SISSA - SCUOLA
INTERNAZIONALE
SUPERIORE
DI STUDI AVANZATI
TRIESTE
Strada Costiera 11

ISAS - INTERNATIONAL SCHOOL
FOR ADVANCED STUDIES

The Renormalization Group in Two–Dimensional Field Theory

Thesis submitted for the degree of

"Magister Philosophiae"

Elementary Particle Sector

Candidate:

Müge Boz

Supervisor:

Professor
Loriano Bonora

Academic Year 1989/90

TRIESTE

Contents

Acknowledgements

I am very deeply indebted to Professor Loriano Bonora for his valuable scientific collaboration and support; kind interest and great patience during all the stages of this work.

Also many hearty thanks to all the dear friends at SISSA for their nice friendship and collaboration.

Introduction

In the development of the theory of critical phenomena the renormalization group ideas played an important role. These ideas are concerned with the long range fluctuation which are the cause of the critical singularities. As the critical point is approached, the correlation lenght infinitely increases. In such a situation the behaviour of correlation functions is independent of the microscopic details of the system and all the correlation functions are identified in the language of the corresponding quantum field theory. These ideas are the basis of the universality and the renormalization group.

The critical behaviour of the statistical system is related to the fixed points of the renormalization group. In the renormalization group approach the critical singularities correspond to the fixed points of the group in the space of effective interactions S. The first stage of the general analysis of the fixed point is the construction of the conformally invariant solution corresponding to the fixed point. The minimal models are the main exact conformally invariant solutions which are related with the degenerate representations of the Virasora algebra with central charge $c<1$. In the sense of Zamolodchikov c can be interpreted as a measure of effective degrees of freedom. It decreases along RG trajectories. As the central charge of the Virasora algebra, c is defined at fixed points only.

However it may be continued to other points in the space of effective interactions S. Zamolodchikov introduces a function c(g), where $g \in S$, such that at any fixed point it is equal to the central charge of the Virasora algebra. Under the action of renormalization group transformations g(t), c(g(t)) decreases monotonically:

$$\frac{d}{dt}\, c(g(t)) \leq 0$$

the equality being satisfied only at fixed points. Therefore it is possible to give a renormalization group meaning to the ordering of the conformal field theory solutions by the magnitude of the central charge c.

The aim of this work can be summarized as the study of the explicit realization of the Zamolodchikov's c-theorem outlined above. Its application to minimal models is also our interest.

The general content of the following chapters is as follows: In chapter 1, the basic framework is set up for understanding the renormalization group method. In chapter 2, a short review of some fundamental concepts in two dimensional field theories is given. In chapter 3 the renormalization group in 2D-conformal field theories is treated including the Zamolodchikov's c-theorem. In chapter 4 some applications of the RG analysis to the 2D-conformal field theories are given.

Chapter 1

The Renormalization Group

In this chapter our aim is to review some aspects of renormalization group and its applications to critical phenomena. The general scheme is as follows:

In the first part we discuss the interrelation between quantum field theory and statistical mechanics (section 1.1.), then continue with the construction of continuum field theory and renormalization (section 1.2). The physical motivation for renormalization and universality will be found in the renormalization group (section 1.3).

In the second part of this chapter we tried to use the renormalization group approach in the sense of Wilson, clearly as a theory of transformation properties of effective Hamiltonians under contraction. Fixed point Hamiltonians remain invariant under this transformation and the transformation properties of such hamiltonians close to fixed point reflect the critical behavior of the system.

The second part is arranged as follows: we begin with an introduction to critical phenomena (section 2.1). The observed scale invariance of the correlation functions suggests that at criticality the effective Hamiltonian is invariant under scale transformations, leading to the renormalization group transformation as a transformation of Hamiltonian under the change of lenght scale (section 2.2). Section 2.3 contains the renormalization group equation with a smooth momentum cut-off, some technical information, with definitions and notations given in the Appendix.

1. Field Theories

1.1 Statistical Mechanics and Quantum Field Theory

Our aim is to discuss the connection between these two subjects as we mentioned before. For the sake of constructing the necessary link, it will be convenient to begin with an illustrative example.

Consider a simple quantum mechanical system described by the Hamiltonian

$$H = \frac{-1}{1} (\sigma_1 - \lambda\sigma_3) \tag{1.1.1}$$

We are in the Hilbert space with observables described by Pauli matrices. This system possesses quantum fluctuations, i.e

$$\langle 0 \mid \sigma_3^2 \mid 0 \rangle \neq \langle 0 \mid \sigma_3 \mid 0 \rangle^2 \tag{1.1.2}$$

Let us study the evolution operator e^{-HT}. We have

$$\langle\, b \mid e^{-HT} \mid a \rangle = \langle\, b\, e^{-H(\Delta T + \Delta T + ..)} \mid a \,\rangle \qquad (1.1.3)$$

where the time interval T is divided into segments such that $N.\Delta T = T$. Then we insert the complete sets of states

$$\langle\, b \mid e^{-HT} \mid a \,\rangle = \sum_{s1=\pm 1} \sum_{s2=\pm 1} \cdots\cdots \sum_{sN\text{-}1=\pm 1} \langle\, b \mid e^{-H\Delta T} \mid S_1 \rangle \langle\, S_1 \mid e^{-H\Delta T} \mid S_2 \,\rangle \ldots$$

$$\ldots \langle\, S_{N\text{-}1} \mid e^{-H\Delta T} \mid a \,\rangle \qquad (1.1.4)$$

Define

$$e^{-V(S,S')} \equiv \langle\, S \mid M \mid S' \,\rangle \equiv \langle\, S \mid e^{-H\Delta T} \mid S' \,\rangle \qquad (1.1.5)$$

We have

$$\langle\, b \mid e^{-HT} \mid a \,\rangle = \sum_{S1=\pm 1} \cdots\cdots \sum_{SN\text{-}1=\pm 1} {}^{-e(V(b,S_1) + V(S_1,S_2) + \ldots V(S_{N\text{-}1},a))} \qquad (1.1.6)$$

This looks like a partition function in statistical mechanics, that of a 1-d Ising model with nearest neighbour couplings. As we calculate V(S, S′) in the small ΔT limit, the most general form of it can be found as:

$$V\,(S,S') = \frac{K}{2}\,(S\text{-}S')^2 + \frac{h}{2}(S+S') \qquad (1.1.7)$$

Also we have

$$e^{-2K} = \frac{\Delta T}{L} \quad e^{h} = 1 + \lambda \frac{\Delta T}{L} \tag{1.1.8}$$

This representation is called as the path integral representation.

Consider the partition function

$$Z = \text{Tr}\, e^{-HT} \tag{1.1.9}$$

Inserting states, we see that this is the 1-d Ising model with periodic boundary conditions. The free energy per site of this system is

$$F = \lim_{N \to \infty} \left(-\frac{1}{N} \text{Ln}Z\right) = E_o \tag{1.1.10}$$

where E_o is the ground state energy. So the statistical mechanical free energy maps on to the quantum ground state energy.

Now let us focus our attention on the two spin correlation function.

$$\langle S_o\, S_r \rangle = \sum_{S_1 .. S_N = \pm 1} \exp\left(-\sum_i V(Si, S_{i+1})\right) S_o\, S_r / Z \tag{1.1.11}$$

In quantum mechanics this has the meaning:

$$\langle S_o\, S_r \rangle = \text{Tr}\left(\sigma_3\, e^{-Hr}\, \sigma_3\, e^{-H(T-r)}\right) / \text{tr}\, e^{-HT} \tag{1.1.12}$$

Inserting the eigenstates of H and taking $T \to \infty$ we find:

$$\langle S_o\, S_r \rangle = | \langle o \mid \sigma_3 \mid o \rangle |^2 + e^{-(E_1 - E_o)r} | \langle o \mid \sigma_3 \mid 1 \rangle |^2 \tag{1.1.13}$$

But as $t \to it$, this corresponds to

$$\langle o \mid T(\sigma_3(t)\, \sigma_3(o)) \mid o \rangle \tag{1.1.14}$$

where T is the time ordered product, σ_3 (t) = e^{iHt} σ_3 e^{-iHt} and $r = |t|$. Correlation functions are related to the vacuum expectation values of the time ordered products. The dimensionless decay lenght of the correlation function (1.1.13) is called the correlation lenght which is defined as.

$$\Delta T \xi = \frac{1}{\Delta} \quad , \qquad \Delta = E_1 - E_o \tag{1.1.15}$$

Now we are turning to quantum field theory in which we have an infinite number of quantum mechanical systems, one associated to each point in space. We will discretize the space into a lattice and consider a finite volume.

Let us begin with a simple example: A two level quantum system at each point of one-dimensional space with a Hamiltonian is given by

$$H = - \sum_n \left[\sigma_1 (n) + \lambda \sigma_3 (n) \sigma_3 (n+1) \right] \tag{1.1.16}$$

Although this is a soluble model, we study its properties in perturbation theory for the moment. When $\lambda = 0$, H is diagonal in basis where σ_1 is diagonal. If we label the states

$$\sigma_1 | \uparrow \rangle = | \uparrow \rangle \qquad \sigma_1 | \downarrow \rangle = | \downarrow \rangle \tag{1.1.17}$$

The ground state of H is

$$| 0 \rangle = | \uparrow \uparrow ... \uparrow \rangle \qquad \langle 0 | H | 0 \rangle = -N = E_o \tag{1.1.18}$$

The first excited states are N-fold degenerate (N is the number of sites) and correspond to flipping one spin

$$| r \rangle = | \uparrow \uparrow .. \uparrow \underset{r}{\downarrow} \uparrow ... \uparrow \rangle \qquad \langle r | H | r \rangle = -N + 2 = E_o + 2 \tag{1.1.19}$$

They are at a finite energy above E_o. These states correspond to a particle localized at site r. The first order perturbation in λ is diagonalized by the states

$$| k \rangle = \sum_r e^{ikr} | r \rangle \ , \quad E_k = \langle k | H | k \rangle = E_o + 2 - 2\lambda \, \mathrm{Cos} k \tag{1.1.20}$$

These are the one particle states with momentum k, reflecting the discrete translation invariance of H.

$$T \sigma_i (n) T^{-1} = \sigma_i (n+1) \qquad [T, H] = 0 \tag{1.1.21}$$

The one-particle state is above the ground state by an amount Δ, where

$$\Delta = E_{k=0} - E_o = 2 - 2\lambda \ . \tag{1.1.22}$$

which is the energy gap.

