ISAS - INTERNATIONAL SCHOOL

FOR ADVANCED STUDIE

Algebraically Integrable System

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Introduction.

A Hamiltonian system, with Hamiltonian H on 2n dimensional symplectic manifold (M,w) is called Liouville integrable if:
i) there is a smooth proper surjection

$$f :\to B \subset R^n$$

such that $df_1 \wedge ... \wedge df_n \neq 0$, where the f_i 's are the components of the vector f and $\{f_i, f_j\} \equiv \omega(X_{f_i}, X_{f_j}) = 0$
ii) the Hamiltonian is a function of f_i .

In these hypothesis the level sets of f are Lagrangian submanifold of M, i.e. $\forall b \in Bw \mid_{[f^{-1}(b)]} = 0$ and the connected components of f^{-1} (b) are real tori R^n/Z^n. Moreover, there is a neighborhood of U of the real torus with coordinate $(\varphi_1 ... \varphi_n, J_1 ... J_n)$ and $(\varphi_1 ... \varphi_n)$ are modulo Z such that:

$$\omega = d\varphi_i \wedge dJ_i$$

and $f_i = f_i(J_1 ... J_n)$.

A Liouville system (M, w, f) is an algebraically integrable system if there exist a smooth algebraic variety $\tilde{M}$, a symplectic structure $\tilde{\omega}$ on $\tilde{M}$ and a surjective proper map $\tilde{f} : \tilde{M} \to \tilde{B} \subset \mathbf{C}^n$, $\tilde{H} : \tilde{M} \to \mathbf{C}^n$ holomorphic such that M is a real component of $\tilde{M}$ and $\tilde{\omega} \mid_M = \omega$, $\tilde{H} \mid_M = H$.
In such condition the fibers of $\tilde{f}$ are abelian varieties or extensions of these.

The problem that naturally rises is: how to characterize the algebraically integrable systems?
The first type of these systems are those that admit a complex extension in which the real invariant tori go in abelian varieties.
Another ones are those that admits a, dependent by an external parameter, Lax pair. To these systems are associated some algebraic curve C, and in some cases a divisor on C such that its temporal evolution goes, via the

Abel's map, to a linear evolution on the Jacobian of C, or better on the real sub-manifold of this one.

For all the known examples of integrable systems there exist a Lax pair, and in this sense they are all algebraically integrable.

Note that if a system is algebraically integrable it doesn't necessary admit a Lax pair, because the Jacobian varieties are a subset of Abelian varieties.

Lax pair.

Let $g \subset gl(r, C)$ be a Lie algebra and (M, ω) a symplectic manifold. A Lax matrix

$$L: \ U \subset M \rightarrow g$$

is a injective map with its tangent map.
An Hamiltonian system admits a Lax pair if exists such a L and the Hamilton equation are equivalent on U to:

$$\dot{L} = [L, B]$$

where $B: \ U \rightarrow g$.

If (L, B) is a Lax pair, let $C \in g$ be a constant matrix that commute with B, than set $L_\xi = L + P(\xi, C)$ and $B_\xi = B + Q(\xi, C)$, where $P(\xi, C)$ and $Q(\xi, C)$ are rational functions of $\xi \in \mathbf{C}$ with coefficients C, so:

$$\dot{L}_\xi = [L_\xi, B_\xi]$$

is also equivalent to the Hamiltonian equation.
Using this ambiguity in the choice of the pair we can suppose that the pair depends on an external complex parameter. At this pair we can associate an algebraic curve $C \subset P^2$ such that the affine points satisfy the equation:

$$R(\xi, s) = det|L_\xi - sI|, \quad \xi, s \in \mathbf{C}$$

Because the eigenvalues of L_ξ don't depend on t, than the algebraic curve is constant (isospectral curve).
In general, if this curve is irreducible, than there exists an holomorphic map

between a compact Riemann surface and C, that is injective on the inverse image of the regular points. C is said the normalization of $\tilde{C}$; in the follow we confuse C with $\tilde{C}$.

