SISSA - SCUOLA
INTERNAZIONALE
SUPERIORE
DI STUDI AVANZATI
TRIESTE
Strada Costiera 11

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Topological Twist in Four Dimensions

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Elementary Particle Sector

Candidate:
Damiano Anselmi

Supervisor:
Prof. Pietro Frè

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Contents

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Chapter 1

Introduction

We show that the BRST quantum version of D=4 N=2 supergravity can be topologically twisted, to yield a formulation of topological gravity in four dimensions. The topological BRST complex is just a rearrangement of the old BRST complex, that partly modifies the role of physical and ghost fields: indeed, the new ghost number turns out to be the sum of the old ghost number plus a suitable internal U(1) charge, that we name R-duality. Furthermore, the action of N=2 supergravity is retrieved from topological gravity by choosing a gauge fixing that reduces the space of physical states to the space of gravitational instanton configurations. Twisting pure N=2 supergravity, one obtains pure topological gravity. The corresponding instanton configurations are the self-dual spin connections. As it stands, the theory we discuss may prove useful in describing gravitational instantons moduli-spaces. The descent equations relating the topological observables are explicitly exhibited and discussed. Then we consider the coupling of vector multiplets to N=2 supergravity. We show that in the minimal case, namely when the special geometry prepotential $F(X)$ is a quadratic polynomial, there exisits a so far unknown on shell $U(1)$ symmetry (R-duality) which is suitalbe for the twist. R-duality is a generalization of the chiral-dual on shell symmetry of N=2 pure supergravity and of the R-symmetry of N=2 super Yang-Mills theory. Thanks to this, the theory can be topologically twisted and topologically shifted, precisely as pure N=2 supergravity, to yield a natural coupling of topological gravity to topological Yang-Mills theory. The gauge-fixing condition that emerges from the twisting is the self-duality condition on the gauge field-strength and on the spin connection. Hence our theory reduces to intersection theory in the moduli-space of gauge instantons living in gravitational instanton backgrounds. We remark that, for deep properties of the parent N=2 theory, the topological Yang-Mills theory we obtain by taking the flat space limit of our gravity coupled Lagrangian is different from the Donaldson theory constructed by Witten. Whether this difference is substantial and what its geometrical implications may be is yet to be seen.

We also discuss the topological twist of the hypermultiplets leading to topological quaternionic σ-models. The instantons of these models, named by us hyperinstantons, correspond to a notion of triholomorphic mappings discussed in the last part of the thesis.

In conclusion, the topological twist of the complete N=2 theory defines intersection theory in the moduli space of gauge instantons plus gravitational instantons plus hyperinstantons. This is possibly a new subject for further mathematical investigation.

Chapter 2

Aim and General Set-up

Recently, Topological Field Theories [1] have attracted a lot of interest, both for their own sake and in connection with string theory. Their general feature is that of recasting intersection theory in the moduli-space of some suitable geometrical structure into the language of standard quantum field-theory, specifically into the framework of the path-integral. Indeed the point-independent correlation functions of these peculiar field-theories represent intersection integrals of cohomology classes in the given moduli-space.

Particularly interesting, because of their relation with N=2 superconformal theories [2] and with Calabi-Yau moduli spaces [3] are topological theories in D=2 [4]. A lot of attention has been devoted to topological sigma models in two dimensions [5]. In this case one probes the moduli-space of holomorphic mappings from the world-sheet to a complex target space. Theories that have a close relation with topological sigma-models are the topological versions of N=2 Landau-Ginzburg models [4]. They have provided an interesting arena for the study of the moduli-spaces associated with Calabi-Yau manifolds [3], a topic of primary interest in connection with the effective Lagrangians of superstring models. In a different, but closely related set up, the coupling of topological matter multiplets to topological 2D gravity [6] has been used to investigate non critical string theories and relations have been established with the integrable hierarchies discovered in matrix models [7].

In two dimensions the relation between N=2 supersymmetry and topological field-theory is established via a topological twist that redefines a new Lorentz group $SO(2)'$ as the diagonal of the old Lorentz group with the U(1) automorphism group of the supersymmetry algebra [8]. In particular it implies that a whole class of N=2 correlation functions is topological in nature and, as such, both independent of the space-time points where the operators are localized and exactly calculable with geometrical techniques [9].

Notwithstanding the interest of the D=2 case, topological theories are worth considering also in four-dimensions. Actually they were originally introduced in D=4 with the discovery by Witten of topological Yang-Mills theory [10] and of its relation with the mathematical theory of Donaldson invariants [11] and with N=2 super Yang-Mills theory. Indeed Witten's original form of Topological Yang-Mills theory, which is already gauged fixed, was obtained via a suitable twist from D=4, N=2 Yang-Mills theory. The twist consists of the redefinition of a new Lorentz group $SO(4)' = SU(2)_L \otimes SU(2)'_R$ where the factor $SU(2)'_R$ is the diagonal of the old $SU(2)_R$ with the $SU(2)_I$ automorphism group of the supersymmetry algebra. The general BRST-approach to this theory was developped only later by Beaulieu and Singer [12], uncovering some of the subtleties hidden in Witten's twist approach.

From the experience of this example a general lesson is anyhow learnt: just as in D=2, also in D=4 any N=2 theory is liable to a topological twist and, as such, it should contain a topological sector where the correlation functions are independent of the space-time points and exactly calculable. In particular this should apply to N=2 supergravity, whose topological twist must yield a gauge-fixed version of D=4 topological gravity. It should also apply to the hypermultiplets, that are the D=4 counterparts of the N=2 Wess-Zumino multiplets. Actually, to state the conjecture in its most general form, the entire matter coupled N=2 supergravity, whose general form has been obtained in [13], further generalizing the results of conformal tensor calculus [14], should be liable to a topological twist and have a topological sector.

Before addressing some of the technical and conceptual details of our derivation, let us spend few words on motivations. They are essentially three:

i) The construction and the analysis of a well founded four-dimensional topologically gravity may furnish a gravitational analogue of Donaldson theory. In other words, it may provide a new tool to study intersection theory on the moduli space of gravitational instantons.

ii) The topological interpretation should provide new calculational tools in N=2 supergravity.

iii) The special Kähler geometry [15] of Calabi-Yau moduli-space is related, as we already recalled, to D=2 topological field-theories. On the other hand, it also follows from the requirement of N=2 supersymmetry in D=4. From the superstring point of view, this is understood in terms of the h-map [16], stating that on the same Calabi-Yau manifold we can compactify both the heterotic and the type II string. The latter has N=2 matter coupled supergravity as an effective lagrangian. Hence the topological interpretation of this theory should shed new light on the relation between topological field-theories in two and in four dimensions.

Let us now outline the conceptual set up and the contents of our thesis.

Our forst purpose is to show that the topological twist of N=2 pure supergravity defines a gauge-fixed version of pure topological gravity where the gauge-fixing condition is $\omega^{-ab} = 0$, ω^{-ab} denoting the antiselfdual part of the spin connection. To this effect we utilize the BRST-approach [17], having, as final goal, the comparison of the abstract gauge-theory a la Beaulieu-Singer [12] with the gauge-fixed approach a la Witten [10].

Our viewpoint on the construction of a BRST-theory is the following. First one singles out the classical symmetries and constructs an abstract BRST-algebra involving only the classical fields and the ghosts, with the exclusion of the antighosts. We name this algebra the *gauge-free* BRST algebra, since, at this level no commitment is made on the gauge fixing terms and on the lagrangian. Next, in the BRST-algebra, one introduces the antighosts and the auxiliary fields. The choice of these latter is motivated by the gauge fixings one wants to consider. Finally one constructs the BRST quantum action with the given gauge-fixings.

In the case of topological gravity the gauge-free BRST algebra is the specialization to the Poincaré group of the gauge-free algebra for a topological Yang-Mills theory. The general form of this algebra is [12]:

$$\begin{aligned} sA &= -\,(\mathcal{D}c + \psi\,), \\ sc &= \phi - \frac{1}{2}\,[\,c\,,c\,], \\ sF &= \mathcal{D}\psi - [\,c\,,F\,], \\ s\psi &= \mathcal{D}\phi - [\,c\,,\psi\,], \\ s\phi &= -\,[\,c\,,\phi\,], \end{aligned} \tag{2.1}$$

where $A = A_\mu dx^\mu$ is the classical 1-form gauge-field, c are the 0-form ghosts (corresponding to ordinary gauge transformations $\delta A_\mu = \mathcal{D}_\mu \varepsilon$), $\psi = \psi_\mu dx^\mu$ is the 1-form ghost associated with the topological symmetry ($\delta A_\mu = u_\mu$) and ϕ is the 0-form ghost for ghosts that has ghost number $g = 2$, while the previous ghosts have $g = 1$. All fields are Lie algebra-valued. The BRST operation s in (2.1) is manifestly nilpotent ($s^2 = 0$) and anticommutes with the exterior derivative ($sd + ds = 0$).

One important ingredient of our discussion will be the relation between the gauge-free BRST algebra for the ordinary theory and for the topological theory. It can be understood in general terms as it follows. As it is more explicitly discussed in section II, one can extend the concept of differential forms to that of *ghost-forms*, by setting

$$\hat{A} = A + c, \tag{2.2}$$

where A and c are the (1,0) and (0,1) parts of $\hat{A}$ (a generic object of form degree f and ghost number g will be described by (f, g)). One can also extend the concept of exterior differentiation defining

$$\hat{d} = d + s, \tag{2.3}$$

where d and s are the (1,0) and (0,1) parts of $\hat{d}$. With these notations one finds that, expanding the extended field-strenght

$$\hat{F} = \hat{d}\hat{A} + \frac{1}{2}\left[\,\hat{A}\,,\,\hat{A}\,\right] \tag{2.4}$$

in its (f, g) sectors, the following identifications are possible: $\hat{F}_{(2,0)} = F$, $\hat{F}_{(1,1)} = \psi$ and $\hat{F}_{(0,2)} = \phi$. Indeed the first two equations in (2.1) amount precisely to these identifications, while the last three are sectors of the extended Bianchi identity

$$\hat{d}\hat{F} + \left[\,\hat{A}\,,\,\hat{F}\,\right] = 0. \tag{2.5}$$

Hence the gauge-free topological BRST algebra corresponds to a parametrization of the extended curvature $\hat{F}$ where no constraints are imposed on the extra components $\hat{F}_{(1,1)}$ and $\hat{F}_{(0,2)}$.

On the other hand the ordinary gauge-free BRST algebra

$$\begin{aligned} sA &= -\mathcal{D}c, \\ sc &= -\frac{1}{2}\,[\,c\,,c\,]\,, \\ sF &= -\,[\,c\,,F\,]\,, \end{aligned} \tag{2.6}$$

correspond to imposing the horizontality conditions $\hat{F}_{(1,1)} = 0$ and $\hat{F}_{(0,2)} = 0$.

This is interpreted in the framework of rheonomy [18] as follows: the topological BRST algebra is the *off-shell* solution of the extended Bianchi identity (2.5) where all the outer components are kept on equal footing with the inner ones. The ordinary gauge-free BRST algebra is instead provided by the *quantum rheonomic* solution of the extended Bianchi identity (2.5) . By definition the *quantum rheonomic parametrization* is obtained from the *classical rheonomic* parametrization by replacing the classical cotangent basis of differential forms with the corresponding extended one. In this way the components of the extended curvatures in the extended basis are the same as the components of the classical curvatures in the classical basis. For instance in the case of Yang-Mills theory the classical basis is given by A and V^a, the last being the vierbein; the classical rheonomic parametrization is:

$$F = F_{ab}\,V^a \wedge V^b, \tag{2.7}$$

so that the quantum rheonomic parametrization is

$$\hat{F} = F_{ab}\,\hat{V}^a \wedge \hat{V}^b = F_{ab}\,V^a \wedge V^b. \tag{2.8}$$

Indeed, in this case $\hat{V}^a = V^a$, the ghost part being attached only to the gauge field A, according to (2.2), since only the gauge transformations are symmetries, not the diffeomorphisms.

In the case of pure gravity the classical curvatures are [18]

$$\begin{aligned} R^a &= \mathcal{D}V^a = dV^a - \omega^a{}_b \wedge V^b, \\ R^{ab} &= d\omega^{ab} - \omega^a{}_c \wedge \omega^{cb}. \end{aligned} \tag{2.9}$$

Their classical rheonomic parametrization is

$$\begin{aligned} R^a &= 0, \\ R^{ab} &= R^{ab}_{cd}\,V^c \wedge V^d, \end{aligned} \tag{2.10}$$

so that the corresponding quantum rheonomic parametrization is

$$\begin{aligned} \hat{R}^a &= 0, \\ \hat{R}^{ab} &= R^{ab}_{cd}\,\hat{V}^c \wedge \hat{V}^d. \end{aligned} \tag{2.11}$$

This time the vielbein being quantum extended

$$\hat{V}^a = V^a + \varepsilon^a. \tag{2.12}$$

Eq.s (2.11) lead to the BRST algebra associated with diffeomorphisms and Lorentz rotations. On the other hand, if we relax (2.11) and we keep all the outer components of $\hat{R}^a$ and $\hat{R}^{ab}$ as independent fields, we obtain a gauge-free BRST algebra that includes also the ghosts for the topological symmetry $\delta V^a_\mu = \xi^a_\mu$, ξ^a_μ being an arbitrary infinitesimal vierbein. This is our definition of gravitational topological BRST algebra.

We want to compare it with the BRST algebra associated with twisted N=2 supergravity. Indeed in order to make a successful twist we must already start at the quantum BRST-level. In fact, from a formal field-theoretic point of view the general framework of topological field-theories is that of geometrical BRST-quantization [17]. One deals with a classical Lagrangian that has a very large symmetry, such as the group of continuous deformations of a gauge-connection or of a metric and which, therefore, is a topological-invariant-density (i.e. some characteristic class of some fibre-bundle). To this symmetry one applies the standard BRST quantization scheme and, in this way, one obtains a topological BRST-cohomology, namely a double elliptic complex involving both the standard exterior derivative $d^2 = 0$ and a second nilpotent operator (the Slavnov operator $s^2 = 0$) that anticommutes with the first: $sd + ds = 0$. The true geometrical and physical content of the theory emerges when one fixes the gauge: indeed the gauge fixing condition is, normally, some kind of self-duality condition that reduces the space of physical states to the space of suitable *instantons*.

In this perspective the relevance of the topological twist is appreciated. The twist, discovered by Witten [10], extracts a topological field-theory with its gauge already fixed to a suitable instanton condition from an N=2 supersymmetric ordinary field-theory. Actually, as already noted, the very first example of topological field-theory, namely Donaldson theory, was constructed in this way starting from N=2 super Yang-Mills theory. The basic ingredients of the twist procedure are:

i) the possibility of changing the spins of the fields, by redefining a new Lorentz group as the diagonal of the old one (or a factor thereof) with an internal symmetry group, in such a way that, after the twist, the top spin boson of each supersymmetric multiplet and one of its fermionic partners acquire the same spin in the new theory;

ii) the existence of an additional U(1)-symmetry of the old theory, such that, redefining also the ghost number as the old one plus this particular U(1)-charge, the anticommuting partners of the bosons, that have acquired the same spin in the twist procedure, have, in the new theory, ghost number one, while their bosonic partners remain with ghost number zero. In this way the old fermions become the ghost associated with the topological symmetry.

The twist not only provides a constructive procedure for topological field-theories but also illuminates the topological character of a sector of the parent theory. This way of thinking has been most successfully implemented in two-dimensions. There the (Euclidean) Lorentz group is SO(2) and it can be easily redefined by taking its diagonal with the U(1) automorphism group of N=2 supersymmetry. In this simple case, the same U(1) provides also the charge to shift the ghost numbers. The result, as already mentioned, is given by either the topological sigma-models, or the topological Landau-Ginzburg models, or their coupling to topological 2D gravity. The topological sector of the original N=2 theory that is unveiled by this twist procedure is that of the chiral correlation functions.

In four-dimensions the twist procedure relies once more on the properties of N=2 supersymmetry, but involves many more subtleties, so that the programme of topologically twisting all N=2, D=4 theories needs deeper thinking. This programme has been started in [19, 20] by twisting pure and matter coupled N=2 supergravity. The present thesis is mainly based on the results of Ref.s [19, 20].

The result of the twist of N=2 supergravity minimally coupled to vector multiplets is given by a D=4 topological Yang-Mills theory coupled to topological D=4 gravity, the space of physical states being the moduli-space of gauge-instantons living in the background of gravitational instantons. One of the properties of this theory is that it does not seem to reduce to Donaldson theory in the limit where the gravitational coupling is switched off. Hence it seems to define a different topological Yang-Mills theory. Whether this difference is substantial or not is still to be clarified; anyhow, it is not accidental, rather it is deeply rooted in the properties of N=2 supersymmetry.

Some of the subtleties one encounters in twisting N=2,D=4 theories relate to the second item of the twisting programme, namely to the identification of the U(1) symmetry needed to shift the ghost-number. This identification is involved with the non-linear sigma model structure of the original N=2 theory, in particular with the special Kähler geometry of the vector multiplet coupling. We find out that the required U(1)-symmetry, named by us R-duality, exists, in the supergravity coupled case, if the Special Kähler manifold is chosen to be $SU(1,n)/SU(n) \times U(1)$, the so named minimal coupling case. In the flat case the needed U(1) also exists, as Witten construction shows, if the minimal coupling is selected. The point is that the minimal coupling in flat space and in curved space correspond to different unequivalent sigma model geometries: the flat C^n-manifold versus the special Kähler manifold $SU(1,n)/SU(n) \times U(1)$. This shows how the flat space limit of the gravity coupled topological Yang-Mills theory is in principle different from Donaldson theory as constructed by Witten.

Various subtleties of the D=4 topological twist are already encountered when studying the twist of pure N=2 supergravity. Indeed the greater complexity of N=2 supergravity with respect to N=2 super Yang-Mills forces us [19] to generalize the procedure of topological twist as introduced by Witten [10] in N=2 super Yang-Mills and at the same time lead us to reach a deeper understanding of its structure. In particular, we stress that the twist acts only on the Lorentz indices and not on the space-time indices [19] and this is quite natural in the formalism of differential forms. This feature of the twist avoids the problem encountered by Witten in Ref. [10], namely that the twisting procedure is meaningful only when space-time is R^4. We shall come back on this aspect extensively in the second part of the thesis.

When passing to the analysis of the topological sector of N=2 matter coupled supergravity, one soon realizes that there are some aspects of the twist that still need a deeper understanding. In particular, as we already pointed out, a fundamental question is the following: what is, in general terms, the $U(1)$ symmetry that leads to the ghost number of the topological version of a given theory? In N=2 super Yang-Mills, as well as in N=2 pure supergravity there is only one $U(1)$ internal symmetry (apart from global dimensional rescalings, that are not relevant to our discussion) and so either it works or not. Fortunately it works. However, in N=2 supergravity coupled to vector multiplets, there can be more that one internal $U(1)$; think for example of the $U(1)$ Kähler transformation or some $U(1)$ subgroup of the group of duality transformations [21] (at least when the vectors are not gauged). Anyway neither of these two known possibilities has the correct properties to become a ghost number and further on we show that indeed they cannot do the job. On the other hand one expects that a twist is possible, since the theory of topological gravity coupled to topological Yang-Mills should exist. In Ref. [19] we have shown how to produce a gauge-free algebra and generic observables for *any* topological theory and it would be very surprising to find that it is impossible to choose any kind of instantons to fix the topological symmetry and a gauge fermion to give a lagrangian to the theory. So, our work on the twist of matter coupled N=2 supergravity starts with the belief that if a suitable $U(1)$ internal charge is missing, this is because it is not known and not because it does not exist. As anticipated, it will be named R-duality, for reasons that we shall explain. First we shall define it and this will lead us to single out the basic properties an internal $U(1)$ symmetry should have in order to give ghost number. Then we shall explicitly prove invariance of the minimally coupled theory under this symmetry.

Our logical development is the following.

In chapter 3 we consider pure N=2 supergravity in the rheonomy framework and we construct its BRST quantization. In particular we discuss its gauge-free BRST algebra prior to the introduction of antighosts. Then we define D=4 topological gravity along the lines discussed above and we introduce the gauge-free topological BRST algebra. We also discuss the descent equations arising from the topological observables associated with the Pontriagin and Euler characteristic classes. Furthermore we discuss the topological twist of pure N=2 supergravity, introducing also the concept of *topological shift* that is instrumental for a correct interpretation of the resulting theory. We identify the ghosts and antighosts and from the latter identification we conclude that the gauge-fixing implicit in the theory is $\omega^{-ab} = 0$. Finally we show that the action of pure N=2 supergravity can be obtained as a topological term plus the BRST-variation of a gauge fermion Ψ that implements the gauge-fixing $\omega^{-ab} = 0$. Some subtleties related with the redundancy of this gauge-fixing and with the appearance of extraghosts are also discussed.

Chapter 4 begins with some general remarks on the possibility that minimal N=2 matter coupled supergravity possesses the desired internal $U(1)$ symmetry (R-duality). Then, after having recalled the structure of N=2 matter coupled supergravity in the rheonomy framework, we fully determine R-duality and prove that it is indeed an on shell symmetry of the theory. We then present the topologically twisted-topologically shifted theory (the gauge-free algebra, the complete BRST algebra, the explicit matching between them, the topological gauge-fixings, the observables, the gauge-fermion).

In chapter 5 we discuss the twist of quaternionic matter multiplets coupled to N=2 supergravity and along with this discussion, we summarize all the steps of the twisting procedure in four dimensions, improved by the experience of the present work.

Finally, chapter 6 contains our conclusions and outlook.

Chapter 3

Twisted Pure N=2 Supergravity

3.1 Introduction

In this chapter we study the topological twist of pure N=2 supergravity. As anticipated, it is convenient to work on the BRST-quantum version of N=2 supergravity, which is constructed in section 3.2. It is convenient to employ the formalism of differential forms and rheonomic parametrizations [18]. The concepts of rheonomy is applied to the construction of the BRST-quantum version of the theory in the way explained in Ref. [17]. Before performing the twist of this theory, we exhibit (section 3.3) the gauge-free algebra of topological gravity and the related observables. In section 3.4 the explicit twist is done (giving a gauge-fixed BRST algebra), while in section 3.5 we give the precise correspondence between the gauge-free algebra and the gauge-fixed algebra. Finally, in section 3.6 we show that the topologically twisted-shifted lagrangian of N=2 supergravity is BRST exact and we explicitly analyze some details (in particular, the redundancy of the gauge-fixing conditions of the topological symmetry).

3.2 BRST-Quantum Version of Pure D=4 N=2 Supergravity

D=4 N=2 pure supergravity is described by the following curvatures [18]

$$\begin{aligned} R^a &= \mathcal{D}V^a - \frac{i}{2}\bar{\psi}_A \wedge \gamma^a \psi_A = dV^a - \omega^a{}_b \wedge V^b - \frac{i}{2}\bar{\psi}_A \wedge \gamma^a \psi_A, \\ R^{ab} &= d\omega^{ab} - \omega^a{}_c \wedge \omega^{cb}, \\ \rho_A &= \mathcal{D}\psi_A = d\psi_A - \frac{1}{2}\omega^{ab} \wedge \sigma_{ab}\psi_A, \\ R^{\otimes} &= F + \epsilon_{AB}\bar{\psi}_A \wedge \psi_B, \end{aligned} \tag{3.1}$$

where Lorentz indices are denoted by latin letters, V^a is the one form representing the vierbein, ω^{ab} the one form representing the spin connection, ψ_A (A=1,2) is the couple of gravitinos (one forms as well), while $F \equiv dA$, A being the one form representing the graviphoton. d denotes the operation of exterior derivative, while $\mathcal{D}$ represents the covariant exterior derivative. Finally, $\sigma^{ab} \equiv \frac{1}{4}[\gamma^a, \gamma^b]$ and ϵ_{AB} is the completely antisymmetric tensor with two indices.