In the $\lambda \to \infty$ limit, H is diagonal in the basis where σ_3 is diagonal

$$\sigma_3 | \uparrow \rangle = | \uparrow \rangle \qquad \sigma_3 | \downarrow \rangle = - | \downarrow \rangle \tag{1.1.23}$$

There are two degenerate ground states

$$| \uparrow \uparrow ... \uparrow \rangle \quad \text{and} \quad | \downarrow \downarrow ... \downarrow \rangle \tag{1.1.24}$$

These states are carried into each other by the action of the symmetry operator U

$$U = \prod_n \sigma_1 (n) \qquad [U, H] = 0 \tag{1.1.25}$$

But this is in contrast with the λ small ground state $|0\rangle$, (1.18) in which $U|0\rangle = |0\rangle$. We can say that at large λ, U is a spontaneously broken symmetry. The ground state is not invariant under the symmetries of H.

Now, we would like to derive a path integral representation of a system like (1.1.16) following the steps used in the Q.M. system. Complete sets of states are described by

$$| \{ S_x \} \rangle \quad \text{where} \quad x = 1, ...N \quad \text{and } S_i = \pm 1 \tag{1.1.26}$$

Inserting such a set at each time interval ΔT

$$\langle a | e^{-HT} | b \rangle = \sum_{\{s(x,t)=\pm 1\}} \langle a | e^{-H\Delta T} | \{ S(x,1)\} \rangle \langle \{ S(x,1) \; e^{-H\Delta T} | \rangle .. | b \rangle \tag{1.1.27}$$

Define the transfer matrix

$$\langle \{ S(x)\} | e^{-H\Delta T} | \{ S'(x)\} \rangle = \langle \{ S(x)\} | M | \{ S'(x)\} \rangle \equiv e^{-V(\{S(x)\},\{S'(x)\})} \tag{1.1.28}$$

Then in the ΔT small limit V is:

$$V(\{ S(x)\}, \{ S'(x)\}) = \sum_x \frac{K_t}{2} (S(x) - S'(x))^2 + \frac{K_x}{2} (S(x) - S(x+1))^2 \tag{1.1.29}$$

with $e^{-2K_t} \approx \Delta T$ $K_x = \lambda\Delta T$, $\Delta T \to 0$. Noting that $\Delta T \to 0$ implies $K_t \rangle\rangle K_x$. So equation (1.1.27) is just the partition function for a two dimensional Ising model. The existence of gap (1.1.22) implies that the spin spin correlation function of this model decays exponentially

$$\langle S(t,x)\, S(t',x)\rangle \sim e^{-\xi^{-1} | t-t' |} \quad \Delta T \xi = \frac{1}{\Delta} \tag{1.1.30}$$

where $| t-t' |$ is the (dimensionless) number of lattice spacings between the two points. At a specific value of λ, $\lambda = 1$, the energy gap vanishes (1.1.22). This implies non-exponential decay-of the spin-spin correlation functions. In fact

$$\langle S(t,x)\, S(t', x)\rangle \sim \frac{1}{| t - t' |^{\eta}} \tag{1.1.31}$$

1.2 Continiuum limit and renormalization

We try to study the problem of constructing a quantum field theory. We try to realize the notion of having a QM system at each point of the spatial continuum.

In the previous section by introducing a lattice, we considered a finite volume in space. Now we try to work on the limiting process of removing it

We introduce a free scalar field theory. The Hamiltonian on a spatial lattice is

$$H = \frac{1}{2} \sum_x \left(-\frac{\partial^2}{\partial \varphi(x)^2}\right) + \sum_x \frac{1}{2} (\varphi(x+\mu) - \varphi(x))^2 + \frac{\mu^2 \varphi}{2} \sum_x \varphi(x)^2 \tag{1.2.1}$$

The isotropic Eucledian path integral description of the system is:

$$Z = \langle 0 |\, e^{-HT}\, | 0 \rangle =$$

$$= \int_{-\infty}^{\infty} \prod_x d\varphi(x) \exp\left[-\frac{1}{2} \sum_x (\varphi(x+\mu)-\varphi(x))^2 + \frac{r}{2} \sum_x \varphi(x)^2\right]$$

$$= \int_{\varphi} \exp [\sum_{x} \varphi(x)\, \varphi(x+\mu) - (\frac{2d+r}{2}) \sum_{x} \varphi(x)^2]$$

(1.2.2)

Rescaling of the fields lead to

$$Z \,\alpha \int_{\varphi} \exp [\kappa \sum_{x} \varphi(x)\, \varphi(x+\mu) \,\text{-}\sum_{x} \varphi(x)^2] \qquad \kappa = \frac{1}{2d+r}$$

(1.2.3)

This model can also be solved exactly. Let us introduce the Fourier transformed fields and try to study its solution let us introduce

$$\varphi(k) = \sum_{x} e^{ikx}\, \varphi(x) \tag{1.2.4}$$

$$Z = \int_{\varphi(k)} \exp [- \frac{1}{2} \sum_{k} \varphi(k)\, \varphi(-k) \, (\sum_{\mu} (2\text{-}2\cos k\mu) \; + r)]$$

(1.2.5)

The theory is described by decoupled normal modes $\varphi(k)$. The Gaussian integration gives

$$\langle \varphi(x)\, \varphi(x') \rangle = \frac{1}{(2\pi)^d} \int_{k} \frac{e^{ik(x-x')}}{\sum_{\mu}(2\text{-}2\text{-}\cos k\mu) + r}$$

(1.2.6)

The asymptotic decay of this correlation function is

$$\langle \varphi(x)\, \varphi(x') \rangle \sim e^{-\sqrt{r}\, | x-x' |} \qquad , \xi = \frac{1}{\sqrt{r}}$$

(1.2.7)

or

$$\xi \sim \text{constant x } | \kappa - \kappa_c |^{-1/2}$$

(1.2.8)

So as $r \to 0$, ξ diverges. There is a critical point in this system $\kappa = 1/2d$. We say the system has a critical exponent $\nu = 1/2d$ describing the correlation lenght divergence.

Now we can define a limit in which the lattice structure disappears. Suppose we are constructing a quantum field theory with a physical lenght scale

$$l_{phys} = \frac{1}{m_{phys}} \tag{1.2.9}$$

in the continuum.

We do this by assigning a lattice spacing a to our lattice model and define

$$l_{\,phys} = a\,\xi(\kappa) \tag{1.2.10}$$

Now let us consider the case: (a → 0. As the lenght l is fixed). To make a lenght constant as the lattice is becoming smaller and smaller, the lenght should cover more and more lattice spacing. This requires adjusting κ so that ξ diverges. The dependence of κ on a is so chosen as to satisfy (1.2.9)

$$(\text{const})\,.a.\quad (\kappa(a) - \kappa_c)^{-1/2} = l_{phy} \tag{1.2.11}$$

so

$$\kappa(a) = (\frac{l_{phys}}{a} \Big/ \text{const})^{-2} + \kappa_c \tag{1.2.12}$$

An important consequence of this scaling limit is the universality of the resulting theory, because the limiting theory does not consider the details of the cut off and wide varieties of different short distance formulations are expected to yield the same result. For example, suppose that some second nearest neighbors coupling is added to (1.2.2)

$$S = \frac{1}{2} \sum_x ((\varphi(x+\mu)\ \varphi(x))^2 + \alpha\ (\varphi\ (x+2\mu) - \varphi(x)^2) + \frac{r'}{2} \sum_x \varphi(x)^2 \tag{1.1.13}$$

Fourier transforming leads to:

$$S = \int_k \varphi_k\, \varphi_{-k}\, (\sum_\mu (2\text{-}2\cos k\mu) + \alpha(2 - 2\cos 2k\mu) + r') \tag{1.1.14}$$

For small r, we concentrate on large distance and small K then

$$S \approx \int_k \varphi_k\, \varphi_{-k}\, (k^2\, (1 + 4\alpha) + r') \tag{1.1.15}$$

Rescaling φ leads to

$$S = \int_k \varphi_k\, \varphi_{-k}\, (k^2 + r) \qquad r = \frac{r'}{(1+4\alpha)} \tag{1.1.16}$$

we see that the model has the same low momentum, long distance behaviour independent of α, as long as r is adjusted correctly.

This phenomenon originates the universality of the critical behaviour; as the correlation lenght is long enough, all the interaction Hamiltonians show the same critical behaviour.

Another important point about the scaling limit is that: to make correlation functions well defined we have to make another adjustment-wave function renormalization or spin rescaling. To see this let us examine $\langle \varphi(x)\varphi(0)\rangle$ for physical distance x $\langle\langle l_{phys}$ in the scaling limit. A physical distance $x = na$ where $n \to \infty$ as $a \to 0$. If we identify

$$\langle \varphi(x)\,\varphi(0)\rangle = \langle \varphi(x)\,\varphi(0)\rangle \sim \int d^d k\ \frac{e^{ik\cdot n}}{k^2} \sim \frac{1}{n^{d-2}} \quad , \; n \langle\langle \frac{1}{\sqrt{r}} \tag{1.2.17}$$

As $a \to 0$, $n \to \infty$ and $\langle \varphi(x)\ \varphi(0)\rangle$ vanishes. So we have to define physical fields.

$$\varphi(x) = a^{-(d-2)/2}\,\varphi(x)$$

then

$$\langle \varphi(x)\,\varphi(0)\rangle \sim \frac{1}{a^{d-2}n^{d-2}} \sim \frac{1}{x^{d-2}} \tag{1.2.18}$$

which is finite for $a \to 0$. The physical reasoning for the renormalization of this wave function and the universality will be found in the renormalization group.

1.3 An Introduction to the Renormalization Group

We discussed how to get a continuum theory from one defined on a lattice using a simple model. If we wish to do the same thing for more complicated theories, the main difficulty will be the number of coupled degrees of freedom. In the path integral there are lots of integrals to do

$$\int \{ d\varphi_i \}\ e^{-S(\{\varphi_i\})} \tag{1.3.1}$$

The idea of the renormalization group is to solve the theory by integrating out degrees of freedom a few at a time. Each step of the procedure would be finite and well defined.

Let us take a lattice and form 2x2 blocks. To each block associate a « block spin ».

$$\varphi_B(x) = \frac{\varphi_1(x) + \varphi_2(x) + \varphi_3(x) + \varphi_4(x)}{c} \tag{1.3.3}$$

representing the average behaviour of degrees of freedom in that block. The integration over the original φ's, with the block variable fixed, leads to

$$e^{S'(\{\varphi_B\})} = \int_{\varphi} \prod_{\text{blocks}} \delta(\varphi_B(x) - \frac{\varphi_1(x) + \varphi_2(x) + \varphi_3(x) + \varphi_4(x)}{c}) \, e^{S(\{\varphi\})} \tag{1.3.4}$$

This calculation could be possible although the original integral is not, since all the long wavelenght modes are stopped by the δ-function kernel. This tells us that there will be only short-range interactions in $S'(\varphi_B\})$. The couplings in $S(\{\varphi)\}$ characterize the strenght of interaction at a, while in $S'(\{\varphi_B\})$ they describe the strenght of interaction at a scale 2a.

