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ON THE FIRST COHOMOLOGY MODULE OF RANK TWO VECTOR BUNDLES ON PROJECTIVE SPACE

Thesis submitted for the degree of
"Magister Philosophiæ"

CANDIDATE
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SUPERVISOR
Prof. Philippe Ellia

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Introduction

The present paper arose in the attempt of solving the following problem, which, as far as we know, is still open:

Problem 1 : Given a rank two vector bundle E on $\mathbb{P}^3$ ($\mathbb{P}^3 = \mathbb{P}^3_k$, the projective 3-space over an algebraically closed field k of characteristic zero), determine whether the first cohomology module $H^1{}_*(E) := \oplus_{t\in Z} H^1(E(t))$ is connected (i.e., has no gaps in the grading (see §0)).

For rank two vector bundles on $\mathbb{P}^2$ this problem is solved: in 3.1 we show that $H^1{}_*(E)$ is always connected. Unfortunately, the method used in this case does not apply to vector bundles on $\mathbb{P}^3$ (see proof of 3.1). On the other hand, rank two vector bundles on $\mathbb{P}^3$ are closely related to curves in $\mathbb{P}^3$, precisely, to subcanonical curves. Moreover, if C is a curve associated with a vector bundle E, the graded modules $H^1{}_*(E)$ and $M(C) := \oplus_{t\in Z} H^1(\mathcal{J}_C(t))$ - the Hartshorne-Rao module of C -, are isomorphic up to shift. Thus, Problem 1 is equivalent to the following:

Problem 1' : Determine whether the Hartshorne-Rao module of a subcanonical curve in $\mathbb{P}^3$ is connected.

It is known, after Rao's paper [R1], that the Hartshorne-Rao module characterizes liaison classes of curves in $\mathbb{P}^3$ and that each graded, finite length module over the polynomial ring $k[x_0,x_1,x_2,x_3]$ is isomorphic, up to shift, to the Hartshorne-Rao module of a nonsingular curve. Anyway, in the same paper, Rao shows that there exist liaison classes which do not contain any subcanonical curve (Example 3.3 in [R1]). Hence it makes sense to ask for problem 1'.

Here we provide partial answers to Problems 1 and 1'. A first result is the following:

Theorem 2.2: *Let* C *be an integral* e-*subcanonical curve. If* $e \leq 6$, *then the Hartshorne-Rao module of* C *is connected.*

The method of the proof is "alla Castelnuovo", in the sense that we reduce the problem to studying a property of the general plane section of the curve. This technique has already been applied in our previuos paper [B], in order to show that for every integral curve with speciality index less or equal to three the Hartshorne-Rao module is connected (Thm. 2.1 in [B]). Here, in §1, we give some other sufficient conditions for the connectedness of the Hartshorne-Rao module of integral curves (1.3 and 1.4), thus completing in some way the information given in [B].

It is clear that Theorem 2.2 provides a partial answer to Problem 1', but not to Problem 1. Indeeed, given a vector bundle E, one does not know a priori for which integer m E(m) has a section vanishing along an integral curve; on the other hand, the invariant e of the curve does depend on m. Nonetheless, we can adapt the method used for curves to vector bundles. The basic tools, in the case of curves, are Castelnuovo-Mumford lemma ([M], Lecture 14) and the so called "generalized trisecant lemma" due to Laudal-Gruson-Peskine ([GP1]); besides, we use the numerical character of the general plane section of a curve to compute the dimensions of some cohomology groups. In dealing with vector bundles, Laudal-Gruson-Peskine lemma is replaced by the restriction theorems of Maruyama and Barth; also, we use some properties of the spectrum of a vector bundle. Unfortunately, what is missing here to push the method further is a "good" vanishing for the cohomology of the restriction of a vector bundle on $\mathbb{P}^3$ to a general plane.

The result we get is the following:

Theorem 3.9: *Let* E *be a semistable (resp., stable), rank two vector bundle on* $\mathbb{P}^3$ *with Chern classes* $-1 \leq c_1 \leq 0$, c_2. *If* $c_2 \leq 14 - 2c_1$ *(resp.,* $c_2 \leq 18 + 2c_1$*), then* $H^1{}_*(E)$ *is connected.*

We also show (3.8) that $H^1{}_*(E)$ is connected if the spectrum of E is "flat" enough with respect to c_2.

For the time being, we do not know any example of a rank two (stable) vector bundle on $\mathbb{P}^3$ with a not connected first cohomology module. If such bundles exist, however, they do not satisfy the simplifying assumption of Barth - that is, that the graded module $H^1{}_*(E)$ is generated in negative degrees (see 3.5, 3.6).

I am very grateful to Philippe Ellia for suggesting the problem and providing several remarks which have greatly contributed to this paper. I would also thank the referee for pointing out a simplification of my original proof of Proposition 3.1, which, in this way, does not require the characteristic zero assumption.

§0 Notations.

