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The Geometrical meaning of the Quantum Correction

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1 Introduction

In his works [1] Witten has given the general framework for calculating correlation functions in two dimensional topological sigma models (TSM). It turned out that on Kähler manifolds all the calculations can be performed from the axioms without needing to resort to any explicit form of the action. The observables of a topological sigma model are in 1-1 correspondence with the De Rham cohomology classes of the target space of the model, but this correspondence is only a morphism of vector spaces (actually of groups) and violates the ring structures.

More explicitly to each cohomology class A_i of the target space T it corresponds an observable $O_{A_i}(x)$ but for the purpose of calculating correlation functions as

$$< O_{A_1}(x_1), ..., O_{A_n}(x_n) > \qquad (1)$$

(the subscript indicates the genus of the world-sheet - in this thesis we restrict our attention to genus 0) we have to take account of all the "instanton corrections".

In other words the contribution to (1) due to topologically trivial instantons (i.e. to constant maps) is given by:

$$< O_{A_1}(x_1), ..., O_{A_n}(x_n) >= \sharp T_1 \bigcap ..... \bigcap T_n \qquad (2)$$

where T_i is the Poincare' dual of A_i and the intersection has to be taken in the target manifold T, but the contribution due to topologically non trivial instantons has the same form as (2) where the intersection has to be taken in a suitable moduli space.

The idea is to define the classes A_i on the moduli space of the maps of fixed degree from the world-sheet to the target by pull-back and then take the intersections there. When the dimension of the moduli space is the same as the degree of the form:

$$A_1 \wedge ..... \wedge A_n$$

[1] E. Witten, Commun. Math. Phys. 117 (88) 353, ibid 118 (88) 411, Nucl. Phys. B340 (90) 281

we have a contribution to the correlation function.

It is a general feature of these models [2] that in this way we obtain a "correction" of the ring structure of the cohomology of the target space. In particular the new ring has the structure of a "Frobenius algebra" [3] and so is endowed with a non degenerate metric.

What we are going to do now is to see an easy example where all the construction works very well.

2 Topological sigma model on P^1

Let us consider a topological sigma model on the Riemann sphere. The moduli space M_d of instantons of winding number d has (complex) dimension $2d+1$, as follows from the general form of a degree d map of the complex plane:

$$w(x) = \frac{a\prod(x-b_i)}{\prod(x-c_i)} \tag{3}$$

which is defined by the position of its $d+1$ poles and zeros and by a normalisation term.
To do intersection theory, we should really have a compact space, $\bar{M}_d$, for the instanton moduli space.
It is instructive to see how the compactification arises in the case of single coverings of the rational curve.

As they correspond to reparametrization of $\mathbf{P}^1$ they are represented by elements of SL(2) which naturally compactifies to $\mathbf{P}^3$.
In other words for single coverings (3) becomes

$$w(x) = \frac{ax+b}{cx+d} \tag{4}$$

where the complex parameters can be chosen to satisfy the equation:

$$ad - bc = 1$$

[2]R. Dijgkgraaf, E. Witten Nucl. Phys B342 (90) 486; D. Gepner Commun. Math. Phys. 141 (91) 381; K. Intriligator HUPT-91/A041

[3]B. A. Dubrovin Commun. Math. Phys. 145 (92) 195

The space of parameters compactifies to $\mathbf{P}^1$ and the "compactification divisor" is isomorphic to the variety $\mathbf{P}^1 \times \mathbf{P}^1$ defined by the quadric

$$ad - bc = 0$$

In general the compactification of M_d is $\mathbf{P}^{2d+1}$.