The result is that the change in effective coupling with scale is only determined by fluctuations around that scale, not by fluctuations around all the degrees of freedom. Hence such renormalizations should be calculable. To determine the other properties of these block transformations, let us integrate over φ_B on both sides of (1.3.4) we find

$$\langle \varphi_B(x)\, \varphi_B(y) \rangle_S = \langle\left(\frac{\varphi_1(x) + \varphi_2(x) + \varphi_3(x) + \varphi_4(x)}{c} \right) \cdot \left(\frac{\varphi_1(y) + \varphi_2(y) + \varphi_3(y) + \varphi_4(y)}{c} \right)\rangle \tag{1.3.5}$$

Correlation functions of the block variables in the block action are just complicated correlation functions of the original variables here we notice that the correlation of the block variable will extend over the same range as the original ones. But in terms of block lattice spacings this will be half as big a range.

$$\xi(S') = \frac{\xi(S)}{2} \tag{1.3.6}$$

So this transformation reduces the number of degrees of freedom coupled together. If we took a system with a very large ξ, the treatment of this system with renormalization group repeatedly could allow us to change it into one with a very short correlation lenght which might be treated by expansion techniques.

2. The Renormalization Group Approach to the Critical Phenomena

In this section we try to deepen our insight into the understanding of critical phenomena via the renormalization group approach. For convinience we begin with a short summary on some aspects of critical phenomena and then by constructing the necessary connections we shall obtain the renormalization group equations in the Wilson's sense which will serve as an example throughout this section.

2.1 Order Parameter, Critical Exponents

We describe the system by a Hamiltonian which is a function of local variables s(r). s(r) is chosen such that the expectation value m = <s(r)> is the order parameter and in a homogenous system s(r) is independent of r. Most of the phase transition can be determined by such a parameter. In many cases the Hamiltonian is invariant under certain transformations of s(r) like either of the three

reversal	$s(r) \rightarrow -(s(r)$
change of phase	$s(r) \rightarrow e^{i\varphi}\, s(r)$
rotation	$s(r) \rightarrow u(r)\, s(r)$

(2.1.1)

(u is a rotation matrix). This holds for example for the Ising model, superfluid helium and the Heisenberg model respectively. Then we expect $m = \langle s(r) \rangle = 0$. Actually this applies above the critical temperature Tc, whereas below Tc the system is in the ordered phase. m = <s(r)> ≠ 0.

On approaching the critical point from below Tc the order parameter vanishes continuously. This behaviour can be described by

$$m = A_m \, | T - Tc |^{\beta} \qquad \text{for} \qquad T < Tc \tag{2.1.2}$$

Also we review a few more results observed in most model calculations and many experiments. The response of the order parameter to an external symmetry breaking field

$$-h \int d^d r \, s(r) \tag{2.1.3}$$

is the susceptibility

$$\chi = \frac{\partial m}{\partial h} = (KT)^{-1} \int d^d r \, G(r, T) \tag{2.1.4}$$

with the correlation function

$$G(r,T) = \langle s(0)\, s(r) \rangle - \langle s(0) \rangle \langle s(r) \rangle \tag{2.1.5}$$

One observes the power laws

$$\chi(T) = A_\chi \, | T\text{-}Tc |^{-\gamma} \tag{2.1.6}$$

$$G(r, Tc) = c \, r^{-d+2-\eta} \qquad \text{for large } r \tag{2.1.7}$$

where η denotes the deviation from the Ornstein-Zernike behaviour. At $T \neq Tc$ the correlation function (2.1.5) decays like

$$G(r, T) \sim \exp(-r/\xi(T)) \tag{2.1.8}$$

for large distances. One observes the power law

$$\xi(T) = A_\xi \, | T\text{-}Tc |^{-\nu} \tag{2.1.9}$$

And the specific heat behaves, close to Tc, like

$$c(T) = A_c \; | T\text{-}Tc |^{-\alpha} \qquad (2.1.10)$$

The exponents $\alpha, \beta, \nu, \gamma$ are called as the critical exponents.

2.2 Basic Properties of The Renormalization Group

In this section we try to formulate some of the basic renormalization group ideas. As a starting point let us introduce a Hamiltonian of the Landau type

$$H = \int d^d r \, [\frac{1}{2} a(T) \, m^2(r) + \frac{1}{2} c(T) \, (\nabla m(r))^2 + \\ + \frac{1}{4} b(T) \, m^4(r) - hm(r)] \tag{2.2.1}$$

which is the effective Hamiltonian. Here m(r) describes the mean magnetization of the cells, a(T), b(T), c(T) are smooth functions of the temperature. The concept of thinning out the number of variables will be a further guide in the RG process.

Now let us consider some of the basic RG ideas:

i) We consider spins S(r) on a lattice. At the critical point the correlation function decays according to

$$\langle S(0) \, S(r) \rangle_{crit} = \frac{c}{r^{d-2+\eta}} \tag{2.2.2}$$

for large distances. Now we imagine the same object under a different lenght scale. To do this, we divide the sample into cubic cells of lenght b units of lattice spacings in each direction. Then the new correlation function will be of the form:

$$\langle S(0) \, S(r) \rangle = \frac{cb^{2d}}{r^{d-2+\eta}} \tag{2.2.3}$$

ii) The change of the since each cell contains b^d spins lenght scale by factor b produces a spacing of the cells equal to the original lattice spacing

$$r = bR \tag{2.2.4}$$

Furthermore the change of the scale for magnetization by a factor $b^{(d+2-\eta)/2}$ gives:

$$S(r) = b^{(d+2-\eta)/2}\, S_1(R) \tag{2.2.5}$$

Then the asymptotic behaviour of the new spin variables has the form

$$\langle S_1(0)\, S_1(R)\rangle_{\text{crit}} = \frac{c}{R^{d-2+\eta}} \tag{2.2.6}$$

This shows that the correlation function is invariant under the change of the scale (2.2.4) which has the implification that the effective interaction at criticality is invariant with respect to the change of the lenght scale. This procedure which changes the scale of the Hamiltonian (effective interaction) is called the renormalization group procedure and the corresponding transformation is called the renormalization group transformation.

We summarize some requirements and properties of the renormalization group transformation and start with a few definitions. The partition function is

$$Z = \text{trace} \exp(-H) \tag{2.2.7}$$

where

$$H = \beta \mathscr{H} \tag{2.2.8}$$

is the Hamiltonian density with

$$\beta = \frac{1}{KT} \tag{2.2.9}$$

And the density for free energy is

$$F = \frac{\beta \mathcal{F}}{V} = -\frac{1}{V} \text{ Ln trace exp(-H)} \tag{2.2.10}$$

where V is the volume of the system.

The renormalization group transformation consist of

i) A change of the lenght scale by a factor $b = e^{l}$ in all linear dimensions (the partition function is left invariant) Since the volume decreases by a factor e^{-dl} we obtain

$$Fo = e^{-dl} F_l \tag{2.2.11}$$

ii) A transformation and/or elimination of the spin variables which leaves the free energy invariant. This is a process which is aimed at thinning out the number of variables in the Hamiltonian. The renormalization group transforms H_0 into H_l

$$H_l = R_l (Ho) \tag{2.2.12}$$

$$F(Ho) = e^{-dl} F(H_l) \tag{2.2.13}$$

2.3 Renormalization Group Equation

In this section we introduce the RG equation with a smooth momentum cut off. It is a generalization of the RG equation given by

Wilson-Kogut. Some technical parts with the necessary definitions and notations are given in Appendix, Here we rather prefered to reflect its characteristics which will serve as an example throughout this Chapter.

The transformation of the Hamiltonian consists of two parts:

(i) A dilatation transformation: under this transformation the lenght scale changes by a factor b. Here V, q, S and V′, q′, S′ etc, denote the quantities before and after the transformation. Then

$$V = b^d V' \tag{2.3.1.a}$$

$$q' = bq \tag{2.3.1.b}$$

and

$$\frac{1}{\sqrt{V}} \; S_q = \frac{1}{\sqrt{V'}} \; S'_{q'} \tag{2.3.1.c}$$

As we consider the infinitesimal dilatations

$$b = 1 + l \tag{2.3.2.a}$$

$$V = (1+ld)\, V' \tag{2.3.2.b}$$

$$q' = (1+l)\, q \tag{2.3.2.c}$$

$$S_q = (1 + l \, \frac{d}{2})\, S'_{q'} = (1+l\, \frac{d}{2})\, S'_q + lq \, \nabla S'_q \tag{2.3.2.d}$$

The new Hamiltonian in terms of the transformed variables will be of the form:

$$H = V' (1+ld)u_0 + (1+ l \, \frac{d}{2})u_1 S'_0 + \frac{1}{2} \, (1+ld) \int_q u_2(q) \, (S'_q + lq\nabla S'_q) \; .$$

$$. \, (S'_{-q} - lq \, \nabla S'_{-q}) + \ldots \tag{2.3.3}$$

One can observe that under this transformation the old Hamiltonian has changed by $l\mathcal{G}_{dil}H$ where $\mathcal{G}_{dil}$ is the generator of the dilatation.

$$\mathcal{G}_{dil}H = dV \frac{\partial H}{\partial V} + \int_q \left(\frac{d}{2} S_q \frac{\delta H}{\delta S_q} + q\nabla S_q \frac{\delta H}{\delta S_q} \right) \qquad (2.3.4)$$

ii) The transformation of the variables: we replace s_q by

$$S_q = S'_q + l\Psi_q \{S'\} \qquad (2.3.5)$$

where $\Psi_q$is a function which may depend on all Fourier components S'. Then we have

$$H\{S\} = H\{S'\} + l \int_q \Psi_q \{S'\} \frac{\delta H \{S'\}}{\delta S'q} \qquad (2.3.6)$$

Equation (2.3.5) yields

$$\int[dS] = \int[dS'] \frac{\partial\{S\}}{\partial\{S'\}} = \int[dS'] \left(1+l \int_q \frac{\delta\Psi_q\{S'\}}{\delta S'_q}\right) \qquad (2.3.7)$$

Therefore

$$\begin{aligned} z &= \int [dS'] \exp(-H\{S\}) \\ &= \int [dS'] \exp(-H\{S'\} - l\mathcal{G}_{tra}\{\Psi\} H\{S'\}) \end{aligned} \qquad (2.3.8)$$

with the generator $\mathcal{G}_{tra}$ for the transformation of the variables

$$\mathcal{G}_{tra}\{\Psi\} H\{S\} = \int_q \left(\Psi_q \frac{\delta H}{\delta S_q} - \frac{\delta\Psi_q}{\delta S_q}\right) \qquad (2.3.9)$$

The generator $\mathcal{G}$ of the RG consist of the contributions (2.3.4) and (2.3.9)

$$\mathcal{G}\{\Psi\} = \mathcal{G}_{dil}H + \mathcal{G}_{tra}\{\Psi\} H \qquad (2.3.10)$$

and the RG equation reads

$$\frac{dH}{dl} = \mathscr{G}\{\Psi\}\, H \qquad (2.3.11)$$

It can not be expected that every Ψ_q will eliminate the Fourier components for large wave vectors, one is free to change Ψ_q within regions.

