A study of a chiral superfluid

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CANDIDATE

Pietro Donatis

SUPERVISOR

Prof. Roberto Iengo

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Chapter 1.

Anyons

1.1. Anyons and Statistics

A fundamental principle of Quantum Mechanics states that two identical particles must be indistinguishable, *i.e.* the observables of the theory must not depend on the interchanging of two identical particles. Since only the absolute value, and not the phase, of the wavefunction is observable, this may change by a phase as two particles are interchanged:

$$\phi(x_1, x_2) = e^{i\theta}\phi(x_2, x_1) \tag{1.1}$$

The value of the phase θ is what determines the quantum statistics of the particles. Different phases yield different quantum theories although with the same classical limit. In discussing the statistics a crucial point is the topology of the configuration space. In particular it is important its connectedness (see [1], [2], [3], [4], [5]). If the configuration space is not simply connected then, when interchanging two identical particles one must take care of the paths in which the interchanging is performed, since things are different if paths do not belong to the same homotopy class. To be more explicit let us do a simple but illuminating example. Let us consider two indistinguishable particles labelled by A and B, and let us introduce an hard core constraint which prevents the two particles to be on the same point at the same instant. Now let us perform a double adiabatic interchange of the two particles along two different paths belonging two different homotopy classes as in figure 1.

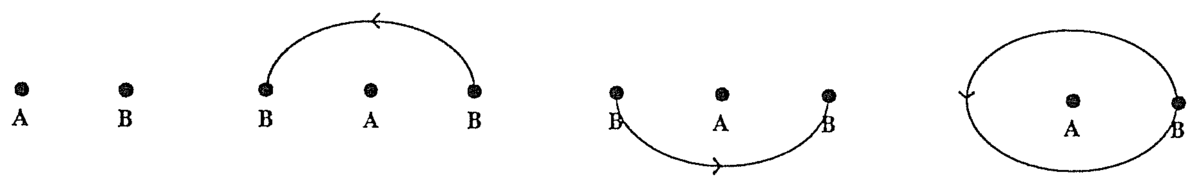

Fig. 1

It is clear that the total effect of this double interchange is taking the particle B along a loop around the particle A. So we have proved that if a particle goes round another the wavefunction acquires a phase 2θ:

$$\phi \rightarrow e^{2i\theta}\phi \tag{1.2}$$

Now as Berry [2] taught us the phase taken by the wave function in adiabatic changes is a quantal effect and it is independent of the path as long as we remain in the same homotopy class. If the dimension of the configuration space is three or higher our loop is homotopic to a point, for the space is simply connected, so the phase $e^{2i\theta}$ has to be 1, that is $\theta = 0,\ \pi(\text{mod } 2\pi)$. We have recovered the usual bosonic and fermionic statistics of three dimensional (and higher) space. Conversely if the configuration space has dimension two a richer structure emerges because the space is now multiply connected and our loop is no more homotopic to a point; therefore the value of θ is no more constrained.

There is some novel feature also in the spin of the particles in 2+1 dimensions which deserves to be mentioned. In rotation-invariant systems in $d \geq 2$ space dimensions the spin s of the particles labels the irreducible projective representations of the rotation group $SO(d)$. The projective representations are in one-to-one correspondence with the irreducible representations of the covering group $\widetilde{SO(d)}$. For $d > 2$ these can be labelled by an integer or half-integer spin. For instance in the case $d=3$ $\widetilde{SO(3)}$ is isomorphic to $SU(2)$ whose representations are known to be labelled by integer or half-integer numbers. For $d=2$ the rotation group $SO(2)$ is isomorphic to the circle S^1 whose covering group is the real line $\mathbf{R}$; the irreducible representations of $\mathbf{R}$ are labeled by real numbers so the spin s can be any real number.

It should not be surprising that in the case where particles can have arbitrary statistics they can also have arbitrary spin, since one should expect that also in two dimensions there should be a connection between spin and statistics. Indeed a spin-statistics theorem in planar QFT has been proved by Fröhlich and Marchetti [6]; the main result is the following equation:

$$s = \frac{\theta}{2\pi}(\mathrm{mod}\,\mathbf{Z}) \tag{1.3}$$

it can be immediately checked that for the cases of bosons and fermions it reproduces the very well known results. We call *anyons* these particles in two-dimensional space that can have *any* spin and statistics [7], [5].

