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Some topics in a model of interface dynamics

Thesis submitted for the degree of

"Magister Philosophiæ"

CANDIDATE

Paolo Buttà

SUPERVISOR

Prof. Gianfausto Dell'Antonio

October 1993

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Introduction

In systems which exhibit phase transitions, distinct thermodynamic phases are spatially separated by interfaces, i.e. sharp transition regions where the order parameter changes very rapidly from one phase to the other. It is obviously of great interest to understand the dynamical behavior of these interfaces. In particular, in nonequilibrium statistical mechanics, it is a very fascinating problem the derivation of the interface dynamics from the "microscopic evolutions" of the system.

Actually this aim is very close to the derivation of hydrodynamics. A fluid is locally in equilibrium and the hydrodynamic parameters vary on a macroscopic scale. Similarly, an interface is extremely flat on the microscopic scale, when averaged with the local equilibrium distribution of the order parameter. In both cases the derivation of the dynamics is based on the separation of space-time scales.

The phenomenological theory of interface motion has been established since a long time (see for example [28] and references there). On the contrary, noticeable results on the derivation from microscopic models are very recent. Moreover, a coherent picture, in a general enough framework and with the standard of rigour of the equilibrium statistical mechanics, is still missing.

We restrict now our attention to the simplest case of interface dynamics. We consider systems whose equilibria are characterized only by the temperature and the order parameter, and with a bulk dynamics that does not conserve the latter. According to the phenomenological theory the picture is as follows (see [25,27,28,29]). First of all, on the macroscopic scale, the transition region is infinitely thin so that one can represent the interface by a smooth surface Σ embedded in $\mathbb{R}^d$, where d is the physical dimension. One introduces the surface tension σ, so that the surface free energy is given by

$$F = \int_{\Sigma} df \, \sigma(\hat{n})$$

where $\hat{n}$ is the local normal to Σ in df. It is postulated that the interface velocity along the local normal is given by

$$v = -\mu \frac{\delta F}{\delta \Sigma} \tag{0.1}$$

where μ is called the mobility of the interface.

Now we further assume that the system is isotropic. In this case, both the surface tension σ and the mobility μ are independent on the local orientation $\hat{n}$ of the interface, so that eq. (0.1) becomes

$$v = \frac{\theta}{R}\hat{n} \tag{0.2}$$

where $1/R$ is equal to $(d-1)$ times the local mean curvature of Σ. According to (0.1), the "transport coefficient" θ should be related to the linear transport coefficient μ and to the thermodynamic quantity σ by the "Einstein relation"

$$\theta = \mu\sigma \tag{0.3}$$

The sign of the mean curvature in (0.2) is chosen such that the velocity v is directed toward the local concavity of the interface. We refer to (0.2) as the *mean curvature equation.*

By the general theory of parabolic equations it is known that eq. (0.2) develops singularities in a finite time. Obviously, for a complete description of the interface dynamics, one has to describe the behavior of the underlying physical system also past the appearance of the singularities. For example, a convex cluster that shrinks to a single point, actually disappears after the shrinking time.

A prototypical microscopic model, corresponding to the previous phenomenological theory, is the ferromagnetic Ising spin system at phase coexistence with a stochastic spin-flip dynamics. This model exhibits phase transition at sufficiently low temperature and with external field $h = 0$. Its equilibria are then described by the temperature and by the order parameter, which is, in this case, the magnetization. Moreover, the Glauber dynamics does not conserve the magnetization so that, by the isotropy of the model, the interfaces should move by the mean curvature flow. We finally recall that the dynamics obeys to the condition of "detailed balance", so that it is reversible with respect to the (equilibrium) Gibbs measure.

One still lacks a derivation of the interface motion by mean curvature from spin systems with spin-flip dynamics and finite range interaction (with the exception of the results in [28] obtained for particular spin models).

Here we consider an Ising spin system with Kac potentials and in the limit of Lebowitz and Penrose. This means that the spin-flip dynamics is governed by a potential J_γ, where γ is a small parameter. The dependence of J_γ on γ is such that its range diverges as γ^{-1}, while the total interaction energy of any spin with all the others remains finite. All the scaling limits of the model are obtained by scaling space and time with functions of γ, in the limit $\gamma \to 0$.

In the context of equilibrium statistical mechanics, the idea of scaling the interaction was introduced by M. Kac, [21], and it was generalized and made precise by J. Lebowitz and O. Penrose (see [15,23,24]). By scaling space with the same parameter as the interaction, they proved that the limiting theory explains the nature of the Van der Waals theory of phase transitions and the origin of the Maxwell rule. Despite of its success, the theory is "dangerously" close to the mean field theory: phase transitions occur independently of the dimension.

Recently, non equilibrium properties for systems with Kac potentials have been studied. In particular, the systematic analysis of Ising spin systems with Glauber dynamics and Kac potentials has been developed in a series of papers ([9,10,11,12,13,14]).

First of all, in [9], it is analyzed the so called *mesoscopic limit*, when the space is scaled with the same parameter γ and the time is not scaled. It is shown that the limiting magnetization density $m(r,t)$ solves the deterministic equation

$$\frac{\partial m}{\partial t} = -m + \tanh\{\beta(J \star m + h)\} \tag{0.4}$$

where "$\star$" denotes convolution, h is the external magnetic field and β the inverse temperature. More precisely, it is proved that the block spin variable of microscopic size $\gamma^{-\alpha}$, with $\alpha < 1$ and centered in a "mesoscopic" point $r \in \mathbb{R}^d$, converges in probability to $m(r,t)$ when $\gamma \to 0$. This deterministic behavior is due to a "mean field effect". Each spin undergoes in a time unit a finite number of random flips. On the other hand, in the limit $\gamma \to 0$, the infinitely many spins in the block variable feel essentially the same potential. Then, due to a law of large numbers that dampens the fluctuations, the block spin variables evolve deterministically.

In order to see the full effect of the stochastic interaction, one has to perform the "macroscopic limits", when also the time is scaled with γ. Then, each spin variable undergoes, in a time unit, many flips (infinitely many when $\gamma \to 0$), so that it reaches a local equilibrium distribution.

There are interesting properties of the system, occurring only on the macroscopic scales, that can be predicted by the long time behavior of the "mesoscopic equation" (0.4). This is the case for the interface dynamics. By scaling diffusively the solution of eq. (0.4) with $h = 0$ and below the critical temperature, one obtains the convergence to a motion by mean curvature, up to the times when the motion is regular (see [12]). It has been also proved that in the bidimensional case the convergence holds at all times (see [5]). On the contrary, there are effects due to the full stochastic interaction that cannot be predicted by eq. (0.4). This is the case of the escape from the nonequilibrium and the successive separation of phases, which occur after quenching a state from high temperature down below the critical value (see [11]). This is also the case when the fluctuations become important, for example in critical phenomena in relation to stochastic quantization (see [2] and [10]) or, below the critical temperature, when the curvature of the clusters becomes too small.

In this thesis we analyze some questions on the derivation of the interface dynamics from the long time behavior of the mesoscopic equation. More precisely, we complete the derivation of the phenomenological theory by proving the validity of the relation (0.3) for this model. Then we prove the convergence of the solution of (0.4) to the motion by mean curvature at all times in the 2-dimensional case. Since in this case the only singularity is the shrinking to a point of a closed curve, we verify that the curve actually disappears past the singularity.

The thesis is organized as follows. In the next section we introduce the model and some preliminary results on the mesoscopic limit (we refer to [9] for details). In the second section we briefly discuss the derivation of the convergence to the motion by mean curvature and the validity of (0.3) (we refer to [12] and [4]). Finally, in the last section, we prove the convergence at all times in the 2D-case (we refer to [5]).

1. Main definitions and the mesoscopic limit

We have divided this section in two subsection. In the first one, we introduce the microscopic model and give its main properties. In the second one, we define the mesoscopic limit and give the main theorem. For details we refer to [9].

1.1 The microscopic model.

As just said, we consider an Ising spin system. Let $\mathbb{Z}^d$ be the unit square lattice of dimension d. A *spin configuration* σ is an element of $\{-1,1\}^{\mathbb{Z}^d}$. Given a subset Δ in $\mathbb{Z}^d$, we denote by σ_Δ the restriction of the configuration σ to Δ, i.e. an element of $\{-1,1\}^\Delta$. Finally, we denote by $\sigma(x)$ the value of σ in $x \in \mathbb{Z}^d$.

A *Kac potential* for the spin system is a function $J_\gamma(x,y)$, $0 < \gamma \le 1$, x and y in $\mathbb{Z}^d$, of the form

$$J_\gamma(x,y) = \gamma^d J(\gamma|x-y|) \tag{1.1}$$

We assume $J(r) = 0$ for all $r > 1$ and $J(r)$ in C^2 when r is in $(0,1)$.

Given a magnetic field $h \in \mathbb{R}$ and a spin configuration σ, we define the energy of σ in Δ as

$$H_\gamma(\sigma_\Delta) = -h\sum_{x\in\Delta}\sigma(x) - \frac{1}{2}\sum_{x\neq y\in\Delta} J_\gamma(x,y)\sigma(x)\sigma(y) \tag{1.2}$$

The energy of σ in Δ plus the interaction energy with the spins in the complement, Δ^c, of Δ, is

$$H_\gamma(\sigma_\Delta|\sigma_{\Delta^c}) = H_\gamma(\sigma_\Delta) - \sum_{x\in\Delta, y\notin\Delta} J_\gamma(x,y)\sigma(x)\sigma(y) \tag{1.3}$$

In the general definition of a Kac potential, the only requirement on J is that it is in $L^1(dr, \mathbb{R}^d)$.