Wilson's choice (which leads to an elimination) is

$$\Psi_q = h(q)\,(S_q - \frac{\delta H}{\delta S_{-q}}) \qquad (2.3.12)$$

where $h(q) = c + 2q^2$. They give an argument [1] that the expression (2.3.12) with a sufficiently large h(q) leads to an elimination of the degrees of freedom S.

In general Ψ_q depends on H as for example in the equation (2.3.12). Therefore one obtains the renormalization group equations non-linear in H.

Here we studied the renormalization group equations in the sense of Wilson. In the following chapters we focus our attention in two dimensional theories and treat the R.G equations in the Callan-Symanzik form.

Chapter 2

Conformal Field Theories

In this chapter our aim is to review some of the basic concepts in two dimensional conformal field theories.

In section 2.1 we formulate the main ideas of the non-Hamiltonian 2D field theory. Following the basic properties of the stress tensor given in chapter 2.2, we conclude this chapter by studying some aspects of degenerate families and minimal models.

2.1 The Algebra of Local Fields

The algebra of local fields is an important hypothesis which assumes the existence of a basis set of local fields. The theory contains a set of mutuallly local fields $A_i(x)$ which can be considered as the elements of the infinite dimensional space $\mathscr{A}$. The set of fields with a countable basis $\{A_i,\ i=0, ...\}$ is complete in a sense that we specify below:

The set $\{A_i,\ i=0,\ 1\ ...\}$ contains an identity operator which is described by the equation

$$\langle IX \rangle = \langle X \rangle \qquad (2.1.1)$$

where X is any product of fields of A

$$X = A_{i1}(x_1) \ldots A_{iN}(x_N) \qquad (2.1.2)$$

For convenience, the basis $\{A_i\}$ is chosen such that

$$\langle A_i \rangle = 0 \text{ for } i \neq 0 \qquad (2.1.3)$$

and also

$$\langle I \rangle = 1 \qquad (2.1.4)$$

The linear operators ∂_z and $\partial_{\bar{z}}$ act in space $\mathscr{A}$.

By definition

$$\langle X\, \partial_z A(z\, \bar{z})\, Y \rangle = \frac{\partial}{\partial z} \langle X A(z, \bar{z})\, Y \rangle \qquad (2.1.5a)$$

$$\langle X\, \partial_{\bar{z}} A(z\, \bar{z})\, Y \rangle = \frac{\partial}{\partial \bar{z}} \langle X A(z, \bar{z}) Y \rangle \qquad (2.1.5b)$$

where we introduced the complex coordinates $(z, \bar{z})$ as:

$$z = x^1 + ix^2$$
$$\bar{z} = x^1 - ix^2 \qquad (2.1.6)$$

Here (x^1, x^2) are the Cartesian coordinates on the plane, x_i corresponds to z_i, $\bar{z}_i$ and $A(x_i) = A(z_i, \bar{z}_i)$.

We denote

$$D = \partial_z \qquad \bar{D} = \partial_{\bar{z}} \qquad (2.1.7)$$

Then

$$DI = \bar{D}I = 0 \qquad (2.1.8)$$

The completeness of the set $\{A_j\}$ means that one can generate any set by the linear action of these operators which is equivalent to the operator algebra

$$A_i(x_1)\, A_j(x_2) = \sum_k C_{ij}^{\ k}(x_1, x_2)\, A_k(x_2) \qquad (2.1.9)$$

where $A_i, A_j \in \mathscr{A}$ and $C_{ij}^{k}(x_1, x_2)$ are the structure constants of the operator algebra. This relation can be considered as a set of correlation functions

$$\langle A_i(x_1) A_j(x_2) X \rangle = \sum_k C_{ij}^{k}(x_1, x_2) \langle A_k(x_2) X \rangle \tag{2.1.10}$$

where X is a product of the fields (2.1.2). The fusion rule (2.1.9) is useful in a sense that one can determine any correlation function by lowering the order of the correlator and reducing it to the one-point functions (2.1.3), (2.1.4). For example

$$\langle A_i(x) A_j(x) \rangle = C_{ij}^{o}(x) = D_{ij}(x) \tag{2.1.11}$$

However it is important that the operator algebra (2.1.9) must be associative, which brings hard restrictions on the structure constants.

The space $\mathscr{A}$ can be decomposed into the subspaces $\mathscr{A} = \mathscr{A}^{(B)} \oplus \mathscr{A}^{(F)}$ of Bose and Fermi fields.

The Eucledian invariance of the theory is guaranteed by the following condition:

There exist a symmetric stress tensor $T^{\mu\nu}(x) = T^{\nu\mu}(x)$ in space $\mathscr{A}^{(B)}$ satisfying the conservation equation $\partial_\mu T^{\mu\nu} = 0$, that is the equation

$$\partial_\mu \langle T^{\mu\nu}(x)\, x \rangle = 0 \tag{2.1.12}$$

for all $x \in R^2 \setminus \{y_i\}$ where y_i are points indicated in (2.1.2). This tensor has three independent components.

$$T = T_{zz} \ ; \ \overline{T} = T_{\bar{z}\bar{z}} \ ; \ \theta = -T_{z\bar{z}} \qquad (2.1.13)$$

Then the conservation equations for $T_{\mu\nu}$ are:

$$\partial_{\bar{z}} T(z, \bar{z}) = \partial_z \theta (z, \bar{z}) \qquad \partial_{\bar{z}} \theta(z, z) = \partial_z \overline{T}(z, \bar{z}) \qquad (2.1.14)$$

The expressions

$$\oint_c \langle [T(\xi,\bar{\xi}) \frac{d\xi}{2\pi i} - \theta(\xi,\bar{\xi}) \frac{d\bar{\xi}}{2\pi i}] A(z, \bar{z}) \rangle = \partial z A(z, \bar{z}) \equiv P A (z, \bar{z}) \qquad (2.1.15 \text{ a})$$

$$\oint_c \langle [\overline{T}(\xi,\bar{\xi}) \frac{d\bar{\xi}}{2\pi i} - \theta(\xi,\bar{\xi}) \frac{d\xi}{2\pi i}] A(z, \bar{z}) \rangle = \partial \bar{z} A(z, \bar{z}) \equiv \overline{P} A (z,\bar{z}) \qquad (2.1.15 \text{ b})$$

and

$$S A (z,\bar{z}) = \oint_c \{[(\xi - z) T(\xi,\bar{\xi}) - (\bar{\xi} - \bar{z}) \theta (\xi,\bar{\xi})] \frac{d\xi}{2\pi i} - $$
$$-[(\bar{\xi} - \bar{z}) \overline{T}(\xi , \bar{\xi}) - (\xi - z) \theta (\xi,\bar{\xi})] \frac{d\bar{\xi}}{2\pi i} \} A(z, \bar{z}) \qquad (2.1.16)$$

define the momentum and spin operators acting on with the following commutation relations:

$$[P, \overline{P}] = 0 \ ; \ [P, S] = P \ ; \ [\overline{P}, S] = \overline{P} \qquad (2.1.17)$$

The space splits into eigenspaces of operator S; $\mathcal{A} = \oplus \mathcal{A}^{(s)}$, where $S\mathcal{A}^{(s)} = s\mathcal{A}^{(s)}$. We assume that the basis $\{A_j\}$ is chosen in such a way that

$$S\,A_j = s\,A_j \tag{2.1.18}$$

the number s_j is called the spin of the field A_j, one can deduce that

$$P : \mathcal{A}^{(s)} \to \mathcal{A}^{(s+1)} \qquad \overline{P} : \mathcal{A}^{(s)} \to \mathcal{A}^{(s-1)} \tag{2.1.19}$$

2.2 Stress Energy Tensor

We know that stress energy tensor satisfies the continuity equation

$$\partial_\mu T^{\mu\nu} = 0 \tag{2.2.1}$$

which guarantees the Eucledian invariance of the theory.

We also know that it is traceless

$$\theta(x) = T_\mu{}^\mu = 0 \tag{2.2.2}$$

which implies the scale invariance of the theory, and it also implies the conformal invariance.

Conformal transformations are the coordinate transformations of the type

$$\xi^a \to \eta^a(\xi) \tag{2.2.3}$$

where a = 1 ... D, with the property such that the metric tensor transforms as

$$g_{ab} \to g'_{ab} = \frac{\partial \xi^{a'}}{\partial \eta^a} \frac{\partial \xi^{b'}}{\partial \eta^b} g_{a'b'} = \rho(\xi)\, g_{ab} \tag{2.2.4}$$

where $\rho(\xi)$ is a function of ξ.

These type of coordinate transformations form the conformal group. In two dimensions the conformal group is infinite dimensional. It is convinient to introduce the complex coordinates

$$z = \xi^1 + i\xi^2$$
$$\bar{z} = \xi^1 - i\xi^2 \tag{2.2.5}$$

In terms of these variables the conformal group of transformations (2.2.3) has the form

$$z \to \xi(z) \quad , \quad \bar{z} \to \bar{\xi}(\bar{z}) \tag{2.2.6}$$

where $\xi(z)$ and $\bar{\xi}(\bar{z})$ are arbitrary analytic functions.

With the condition (2.2.2), it follows that

$$\partial_{\bar{z}} T = 0 \quad , \quad \partial_z \bar{T} = 0 \tag{2.2.7}$$

And

$$\begin{aligned} T &= T(z) \\ \bar{T} &= \bar{T}(\bar{z}) \end{aligned} \tag{2.2.8}$$

The equation (2.2.7) implies that any correlation function of the type

$$\langle T(z)X \rangle = \langle T(z)\, A_1(z_1, \bar{z}_1) \ldots A_N(z_N, \bar{z}_N) \rangle \tag{2.2.9}$$

is a single valued analytic function of z with possible singularities at the points $z_1, z_2 \ldots z_N$ and furthermore the integral

$$\int_c \varepsilon(z) \langle T(z)X \rangle \frac{dz}{2\pi i} \tag{2.2.10}$$

can be written as the sum of residues

$$\sum_{z_i \in \Lambda_c} \int_{c_i} \frac{dz}{2\pi_i} \varepsilon(z) \langle T(z)X \rangle = \sum_{z_i \in \Lambda_c} \langle A_1(z_1, \bar{z}_1) \ldots \delta_\varepsilon A_i(z_i, \bar{z}_i) \ldots \rangle \tag{2.2.11}$$

where $\varepsilon(z)$ is analytic in the domain Λ_c bounded by the contour c. Here the operator.

$$\delta_\varepsilon A(z, \bar{z}) = \oint \frac{d\xi}{2\pi_i} \varepsilon(\xi)\, T(\xi)\, A(z, \bar{z}) \tag{2.2.12}$$

denotes the variation of the field A under infinitesimal conformal transformations.