We work over an algebraically closed field k of characteristic zero. Let C denote a curve in $\mathbb{P}^3_k = \mathbb{P}^3$ - i.e. an equidimensional, locally Cohen-Macaulay, closed subscheme of dimension one - and $\mathcal{I}_C$ its ideal sheaf. We fix once and for all some notations about C:

d_C = the degree of C ; g_C = the arithmetic genus of C ; $s_C = \min\{k \mid H^0(\mathcal{I}_C(k)) \neq 0\}$

$e_C = \max\{k \mid H^2(\mathcal{I}_C(k)) \neq 0\}$, the speciality index ;

Γ_C = the generic plane section of C ; $\sigma_C = \min\{k \mid H^0(\mathcal{I}_\Gamma(k)) \neq 0\}$.

(We will drop the subscript C if no confusion arises).

A curve C is <u>a-subcanonical</u> if $\omega_C \cong \mathcal{O}_C(a)$, where ω_C is the dualizing sheaf. If C is integral, it is easily seen that a = e. Moreover, $h^1(\mathcal{O}_C(e)) = 1$ and computing $h^0(\mathcal{O}_C(e))$ by Riemann-Roch one gets: $g = de/2 + 1$. It follows that, if e is odd, for integral subcanonical curves only even degrees are possible.

We recall that we can associate with Γ a sequence of positive integers $\chi(\Gamma)=(n_0,\dots, n_{\sigma-1})$ called its <u>numerical character</u> (see [GP]) and satisfying the following properties:

i) $n_0 \geq \dots \geq n_{\sigma-1} \geq \sigma$;

ii) $\Sigma_{0\leq i\leq \sigma-1} (n_i-i) = d$;

The numerical character of Γ determines its Hilbert function $h_\Gamma(\cdot)$ by means of the formula: $h_\Gamma(n) = \Sigma_{0\leq i\leq \sigma-1} [(n-i+1)_+ - (n-n_i+1)_+]$ (here $(x)_+= \max\{0,x\}$).

Moreover, if C is an integral curve, $\chi(\Gamma)$ is connected, that is, it verifies the condition:

iii) $n_i \leq n_{i+1}+1$, $0 \leq i \leq \sigma-2$ (see[GP]).

When we refer to an order relation on a set of characters we always mean the lexicographical order.

The <u>Hartshorne-Rao module</u> of C is the graded $k[x_0,x_1,x_2,x_3]$-module defined as: $M(C) := \oplus_{t\in \mathbb{Z}} H^1(\mathcal{I}_C(t))$. M(C) has finite length, moreover, if C is integral, then $H^1(\mathcal{I}_C(t)) = 0$ for $t \leq 0$ and $t \geq d-2$.

For convenience we recall some definitions and lemmas from [B]:

Definition: Let $M = \oplus_{k \in Z} M_k$ be a graded module. We say that $t \in \mathbf{Z}$ is a <u>gap</u> for M if the following conditions occur: $M_{t-1} \neq 0$, $M_t = 0$ and $M_{t+s} \neq 0$ for some $s > 0$.

We say that M is <u>connected</u> if it has no gaps.

Lemma A (Lemma 1.6 of [B]): *Let* C *be a curve and* t *a positive integer such that* $H^1(\mathcal{I}_C(t)) = 0$ *and* $t > e+1$. *Then* $H^1(\mathcal{I}_C(k)) = 0$ *for every* $k > t$.

This is an immediate consequence of Castelnuovo-Mumford lemma; it implies that we can only find gaps t with $t \leq e+1$.

Lemma B (Lemma 1.7 of [B]): Let C be an integral curve and t a positive integer such that $H^1(\mathcal{I}_C(t-1)) \neq 0$ and $H^1(\mathcal{I}_C(t)) = 0$. Then $\sigma \leq t$, $s \leq t+1$ and we have the following cases:

a) if $\sigma = t$, then $d \leq \sigma^2+1$;

b) if $\sigma < t$, then $d \leq t\sigma$;

If, moreover, $d = t^2+1$ and $t \geq 4$, then M(C) is connected.

We will also say that an integer t is a <u>potential gap</u> if $t \leq e+1$, $H^1(\mathcal{I}_C(t-1)) \neq 0$ and $H^1(\mathcal{I}_C(t)) = 0$.

§1 Generalities.

We state a series of preliminary results which we use in the proof of theorem 2.2 but that may also have some interest in their own. In particular, we give some sufficient conditions for the connectedness of the Hartshorne-Rao module of integral curves (1.3 and 1.4), which complete the results of [B].

Remark 1.1: Note that for any curve C it is $n_0 \geq e+3$; indeed, from the exact sequence

$$0 \to \mathcal{I}_C(-1) \to \mathcal{I}_C \to \mathcal{I}_\Gamma \to 0 \qquad (*),$$

one gets $h^1(\mathcal{I}_\Gamma(e+1)) > 0$. On the other hand $h^1(\mathcal{I}_\Gamma(e+1)) = \Sigma_{0\leq i\leq \sigma-1} [(n_i-e-2)_+ - (i-e-2)_+]$, thus: $\Sigma_{0\leq i\leq \sigma-1} (n_i-e-2)_+ > \Sigma_{0\leq i\leq \sigma-1} (i-e-2)_+ \geq 0$ and this implies $n_0 \geq e+3$.