The above curvatures satisfy the following Bianchi identitites [18]

$$\begin{aligned} \mathcal{D}R^a + R^a{}_b \wedge V^b - i\bar{\psi}_A \wedge \gamma^a \rho_A &= 0, \\ \mathcal{D}R^{ab} &= 0, \\ \mathcal{D}\rho_A + \frac{1}{2}R^{ab} \wedge \sigma_{ab}\psi_A &= 0, \\ \mathcal{D}R^{\otimes} + 2\epsilon_{AB}\bar{\psi}_A \wedge \rho_B &= 0. \end{aligned} \tag{3.2}$$

The rheonomic parametrizations of the four curvatures (3.1) that are compatible with the Bianchi identities (3.2), at least on shell, since we do not introduce auxiliary fields, are [18]

$$\begin{aligned} R^a &= 0, \\ R^{ab} &= R^{ab}{}_{cd}V^c \wedge V^d + \bar{\theta}^{ab}_{A|c}\psi_A \wedge V^c - \frac{1}{2}\bar{\psi}_A \wedge \mathcal{F}^{ab}\psi_B\epsilon_{AB}, \\ \rho_A &= \rho^{ab}_A V_a \wedge V_b + \frac{1}{2}i\gamma^a\mathcal{F}_{ab}\psi_B \wedge V^b\epsilon_{AB}, \\ R^{\otimes} &= F_{ab}V^a \wedge V^b, \end{aligned} \tag{3.3}$$

where $\mathcal{F}^{ab} \equiv F^{ab} + \frac{i}{2}\gamma_5 F_{cd}\varepsilon^{abcd}$ and $\bar{\theta}^{ab|c}_A = 2i\bar{\rho}^{c[a}_A\gamma^{b]} - i\bar{\rho}^{ab}_A\gamma^c$ [18], where the square brakets denote antisymmetrization. These parametrizations are found by expanding the curvatures (3.1) in a basis of differential forms in superspace (which can be written as exterior products of V^a and ψ_A) and then imposing the Bianchi identities (3.2) on shell [18].

For completeness, we write here the lagrangian of N=2 supergravity, because it will be useful later on.

$$\begin{aligned} \mathcal{L} = \; & R^{ab} \wedge V^c \wedge V^d\epsilon_{abcd} + 4\bar{\rho}_A \wedge \gamma_5\gamma_a\psi_A \wedge V^a + 2iR^{\otimes} \wedge \bar{\psi}_A \wedge \gamma_5\psi_B\epsilon_{AB} + \\ & -2i\bar{\psi}_A \wedge \psi_B \wedge \bar{\psi}_A \wedge \gamma_5\psi_B - F^{ab}V^c \wedge V^d \wedge R^{\otimes}\epsilon_{abcd} + \\ & +\frac{1}{12}F_{ab}F^{ab}V^i \wedge V^j \wedge V^k \wedge V^l\varepsilon_{ijkl}. \end{aligned} \tag{3.4}$$

N=2 supergravity has an internal $SU(2)_I$ symmetry holding off-shell and an internal $U(1)$ symmetry, which, however, holds only on shell [22] (that is to say it is a symmetry of the equations of motion, but not of the lagrangian). This $U(1)$ internal symmetry combines chirality of the gravitinos with duality of the graviphoton in the following way [22]

$$\begin{aligned} \delta\psi_A &= i\alpha\gamma_5\psi_A, \\ \delta F_{ab} &= -2i\alpha\tilde{F}_{ab} = \alpha\epsilon_{abcd}F^{cd}. \end{aligned} \tag{3.5}$$

One easily verifies that the equations of motion derived from (3.4) are all invariant under the chiral-dual transformations (3.5). The less trivial case is the one of the field equation coming from the variation of the graviphoton A, that is

$$4i\epsilon_{AB}\bar{\rho}_A \wedge \gamma_5\psi_B - \mathcal{D}(F^{ab}V^c \wedge V^d)\epsilon_{abcd} = 0. \tag{3.6}$$

Its variation under (3.5) is the last of the Bianchi identities (3.2) and viceversa, thus proving U(1) on shell invariance (the remaining Bianchi identities are trivially invariant).

The operation of BRST transformation is denoted by s. We introduce ghost-number and s has ghost-number one. In such a way we have two natural gradations: form-number f and ghost-number g. As anticipated in the introduction, a generic object is described by the couple (f,g). When permuting two objects it is the sum $f+g$ that determines the correct sign (but note that some fields, like the gravitinos and their ghosts also have a fermionic number and when permuting two of them, the preceding rule must by suitably amended). So, $f+g$ is a gradation of primary importance. We shall call it the *ghost-form-number*. Any object must have a well defined ghost-form-number and so the first part of BRST quantization consists in extending any differential form Ω (of form-number f, say) to a *ghost-form* $\hat{\Omega}$ of ghost-form-number f. Let

$$\begin{aligned} \hat{V}^a &= V^a + \varepsilon^a, \\ \hat{\omega}^{ab} &= \omega^{ab} + \varepsilon^{ab}, \\ \hat{\psi}_A &= \psi_A + c_A, \\ \hat{A} &= A + c, \end{aligned} \tag{3.7}$$

where ε^a, ε^{ab}, c_A and c (form-number zero, ghost-number one) are the ghosts of diffeomorphisms, Lorentz rotations, supersymmetries and Maxwell transformations, respectively. For the time being, the spin connection is treated as an independent variable: later on we shall go over to second order formalism. It is useful to similarly extend the operation of exterior differentiation, as already mentioned in (2.3). As $d^2 = 0$ and the extension to hatted quantities preserves all the algebraic manipulations (as one can easily convince oneself), we are guaranteed that this property is extended to $\hat{d}^2 = 0$, that is to say

$$\begin{aligned} d^2 &= 0, \\ ds + sd &= 0, \\ s^2 &= 0. \end{aligned} \tag{3.8}$$

In particular we are guaranteed to find a well defined BRST algebra ($s^2 = 0$).

The curvatures are extended to ghost-forms of ghost-form-number two, that are the sum of a (2,0)-piece (the original curvature) plus a (1,1)-term and a (0,2)-term. These extra-terms will be fixed by the rheonomic parametrizations. Let it be

$$\begin{aligned} \hat{R}^a &= R^a + \psi^a + \phi^a, \\ \hat{R}^{ab} &= R^{ab} + \chi^{ab} + \eta^{ab}, \\ \hat{\rho}_A &= \rho_A + \xi_A + \zeta_A, \\ \hat{R}^{\otimes} &= R^{\otimes} + \psi + \phi. \end{aligned} \tag{3.9}$$

The same curvatures can be written by suitably extending the definitions (3.1) (that is to say by replacing nonhatted quantities with the corresponding hatted version). For example,

$$\hat{R}^a = R^a + \psi^a + \phi^a = \hat{d}\hat{V}^a - \hat{\omega}^{ab} \wedge \hat{V}_b - \frac{i}{2}\bar{\hat{\psi}}_A \wedge \gamma^a \hat{\psi}_A. \tag{3.10}$$

After explicit substitution and separation of the various (f, g)-parts, one can read, besides the definition of R^a itself,

$$\begin{aligned} sV^a &= \psi^a - \mathcal{D}\varepsilon^a + \varepsilon^{ab} \wedge V_b + \frac{i}{2}(\bar{\psi}_A \wedge \gamma^a c_A + \bar{c}_A \wedge \gamma^a \psi^A), \\ s\varepsilon^a &= \phi^a + \varepsilon^{ab} \wedge \varepsilon_b + \frac{i}{2}\bar{c}_A \wedge \gamma^a c_A. \end{aligned} \tag{3.11}$$

These are the BRST variations of V^a and ε^a (at least upon fixing ψ^a and ϕ^a). By analysing the remaining curvatures in a similar way, one gets the rest of the BRST algebra. We give only the results. The complete BRST algebra is

$$\begin{aligned} sV^a &= \psi^a - \mathcal{D}\varepsilon^a + \varepsilon^{ab} \wedge V_b + i\bar{c}_A \wedge \gamma^a \psi_A, \\ s\omega^{ab} &= \chi^{ab} - \mathcal{D}\varepsilon^{ab}, \\ s\varepsilon^a &= \phi^a + \varepsilon^{ab} \wedge \varepsilon_b + \frac{i}{2}\bar{c}_A \wedge \gamma^a c_A, \\ s\varepsilon^{ab} &= \eta^{ab} + \varepsilon^a{}_c \wedge \varepsilon^{cb}, \\ s\psi_A &= \xi_A - \mathcal{D}c_A + \frac{1}{2}\varepsilon^{ab}\sigma_{ab}\psi_A, \\ sc_A &= \zeta_A + \frac{1}{2}\varepsilon^{ab}\sigma_{ab}c_A, \\ sA &= \psi - dc - 2\epsilon_{AB}\bar{c}_A \wedge \psi_B, \\ sc &= \phi - \epsilon_{AB}\bar{c}_A \wedge c_B \end{aligned} \tag{3.12}$$

In a similar way one can also analyse the content of the hatted extensions of the Bianchi identities (3.2). One then finds two sets of variations: i) the variations of the curvatures themselves, i. e.

$$\begin{aligned} sR^a &= -\mathcal{D}\psi^a + \varepsilon^{ab} \wedge R_b - R^{ab} \wedge \varepsilon_b - \chi^{ab} \wedge V_b + i\bar{\psi}_A \wedge \gamma^a \xi_A + i\bar{c}_A \wedge \gamma^a \rho_A, \\ sR^{ab} &= -\mathcal{D}\chi^{ab} + \varepsilon^a{}_c \wedge R^{cb} - R^a{}_c \wedge \varepsilon^{cb}, \\ s\rho_A &= -\mathcal{D}\xi_A + \frac{1}{2}\varepsilon^{ab}\sigma_{ab}\rho_A - \frac{1}{2}R^{ab}\sigma_{ab}c_A - \frac{1}{2}\chi^{ab}\sigma_{ab}\psi_A, \\ sR^{\otimes} &= -d\psi - 2\epsilon_{AB}(\bar{\psi}_A \wedge \xi_B + \bar{c}_A \wedge \rho_B), \end{aligned} \tag{3.13}$$

that are consistent with their definitions (3.1) and with (3.12); ii) the variations of the *free parameters* ψ^a, ϕ^a, χ^{ab}, η^{ab}, ξ_A, ζ_A, ψ and ϕ,

$$\begin{aligned} s\psi^a &= -\mathcal{D}\phi^a + \varepsilon^{ab} \wedge \psi_b - \chi^{ab} \wedge \varepsilon_b - \eta^{ab} \wedge V_b + i\bar{\psi}_A \wedge \gamma^a \zeta_A + i\bar{c}_A \wedge \gamma^a \xi_A, \\ s\phi^a &= \varepsilon^{ab} \wedge \phi_b - \eta^{ab} \wedge \varepsilon_b + i\bar{c}_A \wedge \gamma^a \zeta_A, \end{aligned}$$

$$
\begin{aligned}
s\chi^{ab} &= -\mathcal{D}\eta^{ab} + \varepsilon^{ac} \wedge \chi_c{}^b - \chi^{ac} \wedge \varepsilon_c{}^b, \\
s\eta^{ab} &= \varepsilon^{ac} \wedge \eta_c{}^b - \eta^{ac} \wedge \varepsilon_c{}^b, \\
s\xi_A &= -\mathcal{D}\zeta_A + \frac{1}{2}\varepsilon^{ab}\sigma_{ab}\xi_A - \frac{1}{2}\chi^{ab}\sigma_{ab}c_A - \frac{1}{2}\eta^{ab}\sigma_{ab}\psi_A, \\
s\zeta_A &= \frac{1}{2}\varepsilon^{ab}\sigma_{ab}\zeta_A - \frac{1}{2}\eta^{ab}\sigma_{ab}c_A, \\
s\psi &= -d\phi - 2\epsilon_{AB}(\bar{c}_A \wedge \xi_B + \bar{\psi}_A \wedge \zeta_B), \\
s\phi &= -2\epsilon_{AB}\bar{c}_A \wedge \zeta_B.
\end{aligned}
\tag{3.14}
$$

Eq.s (3.14) and (3.13) are the specialization to the case of the BRST quantum algebra of N=2 supergravity of the last three equations in (2.1).

As a next step, we fix the free parameters (ψ^a, ϕ^a, χ^{ab}, η^{ab}, ξ_A, ζ_A, ψ and ϕ) by means of the rheonomic conditions [17, 18]. These conditions state that the parametrizations of the hatted curvatures are obtained by the old ones (see (3.3)) upon substitution of the forms V^a and ψ_A (the basis of forms in superspace) by their hatted quantities. For example, according to this prescription, $\hat{\rho}_A$ is equal to

$$
\hat{\rho}_A = \rho_{A|ab}\hat{V}^a \wedge \hat{V}^b + \frac{1}{2}i\gamma^a\mathcal{F}_{ab}\hat{\psi}_B \wedge \hat{V}^b\epsilon_{AB}. \tag{3.15}
$$

After use of (3.7) and (3.9) and separation of the various (f, g)-parts, one can read the definitions of ξ_A and ζ_A. In a similar way one prodeeds for the other curvatures and free parameters. We report here only the final result, that is

$$
\begin{aligned}
\psi^a &= 0, \\
\phi^a &= 0, \\
\chi^{ab} &= 2R^{ab}{}_{cd}V^c \wedge \varepsilon^d + \bar{\theta}^{ab}_{A|c}(c_A \wedge V^c + \psi_A \wedge \varepsilon^c) - \bar{c}_A\mathcal{F}^{ab}\psi_B\epsilon_{AB}, \\
\eta^{ab} &= R^{ab}{}_{cd}\varepsilon^c \wedge \varepsilon^d + \bar{\theta}^{ab}_{A|c}c_A \wedge \varepsilon^c - \frac{1}{2}\bar{c}_A\mathcal{F}^{ab}c_B\epsilon_{AB}, \\
\xi_A &= 2\rho_{A|ab}\varepsilon^a \wedge V^b + \frac{i}{2}\gamma^a\mathcal{F}_{ab}(c_B \wedge V^b + \psi_B \wedge \varepsilon^b)\epsilon_{AB}, \\
\zeta_A &= \rho_{A|ab}\varepsilon^a \wedge \varepsilon^b + \frac{i}{2}\gamma^a\mathcal{F}_{ab}c_B \wedge \varepsilon^b\epsilon_{AB}, \\
\psi &= 2F_{ab}V^a \wedge \varepsilon^b, \\
\phi &= F_{ab}\varepsilon^a \wedge \varepsilon^b.
\end{aligned}
\tag{3.16}
$$

One can verify that (3.14) are consistent with (3.16) on shell. This requires no further computational work than the one which is required to prove that the rheonomic parametrizations (3.3) are consistent with the Bianchi identities (3.2) [18] (the formal manipulations are the same).

By means of suitable redefinitions one can put formulas (3.12) in a more familiar form, i. e. to write diffeomorphisms in terms of Lie derivatives [17]. To this purpose, let $\varepsilon^\mu = \varepsilon^a V_a^\mu$, V_a^μ being the inverse vierbein, so that $\varepsilon^a = i_\varepsilon V^a$, where i denotes contraction.

The Lie derivative $\mathcal{L}_\varepsilon$ is equal to $i_\varepsilon d - d i_\varepsilon$, where the minus sign is due to the fact that ε is a ghost [17]. If we define $\varepsilon'^{ab} = \varepsilon^{ab} - i_\varepsilon \omega^{ab}$, $c'_A = c_A - i_\varepsilon \psi_A$ and $c' = c - i_\varepsilon A$, then we get

$$\begin{aligned} sV^a &= \mathcal{L}_\varepsilon V^a + \varepsilon'^{ab} \wedge V_b + i\bar{c}'_A \wedge \gamma^a \psi_A, \\ s\psi_A &= \mathcal{L}_\varepsilon \psi_A + \frac{1}{2}\varepsilon'^{ab}\sigma_{ab}\psi_A - \mathcal{D}c'_A + \frac{i}{2}\gamma^a \mathcal{F}_{ab} c'_B \wedge V^b \epsilon_{AB}, \\ sA &= \mathcal{L}_\varepsilon A - dc' - 2\epsilon_{AB}\bar{c}'_A \wedge \psi_B. \end{aligned} \tag{3.17}$$

We see that the variations of the main fields V^a, ψ_A and A are the sum of diffeomorphisms, Lorentz rotations, supersymmetries and Maxwell transformations, as it must be.

The last point reguards the possibility of employing the second order formalism, that is to say of expressing ω^{ab} in terms of the vierbein ($R^a = 0$). For consistency, we must also have $sR^a = 0$, and this gives a condition on χ^{ab}. Consequently, we should expect to have a condition on $s\chi^{ab}$, however $s\chi^{ab}$ turns out to be automatically consistent with (3.12) and so it imposes no further constraint.

3.3 Topological gravity

In this section we discuss the gauge-free BRST algebra of topological gravity. As already pointed out this BRST algebra involves only ghosts (and not antighosts). In the following sections we show that this algebra stands to the twist of the algebra of N=2 supergravity determined in section 3.2 as the Beaulieu-Singer approach [12] stands to the Witten approach [10]. The procedure resembles the construction of the BRST quantum version of supergravity, but the difference is that, according to the discussion of chapter 1 and section 3.2, we impose no rheonomic parametrization. We show that this prescription gives automatically a topological theory. Similarly, (3.12) and (3.14), without imposition of (3.16), are the gauge-free BRST algebra of topological N=2 supergravity.

As before, we define hatted quantities

$$\begin{aligned} \hat{d} &= d + s, \\ \hat{V}^a &= V^a + \varepsilon^a, \\ \hat{\omega}^{ab} &= \omega^{ab} + \varepsilon^{ab}, \\ \hat{R}^a &= R^a + \psi^a + \phi^a, \\ \hat{R}^{ab} &= R^{ab} + \chi^{ab} + \eta^{ab}, \end{aligned} \tag{3.18}$$

but now ψ^a, ϕ^a χ^{ab} and η^{ab} remain independent fields. From the definitions of the curvatures (2.9), extended to hatted expressions as before, and the Bianchi identities

$$\begin{aligned} \mathcal{D}R^a + R^a{}_b \wedge V^b &= 0, \\ \mathcal{D}R^{ab} &= 0, \end{aligned} \tag{3.19}$$

also extended to hatted quantities, one obtains the BRST algebra

$$\begin{aligned}
sV^a &= \psi^a - \mathcal{D}\varepsilon^a + \varepsilon^{ab} \wedge V_b, \\
s\omega^{ab} &= \chi^{ab} - \mathcal{D}\varepsilon^{ab}, \\
s\varepsilon^a &= \phi^a + \varepsilon^{ab} \wedge \varepsilon_b, \\
s\varepsilon^{ab} &= \eta^{ab} + \varepsilon^a{}_c \wedge \varepsilon^{cb}, \\
s\psi^a &= -\mathcal{D}\phi^a + \varepsilon^{ab} \wedge \psi_b - \chi^{ab} \wedge \varepsilon_b - \eta^{ab} \wedge V_b, \\
s\phi^a &= \varepsilon^{ab} \wedge \phi_b - \eta^{ab} \wedge \varepsilon_b, \\
s\chi^{ab} &= -\mathcal{D}\eta^{ab} + \varepsilon^{ac} \wedge \chi_c{}^b - \chi^{ac} \wedge \varepsilon_c{}^b, \\
s\eta^{ab} &= \varepsilon^{ac} \wedge \eta_c{}^b - \eta^{ac} \wedge \varepsilon_c{}^b.
\end{aligned} \tag{3.20}$$

Once more this is the specialization to the case of the Poincaré algebra of Eq.s (2.1). Of course, this algebra is also obtainable by reduction to N=0 of the N=2 algebra of Eq.s (3.12) and (3.14) (with no imposition of (3.16)).

We have used the same symbols as before, for similar, but different, quantities. Whenever necessary, we shall distinguish objects belonging to the BRST algebra of N=2 supergravity (3.12) from those of the BRST algebra of topological gravity (3.20) by an index, which will be 2 in the former case, 0 in the latter. For example, ${\omega_2}^{ab}$ will be the superconnection (coming from ${R_2}^a = 0$), while ${\omega_0}^{ab}$ will be the usual connection (coming from ${R_0}^a = 0$). The transformations (3.12) will be denoted by s_2, the transformations (3.20) by s_0. Similarly, we shall write ${\psi_2}^a$ and ${\psi_0}^a$, ${\phi_2}^a$ and ${\phi_0}^a$, et cetera.

As before, we are guaranteed that $s^2 = 0$, but now $s_0^2 = 0$ holds off-shell (it is the imposition of a rheonomic parametrization holding only on shell that forces $s_2^2 = 0$ to hold only on shell). Let us analyse (3.20) in more detail. As we see, ${\psi_0}^a$ represents the topological ghost and the variation of V^a is equal to the topological variation ${\psi_0}^a$ plus diffeomorphisms plus Lorentz rotations. ${\phi_0}^a$ and ${\eta_0}^{ab}$ are ghosts for ghost, the former corresponding to diffeomorphisms, the latter corresponding to Lorentz rotations. As for ${\chi_0}^{ab}$, in the second order formalism ($R^a = 0$) the condition $s_0 {R_0}^a = 0$ (which can be read from the first formula of (3.13) upon reduction to N=0) implies $\chi_0^{ab} \wedge V_b = -\mathcal{D}_0 {\psi_0}^a - {R_0}^{ab} \wedge \varepsilon_b$, which can be solved in the same manner as the condition defining ω_0^{ab} (i. e. ${\omega_0}^{ab} \wedge V_b = dV^a$). As noted in section 3.2, the fact that ${\chi_0}^{ab}$ depends on the other fields does not impose further constraints and the BRST algebra is well defined. From now on we shall employ the second order formalism.

The procedure here followed to determine a BRST algebra for topological gravity does not introduce any antighost. This is because we are not choosing any particular gauge-fixing. The topological twist, on the other hand, will give automatically a preferred gauge-fixing for the topological symmetry, as we shall see in the following section.

Now we describe the observables of the theory, which are related to the Pontriagin $\mathcal{P} = {R_0}^{ab} \wedge R_{0ab}$ and Euler characteristic classes $\mathcal{E} = {R_0}^{ab} \wedge {R_0}^{cd} \varepsilon_{abcd}$. $s{\phi_0}^a$ and $s{\eta_0}^{ab}$ should be compared with the variation of the ghost for ghost ϕ that appears in (2.1), $s\phi = -[c, \phi]$. As we see, the transformation of ϕ is nothing but a gauge transformation and so all gauge invariants constructed from ϕ are BRST invariants and can lead to the descent equations that give the observables of the theory [10, 12]. In our case it is ${\eta_0}^{ab}$ that has a BRST variation which is only a gauge transformation (Lorentz rotation). ${\eta_0}^{ab}$ is a 4×4 antisymmetric matrix. Any 4×4 matrix has the four invariants $\mathrm{tr}[\eta_0]$, $\mathrm{tr}[{\eta_0}^2]$, $\mathrm{tr}[{\eta_0}^3]$ and $\mathrm{tr}[{\eta_0}^4]$. In our case only $\mathrm{tr}[{\eta_0}^2]$ and $\mathrm{tr}[{\eta_0}^4]$ are nonvanishing. What are the corresponding descent equations and to what topological invariants do they correspond? It will be soon proved that they correspond to the Pontriagin number and to the Euler number.

