_{xxx} = 0,$$

and an inversion $t \to -t$ results in

$$u_t = uu_x + u_{xxx}.$$

1.3 A soliton solution

Now we look for a solution of (3.1) which is translationally invariant[3], that is of the form

$$\tilde{u}(x,t) = u(x - vt).$$

We also require, that u and its derivatives go to zero as $|x - vt| \to +\infty$. For let $\alpha = 1$ and $\beta = 1$ in (3.1) and substitute $\tilde{u}$ in the equation, we obtain

$$-v\tilde{u}_x + \tilde{u}\tilde{u}_x + \tilde{u}_{xxx} = 0.$$

Since this equation involves only x derivatives, we can set $t = 0$. Now integrating with respect to x and imposing the above boundary conditions, we get

$$-v\tilde{u} + 1/2\tilde{u}^2 + \tilde{u}_{xx} = 0$$

multiplication by $2\tilde{u}_x$ and one more integration gives

$$-v\tilde{u}^2 + 1/3\tilde{u}^3 + \tilde{u}_x^2 = 0$$

From this relation and the assumption $\tilde{u}_x(0) = 0$, $\tilde{u}$ can be determined explicitly

$$\tilde{u}(x) = 3v \text{ sech}^2 \left[\frac{1}{2}x\sqrt{v}\right]$$

where v is an arbitrary constant. Therefore a taller soliton moves faster. Furthermore, restoring the time variable, we obtain

$$\tilde{u}(x,t) = 3v \text{ sech}^2[\frac{1}{2}\sqrt{v}(x - vt)] \tag{4.1}$$

Notice that all solitons travel to the right, but if we change v to $-v$, we will not get a soliton moving to the left, but an oscillating wave.

1.4 Uniqueness of the solutions

Before considering the integrability and studying the initial value problem for the KdV equation, we will show that the KdV equation admits unique solutions for given initial conditions, and so resulting unicity for the soliton solutions[3]. Let us assume that $u(x,t)$ and $\tilde{u}(x,t)$ represent two solutions to the KdV equation satisfying the same initial conditions. That is

$$u_t + uu_x + u_{xxx} = 0 \tag{5.1}$$

and

$$\tilde{u}_t + \tilde{u}\tilde{u}_x + \tilde{u}_{xxx} = 0 \tag{6.1}$$

with

$$u(x,0) = \tilde{u}(x,0) = f(x) \tag{7.1}$$

Then subtracting (6.1) from (5.1), we have

$$(u - \tilde{u})_t = -\alpha\big[uu_x - \tilde{u}\tilde{u}_x\big] - \beta\big[u - \tilde{u}\big]_{xxx}.$$

Now add and subtract $\alpha u\tilde{u}_x$ and define $w(x,t) = u(x,t) - \tilde{u}(x,t)$ and write

$$w_t = uw_x + w\tilde{u}_x + w_{xxx}.$$

Remember that u and $\tilde{u}$ vanish at spatial infinity. Multiplying the last equation by w and integrating over x, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{+\infty}\frac{1}{2}w^2 dx = \int_{-\infty}^{+\infty} w^2(\tilde{u}_x - \frac{1}{2}u_x)dx. \tag{8.1}$$

Next define $E(t) = \int_{-\infty}^{+\infty}\frac{1}{2}w^2(x,t)dx$ and $m = 2\ \max|\tilde{u}_x - \frac{1}{2}u_x|$; we then obtain from (8.1) the inequality

$$\frac{\mathrm{d}E(t)}{\mathrm{d}t} \leq mE(t)$$

or $E(t) \leq E(0)e^{mt}$. Since $E(t)$ is positive semidefinite, it follows that $E(t)$ vanishes if $E(0)$ equals zero. Now from the definition of w and (7.1) follows that $w(x,0) = 0$. This implies that

$$E(0) = \frac{1}{2} \int_{-\infty}^{+\infty} \frac{1}{2} w^2(x,0)dx = 0$$

and consequently $E(t) = 0$, that is $w(x,t) = 0$ for any t, or

$$u(x,t) = \tilde{u}(x,t).$$

Therefore the KdV equation admits unique solutions for given initial conditions.

1.5 Integrability

The study of integrability of an Hamiltonian system dates back to Liouville: *"An Hamiltonian system whose phase space is 2N dimensional is integrable by quadratures if and only if there exist exactly N functionally independent conserved quantities which are in involution (that is, the Poisson brackets of these conserved quantities one with another vanish)"*. In other words, assume that the system is describable by the canonical coordinates and momenta (q_i, p_i), $i = 1, 2, ..., N$. If there exist N globally defined conserved quantities K_i, $i = 1, 2, ..., N$, in involution $\{K_i, K_j\} = 0$, $i, j = 1, 2, ...N$, then there exists a transformation to new variables θ_i, P_i (cyclic action-angle variables) such that the new time evolution is given by

$$\theta_i(t) = f_i t + \alpha_i,$$

and where $K_i = K_i(q_i, p_j) = P_i$, the time evolution being

$$\dot{P}_i = \{P_i, H(P_j)\} = 0$$

and

$$\dot{\theta}_i = \{\theta_i, H(P_j)\} = f_i(P_j) = \text{const.}$$

α_i's are integration constants, fixed by the initial conditions. The scheme for solving the time evolution is therefore given by the diagram

$$\begin{array}{ccc} q_i(0), p_i(0) & \xrightarrow{\text{“direct” step}} & \theta_i(0), P_i(0) \\ & & \big\downarrow \text{“free” time evolut.} \\ q_i(t), p_i(t) & \xleftarrow{\text{“inverse” step}} & \theta_i(t), P_i(t) \end{array}$$

Although we discussed Liouville's theorem in the case of a finite number of degrees of freedom, the same is hoped to go through for a system with an infinite number of degrees of freedom.

1.5.1 The Inverse Scattering Transform method

The above mentioned scheme may be generalized to describe solutions of the KdV equation[4]. Consider first the "direct" step. For take the time independent Schrödinger equation described by

$$\frac{\partial^2 \psi}{\partial x^2} + \left(\frac{1}{2}u(x,t) + \lambda\right)\psi = 0, \tag{9.1}$$

where $u(x,t)$ is the solution of the KdV equation

$$u_t = uu_x + u_{xxx}. \tag{10.1}$$

The variable t in $u(x,t)$ should be considered as a parameter characterizing the potential in the Schrödinger equation (9.1) (so λ and ψ will depend on t).

Solving eq.(9.1) for u and inserting the result in (10.1) yields

$$\lambda_t \psi^2 - \frac{\partial}{\partial x}\left[\psi^2 \frac{\partial}{\partial x}\left(\frac{(\psi_{xxx} - \psi_t + (\frac{1}{2}u - 3\lambda)\psi_x)}{\psi}\right)\right] = 0. \tag{11.1}$$

Integrating this over x and remembering that the wave function ψ vanishes at $x = \pm\infty$, we obtain that the eigenvalues are time independent

$$\lambda_t = 0.$$

Now we go back to eq. (11.1) dropping the first term and integrating the resulting equation twice to yield

$$\psi_t - \psi_{xxx} - \left(\frac{1}{2}u - 3\lambda\right)\psi_x = C\psi. \tag{12.1}$$

So that, the computation of the evolution of ψ in regions where u vanishes is straightforward.

Consider first a discrete eigenvalue $\lambda_n = -k_n^2 < 0$. Then $C = 0$ because we are assuming the normalization $\int \psi_n^2 dx = 1$. Further inserting

$$\psi_n = c_n(t)e^{(-k_n x)} \quad \text{for} \quad x \to \infty,$$

into eq. (12.1), we find

$$c_n = c_0 e^{(-8k_n^3 t)}. \tag{13.1}$$

The analogous coefficients for large negative x grow exponentially in time.

Second for the continuous case which corresponds to the scattering states $\lambda = k^2 > 0$, we choose the wave functions to have the asymptotic form

$$\psi(x,t) \longrightarrow T(k,t)e^{(ikx)}, \quad \text{for } x \to +\infty$$

(14.1)

$$\psi(x,t) \longrightarrow e^{(ikx)} + R(k,t)e^{(-ikx)}, \quad \text{for } x \to -\infty.$$

Here we assume a plane wave incident from the left ($x = -\infty$) and $R(k,t)$ and $T(k,t)$ represent the coefficients of reflection and transmission. Unitarity requires them to satisfy

$$|R(k,t)|^2 + |T(k,t)|^2 = 1$$

We insert eq. (14.1) in eq. (12.1) and equate the coefficients of the two independent solutions at $+\infty$ and at $-\infty$, finding $C = 4ik^3$ and the time evolution of R and T as

$$R(k,t) = R(k,0)e^{(8ik^3t)}$$

$$T(k,t) = T(k,0). \tag{15.1}$$

Therefore, recalling the above diagram we recognize that the first two steps have been completed: the "direct" step, determining the scattering data $(R(k,0), T(k,0), \lambda_n, c_n(0))$, of the Schrödinger equation (9.1) with the potential $\frac{1}{6}u(x,0)$, and the "free" time evolution given by (13.1) and (15.1).

To complete the diagram we need the "inverse" step. But if one has at disposal the scattering data, then the potential u is determined uniquely by solving the Gel'fand-Levitan-Marchenko (GLM) equation. Let $K_t(x,y)$ be the solution of the GLM equation

$$K_t(x,y) + B_t(x,y) + \int_x^{+\infty} dz K_t(x,z)B_t(y,z) = 0 \quad y \geq x$$

$$K_t(x,y) = 0 \quad \text{if } y < x$$

with

$$B_t(x,y) = \frac{1}{2}\int_{-\infty}^{+\infty} dk R(k,t)e^{ik(x+y)} + \sum_{n=1}^{N} c_n(t)e^{-k_n(x+y)}$$

where N is the total number of bound states. Then the potential is obtained from the knowledge of $K(x,y)$ as

$$\frac{1}{6}u(x,t) = 2\frac{\partial}{\partial x}K_t(x,y)|_{y=x}$$

(t is just a parameter here).

We see that all reflectionless potentials are soliton solutions of the KdV equation, that is if the initial solution $u(x,0)$ is a reflectionless potential, or equivalently $R(k,0)=0$, $|T(k,0)|=1$, the solution $u(x,t)$ can be immediately constructed as

$$\frac{1}{6}u(x,t) = 2\frac{d^2}{dx^2}\log[\det(I + C^t(x))]$$

where

$$c^t_{lm}(x) = \frac{c_l(t)c_m(t)}{k_l + k_m}e^{(k_l+k_m)x}.$$

These solutions $u(x,t)$ are solitons.

We have solved the equation versus a similar diagram as in the finite dimensional case, but now for infinite degrees of freedom (as is the case of the KdV equation), where the Liouville theorem is supposed to hold.

$$\begin{array}{ccc} u(x,0) & \xrightarrow{\text{“direct” step via Schroedinger}} & \{R(k,0),\lambda_n,c_n(0)\} \\ & & \Big\downarrow \text{“free” time evolut.} \\ u(x,t) & \xleftarrow{\text{“inverse” step via GLM eq.}} & \{R(k,t),\lambda_n,c_n(t)\} \end{array}$$

Notice that the scattering data is the analogue to action-angle variables . Note also that we have found an infinite number of conserved quantities.

We know that the above solution of the initial-value problem for the KdV equation, is in some sense mysterious, in fact there are many questions unanswered.