Remark 1.2: Let Γ be the general plane section of an integral curve C. One can easily see that if Γ has a steep character - that is, of type $(n_0, n_0-1, n_0-2, ..., n_0-\sigma+1)$ - then it is a complete intersection of type $(\sigma, n_0-\sigma+1)$. In this case, if moreover C has degree $d > 4$ and does not lie on a quadric, C itself is a complete intersection (Corollario 1 of Teorema 4 in [S2]), hence M(C) = 0.

Proposition 1.3: *Let C be an integral curve with* $\sigma \leq 3$, *then* M(C) *is connected.*

Proof: If $\sigma = 1$, by the trisecant lemma C is a plane curve and thus M(C) = 0. If $\sigma = 2$ there are two cases: either the curve lies on a quadric - then it is easy to see that M(C) is connected - or $d \leq 5$ (by Laudal-Gruson-Peskine lemma, see [GP1]), then M(C) has no gaps. If $\sigma = 3$, suppose we have a potential gap t, that is: $t \leq e+1$, $H^1(\mathcal{I}_C(t-1)) \neq 0$ and $H^1(\mathcal{I}_C(t)) = 0$. Then it must be $t \geq 3$ (Lemma B), hence $e \geq 2$. Moreover, $d \leq 10$ if $t = 3$ and $d \leq 3t \leq 3(e+1)$ if $t > 3$ (Lemma B, a), b)). Let us now determine the numerical characters corresponding to these data, that is, the triplets of integers (n_0, n_1, n_2) satisfying conditions i), ii), iii) of §0 and $n_0 \geq e+3$, with the above restrictions on e and d. Writing the triplet as (e+3+a, e+3+a-b, e+3+a-b-c), where a, b and c are non negative integers and b,c $\leq$ 1, condition ii) yields: 3e + 9 + 3a - 2b - c = d + 3. If t = 3 we get: 3e + 3a - 2b - c $\leq$ 4. Taking account of the bounds on e, a, b and c one sees that it must be a = 0, b = 1, e = 2 and c = 1 or c = 0. Thus we obtain two characters: (5,4,3) and (5,4,4). The first one is steep, then M(C) = 0 by Remark 1.2, which contradicts the hypothesis $H^1(\mathcal{J}_C(t-1)) \neq 0$. The second one yields $h^1(\mathcal{J}_\Gamma(4)) = \Sigma_{0\leq i\leq 2}[(n_i-5)_+ - (i-5)_+] = 0$, hence from sequence (*) we get $H^1(\mathcal{J}_C(4)) = 0$ and by Lemma A we conclude that M(C) is connected. If t > 3, condition ii) yields: 3e + 9 + 3a - 2b - c $\leq$ 3e + 6. As before we see that a = 0 and b = c = 1, hence the character is (e+3, e+2, e+1), which is steep. #

Corollary 1.4: *Let* C *be an integral curve of degree* d *and arithmetic genus* g.

(i) if $s_C \leq 3$, *then* M(C) *is connected;*

(ii) if g > G(d,4), *the maximal genus for smooth irreducible curves of degree* d *not lying on a cubic surface, then* M(C) *is connected;*

(we recall that G(d,4) = 3d - 19 *if* d $\leq$ 12, *while for* d > 12 *it is* $G(d,4) = d^2/8 + 1 - 3r(4 - r)/8$, *where* r *is an integer depending on the parity of* d, *namely,* $0 \leq r < 4$ *and* d + r $\equiv$ 0 (mod 4) *(see [GP]));*

(iii) if d $\leq$ 15, *then* M(C) *is connected.*

Proof: (i): it follows from 1.3, since $\sigma_C \leq s_C \leq 3$; (ii): if g > G(d,4) it is $s_C \leq 3$ and we conclude by (i). (iii): suppose M(C) is not connected, then from Corollary 2.2 in [B] we know that 2d + 1 $\leq$ g, on the other hand by (ii) we also have g $\leq$ G(d,4). Comparing the values of G(d,4) and 2d + 1 one gets d $\geq$ 16. #

Remark 1.5: We note that from (ii) in 1.4 and from Corollary 2.2 in [B], it follows that M(C) is connected if the genus g is "small" or "very big" with respect to d, namely, if g < 2d + 1 or g > G(d,4).

Proposition 1.6: *Let* C *be an integral curve and suppose* $H^1(\mathcal{J}_C(e)) \neq 0$, $H^1(\mathcal{J}_C(e+1)) = 0$. *Then* $H^1(\mathcal{J}_C(k)) = 0$ *for every* k $\geq$ e+2 *if and only if* n_0 = e+3.