We start by noticing that the proof that the form ${R_0}^{ab} \wedge R_{0ab}$ is closed works with hatted quantities, exactly as with nonhatted ones:

$$\hat{d}(\hat{R}_0{}^{ab} \wedge \hat{R}_{0ab}) = -2\hat{d}\hat{R}_0{}^{ab} \wedge \hat{R}_{0ba} = -2(\hat{\omega}_0{}^{ab} \wedge \hat{R}_{0bc} \wedge \hat{R}_0{}^c{}_a - \hat{R}_0{}^{ab} \wedge \hat{\omega}_{0bc} \wedge \hat{R}_0{}^c{}_a) = 0. \quad (3.21)$$

We have used the hatted Bianchi identity $\hat{\mathcal{D}}\hat{R}_0{}^{ab} = 0$. After explicit substitution and separation of the various (f, g)-parts, one can read the descent equations

$$\begin{aligned} s_0\, \mathrm{tr}[\eta_0 \wedge \eta_0] &= 0, \\ s_0\, \mathrm{tr}[\eta_0 \wedge \chi_0 + \chi_0 \wedge \eta_0] &= -d\, \mathrm{tr}[\eta_0 \wedge \eta_0], \\ s_0\, \mathrm{tr}[\eta_0 \wedge R_0 + \chi_0 \wedge \chi_0 + R_0 \wedge \eta_0] &= -d\, \mathrm{tr}[\eta_0 \wedge \chi_0 + \chi_0 \wedge \eta_0], \\ s_0\, \mathrm{tr}[R_0 \wedge \chi_0 + \chi_0 \wedge R_0] &= -d\, \mathrm{tr}[\eta_0 \wedge R_0 + \chi_0 \wedge \chi_0 + R_0 \wedge \eta_0], \\ s_0\, \mathrm{tr}[R_0 \wedge R_0] &= -d\, \mathrm{tr}[R_0 \wedge \chi_0 + \chi_0 \wedge R_0], \\ 0 &= -d\, \mathrm{tr}[R_0 \wedge R_0], \end{aligned} \quad (3.22)$$

where the trace refers to the Lorentz indices. So, we have the following observables

$$\begin{aligned} \mathcal{O}^{(0)} &= \mathrm{tr}[\eta_0 \wedge \eta_0], \\ \mathcal{O}^{(1)}_\gamma &= \int_\gamma \mathrm{tr}[\eta_0 \wedge \chi_0 + \chi_0 \wedge \eta_0], \\ \mathcal{O}^{(2)}_S &= \int_S \mathrm{tr}[\eta_0 \wedge R_0 + \chi_0 \wedge \chi_0 + R_0 \wedge \eta_0], \\ \mathcal{O}^{(3)}_V &= \int_V \mathrm{tr}[R_0 \wedge \chi_0 + \chi_0 \wedge R_0], \\ \mathcal{O}^{(4)}_{\mathcal{M}} &= \int_{\mathcal{M}} \mathrm{tr}[R_0 \wedge R_0], \end{aligned} \quad (3.23)$$

where $\mathcal{M}$ is the four dimensional manifold where the theory is defined (we suppose $\partial\mathcal{M} = 0$ for simplicity) and γ, S, and V are generic one-, two- and three-dimensional cycles on $\mathcal{M}$. So we have proved that $\mathrm{tr}[{\eta_0}^2]$ corresponds to the Pontriagin number. In precisely the same way, one can deduce descent equations and construct observables associated to the Euler form $\mathcal{E} = {R_0}^{ab} \wedge {R_0}^{cd}\epsilon_{abcd}$. These observables will be denoted by $\tilde{\mathcal{O}}^{(n)}$ and correspond to $\mathrm{tr}[\eta_0 \wedge \tilde{\eta_0}]$. As $\mathrm{tr}[{\eta_0}^4] = \frac{1}{16}(\mathrm{tr}[\eta_0 \wedge \tilde{\eta_0}])^2 + \frac{1}{2}(\mathrm{tr}[{\eta_0}^2])^2$, we see that we have exhausted the two invariants discussed before.

3.4 Topological twist of N=2 supergravity

The topological twist of N=2 supergravity is performed in a similar way as the topological twist of Yang Mills theories [10]. Nevertheless, some generalizations and specifications are needed. We identify the internal symmmetry group $SU(2)_I$ with $SU(2)_R$, the right handed part of the Lorentz group, that is to say we define a twisted $SU(2)'_R$ as the diagonal subgroup of $SU(2)_R \otimes SU(2)_I$. Let us fix a bit of notation. Every field will be classified, before the twist, by an expression like ${}^c(L,R,I)^g_f$, where L, R and I are the representation labels for $SU(2)_L$, $SU(2)_R$ and $SU(2)_I$ respectively, c is the $U(1)$ charge, g is the ghost number and f is the form degree. Some fields (the graviphoton and the corresponding ghosts) have not a well defined $U(1)$ charge and so c will be replaced by a dot in these cases. After the twist, each field will be denodet by $(L,R')^{g+c}_f$, where $R' = R \otimes I$. The new ghost number is the sum of the old ghost number and the old $U(1)$ charge. So, for some fields the new ghost number is not defined off-shell, but only on shell. However, we do not think this is a problem, rather one of the new features of ghost number conservation in topological theories. We note that ghost number conservation has particular features even in twisted Yang Mills theories [10], because the chiral anomaly of the untwisted theory appears as a ghost number anomaly in the twisted version of the theory. In two dimensional topological theories, the same phenomenon is represented by the appearance of a charge at infinity after the twist [8]. We think that the new features of ghost number conservation that appear in twisted N=2 supergravity deserve further investigation.

The fields are also characterized by a fermionic number, however it will not play an important role in the twisted theory. We shall explain this fact in the following section.

In Table 3.1 we list the fields of N=2 supergravity and their twisted counterparts. We see that the twisted version of ψ_A has a $(\frac{1}{2},\frac{1}{2})^1_1$ component. This is substantially the ghost of topological variations of the vierbein (the exact identification will be given in the following section). The components $(0,1)^{-1}_1$ and $(0,0)^{-1}_1$ become the corresponding antighosts. The variation of the $(0,1)^{-1}_1$ component, in particular, gives the gauge-fixing of the topological symmetry, precisely as in Yang-Mills theories. The twisted version of A represents ghosts for ghosts and antighosts for ghosts. This is because the tensor F^{ab} has two components of $U(1)$ charge ± 2, and so the twisted version of F^{ab} has two components of ghost number ± 2.

In Table 3.2 we list the antighosts and Lagrange multipliers of N=2 supergravity and their twisted counterparts. $\bar{\varepsilon}^a$ and $\bar{\varepsilon}^{ab}$ are the antighosts of diffeomorphisms and Lorentz rotations, respectively; π^a and π^{ab} are the corresponding Lagrange multipliers; $\bar{c}^*_A$ are the antighosts of supersymmetries and P_A are their Lagrange multipliers; $\bar{c}$ and P are the antighost and Lagrange multiplier of the Maxwell gauge-symmetry. In Table 3.3 (at the end of this chapter) we give a summary of all the fields involved in the BRST quantum algebra of N=2 supergravity, their twisted version and their meaning.

The explicit twist can be realized by interpreting the internal indices A, B as dotted indices $\dot{\alpha}$, $\dot{\beta}$. Refer to the appendix (chapter 7) for the notation. The left handed and

Table 3.1: Topological twist

Field	Before the twist	After the twist
V^a	${}^0(\frac{1}{2},\frac{1}{2},0)^0_1$	$(\frac{1}{2},\frac{1}{2})^0_1$
ε^a	${}^0(\frac{1}{2},\frac{1}{2},0)^1_0$	$(\frac{1}{2},\frac{1}{2})^1_0$
ε^{ab}	${}^0(1,0,0)^1_0 \oplus {}^0(0,1,0)^1_0$	$(1,0)^1_0 \oplus (0,1)^1_0$
ψ_A	${}^1(\frac{1}{2},0,\frac{1}{2})^0_1 \oplus {}^{-1}(0,\frac{1}{2},\frac{1}{2})^0_1$	$(\frac{1}{2},\frac{1}{2})^1_1 \oplus (0,1)^{-1}_1 \oplus (0,0)^{-1}_1$
c_A	${}^1(\frac{1}{2},0,\frac{1}{2})^1_0 \oplus {}^{-1}(0,\frac{1}{2},\frac{1}{2})^1_0$	$(\frac{1}{2},\frac{1}{2})^2_0 \oplus (0,1)^0_0 \oplus (0,0)^0_0$
A	${}^{\cdot}(0,0,0)^0_1$	$(0,0)^{\cdot}_1$
c	${}^{\cdot}(0,0,0)^1_0$	$(0,0)^{\cdot}_0$

right handed components of ψ_A are twisted as follows

$$\begin{aligned} \psi_{\alpha A} &\rightarrow \psi_{\alpha \dot{A}}, \\ \psi^{\dot{\alpha}}{}_A &\rightarrow \psi^{\dot{\alpha}\dot{A}}, \end{aligned} \tag{3.24}$$

while $\epsilon_{AB} \rightarrow \epsilon_{\dot{A}\dot{B}} = -\epsilon^{\dot{A}\dot{B}}$. Let us now consider the supersymmetry transformations (which can be read from (3.17) when $\varepsilon^a = 0$ and $\varepsilon^{ab} = 0$)

$$\begin{aligned} \delta V^a &= i\bar{c}_A \wedge \gamma^a \psi_A, \\ \delta \psi_A &= -\mathcal{D}c_A + \frac{i}{2}\gamma^a \mathcal{F}_{ab} c_B \wedge V^b \epsilon_{AB}, \\ \delta A &= -2\epsilon_{AB}\bar{c}_A \wedge \psi_B. \end{aligned} \tag{3.25}$$

We now twist these transormations and specialize the twisted version of the supersymmetry ghost c_A to its $(0,0)^0_0$-component. This component is $C \equiv c_{\dot{\alpha}}{}^{\dot{A}}\delta^{\dot{\alpha}}_{\dot{A}}$ (see the appendix). We set it equal to a constant and precisely $-ie$, for convenience. Here, e is an object that rearranges the form-number, ghost-number and statistics in the correct way and that appears only in the intermediate steps of the twist. It will be called the *broker*. The broker is a zero-form with fermionic statistics and ghost number one. e^2 has even ghost number and Bose statistics, hence it can be set equal to a number and in our notation we normalize it as $e^2 = 1$.

The twisted version of ψ_A consists of a $(\frac{1}{2},\frac{1}{2})^1_1$-component, which will be denoted by $\tilde{\psi}^m = \frac{e}{2}\psi_{\alpha\dot{A}}(\bar{\sigma}^m)^{\dot{A}\alpha}$ (the ghost of topolgical symmetry), a $(0,1)^{-1}_1$-component, which will be denoted by $-e\tilde{\psi}^{ab} = (\bar{\sigma}^{ab})^{\dot{A}}{}_{\dot{\alpha}}\psi^{\dot{\alpha}}{}_{\dot{A}}$ (the antighost corresponding to the gauge breaking of the topological symmetry) and a $(0,0)^{-1}_1$-component, denoted by $\tilde{\psi} = -e\psi_{\dot{\alpha}}{}^{\dot{A}}\delta^{\dot{\alpha}}_{\dot{A}}$ (see Table 3.1 and Appendix A). As an example of the action of the broker e, note that, while $\frac{1}{2}\psi_{\alpha\dot{A}}(\bar{\sigma}^a)^{\dot{A}\alpha}$ is a one-form, is a fermion and has ghost number zero, the true topological ghost $\tilde{\psi}^a$ must be a one-form, with ghost number one and it is a boson.

The transformations (3.25) become

$$\delta V^a = \tilde{\psi}^a,$$

Table 3.2: Twist of antighosts and Lagrange multipliers

Field	Before the twist	After the twist
$\bar{\varepsilon}^a$	${}^0(\frac{1}{2},\frac{1}{2},0)_0^{-1}$	$(\frac{1}{2},\frac{1}{2})_0^{-1}$
$\bar{\varepsilon}^{ab}$	${}^0(1,0,0)_0^{-1} \oplus {}^0(0,1,0)_0^{-1}$	$(1,0)_0^{-1} \oplus (0,1)_0^{-1}$
$\bar{c}_A^*$	${}^{-1}(\frac{1}{2},0,\frac{1}{2})_0^{-1} \oplus {}^1(0,\frac{1}{2},\frac{1}{2})_0^{-1}$	$(\frac{1}{2},\frac{1}{2})_0^{-2} \oplus (0,1)_0^0 \oplus (0,0)_0^0$
$\bar{c}$	${}^{\cdot}(0,0,0)_0^{-1}$	$(0,0)_0^{\cdot}$
π^a	${}^0(\frac{1}{2},\frac{1}{2},0)_0^0$	$(\frac{1}{2},\frac{1}{2})_0^0$
π^{ab}	${}^0(1,0,0)_0^{-1} \oplus {}^0(0,1,0)_0^{-1}$	$(1,0)_0^0 \oplus (0,1)_0^{-1}$
P_A	${}^{-1}(\frac{1}{2},0,\frac{1}{2})_0^0 \oplus {}^1(0,\frac{1}{2},\frac{1}{2})_0^0$	$(\frac{1}{2},\frac{1}{2})_0^{-1} \oplus (0,1)_0^1 \oplus (0,0)_0^1$
P	${}^{\cdot}(0,0,0)_0^0$	$(0,0)_0^{\cdot}$

$$
\begin{aligned}
\delta\tilde{\psi}^a &= \frac{1}{2}F^{+ab} \wedge V_b, \\
\delta\tilde{\psi}^{ab} &= \frac{i}{2}\omega^{-ab}, \\
\delta\tilde{\psi} &= 0, \\
\delta A &= i\tilde{\psi}.
\end{aligned}
\tag{3.26}
$$

These transformations should be compared with those of topological Yang-Mills theories, as found in Ref. [10]. As we see, the topological gauge-fixing is the antiselfdual part of the spin connection. Its vanishing describes the gravitational instantons of the theory of topological gravity that we are studying.

The square of the transformation (3.26) is not zero, but it is a Lorentz rotation with field-dependent parameters $\frac{1}{2}F^{+ab}$. This can be immediately deduced from the fact that $\delta^2 V^a = \delta\tilde{\psi}^a = \frac{1}{2}F^{+ab} \wedge V_b$. A phenomenon like the present one also happens in topological Yang-Mills theories [10]. It is only when dealing with the complete BRST algebra [12] that the square of the transformations is zero (at least on shell).

In [12] we see that the complete BRST symmetry of topological Yang-Mills theories derives from a composition of the BRST symmetry of the untwisted N=2 super Yang-Mills theory and the $(0,0)_0^0$-component of the supersymmetric transformations. We want to consider the analogue of this mechanism in twisted topological gravity. Here we have to deal with the fact that now supersymmetry is a local symmetry and nevertheless we expect that the twisted BRST symmetry is in some sense a composition of the twisted version of the transformations (3.12) and the transformations (3.26). At the same time we need to be sure that the new BRST symmetry closes on shell. We cannot simply specialize the twisted version of supersymmetry transformations to their $(0,0)_0^0$-component, because this would require to set some ghosts equal to zero, thus not guaranteeing $s^2 = 0$. A simple way to overcome all this is to shift the $(0,0)_0^0$-component C of the twisted version of the ghosts c_A (multiplied by $-e$) by a constant, namely $C \to C + i$. In such a way the new BRST transformations of the main fields are the old ones plus the transformations (3.26), as we would like, and closure on shell is automatically assured. The procedure of shifting C will be called *topological shift*. The topological shift should be considered as a mere trick to reach our purpose to define a suitable new BRST symmetry and should not be reguarded as substantial.

The twisted-shifted BRST symmetry will be denoted by s'. It will not be explicitly written down here; we only make some observations. The topological twist and the topological shift make the new BRST transformations appear as follows

$$\begin{aligned} s'V^a &= \tilde{\psi}^a - d\varepsilon^a + \varepsilon^{ab} \wedge V_b + \cdots, & s'\omega^{ab} &= \chi^{ab} - d\varepsilon^{ab} + \cdots, \\ s'\varepsilon^a &= C^a + \cdots, & s'\varepsilon^{ab} &= -\tfrac{1}{2}F^{+ab} + \cdots, \\ s'\eta^{ab} &= 0 + \cdots, & s'\tilde{\psi}^a &= -dC^a + \tfrac{1}{2}F^{+ab} \wedge V_b + \cdots, \\ s'\tilde{\psi}^{ab} &= -dC^{ab} + \tfrac{i}{2}\omega^{-ab} + \cdots, & s'\tilde{\psi} &= -dC + \cdots, \\ s'C^a &= 0 + \cdots, & s'C^{ab} &= \tfrac{i}{2}\varepsilon^{-ab} + \cdots, \\ s'C &= 0 + \cdots, & s'A &= i\tilde{\psi} - dc + \cdots, \\ s'c &= -\tfrac{1}{2} + iC + \cdots. \end{aligned} \tag{3.27}$$

where $\chi^{ab} \wedge V_b = -d\tilde{\psi}^a + \cdots$ and the dots refer to interactions terms, i. e. terms involving products of two or more fields. $C^a = \frac{e}{2}c_{\alpha\dot{A}}(\bar{\sigma}^a)^{\dot{A}\alpha}$, $C^{ab} = -e(\bar{\sigma}^{ab})^{\dot{A}}{}_{\dot{\alpha}}c^{\dot{\alpha}}{}_{\dot{A}}$ and $C = -ec_{\dot{\alpha}}{}^{\dot{A}}\delta^{\dot{\alpha}}_{\dot{A}}$ (see Table 3.1 and the appendix). The BRST transformation of the Maxwell ghost c contains a constant $-\frac{1}{2}$. This constant is inessential and can be suppressed. In fact c appears only in $s'A$ as dc (not even the dots contain c). Consequently, $s'^2 = 0$ is assured even if we write $s'c = iC + \cdots$. The transformations (3.27) will be useful for the computations of the next section; in particular, note that, according to (3.16), $\eta^{ab} = -\frac{1}{2}F^{+ab} + \cdots$, so (3.27) shows that $s'\varepsilon^{ab} = \eta^{ab} + \cdots$, in agreement with (3.20).

3.5 Matching between Twisted N=2 Supergravity and Topological Gravity

In this section we give the correspondence between the transformations s' (i. e. the topologically twisted and topologically shifted version of s_2 (3.12)) and the transformations s_0 (3.20), at least for what reguards the sector that not includes antighosts. First of all, let us compare

$$s_2 V^a = -\mathcal{D}_2\varepsilon^a + \varepsilon^{ab} \wedge V_b + i\bar{c}_A \wedge \gamma^a \psi_A \tag{3.28}$$

with

$$s_0 V^a = \psi_0^a - \mathcal{D}_0\varepsilon^a + \varepsilon^{ab} \wedge V_b. \tag{3.29}$$

Let us put $\omega_2^{ab} = \omega_0^{ab} - A^{ab}$, where A^{ab} is determined by the condition $A^{ab} \wedge V_b = \frac{i}{2}\bar{\psi}_A \wedge \gamma^a \psi_A$. We can identify the two variations of V^a (i. e. impose $s_2 V^a = s_0 V^a$) if we put

$$\psi_0{}^a = i\bar{c}_A \wedge \gamma^a \psi_A - A^{ab} \wedge \varepsilon^b = \tilde{\psi}^a + \cdots. \tag{3.30}$$

As we see, there is no need to make the topological twist and the topological shift explicit. By comparing

$$s_2\varepsilon^a = \varepsilon^{ab}\wedge\varepsilon_b + \frac{i}{2}\bar{c}_A\wedge\gamma^a c_A \tag{3.31}$$

with

$$s_0\varepsilon^a = {\phi_0}^a + \varepsilon^{ab}\wedge\varepsilon_b, \tag{3.32}$$

we deduce

$${\phi_0}^a = \frac{i}{2}\bar{c}_A\wedge\gamma^a c_A = C^a + \cdots. \tag{3.33}$$

By comparing $s_2\varepsilon^{ab}$ and $s_0\varepsilon^{ab}$, one deduces ${\eta_2}^{ab} = {\eta_0}^{ab}$. After making these identifications, one can check by a direct but tedious computation that $s_2{\psi_0}^a$, $s_2{\phi_0}^a$, $s_2{\chi_0}^{ab}$ $s_2{\eta_0}^{ab}$ and $s_2{\omega_0}^{ab}$ automatically match with the corresponding s_0-transformations.

Let us make a comment to the fact that the above identifications involve bilinear terms in the fields. This problem is promptly solved by the topological shift that reduces the above products of fields to a term linear in the fields plus interactions. The presence of these interactions is the consequence of the already noted fact that, since supersymmetry is local, we cannot specialize it to its $(0,0)^0_0$ component. Moreover, a little insight shows that the appearance of the bilinear terms is all but a problem. In fact, we must remember that in the topological twist chirality adds to ghost number; since the commutation properties between s and the fields is regulated by ghost-form number, it could happen, in general, that a field changes its properties of commutation with s during the twist. This would be surely dangerous, because it is important to preserve all formal manipulations to guarantee $s^2 = 0$ on shell. Having to deal with bilinear terms, we are sure that the commutation properties do not change, since chirality is always even. An analogous observation can be done about fermion number; furthermore, since in the twisted theory fermion number has no importance, we can simply forget about it. By means of bilinear terms it is also possible to define the antighosts ${\psi_0}^{ab}$ and ψ_0, at least up to interaction pieces. The bilinear terms corresponding to them are $\bar{c}_A\wedge\sigma^{ab}\frac{1-\gamma_5}{2}\psi_B\epsilon_{AB}$ and $\bar{c}_A\wedge\psi_B\epsilon_{AB}$.

The above identifications permit to get explicitly the observables of the twisted theory, by simply taking the definitions (3.23), rewriting them in the twisted notation and shifting the gost C. All that is needed is ${\chi_0}^{ab}$ and ${\eta_0}^{ab}$, which are given by

$$\begin{aligned}{\eta_0}^{ab} &= R_2^{ab}{}_{cd}\varepsilon^c\wedge\varepsilon^d + \bar{\theta}^{ab}_{A|c}c_A\wedge\varepsilon^c - \frac{1}{2}\bar{c}_A\mathcal{F}^{ab}c_B\epsilon_{AB}, \\ {\chi_0}^{ab}\wedge V_b &= -\mathcal{D}_0\psi_0^a - R_0^{ab}\wedge\varepsilon_b. \end{aligned} \tag{3.34}$$

3.6 The lagrangian of topological gravity

In this section we discuss the twisted-shifted version of the lagrangian (3.4) of N=2 supergravity. In particular, we want to show that it can be written as the BRST variation of a gauge fermion Ψ. We shall be satisfied of a gauge fermion Ψ that reproduces the kinetic terms of the twisted-shifted N=2 supergravity lagrangian. In fact any gauge fermion can be corrected by adding to it interaction terms and indeed, even if they are not explicitly written, they are in general required in perturbation theory (a similar remark is made in [17]). In any case, the main requirement for a good gauge fermion is that it must remove all degeneracies of the kinetic terms and permit the definition of propagators, so we are justified in concentrating our attention on the kinetic terms, a restriction that simplifies considerably the computational effort.

First of all, we re-write the gravitational action in the second order formalism in a convenient way. As a matter of fact, one easily verifies that

$$\begin{aligned}\mathcal{L} &= R^{ab} \wedge V^c \wedge V^d \epsilon_{abcd} = \\ &= 2\omega^{ab} \wedge \omega^c{}_e \wedge V^e \wedge V^d \epsilon_{abcd} + \\ &\quad -\omega^{ae} \wedge \omega_e{}^b \wedge V^c \wedge V^d \epsilon_{abcd} + d(\omega^{ab} \wedge V^c \wedge V^d \epsilon_{abcd}),\end{aligned} \tag{3.35}$$

and that

$$\begin{aligned}\mathcal{A} &\equiv 4\omega^{-ab} \wedge \mathcal{M}_{ab,cd} \wedge \omega^{-cd} \equiv \\ &\equiv 4\omega^{-ab} \wedge \omega^{-c}{}_e \wedge V^e \wedge V^d \epsilon_{abcd} - 2\omega^{-ae} \wedge \omega_e^{-b} \wedge V^c \wedge V^d \epsilon_{abcd} = \\ &= \mathcal{L} - d(\omega^{ab} \wedge V^c \wedge V^d \epsilon_{abcd} - 2iV^a \wedge dV_a).\end{aligned} \tag{3.36}$$

In other words, we have written the gravitational lagrangian as quadratic in the anti-selfdual part of the spin connection, which is our gauge-fixing, plus a total derivative. $\mathcal{M}_{ab,cd}$ is a two form and is independent from derivatives of the vierbein. This way of expressing the gravitational lagrangian (up to a topological term) should be compared with the expression $-\frac{1}{4}\mathrm{tr}[F^-_{\mu\nu} F^{-\mu\nu}]$ for the lagrangian of Yang-Mills theories, which is the square of the gauge-fixing of topological Yang-Mills theories [12]. Making space-time components explicit, we can write

$$\mathcal{A} \equiv 4\omega_\mu^{-ab} \wedge \mathcal{M}_{\nu,\rho ab,cd} \wedge \omega_\sigma^{-cd} \epsilon^{\mu\nu\rho\sigma}\, d^4x. \tag{3.37}$$

$\mathcal{M}_{\nu,\rho ab,cd}$ is a matrix which is antysimmetric and antiselfdual in ab and cd. One can easily verify that there exist only two such matrices in flat space, namely the identity $\mathcal{I}_{\nu,\rho ab,cd} = \frac{1}{4}\eta_{\nu\rho}(\eta_{ac}\eta_{bd} - \eta_{ad}\eta_{bc} + i\epsilon_{abcd})$, where $\eta_{ab} = \mathrm{diag}(1, -1, -1, -1)$, and $\mathcal{M}_{\nu,\rho ab,cd}$ itself. Furthemore, $\mathcal{M}$ is invertible (one proves that $\mathcal{M}^2$ is not proportional to $\mathcal{M}$, so it must be a nontrivial linear combination of $\mathcal{M}$ and $\mathcal{I}$).