We define now the *Glauber dynamics*. Let $\beta > 0$ be the "inverse temperature", then for any $\gamma > 0$ the Glauber dynamics is the unique Markov process with state space $\{-1,1\}^{\mathbb{Z}^d}$ and generator L_γ, where L_γ is the unique extension of the operator which acts on the cylinder functions f as

$$L_\gamma f(\sigma) = \sum_{x\in\mathbb{Z}^d} c_\gamma(x,\sigma)[f(\sigma^x) - f(\sigma)] \tag{1.4}$$

In (1.4) we denote by σ^x the spin configuration obtained from σ by flipping the spin at x, i.e.

$$\sigma'(y) = \begin{cases} \sigma(y) & \text{if } y \neq x \\ -\sigma(x) & \text{if } y = x \end{cases} \tag{1.5}$$

while the "flip rate" $c_\gamma(x,\sigma)$ of the spin at x in the configuration σ is given by

$$c_\gamma(x,\sigma) = \frac{e^{-\beta h_\gamma(x)\sigma(x)}}{e^{+\beta h_\gamma(x)} + e^{-\beta h_\gamma(x)}} \tag{1.6a}$$

$$h_\gamma(x) = h + (J_\gamma \circ \sigma)(x), \qquad (J_\gamma \circ \sigma)(x) = \sum_{y \neq x} J_\gamma(x,y)\sigma(y) \tag{1.6b}$$

The existence and the uniqueness of the above Markov process is proved in [26]. The canonical space of realizations of this process is the Skorohod space of cadlag trajectories $D(\mathbb{R}_+, \{-1,1\}^{\mathbb{Z}^d})$. We denote by σ_t the spin configuration at time t of the process whose value in x, $\sigma(x,t) = \sigma_t(x)$, is then a random variable.

By our choice (1.6), the flip rate $c_\gamma(x,\sigma)$ verify the so called "detailed balance" condition

$$\frac{c_\gamma(x,\sigma^x)}{c_\gamma(x,\sigma)} = e^{-\beta[H_\gamma((\sigma^x)_\Lambda) - H_\gamma(\sigma_\Lambda)]} \tag{1.7}$$

where Λ is any set that contains x and such that the spin at x does not interact with those in Λ^c. This condition implies that the Glauber evolution is reversible with respect to any *Gibbs measure* $\mu_{\beta,h,\gamma}$, with Kac potential J_γ, magnetic field h and inverse temperature β. Then L_γ is a self-adjoint operator in the probability space $(\mu_{\beta,h,\gamma}, \{-1,1\}^{\mathbb{Z}^d})$ and, consequently, $\mu_{\beta,h,\gamma}$ is an invariant probability with respect to the Glauber evolution. We recall that a Gibbs measure $\mu_{\beta,h,\gamma}$ is any measure on $\{-1,1\}^{\mathbb{Z}^d}$ which satisfies the DLR equations: namely, for all $x \in \mathbb{Z}^d$ and spin configuration σ, one has

$$\mu_{\beta,h,\gamma}\Big(\sigma(x)\big|\{\sigma(y), y \neq x\}\Big) = \frac{e^{\beta h_\gamma(x)\sigma(x)}}{e^{+\beta h_\gamma(x)} + e^{-\beta h_\gamma(x)}} \qquad \mu_{\beta,h,\gamma} - \text{almost surely} \tag{1.8}$$

where $\mu_{\beta,h,\gamma}(\cdot|\{\sigma(y), y \neq x\})$ is the conditional probability on the σ-algebra generated by the stochastic variables $\{\sigma(y) \,:\, y \neq x\}$

We point out that (1.6a) is not the unique possible choice for the flip rate such that (1.7) is satisfied. As a consequence, there are different evolutions that can be equally used to describe the approach to the equilibrium of the system. The choice (1.6a) leads to a simpler mesoscopic equation.

1.2 Mesoscopic limit and propagation of chaos.

In this subsection we briefly describe the behavior of the Ising spin system in the mesoscopic limit. The microscopic points $x \in \mathbb{Z}^d$ are represented in the mesoscopic space $\mathbb{R}^d$ by the lattice $\gamma\mathbb{Z}^d$ while the time is unchanged (that is $(x,t) \to (r,t) = (\gamma x, t)$). As mentioned in the introduction, for small γ's, the Glauber dynamics in the mesoscopic representation is almost deterministic, with the evolution described by eq. (0.4). The convergence of the Glauber dynamics to a deterministic evolution, as $\gamma \to 0$, holds in a very strong sense. Actually, one can prove the weak convergence of the process, the convergence of all the correlation functions and, finally, the convergence of the block spin variables. In relation to this last result, we recall that the block spin variable is constructed by taking the average of the spins on a square of size $\gamma^{-\alpha}$ with $0 < \alpha < 1$ and centered in a mesoscopic point r. Then, in the limit $\gamma \to 0$, the "mesoscopic size" of the block goes to 0, while the numbers of spin in block goes to ∞. The convergence is proved in the sense of "typical sequences", i.e. in probability. We do not enter in details and we describe here only the result on the correlation functions. We refer to [9] for a more complete description.

We define the *macroscopic profile* a function $m_0(r)$, $r \in \mathbb{R}^d$, such that $|m_0(r)| \leq 1$, for all r. We suppose that m_0 is smooth, e.g. that it is in C^3, with uniformly bounded derivatives. A "microscopic approximation" to m_0 is a family μ^γ, $0 < \gamma \leq 1$, of probability measures on $\{-1,1\}^{\mathbb{Z}^d}$ which "approximates" m_0 in the following sense: μ^γ, for each γ, is a product measure with averages

$$\mathbb{E}_{\mu^\gamma}\left(\sigma(x)\right) = m_0(\gamma x) \tag{1.9}$$

For any positive integer n, let $\mathbb{Z}^{dn}_{\neq}$ be the collection of all the sets $\underline{x} = (x_1, \ldots, x_n)$ in $\mathbb{Z}^d$ with n distinct elements. Then the following theorem holds.

THEOREM 1.1. *Given γ, we denote by $\mathbb{E}^\gamma_{\mu^\gamma}$ the expectation of the law of the Glauber process starting from μ^γ. Then, there exists $a > 0$ such that for any positive integer n,*

$$\lim_{\gamma \to 0} \sup_{t \leq a|\log\gamma|} \sup_{\underline{x} \in \mathbb{Z}^{dn}_{\neq}} \left| \mathbb{E}^\gamma_{\mu^\gamma}\left(\prod_{i=1}^n \sigma(x_i,t)\right) - \prod_{i=1}^n m(\gamma x_i, t)\right| = 0 \tag{1.10}$$

where $m(r,t)$ is the unique solution of (0.4) with initial datum $m(r,0) = m_0(r)$.

The proof of Theorem 1.1 and of all the other results in [9] is based on bounds on some functions, called v-functions, which are special linear combinations of the spin correlation functions. The techniques used are inspired to the usual cluster expansion of equilibrium statistical mechanics. The physical idea is that a weak form of "propagation of chaos" holds in this case. We briefly recall these facts. We consider the "discretized version" of eq. (0.4)

$$\frac{dm^\gamma(x,t)}{dt} = -m^\gamma(x,t) + \tanh\{\beta[(J_\gamma \circ m^\gamma)(x,t) + h]\} \tag{1.11}$$

with initial condition $m^\gamma(x,0) = \mathbb{E}^\gamma_{\mu^\gamma}(\sigma(x))$. This equation is related to the Glauber dynamics since one has

$$\left|\frac{dm^\gamma(x,t)}{dt} - \mathbb{E}_{\nu_t}\left(L_\gamma\sigma(x)\right)\right| \le c\gamma^d \tag{1.12}$$

for a suitable constant c, where ν_t is the product measure on $\{-1,1\}^{\mathbb{Z}^d}$ with averages $\mathbb{E}_{\nu_t}\left(\sigma(x)\right) = m^\gamma(x,t)$ for all $x \in \mathbb{Z}^d$. If propagation of chaos holds, namely if μ^γ_t, the Glauber distribution at time t, remains a product measure, then eq. (1.12) allows to determine, to leading orders in γ, all the spin correlation functions and then the distribution of the process. But the measure μ^γ_t is not a product measure, also if $\mu^\gamma_0 = \mu^\gamma$ is a product measure. What happens is that one can introduce a "distance" between μ^γ_t and the product measure ν_t, that vanishes in the limit $\gamma \to 0$. This distance is given in terms of the v-functions, defined by

$$v^\gamma(\underline{x},t) = \mathbb{E}^\gamma_{\mu^\gamma}\left(\prod_{x\in\underline{x}}[\sigma(x,t) - m^\gamma(x,t)]\right) \tag{1.13}$$

where $\underline{x} \in \mathbb{Z}^{dn}_{\neq}$. All the results are based on the following basic estimate. There are constants c, K and $a > 0$ such that, for all $t \le a|\log\gamma|$, one has

$$\sup_{\underline{x}\in\mathbb{Z}^{dn}_{\neq}} |v^\gamma(\underline{x},t)| \le ce^{Knt}\gamma^{dn/2} \tag{1.14}$$

We point out that the convergence is guaranteed up to times which diverges with γ but that are small in units $|\log\gamma|$. Thus, as in Theorem 1.1, all the results on the mesoscopic limit hold up to times which diverges with γ but that are small in unit $|\log\gamma|$. Using this fact and the proof of convergence to mean curvature motion of eq. (0.4), it is possible to prove that the Ising spin system converges, as $\gamma \to 0$, to a motion by mean curvature under the "macroscopic scaling"

$$(x,t) \to (\xi,\tau) = (\lambda\gamma x, \lambda^2 t)$$

where $\lambda = |\log\gamma|^{1/2}$. We refer to [9] for this result. The existence of a true "hydrodynamic limit", i.e. with λ a free parameter and by performing first the limit $\lambda \to 0$, is still unproved.

2. Motion by mean curvature

We have divided this section into two subsections. In the first one we briefly explain how to recover the motion by mean curvature by scaling the mesoscopic equation (0.4). In the second one we prove the validity of the Einstein relation (0.3) for this model. We thus complete the derivation of the phenomenological picture from the microscopic model.

2.1 Derivation of motion by mean curvature.

We consider the evolution equation (0.4) with ferromagnetic interaction J and external magnetic field $h = 0$, namely

$$\frac{\partial m}{\partial t} = -m + \tanh\{\beta J \star m\} \tag{2.1}$$

where $m = m(r,t)$, $r \in \mathbb{R}^d$, $t \geq 0$, $\beta > 0$, $0 \leq J = J(|r|)$. As in section 1, we assume $J \in C^2$, and $J(|r|) = 0$ for $|r| \geq 1$.

