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# Geometry and dynamics in quotient spaces of $\mathrm{PSL}_2(\mathbb{C})$

Thesis submitted for the degree of
"Magister Philosophiæ"

CANDIDATE
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In a celebrating paper, contained in a volume on "the mathematical heritage of Henri Poincaré", William Thurston raised some questions and projects concerning 3-manifolds and Klenian groups [**Th**]; the $19^{th}$ was:

*Find topological and geometric properties of quotient spaces of arithmetic subgroups of* $PSL_2(\mathbb{C})$. *These manifolds often seem to have special beauty.*

I have recently become acquainted with a work by Alberto Verjovsky [**Ve**] on the quotient space $PSL_2(\mathbb{R})/PSL_2(\mathbb{Z})$, a fascinating interplay between topological dynamics, geometry and arithmetic, where a lucid geometrical viewpoint throws new light in old material dealing with horocycle flows on non-compact homogeneous surfaces, their ergodic properties, convergence of ergodic measures and number theory, namely the Riemann hypothesis in the case of the modular group (Dani, Zagier, Sarnak,...).

Geodesic and horocycle flows on compact surfaces of constant negative curvature are very classical problems in topological dynamics, mainly thanks to the work by Hedlund. Both have a geometrical description, they are motions of unit tangent vectors along lines of constant geodesic curvature (zero in the geodesic case, and maximal without being closed in the horocycle case), and a group-theoretic description, actions of non-compact one-parameter subgroups of $PSL_2(\mathbb{R})$ on $PSL_2(\mathbb{R})/\Gamma$, where $\Gamma$ is a co-compact disctrete subgroup (since the above is the group of orientation preserving isometries of the hyperbolic two-dimensional space). Properties a dynamic systemist is interested on are: ergodicity and mixing for the geodesic flow (since it is Anosov), minimality and unique ergodicity for the horocycle flow. In the case of co-compact discrete subgroups of $PSL_2(\mathbb{R})$, the unique ergodicity has been shown by Furstenberg, by reducing the proof to some $L^2$-estimates of harmonic functions in the unit disk (a model for the hyperbolic plane $\mathbb{H} = SL_2(\mathbb{R})/SO(2)$), connected to ergodic measures invariant under the horocycle flow. In the non-compact case, the horocycle flow cannot be uniquely ergodic, since there exist proper closed subgroups $H$ containing horocycle subgroups, having closed orbits which support finite $H$-invariant measures. This fact is quite general, in our case it is easily seen that for any cusp of the non-compact orbifold $\mathbb{H}/\Gamma$, there exist a one parameter family of closed orbits, each one supporting a probability measure clearly flow-invariant and ergodic. A work by Dani shows that for reasonable groups these measures constitute the all collection of ergodic invariant measures, thus determining the cone of finite invariant measures. The quotient $\mathbb{H}/PSL_2(\mathbb{Z})$, the modular orbifold, contains only the standard cusp at infinity, preserved by the obvious parabolic subgroup of the modular group, translations by integers in the real axis of the upper half plane model for $\mathbb{H}$; there is thus a curve $\{m_y\}_{y>0}$ of ergodic probability measures preserved by the flow, each one supported on a closed orbit of period $y^{-1}$. As the period increases, measures converge weakly to the normalizad Haar measure $\overline{m}$ on $PSL_2(\mathbb{R})/PSL_2(\mathbb{Z})$, and the rate of approach of mean values of smooth functions with compact support is intimately connected with the Riemann hypothesis (this is not so surprising since the modular group contains informations about prime numbers).

All that can be understood by easy geometric constructions, relating the analysis of convergence, the way closed orbits fill the quotient space as the period increases, to lattice point counting, by means of looking at particular characteristic functions on the group.

The group $SL_2(\mathbb{R})$ is also a prototype for what are called homogeneous contact manifolds (it has been shown that the only semisimple groups endowed with a left-invariant contact structure are locally isomorphic with either $SL_2(\mathbb{R})$ or $SO(3)$). Its quotients by co-compact discrete subgroups furnish examples of non-regular compact contact manifolds, i.e. contact manifolds which cannot be seen as $S^1$-bundles over the space of orbits of the characteristic vector field. Indeed, the natural left-invariant contact structure on $SL_2(\mathbb{R})$, gives rise to the characteristic vector field which generates the geodesic flow; in that case it is a topologically transitive Anosov flow, containing dense orbits and a countable number of periodic orbits, too, whose union is dense. Note that orbits of the horocycle flow are legendrian curves, as well as orbits of the $SO(2)$ right action. Perhaps, it would be interesting to investigate for some Bennequin type classification of these contact sructures.