1.2. Anyons and Chern-Simons electrodynamics

In this section we present a very simple way to implement anyons' statistics. The idea is to give an extra phase to the wavefunction by means of an Aharonov-Bohm [1] mechanism. This can be done attaching to every charge a magnetic flux via the minimal coupling of the wavefunction to a Chern-Simons (CS) gauge potential [8]. Let us consider the following action [9]:

$$S = \int d^3x \left\{ \frac{1}{2}\mu\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho + \frac{1}{2m}\phi^\dagger(D_x^2 + D_y^2)\phi + i\phi^\dagger D_0\phi \right\} \tag{1.5}$$

here $D_x = \partial_x - ieA_x$ and $D_0 = \partial_0 + ieA_0$. The first term is the CS term, it can be considered as a gauge invariant mass term for a gauge potential; it should be also noticed that CS term breaks the invariance under parity and time reversal so we can say that it gives a mass μ to the photon at the expense of losing the discrete symmetries P and T.

Note that varying this action with respect to $\phi^\dagger$ one recovers the Schrödinger equation:

$$i\frac{\partial}{\partial t}\phi = -\frac{1}{2m}(D_x^2 + D_y^2)\phi + eA_0\phi \tag{1.6}$$

and varying with respect to A_μ one gets:

$$\mu\epsilon^{\mu\nu\rho}\partial_\nu A_\rho = -j^\mu \tag{1.7}$$

here j^μ is the charge-density current:

$$j^\mu = \left(e\phi^\dagger\phi,\, \frac{e}{2mi}(\phi^\dagger\vec{D}\phi - \phi(\vec{D}\phi)^\dagger)\right) \tag{1.8}$$

The 0-*th* component of equation (1.8) reads:

$$B = -\frac{1}{\mu} j^0 \tag{1.9}$$

Integrating both sides over all the plane one finds:

$$\Phi_B = -\frac{1}{\mu} Q \tag{1.10}$$

Equation (1.10) tells that the net effect of the CS term is to attach to every charge Q the magnetic flux $-\frac{1}{\mu}Q$. Therefore when a particle carries a particle of charge e round another of the same charge, as we discussed in the previous section, it really moves round a magnetic flux and so it gets an Aharonov-Bohm phase and consequently a shift of statistics. This phenomenon is known in the literature also as *statistical transmutation.* The Aharonov-Bohm phase is easily computed:

$$e \oint d\vec{l} \cdot \vec{A} = e \int d^2 r B = e\,\Phi_B = -\frac{1}{\mu} e^2 \tag{1.11}$$

this phase should be added to the statistical phase computed in the previous section:

$$\phi \rightarrow e^{2i(\theta - \frac{e^2}{2\mu})} \phi \tag{1.12}$$

i.e. the presence of the CS term causes a shift of the statistical parameter θ by:

$$\Delta\theta = -\frac{e^2}{2\mu} \tag{1.13}$$

If, for reasons that will become clear in the next section, we rewrite the photon mass μ in terms of a new parameter k as:

$$\mu = \frac{k e^2}{2\pi} \tag{1.14}$$

the shift $\Delta\theta$ can be rewritten as:

$$\Delta\theta = -\frac{\pi}{k} \tag{1.15}$$

so if, for instance, we are dealing with bosons we have $\theta = 0$; and in presence of the CS term the interchanging of two particles gives to the wavefunction the phase:

$$\phi \rightarrow e^{-i\frac{\pi}{k}} \phi \tag{1.16}$$

if, conversely, we are dealing with fermions we have $\theta = \pi$ and so:

$$\phi \longrightarrow e^{-i\pi(1-\frac{1}{k})}\phi \tag{1.17}$$

1.3. Anyon superconductivity and the mean field approximation

In this section we present an approach to the many anyons system that leads us to believe that an anyon gas behaves like a superfluid or, if anyons are charged, like a superconductor [10], [11].
This approach is called mean field approximation and it is based on the assumption of reproducing the effect of the single magnetic flux attached to each particle by a total flux produced by an uniform average magnetic field. This average magnetic field is related to the average density of particles v^2 by the formula:

$$B = -\frac{e}{\mu}v^2 = -\frac{2\pi}{ke}v^2 \tag{1.18}$$

This approximation is of course expected to be reasonable if the number of particles is very high.
Our system is now a great number of charged particles moving in an external uniform magnetic field. Classically they will move along cyclotron orbits of radius:

$$r = -\frac{mv_F}{eB} \tag{1.19}$$

if we take for v_F the nominal velocity of particles on a Fermi surface $v_F = \sqrt{\frac{4\pi v^2}{m^2}}$ we get the relation [11]:

$$\pi r^2 v^2 = k^2 \tag{1.20}$$

so the average number of particles inside a cyclotron orbit is k^2 so the mean field approximation seems viable for large values of k.