We normalize $\int dr J(|r|) = 1$ and consider the case $\beta > 1$. Under these assumptions eq. (2.1) admits two constant solutions $\pm m_\beta$, where m_β is the strictly positive solution of

$$m_\beta = \tanh\{\beta m_\beta\} \tag{2.2}$$

For the spin system in the Lebowitz-Penrose limit, $\beta = 1$ is the critical temperature, while $\pm m_\beta$ are the magnetizations of the pure phases at inverse temperature β. Obviously the system exhibits two different phases, i.e. a phase transition, only for $\beta > 1$. We refer to [15] and [23] for details.

The purpose is to characterize the evolution of an initial datum which has two coexisting phases. More precisely, one considers an initial datum which is close to m_β inside a finite region Λ_0 and to $-m_\beta$ outside; then, the evolved profile has the same structure with Λ_0 replaced by the region obtained moving its boundary by mean curvature. This result holds only in a suitable scaling limit and the convergence is limited to times when the motion by curvature is regular. In the next section we will give a stronger result for the 2D-case. The choice of one single cluster in the previous setting is made for simplicity of notations, but it can be easily avoided. On the contrary, as just said in the introduction, phase separation cannot be predicted from (2.1) and the existence of a "good" initial datum is here assumed.

We start by recalling the definition of motion by mean curvature. Let Γ_0 be a C^2 closed surface embedded in $\mathbb{R}^d$; we denote by Λ_0 the open finite region with boundary Γ_0. The motion by mean curvature starting from Γ_0 is defined by the equation

$$\frac{d\xi}{d\tau} = \frac{\theta}{R}\nu \tag{2.3}$$

where $\xi = \xi(\tau)$ is a point of the surface Γ_τ, $R^{-1} = R^{-1}(\xi)$ is $(d-1)$ times the mean curvature of Γ_τ; θ is a constant and $\nu = \nu(\xi)$ the unit vector normal in Γ_τ in ξ pointing toward the interior of Γ_τ.

Following [12] we call (ξ, τ) the "macroscopic variables" which are related to the "mesoscopic" ones (r, t) by setting:

$$r = \lambda^{-1}\xi; \quad t = \lambda^{-2}\tau \tag{2.4}$$

where λ is the scaling parameter which we suppose much smaller than 1. We then define

$$m^{(\lambda)}(\xi, \tau) = m(\lambda^{-1}\xi, \lambda^{-2}\tau) = m(r, t) \tag{2.5}$$

where $m(r, t)$ is the solution of (2.1) with initial datum

$$m(r, 0) = m_0(\lambda r; \lambda) \tag{2.6}$$

where $m_0(\lambda r; \lambda)$ converges, as $\lambda \to 0$, to m_β inside Λ_0 and to $-m_\beta$ outside of it. The equation for $m^{(\lambda)}$ is

$$\frac{\partial m^{(\lambda)}}{\partial \tau} = \lambda^{-2}\Big\{ - m^{(\lambda)} + \tanh\{\beta J^{(\lambda)} \star m^{(\lambda)}\}\Big\} \tag{2.7}$$

where

$$J^{(\lambda)}(|\xi|) = \lambda^{-d} J(\lambda^{-1}|\xi|) \tag{2.8}$$

The divergent factor λ^{-2} forces the curly bracket term in (2.7) to be small, hence $m^{(\lambda)}$ should be close to a stationary solution of (2.1). Hence we look for a solution close to m_β when ξ is in Λ_τ and to $-m_\beta$ when ξ is outside. Since $J^{(\lambda)}$ is an approximate δ-function, in order to describe the transition region, it is convenient to go back to the mesoscopic coordinates. But in these coordinates the interface is extremely flat, so that it is natural to look for stationary solutions

$$\bar{m}^d(r) = \tanh\{\beta J \star \bar{m}^d(r)\} \tag{2.9}$$

which have a planar symmetry. Therefore, modulo translations, $\bar{m}^d$ depends only on one parameter, $\bar{m}^d(r) = \bar{m}(r \cdot \nu)$, ν a unit vector, where $\bar{m}(\cdot)$ is the solution of the $d = 1$ problem:

$$\bar{m} = \tanh\{\beta \tilde{J} \star \bar{m}\}, \qquad \lim_{x \to \pm\infty} \bar{m}(x) = \pm m_\beta, \qquad \bar{m}(0) = 0 \tag{2.10}$$

with

$$\tilde{J}(x) = \int_{\mathbb{R}^{d-1}} dy\, J(|x^2 + y^2|^{1/2}) \tag{2.11}$$

We briefly summarize the property of the "istanton" $\bar{m}$ (we refer to [12] and [14] for details and proofs):

i) $\bar{m}$ is an antisymmetric strictly increasing function in $\mathcal{C}^2(\mathbb{R})$.

ii) there exists constants α and M, both positive, such that:

$$\lim_{x \to \pm\infty} e^{\alpha|x|} \Big| \bar{m}(x) \mp [m_\beta - \frac{M}{\alpha} e^{-\alpha|x|}] \Big| = 0 \tag{2.12a}$$

$$\lim_{|x| \to +\infty} e^{\alpha|x|} \Big| \bar{m}'(x) - M e^{-\alpha|x|} \Big| = 0 \tag{2.12b}$$

$$\lim_{x \to \pm\infty} e^{\alpha|x|} \Big| \bar{m}''(x) \pm \alpha M e^{-\alpha|x|} \Big| = 0 \tag{2.12c}$$

iii) $\bar{m}$ is the unique stationary solution of (2.1), modulo translation, in the space

$$\mathcal{A}_+ = \{ m \in \mathcal{C}(\mathbb{R}) \mid |m|_\infty \le 1 ;\ \liminf_{x \to \infty} m(x) > 0 \,,\ \limsup_{x \to -\infty} m(x) < 0 \}$$

that is, if $m \in \mathcal{A}_+$ and $m = \tanh\{\beta \tilde{J} \star m\}$, there exists $a \in \mathbb{R}$ such that $m(x) = \bar{m}(x - a)$.

The exponential convergence property (2.12a) makes $\bar{m}$ the right candidate to match the $\pm m_\beta$ magnetizations at both sides of the interface in order to construct the initial datum. The precise result obtained in [12] is the following theorem.

THEOREM 2.1. *Let Γ_τ be the evolution of Γ_0 according to equation (2.3) and $m^{(\lambda)}(\xi, \tau)$ as in (2.5) with initial datum*

$$m_0(\lambda r; \lambda) = \begin{cases} \bar{m}(\lambda^{-1} d(\xi, \Gamma_0)) & \text{if} \quad |d(\xi, \Gamma_0)| \le \lambda^{1-\zeta} \\ sgn(d(\xi, \Gamma_0)) \bar{m}(\lambda^{-\zeta}) & \text{if} \quad |d(\xi, \Gamma_0)| > \lambda^{1-\zeta} \end{cases} \tag{2.13}$$

where $0 < \zeta < 1$ and $d(\xi, \Gamma_0)$ is the signed distance of ξ from Γ_0 (positive when $\xi \in \Lambda_0$).

Let τ_s be the time when (2.3) becomes singular. Then for all $\tau^\star < \tau_s$ there exists $a, b > 0$ such that, for all $\tau \le \tau^\star$ and all λ sufficiently small,

$$|m^{(\lambda)}(\xi, \tau) - sgn(d(\xi, \Gamma_\tau)) m_\beta| \le \lambda^b \tag{2.14}$$

for all ξ such that $|d(\xi,\Gamma_\tau)| \geq \lambda^a$; as in (2.13), $d(\xi,\Gamma_\tau)$ is the signed distance, positive when $\xi \in \Lambda_\tau$, with Λ_τ the finite open region of boundary Γ_τ.

The proof of Theorem 2.1 is obtained by constructing super and sub-solutions of the equation (2.1) which give the desired estimates on the solution. The evolution is studied separately for short times and in small neighbourhoods of the interface. Then the global solution is obtained by using a "patching and iterating" procedure. All this machinery holds in view of the "good properties" of eq. (2.1) (see the "Comparison Theorem" and the "Barrier Lemma" in the next section). The advantage of working locally is that one can consider perturbations of the planar istanton. We refer to [12] for details. On the contrary, we show how to recover locally eq. (2.3) from eq. (2.1) in the linear approximation.

Let ξ_0 be a point in Γ_0 and let $r_0 = \lambda^{-1}\xi_0$. We want to study the evolution (2.1) in a small neighbourhood of r_0 and in a time interval $[0,T]$, $T = \lambda^{-\delta}$, $\delta > 0$ and small. We choose a local frame with the x-axis along the normal to $\lambda^{-1}\Gamma_0$ at r_0 and the other axes along the principal axes of curvature. Then, to the first order in λ, the equation for the surface is $x = x^\star(y)$ with

$$x^\star(y) = \frac{\lambda}{2}\sum_{i=1}^{d-1}\frac{y_i^2}{R_i} \qquad \sum_{i=1}^{d-1}\frac{1}{R_i} = \frac{1}{R} \tag{2.15}$$

R^{-1} is therefore $(d-1)$ times the mean curvature of Γ_0 at ξ_0. According to the choice (2.13), in a neighbourhood of the origin and to the first order in λ, one has

$$m(r,0) = \bar{m}(x) - \frac{\lambda}{2}\Big(\sum_{i=1}^{d-1}\frac{y_i^2}{R_i}\Big)\bar{m}'(x) \tag{2.16}$$