First is unclear why the dynamics of a non linear system can be controlled by a linear one. Secondly it is obscure how one can obtain the conserved quantities and hence conclude about the integrability. All these aspects of the problem have been clarified by numerous scientists by more sophisticated methods which improve the above IST scheme. We want to mention here only (just for brevity) some of the pioneering researchers as C.S. Gardner, L.D. Faddeev, M.D. Kruskal, R.M. Miura, N. Zabusky, V.E. Zakharov. We cannot report here all the known results, because the purpose of this first chapter is to give only the main results and ideas behind the KdV equation.

1.5.2 Infinite conserved quantities

In order to complete the above discussion we show that the KdV equation can be written as an Hamiltonian equation where the Hamiltonian and the fundamental Poisson bracket are given by[5]

$$H(u) = \int_{-\infty}^{+\infty} \left(\frac{1}{3!} u^3(x) - \frac{1}{2} \left(\frac{\partial u}{\partial x} \right)^2 \right) dx$$

$$\{u(x), u(y)\} = \frac{\partial}{\partial x} \delta(x - y)$$

so that

$$u_t = \{u(x), H\} = u u_x + u_{xxx}.$$

As a last topic we derive the first integrals. The central idea in order to construct explicitly infinite conserved quantities of the KdV equation was given by Miura[6], but here we just state some facts inspired from that idea.

A tedious analysis of the problem[5] results in the following recursion relation. Let $v_0 = u$ the solution of the KdV equation and take the n-th quantity v_n to be given by

$$v_n + i\frac{\partial v_{n-1}}{\partial x} + \frac{1}{6}\sum_{m=0}^{n-2} v_{n-m-2}v_m = 0$$

with n, a positive integer.

The n-th conserved density is given as

$$\rho_n = 3(-1)^n v_{2n};$$

then the n-th conserved quantity is

$$K_n = \int_{-\infty}^{+\infty} \rho_n dx,$$

so the first few of them are explicitly given by

$$K_0 = 3\int_{-\infty}^{+\infty} u(x,t)dx,$$

$$K_1 = \frac{1}{2}\int_{-\infty}^{+\infty} u^2(x,t)dx,$$

$$K_2 = \int_{-\infty}^{+\infty} \left(\frac{1}{3!}u^3(x) - \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2\right) dx.$$

One can check that the above infinite number of conserved quantities are in involution[5]

$$\{K_n, K_m\} = 0.$$

Chapter 2

The Kadomtsev - Petviashvili (KP) equation

The fundamental equation of soliton theory was given, as we said, by Korteweg and de Vries in 1895. The equation describes the amplitude of long one-dimensional waves on the surface of a fluid. In physical terms the effects included in the equation are weak non-linearity and dispersion. The extension of this equation to motion in more than one dimension were given by Kadomtsev and Petviashvili[7], who generalized the dispersion relation to give an extra term in the equation due to an extra dimension[8]. They also drew attention to the effect of changing the sign of the dispersion term. They noted that the effect of this was to introduce an instability for transverse perturbations[9]. So if we start from the KdV and generalize it to two space dimensions we obtain

$$(u_t + 6uu_x + u_{xxx})_x + u_{yy} = 0 \tag{1.2}$$

Changing the sign of the dispersion term, as is the case of wave propagation in plasma, the KP equation is obtained

$$(u_t + 6uu_x + u_{xxx})_x - u_{yy} = 0 \tag{2.2}$$

(the linear dispersion relation for these two equations is given by $\omega k = k^4 \pm m^2$, $+$ for (1.2), $-$ for (2.2)).

2.1 Integrability

The complete integrability in the Hamilton Liouville sense for the KdV was demonstrated by Zakharov and Faddeev. The KP equations

$$(u_t + 6uu_x + u_{xxx})_x \pm u_{yy} = 0 \qquad (3.2)$$

seem to play in 2+1 dimensions the role played by the KdV equation in 1+1. If $w = \int_{-\infty}^{x} u(x', y, t)dx'$, the KP equations can be written in an Hamiltonian form[10]

$$H = \int\int_{-\infty}^{+\infty} \left(\frac{1}{2}u_x^2 - u^3 \mp \frac{1}{2}w_y^2\right) dxdy \qquad (4.2)$$

and momentum $P = (P_x, P_y)$

$$P_x = \int\int_{-\infty}^{+\infty} \frac{1}{2}u^2 dxdy, \quad P_y = \int\int_{-\infty}^{+\infty} \frac{1}{2}uw_y dxdy \qquad (5.2)$$

with the Poisson bracket

$$\{f, g\} = \int\int_{-\infty}^{+\infty} \left(\frac{\delta f}{\delta u}\right) \frac{\partial}{\partial x} \left(\frac{\delta g}{\delta u}\right) dxdy.$$

Then $u_t = \{u, H\} = \frac{\partial}{\partial x}(\frac{\delta H}{\delta u})$ become equations (3.2). It is supposed that u goes to zero "fast enough" for $(x^2 + y^2) \rightarrow \infty$. Equations (3.2) imply as a consequence $\int_{-\infty}^{+\infty} u_{yy} dx = 0$ and $\int_{-\infty}^{+\infty} udx = 0$ is imposed.

A countable infinity of constants of motion $K_n = \int \tilde{K}_n dxdy$ does exist, and in particular

$$K_1 = \int \frac{1}{2}udxdy, \quad K_2 = P_x,$$

$$K_3 = P_y, \quad K_4 = H.$$

Actually, the action-angle variables were found and integrability was proved, see for this[10][11].

2.2 Soliton solutions and the Hirota approach.

As we have already mentioned, the KP equation

$$(u_t + 6uu_x + u_{xxx})_x - u_{yy} = 0 \tag{6.2}$$

is the positive dispersion counterpart of the negative dispersion KdV equation. It is obvious that any solution of the KdV equation may be converted into a solution of the KP equation by a change of the variable y to iy'. The introduction of complex numbers requires, of course, that also the parameters in the solution are considered complex, and these parameters must be chosen such that the resulting solution is real[8]. The obtained solution is

$$u = 2\lambda^2 \operatorname{sech}^2[\lambda x + 2\mu y - 4\lambda(\lambda^2 + 3\mu^2)t]$$

where λ and μ are two real parameters. Unfortunately, such a soliton will be unstable since transverse periodic perturbations, with wave number k, cause the soliton to grow without bounds in the small k-limit[9] (more exactly this happens if $k < (\frac{\sqrt{3}}{4})v$, where v is the soliton velocity). In 1977 Manakov *et al.*[12] found a class of localized soliton solutions of (6.2) that are no longer exponential in character, but take the form of rational functions in space and time variables (such solutions are singular for the KdV equation).

A method complementary to IST which enables us to generate soliton solutions, was first proposed by Hirota[13] for the KdV equation. This technique has the advantage of being applicable directly upon the equation. In order to show what was the starting point of Hirota's work let us consider the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0 \tag{7.2}$$

and define $u = w_{x'}$. Substituting and integrating over x (the constant of integration is zero due to the properties of the particular solutions we are looking for) we have

$$w_t + 3w_x^2 + w_{xxx} = 0.$$

The transformation $w = 2\frac{\partial}{\partial x}\log f(x,t)$ reduces the KdV equation to a homogeneous equation in $f(x,t)$ (Hirota, 1971)

$$ff_{xxxx} - 4f_x f_{xxx} + 3f_{xx}^2 + ff_{xt} - f_x f_t = 0. \qquad (8.2)$$

The transformation is known as the Cole-Hopf transformation (Cole 1951 , Hopf 1950). Let us examine this equation. First of all it is quadratic in f. Furthermore, we observe that $f = 1$ is the trivial solution. Next let

$$f(x,t) = 1 + e^{\theta(x,t)}, \qquad (9.2)$$

where $\theta(x,t) = kx + \omega t + \theta_0$. The form (9.2) is an exact solution if $\omega = -k^3$. Hirota noted that the terms in (8.2) were very similar to the Leibniz rule for derivatives of products, except for signs. He defined the new operator[14]

$$D_x^n a \cdot b = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^n a(x)b(x')|_{x=x'}, \qquad (10.2)$$

in term of which the KdV equation (7.2) takes the form

$$D_x(D_t + D_x^3)f \cdot f = 0. \qquad (11.2)$$

The above discussion motivates us to look for a solution of the form

$$f_N = 1 + \sum_{n=1}^{N} \epsilon^n f_N^{(n)} \qquad (12.2)$$

where ϵ is a convenient expansion term. Now, if one substitutes (12.2) into (11.2), one can see that the expansion truncates (that is we obtain the exact solution with the term of first order in ϵ) when $f_N^{(1)}$ is chosen to be of the form

$$f_N^{(1)} = \sum_{i=1}^{N} e^{\theta_i}$$

where $\theta_i = k_i x - k_i^3 t + \theta_i^0$.

The first two multisoliton solutions are given by

$$f_1 = 1 + e^{\theta_1}, \quad f_2 = 1 + e^{\theta_1} + e^{\theta_2} + e^{\theta_1+\theta_2+A_{12}},$$

and the general case was proven by Hirota to be given by

$$f_N = \sum_{\mu=0,1} \exp\left(\sum_{i<j}^{N} A_{ij}\mu_i\mu_j + \sum_{i=1}^{N} \mu_i\theta_i\right) \tag{13.2}$$

where

$$\exp A_{ij} = \left(\frac{k_i - k_j}{k_i + k_j}\right)^2 .$$

We note that the usual single soliton is easily recovered from f_1 to be

$$u(x,t) = \left(\frac{k_1^2}{2}\right) \operatorname{sech}^2\left[\frac{1}{2}\left(k_1 x - k_1^3 t + \theta_1^0\right)\right] . \tag{14.2}$$

The fact that one can recover rational solutions relies on the freedom of choosing the phase constant θ_i^0. For example in (14.2) , if we choose $e^{\theta_1^0} = -1$, we have the "singular" soliton

$$u(x,t) = -\left(\frac{k_1^2}{2}\right) \operatorname{cosech}^2\left[\frac{1}{2}(k_1 x - k_1^3 t)\right] .$$

Passing to the limit $k_1 \to 0$ (i.e., the "long wave" limit) we find[15]

$$u = -\frac{2}{x^2}. \tag{15.2}$$

Or, to be more clear, letting $\alpha = e^{\theta_1^0}$ we have

$$f_1 = 1 + \alpha e^{(k_1 x - k_1^3 t)}.$$

As $k_1 \to 0$ we have $f_1 = 1 + \alpha(1 + k_1 x - k_1^3 t) + O(k_1^2)$. If we take $\alpha = -1$, then

$$f_1 = -k_1 x + O(k_1^2)$$

and we recover (15.2) from the transformation $u = 2\frac{\partial^2}{\partial x^2} \log f$. Now we can go back to the KP equation (6.2). We look for a solution, $u \to 0$ as $|x| \to \infty$, of the form

$$u = 2\frac{\partial^2}{\partial x^2} \log f_N. \qquad (16.2)$$

Inserting (16.2) into (6.2) yields

$$(D_x D_t + D_x^4 - D_y^2) f_N \cdot f_N = 0. \qquad (17.2)$$

The one-, and two-solitons solutions are given by

$$f_1 = 1 + e^{\theta_1}, \quad f_2 = 1 + e^{\theta_1} + e^{\theta_2} + e^{\theta_1 + \theta_2 + A_{12}},$$

where $\theta_i = k_i(x + P_i y - (k_i^2 - P_i^2)t) + \theta_i^0$ and

$$\exp A_{ij} = \frac{3(k_i - k_j)^2 + (P_i - P_j)^2}{3(k_i + k_j)^2 + (P_i - P_j)^2},$$

P_i, P_j, being complex parameters.