Proof: First note that in view of Lemma A we just need to prove: $h^1(\mathcal{I}_C(e+2)) = 0$ if and only if $n_0 = e+3$. From sequence (*) we get the isomorphism $H^1(\mathcal{I}_C(e+2)) \cong H^1(\mathcal{I}_\Gamma(e+2))$. Now $h^1(\mathcal{I}_\Gamma(e+2)) = \Sigma_{0\le i\le\sigma-1}[(n_i-e-3)_+ - (i-e-3)_+] = \Sigma_{0\le i\le\sigma-1}(n_i-e-3)_+$, since in our hypothesis $i \le e$ (indeed, by Lemma B one has $\sigma \le e+1$, hence $i \le \sigma-1 \le e$). It follows that $h^1(\mathcal{I}_\Gamma(e+2)) = 0$ if and only if $n_0 \le e+3$. Being also $n_0 \ge e+3$ (Remark 1.1), we obtain our thesis. #

Proposition 1.7: *Let* C *be an integral curve and suppose* $H^1(\mathcal{I}_C(e-1)) \neq 0$, $H^1(\mathcal{I}_C(e)) = 0$. *Then* $h^1(\mathcal{I}_\Gamma(e+1)) = h^2(\mathcal{I}_C(e))$ *and* $h^1(\mathcal{I}_\Gamma(e+2)) = 0$ *if and only if* $H^1(\mathcal{I}_C(k)) = 0$ *for every* $k \ge e+1$.

Proof: Consider the exact sequence: $..H^1(\mathcal{I}_C(t-1)) \to H^1(\mathcal{I}_C(t)) \to H^1(\mathcal{I}_\Gamma(t)) \to H^2(\mathcal{I}_C(t-1)) \to H^2(\mathcal{I}_C(t))$... For $t = e+1$ we get the short exact sequence: $0 \to H^1(\mathcal{I}_C(e+1)) \to H^1(\mathcal{I}_\Gamma(e+1)) \to H^2(\mathcal{I}_C(e)) \to 0$, then $H^1(\mathcal{I}_C(e+1)) = 0$ if and only if $h^1(\mathcal{I}_\Gamma(e+1)) = h^2(\mathcal{I}_C(e))$. If this holds, for $t = e+2$ there is an isomorphism $H^1(\mathcal{I}_C(e+2)) \cong H^1(\mathcal{I}_\Gamma(e+2))$, hence $H^1(\mathcal{I}_C(e+2)) = 0$ if and only if $H^1(\mathcal{I}_\Gamma(e+2)) = 0$. We conclude applying Lemma A. #

Corollary 1.8: *Let* C *be an integral, e-subcanonical curve. Suppose* $H^1(\mathcal{I}_C(e-1)) \neq 0$, $H^1(\mathcal{I}_C(e)) = 0$. *Then* $H^1(\mathcal{I}_C(k)) = 0$ *for every* $k \ge e+1$ *if and only if* $n_0 = e+3$ *and* $n_i < e+3$ *for* $i = 1,\ldots,\sigma-1$.

Proof: It follows at once from 1.6 and 1.7 above and $h^2(\mathcal{I}_C(e)) = h^1(\mathcal{O}_C(e)) = 1$. #

Remark 1.9: If C is a subcanonical curve and $H^1(\mathcal{I}_C(k)) = 0$ for some k, we get an upper bound, depending on k and e, for the degree d. Consider the surjective map: $H^0(\mathcal{O}_{\mathbb{P}^3}(k)) \to H^0(\mathcal{O}_C(k)) \to 0$, computing dimensions we have: $h^0(\mathcal{O}_{\mathbb{P}^3}(k)) =$

$(k+3)(k+2)(k+1)/6 \geq h^0(\mathcal{O}_C(k)) = dk+1-g+h^1(\mathcal{O}_C(k)) = d(k-e/2)+ h^0(\mathcal{O}_C(e-k))$, since $g = de/2+1$. Hence: $d \leq 2[(k+3)(k+2)(k+1)/6 - h^0(\mathcal{O}_C(e-k))] / (2k-e)$.

§2 Subcanonical curves.

Remark 2.1: In [B], Theorem 2.1, we proved that the module M(C) is always connected for integral curves with speciality index $e \leq 3$. Actually, the proof of that theorem consisted of showing that t is not a gap if $t \leq 4$. Alternatively, one can recover this fact from Lemma B, 1.4 (iii) and 1.2 above. We shall assume it in the proof of next theorem.

Theorem 2.2: *Let C be an integral, e-subcanonical curve. If* $e \leq 6$, *then* M(C) *is connected.*

Proof: We prove that if C is subcanonical, then t is not a gap for $5 \leq t \leq e+1$ and $4 \leq e \leq 6$.