Let us see now the most important result obtained from the mean field approach to anyons. The system consist of a fixed, say N, number of particles of charge e and mass m moving in a constant magnetic field. The spectrum of energy is given by the well known Landau levels:

$$E_n = (n + 1/2)\frac{eB}{m} \tag{1.21}$$

If we suppose that our two-dimensional surface is finite and if its area is Σ then the Landau levels have finite degeneracy; the number of particles for each level is:

$$-\frac{eB}{2\pi}\Sigma = \frac{e}{2\pi}\frac{2\pi}{ke}v^2\Sigma = \frac{v^2\Sigma}{k} = \frac{N}{k} \tag{1.22}$$

that is, if the parameter k is an integer the particles fill exactly k Landau levels. If we have k levels exactly filled, we are in a situation very favourable to superconductivity since the energy difference between consecutive Landau levels offers an energetic gap to external perturbation. Beside this, adding or subtracting a real magnetic field causes a redistribution of the particles and the new arrangements, with Landau levels only partially filled, require a bigger energy [11] that is the system refuses an external magnetic field. We have found a sort of Meissner effect. In the next chapter we will start from the mean field theory as a superconducting ground state and will study the elementary excitations.

Chapter 2.

Vortices

2.1. Self-dual vortices

In this chapter we look for elementary excitations of the mean field solution of a superconducting anyon system.
In a fundamental paper on anyon superconductivity [11] these excitations are recognized to be vortices. Their argument goes as follows.
The crucial points are the connection between particle density and the *statistical* CS magnetic field, and the fact that in the mean field solution we have an integer number, k, of Landau levels exactly filled. If we add an external *real* magnetic field we change the degeneracy of the Landau levels so we must excite particles, or holes, across the gap.
Conversely if the Landau levels are not exactly filled there must be a magnetic field accounting for the mismatch between the density of the particles and the degeneracy of the Landau levels. This means that every particle excitation of the system is connected to the presence of a real magnetic field, so that charged particle excitations and vortices are the same thing.
Actually we will take a slightly different point of view: we look for vortex-like semiclassical solutions corresponding to fluctuations of the *statistical* magnetic field instead of introducing a *real* external one. To this aim we propose the following non-relativistic lagrangian density:

$$\mathcal{L} = \frac{ke^2}{4\pi}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho + i\phi^\dagger\partial_0\phi + \frac{1}{2m}\phi^\dagger\vec{D}^2\phi + e(v^2 - |\phi|^2)A_0 - g(v^2 - |\phi|^2)^2 \quad (2.1)$$

here $\vec{D} = \vec{\nabla} - ie\vec{A}$, $e > 0$, $g > 0$.
ϕ is a *superfluid wavefunction* and can be regarded as a Landau-Ginzburg order parameter in the sense that it does not explicitly depends on the positions of the particles but $|\phi|^2$ is the density as a function of the coordinate on the plane.
The first term in (2.1) is a CS one to implement anyons' statistics.

The second term is the usual gauged kinetic term.
The third term is the novel of our approach with respect to the usual CS theory where the magnetic field is proportional to the matter density; indeed varying the corresponding action with respect to the non-dynamical field A_0 we get the constraint:

$$B = \epsilon^{ij}\partial_i A_j = -\frac{2\pi}{ke}(v^2 - |\phi|^2) \tag{2.2}$$

that is B is related to the fluctuation of the density from the mean value v^2, or, in other words, that B is the deviation from the mean value

$$B_0 = -\frac{2\pi}{ke}v^2 \tag{2.3}$$

due to the presence of the matter density $|\phi|^2$.
The fourth term is a potential term which ensures that the vacuum, or unexcited, solutions correspond to the mean field density v^2.

Now we shall prove that there exist self-dual solutions that minimize the total energy of this system which are vortices. We will follow the method of reference [12] based on the classical work of Bogomol'nyi [13]. We look for classical static solutions of the form:

$$\phi(r,\theta) = f(r)\, e^{in\theta} \tag{2.4}$$

here n is a topological invariant corresponding to the *vorticity*, that is how many times the vortex winds round; the elementary vortex has $|n|=1$.
Classical solutions are those that minimize the total energy:

$$E = \int d^2r \left\{ \frac{1}{2m}|\vec{D}\phi|^2 + g(v^2 - |\phi|^2)^2 \right\} \tag{2.5}$$