What seems interesting, is to look for an extension of part of this job to the complex case, where the group is the group of isometries of the three-dimensional hyperbolic space, e.g. for the quotient of $SL_2(\mathbb{C})$ by the Picard group $SL_2(\mathbb{Z}[i])$ or its subgroups.

One good picture of the hyperbolic three-dimensional space is the Poincarè upper half space, $\mathbb{H}^3 = \{(z,t)\ ;\ z \in \mathbb{C}, t \in \mathbb{R}, t > 0\}$, and the group of orientation preserving isometries is $PSL_2(\mathbb{C})$; the action on $\mathbb{H}^3$ may be defined as fractional linear transformations, using quaternionic notations, $q = z + jt$, the formula is

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad : q \mapsto (aq+b)(cq+d)^{-1}$$

The stabilizer of a point is $SO(3)$, thus, as a manifold, $PSL_2(\mathbb{C})$ is the orthonormal frame bundle of $\mathbb{H}^3$. The geodesic flow is now a holomorphic action of $\mathbb{C}^*$, and it can still given the meaning of "holomorphic Anosov flow" [**Gh**], stable and unstable foliations being defined by two holomorphic actions of the complex affine group $Aff(\mathbb{C})$. The geodesic flow comes from the action of homotheties $\mathbb{C}^*$, call $X$ the associated holomorphic vector field; the translation group $\mathbb{C}$ gives rise to the holomorphic vector fields $Y$ and $Z$ generating the strictly unstable and stable foliations, with the obvious commutation relations is $sl_2(\mathbb{C})$.

Take, for instance, the Picard group $\Gamma = SL_2(\mathbb{Z}[i])$; it is generated by

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix};$$

and a fundamental domain for it in $\mathbb{H}^3$ is

$$F = \{x + iy + jt \ ; \ |x| < 1/2, \ 0 \leq y \leq 1/2, \ |z|^2 + t^2 \geq 1\}$$

The quotient $\mathbb{H}^3/\Gamma$ is non-compact but of finite volume, it has only one cusp, the standard one at infinity, and a horosphere near the end, the surface orthogonal to the ray of geodesics from the infinity, is a flat torus $T^2$. The action of the geodesic flow, gives rise to an embedded $\mathbb{C}^* \times T^2$ in $\Gamma \backslash PSL_2(\mathbb{C})$ (coming from a leaf of the unstable foliation), and one would like to investigate the pattern, the way it fills the homogeneous space.

All that, again, can be viewed as a complex version af a left invariant contact structure, since there exists the holomorphic one form $\omega$, vanishing on the strictly stable and unstable vactor fields, taking value one on the generator of the geodesic flow, and such that $\omega \wedge d\omega$ is non-singular. The above tori are "legendrian" holomorphic tori, i.e. elliptic curves.

By the way, the relation with three-manifolds is evident, and we plan to find connections with those manifolds coming from knots in $S^3$ (like the classical "eight knot" example due to Riley), and Thurston's Dehn surgery.

What follows, is a sketch of the situation in the case of the modular group.

The group $SL_2(\mathbb{R})$ acts transitively by isometries on the two-dimensional hyperbolic space $\mathbb{H} = \{z = x + iy \in \mathbb{C}, y > 0\}$, the metric being $(dx^2 + dy^2)y^{-2}$, by means of fractional linear transformations

$$g \cdot z = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = (az + b)/(cz + d)$$

The kernel of this action is $\mathbb{Z}_2 = \{\pm 1\}$, thus it can be easily seen that $PSL_2(\mathbb{R}) = SL_2(\mathbb{R})/\mathbb{Z}_2$ is the full group of orientation preserving isometries of $\mathbb{H}$. Indeed, as the stability group of $i \in \mathbb{H}$ is $SO(2)$, one can realize $PSL_2(\mathbb{R})$ as the unit tangent bundle of $\mathbb{H}$, say $S\mathbb{H} \simeq \mathbb{H} \times S^1$, by means of its action and its differential in a fixed point of $S\mathbb{H}$: i.e.

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto (g \cdot i, 2\beta_g(i))$$

where $\beta_g(z) = \arg(cz + d)$.