<u>Case t = 5</u>: suppose $H^1(\mathcal{J}_C(4)) \neq 0$ and $H^1(\mathcal{J}_C(5)) = 0$, then by 1.3 and Lemma B we can assume $4 \leq \sigma \leq 5$. For $\sigma = 4$ we have $10 \leq d \leq 20$, the numerical characters for these couples (d,σ) are:

$d = 10, \sigma = 4$	(4,4,4,4),
$d = 11, \sigma = 4$	(5,4,4,4),
$d = 12, \sigma = 4$	(5,5,4,4),
$d = 13, \sigma = 4$	(5,5,5,4) or (6,5,4,4),
$d = 14, \sigma = 4$	(5,5,5,5) or (6,5,5,4),
$d = 15, \sigma = 4$	(6,5,5,5) or (6,6,5,4),
$d = 16, \sigma = 4$	(6,6,5,5) or (7,6,5,4),
$d = 17, \sigma = 4$	(6,6,6,5) or (7,6,5,5),
$d = 18, \sigma = 4$	(6,6,6,6) or (7,6,6,5),
$d = 19, \sigma = 4$	(7,6,6,6) or (7,7,6,5)
$d = 20, \sigma = 4$	(7,7,6,6) or (8,7,6,5).

The last one is steep, thus it is not allowed in our hypothesis (Remark 1.2). If $e = 4$ we consider characters with $n_0 \geq 7$ (Remark 1.1), indeed there only remain characters with

$n_0 = 7$ and by 1.6 we conclude that M(C) is connected. If $e \geq 5$ none of the above characters can occur.

For $\sigma = 5$, we have $15 \leq d \leq 26$ and the characters are:

$d = 15, \sigma = 5$ (5,5,5,5,5),

$d = 16, \sigma = 5$ (6,5,5,5,5),

$d = 17, \sigma = 5$ (6,6,5,5,5),

$d = 18, \sigma = 5$ (6,6,6,5,5) or (7,6,5,5,5),

$d = 19, \sigma = 5$ (6,6,6,6,5) or (7,6,6,5,5),

$d = 20, \sigma = 5$ (6,6,6,6,6) or (7,6,6,6,5) or (7,7,6,5,5),

$d = 21, \sigma = 5$ (7,6,6,6,6) or (7,7,6,6,5) or (8,7,6,5,5),

$d = 22, \sigma = 5$ (7,7,6,6,6) or (7,7,7,6,5) or (8,7,6,6,5),

$d = 23, \sigma = 5$ (7,7,7,6,6) or (8,7,6,6,6) or (8,7,7,6,5),

$d = 24, \sigma = 5$ (7,7,7,7,6) or (8,7,7,6,6) or (8,8,7,6,5),

$d = 25, \sigma = 5$ (7,7,7,7,7) or (8,7,7,7,6) or (8,8,7,6,6) or (9,8,7,6,5),

$d = 26, \sigma = 5$ (8,7,7,7,7) or (8,8,7,7,6) or (9,8,7,6,6).

If $e = 4$ it is $d \leq 18$ (Remark 1.9) and $n_0 \geq 7$ (Remark 1.1), then we have only (7,6,5,5,5) and we conclude by 1.6. If $e = 5$ it is $d \leq 22$ and $n_0 \geq 8$, thus only (8,7,6,5,5) and (8,7,6,6,5) are possible and we apply Corollary 1.8. If $e = 6$ it is $n_0 \geq 9$, there are only two characters satisfying this condition, one is (9,8,7,6,5) which is steep, hence not admissible. The other one is (9,8,7,6,6), which occurs for $d = 26$, then by the last statement of Lemma B, M(C) is connected.

Case t = 6: suppose $H^1(\mathcal{J}_C(5)) \neq 0$ and $H^1(\mathcal{J}_C(6)) = 0$, by Lemma A we only need to consider $e \geq 5$ and, by Proposition 1.3 and Lemma B, $4 \leq \sigma \leq 6$. For $\sigma = 4$ we have $10 \leq d \leq 24$, then computing the numerical characters corresponding to these couples (d,σ) one sees that all of them have $n_0 \leq 8$, except for a steep character, namely (9,8,7,6), which is the maximal character obtained for $d = 24$ and $\sigma = 4$. Thus $e = 5$ by 1.1 and we conclude by 1.6. For $\sigma = 5$ it is $15 \leq d \leq 30$. If $e = 5$ it is $d \leq 24$ (Remark 1.9), then computing the characters one sees that the only admissible ones have exactly $n_0 = 8$ and again Proposition 1.6 applies. If $e = 6$, then $d \leq 27$ and there is only one possible character, namely (9,8,7,6,6) - it is the maximal one for the couple (27,5) - then we use Corollary 1.8. For $\sigma = 6$ it is $21 \leq d \leq 37$. Again, if $e = 5$ then $d \leq 24$ and the only character with $n_0 \geq 8$ is (8,7,6,6,6,6) - obtained for $d = 24$, hence we can apply 1.6. If $e = 6$ then $d \leq 27$ and the only character with $n_0 \geq 9$ is (9,8,7,6,6,6) - for $d = 27$, we conclude by 1.8.