Now, if δ is small enough, the linear approximation of (2.1) around $\bar{m}$ is rather accurate to describe the evolution. All this can be stated precisely, see [12]. Here we just make this approximation with no further justification. Then we approximate, for all $t \leq T$ and in a neighbourhood of the origin,

$$m(r,t) = \bar{m}(x) + e^{Lt}\phi_0(r) \qquad \phi_0(r) = \frac{\lambda}{2}\Big(\sum_{i=1}^{d-1}\frac{y_i^2}{R_i}\Big)\bar{m}'(x) \tag{2.17}$$

where L is the linearization of (2.1) around $\bar{m}$:

$$L\phi(r) = -\phi(r) + (1-\bar{m}^2(x))\beta J \star \phi(r) \tag{2.18}$$

We make the transformation

$$\phi = \bar{m}'\psi \qquad \mathcal{L}\psi = \frac{1}{\bar{m}'}L(\bar{m}'\psi)$$

so that

$$\mathcal{L}\psi(r) = \int dr' K(r,r')[\psi(r') - \psi(r)] \tag{2.19}$$

where

$$K(r,r') = (1 - \bar{m}(x)^2)\beta J(|r - r'|)\frac{\bar{m}'(x')}{\bar{m}'(x)} \tag{2.20}$$

Now we are interested in the value of (2.17), at time T, along the normal to $\lambda^{-1}\Gamma_0$ at r_0, that is in $r = (x, 0)$. For the previous transformation we have

$$m((x,0),T) = \bar{m}(x) + \bar{m}'(x)\big(e^{\mathcal{L}T}\psi_0\big)(x,0) \tag{2.21}$$

where $\psi_0(r) = \phi_0(r)/\bar{m}'(x)$. But we can write

$$\big(e^{\mathcal{L}T}\psi_0\big)(x,0) = \int_0^T dt\big(e^{\mathcal{L}t}\mathcal{L}\psi_0\big)(x,0)$$

and, by explicit computation, one has

$$\big(e^{\mathcal{L}T}\psi_0\big)(x,0) = \int_0^T dt\big(e^{\mathcal{L}^{(1)}t} f\big)(x) \tag{2.22}$$

where $\mathcal{L}^{(1)}$ is the operator as in (2.19) in $d = 1$ with interaction $\tilde{J}$ (see (2.11)), while

$$f(x) = -(1 - \bar{m}(x)^2)\beta \int dx' dz J\big(|(x'-x)^2 + z^2|^{1/2}\big)\frac{\bar{m}'(x')}{\bar{m}'(x)}\frac{\lambda}{2}\sum_i \frac{z_i^2}{R_i} \tag{2.23}$$

$\mathcal{L}^{(1)}$ is the generator of a jump Markov process (note that $\int dr' K(r,r') = 1$), and, in [14], it is proved the validity of a Perron-Frobenius theorem for this process. More precisely, one considers the measure

$$\mu(dx) = N\frac{\bar{m}'(x)^2}{1 - \bar{m}(x)^2}dx, \qquad N^{-1} = \int_{\mathbb{R}} dx\, \frac{\bar{m}'(x)^2}{1 - \bar{m}(x)^2} \tag{2.24}$$

It is easy to see that μ is invariant for the process. Then, it is shown that the process converges exponentially to this measure. The rate of convergence of $e^{\mathcal{L}^{(1)}t}$ depends on the starting point x: it takes a time proportional to $|x|$ to reach a neighbourhood of the origin and then it approaches the equilibrium exponentially fast.

Using this property, by (2.22) one has

$$\Big|\big(e^{\mathcal{L}T}\psi_0\big)(x,0) - \mu(f)T\Big| \le \lambda C(|x| + 1)$$

for a suitable constant C and with

$$\mu(f) = \int \mu(dx) f(x) = -\frac{\lambda}{R}\theta$$

where we have defined

$$\theta = \int \mu(dx)\,(1-\bar{m}(x)^2)\beta \int dx' \int_{\mathbb{R}^{d-1}} dy\, J\big(|(x'-x)^2+y^2|^{1/2}\big)\frac{\bar{m}'(x')}{\bar{m}'(x)} y_1^2/2 \tag{2.25}$$

Then, since $T = \lambda^{-\delta}$, in the linear approximation, one has

$$m\big((x,0),T\big) \approx \bar{m}(x) - \frac{\lambda}{R}\theta T \bar{m}'(x) \approx \bar{m}\big(x - \frac{\lambda}{R}\theta T\big) \tag{2.26}$$

By (2.26) we conclude that, in the mesoscopic scale, the interface has moved by $\lambda T\theta/R$ in the time T. Then, in the macroscopic space, the displacement becomes $\lambda^2 T\theta/R$, where $\lambda^2 T$ is just the macroscopic time corresponding to the mesoscopic one T. So, in this approximation, we recover eq. (2.3) from (2.1) with transport coefficient θ given by (2.25).

2.2 The Einstein relation.

As we shall see, it is possible to compute the mobility μ and the surface tension σ independently for this model. We will prove that the value of θ obtained in the previous subsection is equal to the product $\mu\sigma$, so that the Einstein relation (0.3) is indeed verified in this model.

The excess free energy associated to m (see [6,15,23]) is

$$F(m) = \int dr\,(f(m) - f(m_\beta)) + \frac{1}{4}\int dr dr'\, J(r-r')(m(r)-m(r'))^2 \tag{2.27}$$

where

$$\begin{aligned} f(m) &= -\frac{m^2}{2} + \beta^{-1}\frac{1+m}{2}\log\frac{1+m}{2} + \beta^{-1}\frac{1-m}{2}\log\frac{1-m}{2} \\ &= -\frac{m^2}{2} + \beta^{-1}\frac{m}{2}\log\frac{1+m}{1-m} + \beta^{-1}\frac{1}{2}\log(1-m^2)\,; \end{aligned} \tag{2.28}$$

$$\int dr dr'\, J(r-r')(m(r)-m(r'))^2 = 2\int dr\,(m^2 - mJ\star m) \tag{2.29}$$

so that

$$F(m) = \int dr\,(g(m) - g(m_\beta)) \tag{2.30}$$

where

$$g(m) = \beta^{-1}\frac{m}{2}\log\frac{1+m}{1-m} + \beta^{-1}\frac{1}{2}\log(1-m^2) - \frac{1}{2}mJ\star m \tag{2.31}$$

We point out that $F(m)$ is a positive definite functional of m. Then it is well defined for every measurable m, but not finite in general.

Since $\bar{m}$ is interpreted as the interface profile connecting the $\pm m_\beta$ phases, the surface tension can be expressed as

$$\begin{aligned}\sigma &= \lim_{L\to\infty}\frac{1}{(2L)^{d-1}}\lim_{M\to\infty}\int_{-L}^{L}dy_1\cdots\int_{-L}^{L}dy_{d-1}\int_{-M}^{M}dx\,(g(m^*)-g(m_\beta))\\ &= \int dx\,(g(m^*)-g(m_\beta))\end{aligned}\tag{2.32}$$

where

$$m^*(r)=\bar{m}(x)\tag{2.33}$$

if $r=(x,y_1,...,y_{d-1})$ in a coordinate frame with the x-axis orthogonal to the equilibrium interface.

A more microscopic definition of the surface tension (see [28] and references there) involves the computation of the logarithm (normalized by the surface area) of the ratio of two partition functions with different boundary conditions. The second one has boundary conditions + on the two opposite faces of a cube and periodic conditions on the other ones; the first one is defined by conditions + and − instead of + and +. The correct procedure for obtaining the surface tension is to take first the thermodynamic limit, then the limit as $\gamma\to 0$, γ being the scalus parameter in the Kac potential. To my knowledge, there is no proof that this gives rise to the value (2.32). However, if one takes the thermodynamic limit and simultaneously $\gamma\to 0$, in a suitable fashion, then (2.32) can be proven to hold, as it follows from the analysis of [6], and from results recently obtained by Cassandro and Vares.

We compute now the mobility. We look for a planar travelling wave solution of (0.4)

$$m_h(r,t)=m_h(x-v(h)t)\tag{2.34}$$

for small h. E. Orlandi and L. Triolo (private communication) have shown the existence of a solution for small h which is close to the $h=0$ stationary solution m^*. Avoiding the uniqueness problem we take this solution and expand

$$v(h)=v_1h+O(h^2),\qquad m_h=\bar{m}+h\psi+O(h^2)\tag{2.35}$$

From (0.4), at first order in h, we obtain the following identity

$$-v_1\bar{m}'=-\psi+(1-\bar{m}^2)\beta\tilde{J}\star\psi+\beta(1-\bar{m}^2)\tag{2.36}$$

Equation (2.10) implies also

$$(1 - \bar{m}^2)\beta\tilde{J} \star \bar{m}' = \bar{m}' \tag{2.37}$$

We multiply both sides of (2.36) by $\bar{m}'(x)/(1-\bar{m}(x)^2)$ and then integrate; using (2.37) we obtain

$$v_1 = -2N\beta m_\beta \tag{2.38}$$

But, by the definition of the mobility, it must be

$$v(h) = -2m_\beta \mu h + O(h^2) \tag{2.39}$$

Equations (2.38) and (2.39) imply then

$$\mu = N\beta \tag{2.40}$$

We can now prove (0.3). By (2.25) and the definition (2.24) of the invariant measure, we have the following expression for θ

$$\theta = \frac{1}{2}N\beta \int dx\, \bar{m}'(x) \int dx'dy\, J\left(|(x'-x)^2 + y^2|^{1/2}\right)\bar{m}'(x')y_1^2 \tag{2.41}$$

From (2.33) and the definition (2.10) of $\bar{m}$, it follows easily that

$$g(m^\star)(r) = \frac{1}{2}(\bar{m}\tilde{J} \star \bar{m} + \beta^{-1}\log(1-\bar{m}^2))(x) \tag{2.42}$$

Using (2.37), from (2.42) one has

$$\frac{d}{dx}g(m^\star) = \frac{1}{2}(\bar{m}'\tilde{J} \star \bar{m} - \bar{m}\tilde{J} \star \bar{m}') \tag{2.43}$$

Integrating by parts in (2.32) and by (2.40), we obtain

$$\begin{aligned} \mu\sigma &= -\frac{1}{2}N\beta \int dx\, x\frac{d}{dx}g(m^\star) \\ &= \frac{1}{2}N\beta \int dxdx'(x'-x)\bar{m}'(x)\tilde{J}(x-x')\bar{m}(x') \end{aligned} \tag{2.44}$$

In order to compare (2.41) with (2.44), it is convenient to eliminate the dependence on the y_1 variable to have θ expressed in terms of $\bar{m}$ and $\tilde{J}$. We note that

$$\frac{\partial}{\partial x'}J\left(|(x'-x)^2 + y^2|^{1/2}\right) = \frac{x'-x}{y_1}\frac{\partial}{\partial y_1}J\left(|(x'-x)^2 + y^2|^{1/2}\right) \tag{2.45}$$

Integrating by parts in dx', using (2.45), and then integrating by parts in dy_1, we finally obtain

$$\theta = \frac{1}{2}N\beta \int dxdx'(x'-x)\bar{m}'(x)\tilde{J}(x-x')\bar{m}(x') \tag{2.46}$$

Equation (0.3) follows then by (2.44) and (2.46).