At first sight we see that finiteness of the energy requires

$$\lim_{r\to\infty} f(r) = v \tag{2.6}$$

according to the fact that matter density should be v^2 everywhere but in the region of the vortex.
We can choose for the gauge field the form, see equation (2.2):

$$\begin{cases} A_r = 0 \\ A_\theta = -\dfrac{2\pi}{ke}\dfrac{1}{r}\displaystyle\int_0^r dr'r'\left(v^2 - f^2(r')\right) \end{cases} \tag{2.7}$$

since in polar coordinates we have

$$B = \frac{1}{r}\partial_r(rA_\theta) - \partial_\theta A_r \tag{2.8}$$

With this choice the total energy becomes:

$$E = \int d^2r \left\{ \frac{1}{2m}\left[(\partial_r f)^2 + \frac{1}{r^2}(n - erA_\theta)^2 f^2\right] + g(v^2 - |\phi|^2)^2 \right\} \tag{2.9}$$

from this equation we get a second requirement for the energy to be finite:

$$\lim_{r\to\infty} A_\theta(r) = \frac{n}{er} \to 0 \tag{2.10}$$

For our particular case this yields:

$$\frac{2\pi}{ke}\frac{1}{r}\int_0^\infty dr\, r\Big(v^2 - f^2(r)\Big) = -n \tag{2.11}$$

which is nothing but the quantization of the magnetic flux in integer factors of $\frac{2\pi}{e}$. The problem now is to minimize the total energy; for this purpose it is useful the following identity [9]:

$$|\vec{D}\phi|^2 = |(D_x \pm iD_y)\phi|^2 \pm \frac{m}{e}\vec{\nabla} \wedge \vec{J} \pm eB|\phi|^2 \tag{2.12}$$

here $\vec{J}$ is the usual current

$$\vec{J} = \frac{e}{2mi}\left[\phi^\dagger \vec{D}\phi - \phi(\vec{D}\phi)^\dagger\right] \tag{2.13}$$

Then we have:

$$E = \int d^2r \left\{ \frac{1}{2m}|(D_x - iD_y)\phi|^2 - \frac{1}{2e}\vec{\nabla} \wedge \vec{J} - \frac{e}{2m}B|\phi|^2 + g(v^2 - |\phi|^2)^2 \right\} \tag{2.14}$$

Notice that we have chosen the lower (negative) sign in identity (2.12). We can rewrite the third term as:

$$-\frac{e}{2m}B|\phi|^2 = \frac{e}{2m}B(v^2 - |\phi|^2) - \frac{ev^2}{2m}B = -\frac{\pi}{mk}(v^2 - |\phi|^2)^2 - \frac{ev^2}{2m}\vec{\nabla} \wedge \vec{A} \tag{2.15}$$

so we are left with:

$$E = \int d^2r \left\{ \frac{1}{2m}|(D_x - iD_y)\phi|^2 - \frac{1}{2}\vec{\nabla} \wedge \Big(\frac{1}{e}\vec{J} + \frac{ev^2}{m}\vec{A}\Big) + \Big(g - \frac{\pi}{mk}\Big)(v^2 - |\phi|^2)^2 \right\} \tag{2.16}$$

Let us compute the contribution of the second term:

$$\frac{1}{2}\int d^2r\,\vec{\nabla}\wedge\Big(\frac{1}{e}\vec{J}+\frac{ev^2}{m}\vec{A}\Big)=\frac{1}{2}\oint_{C\infty}d\vec{l}\cdot\Big(\frac{1}{e}\vec{J}+\frac{ev^2}{m}\vec{A}\Big)=$$

$$=\frac{1}{2}\int\Big\{rd\theta\Big(\frac{1}{e}\vec{J}^{\infty}_{\theta}+\frac{ev^2}{m}\vec{A}^{\infty}_{\theta}\Big)\Big\}\tag{2.17}$$

Now we have:

$$J_\theta=\frac{e}{2mi}\Big[\phi^\dagger(\frac{1}{r}\partial_\theta-ieA_\theta)\phi-\phi(\frac{1}{r}\partial_\theta+ieA_\theta)\phi^\dagger\Big]=$$

$$=\frac{e}{2mi}\Big[f^2\Big(\frac{in}{r}-ieA_\theta\Big)-f^2\Big(-\frac{in}{r}+ieA_\theta\Big)\Big]=\frac{e}{m}f^2\Big(\frac{n}{r}-eA_\theta\Big)\to 0\tag{2.18}$$

So

$$\frac{1}{2}\int d^2r\,\vec{\nabla}\wedge\Big(\frac{1}{e}\vec{J}+\frac{ev^2}{m}\vec{A}\Big)=\frac{1}{2}r\int d\theta\,\frac{ev^2}{m}\frac{n}{er}=\frac{\pi v^2}{m}n\tag{2.19}$$