Case t = 7: suppose $H^1(\mathcal{I}_C(6)) \neq 0$ and $H^1(\mathcal{I}_C(7)) = 0$, then we have to consider only $e = 6$ and $4 \leq \sigma \leq 7$. For $\sigma = 4$ we have $10 \leq d \leq 28$ (Lemma B) and computing the characters one sees that the only possible ones have $n_0 = 9$, thus we can apply 1.6. For $\sigma = 5$ and $\sigma = 6$ we consider characters with $d \leq 30$ (Remark 1.9), then again the only possible ones have exactly $n_0 = 9$ and we conclude as above. Finally, for $\sigma = 7$ and $d \leq 30$ there are no characters with $n_0 \geq 9$. This concludes our proof. #

§3 Vector bundles.

We recall that a rank two vector bundle E on $\mathbb{P}^3$, with a section $s \in H^0(E)$ which vanishes in codimension two, determines a locally complete intersection (l.c.i. for short), subcanonical curve C, with $e_C = c_1(E) - 4$. Viceversa, we can associate with any l.c.i. subcanonical curve C a rank two vector bundle on $\mathbb{P}^3$ and a section of this bundle such that C is exactly its zero locus. The section gives an exact sequence:

$$0 \to \mathcal{O}_{\mathbb{P}3} \to E \to \mathcal{I}_C(k) \to 0$$

and the Chern classes of E are: $c_1(E) = k$, $c_2(E) = \deg C$. From this sequence one sees that $H^1{}_*(E) := \oplus_{t \in Z} H^1(E(t)) \cong M(C)(k)$, thus the connectedness of M(C) is equivalent to that of $H^1{}_*(E)$. Anyway, Theorem 2.2 above becomes meaningless when translated in terms of vector bundles. Indeed, given a vector bundle F we do not know a priori for which integer k F(k) has a section vanishing along an integral curve. On the other hand, the method of considering the restriction to a generic plane can be applied to semistable (stable) vector bundles - thanks to the restriction theorems of Maruyama and Barth - and it yields sufficient conditions for $H^1{}_*(E)$ to be connected.

We shall always deal with normalized vector bundles, that is with first Chern class c_1 in the range $-1 \leq c_1 \leq 0$.

Let us first consider rank two vector bundles on $\mathbb{P}^2$. In this case the first cohomology modules are completely characterized. Indeed, A.P.Rao showed that a finite length $k[x_0,x_1,x_2]$-module is the first cohomology module of some rank two vector bundle on $\mathbb{P}^2$ if and only if it has the least possible number of relations among its minimal generators, namely, $r = s + 2$, where s is the number of minimal generators and r is the number of relations among them (see Prop.1.1 and Cor. 1.3 in [R2]). As a consequence of this we obtain the following:

Proposition 3.1: *Let* E *be a rank two vector bundle on* $\mathbb{P}^2$. *Then* $H^1{}_*(E)$ *is connected.*

Proof: Suppose $H^1{}_*(E)$ is not connected, that is, there exists a $t \in \mathbf{Z}$ such that $H^1(E(t-1)) \neq 0$, $H^1(E(t)) = ... = H^1(E(t+h)) = 0$ and $H^1(E(t+h+1)) \neq 0$ for some integer $h \geq 0$. Then $H^1{}_*(E)$ has two "connected components": $M_1 = \oplus_{k \leq t-1} H^1(E(k))$ and $M_2 = \oplus_{k \geq t+s+1} H^1(E(k))$. Let m_i be the number of minimal generators of M_i $(i = 1, 2)$, then M_i has at least $m_i + 2$ minimal relations (indeed, consider a minimal presentation of a finite length $k[x_0,x_1,x_2]$-module M: $L_1 \to L_0 \to M \to 0$; the sheafified version of this gives an exact sequence $0 \to \mathcal{F} \to \mathcal{L}_1 \to \mathcal{L}_0 \to 0$, where $\mathcal{F}$ is the sheafified kernel of the map $L_1 \to L_0$. Since $H^1{}_*(\mathcal{F}) \cong M$, rank$\mathcal{F}$ is at least 2 if $M \neq 0$, hence rank$L_1 \geq$ rank$L_0 + 2$). It follows that $H^1{}_*(E)$ has $m_1 + m_2$ minimal generators and at least $m_1 + m_2 + 4$ relations among them, which contradicts the minimality of the number of relations. #

Notice that this result holds also in characteristic different from zero.

If $ch(k) = 0$ we know moreover that, for $t \geq -c_1$, the function $h^1(E(t))$ is strictly decreasing to zero, indeed we have:

Lemma 3.2: *Let* E *be a semistable, rank two, vector bundle on* $\mathbb{P}^2$. *Then, for any integer* $t \geq -c_1$, *either* $h^1(E(t-1)) > h^1(E(t))$ *or* $h^1(E(t-1)) = h^1(E(t)) = 0$.