First of all, we introduce a Lagrange multiplier B^{ab} for the topological symmetry (a one form, antisymmetric and antiselfdual in ab), such that $s'\psi^{ab} = B^{ab}$ and $s'B^{ab} = 0$. In this section we omit the subscript 0 in ${\psi_0}^a$, ${\psi_0}^{ab}$ and ψ_0. By comparing with the old expression for $s'\psi^{ab}$, (3.27), we see that the linear part of the gauge-fixing term is $\frac{i}{2}\omega^{-ab} - dC^{ab}$, that is to say there is a ghost term besides the expected term $\frac{i}{2}\omega^{-ab}$. We shall explain in a short time the reason for this presence. In any case, we expect that the gauge-fermion Ψ contains a term

$$\Psi_1 = 16i(iB^{ab} + \omega^{-ab} + 2idC^{ab}) \wedge \mathcal{M}_{ab,cd} \wedge \psi^{cd}. \tag{3.38}$$

Indeed, the BRST variation of Ψ_1, i. e. $s'\Psi_1$, contains a term

$$-16i(iB^{ab}+\omega^{-ab}+2idC^{ab})\wedge\mathcal{M}_{ab,cd}\wedge B^{ab}, \tag{3.39}$$

which, upon integration over B^{ab} gives

$$4(\omega^{-ab}+2idC^{ab})\wedge\mathcal{M}_{ab,cd}\wedge(\omega^{-cd}+2idC^{cd}). \tag{3.40}$$

So, the gravitational lagrangian (3.35) is correctly reproduced (at least up to a topological term), but there are two more terms, namely $16idC^{ab}\wedge\mathcal{M}_{ab,cd}\wedge\omega^{-cd}$ and $-16dC^{ab}\wedge\mathcal{M}_{ab,cd}\wedge dC^{cd}$. The first one is zero (or better, it is a total derivative), because

$$d(\mathcal{M}_{ab,cd}\wedge\omega^{-cd})\equiv 0, \tag{3.41}$$

as can be promptly checked. The second term looks like, at first sight, the kinetic lagrangian for the ghosts C^{ab}, however this is not true, because the kinetic part of $-16dC^{ab}\wedge\mathcal{M}_{ab,cd}\wedge dC^{cd}$ turns out to be zero. From these remarks we can deduce two considerations:

i) the gauge-fixing $\omega^{-ab}=0$ is redundant, because the one forms ω^{-ab} are not independent, but are related by the condition (3.41), which holds identically, without imposition of $\omega^{-ab}=0$;

ii) the ghost C^{ab} is an extraghost, i. e. a ghost the presence of which is due to a redundancy of the gauge-fixing; of course, it is associated to the redundancy (3.41) of the topological gauge-fixing conditions $\omega^{-ab}=0$.

A good treatment of such nontrivial ghosts of vanishing ghost number can be found in Ref. [23], where the case of the antisymmetric tensor, call it $B_{\mu\nu}$, is explicitly exhibited. In that case the BRST variation of the antighost $\bar{C}_\mu$ ($\delta\bar{C}_\mu=\partial^\nu B_{\mu\nu}-\partial_\mu c_1$) contains, as well as the expected gauge-fixing term, $\partial^\nu B_{\mu\nu}$, a term involving the extraghost c_1 and giving information about the redundancy (which is $\partial_\mu\partial_\nu B^{\mu\nu}\equiv 0$), precisely as it happens in our case. However, in the simple example of the antisymmetric tensor $B_{\mu\nu}$ the analogous term of $-16dC^{ab}\wedge\mathcal{M}_{ab,cd}\wedge dC^{cd}$ does give the kinetic lagrangian of the extraghost c_1 and so there is no further problem. In our case, instead, this does not happen (the reason is the richness of symmetries of our theory, in particular local supersymmetry). Since, as previously noted, only one matrix with the properties of $\mathcal{M}$ exists besides $\mathcal{M}$ itself, that is the identity $\mathcal{I}$, there is little to do: to give a kinetic term to C^{ab}, it is necessary to have a further extraghost, say C^{*ab} (of ghost number zero, antiselfdual in ab) and a kinetic term $dC^{*ab}\wedge\mathcal{I}_{ab,cd}\wedge dC^{cd}$, that is to say $C^{*ab}\Box C_{ab}$ plus interactions. However, since we are only reinterpreting a theory and we cannot construct it by hand, such a field and such a kinetic term must already be present. In particular C^{*ab} can only come from the twist of the antighost $\bar{c}^*_A$ of N=2 local supersymmetry and as a matter of fact, Table 3.2 shows that such a field is indeed present and it is precisely the $(0,1)^0_0$-component of the twisted version of $\bar{c}^*_A$. So, all we have to do is to check that the gauge-fermion that breaks supersymmetry, say Ψ_S, gives the correct kinetic lagrangian for C^{ab} and C^{*ab}. Let us choose the most common expression for Ψ_S, i. e.

$$\Psi_S\equiv\bar{c}^*_A\not{D}(\gamma^a V^\mu_a\psi_{\mu A}+\alpha P_A), \tag{3.42}$$

where P_A is the lagrange multiplier of supersymmetries ($s'c^*_A = P_A$, $s'P_A = 0$) and α is a constant that is usually determined in order to conveniently simplify the kinetic term of gravitinos (α will be of no importance for our purposes). In any case, the BRST variation of Ψ_S contains a term of the kind

$$\bar{c}^*_A \partial\!\!\!/ \gamma^a V^\mu_a \partial_\mu c_A. \tag{3.43}$$

As we expect, after the twist, the quadratic term in $C^{*ab} - C^{cd}$ is $C^{*ab}\Box C_{ab}$.

Let us now come back to the analysis of the BRST variation of the gauge-fermion Ψ_1, (3.38). The terms $-16i(iB^{ab} + \omega^{-ab} + 2idC^{ab}) \wedge s'(\mathcal{M}_{ab,cd}) \wedge \psi^{cd}$ are only interaction terms and so we discard them. Then there are the terms $16is'(\omega^{-ab} + 2idC^{ab}) \wedge \mathcal{M}_{ab,cd} \wedge \psi^{cd}$. By looking at (3.27), one sees that the term with dC^{ab} in Ψ_1 is required in order to restore invariance under Lorentz rotations (i. e. in order to avoid kinetic terms like $d\varepsilon^{-ab} \wedge \mathcal{M}_{ab,cd} \wedge \psi^{cd}$). So, the only kinetic term coming from Ψ_1 that remains to be discussed is $16i\chi^{-ab} \wedge \mathcal{M}_{ab,cd} \wedge \psi^{cd}$. This term reproduces the twisted version of the Rarita-Schwinger action, precisely the part that contains ψ^a and ψ^{ab}, which turns out to be

$$16d\psi^a \wedge \psi_{ab} \wedge V^b. \tag{3.44}$$

The remaining piece of the Rarita-Schwinger action, namely

$$-8d\psi^a \wedge \psi \wedge V_a, \tag{3.45}$$

can be retrieved by means of a further piece Ψ_2 to be added to the gauge fermion Ψ_1. Ψ_2 must also give account of the kinetic term of the graviphoton and turns out to be (remember $\eta^{ab} = -\frac{1}{2}F^{+ab} + \cdots$ and $R^\otimes = dA + \cdots$)

$$\Psi_2 = 8iR^\otimes \wedge \psi^a \wedge V_a + \frac{2}{3}\eta^{ab}\varepsilon_{ab}V^i \wedge V^j \wedge V^k \wedge V^l \varepsilon_{ijkl}. \tag{3.46}$$

Summarizing, the total gauge-fermion is

$$\begin{aligned} \Psi &= 16i(iB^{ab} + \omega^{-ab} + 2idC^{ab}) \wedge \mathcal{M}_{ab,cd} \wedge \psi^{cd} + \\ &\quad +8iR^\otimes \wedge \psi^a \wedge V_a + \frac{2}{3}\eta^{ab}\varepsilon_{ab}V^i \wedge V^j \wedge V^k \wedge V^l \varepsilon_{ijkl}, \end{aligned} \tag{3.47}$$

plus the usual terms that break diffeomorphisms, Lorentz rotations, supersymmetries (this one being in part already discussed) and Maxwell gauge-symmetry.

Table 3.3: Summary of the fields of N=2 supergravity and their twisted versions

Field	Meaning	Classification	Tw.-field	Twisted meaning	Tw.-classification
V^a	vierbein	${}^0(\frac{1}{2},\frac{1}{2},0)^0_1$	V^a	vierbein	$(\frac{1}{2},\frac{1}{2})^0_1$
ψ_A	gravitinos	${}^1(\frac{1}{2},0,\frac{1}{2})^0_1 \oplus {}^{-1}(0,\frac{1}{2},\frac{1}{2})^0_1$	ψ^a	topological ghost	$(\frac{1}{2},\frac{1}{2})^1_1$
			ψ^{ab}	topological antighost	$(0,1)^{-1}_1$
			ψ	antighost	$(0,0)^{-1}_1$
A	graviphoton	${}^{\cdot}(0,0,0)^0_1$	A	ghosts for ghosts	$(0,0)^{\cdot}_1$
ε^a	ghost	${}^0(\frac{1}{2},\frac{1}{2},0)^1_0$	ε^a	ghost	$(\frac{1}{2},\frac{1}{2})^1_0$
π^a	L. multiplier	${}^0(\frac{1}{2},\frac{1}{2},0)^0_0$	π^a	L. multiplier	$(\frac{1}{2},\frac{1}{2})^0_0$
$\bar{\varepsilon}^a$	antighost	${}^0(\frac{1}{2},\frac{1}{2},0)^{-1}_0$	$\bar{\varepsilon}^a$	antighost	$(\frac{1}{2},\frac{1}{2})^{-1}_0$
ε^{ab}	ghost	${}^0(1,0,0)^1_0 \oplus {}^0(0,1,0)^1_0$	ε^{ab}	ghost	$(1,0)^1_0 \oplus (0,1)^1_0$
π^{ab}	L. multiplier	${}^0(1,0,0)^{-1}_0 \oplus {}^0(0,1,0)^{-1}_0$	π^{ab}	L. multiplier	$(1,0)^{-1}_0 \oplus (0,1)^{-1}_0$
$\bar{\varepsilon}^{ab}$	antighost	${}^0(1,0,0)^{-1}_0 \oplus {}^0(0,1,0)^{-1}_0$	$\bar{\varepsilon}^{ab}$	antighost	$(1,0)^{-1}_0 \oplus (0,1)^{-1}_0$
c_A	ghost	${}^1(\frac{1}{2},0,\frac{1}{2})^1_0 \oplus {}^{-1}(0,\frac{1}{2},\frac{1}{2})^1_0$	C^a	ghost for ghost	$(\frac{1}{2},\frac{1}{2})^2_0$
			C^{ab}	extraghost	$(0,1)^0_0$
			C	extraghost	$(0,0)^0_0$
P_A	L. multiplier	${}^{-1}(\frac{1}{2},0,\frac{1}{2})^0_0 \oplus {}^1(0,\frac{1}{2},\frac{1}{2})^0_0$	P^a	antighost	$(\frac{1}{2},\frac{1}{2})^{-1}_0$
			P^{ab}	ghost	$(0,1)^1_0$
			P'	ghost	$(0,0)^1_0$
$\bar{c}^*_A$	antighost	${}^{-1}(\frac{1}{2},0,\frac{1}{2})^{-1}_0 \oplus {}^1(0,\frac{1}{2},\frac{1}{2})^{-1}_0$	C^{*a}	antighost for ghost	$(\frac{1}{2},\frac{1}{2})^{-2}_0$
			C^{*ab}	extraghost	$(0,1)^0_0$
			C^*	extraghost	$(0,0)^0_0$
c	ghost	${}^{\cdot}(0,0,0)^1_0$	c	ghost	$(0,0)^{\cdot}_0$
P	L. multiplier	${}^{\cdot}(0,0,0)^0_0$	P	ghost	$(0,0)^{\cdot}_0$
$\bar{c}$	antighost	${}^{\cdot}(0,0,0)^{-1}_0$	$\bar{c}$	ghost	$(0,0)^{\cdot}_0$

Chapter 4

Twisted Minimally Coupled N=2 Supergravity

4.1 Introduction

In this chapter we study the topological twist of N=2 supergravity minimally coupled to n vector multiplets. The matter vectors can be arbitrarily gauged, but the graviphoton cannot be gauged. In section 4.2 we make some general observations about the internal $U(1)$ charge that should be added to ghost number to define the ghost number of the topological theory. In section 4.3 we recall N=2 supergravity minimally coupled to vector multiplets in the rheonomy framework. Soon after this (section 4.4), we work out the precise definition of R-duality and show that it is an on shell symmetry of the theory. Next we topologically twist and topologically shift the BRST-quantum version of the theory (section 4.5), thus finding the BRST algebra of the topological theory. We also discuss the gauge-free algebra and give the complete identification with the gauge-free subalgebra of the BRST algebra that comes from the twist. We show the observables and the find the gauge-fermion.

4.2 General remarks on R-duality

In this section we discuss the possibility that minimal N=2 matter coupled supergravity is R-duality invariant. This internal $U(1)$ charge will add to the ghost number to define the ghost number of the topologically twisted theory. Thus we shall be able to extend the procedure of topological twist and topological shift of chapter 3 in a rather direct way.

Let us first make some simple remarks about the properties of the chiral-dual invariance displayed by N=2 simple supergravity. These properties will guide us in finding the desired generalization to the matter coupled case. We use the same notation of chapter

3. Consider the Bianchi identity of the graviphoton A, that is

$$\mathcal{D}R^{\otimes} + 2\epsilon_{AB}\bar{\psi}_A \wedge \rho_B = 0, \tag{4.1}$$

its equation of motion,

$$4i\epsilon_{AB}\bar{\rho}_A \wedge \gamma_5\psi_B - \mathcal{D}(F^{ab}V^c \wedge V^d)\epsilon_{abcd} = 0, \tag{4.2}$$

the rheonomic parametrization of the graviphoton curvature $R^{\otimes}$,

$$R^{\otimes} = F_{ab}V^a \wedge V^b, \tag{4.3}$$

and the on shell chiral-dual transformation, i. e.

$$\begin{aligned} \hat{\delta}\psi_A &= i\gamma_5\psi_A \\ \hat{\delta}F_{ab} &= -2i\tilde{F}_{ab} = \epsilon_{abcd}F^{cd}. \end{aligned} \tag{4.4}$$

In chapter 3 it was noted that the chiral-dual variation of the Bianchi identity is the equation of motion and *viceversa*. This is evident if we re-write the Bianchi identity of the graviphoton and its equation of motion in the following form

$$\begin{aligned} d[R^{\otimes} - \epsilon_{AB}\bar{\psi}_A \wedge \psi_B] &= 0, \\ d[\epsilon_{abcd}F^{ab}V^c \wedge V^d - 2i\epsilon_{AB}\bar{\psi}^A \wedge \gamma_5\psi^B] &= 0. \end{aligned} \tag{4.5}$$

Moreover, let us see what is the condition for the transformation (4.4) to be well defined, i. e. what is required for the existence of a $\hat{\delta}A$ compatible with (4.4). One immediately finds

$$\begin{aligned} \epsilon_{abcd}F^{cd}V^a \wedge V^b &= \hat{\delta}[F_{ab}V^a \wedge V^b] = \\ = \hat{\delta}R^{\otimes} &= d\hat{\delta}A + 2i\epsilon_{AB}\bar{\psi}_A \wedge \gamma_5\psi_B. \end{aligned} \tag{4.6}$$

So, $\epsilon_{abcd}F^{cd}V^a \wedge V^b - 2i\epsilon_{AB}\bar{\psi}_A \wedge \gamma_5\psi_B$ must be an exact form and we focus on the case in which a necessary and sufficient condition for this to be true is that the form is closed, i. e. $d[\epsilon_{abcd}F^{cd}V^a \wedge V^b - 2i\epsilon_{AB}\bar{\psi}_A \wedge \gamma_5\psi_B] = 0$. This is precisely the equation of motion for the graviphoton (4.2). Consequently, the $U(1)$ transformation is defined on shell and only on shell. This way of reasoning is a natural generalization of the well known case of electromagnetism and it will directly extend to N=2 matter coupled supergravity.

What do we expect R-duality to be like? Obviously, it should reduce to the known results both on the gravitational multiplet when matter is suppressed and on the vector multiplets when gravity is switched off. In other words, it should be a dual transformation on the graviphoton (that is why we call it *duality*), a chiral transformation on the fermions and should leave the graviton and the matter vectors inert. The scalars of the vector multiplets should have charges $+2$ and -2. Consequently, on the fields of the vector multiplets the symmetry we are seeking should act as the usual internal $U(1)$ symmetry of N=2 super Yang-Mills, which is an R-symmetry [24]. Finally, it should be possible to gauge the matter vectors (but not the graviphoton) while preserving the symmetry.

We expect R-duality not to be present in the most general case, i. e. with any special Kähler manifold, but only in the simplest case, namely for minimal coupling [25]. This is suggested by the fact that something similar seems to happen even in the case of flat N=2 super Yang-Mills theory. As a matter of fact, the theory involves the choice of an arbitrary flat special geometry prepotential $F(X)$, which is a holomorphic homogeneous function of degree two of the simplectic sections X_Λ [26]. As a result, the lagrangian involves a coupling matrix $f^{ij}(z)$, which, in flat coordinates $z_i = \frac{X_i}{X_0}$, depends holomorphically on the scalars z_i and is given by the second derivative of F, $f^{ij}(z) = \frac{\partial}{\partial z_i}\frac{\partial}{\partial z_j}F(X(z))$ [26]. The kinetic lagrangian of the vectors has the following form

$$F^i_{\mu\nu}F^{j\mu\nu}\mathrm{Re}\, f^{ij} - \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F^i_{\mu\nu}F^j_{\rho\sigma}\mathrm{Im}\, f^{ij}. \tag{4.7}$$

Only when $f^{ij}(z) = \delta^{ij}$, namely when F is quadratic, there is an evident R-invariance, since if z has a nonvanishing charge, then the only neutral holomorphic function of z is the constant. In other words, the topological twist appears to be possible only in one case, although the negative result that R-symmetry is barred in nonminimal coupling has not been established in a conclusive way. Indeed, we shall prove that R-duality exists in minimal matter coupled N=2 supergravity, but we shall not prove that this is the only possible case. There could be some unexpected field redefinitions that make it work in more general cases, even if they presumably cannot make it suitable for a topological twist. Uniqueness remains, for the time being, just our conjecture.

We recall that in topological Yang-Mills theory the chiral anomaly becomes ghost number anomaly after the twist and can be described by saying that the functional measure has a definite nonvanishing ghost number. Consequently, only the amplitudes of observables that have a total ghost number opposite to this value can be nontrivial. These features of ghost number are present also in topological gravity with or without matter. In Ref. [27] it is shown that the dual invariance of Maxwell theory in external gravity is anomalous. In topological gravity we thus expect a ghost number anomaly which is due not only to the anomalous chiral behaviour of the fermions, but also to the anomalous dual behaviour of the graviphoton. In other words one has to take care of the zero modes of the graviphoton, besides those of the fermions.

Let us now derive some a *priori* information about R-duality. As in Ref. [19] to each field of the theory we assign a set of labels ${}^c(L, R, I)^g_f$, where L is the representation of $SU(2)_L$, R is the representation of $SU(2)_R$, I is the representation of $SU(2)_I$, c is the $U(1)_I$ charge, g the ghost number and f the form number. If the twist acts on $SU(2)_R$, then after the twist we have objects described by $(L, R\otimes I)^{g+c}_f$. In this case the left handed components of gravitinos and gauginos must necessarily have $U(1)_I$ charge $+1$, since they are the only fermions that have the correct spin content to give the topological ghosts after the twist. For example, the left handed components of the gravitinos are characterized by $(\frac{1}{2}, 0, \frac{1}{2})^0_1$ and give $(\frac{1}{2}, \frac{1}{2})_1$ after the twist, and the vierbein V^a is also a

$(\frac{1}{2},\frac{1}{2})_1$ object. Similarly, the left handed components of the gauginos become $(\frac{1}{2},\frac{1}{2})_0$ after the twist: let us call them λ_a. The vector bosons, however, are Lorentz scalars, so they give $(0,0)^0_1$. Consequently, the correct topological ghosts can only be $\lambda_a V^a$.

The charge of the right handed components of gravitinos and gauginos is fixed to be -1 by the fact that they are the natural candidates to become the topological antighosts, as far as their Lorentz transformation properties are concerned. As a check, we can also see that the charge of the right handed gravitinos is independently fixed by the following argument to the value $c = -1$. The supersymmetry charges must also transform. In fact, the right handed components of the supersymmetry ghosts, which are the ghost partners of the right handed gravitinos and so must have the same charge, are characterized by $(0,\frac{1}{2},\frac{1}{2})^1_0$ and give $(0,1)_0 \oplus (0,0)_0$ after the twist. This is the only possibility to obtain a scalar zero form from the supersymmetry ghosts and we recall [19] that the $(0,0)_0$ component must be topologically shifted by a constant in order to define the BRST symmetry of the topological theory. This implies $g + c = 0$ for the right handed components of the supersymmetry ghosts, and so $c = -1$.

We conclude that on any of the so far considered fermions, collectively denoted by λ (supersymmetry ghosts included), R-duality acts as follows

$$\begin{aligned} \hat{\delta}\lambda_L &= \lambda_L \\ \hat{\delta}\lambda_R &= -\lambda_R, \end{aligned} \tag{4.8}$$

where $\hat{\delta}$ denotes R-duality and λ_L, λ_R are the left and right handed components, respectively. This automatically rules out the $U(1)$ Kähler transformation as a candidate for R-duality, since the $U(1)$ Kähler charges of the gaugino and gravitino left handed components are opposite to each other [13]. Note that the previous reasonings are not applicable to the case of hypermultiplets. Indeed, we shall find that the left handed components of the spinors contained in these multiplets have charge -1, while the right handed ones have charge $+1$ (chapter 5).

Once we have fixed the charges of the fermions, the R-duality transformations of the bosons are uniquely fixed by requiring on shell consistency with supersymmetry, δ_ε, i. e.

$$[\hat{\delta}, \delta_\varepsilon] = 0. \tag{4.9}$$

Before giving the complete result obtained from this requirement, we recall the structure of N=2 matter coupled supergravity.