Taking $e^{\theta_i^0} = -1, \quad k_i \to 0$ (with $\frac{k_1}{k_2} = O(1)$) we find

$$f_1 = -k_1 \eta_1 + O(k_1^2)$$

$$f_2 = k_1 k_2 \left(\eta_1 \eta_2 - \frac{12}{(P_1 - P_2)^2} \right) + O(k^3)$$

where $\eta_i = x + P_i y + P_i^2 t$.

Then a non singular solution is obtained from f_2 when $P_2 = P_1^*$. In this case we have

$$\tilde{f}_2 = \eta_1 \eta_1^* - \frac{12}{(P_1 - P_1^*)^2}.$$

Letting $P_1 = P_R + iP_I$ we have

$$u = 2\frac{\partial^2}{\partial x^2} \log \left[(x' + P_R y')^2 + P_I^2 y'^2 + \frac{3}{P_I^2} \right], \tag{18.2}$$

where $x' = x - (P_R^2 + P_I^2)t$ and $y' = y + 2P_R t$.

Hence we have a permanent rational soliton solution decaying as $O(\frac{1}{x^2}, \frac{1}{y^2})$ for $|x|, |y| \to \infty$. In general when $N = 2M$ this method yields formulae for a rational M soliton solution[15]. The method used here unfortunately does not give strong evidence about the role of these solutions in the general initial value problem. Actually, this criticism can be raised for every method that is used nowadays for solving the above non linear evolution equations. Concretely, a lot of methods have been developed to solve just a class of equations, as for example the KdV one. But these methods remain confined to certain equations and its applicability for other classes is hard to see, when not impossible.

The lack of a general theory of solving non linear partial differential equations (PDEs) gives the numerical approach of fundamental importance in order to reach the solution. An indicative example is the decisive work of Zabusky and Kruskal (1965) in the case of the KdV. Obviously, a non linear term in a PDE gives problems also from the numerical point of view, but in some sense, inside any numerical theory, there exists a "general experienced way" how to handle the non linearity.

Chapter 3

Fourier spectral methods

The basic idea in any numerical method to solve a differential equation is to discretize the given continuous problem with infinitely degrees of freedom to obtain a discrete problem, or a system of equations with only a finite number of (many) unknowns that may be solved using a computer. There are basically two steps to achieve a numerical approximated solution $u_N(x)$ of a solution $u(x)$ of a differential equation. First, an appropriate finite or discrete representation of the solution must be chosen. This may take the form of an interpolating function between the values $u(x_j)$ at some suitable points x_j, or series coefficients in the finite representation

$$u_N(x) = \sum_{k=o}^{N} a_k \varphi_k(x)$$

with given expansion functions $\varphi_k(x)$.

The second step is to obtain equations for the discrete values $u_N(x_j)$ or the coefficients a_k, from the original equation. In the case of a differential equation, this second step involves finding an approximation for the differential operator in terms of the grid point values of u_N or, equivalently, the expansion coefficients.

3.1 The Fourier expansion

The set of functions

$$\varphi_k(x) = e^{ikx} \qquad (1.3)$$

form an orthogonal system over the interval $(0, 2\pi)$:

$$\int_0^{2\pi} \varphi_k(x)\varphi_l(x)^* dx = 2\pi\delta_{kl}$$

For a complex valued function u defined on $(0, 2\pi)$, we introduce the Fourier coefficients of u as

$$\hat{u}_k = \frac{1}{2\pi}\int_0^{2\pi} u(x)e^{-ikx}dx, \quad k = 0, \pm 1, \pm 2, ... \tag{2.3}$$

(The integrals exist for any function which is integrable in the sense of Lebesgue). The Fourier series of the function u is defined as

$$Su = \sum_{k=-\infty}^{+\infty} \hat{u}_k \varphi_k. \tag{3.3}$$

It represents the formal expansion of u in terms of the Fourier orthogonal system. In order to make this expansion rigorous, one has to cope with three problems:

(i) when and in which sense is the series convergent;

(ii) what is the relation between the series and the function u;

(iii) how rapidly does the series converge.

The basic issue is how u is approximated by the sequence of trigonometric polynomials

$$P_N u(x) = \sum_{k=-N}^{N-1} \hat{u}_k e^{ikx}. \tag{4.3}$$

In most cases, the most important characterization of the approximation is the number of degrees of freedom. Equation (4.3) corresponds to $2N$ degrees of freedom and we shall refer to it as the $2N$ order truncated Fourier series of u. Points (i), (ii) and (iii) have been subject of a thorough mathematical investigation. We review here only those basic results relevant to the application of spectral methods to PDEs.

We recall the following results about convergence of Fourier series. (Hereafter, a function u defined in $(0, 2\pi)$ will be called periodic if $u(0^+)$ and $u(2\pi^-)$ exist and are equal).

(a) If u is continuous, periodic and of bounded variation on $[0, 2\pi]$, then Su is uniformly convergent to u, i.e.,

$$\max_{x\in[0,2\pi]} |u(x) - P_N u(x)| \to 0, \quad as\ N \to \infty.$$

(b) If u is of bounded variation on $[0, 2\pi]$, then $P_N u$ converges pointwise to $\frac{(u(x^+)+u(x^-))}{2}$ for any $x \in [0, 2\pi]$.

(c) If u is continuous and periodic, its Fourier series does not necessarily converge at every point $x \in [0, 2\pi]$.

The series Su is said to be convergent in the mean (or L^2-convergent) to u if

$$\int_0^{2\pi} |u(x) - P_n u(x)|^2 \, dx \to 0, \quad \text{as} \quad N \to \infty.$$

Clearly, the convergence in the mean can be defined only for square integrable functions. Henceforth, we assume that $u : (0, 2\pi) \to \mathbb{C}$ such that $|u|^2$ is Lebesgue-integrable over $(0, 2\pi)$. $L^2(0, 2\pi)$ is a complex Hilbert space with inner product

$$(u, v) = \int_0^{2\pi} u(x)v(x)^* \, dx \tag{5.3}$$

and norm

$$\|u\| = \left(\int_0^{2\pi} |u(x)|^2 \, dx \right)^{\frac{1}{2}}. \tag{6.3}$$

Let S_N be the space of the trigonometric polynomials of degree N defined as

$$S_N = \text{span}\{\varphi_k = e^{ikx} | -N \le k \le N-1\}.$$

Then by the orthogonality relation one has

$$(P_N u, v) = (u, v) \qquad \forall v \in S_N.$$

This shows that $P_N u$ is the orthogonal projection of u upon the space of the trigonometric polynomials of degree N. Equivalently, $P_N u$ is the closest element to u in S_N with respect to the norm (6.3). The L^2-convergence does not imply the pointwise convergence of $P_N u$ to u at any point in $[0, 2\pi]$. However, a non trivial result[16] asserts that $P_N u(x)$ converges to $u(x)$ as $N \to \infty$ for any x outside a set of zero measure in $[0, 2\pi]$.

We deal now with the problem of the speed of convergence of the Fourier series. First of all, Notice that by the Parseval identity one has

$$\|u - P_N u\| = \left(2\pi \sum_{|k|>N} |\hat{u}_k|^2 \right)^{\frac{1}{2}} .$$

On the other hand, if u is sufficiently smooth,

$$\max_{x \in [0,2\pi]} |u(x) - P_N u(x)| \leq \sum_{|k|>N} |\hat{u}_k|.$$

So the size of the error created by replacing u with $P_N u$ depends upon how fast the Fourier coefficients of u decay to zero. This depends on the periodicity and regularity properties of u in the domain $(0, 2\pi)$. Indeed, if u is continuously differentiable in $[0, 2\pi]$, then for $k \neq 0$

$$2\pi \hat{u}_k = \int_0^{2\pi} u(x) e^{-ikx}\, dx = -\frac{1}{ik}(u(2\pi^-) - u(0^+)) + \frac{1}{ik} \int_0^{2\pi} u'(x) e^{-ikx}\, dx$$

Hence $\hat{u}_k = O(k^{-1})$. If now u' is itself continuously differentiable in $[0, 2\pi]$, the last integral is 2π times the k-th Fourier coefficient of u', hence it decays like k^{-1}. It follows that $\hat{u}_k = O(k^{-2})$ if and only if $u(2\pi^-) = u(0^+)$. Iterating this argument, one proves that if u is m-times continuously differentiable in $[0, 2\pi]$ $(m \geq 1)$, and if $u^{(j)}$ (the j-th derivative of u) is periodic for all $j \leq m - 1$, then

$$\hat{u}_k = O(k^{-m}), \qquad k = \pm 1, \pm 2, ...;$$

As a corollary we conclude that the k-th Fourier coefficient of a function which is infinitely differentiable and periodic with all its derivatives on $[0, 2\pi]$ decays faster than any negative power of k.

3.2 Fourier spectral methods

Now we are ready for the second step of the numerical approach, that is how to approximate our differential operators of a given equation according to the representation of the approximated solution of the problem. Spectral methods[17][18] may be viewed as an extreme development of the class of the discretization schemes for differential equations known generically as the methods of weighted residuals (MWR). The key elements are the trial functions and the test functions.

The trial functions are used as the basis functions for a truncated series expansion of the solution. The test functions are used to ensure that the differential equation is satisfied as closely as possible by the truncated series expansion. So if we define $\tilde{u}$ as our approximated solution of the differential equation $L(u) = 0$, and define the residual $r(x, t) = L(\tilde{u})$, the goal is to minimize the residual or, equivalently, make the residual satisfy a suitable orthogonality condition with respect to each of the test functions. The choice of trial functions is one of the features which distinguish spectral methods from finite-element and finite-difference methods.

The trial functions for spectral methods are infinitely differentiable globally defined functions, whereas for the last two methods local trial functions are used. The choice of test functions distinguishes between the three most commonly used spectral schemes, namely, the Galerkin, collocation, and tau methods.

In the Galerkin approach, the test functions are the same as the trial functions. They are, therefore, infinitely smooth functions which individually satisfy the boundary conditions. The differential equation is enforced by requiring that the integral of the residual times each test function be zero. In the collocation approach the test functions are translated Dirac delta functions centered at special so called collocation points. Spectral tau methods are similar to Galerkin methods in the way that the differential equation is enforced , however none of the test functions needs to satisfy the boundary conditions. A supplementary set of equations is used to enforce the boundary conditions.

However, the concurrence of more representations is a common feature of most spectral methods. For example, in collocation spectral methods the primary unknowns are the solution-values at the collocation points (physical unknowns), but, it is customary to use expansion coefficients (spectral unknowns) in order to calculate effectively the derivatives by recurrence relations, and then go back to physical unknowns. On the other hand, for those non linear problems whose primary unknowns are the spectral ones, the nonlinear terms give rise to convolution sums whose direct calculation is very impractical, and therefore should be avoided. On the contrary one can go to physical unknowns, computing the non linear terms by simple multiplications, and only then go back to the spectral representation.