If ω is the unique non trivial homology class of $\mathbf{P}^1$ (the hyperplane class) then its pull-back under the "universal instanton"

$$i_d : \bar{M}_d - -- \rightarrow \mathbf{P}^1 \qquad i(w) = w(0)$$

(which obviously is only a rational map defined on M_d) gives a linear relation between the coefficients of $w(x)$ (see(3)).
In other words the pull-back of the hyperplane class of $\mathbf{P}^1$ is the hyperplane class of $\bar{M}_d = \mathbf{P}^{2d+1}$.
Finally the correction to the correlation function (1) due to instantons of degree d

$$< O_{A_1} .....O_{A_n} >_d \qquad (5)$$

is non zero if and only if $n = 2d + 1$ and in this case is equal to one.

Summarizing, what we have found is the following:

$$< O_{A_1} .....O_{A_n} >_d = \begin{cases} 1 & \text{if } n = 2d+1 \\ 0 & \text{otherwise} \end{cases} \qquad (6)$$

It is clear now that all the instanton corrections (6) can be encoded in the relation $\omega^2 = 1$ which can be viewed as a deformation (the "quantum correction") of the defining relation of the cohomology ring of the target.
But it is also clear which is the geometrical meaning of $\omega^2 = 1$. To understand this point let us return to the example of single coverings of $\mathbf{P}^1$. The definition of w (4) shows that the the instantons which belong to the compactification divisor are constant functions which map $\mathbf{P}^1$ into the point of the target defined by the ratio between the vectors (a, b) and (c, d). Moreover to give a point on $\mathbf{P}^1 \times \mathbf{P}^1$ we have to specify not only a constant map but also the point of the world-sheet where $w(x)$ is not defined. If we constrain this point to be the origin of $\mathbf{P}^1$, then we have to put $b = d = 0$ and the

constant instantons obtained in this way are in 1-1 correspondence with the points of the target. Finally this locus of constant maps into the moduli space $\bar{M}_1$ is exactly the cycle ω^2 where ω is the hyperplane class, so we can read the quantum correction as the relation which allows us to go from $\bar{M}_1$ to $\bar{M}_0$ when we calculate a correlation function:

$$< P(\omega)\omega^2 >_{M_1} = < P(\omega) >_{M_0}$$

More generally ω^2 is the locus of the moduli space $\bar{M}_{d-1}$ into $\bar{M}_d$.

In the rest of this thesis we describe how this construction generalizes to sigma models on Grassmannians.

3 Topological sigma models on Grassmann manifolds and the quot scheme

In the example of the $\mathbf{P}^1$-TSM we have seen that the quantum correction describe how instantons of degree d "live" in the moduli space of instantons of degree $d+1$. It turned out that the maps of lower degree (counted with multiplicity) generate all the compactification locus of the moduli space and the quantum correction isolate into this a cycle in such a way that the multiplicity of of the maps of lower degree is eliminated.

Now it is clear that to generalize this framework to a Grassmann sigma model we have first to compactify the moduli space of rational maps into the Grassmannian.

A nice compactification of this space is described in [4].

In what follows we recall some definitions and properties of the Stromme construction.

The datum of a morphism

$$\pi : \mathbf{P}^1 \longrightarrow G(s, V)$$

[4]S. Stromme in: Space curves, F. Ghione et al. (Eds.), Springer LNM (87)

is equivalent to the datum of a locally free quotient of $\pi^* V$ of rank r and degree d, which is defined by pulling back the tautological sequence on $G(s,V)$:

$$0 \longrightarrow S \longrightarrow V \longrightarrow Q \longrightarrow 0$$

This means that compactify the variety $Hom_d(\mathbf{P}^1, G)$ is equivalent to compactify the space of all free quotients of the trivial bundle V on $\mathbf{P}^1$ of rank r and degree d.
There is a general construction due to Grothendieck [5] which provides such a compactifications.

It is a general fact that a family of locally free quotients of a trivial bundle can occasionally degenerate into quotients which are not longer locally free. The idea is to add such degenerate quotients if they have the same "asymptotic behaviour" under twists of the general member of the family to which they belong.
The right notion which encode this behaviour is the Hilbert polynomial H(m):

$$H_Q(m) = \chi(\mathbf{P}^1, Q(m)) = (m+1)r + 1$$

where Q is a quotient on $\mathbf{P}^1$, $Q(m) = Q \otimes O(m)$ and r and d are by definition the rank and the degree of Q.