4.3 Minimally Coupled N=2 supergravity

By definition, N=2 supergravity minimally coupled to n vector multiplets corresponds to the case where the special Kähler manifold spanned by the vector multiplet scalars is the homogeneous manifold $\mathcal{M} = \frac{SU(1,n)}{SU(n)\otimes U(1)}$. In the language of holomorphic prepotentials this corresponds to the choice $F(X) = \frac{1}{4}({X_0}^2 - \sum_{i=1}^n {X_i}^2)$. An easy way to obtain the explicit form of this theory, in the rheonomy framework that we use throughout the paper, is by truncation of N=3 matter coupled supergravity [18, 28]. If we are interested in the case of just one vector multiplet, it is more convenient to truncate pure N=4 $SO(4)$ supergravity [29]. As a matter of fact, we first tested our conjectures using this trick (which we do not discuss here) and, after having found that their were correct, we extended them to n vector multiplets in the way we now present.

The gravitational multiplet is $(V^a, \psi_A, \psi^A, A_0)$ (the index A taking the values 1, 2), where V^a is the vierbein, ψ_A are the gravitino left handed components ($\gamma_5\psi_A = \psi_A$), ψ^A are the right handed ones ($\gamma_5\psi^A = -\psi^A$) and A_0 is the graviphoton. The n vector multiplets are labelled by an index $i = 1, \ldots n$ and are denoted by $(A_i, \lambda_i^A, \lambda_A^i, z_i, \bar{z}^i)$, A_i being the vector bosons, λ_i^A the gaugino left handed components, λ_A^i the right handed ones, z_i and $\bar{z}^i$ the scalars. Vierbein, gravitinos, graviphoton and vector bosons are 1-forms, all the other fields being 0-forms.

A special Kähler manifold $SK(n)$ is a Hodge Kähler manifold providing the base manifold for a flat $Sp(2n+2)$ simplectic vector bundle $\mathcal{S} \xrightarrow{\pi} SK(n)$, whose holomorphic sections $(X_\Lambda, \frac{\partial F}{\partial X_\Lambda})$, $\Lambda = 0, 1 \ldots n$, are given in terms of a prepotential $F(X)$, homogeneous of degree two in the $n+1$ variables $X_\Lambda(z)$ (z belonging to $SK(n)$). It is common to introduce the following expressions

$$\begin{aligned}
F^{\Lambda\Sigma} &= \partial^\Lambda \partial^\Sigma F(X), \\
N^{\Lambda\Sigma} &= F^{\Lambda\Sigma} + \bar{F}^{\Lambda\Sigma}, \\
G &= -\ln(N^{\Lambda\Sigma} X_\Lambda \bar{X}_\Sigma), \\
L_\Lambda &= e^{\frac{G}{2}} X_\Lambda, \\
f_\Lambda^i &= \partial^i L_\Lambda + \frac{1}{2} G^i L_\Lambda, \\
\mathcal{N}^{\Lambda\Sigma} &= -\bar{F}^{\Lambda\Sigma} + \frac{1}{N^{\Delta\Gamma} L_\Delta L_\Gamma} N^{\Lambda\Pi} L_\Pi N^{\Sigma\Xi} L_\Xi,
\end{aligned} \tag{4.10}$$

where G is the Kähler potential, $\partial^\Lambda = \frac{\partial}{\partial X_\Lambda}$, $\partial^i = \frac{\partial}{\partial z_i}$, $G^i = \partial^i G$.

In the minimal case, if we use the special coordinates $z_\Lambda = \frac{X_\Lambda}{X_0}$ ($z_0 = 1$) and furthermore we impose $X_0 \equiv 1$, then $F(z) = \frac{1}{4}(1 - \sum_{i=1}^n z_i z_i)$ and

$$\begin{aligned}
F^{\Lambda\Sigma} &= = \frac{1}{2}\eta^{\Lambda\Sigma} = \frac{1}{2}\mathrm{diag}(1, -1, \ldots - 1), \\
N^{\Lambda\Sigma} &= \eta^{\Lambda\Sigma}, \\
G &= -\ln a, \\
L_\Lambda &= \frac{z_\Lambda}{\sqrt{a}}, \\
f_\Lambda^i &= \begin{pmatrix} f_0^i \\ f_j^i \end{pmatrix} = \frac{1}{a\sqrt{a}} \begin{pmatrix} \bar{z}^i \\ a\delta_j^i + z_j \bar{z}^i \end{pmatrix}, \\
\mathcal{N}^{\Lambda\Sigma} &= \begin{pmatrix} \mathcal{N}^{00} & \mathcal{N}^{0j} \\ \mathcal{N}^{i0} & \mathcal{N}^{ij} \end{pmatrix} = \frac{1}{2(1 - z_i z_i)} \begin{pmatrix} 1 + z_l z_l & -2z_j \\ -2z_i & \delta_{ij}(1 - z_l z_l) + 2 z_i z_j \end{pmatrix},
\end{aligned} \tag{4.11}$$

where $a = 1 - z_i \bar{z}^i$.

In the notation of N=3 matter coupled supergravity [18, 28], the manifold $\frac{G}{H} = \frac{SU(3,n)}{SU(3)\otimes SU(n)\otimes U(1)}$ (which becomes $\mathcal{M} = \frac{SU(1,n)}{SU(n)\otimes U(1)}$ when truncating to N=2), is described by a matrix $L_\Lambda{}^\Sigma(z,\bar{z})$ that depends on the coordinates $z_i^A, \bar{z}_i^A \equiv z^i_A$, where $A = 1,2,3$, $i = 1,\dots n$, $\Lambda = (A,i)$. The N=2 truncation is realized by setting to zero the fermions that have index $A = 3$, the bosons with $A = 1,2$, the spin 1/2 of the N=3 graviton multiplet and the $SU(3)$-singlet spin 1/2 fields of the vector multiplets. The L matrix is [18, 28]

$$L_\Lambda{}^\Sigma(z,\bar{z}) = \begin{pmatrix} L_1{}^1 & L_1{}^2 & L_1{}^3 & L_1{}^j \\ L_2{}^1 & L_2{}^2 & L_2{}^3 & L_2{}^j \\ L_3{}^1 & L_3{}^2 & L_3{}^3 & L_3{}^j \\ L_i{}^1 & L_i{}^2 & L_i{}^3 & L_i{}^j \end{pmatrix} = \frac{1}{\sqrt{a}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \bar{z}^j \\ 0 & 0 & z_i & M_i{}^j \end{pmatrix}, \tag{4.12}$$

where $M_i{}^j = \sqrt{a}\delta_i^j + \frac{z_i \bar{z}^j}{|z|^2}(1-\sqrt{a})$. The correspondence with the N=2 notation is the following

$$L_\Lambda{}^\Sigma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & L_0 & f_0^k (g^{-\frac{1}{2}})_k{}^j \\ 0 & 0 & L_i & f_i^k (g^{-\frac{1}{2}})_k{}^j \end{pmatrix}, \tag{4.13}$$

where $(g^{-\frac{1}{2}})_i{}^j = \sqrt{a}\delta_i^j + \frac{z_i \bar{z}^j}{|z|^2}(a - \sqrt{a})$. Note that $\frac{1}{a} M_i{}^k \frac{1}{a} M_k{}^j = g_i{}^j \equiv \partial_i \partial^j G$, where $\partial_i = \partial^{i*}$; $g_i{}^j$ is the metric tensor of the Kähler manifold $\mathcal{M}$. We thus define $\frac{1}{a} M_i{}^j = (g^{\frac{1}{2}})_i{}^j$, and $aM^{-1}{}_i{}^j = (g^{-\frac{1}{2}})_i{}^j$.

The N=2 truncation of the $\frac{G}{H}$ connection $\Omega_\Lambda{}^\Sigma$ is

$$\Omega_\Lambda{}^\Sigma = (L^{-1})_\Lambda{}^\Pi (dL_\Pi{}^\Sigma + g f_\Pi{}^{\Delta\Gamma} A_\Delta L_\Gamma{}^\Sigma) \equiv \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -iQ & P^j \\ 0 & 0 & P_i & Q_i^j + \frac{i}{n}\delta_i^j Q \end{pmatrix}. \tag{4.14}$$

In particular, Q is the gauged Kähler connection and P^i is the gauged vierbein on $\mathcal{M}$,

$$\begin{aligned} Q &= -\frac{i}{2}(G^i \nabla z_i - G_i \nabla \bar{z}^i), \\ P_i &= (g^{\frac{1}{2}})_i{}^j \nabla z_j \end{aligned} \tag{4.15}$$

and $P^i = P_i^*$. From now on, let Λ take only the values $(A = 3, i = 1,\dots n)$. For convenience, the index 3 will be eventually replaced by a 0 or simply omitted, when there can be no misunderstanding.

At this point, truncating the N=3 curvature definitions (see Eq.s (IV.7.46) and (IV.7.48) of Ref. [18]), we obtain the N=2 curvature definitions already adapted to the minimal coupling.

$$\begin{aligned}
R^a &= dV^a - \omega^{ab} \wedge V_b - i\bar{\psi}_A \gamma^a \wedge \psi^A \equiv \mathcal{D}V^a - i\bar{\psi}_A \wedge \gamma^a \psi^A, \\
R^{ab} &= d\omega^{ab} - \omega^{ac} \wedge \omega_c{}^b, \\
\rho_A &= d\psi_A - \frac{1}{4}\omega^{ab}\gamma_{ab} \wedge \psi_A + \frac{i}{2}Q \wedge \psi_A = \mathcal{D}\psi_A + \frac{i}{2}Q \wedge \psi_A \equiv \nabla\psi_A, \\
\rho^A &= d\psi^A - \frac{1}{4}\omega^{ab}\gamma_{ab} \wedge \psi^A - \frac{i}{2}Q \wedge \psi^A = \mathcal{D}\psi^A - \frac{i}{2}Q \wedge \psi^A \equiv \nabla\psi^A, \\
F_\Lambda &= dA_\Lambda + f_\Lambda{}^{\Omega\Delta} A_\Omega \wedge A_\Delta + \epsilon_{AB} L_\Lambda \bar{\psi}^A \wedge \psi^B + \epsilon^{AB} \bar{L}_\Lambda \bar{\psi}_A \wedge \psi_B, \\
\nabla\lambda_{iA} &= d\lambda_{iA} - \frac{1}{4}\omega^{ab} \wedge \gamma_{ab}\lambda_{iA} + \frac{i}{2}\left(1 + \frac{2}{n}\right) Q\lambda_{iA} + Q_i{}^j \lambda_{jA}, \\
\nabla\lambda^{iA} &= d\lambda^{iA} - \frac{1}{4}\omega^{ab} \wedge \gamma_{ab}\lambda^{iA} - \frac{i}{2}\left(1 + \frac{2}{n}\right) Q\lambda^{iA} + Q^i{}_j \lambda^{jA}, \\
\nabla z_i &= dz_i + gA_\Lambda k_i^\Lambda(z), \\
\nabla \bar{z}^i &= d\bar{z}^i + gA_\Lambda k^{i\Lambda}(\bar{z}),
\end{aligned} \tag{4.16}$$

where $\gamma_{ab} = \frac{1}{2}[\gamma_a, \gamma_b]$ and $Q^i{}_j = (Q_i{}^j)^*$. $k_{\Lambda i}(z)$ and $k^i_\Lambda(\bar{z})$ are respectively the holomorphic and antiholomorphic Killing vectors generating the special Kähler manifold isometries. The explicit expression of these Killing vectors can be read from Eq.s (4.14) and (4.15), isolating the term proportional to A_Λ in the definition of $P_i = (g^{\frac{1}{2}})_i{}^j(dz_j + gA_\Lambda k_j^\Lambda(z))$. One finds $k_i^\Lambda(z) = f_i{}^{\Lambda k} z_k$ in the case in which only the matter vectors are gauged (this point will be justified in the following section). In the N=2 notation it is useful to introduce the new definitions

$$\begin{aligned}
\lambda_i^A &= -\epsilon^{AB}(g^{-\frac{1}{2}})_i{}^j \lambda_{jB}, \\
\lambda_A^i &= -\epsilon_{AB}(g^{-\frac{1}{2}})_j{}^i \lambda^{jB}.
\end{aligned} \tag{4.17}$$

Since z and $\bar{z}$ will be shown to have opposite R-duality charges, the matrix $g^{\frac{1}{2}}$ is R-duality invariant and so the above definitions do not change the R-duality transformation properties of the fermions. Formulae (4.17) are determined in such a way as to match the following rheonomic parametrizations

$$\begin{aligned}
P_i &= P_{i|a} V^a + \epsilon^{AB} \bar{\lambda}_{iA} \psi_B, \\
\nabla z_i &= Z_{i|a} V^a + \bar{\lambda}_i^A \psi_A,
\end{aligned} \tag{4.18}$$

that appear in the N=3 and N=2 formulations, respectively. In the N=2 notation the gaugino curvatures are

$$\begin{aligned}
\nabla\lambda_i^A &= \mathcal{D}\lambda_i^A - \frac{i}{2}Q\lambda_i^A - \Gamma_i{}^j \lambda_j^A, \\
\nabla\lambda_A^i &= \mathcal{D}\lambda_A^i + \frac{i}{2}Q\lambda_A^i - \Gamma^i{}_j \lambda_A^j,
\end{aligned} \tag{4.19}$$

where $\Gamma_i{}^j = -(g^{-1})_i{}^l(\partial^j g_l{}^k)\nabla z_k - gA_\Lambda \partial^j k_i^\Lambda$ is the gauged Levi-Civita holomorphic connection on $\mathcal{M}$ and $\Gamma^i{}_j = (\Gamma_i{}^j)^*$.

In the variables λ_{iA}, P_i inherited from the N=3 truncation, the standard N=2 Bianchi identities (see Eq.s (3.35) of Ref. [13]) take the following form

$$\begin{aligned}
\mathcal{D}R^a + R^{ab}\wedge V_b + i\bar{\rho}^A \wedge \gamma^a\psi_A - i\bar{\psi}^A \wedge \gamma^a \rho_A &= 0,\\
\mathcal{D}R^{ab} &= 0,\\
\nabla\rho_A + \frac{1}{4}R^{ab}\wedge\gamma_{ab}\psi_A - \frac{i}{2}K\wedge\psi_A &= 0,\\
\nabla\rho^A + \frac{1}{4}R^{ab}\wedge\gamma_{ab}\psi^A + \frac{i}{2}K\wedge\psi^A &= 0,\\
\nabla F_\Lambda - f^i_\Lambda \nabla z_i \epsilon_{AB}\bar{\psi}^A\wedge\psi^B - \bar{f}_{\Lambda i}\nabla\bar{z}^i\epsilon^{AB}\bar{\psi}_A\wedge\psi_B + &\\
+2L_\Lambda\epsilon_{AB}\bar{\psi}^A\wedge\rho^B + 2\bar{L}_\Lambda\epsilon^{AB}\bar{\psi}_A\wedge\rho_B &= 0,\\
\nabla^2\lambda_{iA} + \frac{1}{4}R^{ab}\wedge\gamma_{ab}\lambda_{iA} - R_i{}^j\lambda_{jA} - \frac{i}{2}\left(1+\frac{2}{n}\right)K\lambda_{iA} &= 0,\\
\nabla^2\lambda^{iA} + \frac{1}{4}R^{ab}\wedge\gamma_{ab}\lambda^{iA} - R^i{}_j\lambda^{jA} + \frac{i}{2}\left(1+\frac{2}{n}\right)K\lambda^{iA} &= 0,\\
\nabla P_i = dP_i + Q_i{}^j\wedge P_j + i\left(1+\frac{1}{n}\right)Q\wedge P_i &= 0,\\
\nabla P_i = dP^i + Q^i{}_j\wedge P^j - i\left(1+\frac{1}{n}\right)Q\wedge P^i &= 0,
\end{aligned} \tag{4.20}$$

where $K = dQ$, $R_i{}^j = dQ_i{}^j + Q_i{}^k\wedge Q_k{}^j$ and $R^i{}_j = (R_i{}^j)^*$. The rheonomic parametrizations are

$$\begin{aligned}
R^a &= 0,\\
R^{ab} &= R^{ab}{}_{cd}V^c\wedge V^d - i\bar{\psi}_A(2\gamma^{[a}\rho^{A|b]c} - \gamma^c\rho^{A|ab})\wedge V_c +\\
&\quad -i\bar{\psi}^A(2\gamma^{[a}\rho_A{}^{|b]c} - \gamma^c\rho_A{}^{|ab})\wedge V_c + 2G^{-ab}\epsilon^{AB}\bar{\psi}_A\wedge\psi_B +\\
&\quad +2G^{+ab}\epsilon_{AB}\bar{\psi}^A\wedge\psi^B + \frac{i}{4}\varepsilon^{abcd}\bar{\psi}_A\wedge\gamma_c\psi^B(2\bar{\lambda}_{iB}\gamma_d\lambda^{iA} - \delta^A_B\bar{\lambda}_{iC}\gamma_d\lambda^{iC}),\\
\rho_A &= \rho_{A|ab}V^a\wedge V^b - 2i\epsilon_{AB}G^+_{ab}\gamma^a\psi^B\wedge V^b + \frac{i}{4}\psi_B\bar{\lambda}^{iB}\gamma^a\lambda_{iA}\wedge V_a +\\
&\quad +\frac{i}{8}\gamma_{ab}\psi_B\left(2\bar{\lambda}^{iB}\gamma^a\lambda_{iA} - \delta^B_A\bar{\lambda}^{iC}\gamma^a\lambda_{iC}\right)\wedge V^b,\\
\rho^A &= \rho^A_{ab}V^a\wedge V^b - 2i\epsilon^{AB}G^-_{ab}\gamma^a\psi_B\wedge V^b + \frac{i}{4}\psi^B\bar{\lambda}_{iB}\gamma^a\lambda^{iA}\wedge V_a +\\
&\quad +\frac{i}{8}\gamma_{ab}\psi^B\left(2\bar{\lambda}_{iB}\gamma^a\lambda^{iA} - \delta^A_B\bar{\lambda}_{iC}\gamma^a\lambda^{iC}\right)\wedge V^b,\\
F_\Lambda &= F^{ab}_\Lambda V_a\wedge V_b + i(f^i_\Lambda\bar{\lambda}^A_i\gamma^a\psi^B\epsilon_{AB} + \bar{f}_{\Lambda i}\bar{\lambda}^i_A\gamma^a\psi_B\epsilon^{AB})\wedge V_a,\\
\nabla\lambda_{iA} &= \nabla_a\lambda_{iA}V^a + iP_{i|a}\gamma^a\psi^B\epsilon_{AB} + G^{+ab}_i\gamma_{ab}\psi_A + gC_i\psi_A,\\
\nabla\lambda^{iA} &= \nabla_a\lambda^{iA}V^a + iP^i_{|a}\gamma^a\psi_B\epsilon^{AB} + G^{-i}_{ab}\gamma^{ab}\psi^A + gC^i\psi^A,\\
\nabla z_i &= Z_{i|a}V^a + \bar{\lambda}^A_i\psi_A,
\end{aligned}$$

$$\nabla \bar{z}^i = \bar{Z}^i_{|a} V^a + \bar{\lambda}^i_A \psi^A. \tag{4.21}$$

where $C^i = (L^{-1})_3{}^k L_j{}^i L_l{}^3 f_k{}^{jl}$ are obtained from the N=2 truncation as particular instances of the N=3 boosted structure constants, $C_i = (C^i)^*$, $\bar{f}_{\Lambda i} = (f^i_\Lambda)^*$. G^{+ab}, G^{+ab}_i, G^-_{ab} and G^{-i}_{ab} are determined by the equation

$$\frac{1}{2} F^{ab}_\Lambda + L_\Lambda{}^3 G^{-ab} + L_\Lambda{}^i G^{+ab}_i + (L^{-1})_3{}^\Pi J_{\Pi\Lambda} G^{+ab} - (L^{-1})_i{}^\Pi J_{\Pi\Lambda} G^{i-ab} = 0, \tag{4.22}$$

where $J_{\Lambda\Pi}$ is the $SU(1,n)$-invariant metric

$$J_{\Lambda\Pi} = \begin{pmatrix} 1 & 0 \\ 0 & -\delta_{ij} \end{pmatrix}. \tag{4.23}$$

One finds

$$\begin{aligned} G^{+ab} &= -\frac{1}{4}\sqrt{a}(1+2\bar{\mathcal{N}})^{0\Lambda} F^{+ab}_\Lambda, \\ G^{+ab}_i &= -\frac{1}{4}\sqrt{a}(g^{\frac{1}{2}})_i{}^j (1+2\bar{\mathcal{N}})_j{}^\Lambda F^{+ab}_\Lambda + \frac{1}{a}\bar{z}^2 z_i G^{+ab} + \frac{1}{2\sqrt{a}} z_i \bar{z}^k F^{+ab}_k. \end{aligned} \tag{4.24}$$

and $G^-_{ab} = (G^+_{ab})^*$ and $G^{-i}_{ab} = (G_{iab}{}^+)^*$. The rheonomic parametrizations are on-shell consistent with the Bianchi identities (4.20).