Hereafter we will consider spectral Galerkin and spectral collocation methods with trigonometric polynomials as trial functions. In this case periodic boundary conditions are automatically satisfied. In the last section of this chapter we consider a model equation to give an example of Fourier Galerkin and Fourier collocation methods[18]

$$u_t - \Delta u + f(u, \nabla u) = 0, \tag{7.3}$$

obviously the above equation must be supplemented with an initial condition

$$u(x, 0) = u_0(x).$$

3.2.1 Spectral projection of the model equation

Equation (7.3) can be written as

$$u_t + Lu + G(u) = 0 \tag{8.3}$$

where the non linear term $G(u)$ contains u and some of its spatial derivatives. The discretization process consists of defining a space X_N of trial functions, a space Y_N of test functions, discrete approximations G_N and L_N respectively, and an orthogonal projection operator Q_N from a suitable Hilbert space which maps X_N onto the space Y_N. The weighted residual minimization statement is equivalent to the application of the orthogonal projection operator. It may be expressed as

$$Q_N(u_t^N + L_N u^N + G_N(u^N)) = 0, \quad u^N \in X_N$$

or, in variational form as $u^N \in X_N$

$$(u_t^N + L_N u^N + G_N(u^N), v) = 0, \quad \forall v \in Y_N$$

3.2.2 The Fourier Galerkin method

We look for a solution which is periodic in space in the interval $(0, 2\pi)$. The trial space X_N is S_N, the set of all trigonometric polynomials of degree $\leq N$. The approximate function u^N is represented as the truncated Fourier series

$$u^N(x,t) = \sum_{k=-N}^{N-1} \hat{u}_k(t) e^{ikx}. \tag{9.3}$$

In this method the coefficients $\hat{u}_k(t), \quad k = -N, ..., N-1$ may be cosidered as the fundamental unknowns. A set of ordinary differential equations (ODEs) for the

$\hat{u}_k(t)$ are obtained by requiring that the residual of (7.3) be orthogonal to all the test functions in $Y_N = S_N$

$$\int_0^{2\pi} (u_t^N - \Delta u^N + f(u^N, \nabla u^N))e^{-ikx} dx = 0, \quad k = -N, ..., N-1. \tag{10.3}$$

Due to the orthogonality property of the test and trial functions one obtains

$$\frac{\partial \hat{u}_k}{\partial t} + k^2 \hat{u}_k + (f(\widehat{u^N, \nabla} u^N))_k = 0, \quad k = -N, ..., N-1, \tag{11.3}$$

where

$$(f(\widehat{u^N, \nabla} u^N))_k = \frac{1}{2\pi} \int_0^{2\pi} f(u^N, \nabla u^N) e^{-ikx} dx. \tag{12.3}$$

The initial conditions are clearly

$$\hat{u}_k(0) = \frac{1}{2\pi} \int_0^{2\pi} u_0(x) e^{-ikx} dx. \tag{13.3}$$

This system of ordinary differential equations is typically discretized in time by a method which is explicit for the non linear term and implicit for the linear one [18][19]. The wave number $k = -N$ appears unsymmetrically in this approximation. If $\hat{u}_{-N}$ has a non-zero imaginary part, then u^N is not a real-valued function. This can lead to a number of difficulties and it is advisable in practice simply to enforce the condition that $\hat{u}_N$ is zero. This can be automatically avoided if one approximates the solution with an odd number of modes, however we want the most widely used routine of Fourier transform[20] (Fast Fourier Transform, FFT) which requires an even number of modes and we want to describe spectral methods in a way that correspond directly to the way they are implemented. Observe that the non linear term in (11.3) normally gives rise to convolution sums from the integral (12.3), whose direct calculations is not convenient. Fortunately, transform methods allow this term to be evaluated computing the non linear term by simple multiplications of the solution-values at collocation points.

3.2.3 The Fourier collocation method

We again assume periodicity on $(0, 2\pi)$ and take $X_N = S_N$, but now think of the approximate solution u^N as represented by its values at the grid points $x_j = \frac{\pi j}{N}$, $j = 0, 1, ..., 2N-1$. The grid values of u^N are related to its discrete Fourier coefficients by

$$u^N(x_j) = \sum_{k=-N}^{N-1} \tilde{u}_k^N e^{ikx_j} \quad j = 0, 1, ..., 2N-1, \tag{14.3}$$

where

$$\tilde{u}_k^N = \frac{1}{2N} \sum_{j=0}^{2N-1} u^N(x_j) e^{-ikx_j} \tag{15.3}$$

are the discrete Fourier coefficients.

For the collocation method[18] we require that (7.3) is satisfied at these points, i.e.,

$$\left[\frac{\partial u^N}{\partial t} - \Delta u^N + f(u^N, \nabla u^N)\right]_{x=x_j} = 0, \quad j = 0, 1, ..., 2N-1. \tag{16.3}$$

Initial conditions are obviously $u^N(x_j, 0) = u_0(x_j)$.

Let us introduce the Fourier collocation derivative of u^N, $\mathfrak{D}_N u^N$, to be distinguished from the true spectral derivative of u, which we refer to as the Fourier Galerkin derivative. So we define

$$(\mathfrak{D}_N u^N)(x_j) = \sum_{k=-N}^{N-1} (ik\tilde{u}_k^N) e^{ikx_j} \tag{17.3}$$

Therefore the linear operator $-\Delta_N u^N$ is $-\mathfrak{D}_N^2 u^N$. The non linear term is discretized as $f_N(u^N, \mathfrak{D}_N u^N)$. The orthogonal projection is already expressed by (16.3); it is taken from $C^0([0, 2\pi])$ into S_N with respect to the discrete inner product

$$(u, v)_N = \frac{\pi}{N} \sum_{j=0}^{2N-1} u(x_j) v(x_j)^*,$$

which coincides with the inner product (5.3) if $u, v \in S_N$. (Again an explicit/implicit time-discretization is employed.)

3.3 Convergence, consistency and stability

Whatever numerical approach is chosen, one would ideally like to show that the method is convergent, i.e., the global error for some fixed and finite value of t tends to zero as the step lengths (nodal separations) tend to zero, or as the number of basis functions tend to infinity. A consistent method is one in which the truncation error tends to zero when the discretization parameter N tends to infinity. A scheme will be called stable if it is possible to control the discrete solution by the data in a way independent of the parameter N. This means that a suitable norm of the discrete solution is bounded by a constant multiple of an appropriate norm of the data, and all the norms involved do not depend on N.

Here, we present a simple example: the stability of the Fourier Galerkin method for the wave equation and we refer to the book of Canuto *et al.* (1988) for a complete discussion about stability and convergence.

3.3.1 An example: the wave equation

The hyperbolic problem

$$\begin{cases} \frac{\partial u}{\partial t} - \frac{\partial u}{\partial x} = 0, & \text{in } 0 \le x \le 2\pi, \quad t > 0; \\ u(x,t) = u(x+2\pi, t); & \\ u(x,0) = u_0(x), & \text{in } 0 \le x \le 2\pi. \end{cases} \tag{18.3}$$

is discretized by

$$\int_0^{2\pi} \left[\frac{\partial u^N}{\partial t} - \frac{\partial u^N}{\partial x} \right] v(x)^* dx = 0 \quad \forall v \in S_N, \tag{19.3}$$

and

$$u^N(x,t) = \sum_{k=-N}^{N-1} \hat{u}_k(t) e^{ikx}.$$

For any $t > 0$, let us set $v(x) = u^N(x,t)$ in (19.3). An integration by parts yields

$$\mathrm{Re} \int_0^{2\pi} \frac{\partial u^N}{\partial x} u^{N*} dx = \frac{1}{2}\{|u^N(2\pi,t)|^2 - |u^N(0,t)|^2\} = 0$$

by the periodicity condition. It follows that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int_0^{2\pi} |u^N(x,t)|^2 dx = \mathrm{Re} \int_0^{2\pi} \frac{\partial u^N}{\partial x} u^{N*} dx = 0,$$

i.e., the L^2-norm (in space) of the spectral solution is constant in time. Therefore for any $t > 0$

$$\int_0^{2\pi} |u^N(x,t)|^2 dx = \int_0^{2\pi} |P_N u^N(x,0)|^2 dx =$$

$$\int_0^{2\pi} |P_N u_0(x)|^2 dx \le \int_0^{2\pi} |u_0(x)|^2 dx.$$

Then the above Galerkin scheme is stable.

For the convergence, we just provide an estimation of the error[18] between the exact and the spectral solution. For all $t > 0$, we have

$$\int_0^{2\pi} |u(x,t) - u^N(x,t)|^2 dx \le C \cdot N^{-2m} \int_0^{2\pi} \left| \frac{\partial^m u}{\partial x^m}(x,t) \right|^2 dx.$$

Chapter 4

Fourier spectral methods for the KdV and KdV Hirota equation

In this chapter we will report some recent results about the numerical approximation by Fourier spectral methods to the KdV equation with periodic solutions,

$$\begin{cases} u_t + uu_x + \alpha u_{xxx}, & x \in \mathbb{R}, \quad t > 0; \\ u(x + 2\pi, t) = u(x, t); \\ u(x, 0) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{1.4}$$

Then we will consider, briefly, the numerical experiment of Zabusky and Kruskal[1]. We have repeated their experiment using a Fourier Galerkin scheme as explained in paragraph 3.2.2. The present calculations were intended only for graphical use. The end of the chapter will be devoted to the development a new method, that is, a numerical scheme for the Hirota version (11.2) of the KdV equation. This method seems to be fruitful especially for those problems where one wants to handle KdV singular solutions, whose points of singularity are just points where the solution of (11.2) reaches zero.

4.1 Fourier spectral methods for the KdV equation

In the last decade Fourier spectral methods have been used in many applications of the KdV equation[21]. The classical Fourier Galerkin scheme has been used, as well as the Fourier collocation method[18]. We cannot report here all these applications, neither we can do a comparison between the classical finite differences and the most powerful spectral schemes. So we restrict ourselves to report just few significant results (for a complete discussion and extended results see Ref. [22]). Hereafter, we will not be concerned with any time discretization of the KdV equation. However, we recall that the semi-implicit time advancing schemes are customarily used for such kind of equations, and for finite time intervals they are stable without any restriction on the time and space discretization parameters[23][24].

Working with 2π-periodic function, we introduce the periodic Sobolev space defined over $]0, 2\pi[$. We first recall the definition of classical Sobolev spaces. We set

$$L^2(0,2\pi) = \{f :]0,2\pi[\rightarrow \mathbb{C}, \quad \|f\| = \left[\int_0^{2\pi} |f(x)|^2 dx\right]^{\frac{1}{2}} < \infty\} \tag{2.4}$$

And we denote its scalar product by $(\cdot,\cdot)$. Now, for any integer $r > 0$, we set

$$H^r(0,2\pi) = \{f \in L^2(0,2\pi), \quad \|f\|_r = \left[\sum_{j=0}^{r} \|\frac{\partial^j f}{\partial x^j}\|^2\right]^{\frac{1}{2}} < \infty\}$$

Next we consider the subspace $C_\sharp^\infty(0,2\pi)$ of $C^\infty(0,2\pi)$ of all functions that are 2π-periodic together with all their derivatives. And finally denote with $H_\sharp^r(0,2\pi)$ the closure of $C_\sharp^\infty(0,2\pi)$ in $H^r(0,2\pi)$.