3. Convergence at all times in the 2D-case

As just mentioned, for a generic initial datum, the motion by mean curvature is regular only for finite times; on the other hand, the microscopic evolution is well defined for all times. It is therefore an obvious problem the description of the system past the appearance of singularities (see for example [1,8,16,17,22]).

The simplest singularity for the motion by mean curvature of a closed regular surface is the shrinking to a point. This is the only singularity which appears in the 2-dimensional case; more precisely, any closed curve embedded in $\mathbb{R}^2$ moving by curvature, becomes convex and then shrinks to a point in a finite time. For closed surfaces embedded in $\mathbb{R}^3$ a sufficient condition to have such a singularity is the uniform convexity of the initial datum (see [20]).

In this section we analyze eq. (2.1) in the 2-dimensional case at all times. We choose the initial datum as in Theorem 2.1; we then prove that, past the appearance of the singularity, the magnetization profile is close, in the previous scaling, to the negative magnetization $-m_\beta$. So we verify, as physically expected, that a cluster of one phase, shrinking to a point, actually disappears. Here we consider the case $d = 2$ because this kind of singularity exhausts all the possibilities. However, it will be clear that the result holds also for $d > 2$, provided the initial datum is such that the limiting motion becomes singular just by the shrinking of the surface to a point. Similar results for the Allen-Cahn equation are proven in [7]. More precisely, we are going to prove the following theorem.

THEOREM 3.1. *Let $d = 2$ so that $\tau_s = \tau_e$, the extinction time for the curve Γ_τ. We fix the origin where shrinking takes place. Then, under the same hypothesis of Theorem 2.1, there exist positive numbers η and q such that, for all λ small enough,*

$$|m^{(\lambda)}(\xi,\bar\tau) + m_\beta| \le \lambda^\eta \qquad \forall\, \xi \in \mathbb{R}^2 \qquad \forall\, \bar\tau > \tau_e \tag{3.1}$$

so that

$$\lim_{\lambda\to 0} m^{(\lambda)}(\xi,\bar\tau) = -m_\beta \qquad \forall\, \xi \in \mathbb{R}^2 \qquad \forall\, \bar\tau > \tau_e$$

and

$$|m^{(\lambda)}(\xi,\tau_e) + m_\beta| \le \lambda^\eta \qquad \forall\, \xi \in \mathbb{R}^2 \ :\ |\xi| > \lambda^q \tag{3.2}$$

The proof of Theorem 2.1 is obtained by constructing super and subsolutions of eq. (2.1) which give the desired estimates on the solution. In our case as sub-solution can be taken the constant one $-m_\beta$ (in fact, by the monotonicity of $\bar{m}$ and the "Comparison Theorem", see below, $m^{(\lambda)}(\xi,\tau) \geq -m_\beta$). We will need two supersolutions for each value of $\bar{\tau}$ in order to obtain (3.1). Since $\bar{\tau}$ can be taken arbitrarily close to τ_e, (3.2) can be proven using the same supersolutions. In the next subsection we give some properties of the motion by mean curvature which suggest the form of the required supersolutions to the 0^{th}-order in λ.

3.1 Some basic properties of the motion by mean curvature.

We denote with $C(\xi, R)$ the circle centered in ξ of radius R and with $B(\xi, R)$ the open ball of boundary $C(\xi, R)$.

Let Γ_0 be a C^2 closed curve embedded in $\mathbb{R}^2$. As already mentioned, under evolution by mean curvature Γ_τ first becomes convex and then shrinks to a point at a finite time τ_e. It is known that the flow by mean curvature makes convex curves circular. More precisely, if $\widetilde{\Gamma}_\tau$ is the homothetic expansion of Γ_τ such that $|\widetilde{\Gamma}_\tau| = |\Gamma_0|$, then $\widetilde{\Gamma}_\tau$ converges, in the C^2 sup-norm, to the circle of length $|\Gamma_0|$ centered in the shrinking point. We refer to [18] and [19] for details. An easy consequence of these facts is the following lemma.

LEMMA 3.2. *Let Γ_0 be a C^2 closed curve embedded in $\mathbb{R}^2$; we fix the origin in the point where shrinking takes place. Then for any $\epsilon > 0$ there exists $\tau' < \tau_e$ and $0 < R_1 < R_2$ such that $R_2 - R_1 < \epsilon$ and*

$$B(0, R_1(\tau)) \subseteq \Lambda_\tau \subseteq B(0, R_2(\tau)) \qquad \forall \tau' \leq \tau < \tau_e \tag{3.3}$$

where $R_i(\tau)$ $(i = 1, 2)$ solves the equation

$$dR_i(\tau) = -\frac{\theta}{R_i(\tau)} d\tau \qquad R_i(\tau') = R_i \tag{3.4}$$

Proof. The convergence of Γ_τ to a circle, in the sense explained before, implies that there exists $\tau' < \tau_e$ and $R_1 < R_2$ such that

$$B(0, R_1) \subseteq \Lambda_{\tau'} \subseteq B(0, R_2); \qquad R_2 - R_1 < \epsilon \tag{3.5}$$

But it is a simple geometrical property of the flow by mean curvature that curves which do not intersect at a given time, cannot intersect if moved by curvature for all later times for which the motion is regular; then the lemma follows from (3.5). $\square$

Let now τ_{R_i} be such that $R_i(\tau_{R_i}) = 0$; eq. (3.4) imply that

$$\tau_{R_i} = \tau' + \frac{{R_i}^2}{2\theta} \tag{3.6}$$

Then

$$\tau_{R_2} - \tau_{R_1} = \frac{{R_2}^2 - {R_1}^2}{2\theta} = \frac{(R_1 + \epsilon)^2 - {R_1}^2}{2\theta} = \epsilon \frac{2R_1 + \epsilon}{2\theta} \tag{3.7}$$

Since R_1 is uniformly bounded in ϵ, eq. (3.7) gives $\tau_{R_2} - \tau_{R_1} = O(\epsilon)$. But from Lemma 3.2, $\tau_{R_1} \leq \tau_e \leq \tau_{R_2}$, so that we have

COROLLARY 3.3. *Under the same hypothesis of Lemma 3.2, for any $\bar{\tau} > \tau_e$ there exists a time $\tilde{\tau} < \tau_e$ and a radius $\tilde{R}$ such that*

$$\Lambda_\tau \subseteq B(0, \tilde{R}(\tau)) \qquad \forall \tilde{\tau} \leq \tau < \tau_e ; \qquad \qquad \tau_{\tilde{R}} \doteq \tilde{\tau} + \frac{\tilde{R}^2}{2\theta} < \bar{\tau} \tag{3.8}$$

where $\tilde{R}(\tau)$ solves

$$d\tilde{R}(\tau) = -\frac{\theta}{\tilde{R}(\tau)} d\tau, \qquad \tilde{R}(\tilde{\tau}) = \tilde{R} \tag{3.9}$$

From Corollary 3.3 one derives

PROPOSITION 3.4. *Under the same hypothesis of Lemma 3.2, for any $\bar{\tau} > \tau_e$ there exists a time $\tau^\star < \tau_e$ and a radius $R^\star$ such that*

$$\Lambda_\tau \subseteq B(v^\star(\tau - \tau^\star)\hat{n}, R^\star) \qquad \qquad \forall \tau^\star \leq \tau < \tau_e \tag{3.10}$$

and

$$\tau_0 \doteq \inf\{\tau \,|\, 0 \notin B(v^\star(\tau - \tau^\star)\hat{n}, R^\star)\} < \bar{\tau} \tag{3.11}$$

where $v^\star = \theta / R^\star$ and $\hat{n}$ is any unit vector in $\mathbb{R}^2$.

Proof. According to Corollary 3.3, for any $\tilde{\tau} \leq \tau^\star < \tau_e$, by setting $\tilde{R}(\tau^\star) = R^\star$, we have

$$\Lambda_\tau \subseteq B(0, \tilde{R}(\tau)) \subseteq B(v^\star(\tau - \tau^\star)\hat{n}, R^\star) \qquad \qquad \forall \tau^\star \leq \tau < \tau_e \tag{3.12}$$

where the last inclusion follows from the fact that $v^\star$ is just the velocity of $C(0, \tilde{R}(\tau))$ at $\tau = \tau^\star$. On the other hand, by integration of eq. (3.9), it is easy to see that

$$\tau_0 = \tau^\star + \frac{R^\star}{v^\star} = \tau_{\tilde{R}} + (\tau_{\tilde{R}} - \tau^\star) \tag{3.13}$$

so that (3.11) holds if we choose $\tau^\star$ such that $\tau_{\tilde{R}} - \tau^\star < \bar{\tau} - \tau_{\tilde{R}}$. □

In order to construct supersolutions, we need now the following notion.

The biased motion by curvature : With the same notation as in the previous section, for any h real, we define the h-biased motion $\Gamma_\tau^{(h)}$ of Γ_0 so that the points of $\Gamma_\tau^{(h)}$ satisfy the equation

$$\frac{d\xi^{(h)}}{d\tau} = \left(\frac{\theta}{R} - h\right) \nu \tag{3.14}$$

The following lemma holds (see [12] and references there).