We can now write down the lagrangian of N=2 supergravity minimally coupled to n vector multiplets.

$$\mathcal{L} = \mathcal{L}_{kin} + \mathcal{L}_{Pauli} + \mathcal{L}_{torsion} + \mathcal{L}_{4Fermi} + \Delta\mathcal{L}_{gauging} + \Delta\mathcal{L}_{potential}, \tag{4.25}$$

where

$$\begin{aligned} \mathcal{L}_{kin} &= \varepsilon_{abcd} R^{ab} \wedge V^c \wedge V^d - 4(\bar{\psi}^A \wedge \gamma_a \rho_A + \bar{\rho}^A \wedge \gamma_a \psi_A) \wedge V^a + \\ &\quad -\frac{i}{3} g_i{}^j (\bar{\lambda}^A_j \gamma_a \nabla \lambda^i_A + \bar{\lambda}^i_A \gamma_a \nabla \lambda^A_j) \wedge V_b \wedge V_c \wedge V_d \varepsilon^{abcd} + \\ &\quad +\frac{2}{3} g_i{}^j [\bar{Z}^i_{|a}(\nabla z_j - \bar{\lambda}^A_j \psi_A) + Z_{j|a}(\nabla \bar{z}^i - \bar{\lambda}^i_A \psi^A)] \wedge V_b \wedge V_c \wedge V_d \varepsilon^{abcd} + \\ &\quad +\frac{1}{6}(\bar{\mathcal{N}}^{\Lambda\Sigma} F^{+ab}_\Lambda F^+_{\Sigma ab} + \mathcal{N}^{\Lambda\Sigma} F^{-ab}_\Lambda F^-_{\Sigma ab} + \\ &\quad - g_i{}^j \bar{Z}^i_{|a} Z_{j|}{}^a) \varepsilon_{cdef} V^c \wedge V^d \wedge V^e \wedge V^f + \\ &\quad -4i(\bar{\mathcal{N}}^{\Lambda\Sigma} F^{+ab}_\Lambda - \mathcal{N}^{\Lambda\Sigma} F^{-ab}_\Lambda) \wedge (F_\Sigma + \\ &\quad -i(f^i_\Sigma \bar{\lambda}^A_i \gamma^c \psi^B \epsilon_{AB} + \bar{f}_{\Sigma i} \bar{\lambda}^i_A \gamma^c \psi_B \epsilon^{AB}) \wedge V_c) \wedge V_a \wedge V_b, \\ \mathcal{L}_{Pauli} &= -4i F_\Lambda \wedge (\mathcal{N}^{\Lambda\Sigma} L_\Sigma \epsilon_{AB} \bar{\psi}^A \wedge \psi^B - \bar{\mathcal{N}}^{\Lambda\Sigma} \bar{L}_\Sigma \epsilon^{AB} \bar{\psi}_A \wedge \psi_B) + \\ &\quad +4F_\Lambda \wedge (\bar{\mathcal{N}}^{\Lambda\Sigma} f^i_\Sigma \bar{\lambda}^A_i \gamma_a \psi^B \epsilon_{AB} - \mathcal{N}^{\Lambda\Sigma} \bar{f}_{\Sigma i} \bar{\lambda}^i_A \gamma_a \psi_B \epsilon^{AB}) \wedge V^a + \\ &\quad -2i g_i{}^j (\nabla z_j \wedge \bar{\lambda}^i_A \gamma_{ab} \psi^A - \nabla \bar{z}^i \wedge \bar{\lambda}^A_j \gamma_{ab} \psi_A) \wedge V^a \wedge V^b, \\ \mathcal{L}_{torsion} &= R^a \wedge V_a \wedge g_i{}^j \bar{\lambda}^i_A \gamma_b \lambda^A_j \wedge V^b, \end{aligned}$$

$$
\begin{aligned}
\mathcal{L}_{4Fermi} &= i(W\epsilon_{AB}\bar{\psi}^A \wedge \psi^B \wedge \epsilon_{CD}\bar{\psi}^C \wedge \psi^D - \bar{W}\epsilon^{AB}\bar{\psi}_A \wedge \psi_B \wedge \epsilon^{CD}\bar{\psi}_C \wedge \psi_D) + \\
&\quad -2ig_i{}^j\bar{\lambda}^i_A\gamma_a\lambda^B_j\bar{\psi}_B \wedge \gamma_b\psi^A \wedge V^a \wedge V^b + \\
&\quad +i(W_{ij}\epsilon^{AB}\bar{\lambda}^i_A\gamma_a\psi_B \wedge V^a \wedge \epsilon^{CD}\bar{\lambda}^j_C\gamma_b\psi_D \wedge V^b + \\
&\quad -W^{ij}\epsilon_{AB}\bar{\lambda}^A_i\gamma_a\psi^B \wedge V^a \wedge \epsilon_{CD}\bar{\lambda}^C_j\gamma_b\psi^D \wedge V^b) + \\
&\quad +\frac{1}{18}\epsilon_{abcd}V^a \wedge V^b \wedge V^c \wedge V^d g_i{}^j\bar{\lambda}^i_A\gamma^m\lambda^A_j g_k{}^l\bar{\lambda}^k_B\gamma_m\lambda^B_l , \\
\Delta\mathcal{L}_{gauging} &= \frac{2i}{3}g(\bar{\lambda}^A_i\gamma^a\psi^B W^i_{AB} + \bar{\lambda}^i_A\gamma^a\psi_B W^{AB}_i) \wedge V^b \wedge V^c \wedge V^d\varepsilon_{abcd} + \\
&\quad +\frac{1}{6}g(M^{ij}\bar{\lambda}^A_i\lambda^B_j\epsilon_{AB} + M_{ij}\bar{\lambda}^i_A\lambda^j_B\epsilon^{AB})\varepsilon_{abcd}V^a \wedge V^b \wedge V^c \wedge V^d , \\
\Delta\mathcal{L}_{potential} &= -\frac{1}{12}g^2 g_i{}^j W^i_{AB}W^{AB}_j\varepsilon_{abcd}V^a \wedge V^b \wedge V^c \wedge V^d ,
\end{aligned}
\tag{4.26}
$$

where $W = 2L_\Lambda L_\Sigma \mathcal{N}^{\Lambda\Sigma}$, $W^{ij} = 2\bar{\mathcal{N}}^{\Lambda\Sigma} f^i_\Lambda f^j_\Sigma$ and $W_{ij} = (W^{ij})^*$, while $M^{ij} = k^{l\Lambda} f^{[i}_\Lambda g_l{}^{j]}$ and $M_{ij} = (M^{ij})^*$, $W^i_{AB} = \epsilon_{AB}k^{i\Lambda}L_\Lambda$, $W^{AB}_i = (W^i_{AB})^*$. The lagrangian in Eq.s (4.25) and (4.26) agrees with the lagrangian (4.13) of Ref. [13] upon suppression of the hypermultiplets and up to $\mathcal{L}_{4Fermi}$ and the second term of $\Delta\mathcal{L}_{gauging}$, that were not calculated in [13]. Indeed, the very reason why we have performed the above described N=2 truncation of the N=3 theory was that of obtaining these terms without calculating them explicitly. Our purpose is that of checking R-duality in the minimal coupling, however, as a byproduct, we have also obtained the complete form of the lagrangian of N=2 supergravity coupled to vector multiplets for an arbitrary choice of the special Kähler manifold. All the objects entering (4.26) have already been interpreted in a general N=2 setup (in which the graviphoton can be gauged). As a matter of fact, the N=3 theory does not admit the most general gauging of the vectors [18, 28], but it surely admits any gauging of the matter vectors. Even if the minimal N=2 theory exists in any case, the truncation from N=3 can only give the minimal N=2 theory in which the graviphoton is not gauged.

As promised, in the following section we define R-duality and prove that it is indeed an on-shell symmetry of the above theory.

4.4 R-duality for N=2 matter coupled supergravity

Now, starting form the R-duality transformation properties of the fermions, as derived in section 4.2, we determine the transformations of the bosons by simply requiring $[\hat{\delta}, \delta_\varepsilon] = 0$ on-shell, if δ_ε is the supersymmetry transformation with parameters ε (let ε_A and ε^A be the left and right handed components, respectively). The supersymmetry transformations can be read in the usual way from the rheonomic parametrizations (4.18) and (4.21). In any case, their explicit expression will be written down later on in the context of the BRST-quantization of the theory (see formula (4.41)). So, we start from

$$
\begin{aligned}
&\hat{\delta}\psi_A = \psi_A, \quad \hat{\delta}\varepsilon_A = \varepsilon_A, \quad \hat{\delta}\lambda^A_i = \lambda^A_i, \\
&\hat{\delta}\psi^A = -\psi^A, \quad \hat{\delta}\varepsilon^A = -\varepsilon^A, \quad \hat{\delta}\lambda^i_A = -\lambda^i_A.
\end{aligned}
\tag{4.27}
$$

First of all, consistency of R-duality with supersymmetry states that, if a field ϕ has an R-duality charge equal to q, then $\delta_\varepsilon\phi$ has the same charge q and viceversa. It is immediate to see that $\hat{\delta}\delta_\varepsilon V^a = 0$ and so we deduce $\hat{\delta}V^a = 0$. This is good, because in our mind, R-duality is to become ghost number and the vierbein should remain of zero ghost number together with all the matter vectors. Similarly, $\hat{\delta}\delta_\varepsilon z_i = 2\delta_\varepsilon z_i$, requiring $\hat{\delta}z_i = 2z_i$. An analogous reasoning gives, when applied to $\bar{z}^i$, $\hat{\delta}\bar{z}^i = -2\bar{z}^i$, thus confirming that z_i and $\bar{z}^i$ have opposite charges. This immediately rules out the possibility that the $U(1)$ symmetry we are looking for might be a subgroup of the group of duality transformations [21, 18]. Indeed, in that case z_i and $\bar{z}^i$ would have the same charge. This is welcome, because, if $U(1)_I$ were a subgroup of the duality group, we could not maintain the symmetry in the presence of gauging, as, on the contrary, we expect to be able to do. We immediately see that the Kähler potential G is invariant, as well as the metric $g_i{}^j$ (note that this fact would not hold true in the nonminimal case). It remains to find the transformation properties of the vector bosons. Let us concentrate on the ungauged case ($g = 0$) for the moment. One can verify that $[\hat{\delta}, \delta_\varepsilon]\psi_A = 0$ and $[\hat{\delta}, \delta_\varepsilon]\psi^A = 0$ imply

$$\hat{\delta}G^{\pm ab} = \pm 2G^{\pm ab}, \tag{4.28}$$

while $[\hat{\delta}, \delta_\varepsilon]\lambda_i^A = 0$ and $[\hat{\delta}, \delta_\varepsilon]\lambda^i_A = 0$ imply

$$\hat{\delta}G_i^{+ab} = 0, \quad \hat{\delta}G^{i-ab} = 0\,. \tag{4.29}$$

respectively. Equations (4.28) and (4.29) form a linear system of equations in $\hat{\delta}F_\Lambda^{\pm ab}$, in which the number of unknowns equals the number of equations. The unique solution is

$$\begin{array}{ll} \hat{\delta}F^{0+ab} = 4\bar{\mathcal{N}}^{0\Lambda}F_\Lambda^{+ab}, & \hat{\delta}F_i^{+ab} = 0, \\ \hat{\delta}F^{0-ab} = -4\mathcal{N}^{0\Lambda}F_\Lambda^{-ab}, & \hat{\delta}F_i^{-ab} = 0. \end{array} \tag{4.30}$$

The graviphoton is thus transformed in a way that resembles the duality transformations and this forbids its gauging it if we want R-duality. Consequently, when considering the gauged case, we must assume that only the matter vectors are gauged, i. e. $f_\Lambda^{\Sigma\Omega} = 0$ whenever one of the indices Λ, Σ, Ω takes the value zero. There is no restriction, on the contrary, on the gauge group of the matter vectors.

Let us rewrite the rheonomic parametrization of the vectors and the definition of their curvatures

$$\begin{aligned} F_\Lambda &= F_\Lambda^{ab}V_a \wedge V_b + i(f_\Lambda^i\bar{\lambda}_i^A\gamma^a\psi^B\epsilon_{AB} + \bar{f}_{\Lambda i}\bar{\lambda}^i_A\gamma^a\psi_B\epsilon^{AB}) \wedge V_a, \\ F_\Lambda &= dA_\Lambda + f_\Lambda{}^{\Omega\Delta}A_\Omega \wedge A_\Delta + \epsilon_{AB}L_\Lambda\bar{\psi}^A \wedge \psi^B + \epsilon^{AB}\bar{L}_\Lambda\bar{\psi}_A \wedge \psi_B. \end{aligned} \tag{4.31}$$

These expressions show that, under the above conditions on the structure constants $f_\Lambda^{\Sigma\Omega}$, the transformations $\hat{\delta}F_i^{+ab} = 0$ and $\hat{\delta}F^{i-ab} = 0$ imply $\hat{\delta}A_i = 0$, i. e. all the matter vectors are inert under R-duality (they will have ghost number zero after the twist and this is good in order to recover topological Yang-Mills theory).

Summarizing, R-duality acts on-shell as follows

$$\begin{array}{ll} \hat{\delta}V^a = 0, & \\ \hat{\delta}\psi_A = \psi_A, & \hat{\delta}\psi^A = -\psi^A, \\ \hat{\delta}F^{0+ab} = 4\bar{\mathcal{N}}^{0\Lambda}F_\Lambda^{+ab}, & \hat{\delta}F^{0-ab} = -4\mathcal{N}^{0\Lambda}F_\Lambda^{-ab}, \\ \hat{\delta}A_i = 0, & \\ \hat{\delta}\lambda_i^A = \lambda_i^A, & \hat{\delta}\lambda^i_A = -\lambda^i_A, \\ \hat{\delta}z_i = 2z_i, & \hat{\delta}\bar{z}^i = -2\bar{z}^i. \end{array} \tag{4.32}$$

One easily checks that formulas (4.28), (4.29), (4.30) and (4.32) are still valid when all the vectors but the graviphoton are gauged.

What about $\hat{\delta}A_0$? As in all duality-type transformations, $\hat{\delta}A_0$ should be meaningful only on-shell (see section 4.2). In fact, (4.30) and (4.31) imply (using the explicit expressions (4.11))

$$\begin{aligned} \hat{\delta}F_0^{ab}V_a\wedge V_b &= 4(\bar{\mathcal{N}}_0^\Lambda F_\Lambda^{+ab} - \mathcal{N}_0{}^\Lambda F_\Lambda^{-ab})V_a\wedge V_b = \\ &= \hat{\delta}[dA_0 + \epsilon_{AB}L_0\bar{\psi}^A\wedge\psi^B + \epsilon^{AB}\bar{L}_0\bar{\psi}_A\wedge\psi_B + \\ &\quad -i(f_0^i\bar{\lambda}_i^A\gamma^a\psi^B\epsilon_{AB} + \bar{f}_{0i}\bar{\lambda}^i_A\gamma^a\psi_B\epsilon^{AB})\wedge V_a] = \\ &= d\hat{\delta}A_0 - 4\mathcal{N}_0{}^\Lambda L_\Lambda\epsilon_{AB}\bar{\psi}^A\wedge\psi^B + 4\bar{\mathcal{N}}_0^\Lambda\bar{L}_\Lambda\epsilon^{AB}\bar{\psi}_A\wedge\psi_B + \\ &\quad -4i(\bar{\mathcal{N}}_0^\Lambda f_\Lambda^i\bar{\lambda}_i^A\gamma^a\psi^B\epsilon_{AB} - \mathcal{N}_0{}^\Lambda\bar{f}_{\Lambda i}\bar{\lambda}^i_A\gamma^a\psi_B\epsilon^{AB})\wedge V_a. \end{aligned} \tag{4.33}$$

Imposing $d^2\hat{\delta}A_0 = 0$, we get

$$\begin{aligned} d[(\bar{\mathcal{N}}_0^\Lambda F_\Lambda^{+ab} - \mathcal{N}_0{}^\Lambda F_\Lambda^{-ab})V_a\wedge V_b + \mathcal{N}_0{}^\Lambda L_\Lambda\epsilon_{AB}\bar{\psi}^A\wedge\psi^B - \bar{\mathcal{N}}_0^\Lambda\bar{L}_\Lambda\epsilon^{AB}\bar{\psi}_A\wedge\psi_B + \\ +i(\bar{\mathcal{N}}_0^\Lambda f_\Lambda^i\bar{\lambda}_i^A\gamma^a\psi^B\epsilon_{AB} - \mathcal{N}_0{}^\Lambda\bar{f}_{\Lambda i}\bar{\lambda}^i_A\gamma^a\psi_B\epsilon^{AB})\wedge V_a] = 0. \end{aligned} \tag{4.34}$$

One can easily verify that this is the equation of motion of the graviphoton as derived from the lagrangian (4.25). Furthermore, the R-duality variation of the A_0 equation of motion is proportional to the A_0-Bianchi identity and viceversa. It is easily checked that the other curvatures of (4.16) and the remaining Bianchi identities of (4.20) transform correctly, so the last step in order to establish R-duality of the theory is the proof of invariance for the remaining field equations.

The equations of motion of the vector bosons can be written in the following form

$$dS^\Lambda + 2f_\Delta^{\Lambda\Sigma}A_\Sigma\wedge S^\Delta + R^\Lambda = 0, \tag{4.35}$$

where S^Λ is, by definition, the coefficient in the lagrangian of the field strength F_Λ, namely

$$\begin{aligned} S^\Lambda = \; & (\bar{\mathcal{N}}^{\Lambda\Sigma}F_\Sigma^{+ab} - \mathcal{N}^{\Lambda\Sigma}F_\Sigma^{-ab})V_a\wedge V_b + \mathcal{N}^{\Lambda\Sigma}L_\Sigma\epsilon_{AB}\bar{\psi}^A\wedge\psi^B - \bar{\mathcal{N}}^{\Lambda\Sigma}\bar{L}_\Sigma\epsilon^{AB}\bar{\psi}_A\wedge\psi_B + \\ & +i(\bar{\mathcal{N}}^{\Lambda\Sigma}f_\Sigma^i\bar{\lambda}_i^A\gamma^a\psi^B\epsilon_{AB} - \mathcal{N}^{\Lambda\Sigma}\bar{f}_{\Sigma i}\bar{\lambda}^i_A\gamma^a\psi_B\epsilon^{AB})\wedge V_a, \end{aligned} \tag{4.36}$$

and R^Λ is the remainder that comes from the $\frac{\delta}{\delta A_\Lambda}$-variation of those terms that are manifestly R-duality invariant and do not depend on the graviphoton A_0. Since one can easily verify that $\hat{\delta}S^\Lambda$ vanishes whenever $\Lambda \neq 0$ (to this purpose, note that $\hat{\delta}(\bar{\mathcal{N}}^{i\Sigma}F_\Sigma^{+ab}) = \hat{\delta}(\mathcal{N}^{i\Sigma}F_\Sigma^{-ab}) = 0$ and use the explicit expressions (4.11)), then the field equations of the matter vectors are all R-duality invariant.

In order to prove R-duality invariance of the remaining field equations, we note that it is not necessary to study the entire lagrangian $\mathcal{L}$ (4.25), because various terms can give only contributions with the correct $\hat{\delta}$-transformation properties. These are precisely the R-duality invariant terms of $\mathcal{L}$ that do not depend on A_0. On the other hand, since $\hat{\delta}F_0^{ab}$ depends on all the fields, we cannot neglect a term $\Delta\mathcal{L}$ only because it is $\hat{\delta}$-invariant ($\hat{\delta}\Delta\mathcal{L} = 0$) if it contains A_0. Indeed, if ϕ is a field of charge q ($\hat{\delta}\phi = q\phi$; we can take $\phi \neq A_0$ since the A_0-equation has already been studied), then the contributions to its field equation (i. e. $\frac{\partial}{\partial\phi}\Delta\mathcal{L}$) must have charge $-q$ in order to transform correctly ($\hat{\delta}\frac{\partial}{\partial\phi}\Delta\mathcal{L} = -q\frac{\partial}{\partial\phi}\Delta\mathcal{L}$) and it must happen that

$$\left[\hat{\delta}, \frac{\partial}{\partial\phi}\right]\Delta\mathcal{L} = -q\frac{\partial}{\partial\phi}\Delta\mathcal{L}. \tag{4.37}$$

For this to be true it is sufficient (and necessary, if $\Delta\mathcal{L}$ has not a special form) to have

$$\left[\hat{\delta}, \frac{\partial}{\partial\phi}\right]\phi' = -q\frac{\partial}{\partial\phi}\phi', \tag{4.38}$$

for all fields ϕ'. However, this is not true for $\phi' = F_0^{ab}$ and so, if $\Delta\mathcal{L}$ depends on A_0 one should analyze it explicitly. Summarizing, it is sufficient to test R-duality invariance of the contributions to the field equations that come from the terms of the lagrangian either containing A_0 or not $\hat{\delta}$-invariant. This part of the lagrangian is given by

$$\begin{aligned}
\Delta\mathcal{L} \equiv\; & \frac{1}{6}(\bar{\mathcal{N}}^{\Lambda\Sigma}F_\Lambda^{+ab}F_{\Sigma ab}^{+} + \mathcal{N}^{\Lambda\Sigma}F_\Lambda^{-ab}F_{\Sigma ab}^{-})\varepsilon_{cdef}V^c \wedge V^d \wedge V^e \wedge V^f + \\
& -4i(\bar{\mathcal{N}}^{\Lambda\Sigma}F_\Lambda^{+ab} - \mathcal{N}^{\Lambda\Sigma}F_\Lambda^{-ab}) \wedge V_a \wedge V_b \wedge (F_\Sigma + \\
& -i(f_\Sigma^i\bar{\lambda}_i^A\gamma^c\psi^B\epsilon_{AB} + \bar{f}_{\Sigma i}\bar{\lambda}_A^i\gamma^c\psi_B\epsilon^{AB}) \wedge V_c), \\
& -2i\frac{1}{\sqrt{a}}F^0 \wedge (\epsilon_{AB}\bar{\psi}^A \wedge \psi^B - \epsilon^{AB}\bar{\psi}_A \wedge \psi_B) + \\
& -\frac{2}{a\sqrt{a}}F^0 \wedge (\bar{z}^i\bar{\lambda}_i^A\gamma_a\psi^B\epsilon_{AB} - z_i\bar{\lambda}_A^i\gamma_a\psi_B\epsilon^{AB}) \wedge V^a + \\
& +\frac{i}{a}(\epsilon_{AB}\bar{\psi}^A \wedge \psi^B \wedge \epsilon_{CD}\bar{\psi}^C \wedge \psi^D - \epsilon^{AB}\bar{\psi}_A \wedge \psi_B \wedge \epsilon^{CD}\bar{\psi}_C \wedge \psi_D) + \\
& -\frac{i}{a^3}(z_iz_j\epsilon^{AB}\bar{\lambda}_A^i\gamma_a\psi_B \wedge V^a \wedge \epsilon^{CD}\bar{\lambda}_C^j\gamma_b\psi_D \wedge V^b + \\
& -\bar{z}^i\bar{z}^j\epsilon_{AB}\bar{\lambda}_i^A\gamma_a\psi^B \wedge V^a \wedge \epsilon_{CD}\bar{\lambda}_j^C\gamma_b\psi^D \wedge V^b),
\end{aligned} \tag{4.39}$$

where W and W^{ij} have been replaced by their explicit expressions in terms of z_i, and $\bar{z}^i$ and, after replacement, the manifestly $\hat{\delta}$-invariant terms not containing A_0 have been deleted. At this point, the check that the contributions to the field equations of the fermions, the vierbein and the scalars transform correctly is rather direct and we leave it to the reader. We thus conclude that

Proposition. *N=2 supergravity minimally coupled to n vector multiplets gauging an arbitrary n dimensional group (in which the graviphoton is not gauged), is on-shell R-duality invariant*[1].

The possibility that R-duality exists also in the N=3 theory or in more extended supergravity theories as well as the possibility to have it in N=2 matter coupled supergravity in nonminimal cases (even if, we presume, it might not be suitable for a topological twist) remain open problems. Here we have restricted our attention to that internal $U(1)$ symmetry that was relevant to our purposes, that is the topological twist.

We have so far neglected the coupling of matter hypermultiplets to N=2 supergravity, since it is immediately verified that the generalization of R-duality due to the presence of them is trivial. The scalars have 0 charge, however the left handed components of fermions must have -1 charge and the right handed components must have $+1$ charge, differently from the case of the other fermions so far encountered. The twist is by no means trivial. As a matter of fact, it turns out that it is interesting as we shall see in the next chapter.

4.5 Topological twist of the minimal theory

In this section, we discuss the twisted topological theory. First of all, let us note that the gauge-free algebra (i. e. the minimal BRST algebra, with neither antighosts nor gauge-fixings, nor Lagrange multipliers) is simply the tensor product of the gauge-free algebras for topological gravity (3.20) and topological Yang-Mills (2.1), that is to say

$$\begin{aligned}
sA &= -\nabla c - \psi, \\
sc &= \phi - \frac{1}{2}[c,c], \\
s\psi &= \nabla\phi - [c,\psi], \\
s\phi &= -[c,\phi], \\
sV^a &= \psi^a - \mathcal{D}_0\varepsilon^a + \varepsilon^{ab}\wedge V_b, \\
s\omega_0^{ab} &= \chi^{ab} - \mathcal{D}_0\varepsilon^{ab}, \\
s\varepsilon^a &= \phi^a + \varepsilon^{ab}\wedge\varepsilon_b, \\
s\varepsilon^{ab} &= \eta^{ab} + \varepsilon^a{}_c\wedge\varepsilon^{cb}, \\
s\psi^a &= -\mathcal{D}_0\phi^a + \varepsilon^{ab}\wedge\psi_b - \chi^{ab}\wedge\varepsilon_b - \eta^{ab}\wedge V_b, \\
s\phi^a &= \varepsilon^{ab}\wedge\phi_b - \eta^{ab}\wedge\varepsilon_b, \\
s\chi^{ab} &= -\mathcal{D}_0\eta^{ab} + \varepsilon^{ac}\wedge\chi_c{}^b - \chi^{ac}\wedge\varepsilon_c{}^b, \\
s\eta^{ab} &= \varepsilon^{ac}\wedge\eta_c{}^b - \eta^{ac}\wedge\varepsilon_c{}^b.
\end{aligned} \tag{4.40}$$

[1]Note that for an N=2 theory without hypermultiplets, the statement that a certain vector is not gauged is equivalent to the statement that it corresponds to a $U(1)$ subgroup of the full gauge group.

We have grouped the n matter vectors A_i into the column $A=(A_i)$. Similarly, $\psi=(\psi_i)$, $\phi=(\phi_i)$ and $c=(c_i)$.

The observables and the corresponding descent equations can be derived from the hatted extensions of the identities $d\,\mathrm{tr}[F\wedge F]=0$, $d\,\mathrm{tr}[R\wedge R]=0$ and $d\,\mathrm{tr}[R\wedge\tilde{R}]=0$, in the usual way (see section 3.3).