It is well known that the family

$$\varphi_k(x) = \frac{1}{\sqrt{2\pi}} e^{ikx}, \quad k \in \mathbb{Z}$$

is orthonormal and complete in $L^2(0,2\pi)$. Thus a natural approximation of $L^2(0,2\pi)$ by periodic functions will consist of the space defined by

$$S_N = \text{span}\{\varphi_k, -\frac{N}{2} \leq k \leq \frac{N}{2}\} \quad \forall N \in 2N. \tag{3.4}$$

Denote by P_N the operator

$$P_N g = \sum_{k=-\frac{N}{2}}^{\frac{N}{2}} \hat{g}_k \varphi_k, \quad \forall g \in L^2(0,2\pi), \tag{4.4}$$

with

$$\hat{g}_k = \int_0^{2\pi} g(x)\varphi_k(x)^* dx, \quad k \in \mathbb{Z}.$$

A spatial approximation (continuous in time) of problem (1.4) based on the Fourier Galerkin method reads as follows:

Find a mapping $u^N : [0,T] \to S_N$ such that $\forall \psi \in S_N$, $\forall t$, $0 \leq t \leq T$,

$$\begin{cases} (u_t^N + u^N u_x^N + \alpha u_{xxx}^N, \psi) = 0; \\ u^N(0) = P_N u^0. \end{cases} \tag{5.4}$$

This entails a non linear system of ODEs for the Fourier coefficients $\hat{u}_k^N(t)$ of the solution u^N. We present now the main properties enjoyed by the above Fourier Galerkin approximation. They are concerned with the concepts of conservation, stability, uniqueness and convergence.

Lemma 1.4

There exists a unique solution u^N to problem (5.4). Moreover this solution conserves the three first energy integrals of the KdV equation, namely

$$(i) \qquad \frac{\partial}{\partial t}\left[\int_0^{2\pi} u^N(x,t)dx\right] = 0$$

(ii) $$\frac{\partial}{\partial t}\left[\int_0^{2\pi} |u^N(x,t)|^2 dx\right] = 0$$

(iii) $$\frac{\partial}{\partial t}\left[\int_0^{2\pi} \left(\alpha(\frac{\partial u^N}{\partial x})^2 - \frac{u^{N3}}{3}\right)(x,t)dx\right] = 0.$$

Next for the stability we have

Lemma 2.4

Assume that u^0 belongs to $H^1_\sharp(0,2\pi)$. Then there exists a constant $c > 0$, independent of N such that for any t, $0 \le t \le T$

$$\|u^N(\cdot,t)\|_1 \le c,$$

where $\|\cdot\|_1$ is given by

$$\|g\|_1^2 = \sum_{k=-\frac{N}{2}}^{\frac{N}{2}} (1+k^2)\hat{g}_k, \quad \forall g \in S_N.$$

We now turn to the convergence estimate for the Galerkin approximation of the KdV equation.

Theorem 1.4

Assume that $u^0 \in H^m_\sharp(0,2\pi)$, for some integer $m \ge 2$. Then there exists a constant $c > 0$ independent of N such that for any t , $0 \le t \le T$

$$\|u(\cdot,t) - u_N(\cdot,t)\| \le cN^{1-m}.$$

For the proof of the two lemmas and of the theorem see Ref. [22].

Despite its mathematical interest, the Fourier Galerkin method has been generally abandoned in the applications in favour of the Fourier collocation method.

In fact, the latter method allows a very efficient treatment of the non linear term $u\frac{\partial u}{\partial x}$. For this last method Lemma 1.4 and the above Theorem 1.4 have been proved[22], but not the conservation of the second integral (ii).

4.2 The solitons of Zabusky and Kruskal revisited

Zabusky and Kruskal[1] (ZK) discovered the soliton by numerically investigating periodic solutions of the KdV equation. Subsequently[25] the exact solution of the KdV equation on the infinite line were found.

They considered the KdV equation

$$u_t + uu_x + \delta^2 u_{xxx} = 0, \qquad x \in [0, 2\pi] \tag{6.4}$$

using $\delta^2 = 4.8 \times 10^{-4}$ and the (approximated) periodic initial condition

$$u_0(x) = \cos \pi x. \tag{7.4}$$

Thus, initially, $\max |\delta^2 u_{xxx}| / \max |uu_x| = 0.004$, so the third term can be neglected and one deals with the equation $u_t + uu_x = 0$. Its solution is given by the implicit relation

$$u = \cos \pi (x - ut).$$

They used the finite difference scheme, resulting in the iteration procedure

$$u_m^{n+1} = u_m^{n-1} - \frac{\Delta t}{3\Delta x}(u_{m+1}^n + u_m^n + u_{m-1}^n)(u_{m+1}^n - u_{m-1}^n) -$$

$$\frac{\Delta t}{\Delta x^3}(u_{m+2}^n - 2u_{m+1}^n + 2u_{m-1}^n - u_{m-2}^n).$$

where $u_m^n = u(n\Delta t, m\Delta x)$, Δt is the time step and Δx the mesh size.

The linear stability requirement for this scheme is that

$$\frac{\Delta t}{\Delta x^3} \leq (4 + \Delta x^3 |u_{max}|)^{-1},$$

whereas within the Fourier Galerkin calculation the stability limit is

$$\frac{\Delta t}{\Delta x^3} \leq \frac{3}{2\pi^2}.$$

In Fig. 1 we graphically display the result of Zabusky and Kruskal. They have used $\Delta t = 0.0005$ and $\Delta x = 0.01$. Fig. 2 shows our corresponding results obtained using a Fourier Galerkin scheme for the same equation (6.4) and the approximated initial solution (7.4). We have employed 128 mesh points, that is $\Delta x = 0.015$ and $\Delta t = 0.01$.

Figure 1

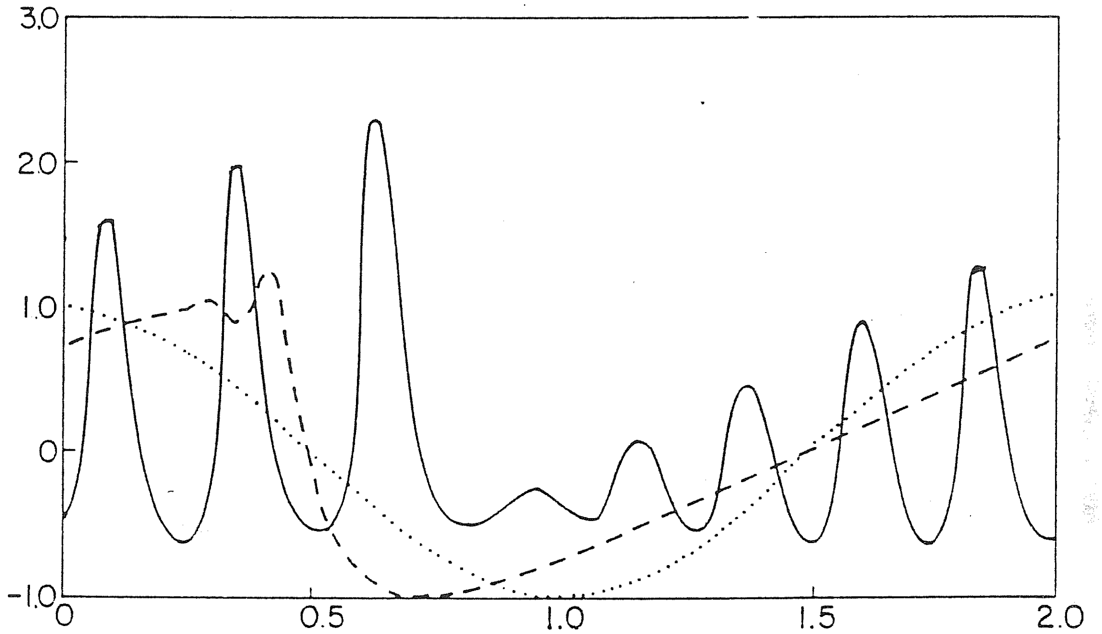
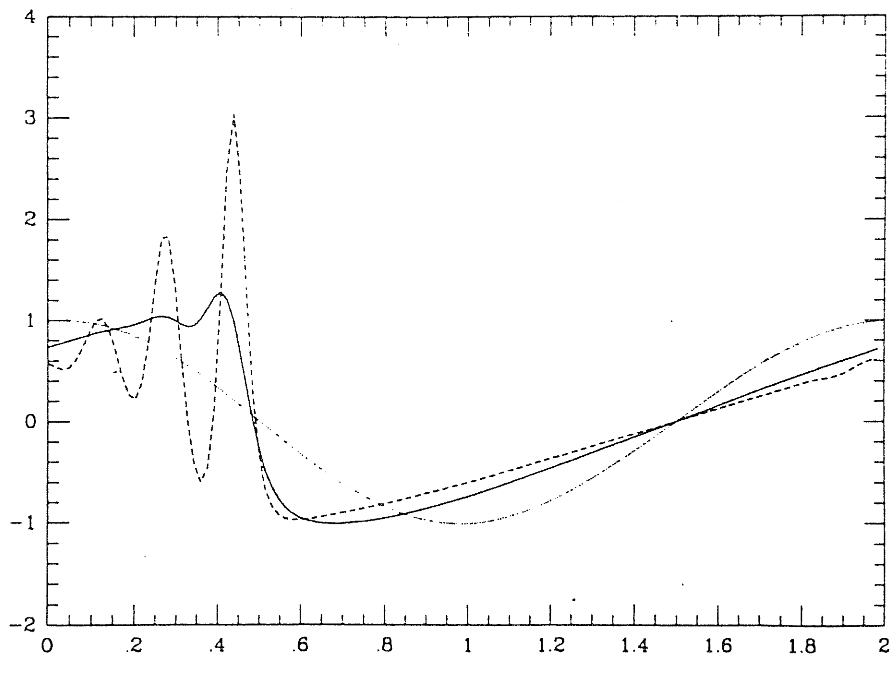


Figure 2



4.3 A new numerical approach using the KdV Hirota equation

As we have mentioned before, R. Hirota found in 1971 a direct method to solve the KdV equation. He started from the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0, \tag{8.4}$$

and substituted for u the transformation

$$u = 2\frac{\partial^2}{\partial x^2} \log f \tag{9.4}$$

finding the homogeneous "KdV Hirota equation" in $f(x,t)$

$$(D_t D_x + D_x^4)\ f \cdot f = 0 \tag{10.4}$$

that is, written explicitly

$$f f_{xxxx} - 4f_x f_{xxx} + 3f_{xx}^2 + f f_{xt} - f_x f_t = 0.$$

The soliton solutions are given by the formula (13.2) which we report here

$$f_N = \sum_{\mu=0,1} \exp\left(\sum_{i<j}^{N} A_{ij}\mu_i\mu_j + \sum_{i=1}^{N} \mu_i\theta_i\right). \tag{12.4}$$

the definition of A_{ij} and of θ_i is given in section 2.2.

Notice that the transformation (9.4) can be rewritten as

$$u = 2\frac{f_{xx}f - f_x^2}{f^2}. \tag{13.4}$$

One sees immediately that if the function $f(x,t)$ has some points of its (x,t) domain where it is zero, then these could be singular points of u. Notice that the

KdV equation admits singular solutions, and these solutions can be constructed using Schur polynomials[26]. This observation suggests that a natural choice in order study theoretically and numerically the properties of the singular solutions of the KdV equation, is to consider the KdV Hirota equation (10.4).

We admit that equation (10.4) is a non linear equation and contains terms with derivatives of higher order compared to those appearing in the KdV equation (8.4). However, despite this, there are in principle no problems to handle a function with zeros, whereas it is difficult to deal with a function which has points where it grows to infinity.

A further motivation to proceed in this direction is the recent work of E. Brezin *et al.*[27] (1990) where the authors look for some poles of the solution of a Painleve' non linear partial differential equation. There they study the equation

$$x = f^3 + f f_{xx} + \frac{1}{2} f_x^2 + \frac{1}{10} f_{xxxx}$$

with asymptotic boundary conditions (this is a steady problem).