LEMMA 3.5. *Let $\tau^\star > 0$ be strictly smaller than the maximum time for which the unbiased motion by curvature is regular. Then there are h_0 and c so that $\Gamma_\tau^{(h)}$ exists and it is regular for all $|h| \le h_0$ and all $\tau \le \tau^\star$. Furthermore if $\xi^{(h)}(\tau)$ and $\xi(\tau)$ verify (3.14) and, respectively, (2.3) with $\xi^{(h)}(0) = \xi(0)$, then*

$$\left|\xi^{(h)}(\tau) - \xi(\tau)\right| \le ch \tag{3.15}$$

The constant c is independent of the starting point in Γ_0.

We denote by $\Lambda_\tau^{(h)}$ the open finite region of boundary $\Gamma_\tau^{(h)}$. Using Corollary 3.3, Proposition 3.4 and Lemma 3.5, we finally obtain:

PROPOSITION 3.6. *With $\bar\tau$, $\tau^\star$ as in Corollary 3.3 and Proposition 3.4 and $h > 0$, one has:*

$$\Lambda_\tau^{(h)} \subseteq B(0, \tilde R^{(h)}(\tau)) \qquad \forall\, \bar\tau \le \tau < \tau_e \tag{3.16}$$

where $\tilde R^{(h)}(\tau)$ solves

$$d\tilde R^{(h)}(\tau) = \left(h - \frac{\theta}{\tilde R^{(h)}(\tau)}\right) d\tau, \qquad \tilde R^{(h)}(\bar\tau) = \tilde R + ch \tag{3.17}$$

and

$$\Lambda_\tau^{(h)} \subseteq B(v_h^\star(\tau - \tau^\star)\hat n, R^\star + ch) \qquad \forall\, \tau^\star \le \tau < \tau_e \tag{3.18}$$

with $v_h^\star = \theta/(R^\star + ch) - h$. Moreover

$$\tau_0^{(h)} \doteq \inf\{\tau \,|\, 0 \notin B(v_h^\star(\tau - \tau^\star)\hat n, R^\star + ch)\} = \tau_0 + O(h) \tag{3.19}$$

so that, for all h small enough, it is $\tau_0^{(h)} < \bar\tau$.

The proof of (3.16) is based on the fact that the h-biased motion has geometrical properties similar to those of the unbiased one. Eq. (3.18) follows from (3.16); in fact, by Lemma 3.5,

$$0 < \tilde R^{(h)}(\tau) - \tilde R(\tau) \le ch \tag{3.20}$$

so that

$$\tilde R^{(h)}(\tau^\star) \le R^\star + ch\,; \qquad v_h^\star \le \frac{\theta}{\tilde R^{(h)}(\tau^\star)} - h \tag{3.21}$$

from which (3.18) is clear. Finally we compute:

$$\tau_0^{(h)} = \tau^\star + \frac{(R^\star + ch)^2}{\theta - R^\star h - ch^2} = \tau_0 + O(h) \tag{3.22}$$

3.2 Supersolutions.

As just mentioned, in [12] Theorem 2.1 is proved by constructing super and sub-solutions. We will construct new supersolutions starting from the ones given in [12]. We recall first their construction. Choose δ and R_0 as follows:

$$1/40 < \delta < 1/20; \quad 2 - 10\delta > \alpha R_0 > 3/2 \tag{3.23}$$

with α as in (2.12). For λ sufficiently small, it is defined

$$m^\star(\xi,\tau) = \begin{cases} \bar{m}(\lambda^{-1} d(\xi, \Gamma_\tau^{(h)})) & \text{if} \quad |d(\xi,\Gamma_\tau^{(h)})| \le R_0 \lambda |\log \lambda| \\ \text{sgn}(d(\xi,\Gamma_\tau^{(h)}))m_\beta + \lambda^{3/2} & \text{if} \quad |d(\xi,\Gamma_\tau^{(h)})| > R_0\lambda|\log\lambda| \end{cases} \tag{3.24}$$

where $h = \lambda^{\delta/2}$. Then one has

$$m^{(\lambda)}(\xi,\tau) \le m^\star(\xi,\tau) \qquad \forall\, \xi \in \mathbb{R}^d \qquad \forall\, \lambda^2 T \le \tau \le \tau^\star \tag{3.25}$$

where $T = \lambda^{-\delta}$ and $\tau^\star$ any time strictly smaller than τ_e. Similarly a subsolution is constructed and Theorem 2.1 is proved with $a = 1/80$ and $b = 3/2$.

For proving our theorem, we construct two supersolutions for each time $\bar\tau$. Let $\bar\tau > \tau_e$ be assigned. From Proposition 3.6 there are $\tau^\star$ and $R^\star$ such that (3.18) and (3.19) hold (we always fix the origin in the singularity point). In order to prove (3.1) it is sufficient to define the two supersolutions only for $\tau \ge \tau^\star$. Let $\hat\xi_1$ be the unit vector along the ξ_1-axis; we define for λ sufficiently small and $\tau \ge \tau^\star$:

$$m^\star_\pm(\xi,\tau) = \begin{cases} \bar{m}(\lambda^{-1} d(\xi, C_{\tau,\pm}^{(h)})) & \text{if} \quad |d(\xi,C_{\tau,\pm}^{(h)})| \le \bar R_0 \lambda |\log \lambda| \\ \text{sgn}(d(\xi,C_{\tau,\pm}^{(h)}))m_\beta + \lambda^{3/2-\gamma} & \text{if} \quad |d(\xi,C_{\tau,\pm}^{(h)})| > \bar R_0\lambda|\log\lambda| \end{cases} \tag{3.26}$$

where

$$C_{\tau,\pm}^{(h)} \doteq C(\mp v_h^\star(\tau-\tau^\star)\hat\xi_1, R^\star + ch)\,; \qquad h = \lambda^{\delta/2} \tag{3.27}$$

with δ as before and γ, $\bar R_0$ such that:

$$3/2 > \alpha \bar R_0 > 3/2 - \gamma\,; \qquad 0 < \gamma < 5\delta \tag{3.28}$$

In the next sections we will prove that for all λ small enough one has

$$m^{(\lambda)}(\xi,\tau) \le m^\star_\pm(\xi,\tau) \qquad \forall\, \xi \in \mathbb{R}^2 \qquad \forall\, \tau \ge \tau^\star + \lambda^2 T \tag{3.29}$$

We conclude this section by showing that Theorem 3.1 follows easily from (3.29).

Proof of Theorem 3.1 under condition (3.29). Let us consider the half-planes

$$P_+ = \{ \xi \in \mathbb{R}^2 \,|\, \xi \cdot \hat{\xi}_1 \geq 0 \}; \qquad P_- = \{ \xi \in \mathbb{R}^2 \,|\, \xi \cdot \hat{\xi}_1 \leq 0 \} \tag{3.30}$$

By (3.19) with $h = \lambda^{\delta/2}$, for λ small enough,

$$\bar{B}^{(h)}_{\bar{\tau},\pm} \cap P_\pm = \phi \tag{3.31}$$

where $\bar{B}^{(h)}_{\tau,\pm}$ is the closed ball of boundary $C^{(h)}_{\tau,\pm}$. So, by definition (3.26), if λ is small enough, we have

$$m^\star_\pm(\xi, \bar{\tau}) = -m_\beta + \lambda^{3/2-\gamma} \qquad \forall\, \xi \in P_\pm \tag{3.32}$$

Then (3.1) follows from (3.29) and (3.32) with $\eta = 3/2 - \gamma$.

It is clear that $|\xi'| \leq R^\star + ch$ for all $\xi' \in C^{(h)}_{\tau_e,\pm} \cap P_\pm$. Then

$$Q_\pm \doteq \{ \xi \in P_\pm \,|\, |\xi| > R^\star + ch + \bar{R}_0 \lambda |\log \lambda| \} \subset \{ \xi \in P_\pm \,|\, d(\xi, C^{(h)}_{\tau_e,\pm}) < -\bar{R}_0 \lambda |\log \lambda| \}$$

so that $m^\star_\pm(\xi, \tau_e) = -m_\beta + \lambda^{3/2-\gamma}$ for all $\xi \in Q_\pm$. We choose then $R^\star = \lambda^\epsilon$ with $\epsilon < \delta/2$. By looking at the proof of (3.29), it is not difficult to see that it holds also with this choice, so that one has

$$m^\star_\pm(\xi, \tau_e) = -m_\beta + \lambda^{3/2-\gamma} \qquad \forall\, \xi \in P_\pm \;:\; |\xi| > ch + \lambda |\log \lambda| + \lambda^\epsilon \tag{3.33}$$

Then (3.2) follows from (3.29) and (3.33) with $q < \epsilon$. □

3.3 Proof of (3.29) through an iterative procedure.

In this subsection we prove the basic estimate (3.29). We follows the same technique used in [12] to prove (3.25). The idea is to localize the analysis by studying the evolution for short times and in small neighbourhoods of the interface, and then to match and to iterate the procedure in order to obtain global estimates. First of all, we need the validity of (3.29) for $\tau = \tau^\star$.