Therefore the total energy is:

$$E=-\frac{\pi v^2}{m}n+\int d^2r\left\{\frac{1}{2m}|(D_x-iD_y)\phi|^2+\Big(g-\frac{\pi}{mk}\Big)(v^2-|\phi|^2)^2\right\}\tag{2.20}$$

From this expression we can learn several things. Firstly we note that the energy E is positive definite, see equation (2.5). We will always study the case where $g\geq\frac{\pi}{mk}$[1]. Therefore the integral on the r.h.s. of equation (2.20) is positive or zero. In the case of $n<0$ (which we call *antivortex* case) we derive the inequality:

$$E\geq\frac{\pi v^2}{m}|n|\tag{2.21}$$

We will see in the next chapter that the small oscillations of the Lagrangian (3.38) (which is the *basic Lagrangian* for our study, and is exactly the same as (2.1)) have a gap $\mathcal{E}=\frac{2\pi}{mk}v^2$, which turn out to be the same as the gap between Landau levels in the anyon mean field theory. For us $k\geq 2$, therefore we see that for $n<0$ the energy of the vortex excitation is:

$$E_V\geq\mathcal{E}\tag{2.22}$$

[1] We will see in the following chapter that the right value of g which reproduces the correct "mean field" energy is $g=\frac{\pi}{m}(1-\frac{1}{k})$ and we see that $g=\frac{\pi}{m}(1-\frac{1}{k})\geq\frac{\pi}{mk}$, equality holding for $k=2$.

We will see numerically in section 2.3 that for $n > 0$ E_V is much higher than the corresponding one for $n < 0$ therefore equation (2.22) implies that the vortices do indeed correspond to higher energy excitations in the spectrum.
Let us now go on taking the case $n < 0$ (for which the energy is minimal) and consider the special case $g = \frac{\pi}{mk}$, where the bound (2.21) is saturated. In this case the equations simplify because the energy becomes:

$$E = \frac{\pi v^2}{m}|n| + \frac{1}{2m}\int d^2r \left\{|(D_x - iD_y)\phi|^2\right\} \tag{2.23}$$

which is minimal for:

$$(D_x - iD_y)\phi = 0. \tag{2.24}$$

Following Jackiw and Weinberg [12] we call equation (2.24) self-dual condition. Notice that there cannot be any solution of (2.24) for $n > 0$, because E, which is positive, would turn out to be negative (see equation (2.20)).
Let us now solve the equation (2.24). In polar coordinates it is written:

$$\frac{\partial f}{\partial r} + \frac{n}{r}f - eA_\theta f = 0. \tag{2.25}$$

If we introduce the auxiliary variable

$$a = -n + erA_\theta \tag{2.26}$$

with the properties:

$$\begin{aligned} a(0) &= -n > 0 \\ \lim_{r\to\infty} a(r) &= 0 \end{aligned} \tag{2.27}$$

equation (2.25) is equivalent to the system:

$$\begin{cases} \partial_r f = \dfrac{1}{r}af \\ \partial_r a = -\dfrac{2\pi}{k}r\left(v^2 - f^2\right) \end{cases} \tag{2.28}$$

This non-linear couple of differential equations has no analytic solution; so we will study its asymptotic behaviours and then solve it numerically.
As $r \to \infty$ the equations can be linearized defining $F = (v - f) \to 0$:

$$\begin{cases} \partial_r F = -\dfrac{v}{r}a \\ \partial_r a = -\dfrac{4\pi v}{k}rF \end{cases} \tag{2.29}$$

performing a derivative of both equations with a little algebra we get to:

$$\begin{cases} r^2\partial_r^2 F + r\partial_r F - \dfrac{4\pi v^2}{k} r^2 F = 0 \\ r^2\partial_r^2 a - r\partial_r a - \dfrac{4\pi v^2}{k} r^2 a = 0 \end{cases} \tag{2.30}$$

these equations define the following Bessel functions:

$$\begin{cases} F(r) = \alpha K_0\left(\sqrt{\dfrac{4\pi}{k}}\, vr\right) \\ a(r) = \alpha\sqrt{\dfrac{4\pi}{k}}\, rK_1\left(\sqrt{\dfrac{4\pi}{k}}\, vr\right) \end{cases} \tag{2.31}$$

here

$$\lim_{z\to\infty} K_0(z) = \sqrt{\frac{\pi}{2z}}\, e^{-z} \qquad\qquad K_1(z) = -K_0'(z) \tag{2.32}$$

and α is an arbitrary constant.