The BRST algebra of N=2 matter coupled supergravity can be found as explained in section 3.2, that is to say by extending all differential forms to ghost forms. We report only the final result, that, together with the translation ghosts ε^a, the Lorentz ghosts ε^{ab}, the supersymmetry ghosts c_A, c^A, involves also the gauge ghosts c^Λ.

$$
\begin{aligned}
sV^a &= -\mathcal{D}\varepsilon^a+\varepsilon^{ab}\wedge V_b+i(\bar{\psi}_A\wedge\gamma^a c^A+\bar{c}_A\wedge\gamma\psi^A),\\
s\varepsilon^a &= \varepsilon^{ab}\wedge\varepsilon_b+i\bar{c}_A\wedge\gamma^a c^A,\\
s\omega^{ab} &= -\mathcal{D}\varepsilon^{ab}+2R^{ab}{}_{cd}V^c\varepsilon^d+i(\varepsilon_c\bar{\psi}_A+V_c\bar{c}_A)(2\gamma^{[a}\rho^{A|b]c}-\gamma^c\rho^{A|ab})+\\
&\quad +i(\varepsilon_c\bar{\psi}^A+V_c\bar{c}^A)(2\gamma^{[a}\rho_A{}^{|b]c}-\gamma^c\rho_A{}^{|ab})+4G^{-ab}\epsilon^{AB}\bar{\psi}_A\wedge c_B+\\
&\quad +4G^{+ab}\epsilon_{AB}\bar{\psi}^A\wedge c^B+\frac{i}{4}\varepsilon^{abcd}(\bar{\psi}_A\wedge\gamma_c c^B+\bar{c}_A\wedge\gamma_c\psi^B)(2\bar{\lambda}_{iB}\gamma_d\lambda^{iA}-\delta^A_B\bar{\lambda}_{iC}\gamma_d\lambda^{iC}),\\
s\varepsilon^{ab} &= \varepsilon^a{}_c\wedge\varepsilon^{cb}+R^{ab}{}_{cd}\varepsilon^c\varepsilon^d-i\bar{c}_A(2\gamma^{[a}\rho^{A|b]c}-\gamma^c\rho^{A|ab})\varepsilon_c+\\
&\quad -i\bar{c}^A(2\gamma^{[a}\rho_A{}^{|b]c}-\gamma^c\rho_A{}^{|ab})\varepsilon_c+2G^{-ab}\epsilon^{AB}\bar{c}_A\wedge c_B+\\
&\quad +2G^{+ab}\epsilon_{AB}\bar{c}^A\wedge c^B+\frac{i}{4}\varepsilon^{abcd}\bar{c}_A\wedge\gamma_c c^B(2\bar{\lambda}_{iB}\gamma_d\lambda^{iA}-\delta^A_B\bar{\lambda}_{iC}\gamma_d\lambda^{iC}),\\
s\psi_A &= -\mathcal{D}c_A+\frac{1}{4}\varepsilon^{ab}\gamma_{ab}\wedge\psi_A-\frac{i}{2}Q\wedge c_A-\frac{i}{2}Q_{(0,1)}\wedge\psi_A+2\rho_{A|ab}V^a\wedge\varepsilon^b+\\
&\quad -2i\epsilon_{AB}G^+_{ab}\gamma^a(c^B\wedge V^b+\psi^B\wedge\varepsilon^b)+\frac{i}{4}(c_BV_a+\psi^B\varepsilon_a)\bar{\lambda}^{iB}\gamma^a\lambda_{iA}+\\
&\quad +\frac{i}{8}\gamma_{ab}(c_BV^b+\psi_B\varepsilon^b)\left(2\bar{\lambda}^{iB}\gamma^a\lambda_{iA}-\delta^B_A\bar{\lambda}^{iC}\gamma^a\lambda_{iC}\right),\\
sc_A &= \frac{1}{4}\varepsilon^{ab}\gamma_{ab}\wedge c_A-\frac{i}{2}Q_{(0,1)}\wedge c_A+\rho_{A|ab}\varepsilon^a\wedge\varepsilon^b-2i\epsilon_{AB}G^+_{ab}\gamma^a c^B\wedge\varepsilon^b+\\
&\quad +\frac{i}{4}c_B\bar{\lambda}^{iB}\gamma^a\lambda_{iA}\wedge\varepsilon_a+\frac{i}{8}\gamma_{ab}c_B\left(2\bar{\lambda}^{iB}\gamma^a\lambda_{iA}-\delta^B_A\bar{\lambda}^{iC}\gamma^a\lambda_{iC}\right)\wedge\varepsilon^b,\\
s\psi^A &= -\mathcal{D}c^A+\frac{1}{4}\varepsilon^{ab}\gamma_{ab}\wedge\psi^A+\frac{i}{2}Q\wedge c^A+\frac{i}{2}Q_{(0,1)}\wedge\psi^A+2\rho^A_{|ab}V^a\wedge\varepsilon^b+\\
&\quad -2i\epsilon^{AB}G^-_{ab}\gamma^a(c_B\wedge V^b+\psi_B\wedge\varepsilon^b)+\frac{i}{4}(c^BV_a+\psi^B\varepsilon_a)\bar{\lambda}_{iB}\gamma^a\lambda^{iA}+\\
&\quad +\frac{i}{8}\gamma_{ab}(c^BV^b+\psi^B\varepsilon^b)\left(2\bar{\lambda}_{iB}\gamma^a\lambda^{iA}-\delta^A_B\bar{\lambda}_{iC}\gamma^a\lambda^{iC}\right),\\
sc^A &= \frac{1}{4}\varepsilon^{ab}\gamma_{ab}\wedge c^A+\frac{i}{2}Q_{(0,1)}\wedge c^A+\rho^A_{|ab}\varepsilon^a\wedge\varepsilon^b-2i\epsilon^{AB}G^-_{ab}\gamma^a c_B\wedge\varepsilon^b+\\
&\quad +\frac{i}{4}c^B\bar{\lambda}_{iB}\gamma^a\lambda^{iA}\wedge\varepsilon_a+\frac{i}{8}\gamma_{ab}c^B\left(2\bar{\lambda}_{iB}\gamma^a\lambda^{iA}-\delta^A_B\bar{\lambda}_{iC}\gamma^a\lambda^{iC}\right)\wedge\varepsilon^b,\\
sA_\Lambda &= -dc_\Lambda-2f_\Lambda{}^{\Omega\Delta}A_\Omega\wedge c_\Delta-2\epsilon_{AB}L_\Lambda\bar{\psi}^A\wedge c^B-2\epsilon^{AB}\bar{L}_\Lambda\bar{\psi}_A\wedge c_B+\\
&\quad +2F^{ab}_\Lambda V_a\wedge\varepsilon_b+i(f^i_\Lambda\bar{\lambda}^A_i\gamma^a c^B\epsilon_{AB}+\bar{f}_{\Lambda i}\bar{\lambda}^i_A\gamma^a c_B\epsilon^{AB})\wedge V_a+
\end{aligned}
$$

$$
\begin{aligned}
& +i(f^i_\Lambda \bar\lambda^A_i \gamma^a \psi^B \epsilon_{AB} + \bar f_{\Lambda i} \bar\lambda^i_A \gamma^a \psi_B \epsilon^{AB}) \wedge \varepsilon_a, \\
sc_\Lambda &= -f_\Lambda{}^{\Omega\Delta} c_\Omega \wedge c_\Delta - \epsilon_{AB} L_\Lambda \bar c^A \wedge c^B - \epsilon^{AB} \bar L_\Lambda \bar c_A \wedge c_B + F^{ab}_\Lambda \varepsilon_a \wedge \varepsilon_b + \\
& +i(f^i_\Lambda \bar\lambda^A_i \gamma^a c^B \epsilon_{AB} + \bar f_{\Lambda i} \bar\lambda^i_A \gamma^a c_B \epsilon^{AB}) \wedge \varepsilon_a, \\
s\lambda_{iA} &= \frac{1}{4}\varepsilon^{ab} \wedge \gamma_{ab} \lambda_{iA} - \frac{i}{2}\left(1+\frac{2}{n}\right) Q_{(0,1)} \wedge \lambda_{iA} - Q_{(0,1)i}{}^j \lambda_{jA} + \nabla_a \lambda_{iA} \varepsilon^a + \\
& +i P_{i|a} \gamma^a c^B \epsilon_{AB} + G_i^{+ab} \gamma_{ab} c_A + g C_i c_A, \\
s\lambda^{iA} &= \frac{1}{4}\varepsilon^{ab} \wedge \gamma_{ab} \lambda^{iA} + \frac{i}{2}\left(1+\frac{2}{n}\right) Q_{(0,1)} \wedge \lambda^{iA} - Q_{(0,1)}{}^i{}_j \lambda^{jA} + \nabla_a \lambda^{iA} \varepsilon^a + \\
& +i P^i_{|a} \gamma^a c_B \epsilon^{AB} + G^{-i}_{ab} \gamma^{ab} c^A + g C^i c^A, \\
sz_i &= -g c_\Lambda k_i^\Lambda(z) + Z_{i|a} \varepsilon^a + \bar\lambda_i^A c_A, \\
s\bar z^i &= -g c_\Lambda k^{i\Lambda}(\bar z) + \bar Z^i_{|a} \varepsilon^a + \bar\lambda^i_A c^A.
\end{aligned}
\tag{4.41}
$$

In Eq. (4.41) $Q_{(0,1)}$ and $Q_{(0,1)_i}{}^j$ are obtained by the one-forms Q and $Q_i{}^j$ upon substitution of ∇z_i with $Z_{i|a}\varepsilon^a + \bar\lambda_i^A c_A$, ∇z_i with $\bar Z^i_{|a}\varepsilon^a + \bar\lambda^i_A c^A$ and of A_Λ with c_Λ. In particular, $Q_{(0,1)} = -\frac{i}{2}(G^i Z_{i|a} - G_i \bar Z^i_{|a})\varepsilon^a - \frac{i}{2}(G^i \bar\lambda_i^A c_A - G_i \bar\lambda^i_A c^A)$.

The BRST algebra of the twisted theory is the above algebra when one implements the topological twist and the topological shift, as explained in section 3.4. From now on, when we shall refer to the above algebra, this implementation will be understood. The explicit twist is realized as follows

$$
\begin{aligned}
\psi_{\alpha A} &\to \psi_{\alpha \dot A}, & \psi^{\dot\alpha A} &\to \psi^{\dot\alpha \dot A}, \\
\lambda_{i\alpha A} &\to \lambda_{i\alpha \dot A}, & \lambda^{i\dot\alpha A} &\to \lambda^{i\dot\alpha \dot A}, \\
\epsilon_{AB} &\to \epsilon_{\dot A \dot B}, & \epsilon^{AB} &\to -\epsilon^{\dot A \dot B},
\end{aligned}
\tag{4.42}
$$

while the topological shift is obtained by

$$
c^{\dot\alpha \dot A} \to -\frac{i}{2} e \varepsilon^{\dot\alpha \dot A} + c^{\dot\alpha \dot A},
\tag{4.43}
$$

where e is the broker.

We now rewrite the most relevant twisted-shifted BRST transformations up to non-linear terms. To this purpose, note that, when z_i and $\bar z^i$ tend to zero, then $a \to 1$; $L_0, \bar L_0 \to 1$; $f_i{}^j \to \delta_i^j$; $(g^{\frac{1}{2}})_i{}^j \to \delta_i^j$; $G^+_{ab} \to -\frac{1}{2}F^{+0}_{ab}$; $G^{i-}_{ab} \to -\frac{1}{2}F^{i-}_{ab}$. Let us define (note that the gauginos are expressed in the N=3 notation, namely λ_{iA}, λ^{iA})

$$
\begin{aligned}
\tilde\psi^a &= \tfrac{e}{2}\psi_{\alpha \dot A}(\bar\sigma^a)^{\dot A\alpha}, & \tilde\psi^{ab} &= -e(\bar\sigma^{ab})^{\dot A}{}_{\dot\alpha}\psi^{\dot\alpha}{}_{\dot A}, & \tilde\psi &= -e\psi_{\dot\alpha}{}^{\dot A}\delta^{\dot\alpha}_{\dot A}, \\
C^a &= \tfrac{e}{2}c_{\alpha \dot A}(\bar\sigma^a)^{\dot A\alpha}, & C^{ab} &= -e(\bar\sigma^{ab})^{\dot A}{}_{\dot\alpha}c^{\dot\alpha}{}_{\dot A}, & C &= -e c_{\dot\alpha}{}^{\dot A}\delta^{\dot\alpha}_{\dot A}, \\
\lambda_i &= \tfrac{e}{2}\lambda_{i\alpha \dot A}(\bar\sigma^a)^{\dot A\alpha}V_a, & \lambda^{iab} &= -e(\bar\sigma^{ab})^{\dot A}{}_{\dot\alpha}\lambda^{i\dot\alpha}{}_{\dot A}, & \tilde\lambda^i &= -e\lambda^i{}_{\dot\alpha}{}^{\dot A}\delta^{\dot\alpha}_{\dot A}.
\end{aligned}
\tag{4.44}
$$

Up to nonlinear terms, we obtain

$$
\begin{array}{rlrl}
sV^a = \tilde{\psi}^a - d\varepsilon^a + \varepsilon^{ab} \wedge V_b, & & s\varepsilon^a = C^a, & \\
s\varepsilon^{ab} = -\frac{1}{2} F_0^{+ab}, & & s\tilde{\psi}^a = -dC^a + \frac{1}{2} F^{+ab}_{0} \wedge V_b, & \\
s\tilde{\psi}^{ab} = -dC^{ab} + \frac{i}{2} \omega^{-ab}, & & s\tilde{\psi} = -dC, & \\
sC^a = 0, & & sC^{ab} = \frac{i}{2} \varepsilon^{-ab}, & \\
sC = 0, & & s\lambda_i = \frac{1}{2} dz_i, & \qquad (4.45) \\
s\lambda^{iab} = iF_i^{-ab}, & & s\tilde{\lambda}^i = 0, & \\
sA_i = -dc_i + \lambda_i, & & sc_i = -\frac{1}{2} z_i, & \\
sz_i = 0, & & s\bar{z}^i = \frac{i}{2} \tilde{\lambda}^i, & \\
sA_0 = i\tilde{\psi} - dc_0, & & sc_0 = -\frac{1}{2} + iC. &
\end{array}
$$

Here $F^{-ab} = \frac{1}{2}(F^{ab} + \frac{i}{2}\varepsilon^{abcd}F_{cd})$.

From Eq.s (4.44) and (4.45) we can directly identify what are the topological ghosts, the topological antighosts (up to interaction terms) and the topological gauge-fixings. More generally, one retrieves the topological meaning of the twisted versions of all the fields of the original theory. $\tilde{\psi}^a$ are the topological ghosts associated to the graviton, λ_i those associated to the matter vectors; the corresponding topological antighosts are $\tilde{\psi}^{ab}$ and λ^{iab}, respectively. The ghosts for ghosts are C^a, F_0^{+ab} and z_i, respectively for diffeomorphisms, Lorentz rotations and gauge transformations. $\bar{z}^i$ are antighosts for ghosts, while C^{ab} and C are extraghosts. Let us discuss the gauge-fixings. They involve complicated expressions depending on the various fields (even in the topological σ-model in two dimensions [5] one finds convenient to impose a topological gauge-fixing depending on the ghosts), but they can be equivalently read when all the ghosts are set to zero, because in the minimum of the BRST action all the ghosts are zero by definition. To this purpose, the interaction terms are negligible (they always contain ghosts). Our expectations are confirmed: the theory does indeed describe Yang-Mills instantons $F_i^{-ab} = 0$ in a background gravitational instanton $\omega^{-ab} = 0$ (the Wick rotation to the Euclidean is of course understood)[2].

We note that there are more observables than those we have constructed by means of the minimal BRST algebra (4.40). They involve also antighosts. In fact there is another noticeable differential form which is closed but not exact and which could be a source of nontrivial observables, namely the Kähler form K. In fact the Kähler potential G exists only locally and $K = dQ$ is only a local statement. The associated descent equations still give observables, however so far we have not revealed their deep meaning (if any). The Kähler form and its extended version are constructed with both ghosts and antighosts, while one usually uses only ghosts. We must remark that the topological Yang-Mills theory we have found is *not* exactly Witten's topological Yang-Mills theory coupled to gravity. In fact, Witten's theory is described by a flat Kähler manifold (and

[2] Note that the BRST variation of the topological gravitational antighost $\tilde{\psi}^{ab}$ contains, in addition to the gauge-fixing ω^{-ab}, also the derivative of the extraghost C^{ab}. As explained in section 3.6, this is due to the redundancy of the gauge conditions $\omega^{-ab} = 0$.

Q exists globally, so K is not interesting), while our theory corresponds to $\frac{SU(1,n)}{SU(n)\otimes U(1)}$ and K cannot be globally exact [13], so it cannot be *a priori* discarded. One has

$$K = ig_i{}^j \nabla z_j \wedge \nabla \bar{z}^i + \frac{i}{2} g(G_i k^{i\Lambda} - G^i k_i^\Lambda)(dA_\Lambda + f_\Lambda{}^{\Sigma\Pi} A_\Sigma \wedge A_\Pi). \tag{4.46}$$

The descent equations derived from $\hat{d}\hat{K} = 0$ give the following observables

$$\begin{aligned} \mathcal{O}^{(0)} &= K_{(0,2)}, \\ \mathcal{O}^{(1)}_\gamma &= \int_\gamma K_{(1,1)}, \\ \mathcal{O}^{(2)}_S &= \int_S K, \end{aligned} \tag{4.47}$$

where γ and S are one- and two-dimensional cycles, while

$$\begin{aligned} K_{(0,2)} &= ig_i{}^j(Z_{j|a}\varepsilon^a + \bar{\lambda}_j^A c_A) \wedge (\bar{Z}^i_{|a}\varepsilon^a + \bar{\lambda}^i_B c^B) + \\ &\quad -\frac{i}{2} g(G_j k^{j\Lambda} - G^j k_j^\Lambda)(\epsilon_{AB} L_\Lambda \bar{c}^A \wedge c^B + \epsilon^{AB}\bar{L}_\Lambda \bar{c}_A \wedge c_B - F_\Lambda^{ab}\varepsilon_a \wedge \varepsilon_b + \\ &\quad -i(f^i_\Lambda \bar{\lambda}^A_i \gamma^a c^B \epsilon_{AB} + \bar{f}_{\Lambda i}\bar{\lambda}^i_A \gamma^a c_B \epsilon^{AB}) \wedge \varepsilon_a), \\ K_{(1,1)} &= ig_i{}^j(Z_{j|a}\varepsilon^a + \bar{\lambda}_j^A c_A) \wedge \nabla \bar{z}^i + ig_i{}^j \nabla z_j(\bar{Z}^i_{|a}\varepsilon^a + \bar{\lambda}^i_A c^A) + \\ &\quad -\frac{i}{2} g(G_j k^{j\Lambda} - G^j k_j^\Lambda)(2\epsilon_{AB} L_\Lambda \bar{\psi}^A \wedge c^B + 2\epsilon^{AB}\bar{L}_\Lambda \bar{\psi}_A \wedge c_B + \\ &\quad -2F_\Lambda^{ab} V_a \wedge \varepsilon_b - i(f^i_\Lambda \bar{\lambda}^A_i \gamma^a c^B \epsilon_{AB} + \bar{f}_{\Lambda i}\bar{\lambda}^i_A \gamma^a c_B \epsilon^{AB}) \wedge V_a + \\ &\quad -i(f^i_\Lambda \bar{\lambda}^A_i \gamma^a \psi^B \epsilon_{AB} + \bar{f}_{\Lambda i}\bar{\lambda}^i_A \gamma^a \psi_B \epsilon^{AB}) \wedge \varepsilon_a). \end{aligned} \tag{4.48}$$

The correspondence between the gauge-free algebra (4.40) and the complete BRST algebra (4.41) is realized by the following identifications

$$\begin{aligned} \psi^a &= i(\bar{c}_A \wedge \gamma^a \psi^A + \bar{\psi}_A \wedge \gamma^a c^A) - A^{ab} \wedge \varepsilon^b = \tilde{\psi}^a + \cdots, \\ \phi^a &= i\bar{c}_A \wedge \gamma^a c^A = C^a + \cdots, \\ \chi^{ab} &= sA^{ab} - A^{ac}\varepsilon_c{}^b + \varepsilon^{ac}A_c{}^b + 2R^{ab}{}_{cd}V^c\varepsilon^d + i(\varepsilon_c \bar{\psi}_A + V_c \bar{c}_A)(2\gamma^{[a}\rho^{A|b]c} - \gamma^c \rho^{A|ab}) + \\ &\quad +i(\varepsilon_c \bar{\psi}^A + V_c \bar{c}^A)(2\gamma^{[a}\rho_A{}^{|b]c} - \gamma^c \rho_A{}^{|ab}) + 4G^{-ab}\epsilon^{AB}\bar{\psi}_A \wedge c_B + \\ &\quad +4G^{+ab}\epsilon_{AB}\bar{\psi}^A \wedge c^B + \frac{i}{4}\varepsilon^{abcd}(\bar{\psi}_A \wedge \gamma_c c^B + \bar{c}_A \wedge \gamma_c \psi^B)(2\bar{\lambda}_{iB}\gamma_d \lambda^{iA} - \delta^A_B \bar{\lambda}_{iC}\gamma_d \lambda^{iC}), \\ \psi_i &= 2\epsilon_{AB} L_i \bar{\psi}^A \wedge c^B + 2\epsilon^{AB}\bar{L}_i \bar{\psi}_A \wedge c_B + \\ &\quad -2F_i^{ab} V_a \wedge \varepsilon_b - i(f_i^j \bar{\lambda}_j^A \gamma^a c^B \epsilon_{AB} + \bar{f}_{ij}\bar{\lambda}^j_A \gamma^a c_B \epsilon^{AB}) \wedge V_a + \\ &\quad -i(f_i^j \bar{\lambda}_j^A \gamma^a \psi^B \epsilon_{AB} + \bar{f}_{ij}\bar{\lambda}^j_A \gamma^a \psi_B \epsilon^{AB}) \wedge \varepsilon_a = -\lambda_i + \cdots, \\ \phi_i &= -\epsilon_{AB} L_i \bar{c}^A \wedge c^B - \epsilon^{AB}\bar{L}_i \bar{c}_A \wedge c_B + F_i^{ab}\varepsilon_a \wedge \varepsilon_b + \\ &\quad +i(f_i^j \bar{\lambda}_j^A \gamma^a c^B \epsilon_{AB} + \bar{f}_{ij}\bar{\lambda}^j_A \gamma^a c_B \epsilon^{AB}) \wedge \varepsilon_a = -\frac{1}{2} z_i + \cdots, \\ \eta^{ab} &= R^{ab}{}_{cd}\varepsilon^c \varepsilon^d - i\bar{c}_A(2\gamma^{[a}\rho^{A|b]c} - \gamma^c \rho^{A|ab})\varepsilon_c - i\bar{c}^A(2\gamma^{[a}\rho_A{}^{|b]c} - \gamma^c \rho_A{}^{|ab})\varepsilon_c + \end{aligned}$$

$$+2G^{-ab}\epsilon^{AB}\bar{c}_A \wedge c_B + 2G^{+ab}\epsilon_{AB}\bar{c}^A \wedge c^B +$$
$$+\frac{i}{4}\varepsilon^{abcd}\bar{c}_A \wedge \gamma_c c^B(2\bar{\lambda}_{iB}\gamma_d\lambda^{iA} - \delta^A_B\bar{\lambda}_{iC}\gamma_d\lambda^{iC}) = -\frac{1}{2}F_0^{+ab} + \cdots, \quad (4.49)$$

where $A^{ab} \wedge V_b = i\bar{\psi}_A \wedge \gamma^a\psi^A$ and the dots stand for nonlinear corrections.