LEMMA 3.7. *For λ small enough*

$$m^{(\lambda)}(\xi, \tau^\star) \leq m^\star_\pm(\xi, \tau^\star) \qquad \forall\, \xi \in \mathbb{R}^2 \tag{3.34}$$

Proof. By (3.25) it is sufficient to prove that for λ small enough

$$m^\star(\xi, \tau^\star) \leq m^\star_\pm(\xi, \tau^\star) \qquad \forall\, \xi \in \mathbb{R}^2 \tag{3.35}$$

By (3.27) it follows that $C^{(h)}_{\tau^*,\pm} = C(0, R^\star + ch)$ so that (3.18) implies $\Lambda^{(h)}_{\tau^*} \subseteq B^{(h)}_{\tau^*,\pm}$, the open ball of boundary $C^{(h)}_{\tau^*,\pm}$, and then $d(\xi, \Gamma^{(h)}_{\tau^*}) \leq d(\xi, C^{(h)}_{\tau^*,\pm})$. Using the fact that $R_0 > \bar{R}_0$, we have then the following possibilities:

i) $d(\xi, C^{(h)}_{\tau^*,\pm}) > \bar{R}_0 \lambda |\log \lambda|$; $d(\xi, \Gamma^{(h)}_{\tau^*}) > R_0 \lambda |\log \lambda|$.
ii) $d(\xi, C^{(h)}_{\tau^*,\pm}) > \bar{R}_0 \lambda |\log \lambda|$; $d(\xi, \Gamma^{(h)}_{\tau^*}) \leq R_0 \lambda |\log \lambda|$.
iii) $-R_0 \lambda |\log \lambda| \leq d(\xi, \Gamma^{(h)}_{\tau^*}) \leq d(\xi, C^{(h)}_{\tau^*,\pm}) \leq \bar{R}_0 \lambda |\log \lambda|$.
iv) $d(\xi, C^{(h)}_{\tau^*,\pm}) \geq -\bar{R}_0 \lambda |\log \lambda|$; $d(\xi, \Gamma^{(h)}_{\tau^*}) < -R_0 \lambda |\log \lambda|$.
v) $d(\xi, C^{(h)}_{\tau^*,\pm}) < -\bar{R}_0 \lambda |\log \lambda|$; $d(\xi, \Gamma^{(h)}_{\tau^*}) < -R_0 \lambda |\log \lambda|$.

In the case iii), inequality (3.35) follows from the monotonicity of $\bar{m}$. In cases i) and v) it follows from the inequality $\pm m_\beta + \lambda^{3/2} \leq \pm m_\beta + \lambda^{3/2-\gamma}$ (λ small enough). Finally, in the cases ii) and iv) we use the inequalities:

$$\begin{cases} \bar{m}(\lambda^{-1} d(\xi, \Gamma^{(h)}_{\tau^*})) \leq \bar{m}(R_0 |\log \lambda|) \leq m_\beta + A\lambda^{\alpha R_0} \leq m_\beta + \lambda^{3/2-\gamma} \\ \bar{m}(\lambda^{-1} d(\xi, C^{(h)}_{\tau^*,\pm})) \geq \bar{m}(-\bar{R}_0 |\log \lambda|) \geq -m_\beta + B\lambda^{\alpha \bar{R}_0} \geq -m_\beta + \lambda^{3/2} \end{cases} \tag{3.36}$$

with A, B suitable constants and λ small. The inequalities (3.36) follow from the monotonicity and the convergence property (2.12a) of $\bar{m}$, for the choice of parameters (3.23) and (3.28). □

In the next section we will prove the following lemma.

LEMMA 3.8. *For $k \in \mathbb{Z}_+$, let $t_k = \lambda^{-2}\tau^\star + kT$, with $T = \lambda^{-\delta}$ as in (3.25). Let $m^{\pm}_{(k)}(r,t)$ be the solution of (2.1) for $t \geq t_k$ with initial datum $m^{\pm}_{(k)}(r, t_k) = m^\star_\pm(\lambda r, \lambda^2 t_k)$ for all $r \in \mathbb{R}^2$. Then, for all λ small enough,*

$$m^{\pm}_{(k)}(r, t_{k+1}) \leq m^\star_\pm(\lambda r, \lambda^2 t_{k+1}) \qquad \forall r \in \mathbb{R}^2 \tag{3.37}$$

We conclude this section by proving (3.29) using Lemmas 3.7, 3.8 and the following theorem, proved in [14].

THE COMPARISON THEOREM. *Let $u(r,t)$ and $v(r,t)$ be two solutions of (2.1) for $t \geq t_0$, such that $u(\cdot, t_0) \geq v(\cdot, t_0)$. Then $u(\cdot, t) \geq v(\cdot, t)$ for all $t \geq t_0$.*

By Lemma 3.7 and the Comparison Theorem we have

$$m^{(\lambda)}(\xi, \tau_1) \leq m^{\pm}_{(0)}(r, t_1)$$

where $\tau_1 = \lambda^2 t_1 = \tau^\star + \lambda^{2-\delta}$. Using now Lemma 3.8 we obtain

$$m^{(\lambda)}(\xi, \tau_1) \leq m^\star_\pm(\xi, \tau_1) \tag{3.38}$$

We can now repeat our estimates, starting at time τ_1. By iteration, we have

$$m^{(\lambda)}(\xi,\tau_k) \le m^\star_\pm(\xi,\tau_k) \qquad \forall\, k \in \mathbb{Z}_+ \tag{3.39}$$

where $\tau_k = \tau^\star + k\lambda^{2-\delta}$. As it will be clear in the proof of Lemma 3.8, (3.37) holds also if we replace T with χT for any $1 \le \chi \le 2$; then (3.39) holds for any time $\tau \ge \tau_1$, that is (3.29) holds. □

3.4 Proof of Lemma 3.8 via localization.

We prove (3.37) for the (+)-supersolution and for $k = 0$; in fact the proof for $k > 0$ is exactly the same, while, by symmetry, (3.37) follows for the (−)-supersolution. We prove that, for λ sufficiently small,

$$m_0^+(r,t_1) \le m^\star_+(\lambda r, \lambda^2 t_1) \qquad \forall\, r \in \mathbb{R}^2 \tag{3.40}$$

The proof of (3.40) is very similar to the analogous inequality for the supersolution $m^\star(\xi,\tau)$ in [12]; in fact, we will be able to make use of some basic estimates given there.

As just mentioned, the idea is to localize the analysis. First of all we need the following lemma, proved in [14].

THE BARRIER LEMMA. *There exist V and c_1, both positive, such that, if $u(r,t)$ and $v(r,t)$ solve (2.1) and $u(r,0) = v(r,0)$ for all $|r| \le VS$, then*

$$|u(0,S) - v(0,S)| \le c_1 e^{-S}$$

To simplify notation, we translate times such that $\tau^\star = 0$; then (3.40) can be rewritten as

$$m_+(r,T) \le \tilde{m}(r,T) \qquad \forall\, r \in \mathbb{R}^2 \tag{3.41}$$

where $m_+(r,t)$ solves (2.1) with initial datum $\tilde{m}(r,0)$ and

$$\tilde{m}(r,t) = \begin{cases} \bar{m}(d(r,\lambda^{-1}C_{\lambda^2 t})) & \text{if} \quad |d(r,\lambda^{-1}C_{\lambda^2 t})| \le \bar{R}_0 |\log\lambda| \\ \operatorname{sgn}(d(r,\lambda^{-1}C_{\lambda^2 t}))m_\beta + \lambda^{3/2-\gamma} & \text{if} \quad |d(r,\lambda^{-1}C_{\lambda^2 t})| > \bar{R}_0 |\log\lambda| \end{cases} \tag{3.42}$$

where $\lambda^{-1}C_{\lambda^2 t} = \lambda^{-1}C(-v_h^\star \lambda^2 t\hat{\xi}_1, R^\star + ch) = C(-v_h^\star \lambda t\hat{x}_1, \lambda^{-1}(R^\star + ch))$, with $\hat{x}_1$ the unit vector along the x_1-axis in the "mesoscopic space". We denote by $B_{\lambda^2 t}$ the ball of boundary $C_{\lambda^2 t}$. We verify (3.41) separately in the two regions:

$$A = \{\, r \in \mathbb{R}^2 \,|\, |d(r,\lambda^{-1}C_0)| \le 2VT \,\}; \qquad \bar{A} = \{\, r \in \mathbb{R}^2 \,|\, |d(r,\lambda^{-1}C_0)| \ge 2VT \,\}$$

a) Estimate away from the interface. We suppose $r \in \lambda^{-1}B_0 \cap \bar{A}$; for λ small enough $VT > \bar{R}_0|\log\lambda|$ so that

$$m_+(\tilde{r},0) = \tilde{m}(\tilde{r},0) = m_\beta + \lambda^{3/2-\gamma} \qquad \forall \tilde{r} \ : \ |\tilde{r}-r| \le VT$$

From the Barrier Lemma it follows that

$$|m_+(r,T) - m(T)| \le c_1 e^{-T} \tag{3.43}$$

where $m(t)$ solves

$$\begin{cases} \dot{m}(t) = -m(t) + \tanh\{\beta m(t)\} \\ m(0) = m_\beta + \lambda^{3/2-\gamma} \end{cases} \tag{3.44}$$

But (3.44) implies that there exist a' and b', both positive, such that

$$|m(t) - m_\beta| \le a' e^{-b' t} \tag{3.45}$$

From (3.43) and (3.45) we finally obtain

$$|m_+(r,T) - m_\beta| \le a' e^{-b'T} + c_1 e^{-T} \le \lambda^{3/2-\gamma} \tag{3.46}$$

for λ small enough. On the other hand the displacement of $\lambda^{-1}C_{\lambda^2 t}$ in the time T is of order $\lambda^{1-\delta}$; therefore, for λ sufficiently small,

$$|d(r,\lambda^{-1}C_{\lambda^2 T})| \ge |d(r,\lambda^{-1}C_0)| - \text{const.}\lambda^{1-\delta} > \bar{R}_0|\log\lambda|$$

so that $\tilde{m}(r,T) = m_\beta + \lambda^{3/2-\gamma}$ for $r \in \lambda^{-1}B_0 \cap \bar{A}$, and then, by (3.46), $m_+(r,T) \le \tilde{m}(r,T)$. In the same manner we work in the case $r \in (\mathbb{R}^2 \backslash \lambda^{-1}B_0) \cap \bar{A}$.

b) Estimates close to the interface. In order to prove

$$m_+(r,T) \le \tilde{m}(r,T) \qquad \forall r \in A \qquad \lambda \text{ small} \tag{3.47}$$

we look for an upper bound of $m_+(r,T)$ and a lower bound of $\tilde{m}(r,T)$.