For $r \to 0$ we have:

$$\begin{cases} f(r) = A_n r^n + \cdots \\ a(r) = n - \dfrac{\pi}{k} v^2 r^2 + \cdots \end{cases} \tag{2.33}$$

A_n being an arbitrary constant.

We solved the equation numerically (with DO2GAF-NAG Fortran Library Routine) using the rescaled adimensional variables:

$$\xi = \sqrt{\frac{4\pi}{k}}\, vr \quad G(\xi) = \frac{1}{v} f(r) \quad a(\xi) = a(r) \tag{2.34}$$

such that equation (2.28) becomes:

$$\begin{cases} G'(\xi) = \dfrac{1}{\xi} a(\xi) G(\xi) \\ a'(\xi) = -\dfrac{1}{2}\xi\left(1 - G^2(\xi)\right) \end{cases} \tag{2.35}$$

and we have found the solutions plotted in tables 1 and 2.

It is also possible to find the dependence of the size of the antivortex on n. Indeed remember that the energy of these antivortex solution is:

$$E_V = \frac{\pi v^2}{m} |n| \tag{2.36}$$

and that it can also be rewritten as:

$$E_V = -\frac{ev^2}{2m}\int d^2r\,\vec{\nabla}\wedge\vec{A} = -\frac{ev^2}{2m}\Phi_B = -\frac{ev^2}{2m}B\cdot\Sigma \tag{2.37}$$

here Φ_B is the flux of the magnetic field B through the surface Σ. But notice that B is different from zero only in the region where $|\phi|^2$ is different from v^2 *i.e.* in the region where the antivortex is, so that Σ can be approximately regarded as the area occupied by the antivortex.
Now if we suppose that the region where $|\phi|^2$ passes from the value 0 to the value v^2 is small with respect to the area occupied by the antivortex, we can say that in Σ the magnetic field takes the value:

$$B_0 = -\frac{2\pi}{ke}v^2 \tag{2.38}$$

so comparing the two expressions for E_V (2.36) and (2.37) we get for Σ the value:

$$\Sigma = \frac{k}{v^2}|n| \tag{2.39}$$

If we call R_n the radius of the n-antivortex we get:

$$R_n = \sqrt{\frac{k|n|}{\pi v^2}} \tag{2.40}$$

Passing to the adimensional quantities we find:

$$\Xi_n = 2\sqrt{|n|} \tag{2.41}$$

The comparison between this theoretical prediction and the radii of the antivortex for different values of n, measured in correspondence where $|\phi| = v/2$, is plotted in table 3.

Now let us turn to the discussion of the vortex solution with $n > 0$. Let us consider the following parity transformation:

$$\begin{cases} x \longrightarrow -x \\ y \rightarrow y \end{cases} \tag{2.42}$$

or, in polar coordinates:

$$\begin{cases} r \longrightarrow r \\ \theta \longrightarrow \pi - \theta \end{cases} \tag{2.43}$$

Applying this transformation to (2.4) we get:

$$\phi(r,\theta) = f(r)\, e^{in\theta} \rightarrow f(r)\, e^{in(\pi-\theta)} = (-1) f(r)\, e^{-in\theta} \tag{2.44}$$

that is the parity transformation maps a vortex in an antivortex.
But *notice* that our lagrangian density (2.1) is not invariant under (2.42) since the CS term changes its sign. This means that self-dual vortices and antivortices are solutions of *different* theories related by a parity transformation. We will see in the following that there exist *non-self-dual* solutions both for $n > 0$ and for $n < 0$ and that there exist both self-dual solutions if the theory is parity invariant.

The antivortex solutions describe hole-like excitations. This can be easily seen computing the variation of the number of particles due to an elementary excitation (*i.e.* with $|n| = 1$) integrating the difference $(v^2 - |\phi|^2)$ over all the plane:

$$\int d^2r\, (v^2 - |\phi|^2) = -\frac{ke}{2\pi} \int d^2r\, B = -kn \tag{2.48}$$

So we see that for the elementary antivortex case $(n = -1)$ we have a decrease of k particles.
We expect that the fundamental antivortex excitation should correspond to a fluctuation of the magnetic field such that the degeneracy of the Landau levels, equation (1.22), is lowered by one unit; therefore since all the k exactly filled levels lose precisely one particle so the total decrease is k.