Now let us consider the conformal properties of T under the infinitesimal transformations

$$z \rightarrow z + \varepsilon(z) \qquad (2.2.13)$$

One can find that

$$\delta_\varepsilon T(z) = \varepsilon(z)\, T'(z) + 2\varepsilon'(z)\, T(z) + \frac{c}{12}\, \varepsilon'''(z) \qquad (2.2.14)$$

where prime denotes the z-derivation. At this point the constant c can be treated as a parameter and it can take positive values which results from reality condition of the stress energy tensor in the Euclidean space.

Next we consider the operator product expansion of the product $T(\xi)\, A(z, \bar{z})$:

$$T(\xi)\, A(z, \bar{z}) = \sum_{n=-\infty}^{\infty} (\xi - z)^{-n-2} L_n A(z, \bar{z}) \qquad (2.2.15)$$

where $L_n A \in \mathcal{A}$. The operators L_n generate the variation δ_ε regarded in (2.2.12) with $\varepsilon(\xi) = (\xi - z)^{n+1}$. Similary the operator $\bar{L}_n$ is defined as

$$\bar{T}(\xi)\, A(z, \bar{z}) = \sum_{n=-\infty}^{\infty} (\bar{\xi} - \bar{z})^{-n-2} \bar{L}_n A(z, \bar{z}) \qquad (2.2.16)$$

In particular

$$L_{-1} = \partial_z \quad ; \bar{L}_{-1} = \partial_{\bar{z}} \qquad (2.2.17)$$

The relation of L_0 and $\bar{L}_0$ with the spin operator (2.1.16) can be given as

$$S = L_0 - \bar{L}_0 \qquad (2.2.18)$$

and the dilation D, which is equivalent to the matrix of anomalous dimension, is connected to L_0 and $\bar{L}_0$ by

$$\Gamma = D = L_0 + \bar{L}_0 \qquad (2.2.19)$$

Then it follows that

$$L_0 T = 2T \ ; \ \bar{L}_0 \bar{T} = 2\bar{T} \ ; \ L_0 \bar{T} = \bar{L}_0 T = 0 \qquad (2.2.20)$$

The operators L obey the commutation relations of the Virasoro algebra.

$$[L_m, L_n] = (m-n)\, L_{m+n} + \frac{c}{12}\,(m^3 - m)\, \delta_{m+n,\,0} \qquad (2.2.21)$$

where the constant c is called the central charge which characterizes the given conformal field theory.

The two point function $\langle\, T(z)\, T(0)\,\rangle$ is determined by the central charge value

$$\langle\, T(z)\, T(0)\,\rangle = \frac{c}{2z^4} \qquad (2.2.22)$$

which again shows that $c > 0$, where we have used the operator product of T(z) with itself:

$$T(\xi)\, T(z) = \frac{c}{2(\xi - 2)^4} + \frac{2}{(\xi - 2)^2}\, T(z) + \frac{1}{\xi - 2}\, \partial_z T(z) + \ldots \qquad (2.2.23)$$

2.3 Degenerate Families and Minimal Models

In the algebra of local fields, there are some distinct fields transforming in the following way

$$\Phi_n(z, \bar{z}) \rightarrow \left(\frac{d\xi}{dz}\right)^{\Delta_n} \left(\frac{d\bar{\xi}}{d\bar{z}}\right)^{\bar{\Delta}_n} \Phi_n(\xi, \bar{\xi}) \tag{2.3.1}$$

under the substitutions (2.2.6). Here Δ_n and $\bar{\Delta}_n$ are the dimensions of the field and the combinations $d_n = \Delta_n + \bar{\Delta}_n$ and $s_n = \Delta_n - \bar{\Delta}_n$ are the anomalous scale dimension and the spin of the field Φ_n respectively.

A conformal family $[\Phi_n]$ of a primary field Φ_n, besides itself Φ_n, contains infinitely many other secondary fields (descendants) which are given as

$$\Phi_n^{(-k1 \ldots -kN)}(z) = L_{-k1}(z) \ldots L_{-kN}(z)\, \Phi_n(z) \tag{2.3.2}$$

with

$$L_{-k}(z) = \oint d\xi \frac{T(\xi)}{(\xi - z)^{k+1}} \tag{2.3.3}$$

and these fields have the dimensions

$$\Delta_n^{(k1 \ldots kN)} = \Delta_n + k_1 + \ldots k_N \tag{2.3.4}$$

Let us define the primary state as

$$|\, n \rangle = \Phi_n(0)\, |\, 0 \rangle \tag{2.3.5}$$

one can obtain

$$L_m \mid n \rangle = 0 \quad \text{if} \quad n > 0$$

$$L_o \mid n \rangle = \Delta_n \mid n \rangle \tag{2.3.6}$$

On the other hand application of L_m, $m < 0$ gives

$$L_{-k1} \ldots L_{-kN} \mid n \rangle = \Phi_n^{(-k1\ldots-kN)} (o) \mid 0 \rangle \tag{2.3.7}$$

Therefore the representations of the Virasoro algebra (2.3.3) correspond to some spaces V_n, those spaces are called Verma modules.

There exists some special values of Δ_n in which case the representation of Virasoro algebra becomes reducible. For these values the space V_Δ contains special kind of vectors $\mid \chi \rangle \in V_\Delta$ such that the equations

$$L_n \mid \chi \rangle = 0 \quad \text{if} \quad n > 0$$
$$L_o \mid \chi \rangle = (\Delta + K) \mid \chi \rangle \tag{2.3.8}$$

are satisfied and they are called null vectors. A conformal family containing fields which give rise to such vectors is called a degenerate conformal family.

All the special values of Δ, which is equivalent to the reducible representations $V\Delta$, can be given by the Kac formula. These values are

$$\Delta_{(p,q)} = \Delta_o + (\frac{1}{2}\alpha + p + \frac{1}{2} \alpha.q)^2 \tag{2.3.9}$$

where

$$\Delta_o = \frac{1}{24}\ (c-1) \qquad (2.3.10)$$

$$\alpha_\pm = \frac{\sqrt{1-c} \pm \sqrt{25-c}}{\sqrt{24}} \qquad (2.3.11)$$

Since the complex dimensions of $\Delta_{p,q}$ are not acceptable, the values of c should be limited in the domain

$$0 < c \leq 1 \qquad (2.3.12)$$

As this domain is regarded, the work of Friedan, Qui and Shenker tells us that the unitarity condition leads to a selection rule for the representations of the Virasoro algebra, so the allowed values of the central charge within the interval (2.3.12) are

$$c_m = 1 - \frac{6}{m(m+1)} \qquad m \geq 2 \qquad (2.3.13)$$

provided that

$$\Delta(p, q) = \frac{[p(m+1) - qm]^2}{4m(m+1)} \qquad (2.3.14)$$

where

$$1 \leq p \leq m-1 \quad , \qquad 1 \leq q \leq p \qquad (2.3.15)$$

These formulas determine all the values of c and Δ, and result in the degenerate representations of the algebra. The primary fields corresponding to the degenerate conformal families are denoted as

$$\varphi(p, q) = \varphi(p - q, p + 1 - m) \qquad (2.3.16)$$

where

$$p = 1, 2 \ldots m\text{-}1$$
$$q = 1, 2 \ldots m \qquad (2.3.17)$$

The parametrization of c and $\Delta(p, q)$ describe the minimal models M_m, m = 3, 4 ...in which all primary fields are degenerate.

The general fusion rules for the degenerate fields have the form

$$\Psi(p_1,q_1)\,\Psi(p_2,q_2) = \sum_{k=|p_1-p_2|+1}^{p_1+p_2-1} \; \sum_{l=|q_1-q_2|+1}^{q_1+q_2-1} [\,\Psi\,(k,l)] \qquad (2.3.18)$$

where k(l) is even if $p_1 + p_2$ $(q_1 + q_2)$ is odd So the degenerate fields form the closed operator algebra which leads to the idea of minimal theories in which all the primary fields are degenerate. And these minimal theories correspond to the two dimensional models which can be identified with the critical and multicritical points of two dimensional statistical system, such as the Ising model, the 3-state potts model and the corresponding tricritical models.

Chapter 3

Renormalization Group Analysis in Two Dimensional Field Theory

As mentioned before, one of the main purposes of this work is to study the general properties of the renormalization group in 2D relativistic field theory and especially realization of the Zamolodchikov's c-theorem, establishing the close connection between the two dimensional field theory and the renormalization group structure.

From this point of view, we start by reviewing the general properties of renormalization group in relativistic 2D-field theory and obtain the renormalization group equation in the Callan-Symanzik form (section 3.1).

Section 3.2 contains the explicit realization of the Zamolodchikov's c-theorem.

3.1 The Renormalization Group Equations in The Callan-Symanzik Form

In the Lagrangian formulation of the theory the correlation functions of $A_i(x)$ are defined by the functional integrals which are of the following form:

$$\int A_1(x_1) \dots A_N(x_N) \exp(-H(\varphi)) D\varphi \qquad (3.1.1)$$

where φ is a set of fundamental fields, Ai (x) are local functions of $\varphi(x)$ and of derivates $\partial_\mu\varphi$, $\partial_\mu\partial_\nu\varphi$ etc., and $H(\varphi)$ is the Euclidean action which is an integral of the local density.

$$H(\varphi) = \int \mathscr{H}(\varphi(x), \partial_\mu\varphi(x))\, d^2x \qquad (3.1.2)$$

It is assumed in (3.1.1) that $\mathscr{H}(\varphi)$ includes a term which ensures the normalization of the distribution function so that the factor z^{-1} is not written.