The compactification of $Hom_d(\mathbf{P}^1, G)$ is obtained by adding all the degenerate quotients which belong to families whose general member is a locally free quotient of rank r and degree d and where all the members of the family have the same rank and degree (in a word: all the degenerate quotients of flat families [6]).

It is a general fact that in such a way we find a compactification R_d of $Hom_d(\mathbf{P}^1, G)$ and that it is equipped with a "universal short exact sequence" on $\mathbf{P}^1 \times R_d$:

$$0 \longrightarrow A \xrightarrow{j} V \longrightarrow B \longrightarrow 0 \qquad (7)$$

which has the following property: for any variety T, the set of morphisms s:

$$s : T \longrightarrow R_d$$

[5] A. Grothendieck : Sem. Bourbaki 221 (60-61)
[6] R. Hartshorne: Algebraic Geometry GTM

is in 1-1 correspondence to the set of short exact sequences on $\mathbf{P}^1 \times T$ of the form of (7) where the Hilbert polynomial on the fibres of the projection

$$\pi_T : \mathbf{P}^1 \times T \longrightarrow T$$

is fixed and equals those of the fibres of

$$\pi_{R_d} : \mathbf{P}^1 \times R_d \longrightarrow R_d$$

This framework provides us with a projective variety R_d of dimension $nd + r(n-r)$ which is irreducible, rational and non singular [7].
Moreover when we fix a point on P we get a rational map

$$i_d : R_d - - \rightarrow G(s, V)$$

which define, by closure, the pull-back of the cycles of $G(s, V)$.

In what follows we will see how a suitable intersection of such cycles select into R_d a locus birational to R_{d-1} as predicted by the quantum correction of the Grassmanniann cohomology ring which we are going to describe in the next section.

4 The quantum correction of the grassmannian ring

We rapidly recall the structure of the cohomology ring of a Grassmann manifold.
On $G(s, n)$ is defined the tautological sequence:

$$0 \longrightarrow S \longrightarrow V \longrightarrow Q \longrightarrow 0$$

which gives the the obvious relation between the total Chern classes of S and Q:

$$sc = 1 \tag{8}$$

[7]Stromme cit.

where

$$s = 1 + s_1 + ... + s_s \quad and \quad c = 1 + c_1 + ... + c_r$$

and s_i are called the Segre classes and c_i the Chern classes of G(s,n).
The cohomology ring $H^*(G)$ of $G(s,n)$ is generated by the Chern classes and so is isomorphic to a quotient of the polynomial ring which they generate modulo a suitable ideal.
It is easy to verify the following relation:

$$H^*(G) = \frac{\mathbf{C}[c_i]}{\{s_{s+1}, ..., s_n\}} \qquad (9)$$

where the s_i are defined formally by:

$$\sum s_i t^i = \frac{1}{\sum c_i t^i} \qquad (10)$$

In particular it is clear from (8) that

$$c_r s_s = 0$$

The quantum correction of the Grassmannian ring says that the last equation must be corrected as follows [8]:

$$c_r s_s = 1$$

In the paper [9] we show that the actual reason of this correction of the cohomology ring of the Grassmannian has the same origin as in the $\mathbf{P}^1$-TSM.
What we have proved is the following:

Proposition: if we define the cycles t_i^* and m_j^*

$$t_i^* = p_*(\Gamma \bigcap i_d^{-1}(s_i^*))^-$$

$$m_i^* = p_*(\Gamma \bigcap i_d^{-1}(c_i^*))^-$$

where the closure has to be taken in $R_d \times G$, Γ is the graph of i_d and p is the projection on the first factor, then

[8]Intriligator cit.
[9]D. Franco, C. Reina in preparation

the locus $t_s^* \bigcap m_r^*$ in R_d, similarly to what happens in the $\mathbf{P}^1$ model, is birationally equivalent to R_{d-1}.
In other words the intersection $t_s^* \bigcap m_r^*$ isolate into the compactification locus of R_d an unique degenerate quotient of degree d for any instanton of degree $d-1$.
Here we give a sketch of the proof referring to [10] for more details.