We cannot go into details, but the solution of the above Painleve' equation is used as initial solution of a KdV equation in the study of two dimensional gravity where KdV equations appear. In other words they investigate the regularity properties of an initial solution of a KdV equation. Here we present a method which enables us to investigate the regularity properties of the evolved in time solution of the KdV equation (If you had the patience to follow the argument up to this part you deserve a good coffee, which I will be glad to offer).

4.3.1 Fourier collocation scheme for the KdV Hirota equation

In section 3.1 we dedicated just few words to the Fourier expansion. There is shown how the size of the error, created by replacing u with $P_N u$, is strictly related to periodicity and regularity properties of u. The error is negligible when the function u is periodic together with all its derivatives up to a high order. A Fourier spectral method works satisfactory in all cases where the truncated Fourier series expansion approximates the function with a negligible error. That is, in the case of periodic functions. If a function is not periodic it can be considered as periodic if approaching the boundary of its domain it goes to zero quickly enough, such that the values of the function and of its derivatives can be approximated by zero, and so considered periodic.

Even if the existence of periodic solutions of the KdV equation has been proved, all the explicitly known solutions are non periodic in the space dimension. In addition the classical soliton solution (4.1) corresponds in the Hirota version to

$$f(x,t) = 1 + e^{kx - k^3 t + \theta_0},$$

and this solution is obviously different from zero whatever computational domain we use.

But, just for brevity (otherwise we have to introduce at least Chebyshev spectral methods), we still want to use Fourier spectral methods. We cannot try to repeat the numerical experiment of Zabusky and Kruskal (where an initial periodic solution is used), because this solution has sense for a KdV equation where the third derivative is negligible, at least initially. But for the KdV Hirota equation this is not the case. So, not having available a periodic solution of the KdV equation, we test our method with an initial condition which is periodic and has no zeros in the spatial domain $[0, L]$.

An appropriate initial condition seems to be

$$f_0(x) = \frac{1}{2\pi^2} e^{-\cos \pi x}. \tag{18.4}$$

Now we can formulate correctly our numerical problem:

let us assume periodicity in $(0, 2)$ and approximate f by a discrete function $\tilde{f}$ which takes the values at the grid points $x_j = \frac{j}{N}$, $j = 0, 1, ..., 2N-1$. Then for the collocation method we require that (10.4) be satisfied at these collocation points, i.e.,

$$\begin{cases} \tilde{f}\tilde{f}_{xxxx} - 4\tilde{f}_x\tilde{f}_{xxx} + 3\tilde{f}_{xx}^2 + \tilde{f}\tilde{f}_{xt} - \tilde{f}_x\tilde{f}_t|_{x=x_j} = 0; \\ \tilde{f}(x_j, 0) = f_0(x_j) \end{cases} \tag{19.4}$$

for $j = 0, 1, ..., 2N-1$.

Derivatives with respect to time occur for $\tilde{f}$ and $\tilde{f}_x$ for which we adopt the following discretization

$$\tilde{f}_t(x_j) \to \frac{\tilde{f}^n(x_j) - \tilde{f}^{n-1}(x_j)}{\Delta t},$$

$$\tilde{f}_{xt}(x_j) \to \frac{\tilde{f}_x^{n+1}(x_j) - \tilde{f}_x^n(x_j)}{\Delta t},$$

where $\tilde{f}^n(x_j) = \tilde{f}(x_j, n\Delta t)$, and Δt denotes the time step, n a positive integer.

Substituting this time discretization into (19.4) we find the values of the first spatial derivative for the next step.

$$f_x^{n+1}(x_j) = 2f_x^n(x_j) - f_x^n(x_j)\frac{f^{n-1}(x_j)}{f^n(x_j)} -$$

$$\tag{20.4}$$

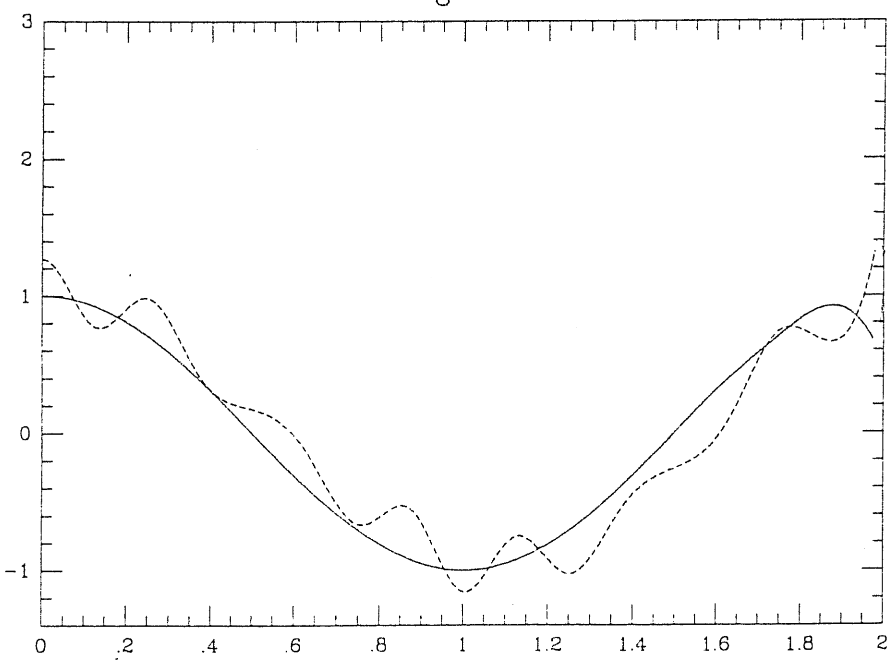
$$\frac{\Delta t}{f^n(x_j)}\left[f^n(x_j)f_{xxxx}^n(x_j) - 4f_x^n(x_j)f_{xxx}^n(x_j) + 3(f_{xx}^n(x_j))^2\right]$$

for $j = 0, 1, ...2N - 1$.

Starting from the initial conditions, and applying (20.4) to them, we recover the corresponding discrete Fourier coefficients of the spatial derivative f_x and then, those of the solution f at the next time step (see 17.3). As last step, using the inverse discrete Fourier transform (DFT^{-1}) we obtain the values of the approximated solution $\tilde{f}(x_j)$ at $t = \Delta t(n+1)$. Therefore by means of (20.4) the initial solution evolves in time when we iterate the above procedure.

We have implemented the algorithm in a computer using the initial solution (18.4), different values of N ($N = 2^m$, m=4,5,6,7) and $\Delta t \leq 10^{-4}$. The used scheme is stable for these values of the parameters. In Fig. 3 we have plotted the value of the solution u of the KdV equation as obtained by the transformation (13.4) from the approximated solution $\tilde{f}$ of the KdV Hirota equation (19.4).

Figure 3



We would like to emphasize the importance of the numerical approach to the KdV Hirota equation in order to study, for example, singular solutions of

KdV equation. The particular scheme proposed here, is just an example of this approach. As we have explained the Fourier spectral method employed is good if one has at disposal an initial periodic solution of the KdV Hirota equation so the aim of the figure is just to say that our algorithm works (starting with $t = 0$ we have after few time steps the solid line, the dashed line corresponds to $t = 1$). In addition, this idea is not restricted to the KdV equation: also the Boussinesq equation, the KP equation, the Sine-Gordon and the Non-linear Schrödinger equation, are just few examples of equations which have a counterpart in a Hirota form[14]. In Appendix III a more complete list of such equations is given.

Chapter 5

A numerical investigation of the Kadomtsev-Petviashvili (KP) equation

Until now we have studied only numerical problems in one space dimension. This has been done just to make lighter the notation and for the sake of brevity. Here we will consider a two-dimensional evolution equation: the KP equation (6.2). We have chosen this equation first because it is the natural counterpart of the KdV equation in two-space dimensions and so it presents the main features of a model equation to present Fourier spectral methods in two dimensions. It has a simple non linear term which involve the solution function and its first and second derivative in the x direction and it possesses dispersive terms such that the concurrence of these two facts make it possible to have soliton solutions. Second, the KP equation can be rewritten using Hirota operators.

Finally, in our knowledge, there is only an article in the literature where the KP equation is investigated numerically[9]. There the authors use a finite difference scheme.

5.1 A Fourier Galerkin scheme for the KP equation

It is easily recognized that any solution of the KdV equation is also a solution of the KP equation. In particular, the soliton solutions of the KdV equation as (4.1) are proved to be unstable solutions[9] of the KP equation

$$(u_t + 6uu_x + u_{xxx})_x - u_{yy} = 0 \tag{1.5}$$

This observation led to the search of stable two-dimensional solitons localized in the space domain. These solutions were found by V.I. Petviashvili in 1975. A modern way in order to construct these solutions is that given by Hirota[14] and reported here in section 2.2 . So a (polynomial) one soliton solution is given by

$$u(x,y,t) = \frac{4(-(x' + P_R y')^2 + P_I^2 y'^2 + \frac{3}{P_I^2})}{((x' + P_R y')^2 + P_I^2 y'^2 + \frac{3}{P_I^2})^2} \tag{2.5}$$

where $x' = x - (P_R^2 + P_I^2)t$ and $y' = y + 2P_R t$. so the soliton moves on the (x, y) plane with a velocity whose the components are $v_x = P_R^2 + P_I^2$ and $v_y = -2P_R$. It has been proved that this solution is stable.

Using the Hirota method one can easily construct M soliton solutions which have polynomial behavior (for a two soliton solution see Appendix II, a fascinating feature of this two soliton solution is that they do not interact).

In the work of Manakov *et al.*, equation (1.5) is studied using a KdV soliton solution as initial solution to the KP equation. They used a Hopscotch finite difference method[9], which was used with success to study the KdV equation. We cannot go into the details of this scheme. Let us say that we have repeated the numerical experiment of Manakov using the same Hopscotch scheme, but we used (2.5) as initial solution. In addition we implemented the popular Leapfrog numerical scheme[20] for our KP equation using the same discretization of the space derivatives and (2.5) as initial solution. In both cases we have experienced a numerical dissipation, that is the presence of dissipative terms due to the discretization of the space derivatives.

5.1.1 The algorithm

Clearly the initial solution (2.5) is not periodic in the (x, y) space, but it is possible to choose a suitable space domain such that the value of (2.5) on the boundary is nearly zero. As we said, in this case Fourier spectral methods work satisfactorily. Let us use as a computational domain the rectangle $(A, B) \times (C, D)$ such that $x \in [A, B]$ and $y \in [C, D]$. Define $L_x = B - A$ and $L_y = D - C$. Our approximated solution $\tilde{u}$ is represented by

$$\tilde{u}(x, y, t) = \sum_{k=-N_x}^{N_x-1} \sum_{m=-N_y}^{N_y-1} \hat{u}_{km}(t) e^{i\frac{2\pi}{L_x}kx} e^{i\frac{2\pi}{L_y}my} \tag{3.5}$$

so the fundamental unknowns are the coefficients $\hat{u}_{km}(t)$, $k = -N_x, ...N_x - 1$ and $m = -N_y, ...N_y - 1$.