In this formulation the variations of the action are described by the stress energy tensor.

$$\delta_\varepsilon H = -\frac{1}{2\pi} \int d^2x\, \partial_\mu\varepsilon_\nu(x)\, T^{\mu\nu}(x) \qquad (3.1.3)$$

under the infinitesimall coordinate transformations.

$$x^\mu \to x^\mu + \varepsilon^\mu(x) \qquad (3.1.4)$$

The variation of the correlation functions (3.1.1) under the transformation law (3.1.4) gives rise to the following relation

$$\sum_{i=1}^{N} \langle A_1(x_1) \dots A_{i-1}(x_{i-1}) \; \delta\varepsilon \; A_i(x_i) A_{i+1}(x_{i+1}) \dots A_N(x_N) \rangle +$$
$$+ \frac{1}{2\pi} \int d^2x \, \partial\mu\varepsilon\nu(x) \langle T^{\mu\nu}(x) A_1(x_1) \dots A_N(x_N) \rangle = 0 \tag{3.1.5}$$

In the non Lagrangian approach this relation is considered as the definition of the linear operators δ_ε acting in $\mathcal{A}$. As we consider the continuity equation, $\partial_\mu T^{\mu\nu} = 0$ this can be written as

$$2\pi\delta_\varepsilon A(x) = -\int_{\partial\Lambda_x} dy^\lambda \, \varepsilon\lambda\mu \in\nu (y) \, T^{\mu\nu}(y) A(x) - \int_{\Lambda_x} d^2y \; \partial\mu \, \varepsilon\nu(y) \, T^{\mu\nu}(y) A(x) \tag{3.1.6}$$

where Λ_x is an arbitrary small region in R^2 including the point x and $\partial\Lambda_x$ is its boundary.

Let us consider the infinitesimal translations

$$\varepsilon\mu(x) = \varepsilon\mu \tag{3.1.7}$$

and rotations

$$\varepsilon\mu(x) = \omega\mu\nu \, x^\nu \quad ; \omega\mu\nu = - \omega\nu\mu \tag{3.1.8}$$

In such a case the corresponding operators are the momentum and spin operators that we introduced before.

$$\delta\varepsilon = i \, \varepsilon^\mu \, P\mu \quad \text{if} \quad \varepsilon^\mu(x) = \varepsilon^\mu$$
$$\delta\varepsilon = i\omega\mu\nu \, (x^\mu P^\nu - \varepsilon^{\mu\nu} \Sigma) \quad \text{if} \quad \varepsilon^\mu(x) = \omega\mu\nu \, x^\nu \tag{3.1.9}$$

Another type of coordinate transformation is the dilation

$$\varepsilon^{\mu}(x) = \frac{1}{2} \; x^{\mu} dt \tag{3.1.9}$$

let D denote the corresponding operator acting in $\mathcal{A}$

$$\delta_{\varepsilon} A(0) = \frac{1}{2} D \;\; A(0)\, dt \tag{3.1.10}$$

for the ε described in (3.1.9). Also for these ε

$$\delta_{\varepsilon} A(x) = \frac{1}{2}\, dt\, (i\, x^{\mu}\, P_{\mu} + D)\, A(x) \tag{3.1.11}$$

Then we obtain (3.1.3) as

$$\sum_{i=1}^{N} \langle \, (x_i \; \frac{\partial}{\partial x^{\mu}_{i}} + D_i)\, A_1\,(x_1) \ldots A_N(x_N) \rangle + \frac{2}{\pi} \int d^2x \, \langle \, \theta(x)\, A_1(x_1) \ldots A_N(x_N) \, \rangle = 0 \tag{3.1.12}$$

Here D_i shows the action of operator (3.1.10) on the field $A_i(x_i)$ and $\theta = 1/4\; T^{\mu}{}_{\mu}$ is the trace of the stress tensor.

The renormalization group is a powerful method which describes the behaviour of the theory under the scale transformations. Here we try to summarize the procedure for determining the renormalization group in two dimensional field theory.

The space of local interactions S is the main concept of this method. S is the action functional of a Eucledian field theory which is an integral of the local density. ($H[\varphi] = \int \mathcal{H}(\varphi) d^2x$ in (3.1.2)) with the locality condition is satisfied.

It is usually assumed that there is a finite ultraviolet cut-off in the theory; in such a case the locality condition can be violated below the scales R_0, where R_0 is the cut-off length. In general the space S is infinite dimensional, however we assume that it can be treated like a finite dimensional manifold. Let $\{ g^a \} = \{ g^1, g^2 ...\}$ be a coordinate system on S, then $\mathscr{H}(\varphi) = \mathscr{H}_g(\varphi)$. We consider the derivatives of the form

$$\Phi_a(x) = -\frac{\partial}{\partial g^a}\mathscr{H}_g(\varphi(x)) \tag{3.1.13}$$

where $\Phi_a \in \mathscr{A}_g^{(0)}$. Here we assume that all the fields Φ_a are spinless, taking care of homogenous and isotropic theories only. Then the subspace $\mathscr{A}_g^{(0)}$ can be considered as a tangent space to S at the point g.

Differentiation of (3.1.1) with respect to g^a gives

$$\frac{\partial}{\partial g^a}\langle A_1(x_1) ... A_N(x_N) \rangle_g = \sum_{i=1}^{N} \langle A_1(x_1) ... A_{i-1}(x_{i-1})\, B_a A_i(x_i) ... \rangle_g + \\ + \int d^2x \langle \Phi_a(x) A_1(x_1) ... A_N(x_N) \rangle_g \tag{3.1.14}$$

Here the operator B_a acts on the field A as:

$$B_a A = \frac{\partial}{\partial g^a} A \tag{3.1.15}$$

We are assuming that the field theories under consideration are renormalizable, then we can write

$$\theta(x) = -\frac{\pi}{2}\beta^a(g)\,\Phi_a(x) \tag{3.1.16}$$

This implies that the trace of stress energy tensor θ, belonging to

$\mathcal{A}_g^{(0)}$, can be spanned by the basis vectors (3.1.13). The coefficients $\beta^a(g)$ are the coordinate components of the vector field β on S. They are called β-functions.

Finally the combination of (3.1.12) and (3.1.14) leads to the renormalization group equations in the Callan-Symanzik form

$$\sum_{i=1}^{N} \langle (x_i^{\mu} \frac{\partial}{\partial x_i^{\mu}} + \Gamma_i (g)) A_1(x_1) \ldots A_N (x_N) \rangle g =$$
$$= \sum_a \beta^a (g) \frac{\partial}{\partial g^a} \langle A_1(x_1) \ldots A_N (x_N) \rangle \quad (3.1.17)$$

where the operator

$$\Gamma = D + \beta^a B_a \quad (3.1.18)$$

is called the anomalous dimension matrix, and it can be shown that

$$\Gamma \Phi_a = \gamma_a{}^b (g) \Phi_b = 2\Phi_a - \frac{\partial \beta^b}{\partial g^a} \Phi_b \quad (3.1.19)$$

This relation guarantees the absence of renormalizations in the components of the stress tensor:

$$\Gamma T_{\mu\nu} = 2 T_{\mu\nu} \quad (3.1.20)$$

The coefficients β^a and the matrix elements of the operator Γ do not depend on R_0 in renormalized theory.

It also follows from (3.1.17) that the two field theories corresponding to two points $g^a(t_1)$ and $g^a(t_2)$ on the same integral curve $g^a(t)$ of the Gell Mann-Low equations

$$dg^a = \frac{1}{2}\beta^a(g)\, dt \tag{3.1.21}$$

differ only by the scale transformation $x^\mu \to x^\mu\, e^{(t1-t2)/2}$.

So, the singularities and the global topological properties of the β-functions play an important role on the scale behaviour of field theory. The simplest singularities of $\beta^a(g)$ correspond to its fixed points g_{*A}:

$$\beta\,(g_{*A}) = 0 \tag{3.1.22}$$

It must be recalled that the critical behaviour of a statistical system is directly connected with the fixed points of the renormalization group.

3.2 Zamolodchikov's c-Theorem

In this section we shall show that in a two dimensional field theory satisfying the positivity condition the renormalization group flow described by equation (3.1.21) has a dissipative character. We shall see that there exists a function C of the coupling constants which is non increasing along renormalization group trajectories which is stationary only at fixed points, and which at a fixed point, is equal to the value of the corresponding theory. For this let us consider the two-point functions.

$$\langle T(z,\bar{z})\, T(o,o) \rangle = \frac{F\ (t)}{z^4}$$

$$\langle T(z,z)\, \theta(o,o) \rangle = \frac{H\ (t)}{z^3\, \bar{z}} \qquad (3.2.1)$$

$$\langle \theta\ (z,z)\, \theta(o,o \rangle = \frac{G\ (t)}{z^2\, \bar{z}^2}$$

where T and θ are the components (2.1.13) of the stress tensor and $t = \text{Log}\ (z\,\bar{z})$.

The conservation of stress-energy tensor says that

$$\partial_{\bar{z}} T - \partial_z \theta = 0 \qquad (3.2.2)$$

Taking the correlation function of the equation with T(o, o) and with θ(o, o) and using (3.2.1) we reach the following two equations

$$\dot{F} = \dot{H} - 3H$$

$$\dot{H} - H = \dot{G} - 2G \qquad (3.2.3)$$

where dot denotes the t-derivative. Eliminating H from above and defining

$$C = 2F + 4H - 6G \qquad (3.2.4)$$

We see that

$$\dot{C} = -12G \qquad (3.2.5)$$

Since

$$G(t) \geq 0 \tag{3.2.6}$$

by the positivity condition, (3.2.5) implies that c(t) is a monotonically decreasing function of t. Moreover there is the equality in (3.2.6) only if $\Theta = \beta^a \Phi_a = 0$, i. e we are at a fixed point, in this case c is a constant.

The function c(t) can be interpreted as a measure of degrees of freedom with significant fluctuations of space size $e^{t/2}$ at a given critical point, naturally it decreases with the increase of t.

If we fix t, say t = 0, then the quantities F, H and G will depend only on the coupling constants g^a, then (3.1.20), (3.2.5) and the renormalization group equations (3.1.17) lead to

$$\beta^a (g) \frac{\partial}{\partial g^a} c(g) = -12\, G_{ab}\, (g)\, \beta^a\, (g)\, \beta^b\, (g) \leq 0 \tag{3.2.7}$$

where

$$G_{ab}(g) = G_{ab}(o, g)$$

$$G_{ab}\, (t, g) = (z\bar{z})^2 \langle \Phi_a\, (z, \bar{z})\, \Phi_b\, (o, o) \rangle \tag{3.2.8}$$

is positive definite because of the positivity condition.

We can deduce from equation (3.2.7) that the renormalization group flow decreases the function c(g). The stationary points of c(g) correspond to the fixed points g_{*A}:

$$\frac{\partial}{\partial g^a} c(g_*) = 0 \Rightarrow \beta^a\, (g_*) = 0 \tag{3.2.9}$$

At the fixed point, namely $g = g_*$, the field θ vanishes which implies the conformal symmetry of the theory.