The locus where the map i_d is not defined can be seen as a determinantal variety [11]. Indeed when we restrict the defining sequence (7) to $\{0\} \times R_d$ we get a quotient sheaf B on R_d which fails to be a bundle exactly where the injection j has rank strictly lower then s (as a bundle map).
It is clear that to those points does not correspond any well defined point in $G(s,n)$.
It is also clear that to a point p of R_d where j has exactly rank $s-1$ we can associate a couple (H_p, f_p) where f_p is a map in $Hom_{d-1}(\mathbf{P}^1, G)$ and H_p is an hyperplane into $f_p(0)$ (H_p is the image in V of the bundle A on the point $(0,p) \in \mathbf{P}^1 \times R_d$).
It can be easily seen [12] that for any point p as before we have:

$$\Gamma \bigcap (\{p\} \times G(s,n)) = \mathbf{P}^{\mathbf{r}} = \{\Lambda \mid H_p \subset \Lambda\} \qquad (11)$$

so that $\mathbf{P}^{\mathbf{r}}$ can be viewed as the "image" under i_d of the point p into the Grassmannian.

Now the Poincare' dual of the classes c_r and s_s into a Grassmann manifold have the following descriptions [13];

$$c_r^* = \{\Lambda \mid v \in \Lambda\} \qquad (12)$$

$$s_s^* = \{\Lambda \mid \Lambda \subset H\}$$

(where H is an hyperplane and v a point of V).
In particular it is clear from (11) that $c_r^* \bigcap s_s^* = 0$ and so $t_s^* \bigcap m_r^*$ is contained into the determinantal locus above (naturally we are taking the set theoretical

[10] D. Franco, C. Reina in preparation
[11] E. Arbarello, M. Cornalba, P. Griffiths, J. Harris: The geometry of algebraic curves Springer Verlag; W. Fulton Intersection Theory Springer Verlag
[12] D. Franco, C. Reina cit
[13] P. Griffiths, J. Harris Principles of algebraic geometry J. Wiley and Sons

intersection which however correspond to the Chow one because, as it will become clear in the following, the cycles intersect transversally). Moreover a couple (H_p, f_p) belongs to $t^*_s \bigcap m^*_r$ if and only if $H_p \subset H$ in fact it is clear that in this case the $\mathbf{P^r}$ image of (H_p, f_p) into $G(s,n)$ intersects both c^*_r and s^*_s.

But now we have found the embedding of R_{d-1} into R_d indeed to a map $f \in R_{d-1}$ we can associate the degenerate point of R_d corresponding to the couple $(f(0) \bigcap H, f)$. This construction fails when $f(0) \subset H$ namely in a closed subvariety of R_{d-1} so what we have found is only a rational map from R_{d-1} into R_d but this does not influence the intersections in maximal codimension (see [14]) □ .

5 The Frobenius algebra of the Grassmann model

In their works [15] Intriligator and Vafa conjectured that the quantum correction of the Grassmannian cohomology ring that arises in topological sigma models on Grassmannians can be described in terms of a suitable topological Landau-Ginzburg model [16] .

The starting point is the simple observation that the defining ideal of the Grassmannian cohomology ring (9) can be "integrate" in a generating function.

Following [17], we define:

$$W(t) = -\log c(-t) = \sum W_i t^i \quad where \quad c(t) = \sum c_i t^i$$

Since our ideal is generated by the derivatives of W.