Proof: Consider the exact sequence: $0 \to E(-1) \to E \to E_L \to 0$, where L is a generic line in $\mathbb{P}^2$. Taking cohomology we get: $...H^0(E(t)) \xrightarrow{r_t} H^0(E_L(t)) \longrightarrow H^1(E(t-1)) \xrightarrow{f_t} H^1(E(t)) \longrightarrow H^1(E_L(t))...$. Since E is semistable, by Grauert - Mülich restriction theorem we have $E_L \cong \mathcal{O}_L \oplus \mathcal{O}_L(c_1)$ hence, for $t \geq -c_1$, $H^1(E_L(t)) = 0$. Thus $h^1(E(t-1)) \geq h^1(E(t))$ and equality holds if and only if the map r_t is surjective. On the other hand, consider the following commutative diagram:

$$\begin{array}{ccc} H^0(E(t)) \otimes H^0(\mathcal{O}_{\mathbb{P}^2}(1)) & \longrightarrow & H^0(E(t+1)) \\ r_t \otimes 1 \downarrow & & r_{t+1} \downarrow \\ g: H^0(E_L(t)) \otimes H^0(\mathcal{O}_L(1)) & \longrightarrow & H^0(E_L(t+1)) \end{array}$$

Since $E_L \cong \mathcal{O}_L \oplus \mathcal{O}_L(c_1)$, g is surjective for $t \geq -c_1$; it is then clear that r_t surjective implies r_{t+1} surjective. This means that if f_t is an isomorphism, then f_k is an isomorphism for every $k > t$ and, by Theorem B of Serre, $h^1(E(t-1)) = h^1(E(t)) = 0$. #

Let us now look at rank two vector bundles on $\mathbb{P}^3$. We have two lemmas which are the analogue of Lemmas A and B (§ 0) for curves:

Lemma 3.3: *Let* E *be a rank two, semistable vector bundle on* $\mathbb{P}^3$. *Suppose that for some integer* $t \geq -1$ *it is* $H^1(E(t)) = H^2(E(t-1)) = 0$, *then* $H^1(E(k)) = 0$ *for any* $k \geq t$.

Proof: This is just an application of Castelnuovo-Mumford theorem. Indeed, $H^3(E(t-2)) \cong H^0(E(-c_1-t-2)) = 0$ for $t \geq -1$, since E is semistable. Hence E is (t+1)-regular and we get the result. #

Lemma 3.4: *Let* E *be a rank two, normalized, semistable (resp., stable with* $c_2 \geq 2$*) vector bundle on* $\mathbb{P}^3$. *Suppose* $H^1(E(t-1)) \neq 0$, $H^1(E(t)) = 0$ *for some* t. *Then* $t \geq 0$ *(resp.,* $t \geq 1$*). More precisely,* $t \geq f(c_1,c_2)$, *with* $f(0,c_2) = \max\{0, -2 + \sqrt{(1+3c_2)}\}$ *(resp.,* $f(0,c_2) = \max\{1, -2 + \sqrt{(1+3c_2)}\}$*) and* $f(-1,c_2) = [-3+\sqrt{(1+12c_2)}]/2$.

Proof: Let H be a generic plane in $\mathbb{P}^3$, from the exact sequence $0 \to E(t-1) \to E(t) \to E_H(t) \to 0$ we get $h^0(E_H(t)) \neq 0$. Hence, if E is semistable, from Maruyama's restriction theorem it follows that $t \geq 0$ (if E is stable with $c_2 \geq 2$, from Barth's restriction theorem it follows that $t \geq 1$). If $c_1 = 0$, Riemann-Roch theorem yields: $\chi(E(t)) = (t+2)[(t+1)(t+3) - 3c_2]/3$. Thus the equation $\chi(E(t)) = 0$ has the three roots: -2, $-2-\sqrt{(1+3c_2)}$, $-2 + \sqrt{(1+3c_2)}$, moreover, if $-2 \leq t \leq -2 + \sqrt{(1+3c_2)}$, then $\chi(E(t)) < 0$. On the other hand, $h^1(E(t)) = 0$ by assumption and $h^3(E(t)) = h^0(E(-t-4)) = 0$ because E is semistable (stable) and $t \geq 0$ ($t \geq 1$). Hence, $\chi(E(t)) = h^0(E(t)) + h^2(E(t)) \geq 0$. It follows that $t \geq \max\{0, -2 + \sqrt{(1+3c_2)}\}$ (resp., $t \geq \max\{1, -2 + \sqrt{(1+3c_2)}\}$). Suppose $c_1 = -1$, by Riemann-Roch $\chi(E(t)) = (2t+3)[(t+1)(t+2) - 3c_2]/6$ and the equation $\chi(E(t)) = 0$ has the three roots: $-3/2$,

$[-3 - \sqrt{(1+12c_2)}]/2$, $[-3+\sqrt{(1+12c_2)}]/2$. Arguing as before we get $t \geq [-3+\sqrt{(1+12c_2)}]/2$.
#

As a straightforward consequence we have:

Corollary 3.5: *Let* E *be a rank two, normalized, semistable vector bundle on* $\mathbb{P}^3$. *If* $H^1{}_*(E)$ *is not connected, then it has a minimal generator in positive degree.*

Remark 3.6: From Corollary 3.5 it follows that for vector bundles which fulfil the simplifying assumption of Barth (see [Ba], top of pag. 211) - that is, such that the module $H^1{}_*(E)$ has no generators of positive degree - the module $H^1{}_*(E)$ is connected. This is the case, for example, of instanton bundles and of stable vector bundles with $c_1 = 0$ and spectrum equal to $\{-1, 0^{c_2-2}, 1\}$ (see for instance [E], Lemma III.3, (i)).
More generally, we show (Prop. 3.8) that $H^1{}_*(E)$ is connected provided that the spectrum of E is "flat" enough with respect to c_2.