Furthermore the two point function (3.2.1) at the fixed point $g = g_{*A}$ can be expressed in terms of the constant c_A, which is the central charge of the Virasoro algebra.

$$\langle\, T(z\bar{z})\, T(oo)\,\rangle_{g_{*A}} = \frac{c_A}{2z^4} \qquad (3.2.10)$$

For $g = g_{*A}$, the functions G and H in (3.2.1) also vanish. Therefore it follows from (3.2.4) that

$$c_A = c(g_{*A}) \qquad (3.2.11)$$

Next, let us consider the $n = 1$ case in the formula (3.2.7). Setting $G_{11} = 1$, equation (3.2.7) leads to

$$\beta(g) = \frac{-1}{12} \frac{d}{dg} c(g) \qquad (3.2.12)$$

Moreover if g_{*0} and g_{*1} are the two fixed points, then corresponding values of the central charges c_0 and c_1 are related as:

$$c_1 - c_0 = \int_{g_{*0}}^{g_{*1}} \beta(g)\, dg \qquad (3.2.13)$$

The main consequence of the above relations is that if the two fixed points g_{*0} and g_{*1} are connected by the renormalization group trajectory so that $g(-\infty) = g_{*0}$ and $g(\infty) = g_1$, then the corresponding constants satisfy the inequality

$$c_0 > c_1 \qquad (3.2.14)$$

Chapter 4

Application of The Renormalization Group Analysis To Conformal Field Theories

We know that the critical behaviour of statistical system is concerned with the fixed points of the renormalization group. In the critical point analysis the first step is to construct the conformally invariant field theory solution corresponding to the fixed point itself. If this solution is know exactly, then using the perturbation theory, one can determine the properties of the renormalization group in the neighborhood of this point.

This section contains some applications of the renormalization group method to the simplest known series of exact conformally invariant solutions which are the minimal models M_p, $p=3,4...$

In section 4.1 the power expansion of the β-functions and of the function c(g) near the fixed point are given, section 4.2 is an application of this perturbative approach to the minimal models in the case $p >> 1$.

Finally section 4.3 contains another application .

4.1 Perturbative Evaluation of c

We find the power series expansions of the β-function at the fixed point $g=0$.

First let us consider the linear part of the expansion. In section 3.1 we defined the spinless fields for $g\neq 0$ (3.1.1)

At $g=0$,

$$\Phi_i^o = \Phi_i \Big|_{g=0} \qquad (4.1.1)$$

where $\Phi_i^o \in \mathcal{A}^{(o)}{}_{g=o}$. We choose the coordinate system in S such that the fields Φ_i^o have well defined dimensions

$$\Delta_i = \bar{\Delta}_i \qquad (4.1.2)$$

and they are orthonormal:

$$\langle \Phi_i(z\bar{z})\, \Phi_j\,(oo)\rangle = \delta_{ij}\,(z)^{-2\Delta_i}\,\,(z)^{-2\Delta_i} \qquad (4.1.3)$$

which corresponds to the choice

$$G_{ij}\,(0) = \delta_{ij} \qquad (4.1.4)$$

Then one can observe that for $g=0$

$$\Gamma(0)\,\Phi_i^o = \gamma_i^j\,(0)\,\Phi_j^o \qquad (4.1.5)$$

where

$$\gamma_i^j\,(0) = \Delta_i\,\delta_i^j \qquad (4.1.6)$$

Furthermore the equation (3.1.19) tells us that

$$\Delta_i\,\delta_i^j\,\Phi_j^o = \delta_i^j\,\Phi_j^o - \frac{\partial\beta^j}{\partial g^i}\,\Phi_j^o \qquad (4.1.7)$$

from which, one gets

$$\beta^i(g) = \varepsilon_i\, g^i + 0(g^2) \qquad (4.1.8)$$

Here

$$\varepsilon_i = 1 - \Delta_i \tag{4.1.9}$$

To calculate the other terms in the expansion we shall use the perturbation theory. We focus on the case where

$$|\varepsilon_i| \sim \varepsilon << 1 \tag{4.1.10}$$

This approximation is useful in a sense that the renormalization group yields non trivial behaviour within the region $g_i \leq \varepsilon$.

Let us consider the equation (3.1.14), in the present case we have

$$\frac{\partial}{\partial g^k} \langle \Phi_i(x)\, \Phi_j(0)\rangle \Big|_{g=0} = \langle (B_k^o\, \Phi_i^o)(x)\, \Phi_j^o\,(0)\rangle + \\ + \langle \Phi_i^o\,(x)\,(B_k^o\, \Phi_j^o\,(0))\,\rangle + \int d^2y\, \langle \Phi_i^o\,(x)\, \Phi_j^o\,(0)\, \Phi_k^o(y)\rangle \tag{4.1.11}$$

Here

$$B_k^o = B_k \Big|_{g=0} \tag{4.1.12}$$

It can be shown that

$$\frac{\partial}{\partial g^k} \langle \Phi_i\,(x)\, \Phi_j\,(0)\,\rangle \Big|_{g=0} = \\ = (x^2)^{-\Delta_i-\Delta_j}\, C_{ijk}\, \{\, \mathcal{J}^j_{ik}\, [(x^2)^{1-\Delta_k} - (x^2)^{\Delta_i-\Delta_j}] + \\ + \mathcal{J}^i_{jk}\, [(x^2)^{1-\Delta_k} - (x^2)^{\Delta_j-\Delta_i}]\} \tag{4.1.14}$$

with the choice of the coordinate system

$$G_{ij} = \delta_{ij} + 0(g^2) \tag{4.2.15}$$

Here

$$\mathcal{J}^j_{ik} = \frac{2\pi}{\varepsilon_i + \varepsilon_k - \varepsilon_j}\,(1 + 0(\varepsilon^3)) \tag{4.1.16}$$

Return back to the equation (3.1.17) again and comparing this with (4.2.14), one obtains

$$\gamma_i^{\ j} = \Delta_i \delta_i^{\ j} + C_i^{\ j}{}_k g^k + 0(g^2) \tag{4.1.17}$$

where

$$C_i^{\ j}{}_k = 2\pi C_{ijk} + 0(\varepsilon^2) \tag{4.1.18}$$

Then, (3.1.19) leads to

$$(\Delta_i \delta_i^{\ j} + C_i^{\ j}{}_k g^k + 0(g^2))\, \Phi_j = \delta_i^{\ j} \Phi_j - \frac{\partial \beta^j}{\partial g^i} \Phi_j \tag{4.1.19}$$

Furthermore

$$\beta^i (g) = \varepsilon_i g^i - 1/2\, C_j^{\ i}{}_k g^j g^k + 0(g^3) \tag{4.1.20}$$

Due to the symmetry

$$C_{ijk} = C_j^{\ i}{}_k \tag{4.1.21}$$

It follows from (3.2.7) that

$$\frac{\partial}{\partial g^i} c(g) = -12\, (\delta_{ij} + 0(g^2))\, (\varepsilon_i g^i - \frac{1}{12} C_{ijk} g^j g^k + 0(g^3)) \tag{4.1.21}$$

Finally, we have

$$c(g) = c_0 - 6\, \varepsilon_i g^i g^i + 2C_{ijk} g^i g^j g^k + 0(g^4) \tag{4.1.22}$$

4.2 The Renormalization Group Approach To The Minimal Models

Let us consider perturbation theory about the minimal models M_p in the case $p \langle\langle 1$. For each of the models there is a primary field

$\varphi = \varphi(1,3)$ with anomalous dimension $d_{(1,3)} = 2\Delta_{(1,3)} = 2\text{-}2\varepsilon$ where ε is a small parameter and its value is

$$\varepsilon = \frac{2}{p+1} \tag{4.2.1}$$

for p ⟨⟨1.

We have the perturbed action

$$\mathcal{H} = \mathcal{H}^{(p)} + g \int \varphi^{(p)}_{(1,3)}(x)\, d^2x \tag{4.2.2}$$

Where $\mathcal{H}^{(p)}$ correspond to the conformal theory M_p and describes the ultraviolet asymptotic form of the theory. We will see that in the region $0 < g \leq \varepsilon$ the infrared asymptotic form of the theory (4.2.2) has also conformal invariance which is described by the model M_{p-1}.

In the expansion of the β-function the first terms are

$$\beta(g) = \varepsilon g - \frac{1}{2}(2\pi C)\, g^2 + 0(g^3) \tag{4.2.3}$$

Here the structure constants c can be obtained from the general formulas [19].

$$C(\varepsilon) = \frac{4}{\sqrt{3}} \frac{(1\text{-}2\varepsilon)^2}{(1\text{-}\varepsilon)(1\text{-}3\varepsilon/2)} \left[\frac{\Gamma(1\text{-}\varepsilon/2)}{\Gamma(1+\varepsilon/2)}\right]^{3/2} \frac{\Gamma^2(1+\varepsilon)}{\Gamma^2(1\text{-}\varepsilon)} \cdot$$

$$\cdot \left[\frac{\Gamma(1+3\varepsilon/2)}{\Gamma(1\text{-}3\varepsilon/2)}\right]^{1/2} \frac{\Gamma(1\text{-}2\varepsilon)}{\Gamma(1+2\varepsilon)}$$

$$= \frac{4}{\sqrt{3}}\left(1\text{-} \frac{3\varepsilon}{2}\right) + 0(\varepsilon^2)) \tag{4.2.4}$$

With this formulation, it has the form

$$\beta(g) = \varepsilon g - \frac{4\pi}{\sqrt{3}}\left(1\text{-}\frac{3\varepsilon}{2}\right) g^2 - \frac{4(2\pi)^2}{3} g^3 + \ldots \tag{4.2.5}$$

There is a fixed point such that

$$g = g_{*1} = \frac{\sqrt{3}}{4\pi} \quad \varepsilon\,(1 + \frac{\varepsilon}{2} \quad + 0(\varepsilon^2)) \tag{4.2.6}$$

This fixed point corresponds to a minimal model M_{p-1}. Since the central charge $c(g_{*1})$ at the fixed point is calculated as

$$c(g_{*1}) = c_p - \frac{3}{2}\varepsilon^3 - \frac{9}{4}\varepsilon^4 \tag{4.2.7}$$

One can observe that this is in accordance with the central charge corresponding to a minimal model M_{p-1}:

$$c(g_{*1}) = c_{p-1} = 1 - \frac{6}{p(p-1)} \tag{4.2.8}$$

The anomalous dimension of the field $\Phi(x,g_{*1})$ in the conformal theory g_{*1} can be obtained from the slope of the β-function.

$$\frac{d\beta}{dg} \Big|_{g=g_*} = -\varepsilon - \varepsilon^2 + \ldots \tag{4.2.9}$$

Thus, we obtained the anomalous dimension $\Delta_{(3,1)}$ in the model M_{p-1}:

$$\Delta = 1 - \varepsilon + \varepsilon^2 \ldots = 1 + \frac{2}{(p-1)} \tag{4.2.10}$$

The crucial point is that in each of the models M_p, there a field $\Phi = \Phi(1,3)$ with anomalous dimension $\Delta_{(1,3)} = 1 - \varepsilon$ where

$$\varepsilon = \frac{2}{p+1} \tag{4.2.11}$$

The fields

$$\Phi(x,0) = \Phi^{(p)}_{(1,3)}\,(x) \tag{4.2.12}$$

4.3 Another Application

We know that in renormalization group theory there is a possibility to construct the properties of the theory at one fixed point in terms of the known properties of the other and Zamolodchikov's theorem is an important result in this direction. When there are two close fixed points, the renormalization group eigenvalue y along the flow connecting the fixed points is small, giving rise to an effective expansion parameter.