If we write

$$c(t) = \prod_1^r (1 + q_i)$$

[14]D. Franco, C. Reina cit

[15]Intriligator cit. ; C. Vafa Mod. Phys Lett. A6 (91) 337

[16]Vafa cit.

[17]Gepner cit.

it can be seen that:

$$W_{n+1} = \sum_1^r \frac{q_i^{n+1}}{n+1}$$

(the q's can be seen as the Chern classes of the line bundles associates via splitting principle to the quotient bundle of the Grassmannian [18]).

The content of the conjecture of Vafa can be summarized in the requirement that the quantum correction of the cohomology ring of $G(s,n)$ is the ring associated to the following potential:

$$Z = W_{n+1} + (-1)^r c_1 \qquad (13)$$

Moreover as this potential can be seen as the defining one for a topological Landau-Ginzburg model, it turns out that all the correlation function (1) can be calculated with the methods of the paper [19] and that from the topological observables we can construct a Frobenius algebra [20].

What we are going to do now is to describe the topological Landau-Ginzburg ring as it arises from the detailed analysis of Intriligator [21] and then see how that ring can be recovered from the geometrical point of view of the last section.
Firstly it is necessary a more precise description of the cohomology ring of the Grassmannian. The elements of this ring are in 1-1 correspondence with Young tableaux with at most s columns of at most r boxes. If μ is any such a tableau, $n(\mu)$ the number of its columns and a_i the length of the i-th one, we can define

$$\Phi_\mu = [a_1, ..., a_{n(\mu)}] = \det_{1\leq i,j\leq n(\mu)} c_{a_i+i-j} \qquad (14)$$

It turns out [22] that these classes generate $H^*(G)$ as a vector space.
The ring structure of $H^*(G)$ is encoded in the Pieri equation :

$$c_i[a_1, ..., a_s] = \sum_{a_i\leq b_i\leq a_{i+1};\sum b_i=\sum a_i+i} [b_1, ..., b_s] \qquad (15)$$

[18] Griffiths, Harris cit.
[19] Vafa cit.
[20] Dubrovin cit.
[21] cit
[22] Griffiths, Harris cit.

As it arises from the analysis of [23] the quantum deformation of the Grassmannian cohomology ring due to the extraterm c_l in the perturbed potential (13) is encoded in a Pieri-like formula very similar to the last one.
The major difference between the two rings lies in the different action of the class c_r (s_s) on the elements of the form (14). Indeed when $n(\mu) < s$ ($n(\mu^*) < r$, where μ^* is the dual tableau of μ) then the action of is the same both in the classical and quantum case:

$$c_r[a_1, \ldots.., a_j] = [a_1, \ldots..a_j, r] \qquad (16-a)$$

$$s_s[a_1, \ldots.., a_j] = [a_1 + 1, \ldots..a_j + 1] \qquad (16-b)$$

but when $n(\mu) = s$ ($n(\mu^*) = r$) the action of c_r (s_s) is equivalent to "deleting" the first row (the last column) of the tableau corresponding to Φ_μ:

$$c_r[a_1, \ldots.., a_s] = [a_1 - 1, \ldots..a_s - 1] \qquad (17-a)$$

$$s_s[a_1, \ldots.., a_j] = [a_1, \ldots..a_{j-1}] \qquad (17-b)$$

Following [24] the cohomology elements of the deformed ring can grouped into orbits under the action of s_s. In each orbit we can pick a convenient representative, the other elements of the orbit being of the form

$$s_s^q \Phi_{[\mu]}$$

The more convenient choice for the orbit representative $[\mu]$ is the element without length s rows in the corresponding tableau.
Now the product of two representatives is described in [25] :

$$\Phi_{\mu_1}\Phi_{\mu_2} = \sum_{\mu_3} N^{\mu_3}_{\mu_1,\mu_2} s_s^{\frac{r(\mu_1)+r(\mu_2)-r(\mu_3)}{s}} \Phi_{\mu_3} \qquad (18)$$

where $r(\mu)$ is is the number of boxes of μ and $N^{\mu_3}_{\mu_1,\mu_2}$ are determinate by the following deformed Pieri formula:

$$c_i[a_1, \ldots.., a_s] = \sum_{a_i \leq b_i \leq a_{i+1};\sum b_i = \sum a_i + i} [b_1, \ldots.., b_s]$$

[23]Intriligator cit.
[24]Intriligator cit.
[25]Gepner cit.

where now $b_i = r$ is the same as $b_i = 0$ [26].
Summarizing the fundamental difference between the two rings lives in the different action of the class c_r on the elements of the form (14).