At this point, if we able to associate to the Lax pair a divisor $D(t)$ on C depending by parameter t, such that, under the Abel Jacobi map, the flow is linear on the Jacobian of C, than the system is algebraically integrable. Respect to this problem M. Adler, P. van Morbeke and D. Mumford [1] show that for an appropriate class of Lax matrices there is a one to one correspondence between an algebraic curve C, two finite sequence of points on C $P_1.....P_n$, $Q_1....Q_m$ a divisor D of degree g (genus of C) regular for this sequence ($dim\ \mathcal{L}\ (D+\sum_{i=1}^{k} P_i - \sum_{i=1}^{k} Q_i) = 0\ \ \forall k$) ; and a Lax matrix up to a conjugation by an appropriate diagonal matrix.
If we fix the matrix B the divisor D evolves in the time according with the Lax equation, under the Abel Jacobi map the corresponding point describes a flow on the Jacobian of C or on a Abelian sub-manifold of JacC.
Every linear flow on JacC is associated with a particular matrix B : $\dot{L} = [L, B]$. The last result has been generalized by Griffiths that has found sufficient conditions about B, in an appropriate class, so that the flow is linear on the JacC or on the PrymC.
Now we suppose that in the Lax pair the B matrix doesn't satisfy the Griffiths's condition, than can we conclude that the flow isn't linear on the Jacobian? Or in other words is there a one to one, up some things, correspondence between a Lax pair and a flow on $C^{(g)} = \frac{C \times C \times .....\times C}{G}$ (and G is the permutation group of g integer)?
To analyze the question confine ourselves for simplicity to hyperelliptic curves by genus equal to degree of freedom of the system. We suppose that is given an algebraic curve whose affine part satisfy the equation $s^2 = P(z)\ s, z \in \mathbf{C}$ and on this curve is given an effective divisor D:

$$D \in Div_0^{+g} = \{D\ effective\ : if\ D = \sum_i^g p_i \ \ i(p_k) \neq i(p_j)\ k \neq j\ p_j \neq \infty\}$$

where i is the hyperelliptic involution.

[1]P. van Morbeke, and D. Mumford Acta Math. 143 (1979), 93-154. M. Adler and P. van Morbeke Advances in Math. 38,(1980) 318-379.

It is known that given a such divisor there exist three polynomial U, V, W such that :

i) $U(z) = \prod_{i=1}^{g}(z - z(p_i))$ monic of degree g

ii) if p_i are distinct, let:

$$V(z) = \sum_{i=1}^{g} s(p_i) \prod_{j \neq i} \frac{(z - z(p_i))}{(z(p_i) - z(p_j))}$$

$V(z)$ is the unique polynomial of degree less then $g-1$ such that $V(z(p_i)) = s(p_i)\ 1 \leq i \leq g$

If p_i has positive multiplicity in D, than:

$$V(z) = \begin{cases} \textit{the unique polynomial of degree } \leq g-1 \ s.\,t. \\ \textit{if } m_D(p_i) = n_i \\ (\frac{d}{dz})^j [V(z) - \mathrm{P}\ (z)] \mid_{z=z(p_i)} = 0 \quad 0 \leq j \leq n_i - 1 \end{cases}$$

by construction $P(z) - V^2(z)$ are divisible by $U(z)$

iii) Define $W(z)\ :\ P(z) - V^2(z) = U(z)W(z)$,the degree of $W(z)$ is $g+1$

.

This correspondence is a bijection and realize the inclusion of $Div_0^{+g}\ in\mathbf{C}^{3g+1}$
The form of the relation:

$$s^2 = P(z) = U(z)W(z) - V^2(z)$$

suggest that this algebraic curve can be considerate as the spectral curve associate with the matrix:

$$L = \begin{pmatrix} V(z) & U(z) \\ W(z) & -V(z) \end{pmatrix}$$

Now, suppose that the divisor D depends on dynamical variables so, it evolves with the time accordingly with the Hamilton equation.

We want to know if it is possible to construct B s. t. under the time evolution:

$$\dot{L} = [B, L]$$

The answers is yes, under the only assumption that the determinant of L is a constant of the motion, which is obvious.
The form of B is:

$$B = \begin{pmatrix} 0 & \frac{\dot{U}}{2V} \\ -\frac{\dot{W}}{2V} & 0 \end{pmatrix}$$

or in an equivalent way:

$$B = \begin{pmatrix} 0 & \frac{\dot{U}}{2V} \\ \frac{\dot{V}-iW}{U} & 0 \end{pmatrix}$$

In term of the matrix L the divisor D can be intrinsically characterized in the following way: calling (f_1, f_2) the components of the eigenvector of the L matrix associated with the eigenvalue s, setting $f_1 = 1$, D is the minimal divisor s. t.