A basis for the Lie algebra $sl_2(\mathbb{R})$ is

$$x = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \; y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} , \; z = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} ;$$

the corresponding left-invariant vector fields, $X, Y$ and $Z$, induce the flows

$$g_t(g) = g \cdot \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}$$

$$h_t^+(g) = g \cdot \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

$$h_t^-(g) = g \cdot \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} .$$

They are called geodesic flow, unstable horocycle flow and stable horocycle flow respectively. In fact, $g_t$ describe the motion of a unit tangent vector in $\mathbb{H}$ along an oriented geodesic $\gamma$ starting from $z \in \mathbb{H}$ (straight $\{x = \text{constant}\}$ lines or semicircles orthogonal to the real axis), $h^+$ and $h^-$ describe the motion of the vector along the two oriented horocycles from $z$, orthogonal to $\gamma$ ($\{y = \text{constant}\}$ lines or circles tangent to the real axis).

The group $SL_2(\mathbb{R})$ is endowed with the standard left-invariant riemannian metric such that $\{X, Y, Z\}$ describe an orthonormal frame, and, since it is unimodular, it has the standard bi-invariant Haar measure $m$ induced by the volume form which takes constant value one in the above framing.

Let us recall the geometrical meaning of Iwasawa's decomposition $SL_2(\mathbb{R}) = \mathcal{N}\mathcal{A}\mathcal{K}$, where $\mathcal{N} = \left\{ \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right), x \in \mathbb{R} \right\}$ in the nilpotent group that generates the horocycle flow by right action, $\mathcal{A} = \left\{ \left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right), a > 0 \right\}$ the diagonal group that generates the geodesic flow, $\mathcal{K} = SO(2) = \left\{ \left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right), \theta \in S^1 \right\}$ the compact circle group. The geodesic flow leaves invariant the splitting $T(SL_2(\mathbb{R})) = E^+ \oplus E^- \oplus E$, where $E^+, E^-, E$ are the line bundles spanned respectively by $Y, Z$ and $X \in sl_2(\mathbb{R})$. The circle group, generated by $W = Y - Z$, has a free action on $SL_2(\mathbb{R})$; the foliations $\mathcal{F}^+$ and $\mathcal{F}^-$, defined by $E^+ \oplus E$ and $E^- \oplus E$, are transverse to $W$, and every leaf intersects each orbit of $W$ in exactly one point.

It is easily seen that both the geodesic and the horocycle flow preserve the Haar measure; moreover, from the commutation relations in the algebra, we can see that $g_t$ dilates the vectors in $E^+$ and contract those in $E^-$, hence it is an Anosov flow: its unstable and stable foliations are $\mathcal{F}^+$ and $\mathcal{F}^-$ respectively. Indeed, these foliations are nothing but left translations of two copies of the affine group $A_2(\mathbb{R}) = \{\lambda : \mathbb{R} \to \mathbb{R} \; ; \; \lambda(r) = ar + b \; , \; a, b \in$

$\mathbb{R}$ , $a > 0\}$ contained in $SL_2(\mathbb{R})$ (note that the flip map sends orbits of $h^+$ to orbits of $h^-$, thus they have the same ergodic properties).

What Furstenberg showed **[Fu]**, is that for co-compact subgroups $\Gamma$, the flow $h^+$ is uniquely ergodic on $\Gamma\backslash SL_2(\mathbb{R})$, with $m$ as unique invariant measure, and the proof, by a standard argument in the theory of measures on homogeneous spaces, amount to show that the natural action of $\Gamma$ on $SL_2(\mathbb{R})/\mathcal{N} \simeq \mathbb{R}^2 - \{0\}$ is uniquely ergodic w.r.t. the Lebesgue measure; this result also yelds minimality of the flow. When the subgroup is not co-compact, but still the quotient space has finite volume (non-uniform lattices), $m$ is again an ergodic invariant measure for both flows, but $h^+$ is no more minimal, neither uniquely ergodic.

$\Gamma\backslash\mathbb{H} = \Gamma\backslash SL_2(\mathbb{R})/SO(2)$ is a non-compact, complete, hyperbolic orbifold of finite area; it is obtained identifying sides of an hyperbolic polygon, whose vertices are conical points, associated to elliptic elements of $\Gamma$ and labelled by a rational number $p/q$ ( $2\pi p/q$ being the angle at which two equal sides have to meet around that point), and at least one cusp, points at infinity associated with parabolic elements (whose angle is zero). These surfaces, compactified, at least in the case of subgroups of the modular group, can be given the structure of a Riemann surface; thus genus, area, cusps and conical points are related by a Gauss-Bonnet formula.