Now we give a short geometrical explanation of the formulae (16-18) following the lines of the proof of the proposition of the last section.

Firstly we recall the definition of the Poincare' dual of the class Φ_μ :

$$\Phi^*_\mu = [a_1, ..., a_s]^* = \{\Lambda \in G(s,n) \mid dim\Lambda \bigcap V_{r+i-a_{r-i+1}} \geq i\} \qquad (19)$$

$$where \qquad V_1 \subset V_2 \subset ..... \subset V_n = \mathbf{C}^n \qquad (20)$$

is a complete flag of $\mathbf{C}^n$.
When $n(\mu) = s$ then $a_1 \neq 0$ and

$$dim\Lambda \bigcap V_{r-a_1} \geq s \Longrightarrow \Lambda \subset V_{n-1}$$

so that $c^*_r \bigcap \Phi^*_\mu = 0$.
Defining

$$\phi^*_\mu = p_*(\Gamma \bigcap i_d^{-1}(\Phi^*_\mu))$$

it is clear that

$$m^*_r \bigcap \phi^*_\mu \subset R_{d-1} \subset R_d$$

where the second inclusion has the meaning said in the proof of the proposition of the last section section.

Now it is easy to see that the intersection $m^*_r \cap \phi^*_\mu$ into R_{d-1} is exactly the cycle $\phi^*_{\mu_1}$ where μ_1 is the tableau obtained by μ deleting the first row. Indeed if the flag (20) is given by a basis $\{e_1, ..., e_n\}$ where e_n is the vector v in the definition of m^*_r (see the proposition), then a couple $(H_p, f_p) \in R_{d-1}$ is in $m^*_r \cap \phi^*_\mu$ if and only if the projection from e_n on $V_{n-1} = H$ of $f_p(0)$ belongs to Φ^*_μ:

$$\pi_{e_n}(f_p(0)) \subset \Phi^*_\mu$$

but the last inclusion is precisely equivalent to to the following conditions;

$$dim f_p(0) \bigcap \bar{V}_{r+1+i-a_{r-i+1}} \geq i$$

[26]Gepner cit.

$$(\bar{V}_j = span\{e_n, e_1, ..., e_{j-1}\})$$

which says that $f_p(0)$ belongs to $\phi^*_{\mu_1}$.

The proof of (17-b) is similar; if $n(\mu^*) = r$ then $a_{n(\mu)} = r$ and (19) says that $\Lambda \supset e_1$.
Again we have:

$$s^*_s \bigcap \Phi^*_\mu = 0 \quad and \quad t^*_s \bigcap \phi^*_\mu \subset R_{d-1} \subset Rd$$

Moreover if we define

$$V'_i = H \bigcap V_{i+1}$$

then it is clear that

$$dim V_i \bigcap span\{H_p, e_1\} \geq i \Longleftrightarrow dim V'_{i-1} \bigcap H_p \geq i - 1$$

$$\Longleftrightarrow dim V'_{i-1} \bigcap f(0) \geq i - 1$$

but this is equivalent to the requirement that $f(0)$ belongs to $\Phi^*_{\mu_2}$ where μ_2 is obtained from μ deleting the last column.