$$(f_2) + i(D) \geq -p_\infty$$

(if the degree of P(z) is odd) where p_∞ is the point of C with coordinate $z = \infty$.
So, given a divisor $D \in Div_0^{+g}$ on the algebraic hyperelliptic curve we can associate a Lax pair modulo conjugation by constant diagonal matrix. The only thing that remains to see is if the flow on the Jacobian of C is linear or not applying the Griffiths's theorem.
Summarizing: given a Lax equation, we are able, at least when the spectral curve is hyperelliptic, to state if the system is algebraically integrable or not. The possibility to generalize this result for any curve depends on if we are so lucky that the inclusion of $D^{+,g}$ in a projective space is realized by something which resembles to a determinant.
In reality, the situation is complicated by fact that there are several Lax pairs which give the same dynamics, and this give rise to several spectral curves. Does it exist a link between the different Lax pairs associated with the same integrable system?

To be able to ceck some correspondance between the integrable system and a divisor D, on a appropriate algebraic curve, we analize the case in which the system is integrable by separation of variables, and as example we

take the Newmann problem.
Let be given a point moving on the S^n $|x| = 1$ under the influence of a quadratic potential $U(x) = \frac{1}{2}\sum_i a_i x_i^2$ with $a_0 \leq a_1 \leq .... \leq a_n$. We define $u_j = u_j(x)$ as a solution of the:

$$A(z)\sum_{\nu=0}^{n} \frac{x_\nu^2}{z-a_\nu} = U(z)$$

where:

$$A(z) = \prod_{\nu=0}^{n}(z-a_\nu) \;\; U(z) = \prod_{j=1}^{n}(z-u_j)$$

In term of the u_j 's the Hamilton-Jacobi equations are solved by separation of variables, and following Moser one finds:

$$S = \frac{1}{2}\sum_{j=1} n \int_{u_j^0}^{u_j} \sqrt{\frac{Q(z)}{-A(z)}}\, dz$$

where $Q(z)$ is given as:

$$Q(z) = z^n + 2\eta_1 z^{n-1} + ... + 2\eta_n.$$

and η_i are constant and $\eta_1 = H(u, \frac{\partial S}{\partial u})$.
We define $\xi_j = \frac{\partial S}{\partial \eta_j}$ the variables conjugate to η_i so, $\dot{\xi}_j = \delta_{j1}$ Now:

$$\xi_i = \frac{1}{2}\sum_{k=1}^{n} \int_{u_k^0}^{u_k} \frac{z^i}{\sqrt{R(z)}}\, dz \;\; i = 1....n$$

and $R(z) = -A(z)Q(z)$
can be interpreted as Abelian integrals on the hyperelliptic algebraic curve $s^2 = R(z)$ of genus n that is the effective degree of freedom of the system.
Let p_i $i = 1....n$ be n point on the curve with coordinate $z_i = u_i(t)$ and $s_i(t) = R(z_i(t))$, we consider the divisor $D(t) = \sum_i p_i(t)$ and construct :

$$\begin{cases} U_{x,y} = \prod_{i=1}^{n}(z-u_i) = A(z)\sum_{\nu=0}^{n}\frac{x_\nu^2}{z-a_\nu} & degU = n = g \\ V_{x,y} = iA(z)\sum_k \frac{x_k \dot{x}_k}{z-a_k} & degV = g-1 \\ W_{x,y} = A(z)(\sum_k \frac{\dot{x}_k}{z-a_k}+1) & degW = g+1 \end{cases}$$

We note that the coordinate of the divisor's point are given by the changing variables in which the Hamilton-Jacobi equation separate.
Now the Lax pair constructed above is equivalent to the system.

Conclusion.

It appear hence that, at least when the system is integrable by separation of variables, it is always possible to set it in the form of Lax and linearize it on the Jacobian of the spectral curve.
The correspondence $(x, y) \rightarrow (U_{x,y}, V_{x,y}, W_{x,y}) \in \mathbf{C}^{3g+1}$ realizes an immersion of $(TS^g)_C$ *in* $\mathbf{C}^{3g+1}$.
One could think that such a kind of immersion for a generic system is that one which realizes the correspondence between the integrable system and the Lax matrix.
One could also think to generalize these results to non hyperelliptic curves.

Another problem consist in how to evaluate the weight of the choice of the Lax pair; in fact, in general, in changing the Lax pair also the algebraic curve, and hence its Jacobian, changes. The question is: does it exist a connection between these different representation? Can one conclude that all these Jacobians intersect themselves in a (Abelian?) subvariety on the real part of which the flow is linear?