Remark 3.7: We recall that if $\Phi(E) = \{k_i\}_{1 \leq i \leq c_2}$ is the spectrum of E, then, by definition, $h^2(E(k)) = h^1(\oplus_{1 \leq i \leq c_2} \mathcal{O}_{\mathbb{P}^1}(k+1+k_i))$ for $k \geq c_1-2$ ([H], Thm. 7.1). It follows that $h^2(E(k)) = 0$ if and only if $k \geq -k^- - 2$, where $k^- = \min\{k_i \mid k_i \in \Phi(E)\}$.

Proposition 3.8: *Let* E *be a rank two, normalized, semistable vector bundle on* $\mathbb{P}^3$. *If* $k^- \geq -f(c_1,c_2) - 1$, *then* $H^1{}_*(E)$ *is connected.*

Proof: Suppose $H^1(E(t-1)) \neq 0$, $H^1(E(t)) = 0$ for some t, then, by 3.4, $t \geq f(c_1,c_2)$. On the other hand, $k^- \geq -f(c_1,c_2) - 1 \geq -t - 1$ implies $h^2(E(t-1)) = 0$ (Rmk. 3.7) and we conclude by 3.3. #

Now, considering the longest possible spectrum for a semistable (resp., stable) vector bundle with given Chern classes, we get a lower bound for k^- and hence a condition on c_1 and c_2 in order that $H^1{}_*(E)$ is connected.

Theorem 3.9: *Let* E *be a semistable (resp., stable), rank two vector bundle on* $\mathbb{P}^3$ *with Chern classes* $-1 \leq c_1 \leq 0$, c_2. *If* $c_2 \leq 14 - 2c_1$ *(resp.,* $c_2 \leq 18 + 2c_1$*), then* $H^1{}_*(E)$ *is connected.*

Proof: Suppose $H^1(E(t-1)) \neq 0$, $H^1(E(t)) = 0$ and $H^2(E(t-1)) \neq 0$ for some t (if $H^2(E(t-1)) = 0$ we conclude by Lemma 3.3). Then $f(c_1,c_2) \leq t \leq -2 - k^-$ (by 3.4 and 3.7). On the other hand, $k^- \geq m(c_1,c_2) := \min\{k_i \mid k_i \in \Psi\}$, where Ψ is the longest possible spectrum for a semistable (resp., stable) vector bundle with Chern classes $c_1 = c_1(E)$, $c_2 = c_2(E)$. Hence, $f(c_1,c_2) \leq t \leq -2 - m(c_1,c_2)$. If $f(c_1,c_2) \leq t = -2 - m(c_1,c_2)$, then $\Phi(E) = \Psi$ and the value $m(c_1,c_2)$ is achieved just by one of the k_i's, that is: $k_1 = m(c_1,c_2)$ and $k_i > m(c_1,c_2)$ for any other i. This yields $h^2(E(t-1)) = 1$ and $h^2(E(t)) = 0$ thus, from the exact sequence $0 \to E(t-1) \to E(t) \to E_H(t) \to 0$, we have $h^1(E_H(t)) = 1$. By 3.2 it follows that $h^1(E_H(k)) = 0$ for $k > t$, thus $H^1(E(k)) = 0$ for any $k > t$, that is, t is not a gap. This means that the allowed interval for t restricts to: $f(c_1,c_2) \leq t \leq -3 - m(c_1,c_2)$. Next step is to compute explicitly the value of $m(c_1,c_2)$. From the properties of the spectrum one can easily see that: if $c_1 = -1$, then $m(c_1,c_2) = -c_2/2$; if $c_1 = 0$ and c_2 is even, then $m(c_1,c_2) = -c_2/2$ (if E is stable, $m(c_1,c_2) = -(c_2-2)/2$); finally, if $c_1 = 0$ and c_2 is odd, then $m(c_1,c_2) = -(c_2+1)/2$ (if E is stable, $m(c_1,c_2) = -(c_2-1)/2$). Now, comparing $f(c_1,c_2)$ and $-3 - m(c_1,c_2)$ we get a bound on c_2: in fact, if c_2 is such that $-3 - m(c_1,c_2) < f(c_1,c_2)$, then there are no gaps in $H^1{}_*(E)$. We obtain the following bounds on c_2: if $c_1 = -1$, then $c_2 \leq 16$; if $c_1 = 0$ and c_2 is even, then $c_2 \leq 14$ (if E is stable $c_2 \leq 18$); if $c_1 = 0$ and c_2 is odd, then $c_2 \leq 13$ (if E is stable $c_2 \leq 17$).

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