More subtle is the reason of the equivalence (16) between the classical and the quantum case.
Let us see (16-a): a simple way to prove it would be to see that $m^*_r \cap \phi^*_\mu$ has not components into the compactification locus. Unfortunately this is not true in general. The reason why we can neglect such components lies im the fact that they do not contribute to the correlation functions. Indeed, as it is clearly seen from the definition (19), when a couple (H_p, f_p) belongs to $m^*_r \cap \phi^*_\mu$ and $n(\mu) < s$, differently to what happens if $n(\mu) = s$, then H_p is not determined. In other words there is a subspace $\mathbf{P}^i$ into the $\mathbf{P}^r$ defined in (11) such that:

$$K \subset \mathbf{P}^i \Longrightarrow (K, f_p) \in m^*_r \bigcap \phi^*_\mu$$

This shows that these components cannot contribute to the calculus of correlation functions because they do not give dimension 0 cycles when we make intersections with other ϕ^*_μ.

In a similar way it can be shown that:

$$s_s[a_1, ..., a_j] = [a_1 + 1, ...a_j + 1]$$

if $n(\mu^*) < r$

$$c_i[a_1, ..., a_s] = \sum_{a_i \leq b_i \leq a_{i+1}; \sum b_i = \sum a_i + i} [b_1, ..., b_s]$$

if $n([a_1, ..., a_j]) < s$.

6 Conclusions

A general feature of the models considered until now is that there is a "stratification" of the moduli spaces:

$$M_1 \subset M_2 \subset ..... \subset M_d \subset M_{d+1} \subset$$

where M_0 is the target and where every M_i is equipped with a rational map to M_0 which can be used to pull-back cohomology classes of the Chow ring. In this way we have a representative $\Phi_{i,d}$ in M_d for every class $\Phi_{0,d}$ in M_0 and we can take intersections of such classes in every moduli space.
Then projecting to the maximal codimension component of the Chow ring, we get "expectation values" on every M_d:

$$< O_{\Phi_1}, ..., O_{\Phi_n} >_d = \sharp\{\Phi_1 \bigcap ..... \bigcap \Phi_n\}$$

if $codim\Phi_1 + ... + codim\Phi_n = dim(M_d)$

$$< O_{\Phi_1}, ..., O_{\Phi_n} >_d = 0$$

otherwise. Moreover we have a total expectation value for every "word" composed with the classes Φ 's:

$$< O_{\Phi_1}, ..., O_{\Phi_n} > = \sum_0^\infty < O_{\Phi_1}, ..., O_{\Phi_n} >_d$$

Then we can define the quantum correction of the Chow ring of the target as the ring generated by the Φ 's modulo the following equivalence relation:

$$P(\Phi_i) \sim Q(\Phi_i) \Longleftrightarrow < P(\Phi_i)H(\Phi_i) > = < Q(\Phi_i)H(\Phi_i) > \bigtriangledown H$$

where P,Q,H are polynomials.

In the examples considered here the cohomology rings have a nice expression as a free ring on the generators modulo defining ideals and the quantum correction may be seen as a deformation of such ideals. Moreover we have an integrate version of both the classical and deformed ideals.

What we hope to do in the future is to generalize what we have done until now for Grassmann manifolds to more general sigma models. For example we think it should be relatively easy to take a flag manifold as target.

The first step of the job, i.e. compactfying moduli spaces, it is easily made. Indeed as for Grassmann targets, a map from $\mathbf{P}^1$ into a flag manifold $F(N_1, ..., N_n; N)$ is a set of locally free quotients (here we have to specify n non decreasing degrees):

$$0 \longrightarrow A_i \longrightarrow \mathbf{C^N} \longrightarrow Q_i \longrightarrow o$$

suitably filtered:

$$0 \longrightarrow A_i \longrightarrow A_{i+1}$$

In other words the datum of a map from $\mathbf{P^1}$ into $F(N_1, .., N_n; N)$ (of multi-degree $(d_1, .., d_n)$) is equivalent to the datum of a diagram as the following on $\mathbf{P^1}$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_{N_n} & \longrightarrow & \mathbf{C}^N & \longrightarrow & Q_{N_n} & \longrightarrow & 0 \\
 & & \uparrow & & \| & & \uparrow & & \\
0 & \longrightarrow & A_{N_{n-1}} & \longrightarrow & \mathbf{C}^N & \longrightarrow & Q_{N_{n-1}} & \longrightarrow & 0 \\
 & & \uparrow & & \| & & \uparrow & & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow & & \| & & \uparrow & & \\
0 & \longrightarrow & A_{N_1} & \longrightarrow & \mathbf{C}^N & \longrightarrow & Q_{N_1} & \longrightarrow & 0
\end{array}$$

(where Q_i has degree d_i on $\mathbf{P^1}$).

Like for Grassmannians, where the compactification of the moduli space can be seen as the zero locus of the morphism $p_d \circ j \circ i_{d-1}$ in the following sequence:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & S_{d-1,G} & \xrightarrow{i_{d-1}} & \mathbf{C}_G^{dn} & \longrightarrow & Q_{d-1,G} & \longrightarrow & 0 \\
 & & & & \downarrow j & & & & \\
0 & \longrightarrow & S_{d,G} \times \mathbf{C}^2 & \longrightarrow & \mathbf{C}_G^{(d+1)n} \times \mathbf{C}^2 & \xrightarrow{p_d} & Q_{d,G} \times \mathbf{C}^2 & \longrightarrow & 0
\end{array}$$

where $G = G_{d-1} \times G_d$ and $G_i = G((i+1)r + d, (i+1)n)$, both the first and the second rows are tautological and j is a canonical map, the compactified family of instantons of multidegree $(d_1, .., d_n)$ is the zero locus of the same morphism in the sequence on $F = F_{d_n - 1} \times F_{d_n}$, where $F_i = F((i+1)N_1 + d_1, ..., (i+1)N_n + d_n, (i+1)N)$:

$$\begin{array}{ccccc} 0 \longrightarrow S_1 \times ... \times S_n & \stackrel{i_{d-1}}{\longrightarrow} & \mathbf{C}^{ndN} & \longrightarrow & Q_1 \times ... \times Q_n \longrightarrow 0 \\ & & \downarrow j & & \\ 0 \longrightarrow S'_1 \times ... \times S'_n \times \mathbf{C}^2 & \longrightarrow & \mathbf{C}^{2+n(d+1)N} & \stackrel{p_d}{\longrightarrow} & Q_1 \times ... \times Q_n \times \mathbf{C}^2 \longrightarrow 0 \end{array}$$

From this description of the compactification we can guess that for a flag sigma model there should be several quantum corrections due to the fact that a family of instantons of fixed multidegree can occasionally degenerate into instantons of different multidegrees.
But if on one hand it should be relatively easy to isolate the degenerate loci in a cohomological way as we made for the Grassmannians, on the other hand it seems very difficult write explicitly the deformed ideal which encode the quantum correction because yet in the classical case the ideal defining the cohomology ring of a flag manifold is much more complicate and there is not (or at least we do not know if there is) an integrated version of it as for the Grassmannian.

Also more complicate is the topological sigma model on targets as G/P, where G is an algebraic group and P is a parabolic subgroup of G. Here we have the same difficulties of the last example plus an embarrassing indetermination in the choice of the compactification scheme.
Indeed a possible choice is relate to the fact that G/P is an algebraic variety (a projective one) so we can canonically ask for a compactification which "stabilizes" the Hilbert polynomial of a rational curve into it. But this polynomial "feels" only the total degree of the curve. So a more precise compactification should arises if we were able to embed G/P in a suitable product of projective spaces (as for the flag manifolds) but we do not know how to do it.
Finally a different approach to the compactification arises when we consider a rational curve into G/P as a bundle on $\mathbf{P}^1$ with "extra structures" and so we may ask for a compactification which preserve such a structure.