There is also an energetic argument supporting this interpretation.
Indeed one should expect that if, starting from the ground state in which the first k Landau levels are exactly filled, one excites a particle over the gap, producing a particle-hole pair, the energy required should be equal to the energy of a particle-hole pair. Therefore one expects that the energy of the antivortex, which we have just discovered to correspond to k holes, should have an energy equal to k times half the energy gap. Now the energy gap between two consecutive Landau levels is $\mathcal{E} = -\frac{e}{m} B_0$ and the energy of the antivortex, for $n = -1$, is

$$E_V = \frac{\pi v^2}{m} \tag{2.49}$$

that remembering (2.38) can be written as:

$$E_V = -\frac{ke}{2m} B_0 \tag{2.50}$$

which is exactly the value expected.

Now we must discuss a problem regarding the conservation of the total number of particles. In fact we have just seen that the antivortices describe a decrease of this number, so how can we cope with the conservation of N? Here we inquire the possibility of compensate the lack of particles caused by the antivortex simply by increasing the asymptotic value of f^2 from v^2 to $\hat{v}^2 = v^2 + \frac{nk}{\Sigma}$ (Σ being the area of the surface). Unfortunately this possibility has a logarithmic divergent energy; in fact we have from (2.7):

$$erA_\theta \simeq \frac{2\pi}{k}\frac{1}{\Sigma}r^2 \tag{2.51}$$

for large r. Therefore the second term in equation (2.9) for the energy behaves like:

$$\frac{1}{r^2}\left(n - \frac{2\pi}{\Sigma}r^2\right)^2 v^2 \tag{2.52}$$

whose integral diverges logarithmically. This lead us to the conclusion that in order to preserve the total number of particles we must require the presence of the vortices beside the antivortices. Indeed in section 2.3 we will find, in a variational way, non-self-dual vortex solutions and we will see that the total number of particles is conserved if the total vorticity is zero.

2.1.1. Currents

In this subsection we compute the electric current density and the total current due to the vortex.
The current density for the vortex solution has been computed in equation (2.18). Using equation (2.25) we get:.

$$\begin{cases} J_r = 0 \\ J_\theta = -\dfrac{e}{m} f\partial_r f \end{cases} \tag{2.53}$$

In table 4 we have plotted the rescaled adimensional quantity:

$$J_\theta(\xi) = \frac{m}{ev^3}\sqrt{\frac{k}{4\pi}}J_\theta(r) \tag{2.54}$$

The total current I passing in the plane is equal to

$$I = \int d\vec{s}\cdot\vec{J} = \int_0^\infty dr\, J_\theta(r) = -\frac{e}{m}\int_0^\infty dr\, f\partial_r f = -\frac{e}{m}\int_0^\infty dr\,\frac{d}{dr}f^2 =$$

$$= -\frac{e}{m}\Big(f^2(\infty) - f^2(0)\Big) = -\frac{e}{m}v^2 \tag{2.55}$$

This current can be interpreted as the Hall current of QHE!
To see this fact let us consider the following hamiltonian:

$$H = |n|\frac{\pi v^2}{m} + \frac{1}{2m}\int d^2r \left\{|\vec{D}\phi|^2 + eB|\phi|^2\right\} \tag{2.56}$$

where $B = -\frac{2\pi}{ke}(v^2 - |\phi|^2)$. It is straightforward to prove that this hamiltonian is equivalent to our original hamiltonian with $g = \frac{\pi}{mk}$. In this form we can interpret the second term in curly brackets in (2.56) as an interaction term between the electric potential $V = \frac{1}{2m}B$ and the electric charge density $e|\phi|^2$. To such a potential will correspond the electric field $\vec{E} = -\vec{\nabla}V$ which in polar coordinates reads:

$$\begin{cases} E_r = -\dfrac{\partial V}{\partial r} = -\dfrac{1}{2m}\dfrac{\partial B}{\partial r} = -\dfrac{2\pi}{mke} f\partial_r f = -\dfrac{2\pi}{ke^2}J_\theta \\ E_\theta = -\dfrac{1}{r}\dfrac{\partial V}{\partial \theta} = 0 \end{cases} \tag{2.57}$$

From the first of the (2.57) we read the value of the Hall conductivity $\sigma = \frac{J_\theta}{E_r}$:

$$\sigma = k\frac{e^2}{2\pi} \tag{2.58}$$

Inserting back the physical constant $\hbar$ this is exactly, apart from the sign, the Hall conductivity for QHE:

$$\sigma_H = k\frac{e^2}{h} \tag{2.59}$$

2.2. Vortices in a parity invariant two dimensional Maxwell electrodynamics

There is an important point we have passed through while discussing the currents in the previous subsection; that is the following equation which can be read off from the first of the (2.57):

$$\frac{1}{2m}\frac{\partial B}{\partial r} = \frac{2\pi}{ke^2}J_\theta \tag{2.60}$$