The modular orbifold (i.e. $\Gamma = SL_2(\mathbb{Z})$), has the standard cusp at infinity, whose stabilizer is

$$\Gamma \supset \Gamma_\infty = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \ ; n \in \mathbb{Z} \right\}$$

and two conical points of indexes $1/2$ and $1/3$. A fundamental domain $F$ in the half-space model for $\mathbb{H}$ is easily seen to be the intersection of the strip $\{\|\Re(z)\| < \frac{1}{2}\}$ with $\{\|z\| > 1\}$, since we know that $PSL_2(\mathbb{Z})$ is generated by

$$z \mapsto -1/z \quad , \qquad z \mapsto z+1$$

The area of the fundamental domain (an hyperbolic triangle with angles $\pi/3, \pi/3$ and $0$), is $\pi/3$, and the volume of $PSL_2(\mathbb{Z})\backslash PSL_2(\mathbb{R})$ is $\pi^2/3$.

Let us fix the notation and call:

$$SM \doteq PSL_2(\mathbb{Z})\backslash PSL_2(\mathbb{R}) \ , \ M \doteq PSL_2(\mathbb{Z})\backslash\mathbb{H}$$

Horocycles perpendicular to the ray of geodesics from the infinity are the straight lines with constant imaginary part, hence associated to the standard cusp there is a one parameter family of closed orbits, namely the collection of the $C_y = \Gamma_\infty\backslash\mathbb{R} \times \{iy\}$, where $y > 0$ and "$y^{-1}$" is the period.

It has been shown by Sarnak, in the more general case where the orbifold has more than one cusp, each one equivalent by coniugation in the group to the standard one, that this family defines an embedded cylinder in $SM$, say $P_\infty$

$$S^1 \times \mathbb{R}_+ \hookrightarrow SM$$

Then $P_\infty$ is the set of periodic points of the flow, and is dense in $SM$. What happens, is that the curves $C_y$ are embedded in $M$ until the period is small enough, i.e. near the cusp; as the period grows they start filling up $M$ in a very uniform way, in the sense that for every open subset $U \subset M$,

$$\frac{\text{lenght}(C_y \cap U)}{\text{lenght}(C_y)} \sim \frac{\text{area}(U)}{\text{area}(M)} \qquad \text{as} \quad y \to 0^+$$

Known results are the following.

Call $\tilde{SM}$ the one-point compactification of $SM$, $\delta_\infty$ the Dirac measure on the cusp, $m_y$ the Borel probability measure which puts uniform mass (w.r.t. the arclenght) on the closed orbit $C_y$, and $\bar{m} = \frac{3}{\pi^2}m$ the normalized Haar measure on $SH$. Obviously these are all ergodic invariant measures; moreover Dani showed that $m_y$ converges weakly to $\delta_\infty$ as $y \to \infty$ and to $\bar{m}$ as $y \to 0$. If $B = C(\tilde{SM})$ is the Banach space of continuous real valued functions on $\tilde{SM}$, and $B^*$ its topological dual endowed with the weak$^*$-topology, the result by Dani is [**Da**]

**Dani's theorem:** *the measures $m_y$ converges to $\bar{m}$ in the weak$^*$-topology as $y \to 0$ ; $\{m_y\}$ and $\bar{m}$ exhaust the collection of ergodic measures of the horocycle flow.*

It was Zagier [**Za**] who realized the connection between the rate of approach of the mean value of a function $f \in C_0^\infty(SM)$ (i.e. smooth with compact support) w.r.t. $m_y$

$$< m_y, f > = \frac{1}{t}\int_0^t f(h_r^+(g))dr = \int_0^1 f(x+iy,0)dx$$

(where $g \in C_y$, $t$ is the period of the $h^+$-orbit and we are in the trivialization in which $PSL_2(\mathbb{R}) \simeq \mathbb{H} \times S^1 \ni (x+iy,\theta)$, and $f$ can be viewed as an automorphic function on $S\mathbb{H}$) to $< \bar{m}, f >$, and the Riemann hypothesis. Sarnak generalized this work in the case of arbitrary non-uniform lattices [**Sa**]; his technology consists in forming a Mellin type transform of the measure $m_y$

$$E(s) = \int_0^\infty m_y y^{s-2} dy \qquad \Re(s) > 1$$

and relate $E(s)$ to some classical Eisenstein series (continuous spectrum eigenfunctions of the laplacian on $\mathbb{H}/\Gamma$). Let

$$G_f(s) \doteq < E(s), f > = \int_0^\infty \int_0^1 f(z,0) y^s \, d\mu(z)$$

where $d\mu(z)$ if the area form in $\mathbb{H}$, $(dx\, dy)/y^2$, and let's expand $f$ in Fourier series w.r.t. the argument in $S^1$

$$f(z,0) = \sum_{n\in\mathbb{Z}} \hat{f}_n(z)$$

$$\hat{f}_n(z) = \frac{1}{2\pi}\int_0^{2\pi} f(z,\theta) e^{-in\theta} \, d\theta$$