If one substitutes the approximated solution (3.5) into (1.5) one encounters some problems only for the non linear term $6(uu_x)_x$ because there convolution sums arise for the spectral coefficients. In this case it is better to use a collocation scheme for the non linear term. That is, suppose to know the spectral unknowns $\hat{u}_{km}(t)$, introduce the grid points (x_i, y_j)

$$x_i = \frac{L_x}{2N_x} i, \qquad i = 0, 1, ..., 2N_x - 1,$$

$$y_j = \frac{L_y}{2N_y} j, \qquad j = 0, 1, ..., 2N_y - 1,$$

and evaluate $\tilde{u}(x_i, y_j, t)$ at these points by means of the discrete Fourier transform (DFT^{-1})

$$\tilde{u}(x_i, y_j, t) = \sum_{k=-N_x}^{N_x-1} \sum_{m=-N_y}^{N_y-1} \hat{u}_{km}(t) e^{i\frac{2\pi}{L_x}kx_i} e^{i\frac{2\pi}{L_y}my_j} . \tag{4.5}$$

We need now the value of $\tilde{u}_x$ at the same grid points. For, starting from the spectral coefficients $ik\hat{u}_{km}(t)$ and again using the inverse discrete Fourier transform we obtain

$$\tilde{u}_x(x_i, y_j, t) = \sum_{k=-N_x}^{N_x-1} \sum_{m=-N_y}^{N_y-1} ik\hat{u}_{km}(t) e^{i\frac{2\pi}{L_x}kx_i} e^{i\frac{2\pi}{L_y}my_j}. \tag{5.5}$$

By a simple multiplication we have for any grid point i, j the product

$$\{\tilde{u}(x_i, y_j, t)\tilde{u}_x(x_i, y_j, t)\};$$

let us indicate it by $(\tilde{u}\tilde{u}_x)(x_i, y_j, t)$.

Finally by direct Fourier transform

$$\widehat{(\tilde{u}\tilde{u}_x)}_{km}(t) = \frac{1}{(2N_x)(2N_y)} \sum_{i=0}^{2N_x-1} \sum_{j=0}^{2N_y-1} (\tilde{u}\tilde{u}_x)(x_i, y_j, t) e^{-i\frac{2\pi}{L_x}kx_i} e^{-i\frac{2\pi}{L_y}my_j}.$$

From this final result the spectral coefficient of the term $(uu_x)_x$ is immediately given

$$6ik\widehat{(\tilde{u}\tilde{u}_x)}_{km}(t).$$

We are ready to write our equation in a Fourier Galerkin form, that is a system of ordinary differential equations (ODE) for the spectral unknowns $\hat{u}_{km}(t)$

$$i(\frac{2\pi}{L_x})k\frac{\partial \hat{u}_{km}}{\partial t} + 6i(\frac{2\pi}{L_x})k\widehat{(\tilde{u}\tilde{u}_x)}_{km} + (\frac{2\pi}{L_x})^4 k^4 \hat{u}_{km} + (\frac{2\pi}{L_y})^2 m^2 \hat{u}_{km} = 0, \tag{6.5}$$

where $k = -N_x, ..., N_x - 1$ and $m = -N_y, ..., N_y - 1$.

In order to solve this system of ODE we use an explicit/implicit time discretization. Discretize the time, $t_n = n\Delta t$, n integer, Δt the time step, and suppose to know the spectral unknowns $\hat{u}_{km}$ at t_n, calling them $\hat{u}_{km}^n$. Then a term is explicit if it is evaluated at t_n. Implicit if it must be evaluated at the new time t_{n+1}.

In our calculation we adopt the following time discretization of equation (5.5)

$$i(\frac{2\pi}{L_x})k(\frac{\hat{u}_{km}^{n+1} - \hat{u}_{km}^n}{2\Delta t}) + 6i(\frac{2\pi}{L_x})k\widehat{(\tilde{u}\tilde{u}_x)}_{km}^n + (\frac{2\pi}{L_x})^4 k^4 \hat{u}_{km}^{n+1} + (\frac{2\pi}{L_y})^2 m^2 \hat{u}_{km}^{n+1} = 0. \tag{7.5}$$

We solve this equation for $\hat{u}_{km}^{n+1}$ and obtain the values of the approximated solution $\tilde{u}$ at $t = t_{n+1}$. In equation (6.5) the spectral unknown $\hat{u}_{00}$ is undetermined. It corresponds to the integral of the approximated solution (3.5) over the rectangle $(A, B) \times (C, D)$ divided by the area of this rectangle. It has been shown[8] that the net volume of the soliton solution (2.5) must be zero (we have verified it numerically) and so we let $\hat{u}_{00}$ equal zero for each time step.

In order to check the convergence and stability properties of the scheme here proposed, we presents some experiments on approximating the exact solution (2.5) of the KP equation. We consider the solution corresponding to the values $P_R = 1$ and $P_I = 10^{-3}$ in (2.5). This is a soliton which attains its maximum at $t = 0$ in $x = 0,\ y = 0$ and moves with a velocity $v_x = 1$ and $v_y = -2$ in the (x, y) plane.

We report in table I the relative error in the maximum norm

$$\xi = \frac{\max_{ij} |\tilde{u}(x_i, y_j, T) - u(x_i, y_j, T)|}{\max_{ij} |\tilde{u}(x_i, y_j, T)|},$$

$$i = 0, 1, ..., 2N_x - 1, \quad j = 0, 1, ..., 2N_y - 1,$$

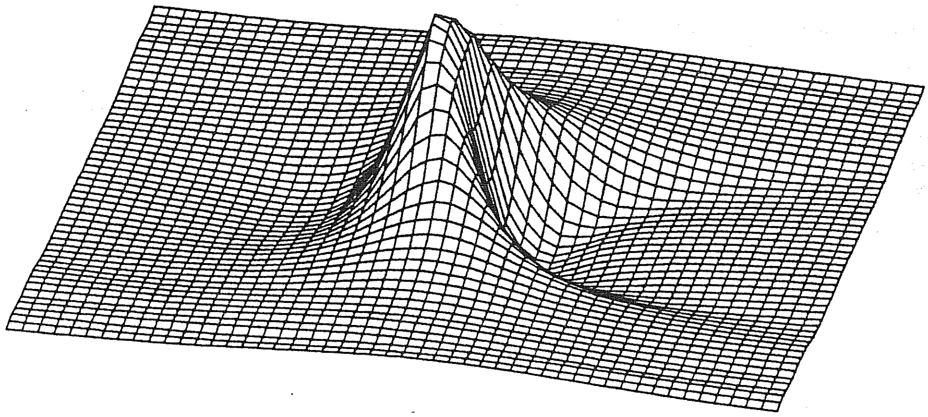
at a time T for different values of N_x and N_y. Our computational space domain is the rectangle $(-17, 17) \times (-25, 25)$ so that the initial solution (2.5) can be considered zero on the boundary together with its partial derivatives. The abscissae $x_i,\ i = 0, 1, ..., 2N_x - 1$ and the ordinate $y_j,\ j = 0, 1, ..., 2N_y - 1$ are exactly the collocation points defined above. We take $T = 1$ as the time level at which the results are displayed.

In Fig. 4 we show graphically the approximated solution computed with our Fourier Galerkin collocation scheme in the case $N_x = 32$ and $N_y = 64$ and time step $\Delta t = 10^{-2}$.

TABLE I. Decimal logarythm of the errors ξ displayed at T=1 for two different values of the time step Δt.

Nx	Ny	$\log_{10}\xi \quad (\Delta t = 10^{-2})$	$\log_{10}\xi \quad (\Delta t = 2\times 10^{-2})$
16	32	−2.86015	−2.81673
32	32	−2.86354	−2.90215
32	64	−2.90080	−2.91017
64	64	−2.90878	−2.92804

Figure 4



Our method is stable and results more efficient than that used by Makhankov *et al.* (1981). In their article the relative error in the maximum norm is of the order of 10^{-2} but they need $\Delta t = 10^{-3}$ to achieve stability, otherwise their scheme is unstable (we have checked it).

Conclusions and Outlook

We would like to summarize here the main results contained in this thesis. Even if the guiding argument throughout this thesis are the spectral methods for the numerical investigation of partial differential equations, we have given just the essentials of particular spectral methods that is the Fourier spectral methods. Obviously, also for these methods we have presented the main features in order to give this method self consistent. We have chosen these methods because they are very efficient and because they provide us, for the first time, numerical methods for which a deep numerical analysis can be done. This is essential when one wants to investigate new problems for which a comparison with old results cannot be done.

We have used these methods in order to investigate old and new problems. So the numerical investigation of the KP equation is an old problem but there is a lack in the literature for it. The only article we know proposes a scheme which works only when one uses the soliton solution of the KdV equation as the initial solution of the KP equation. We have tested their scheme using our non trivial initial solution and experienced some problems. On the other hand we have proposed a Fourier spectral scheme which works satisfactorily even if it cannot be the best one because the investigated solution is not periodic (surely Chebyshev spectral methods will give us better results). In any case, at this time, our scheme is the most efficient we know.

In this thesis we propose a new problem that is the numerical investigation of those non linear equations which can be translated into bilinear equations using generalized Cole-Hopf transformations (see Chapter 2 and Appendix I and III) and the Hirota operators. We suggest this problem because if one takes a solution of the original equation which is singular at some points of its space domain then the analogue soliton solution of the corresponding Hirota equation will have zeros at those points. Therefore the numerical approach to some Hirota equations seems to be fruitful in order to study the singular solutions of the non linear equations which have that Hirota counterpart. We have given an example of how the spectral methods work in the case of the KdV Hirota equation. Notice that the same approach can be done for the KP equation (which has also singular solutions).

Appendix I

We list some properties of the operators D_t, D_x introduced in the text. Define

$$D_t^n D_x^m a \cdot b = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^n \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^m a(x,t)b(x',t')|_{x=x',t=t'}.$$

The following properties are easily seen from the definition.

i) $D_x^m a \cdot 1 = (\frac{\partial}{\partial x})^m a.$

ii) $D_x^m a \cdot b = (-1)^m D_x^m b \cdot a.$

iii) $D_x^m a \cdot a = 0, \quad for\ odd\ m.$

iv) $D_x^m a \cdot b = D_x^{m-1}(a_x \cdot b - a \cdot b_x).$

v) $D_x^m a \cdot a = 2D_x^{m-1} a_x \cdot a, \quad for\ even\ m.$

vi) $D_x D_t a \cdot a = 2D_x a_t \cdot a = 2D_t a_x \cdot a.$

vii) $D_x^m e^{p_1 x} e^{p_2 x} = (p_1 - p_2)^m e^{(p_1+p_2)x}.$

viii) $e^{\epsilon D_x} a(x) \cdot b(x) = a(x+\epsilon) \cdot b(x-\epsilon)$

ix) $D_t(D_x a \cdot b) \cdot ab = D_x(D_t a \cdot b) \cdot ab.$

The following identities are useful to transform nonlinear differential equations into the bilinear forms.

i.a) $\exp(\epsilon \frac{\partial}{\partial x}) \frac{a}{b} = \frac{[\exp(\epsilon D_x) a \cdot b]}{[\cosh\ (\epsilon D_x) b \cdot b]}.$

ii.a) $\frac{\partial}{\partial x}(\frac{a}{b}) = \frac{D_x a \cdot b}{b^2}.$

iii.a) $\frac{\partial^2}{\partial x^2}(\frac{a}{b}) = \frac{D_x^2 a \cdot b}{b^2} - (\frac{a}{b}) \frac{D_x^2 b \cdot b}{b^2}.$

iv.a) $\frac{\partial^3}{\partial x^3}(\frac{a}{b}) = \frac{D_x^3 a \cdot b}{b^2} - 3(\frac{D_x a \cdot b}{b^2}) \frac{D_x^2 b \cdot b}{b^2}.$

v.a) $2\frac{\partial^2}{\partial x^2} \log f = \frac{D_x^2 f \cdot f}{f^2}.$

vi.a) $2\frac{\partial^4}{\partial x^4} \log f = \frac{D_x^4 f \cdot f}{f^2} - 3(\frac{D_x^2 f \cdot f}{f^2})^2.$

Finally we list some properties which are useful to transform the bilinear differential equations into the original form of non linear differential equations.