In this case, there is a way of constructing the renormalization group equations, given the operator product expansion coefficients in the IR unstable fixed point.

Let us consider the action

$$S = S^* - \sum_i \lambda_i \int \Phi_i(r)\, d^2r \tag{4.3.1}$$

where S^* is the action at the IR unstable fixed point and Φ_i is the scaling operator with dimension $y_i = 2 - x_i$

The renormalization group equations constructed by this way in the lowest order are found as:

$$\dot{g}_k = y_k g_k - \pi \sum_{i,j} c_{ijk} g_i g_j + 0(g^3) \tag{4.3.2}$$

where

$$y_i = 2 - x_i \qquad \dot{g}_k = \frac{dg_k}{dl} \tag{4.3.3}$$

To this order, the renormalization group equations can be written as:

$$\dot{g}_k = \frac{\partial}{\partial g_k} \tilde{C}(\{g\}) \tag{4.3.4}$$

where

$$\tilde{C}(\{g\}) = \frac{1}{2} \sum_{\kappa} y_k g_k^2 - \frac{1}{3} \pi \sum_{i,j,k} C_{ijk} g_i g_j g_k \tag{4.3.5}$$

By the virtue of the equation (4.3.4), one can calculate the function C({g}) of the Zamolodchikov's theorem, since it must have the same fixed points as C(g), there is a proportionality between them:

$$C(\{g\}) = c + \alpha \tilde{C}(\{g\}) + 0(g^4) \tag{4.3.6}$$

where α is the proportionality constant and its value can be calculated as

$$\alpha = -6\pi^2 \tag{4.3.7}$$

With this formulation one can find the value of c at the new point. Let us study on the simplest case, with one coupling g_1, assuming that $C_{11j} = 0$ for $j \neq i$. Then the fixed point can be found as

$$g_1 = g_{1*} \equiv \frac{y_1}{\pi c_{111}} + 0(y^2) \tag{4.3.8}$$

At the new fixed point we have

$$C(\{g\}) = c - 6\pi^2\left(\frac{1}{2}\frac{y_1^3}{\pi^2 c_{111}^2} - \frac{1}{3}\pi\frac{C_{111}\, y_1^3}{\pi^3 c_{111}^3}\right) + \ldots \tag{4.3.9}$$

$$C(\{g\}) = c - \frac{y_1^3}{c^2_{111}} + 0(y^4_1) \tag{4.3.10}$$

This result can be applied to the sequence of models with $c<1$ corresponding to the universality class of multicritical Ising models. In these models the operator $\Phi_{1,3}$ with $\Delta = \bar{\Delta} = \Delta_{1,3}$ is always present and it has the dimension

$$x_{1,3} = \frac{((m+1)-3m)^2 - 1}{2m(m+1)} \sim 2 - \frac{4}{m} + 0(m^{-2}) \tag{4.4.1}$$

as $m \to \infty$, and

$$y = 2 - x_{1,3} = \frac{4}{m} + (1/m^2) \tag{4.4.2}$$

The coefficient C_{111} may be found as [19]

$$C_{111} = \frac{4}{\sqrt{3}} (1 + 0(m^{-1})) \tag{4.4.3}$$

Then the value of C at the new point is

$$c(m) - \frac{3}{4^2}(\frac{4}{m})^3 + 0(m^{-4}) = \tag{4.4.4}$$

$$= c(m) - \frac{12}{m^3} + 0(m^{-4}) \tag{4.4.5}$$

$$= c(m-1) + 0(m^{-4}) \tag{4.4.6}$$

Therefore under the perturbation $\Phi_{1,3}$, the renormalization group flows cross over to the theory with the next lowest value of c.

Appendix

Definitions and Notations:

This section contains a collection of definitions for the partition function as a functional integral for the representation of the Hamiltonian and for functional derivates. They will be necessary in the derivation of the RG equation.

The partition function for a system with continuous variables S(r) at lattice points r can be written

$$Z = \prod_r \left(\int d\frac{v}{2\pi} \; S(r) \right) \exp(-H) \qquad \text{(A.1)}$$

where v is the volume per lattice site. Introducing the Fourier components

$$Sq = v \sum_r S(r) \, e^{-iqr} \qquad \text{(A.2)}$$

and

$$S(r) = \frac{1}{V} \sum_q S_q \, e^{iqr} \qquad \text{(A.3)}$$

then the partition function is

$$Z = \int [\, dS \,] \exp(-H) \qquad \text{(A.4)}$$

with

$$\int_q [\, dS \,] = \prod \left(\int \frac{dS_q}{\sqrt{2\pi V}} \right) \qquad \text{(A.5)}$$

Our Hamiltonian is of the form:

$$H = \int d^d r \left[\frac{1}{2} a(T) m^2(r) + \frac{1}{2} c(T) (\nabla m(r))^2 + \frac{1}{4} b(T) m^4(r) - h m(r) \right] \quad (A.6)$$

which can be transformed in terms of the s variables as

$$H = \int d^d r \left[\frac{a}{2} S^2(r) + \frac{c}{2} (\nabla S(r))^2 + \frac{b}{4} S^4(r) - h S(r) \right] \quad (A.7)$$

and in terms of the Fourier components

$$H = \frac{1}{2V} \Sigma_q (a + cq^2) S_q S_{-q} + \frac{b}{4V^3} \Sigma S_{q1} S_{q2} S_{q3} S_{-q1} S_{-q2} S_{-q3} - hS_0 \quad (A.8)$$

The spacing of q-vectors is

$$(\Delta q)^d = \frac{(2\pi)^d}{V} \quad (A.9)$$

Since

$$\sum_q (\Delta q)^d f(q) = \int d^d q \ f(q) \quad (A.10)$$

We have

$$\frac{1}{V} \Sigma f(q) = \int_q f(q) \quad (A.11)$$

where $\int f(q)$ is as defined as:

$$\int_q f(q) = \frac{1}{(2\pi)^d} \int d^d q \, f(q) \quad (A.12)$$

The relation (A.11) becomes an equality in the thermodynamic limit. From this one finds the correspondence of the Kronecker-Delta to the δ-distribution

$$V \delta_{q,0} \wedge (2\pi)^d \delta^d(q) \quad (A.13)$$

So the Hamiltonian (A.8) becomes

$$H = \frac{1}{2} \int_q (a+cq^2)\, S_q\, S_{-q} + \frac{b}{4} \int_{q1q2q3} S_{q1}\, S_{q2}\, S_{q3}\, S_{-q1-q2-q3} - hS_0 \tag{A.14}$$

There is no explicit volume dependence in (A.14), so in general it can be written as

$$H = Vu_0 + u_1 S_o + \frac{1}{2} \int_q u_2(q)\, S_q\, S_{-q} + \frac{1}{3!} \int_{q1q2} u_3(q_1,q_2)\, S_{q1}\, S_{q2}\, S_{-q1-q2} + $$
$$+ \frac{1}{4!} \int u_4(q_1,q_2,q_3)\, S_{q1}\, S_{q2}\, S_{q3}\, S_{-q1-q2-q3} + \ldots \tag{A.15}$$

for translational invariant interactions. The requirement on u_n's is that they are invariant under permutations of the arguments q including $q_n = -q_1 - q_2 \ldots - q_{n-1}$ which as often not explicitly written

$$u_2(q) = u_2(-q)$$
$$u_3(q_1,q_2) = u_3(q_2,q_1) = u_3\,(q_1,-q_1-q_2) \tag{A.16}$$

For a finite volume Hamiltonian we have:

$$H = Vu_0 + u_1 S_0 + \frac{1}{2V} \Sigma u_2(q)\, S_q S_{-q} + \frac{1}{3!V^2} \Sigma\; u_3\,(q_1,q_2)\, S_{q1}\, S_{q2}\, S_{-q1-q2} + \ldots \tag{A.17}$$

References

[1] K. G. Wilson and J. Kogut, Phys. Reports, 12 C 78 (1974).

[2] Patashinskii and Pokrovskii, Fluctuation Theory of Phase transitions, Pergamon Press (1979).

[3] A. A. Belavin, A. M. Polyakov, A. B. Zamolodchikov, N,F. B 241, 333, (1984) J. Stat Phys 34, 763 (1984).

[4] A. B. Zamolodchikov, JETP Lett. 43 (1986) 730.

[5] Zamolodchikov A. B., Zamolodchikov A.I.B., Preprint ITEP, (1989).

[6] Zamolodchikov A. B., Sov J. Nucl. Phys 46(6), (1987), 1096.

[7] Polyakov, Zh Etf 1974, 66, P. 23.

[8] L. P. Kadanoff, Phys Rev Letters, 23, 1430 (1969).

[9] Wilson, Phys. Rev. 1969, 179, p. 1499.

[10] Introduction To The Theory of Quantized Fields Bogoliobov and Shirkov, Wiley, (1980).

[11] Modern Theory of Critical Phenomena S. K. Ma, Benjamin Inc, (1976).

[12] J. L. Cardy and W. W. Ludwig, N. F B 285 [FS 19] (1987) 687.

[13] A. B. Zamolodchikov, Sov J. Nucl. Phys 44 (1986) 529.

[14] J. L. Cardy, in Phase Transitions and Critical PhenomenaVol. 11, C. Domb and J.L. Lebowitz, Ac. Press (1986).

[15] D. A. Castor, J. Martinec, S. H. Shenker, EFI 88-31.

[16] A. W. W. Ludwig, N. F B 285, (1987), 97.

[17] G. Andrews, R. J. Baxter and P. J. Forrester, J-Stat Phys 35 (1984), 193.

[18] D. A. Huse, Phys Rev B 30 (1984) 3908.

[19] I. Dotsenko and Fateev, N. F B 251 (1985) 691.