A computation shows that for any $\gamma \in \Gamma$

$$\hat{f}(n, \gamma \cdot z) = (\varepsilon_\gamma)^{2n} \hat{f}(n,z)$$

$$\varepsilon_\gamma(z) \doteq (cz+d)/|cz+d| \qquad for \quad \gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix}$$

If we change variables and move with $\Gamma$ the fundamental domain $F$ to cover all of $\mathbb{H}$, we get

$$G_f(s) = \sum_{n\in\mathbb{Z}} \int_F E_{2n}(z,s) \hat{f}_n(z) d\mu(z)$$

where

$$E_{2n}(z,s) \doteq \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} (\varepsilon_\gamma(z))^{2n} \, y(\gamma(z))^s$$

Properties of such functions $E_{2n}$ are known from classical works by Selberg in the general case; what turns out in the case of the modular group is that $E(s)$ can be meromorphically continued to all of $\mathbb{C}$ as a distribution on $SM$, invariant under the horocycle flow; it satisfies the functional equation

$$E(s) = \Psi(s,\cdot) * E(1-s)$$

where

$$\Psi(s,\theta) = (\sin\theta/2)^{2s-2} \frac{\zeta(2s-1)}{\zeta(2s)}$$

$\zeta$ is the Riemann zeta-function $\zeta(s) = \sum_{n\geq 1} n^{-s}$, and "$*$" means that $\Psi$ operates on functions by convolution w.r.t. the $\theta$ variable. Thus $G_f$ is meromorphic in $\mathbb{C}$, with no singularities in $\Re(s) > 1/2$, apart from the simple pole at $s = 1$ with residue $< \bar{m}, f >$. A Mellin inversion formula and classical estimates show that if

$$\xi = \sup\{\Re(s)\ ,\ \zeta(s) = 0\}$$

then

$$< m_y, f >= \frac{1}{2\pi i}\int_{\xi-i\infty}^{\xi+i\infty} G_f(s/2)y^{1-s/2}ds$$

Thus $< m_y, f >=< \bar{m}, f > +o(y^{1-\xi/2-\epsilon})$ for any $\epsilon > 0$. In other words, we have

**Zagier's theorem:** *let $f \in C_0^\infty(SM)$, then*

$$< m_y, f >=< m, f > +o(y^{1/2}) \qquad \text{as } y \to 0 \ ;$$

*the error term can be made to be $o(y^{3/4-\epsilon})$ for arbitrary $\epsilon > 0$ if and only if the Riemann hypothesis is true.*

Ideas by Verjovsky give a very simple way to understand these results, and the intimate connection between the way the $C_y$'s fill the modular orbifold and number theory. The key is to choose a particular base for the Borel $\sigma$-algebra of $SM$, and follow the intersection of these sets with the orbit of the horocycle flow as the period increase.

With the aid of the flip map in the above mentioned identification of $SL_2(\mathbb{R}) \simeq S\mathbb{H}$ (we'll shift to the $\mathbb{Z}_2$ quotient at the end just by renormalizing measures) we can identify the stable leaf of $\mathcal{F}^-$ through the identity with the zero section $\mathbb{H} \hookrightarrow S\mathbb{H}$ (it is a copy of the affine group), i.e.

$$L^-(e) = \{h_u^- \circ g_t(e) \ ; \ t, u \in \mathbb{R} \ , \ t > 0\}$$

The idea is to take a family of boxes $B$, which generate the algebra, with a base in that leaf, and fixed "height" $\ell$ (i.e. obtained by acting with $h^+$ on a square in the leaf), in such a way that

$$m(B) = (\text{area of the base}) \cdot \ell$$

Our task is to estimate the numbers $m_y(B)$ as $y \to 0^+$; what is the same, we fix the horocycle with period 1, say $C_1 \subset SM$, and form the horocycles $C_{e^{-t}} \doteq g_t(C_1)$: these orbits will intersect the box $B$ in a certain number $n(t)$ of lines of lenght $\ell$; since the period of $g_t(C_1)$ is $e^t$ (i.e. $y = e^{-t}$)s, and the $m_y$'s are probability measure with support on the closed orbits, what happens is that

$$m_y(B) = n(t)e^{-t}\ell$$

hence, we are left with comparing the behaviour of $n(t)$ for $t$ large with $\bar{m}(B) = \frac{3}{\pi^2}(\text{area})\ell$.