Let $\psi = \frac{a}{b}, \quad u = 2\frac{\partial^2}{\partial x^2} \log b.$

i.b) $\frac{(D_x a \cdot b)}{b^2} = \psi_x.$

ii.b) $\frac{(D_x^2 a\cdot b)}{b^2} = \psi_{xx} + u\psi.$

iii.b) $\frac{(D_x^3 a\cdot b)}{b^2} = \psi_{xxx} + 3u\psi_x.$

iv.b) $\frac{(D_x^4 a\cdot b)}{b^2} = \psi_{xxxx} + 6u\psi_{xx} + (u_{xx} + 3u^2)\psi.$

v.b) $\frac{(D_x^5 a\cdot b)}{b^2} = \psi_{xxxxx} + 10u\psi_{xxx} + 5(u_{xx} + 3u^2)\psi_x.$

Let $u = 2\frac{\partial^2}{\partial x^2}\log f.$

vi.b) $\frac{(D_x^2 f\cdot f)}{f^2} = u.$

vii.b) $\frac{(D_x^4 f\cdot f)}{f^2} = u_{xx} + 3u^2.$

viii.b) $\frac{(D_x^6 f\cdot f)}{f^2} = u_{xxxx} + 15uu_{xx} + 15u^3.$

Appendix II

Using the techniques presented in section 2.2 it is possible to derive the M soliton solution of the KP equation, here we calculate the 2 soliton solution.
Formula (13.2) for $N = 4$ with $\theta_i = x + P_i y - \alpha P_i^2 t$ togheter with the relation

$$B_{ij} = -\frac{12}{(P_i - P_j)^2}$$

gives the two soliton solution of the KP equation (17.2). If $P_3 = P_1^*$ and $P_4 = P_2^*$ we have

$$\theta_3 = \theta_1^*, \qquad \theta_4 = \theta_2^*,$$

and the corresponding solution results to be a stable localized solution of the rational type. Then let

$$P_1 = P_{1r} + i\ P_{1i}, \qquad P_2 = P_{2r} + i\ P_{2i},$$

where $P_{\rho r} = \mathbb{R}e\ P_\rho$ and $P_{\rho i} = \mathbb{I}m\ P_\rho$ for $\rho = 1, 2$.

The two soliton solution of the KP equation is given by

$$f_4 = |\theta_1|^2|\theta_2|^2 - 12\{2\mathbb{R}e\left[\frac{\theta_1^*\theta_2^*}{(P_1 - P_2)^2}\right] + \frac{\theta_2\theta_2^*}{(2i\ P_{1i})^2} + \frac{\theta_1\theta_1^*}{(2i\ P_{2i})^2} + 2\mathbb{R}e\left[\frac{\theta_1^*\theta_2}{(P_1 - P_2^*)^2}\right]\}$$

$$+144\{\frac{1}{(2i\ P_{1i})^2}\frac{1}{(2i\ P_{2i})^2} + \frac{1}{|P_1 - P_2|^4} + \frac{1}{|P_1 - P_2^*|^4}\}.$$

Appendix III

Here we show a list of nonlinear evolution equations and their transformations into bilinear forms.

i) The Korteweg-de Vries equation

$$u_t + uu_x + u_{xxx} = 0, \tag{a.1}$$

is transformed into the bilinear form

$$D_x(D_t + D_x^3)\, ff = 0$$

through the transformation

$$u = 2\frac{\partial^2}{\partial x^2} \log f. \tag{a.2}$$

ii) The Higher order Korteweg-de Vries equation

$$u_t + 45u^2 u_x + 15(u_x u_{xx} + uu_{xxx}) + u_{xxxxx} = 0, \tag{a.3}$$

is transformed through the transformation $(a.2)$ into the bilinear form

$$D_x(D_t + D_x^5)\, ff = 0.$$

iii) The Model equation for shallow water waves

$$u_t - u_{xxt} - 3uu_t + 3u_x \int_x^{\infty} u_t dx' + u_x = 0, \tag{a.4}$$

is transformed through the transformation $(a.2)$ into the bilinear form

$$D_x(D_t + D_x - D_x^2 D_t)\, ff = 0.$$

iv) The Boussinesq equation

$$u_{tt} - u_{xx} - 3(u^2)_{xx} - u_{xxxx} = 0, \qquad (a.5)$$

is transformed through the transformation $(a.2)$ into the bilinear form

$$(D_t^2 - D_x^2 - D_x^4)\, ff = 0.$$

v) The KP equation

$$u_{tx} \pm u_{yy} + 6(uu_x)_x + u_{xxxx} = 0, \qquad (a.6)$$

is transformed through the transformation $(a.2)$ into the bilinear form

$$(D_x D_t \pm D_y^2 + D_x^4)\, ff = 0.$$

vi) The modified Korteweg-de Vries equation

$$u_t + 6u^2 u_x + u_{xxx} = 0, \qquad (a.7)$$

is transformed into the bilinear form

$$(D_t + D_x^3)\, (f + ig)(f - ig) = 0,$$

$$D_x^3\, (f + ig)(f - ig) = 0,$$

through the transformation

$$u = \frac{1}{i}\frac{\partial}{\partial x} \log \frac{(f + ig)}{(f - ig)}.$$

vii) The Sine-Gordon equation

$$u_{xt} = \sin(u) \qquad (a.8)$$

is transformed into the system of bilinear forms

$$D_x D_t g f = f g$$

$$D_x D_t (f f - g g) = 0$$

through the transformation

$$u = \frac{2}{i}\frac{\partial}{\partial x}\log\frac{(f+ig)}{(f-ig)}. \qquad (a.9)$$

vii) The two-dimensional Sine-Gordon equation

$$u_{xx} + u_{yy} - u_{tt} = \sin(u) \qquad (a.10)$$

is transformed through the transformation $(a.9)$ into the system of bilinear forms

$$(D_x^2 + D_y^2 - D_t^2) g f = f g$$

$$(D_x^2 + D_y^2 - D_t^2)(f f - g g) = 0.$$

ix) The nonlinear Schrödinger equation

$$i\psi_t + \psi_{xx} - 2|\psi|^2\psi = 0,$$

is transformed into the system of bilinear forms

$$(iD_t + D_x^2 - \lambda) G F = 0,$$

$$(D_x^2 - \lambda) F F = -2 G G^*$$

where λ is a constant to be determined by the boundary condition on ψ at $|x| = \infty$; through the transformation

$$\psi = \frac{G}{F},$$

F being a real function.

Acknowledgements

I consider this page a personal reflection to remember the story of this thesis, how it evolved and wrote up, I want to remember specially all the people who helped me writing it.

The beginning was in a conversation with Prof. Gian Fausto Dell'Antonio who realized that Applied Mathematics could be a fascinating subject of research to me. Thanks to him I got to know Prof. Alfio Quarteroni who accepted to supervise my thesis. He taught me the basic concepts of Spectral Methods which form the main part of this work. Thanks also to him for helping me concentrating on the important subjects and not drifting off into unresolvable problems. I derived my basic knowledge of numerical analysis from Antonio Lanza, John Miller and Riccardo Valdarnini who raised my knowledge from zero and encouraged my interest on the subject.

Since my way of working is to interact with all people which I can reach, I have to thank all my collegues of the SISSA, but many of them have given more than a small help. I want to remember their essential help to me. First of all I would like to thank Daniele Tommasini, he introduced me into the "TEX" and enforced me to finally start in typing this work. Then I thank Guido Magnano and Xiong Chuan- Sheng who have introduced me to some aspects of the theory of KdV and KP equations. Then I remember Jose' Acosta who gave me some really usable graphic-programs and Nicoletta Bressan for sending me her program for the numerical solution of the KP equation.

I thank also Mauro Orlandini, Andrea Pasquinucci, Enrico Smargiassi, Andrea Ferrante, Miklos Gulacsi and Marina Picek for helping me patiently using the VAX after they taught me correct Fortran. At the end, how he told me to do, I thank Pasquale Pavone for guiding me along my thesis solving many problems of the art of programming and convincing me that "aesthetics have their importance" (because the eye also wants his part).

Bibliography

[1] N.J. Zabusky and M.D. Kruskal, Phys. Rev. Lett, **15** (1965) 240-243

[2] G.L. Jr. Lamb, "*Elements of soliton theory*",Academic Press, New York

[3] P.D. Lax, Commun. Pure and Appl. Math., **21** (1968) 467-490

[4] H. Grosse, preprint UWThPh-1990-10 (1990)

[5] A. Das, "*Integrables Models*", World Scien. Lect. Notes in Physics, Vol. 30

[6] R.M. Miura, J. Math. Phys., **9** (1968) 1202-1204

[7] B.B. Kadomtsev and V.I. Petviashvili, Dokl. Nauk. SSSR, **192** (1970) 539-541

[8] N.C. Freeman, Adv. Appl. Mech., **20** (1980) 1-37

[9] V.G. Makhankov, E.I. Litvinenko and A.B. Shvachka, Comp. Phys. Comm. **2** (1981) 223-232

[10] Z. Jiang, R.K. Bullough and S.V. Manakov, Physica **18D** (1986) 305-307

[11] M.J. Ablowitz and A.I. Nachman, Physica **18D** (1986) 223-241

[12] S.V. Manakov, V.E. Zakharov, L.A. Bordag, A.R.Its and V. B. Matveev, Phys. Lett. A, **63** (1977) 205-206

[13] R. Hirota, Phys. Rev. Lett., **27** (1971) 1192-1194

[14] R. Hirota, Suppl. of Prog. Theor. Phys., **59** (1976) 64-100

[15] M.J. Ablowitz and J. Satsuma, J. Math. Phys.,**19** (1978) 2180-2186

[16] L. Carleson, Acta Math., **116** (1966) 135-157

[17] R.G. Voigt, D. Gottlieb and M.Y. Hussaini, "*Spectral methods for partial differential equations*", SIAM (1984)

[18] C. Canuto, M.Y. Hussaini, A. Quarteroni and T.A. Zang, "*Spectral Methods in Fluid Dynamics*", Springer Verlag (1988)

[19] C.W. Gear, "*Numerical Initial Value Problems in Ordinary Differential Equations*", Prentice-Hall, Englewood Cliffs, N.J.

[20] W.H. Press, B.P. Flannery, S.A. Teukolsky, W.T. Vetterling, "*Numerical Recipes*", Cambridge Univ. Press (1986)

[21] B. Fornberg and F.R.S. Whitham, Phil. Trans. Roy. Soc., **289** (1978) 373-404

[22] Y. Maday and A. Quarteroni, M^2AN **22** (1988) 499-529

[23] T.F. Chan and T. Kerkhoven, SIAM J. Numer. Anal., **22** (1985) 441-454

[24] A. Quarteroni, SIAM J. Numer. Anal., **24** (1987) 323-335

[25] C.S. Gardner, J.M. Greene, M.D. Kruskal, and R.M. Miura, Phys. Rev. Lett., **19** (1967) 1095-1097

[26] E. Date, M. Jimbo, M. Kashiwara and T. Miwa, J. Phys. Soc. Jpn.,**50** (1981) 3806-3813

[27] E. Brezin, E. Marinari and G. Parisi, Phys. Lett. B, **242** (1990) 35-38