Finally, we write the gauge fermion Ψ, the BRST variation of which is the quadratic part of the N=2 lagrangian, after topological twist and topological shift.

$$\begin{aligned}\Psi &= -16i(B^{ab} - i\omega^{-ab} + 2dC^{ab}) \wedge \tilde{\psi}_{ac} \wedge V_b \wedge V^c + 8iF_0 \wedge \psi^a \wedge V_a + \\ &+ \left(\frac{2}{3}\eta^{ab}\varepsilon_{ab} - \frac{1}{6}(M_{iab} - 2iF^-_{iab})\lambda^{iab}\right)\varepsilon_{cdef}V^c \wedge V^d \wedge V^e \wedge V^f + \\ &+\frac{4}{3}\lambda^a_i d\bar{z}^i \wedge \varepsilon_{abcd}V^b \wedge V^c \wedge V^d. \end{aligned} \quad (4.50)$$

Here, B^{ab} and M^{iab} are Lagrange multipliers ($s\tilde{\psi}^{ab} = B^{ab}$, $s\lambda^{iab} = M^{iab}$, $sB^{ab} = 0$, $sM^{iab} = 0$), while λ^a_i is such that $\lambda_i = \lambda^a_i V_a$.

Chapter 5

Hyperinstantons

5.1 Introduction

In this chapter we discuss the topological twist of quaternionic matter multiplets [13] coupled to N=2 supergravity. We shall not develop the entire formalism in full detail, living it for a future publication, but we shall concentrate on some of its relevant aspects. Along the discussion, we shall have occasion to outline some relevant properties of the twisting procedure, that are visible only when dealing with hypermultiplets. These properties contribute to make the difference with the original procedure suggested by Witten [10]. Consequently, we can recapitulate the whole twisting procedure in a completely general way (section 5.2). After this, we analyze in detail (section 5.3) the instantons described by the twisted version of N=2 quaternionic σ-models (hyperinstantons), both in the case when gravity is external and in the case when gravity is dynamical (thus also modifying the gravitational instantons, due to the coupling to hyperinstantons).

5.2 The general structure of the twisting procedure

We already anticipated in chapter 2 that the twisting procedure as described by Witten [10] needs some modifications in order to work correctly. First of all, as shown in chapter 3, the twist acts on the Lorentz group and does not touch the space-time indices. This was straightforward in the case of pure supergravity, since all the fields are one-forms, i. e. they are all on the same footing as far as space-time indices are concerned. Consequently the twist on the Lorentz group works in exactly the same way as the twist proposed by Witten. However, when studying the case of the Yang-Mills theory, one has to face the problem that the vector bosons A_i are one forms and Lorentz scalars, while the gauginos λ_i^A and λ_A^i are zero-forms and Lorentz spinors. If you are in flat space, you can mix Lorentz and Einstein indices and so the twist can work in the way described by Witten. However, Witten himself notes [10] that his method works only in flat space, even if the result is valid in any curved space. If we follow our method, this problem is simply absent. We remain in the most general curved space and act only on the Lorentz indices.

At this point, the twisted vector boson is still a one-form and a Lorentz scalar, while the twisted left handed gaugino $\lambda_i^a \equiv \frac{e}{2}\lambda_{i_{\alpha\dot{A}}}(\bar{\sigma}^a)^{\dot{A}\alpha}$ is a zero-form and a Lorentz vector. From (4.45) you immediately read that the true topological antighost is not simply λ_i^a, but $\lambda_i = \lambda_i^a V_a$, i. e. the object that you obtain from the simple twist (λ_i^a) must be contracted with the vierbein V^a. λ_i is a one-form and a Lorentz scalar, as desired.

In order to show that the contraction with a vielbein plays a substantial role in the twisting procedure, one would like to exhibit a case in which this step is so important that no result can be obtained without it (even in flat space). This is precisely the case of the quaternionic σ-model. The multiplet consists of (q^i, ζ_I, ζ^I), where ζ_I and ζ^I are the left handed and right handed components of the spinors ($I = 1, \ldots 2m$), while q^i are the coordinates of a $4m$-dimensional manifold $\mathcal{Q}(m)$ ($i = 1, \ldots 4m$), with a quaternionic structure, namely possessing three complex structures J^x, $x = 1, 2, 3$, fulfilling the quaternionic algebra. Specifically $\mathcal{Q}(m)$ is a Hyperkähler manifold when gravity is not dynamical (i.e. it is external), while it is a quaternionic manifold when gravity is dynamical. As you see, no field has indices of $SU(2)_I$, i. e. all the fields are singlets under the internal $SU(2)$. Consequently, the usual twisting procedure acts trivially on hypermultiplets: the Lorentz scalars remain Lorentz scalars and the spinors remain spinors. Let us see how the contraction with a suitable vielbein can help when the usual twisting procedure does not give directly the true topological ghosts (i.e. it gives objects with the wrong spin assignment). Since the hypermultiplets are made of zero-forms, the vierbein V^a cannot help us. Fortunately, however, there *is* a vielbein that does the job, namely the quaternionic vielbein $\mathcal{U}_i^{AI}$ ($A = 1, 2$ is an index of $SU(2)_I$) [13]. We can for example take the contraction $\mathcal{U}^i_{AI}\bar{c}^A\zeta^I$, where $\mathcal{U}^i_{AI}$ is the inverse vielbein. After the topological shift, this expression becomes $-\frac{i}{2}e\mathcal{U}^i_{\dot{A}I}\zeta^{\dot{A}I}$, up to interaction terms, and is the natural candidate to become the topological ghost (it is also the *only* candidate). Here is another novelty: the topological ghost is constructed with the *right* handed components of the fermions, *not* the left handed ones. This means that the R-duality charge of ζ^I is $+1$ and that of ζ_I is -1, the opposite of what happens in the other cases that we have studied. This is not completely surprising, because the reasoning of section 4.2 that established the R-duality charges of gravitinos and gauginos was essentially based on the effects of the usual redefinition of $SU(2)_R$ on the representations of the Lorentz group, effects that are absent in the present case. From Ref. [13] one can convince oneself that this is in fact the correct charge assignment.

The general feature of $\mathcal{Q}(m)$ is that its holonomy group $Hol(\mathcal{Q}(m))$ is contained in $SU(2) \otimes Sp(2m)$. This $SU(2)$ is nothing but $SU(2)_I$ [13]. In the Hyperkähler case, the $SU(2)$ part of the spin connection of $\mathcal{Q}(m)$ is flat, while in the quaternionic case its curvature is proportional to $\Omega^x = h_{ik}(J^x)^k_j dq^i \wedge dq^j$, where h_{ij} is the metric of $\mathcal{Q}(m)$. In both cases one can exploit another $SU(2)$, which will be denoted by $SU(2)_Q$, namely the $SU(2)$ factor in the $SU(2) \otimes SO(m)$ maximal subgroup of $Sp(2m)$. We shall see in the next section that the twisting procedure requires also a natural redefinition of $SU(2)_L$, namely

$$SU(2)_L \rightarrow SU(2)'_L = \mathrm{diag}[SU(2)_L \otimes SU(2)_Q]. \tag{5.1}$$

Summarizing, the complete twisting procedure can be divided in the following three steps. Step A corresponds to the redefinitions of $SU(2)_L$, $SU(2)_R$ and ghost number $U(1)_g$

$$\begin{aligned} SU(2)_L &\rightarrow SU(2)'_L = \mathrm{diag}[SU(2)_L \otimes SU(2)_Q], \\ SU(2)_R &\rightarrow SU(2)'_R = \mathrm{diag}[SU(2)_R \otimes SU(2)_I], \\ U(1)_g &\rightarrow U(1)'_g = \mathrm{diag}[U(1)_g \otimes U(1)_I], \\ {}^c(L,R,I,Q)^g_f &\rightarrow (L\otimes Q, R\otimes I)^{g+c}_f, \end{aligned} \tag{5.2}$$

where Q denotes the representation of $SU(2)_Q$. Step B is the correct identification of the topological ghosts (fields with $g+c=1$ from $g=0$, $c=1$) by contraction with a suitable vielbein (if it exists). Step C is the topological shift, namely the shift by a constant (times the broker) of the $(0,0)^0_0$ field coming from the right handed components of the supersymmetry ghosts.

5.3 Topological quaternionic σ-models

We want to analyze the instantons described by the topological varsion of N=2 quaternionic σ-models. One would have to implement the twisting procedure of the previous section on the whole BRST algebra, however, for our purpose of spotting the nature of the instantons, reading the topological gauge-fixings, it is sufficient to consider only those terms in the BRST variations of the fields that correspond to supersymmetries. These terms of the BRST variations are

$$\begin{aligned} \delta q^i &= \mathcal{U}^i_{AI}(\epsilon^{AB}C^{IJ}\bar{c}_B\zeta_J + \bar{c}^A\zeta^I), \\ \delta\zeta_I &= i\mathcal{U}^{BJ}_a\gamma^a c^A\epsilon_{AB}C_{IJ}, \\ \delta\zeta^I &= i\mathcal{U}^{AI}_a\gamma^a c_A, \end{aligned} \tag{5.3}$$

where C_{IJ} is the flat $Sp(2m)$ invariant metric while $\mathcal{U}^{AI}_a$ is the supercovariantized derivative of the quaternionic field q^i with indices flattened both with respect to spacetime and with respect to the quaternionic manifold via the corresponding vielbeins.

$$\mathcal{U}^{AI}_a = V^\mu_a(\mathcal{U}^{AI}_i\partial_\mu q^i - \epsilon^{AB}C^{IJ}\bar{\psi}_{\mu B}\zeta_J - \bar{\psi}_\mu{}^A\zeta^I). \tag{5.4}$$

The topological shift gives, up to nonlinear terms,

$$\begin{aligned} \delta q^i &= -\frac{i}{2}e\mathcal{U}^i_{\dot{A}I}(\zeta^{\dot{A}})^I \equiv \xi^i, \\ \delta(\zeta_\alpha)_I &= \frac{e}{2}\mathcal{U}^{\dot{B}J}_a(\sigma^a)_{\alpha\dot{B}}C_{IJ}, \\ \delta(\zeta^{\dot{A}})^I &= 0, \end{aligned} \tag{5.5}$$

From this equation we realize that the topological symmetry is indeed the expected one for a σ-model, namely the map $q^i : M_{spacetime} \rightarrow \mathcal{Q}(m)$ can be continuously deformed.

The topological ghosts ξ^i are exactly what we anticipated in the previous section. In order to correctly identify the topological antighosts, we have to write the index I as the pair (α, k), where $\alpha = 1,2$ is the doublet index of $SU(2)_Q$ and $k = 1, \dots m$ is the vector index of $SO(m)$, such that $C_{IJ} = C_{(\alpha,k)(\beta,l)}$ takes the form $\epsilon_{\alpha\beta}\delta_{kl}$. The index of $SU(2)_Q$ is written in the same notation as the index of $SU(2)_L$. This automatically implements the identification between $SU(2)_L$ and $SU(2)_Q$ that produces $SU(2)'_L$. Now we write

$$\delta(\zeta_\alpha)_{\beta k} = \frac{e}{2}\mathcal{U}_a^{\dot{B}\gamma l}(\sigma^a)_{\alpha\dot{B}}\epsilon_{\beta\gamma}\delta_{kl} = \frac{e}{2}\mathcal{U}_a^{\dot{B}\gamma k}(\sigma^a)_{\alpha\dot{B}}\epsilon_{\beta\gamma}. \tag{5.6}$$

At this point we can introduce the vielbein $E_i^{ak} \equiv \frac{1}{2}\mathcal{U}_i^{\dot{A}\alpha k}(\sigma^a)_{\alpha\dot{A}}$ and the true topological antighosts $\zeta_k^{+ab} = -e(\sigma^{ab})_\alpha{}^\beta\epsilon^{\alpha\gamma}(\zeta_\beta)_{\gamma k}$ and $\zeta_k = -e\epsilon^{\alpha\beta}(\zeta_\alpha)_{\beta k}$, which, under the Lorentz group transform as $(1,0)$ and $(0,0)$ respectively. One finds

$$\begin{aligned}\delta\zeta^{+abk} &= 2V^{\mu[a}E_i^{b]^+k}\partial_\mu q^i,\\ \delta\zeta^k &= V_a^\mu E_i^{ak}\partial_\mu q^i,\end{aligned} \tag{5.7}$$

where $[ab]^+$ means selfdualization in the indices a, b. Thus we see that *both* ζ_k^{+ab} and ζ_k are topological antighosts (otherwise we would have not enough equations to fix the gauge completely). In the previously studied cases, instead, the $(0,1)$ components were the only topological antighosts, while the $(0,0)$ component permitted to fix the gauge freedom of the topological ghosts (directly related to the gauge freedom of the gauge freedom, which now is missing). Thus, the instantons described by this theory (which we name *hyperinstantons*) are given by the following equations

$$\begin{aligned}V^{\mu[a}E_i^{b]^+k}\partial_\mu q^i &= 0,\\ V_a^\mu E_i^{ak}\partial_\mu q^i &= 0.\end{aligned} \tag{5.8}$$

In a certain sense, Eq.s (5.8) is a condition of holomorphicity of the maps $M_{spacetime} \to \mathcal{Q}(m)$ with respect to the three complex (or almost complex) structures J^x of $\mathcal{Q}(m)$. For this reason we find it proper to name triholomorphic a map q satisfying Eq.s (5.8). In conclusion, in the same way as the instantons of topological σ-models in D=2 are given by holomorphic maps, those of topological σ-models in D=4 are given by triholomorphic maps.

If gravity is external ($\mathcal{Q}(m)$ is Hyperkähler) then the gravitational background should be restricted by the need to have N=2 global supersymmetry, however, the proof that the solutions to the above equations are indeed instantons works for any background and is based on the following identity

$$\begin{aligned}\int_{\mathcal{M}} d^4x\sqrt{-g}\quad g^{\mu\nu}\partial_\mu q^i\partial_\nu q^j h_{ij} = 2\int_{\mathcal{M}} d^4x\sqrt{-g}[(V_a^\mu E_i^{ak}\partial_\mu q^i)^2 + 4(V^{\mu[a}E_i^{b]^+k}\partial_\mu q^i)^2] +\\ -4i\int_{\mathcal{M}} E^{[ak}\wedge E^{b]^-k}\wedge V_a\wedge V_b,\end{aligned} \tag{5.9}$$

where $h_{ij} = 2E_i^{ak}E_j^{bk}\eta_{ab}$ is the metric of $\mathcal{Q}(m)$, while $E^{ak} \equiv E_i^{ak}dq^i$. The form $E^{[ak}\wedge E^{b]^-k}\wedge V_a\wedge V_b$ is proportional to $\Omega^x\wedge V_a\wedge V_b$ (the coefficient, which is a numerical matrix

M_x^{ab} antiselfdual in ab, is not important), where Ω^x are the 2-forms introduced above. Ω_x are closed forms of $\mathcal{Q}(m)$, if $\mathcal{Q}(m)$ is Hyperkähler. Consequently, in such a case the last term of (5.9) is a topological invariant and this completes our proof.

In the case gravity is dynamical ($\mathcal{Q}(m)$ is quaternionic) there exist three forms ω_x such that

$$\begin{aligned} d\Omega_x + \varepsilon_{xyz}\omega^y \wedge \Omega^z &= 0 \quad , \\ d\omega_x + \frac{1}{2}\varepsilon_{xyz}\omega^y \wedge \omega^z &= \Omega_x. \end{aligned} \tag{5.10}$$

The definition of the curvatures changes drastically with respect to the case of pure N=2 supergravity [13], in the sense that the curvature ρ^A of the right handed components of the gravitinos contains a term that modifies the gravitational topological gauge-fixing, after performing the topological twist and the topological shift. This means that the gravitational instantons are no longer described by an antiselfdual spin connection. As a matter of fact, $\rho^A = \mathcal{D}\psi^A + \frac{i}{2}\epsilon^{AB}\epsilon_{CD}(\sigma_x)_B{}^C\omega^x \wedge \psi^D$, where $(\sigma_x)_A{}^B$ are the Pauli matrices and the resulting instantons are given by

$$\omega^{-ab} - \frac{i}{2}M_x^{ab}\omega^x = 0. \tag{5.11}$$

There exist only one matrix with the properties of M_x^{ab}, up to a multiplicative constant, and this constant can be fixed by the fact that $M_x^{ac}M_y^{db}\eta_{cd}\varepsilon^{xy}{}_z = 2iM_x^{ab}$ (see for example section 5 of [30]). The proof that the hyperinstantons that solve Eq.s (5.8) and (5.11) are effectively instantons follows from the fact that the total kinetic lagrangian (Einstein lagrangian plus σ-model kinetic lagrangian) can be written as a sum of squares of the left hand sides of the above equations up to a total derivative

$$\begin{aligned} \mathcal{L}_{kin} &= \varepsilon_{abcd}R^{ab} \wedge V^c \wedge V^d - \frac{1}{6}\varepsilon_{abcd}V^a \wedge V^b \wedge V^c \wedge V^d g^{\mu\nu}h_{ij}\partial_\mu q^i \partial_\nu q^j = \\ &= 4i(\omega^{-ab} - \frac{i}{2}M_x^{ab}\omega^x) \wedge (\omega^-_{ac} - \frac{i}{2}M_{acy}\omega^y) \wedge V_b \wedge V^c + \\ &\quad -\frac{1}{3}\varepsilon_{cdef}V^c \wedge V^d \wedge V^e \wedge V^f[4(V^{\mu[a}E_i^{b]+k}\partial_\mu q^i)^2 + (V_a^\mu E_i^{ak}\partial_\mu q^i)^2] + \\ &\quad +\text{total derivative.} \end{aligned} \tag{5.12}$$

As an example, let us consider the simplest case, namely the case $m=1$, $\mathcal{Q}(1) = H^1$, with the standard flat metric. We have $\mathcal{U}^i_{A\alpha} = (\sigma^i)_{\alpha A}$ and $E_i^a = \delta_i^a$.

The hyperinstantons satisfy

$$\begin{aligned} V^{\mu[a}\partial_\mu q^{b]+} &= 0, \\ V_a^\mu \partial_\mu q^a &= 0. \end{aligned} \tag{5.13}$$

If we further specialize the example, namely we choose flat spacetime metric, we have

$$\begin{aligned} \partial_{[\mu}q_{\nu]^+} &= 0, \\ \partial_\mu q^\mu &= 0. \end{aligned} \tag{5.14}$$

If you imagine that q_μ is an abelian four vector, the hyperinstantons are the selfdual solutions in the Lorentz gauge. But now $\partial_\mu q^\mu = 0$ is a true equation and not a choice of gauge. In particular, all harmonic forms $q = q_\mu dx^\mu$ are solutions (they would be the residual gauge freedom in the interpretation of q_μ as a four potential and so they would not be true solutions).

Chapter 6

Conclusions and Outlook

Having shown that twisted N=2 supergravity is a formulation of D=4 topological gravity it remains to be seen which of the N=2 supergravity correlators are topological and how they are accordingly calculated using some version of intersection theory on instanton moduli-space. We postpone this investigation to a future publication.

We have seen that, with appropriate procedure and relying on an appropriate symmetry (R-duality), all N=2, D=4 theories can be topologically twisted, just as it happens of N=2 theories in two dimensions. This possibility introduces a set of new topological field theories, each of which describes intersection theory in the moduli-space of certain interesting geometrical structures. Some of these structures are, as far as we know, new or at least not well estabilished in the mathematical literature.

To be specific, let us enumerate these theories.

i) The twist of pure N=2 supergravity yields a formulation of topological gravity where the instantons are the metrics with self-dual spin connection.

ii) The twist of N=2 σ-models in flat background, whose target space is a Hyperkähler manifold, introduces the notion of a topological hyperkählerian σ-model, where the appropriate instantons are the triholomorphic maps (hyperinstantons). Correlation functions in this theory will be intersection integrals in the moduli-space of triholomorphic maps: a subject that to our knowledge has not been so far developed and certainly deserves careful investigation.

iii) The twist of N=2 supergravity minimally coupled to vector multiplets yields a topological theory where the instantons are gauge instantons living in the background of gravitational instantons. The moduli-space of these structures is the arena where correlation functions of our theory have to be calculated. Making an analogy with the 2-dimensional world, our theory stands to topological Yang-Mills theory as the topological matter models coupled to topological gravity stand to pure topological minimal models in D=2.

iv) Similarly, twisting N=2 σ-models coupled to N=2 supergravity, one obtains a topological σ-model where the target space is quaternionic and which interacts with topological gravity. The instantons of this theory are interesting objects. They correspond to the quaternionic analogue of triholomorphic maps living in the background of generalized gravitational instantons. The space-time spin connection is no longer selfdual but its antiselfdual part is identified with the $SU(2)$ part of the spin connection on the quaternionic manifold. This is a phenomenon similar to the embedding of the spin connection into the gauge connection occurring in string compactifications.

v) Twisting the complete N=2 matter coupled supergravity, one obtains a topological theory where all the above instantons are fused together: gravitational, gauge and hyperinstantons. To our knowledge, no study of the moduli-space of such structures has been attempted.

vi) Alternatively, one can also study the twist on N=2 hyperkählerian σ-models coupled to N=2 super Yang-Mills. In this case we have the fusion of gauge and hyperinstantons.

Chapter 7

Appendix

Notation and Conventions

In this appendix we give the notation for spinor algebra. The algebra of γ-matrices is represented by

$$\gamma^m = \begin{pmatrix} 0 & \sigma^m \\ \bar{\sigma}^m & 0 \end{pmatrix}, \tag{7.1}$$

where

$$\sigma^0 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{7.2}$$

$$(\bar{\sigma}^m)^{\dot{\alpha}\alpha} = \epsilon^{\dot{\alpha}\dot{\beta}}\epsilon^{\alpha\beta}(\sigma^m)_{\beta\dot{\beta}}, \tag{7.3}$$

and

$$\epsilon^{12} = \epsilon_{21} = 1, \quad \epsilon_{12} = \epsilon^{21} = -1. \tag{7.4}$$

A Lorentz vector v^m is represented by

$$v_{\alpha\dot{\alpha}} = (\sigma^m)_{\alpha\dot{\alpha}} v_m, \tag{7.5}$$

and the inverse formula is

$$v^m = \frac{1}{2} v_{\alpha\dot{\alpha}} (\bar{\sigma}^m)^{\dot{\alpha}\alpha}. \tag{7.6}$$

An antisymmetric tensor F^{ab} is represented by

$$F^{ab} = \frac{1}{2}(F^{+ab} + F^{-ab}) = \frac{1}{2}(f^{+\alpha}_{\beta}(\sigma^{ab})_\alpha{}^\beta + f^{-\dot{\beta}}{}_{\dot{\alpha}}(\bar{\sigma}^{ab})^{\dot{\alpha}}{}_{\dot{\beta}}), \tag{7.7}$$

where

$$\begin{aligned} F^{\pm ab} &= \frac{1}{2}\left(F^{ab} \mp \frac{i}{2}\varepsilon^{abcd}F_{cd}\right), \\ (\sigma^{ab})_\alpha{}^\beta &= \frac{1}{4}(\sigma^a_{\alpha\dot{\alpha}}\bar{\sigma}^{b\dot{\alpha}\beta} - \sigma^b_{\alpha\dot{\alpha}}\bar{\sigma}^{a\dot{\alpha}\beta}), \\ (\bar{\sigma}^{ab})^{\dot{\alpha}}{}_{\dot{\beta}} &= \frac{1}{4}(\bar{\sigma}^{a\dot{\alpha}\alpha}\sigma^b_{\alpha\dot{\beta}} - \bar{\sigma}^{b\dot{\alpha}\alpha}\sigma^a_{\alpha\dot{\beta}}) \end{aligned} \tag{7.8}$$

A generic spinor ψ_A is written as

$$\psi_A = \begin{pmatrix} \psi_{\alpha A} \\ \psi^{\dot{\alpha}}{}_A \end{pmatrix}, \tag{7.9}$$

while

$$\bar{\psi}_A = (\psi^{\alpha}{}_A,\, \psi_{\dot{\alpha} A}), \tag{7.10}$$

and indices are raised and lowered by means of $\epsilon^{\alpha\beta}$ and $\epsilon^{\dot{\alpha}\dot{\beta}}$. See also [31].

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