i) Upper bound of $m_+(r,T)$. In [12] it is proven that

$$m_{(0)}(r,T) \le G(d(r,\lambda^{-1}\Gamma_0);\lambda) \qquad \forall r \ : \ |d(r,\lambda^{-1}\Gamma_0)| \le 3VT \tag{3.48}$$

for λ sufficiently small, with

$$G(x;\lambda) = \bar{m}(x) + \bar{m}'(x)[-\frac{\lambda}{R}T\theta + \lambda C(|x|+1) + c_2\lambda^2 T^5]$$
$$+ [\bar{c}_1 e^{-T} + c_3\lambda^{3/2+\alpha R_0} + c_4\lambda^3 T + c_5\lambda^3 T^{10}]$$

where R^{-1} is the curvature of Γ_0 in any point ξ' such that $d(r, \lambda^{-1}\Gamma_0) = d(r, \lambda^{-1}\xi')$; $\bar{c}_1$, c_2, c_3, c_4, c_5 and C are suitable positive constants. In (3.48) $m_{(0)}(r,t)$ is the solution of (2.1) with initial datum

$$m^\star(\lambda r, 0) = \begin{cases} \bar{m}(d(r, \lambda^{-1}\Gamma_0)) & \text{if} \quad |d(r, \lambda^{-1}\Gamma_0)| \le R_0|\log\lambda| \\ \operatorname{sgn}(d(r, \lambda^{-1}\Gamma_0))m_\beta + \lambda^{3/2} & \text{if} \quad |d(r, \lambda^{-1}\Gamma_0)| > R_0|\log\lambda| \end{cases} \tag{3.49}$$

(see def. (3.24)). In our case $m_+(r,t)$ has initial datum (3.42) with $\bar{R}_0$ such that (3.28) holds. By looking at the proof of (3.48) in [12] it is easy to see that analogous estimates hold for $m_+(r,t)$:

$$m_+(r,t) \le \tilde{G}(x;\lambda) \qquad \forall r \; : \; |x| \le 3VT \tag{3.50}$$

for all λ small enough, where $x \doteq d(r, \lambda^{-1}C_0))$ and

$$\begin{aligned} \tilde{G}(x;\lambda) = \bar{m}(x) + \bar{m}'(x)[-\frac{\lambda}{R}T\theta + \lambda C(|x|+1) + c_2\lambda^2T^5] \\ + [\bar{c}_1 e^{-T} + c_3\lambda^{3/2-\gamma+\alpha\bar{R}_0} + c_4\lambda^3 T + c_5\lambda^3T^{10}] \end{aligned} \tag{3.51}$$

possibly with different values of the constants. We point out that in this case $R = R^\star + ch$ for all r.

ii) Lower bound of $\bar{m}(r,T)$. Now the situation is somewhat different from [12]. In fact in [12] $m^\star(\xi,\tau)$ is constructed by moving Γ_0 with the h-biased motion by mean curvature, while here we construct $m^\star_+(\xi,\tau)$ by moving C_0 with constant velocity.

We note that there exists a positive constant c_6 such that

$$|d(r, \lambda^{-1}C_{\lambda^2T}) - x + v^\star_h T\lambda\cos\beta| \le c_6T^2\lambda^2 \qquad \forall r \; : \; |x| \le \bar{R}_0|\log\lambda| + 1 \tag{3.52}$$

where $\beta = \beta(r)$ is defined by the conditions

$$\cos\beta = \frac{r\cdot\hat{x}_1}{|r|}; \qquad 0 \le \beta < \pi \tag{3.53}$$

(notice that for small λ, $d(0, \lambda^{-1}C_0) \gg \bar{R}_0|\log\lambda|$ and then (3.53) is well defined for all r such that $|x| \le \bar{R}_0|\log\lambda| + 1$). The estimate (3.52) can be easily proved expanding the obvious relation

$$d(r, \lambda^{-1}C_{\lambda^2T}) = \lambda^{-1}R - \sqrt{r^2 + 2v^\star_hT\lambda|r|\cos\beta + (v^\star_h)^2T^2\lambda^2}$$

at $x = \lambda^{-1}R - |r|$ and using the condition on $|x|$ to bound the derivatives.

By Taylor expansion and using (3.52), we obtain

$$\bar{m}(d(r,\lambda^{-1}C_{\lambda^2 T})) \geq \bar{m}(x) - \bar{m}'(x)\left[\left(\frac{\theta}{R} - h\right)\lambda T\cos\beta + c_6\lambda^2 T^2\right] \\ - c_7\lambda^2 T^2 \bar{m}''(x) - c_8\lambda^3 T^3 \qquad \forall r \ : \ |x| \leq \bar{R}_0|\log\lambda| + 1 \tag{3.54}$$

for suitable positive constants c_7, c_8 (we also used the definition of $v_h^\star$). Moreover, again by (3.52), for λ sufficiently small, it holds

$$\begin{cases} |x| > \bar{R}_0|\log\lambda| + 1 & \Longrightarrow \quad |d(r,\lambda^{-1}C_{\lambda^2 T})| > \bar{R}_0|\log\lambda| \\ |x| \leq \bar{R}_0|\log\lambda| - 1 & \Longrightarrow \quad |d(r,\lambda^{-1}C_{\lambda^2 T})| \leq \bar{R}_0|\log\lambda| \end{cases} \tag{3.55}$$

Finally we note that, since $\alpha\bar{R}_0 > 3/2 - \gamma$, (2.12b) implies

$$\lim_{\lambda\to 0} \lambda^{-3/2+\gamma}\bar{m}'\big(\bar{R}_0|\log\lambda| - 1\big) = 0 \tag{3.56}$$

By (3.54), (3.55), (3.56) and definition (3.42), we obtain the required lower bound:

$$\bar{m}(r,T) \geq D(x;\lambda) \qquad \lambda \text{ small} \tag{3.57}$$

where

$$D(x;\lambda) = \left\{\bar{m}(x) - \bar{m}'(x)\left[\left(\frac{\theta}{R} - h\right)\lambda T\cos\beta + c_6\lambda^2 T^2\right] - c_7\lambda^2 T^2\bar{m}''(x) - c_8\lambda^3 T^3\right\} \\ \times \mathbf{1}(|x| \leq \bar{R}_0|\log\lambda| + 1) \\ + [m_\beta \mathrm{sgn}(x) + \lambda^{3/2-\gamma}]\mathbf{1}(|x| > \bar{R}_0|\log\lambda| + 1)$$

where $\mathbf{1}(\cdot)$ is the characteristic function. By (3.50) and (3.57), the estimate (3.47) is proven if

$$\tilde{G}(x;\lambda) \leq D(x;\lambda) \qquad \forall\, |x| \leq 2VT \tag{3.58}$$

for all λ small enough. We analyze the two different cases:

i) $|x| \leq \bar{R}_0|\log\lambda| + 1$: We need that, for small λ,

$$\tilde{G}(x;\lambda) \leq \bar{m}(x) - \bar{m}'(x)\left[\left(\frac{\theta}{R} - h\right)\lambda T\cos\beta + c_6\lambda^2 T^2\right] \\ - c_7\lambda^2 T^2\bar{m}''(x) - c_8\lambda^3 T^3$$

which holds if

$$(1-\cos\beta)\frac{\theta}{R}\lambda T + h\lambda T\cos\beta \geq \lambda C(|x|+1) + c_2\lambda^2 T^5 + c_6\lambda^2 T^2 \\ + c_7\lambda^2 T^2\frac{\bar{m}''(x)}{\bar{m}'(x)} + \frac{1}{\bar{m}'(x)}c_9\lambda^3 T^{10} \tag{3.59}$$

since, for all λ small enough, there exists $c_9 > 0$ such that

$$\bar{c}_1 e^{-T} + c_3 \lambda^{3/2-\gamma+\alpha\bar{R}_0} + c_4 \lambda^3 T + c_5 \lambda^3 T^{10} + c_8 \lambda^3 T^3 \le c_9 \lambda^3 T^{10}$$

(in fact (3.28) implies $3/2 - \gamma + \alpha\bar{R}_0 > 3 - 2\gamma > 3 - 10\delta$). By (2.12) there exists $c > 0$ such that $\bar{m}''(x) \le c\bar{m}'(x)$; since $|x| \le \bar{R}_0 |\log\lambda| + 1$, (3.59) is then implied by

$$(1 - \cos\beta)\frac{\theta}{R}\lambda T + h\lambda T \cos\beta \ge \lambda C(2 + \bar{R}_0 |\log\lambda|) + c_2 \lambda^2 T^5 + c_6 \lambda^2 T^2 + c c_7 \lambda^2 T^2 + c_9 c' \lambda^3 T^{10} \tag{3.60}$$

where c' is a positive constant such that $\bar{m}'(\bar{R}_0|\log\lambda| + 1)^{-1} \le c'\lambda^{-\alpha\bar{R}_0}$, which existence follows from (2.12b). The validity of (3.60) for small λ is guaranteed by our choice of δ and $\bar{R}_0$ in (3.28).

ii) $\bar{R}_0|\log\lambda| + 1 < |x| \le 2VT$: We need, for all λ sufficiently small,

$$m_\beta \mathrm{sgn}(x) + \lambda^{3/2-\gamma} \ge \tilde{G}(x;\lambda)$$

which is implied by

$$\lambda^{3/2-\gamma} \ge (\bar{m}(x) - \mathrm{sgn}(x) m_\beta) + c''\lambda^{1-\delta}\bar{m}'(x) + \lambda\bar{m}'(x)C(1 + |x|) + c'''\lambda^{3-10\delta} \tag{3.61}$$

for suitable positive constants c'' and c'''. By using the properties (2.12) of the instanton $\bar{m}$, it is easy to see that (3.61) is true if it holds

$$\lambda^{3/2-\gamma} \ge b_1 \lambda^{\alpha\bar{R}_0} + b_2 \lambda^{\alpha\bar{R}_0+1-\delta} + b_3 \lambda^{1+\alpha\bar{R}_0}(\bar{R}_0|\log\lambda| + 1) + c'''\lambda^{3-10\delta} \tag{3.62}$$

for some positive constants b_1, b_2 and b_3. But (3.62) is guaranteed, for all sufficiently small λ, by (3.28) and (3.23). □

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