We'll consider boxes from $g \in SL_2(\mathbb{R})$ of the form

$$B_g(a, b, \ell) = \left\{h_u^+ \circ g_t \circ h_v^-(g) \ ; \ |v| < \frac{a}{2}, \ |t| < \frac{b}{2}, \ |u| < \frac{\ell}{2}\right\}$$

Their measure is easily seen to be

$$m\left(B_g(a,b,\ell)\right) = \ell A$$

where $A = a\sinh(b/2)$ is the area of the the central leaf of $\mathcal{F}^-$, called the basis of such a box, namely

$$\left\{ g_t \circ h_v^-(g) \ ; \ |v| < \frac{a}{2},\ |t| < \frac{b}{2} \right\}$$

In order to generate the $\sigma$-algebra, it is enough to consider boxes with basis in the stable leaf $L^-(e)$, and since we want them embedded in $SM$, we'll take basis in the strip $\{z \in \mathbb{H} \ ; \ \Re(z) \in (0,1]\}$. Thus, the base of a generic box will be the set

$$Q = \left\{ |x - x_0| \leq a y_0/2 \ , \ y_0\, e^{-b/2} \leq y \leq y_0\, e^{b/2} \right\}$$

In the above identification, we see that the closed unit horocycle $C_1$ comes from the line $\Lambda_1 = \{(x+i,\pi)\} \subset S\mathbb{H}$, i.e. $C_1 = \Gamma\backslash\Lambda_1$; thus, in order to estimate the measure $m_y = m_{e^{-t}}$, we have to look at the number of intersection points in

$$\Gamma\backslash g_t(\Lambda_0) \cap Q$$

If $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ with $c \neq 0$ (otherwise $\gamma$ would be in $\Gamma_\infty$), since the basis of boxes we're going to consider can be identified with rectangles in $\mathbb{H}$, we see that the only points of intersection with such basis are of the form

$$\frac{a}{c} + i\frac{e^t}{c^2}$$

By the way, if $t = 0$, these are the highest points of the so called Ford disks, disks of unitary hyperbolic area which are images under $PSL_2(\mathbb{Z})$ of the basic horoball $\{z \in \mathbb{H} \ ; \ \Im(z) > 1\}$. The number $n(t)$ is seen to be

$$\begin{aligned} n(t) &= \#\left\{g_t(C_1) \cap Q\right\} \\ &= \#\left\{ a, c \in \mathbb{Z}^+ \ ; \ c \neq 0,\ \{a,c\} = 1,\ \left(\frac{a}{c} + i\frac{e^t}{c^2}\right) \in Q \right\} \end{aligned}$$

where $\{a,c\} \neq 0$ means that $a$ and $c$ are relatively prime. Verjovsky now noted that there exists an area preserving diffeomorphism (up to a factor 2) between the euclidean upper plane $\mathbb{R}^2_+$ and the hyperbolic plane, namely

$$(x,y) \mapsto \left(\frac{x}{y} + \frac{i}{y^2}\right)$$

The preimage of a rectangle in $\mathbb{H}$ is a trapezium whose sides are rays from the origin and $\{y = \text{constant}\}$ lines, and the highest points of the Ford disks are mapped to lattice points $(a, c)$ with $a$ and $c$ relatively prime.

What turns out, is that estimates of $n(t)$ are reduced to lattice point counting in the euclidean plane. Rather long estimates with the use of classical theorems in number theory, lead to

**Verjovsky's theorem:** *there exist open sets $B \subset SM$ and positive constants $K$ depending only on the $B$'s, such that*

$$|m_y(B) - \bar{m}(B)| \leq K y^{1/2} |\log y|$$

*for $y \in (0, 1/2]$; moreover, for every $\beta > 1/2$ we have*

$$\limsup_{y \to 0+} \left( |m_y(B) - \bar{m}(B)|\, y^{-\beta} \right) = \infty.$$

Thus, the exponent 1/2 in Zagier's theorem is "optimal" in some sense, i.e. it cannot be improved if we take characteristic functions of Borel sets (of course, this doesn't disprove the Riemann hypothesis, characteristic functions are not even continuous!).

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