which in cartesian coordinates reads:

$$\vec{\nabla} \wedge B = \frac{4\pi m}{ke^2}\vec{J} \tag{2.61}$$

which is a Maxwell equation provided the following relation between the parameters of the theory holds:

$$m = \frac{ke^2}{4\pi} \tag{2.62}$$

This result suggests that all the work done until this point can be reobtained in the context of a Maxwell, parity invariant, framework where the two distinct theories connected by a parity transformation coexist.
To see how this work let us consider the following lagrangian with a Maxwell term instead of a CS one:

$$L = \int d^2r \left\{ -\frac{1}{4} F_{ij}F_{ij} - \frac{1}{2m}|\vec{D}\phi|^2 - g(v^2 - |\phi|^2)^2 \right\} \tag{2.63}$$

Using the identity (2.12) this can be rewritten as:

$$L = \int d^2r \left\{ -\frac{1}{4} F_{ij}F_{ij} - \frac{1}{2m}|(D_x \pm iD_y)\phi|^2 \mp \frac{1}{2e}\vec{\nabla} \wedge \vec{J} \mp \frac{e}{2m} B|\phi|^2 - \right.$$
$$\left. -g(v^2 - |\phi|^2)^2 \right\} \tag{2.64}$$

We start discussing the case with the lower sign. Integrating (2.61) by parts we get:

$$L = \int d^2r \left\{ -\frac{1}{4} F_{ij}F_{ij} + \frac{2}{m}\phi^\dagger \bar{D} D\phi + \frac{1}{2e}\vec{\nabla} \wedge \vec{J} + \frac{e}{2m} B|\phi|^2 - g(v^2 - |\phi|^2)^2 \right\} \tag{2.65}$$

In equation (2.65) we have introduced the complex notation for the covariant derivatives:

$$D = \frac{1}{2}(D_x - iD_y) \qquad \bar{D} = \frac{1}{2}(D_x + iD_y) \tag{2.66}$$

the equation of motion from this lagrangian are:

$$\frac{\delta L}{\delta \phi^\dagger} = 0 \qquad \Rightarrow \qquad \frac{2}{m}\bar{D}D\phi + \frac{e}{2m}B\phi + 2g(v^2 - |\phi|^2)\phi = 0 \tag{2.67}$$

$$\frac{\delta L}{\delta A_i} = 0 \qquad \Rightarrow \qquad \partial_j F_{ji} = -J_i \tag{2.68}$$

Notice that one possible solution of these equations of motion is:

$$\begin{cases} D\phi = 0 \\ B = -\frac{2\pi}{ke}(v^2 - |\phi|^2) \end{cases} \tag{2.69}$$

provided we take $g = \frac{\pi}{2mk}$ and $m = \frac{ke^2}{4\pi}$ as can be easily checked. These are exactly the equations for ϕ and B found in the previous section (see equations (2.2) and (2.24).
Furthermore the hamiltonian is:

$$H = \int d^2r \left\{ \frac{1}{2}B^2 + \frac{1}{2m}|\vec{D}\phi|^2 + g(v^2 - |\phi|^2)^2 \right\} \tag{2.70}$$

if $B = -\frac{2\pi}{ke}(v^2 - |\phi|^2)$ equation (2.70) can be rewritten as:

$$\begin{aligned} H = \int d^2r \Big\{ & \frac{2\pi^2}{k^2e^2}(v^2 - |\phi|^2)^2 + \frac{1}{2m}|(D_x - iD_y)\phi|^2 - \frac{1}{2e}\vec{\nabla} \wedge \vec{J} - \\ & - \frac{e}{2m}B(|\phi|^2 - v^2) - \frac{ev^2}{2m}B + g(v^2 - |\phi|^2)^2 \Big\} = \\ = \int d^2r \Big\{ & \frac{1}{2m}|(D_x - iD_y)\phi|^2 - \frac{1}{2e}\vec{\nabla} \wedge \vec{J} - \frac{ev^2}{2m}B + \\ & + \left(g + \frac{2\pi^2}{k^2e^2} - \frac{\pi}{mk}\right)(v^2 - |\phi|^2)^2 \Big\} \end{aligned} \tag{2.71}$$

which is identical to equation (2.23) provided:

$$g = \frac{\pi}{mk} - \frac{2\pi^2}{k^2e^2} = \frac{\pi}{2mk} \tag{2.72}$$

Notice that g has no more the value $\frac{\pi}{mk}$ it had in the previous section; the reason is simply that now we have a contribution to the energy also from the Maxwell term.
Now let us discuss the case in which the identity (2.12) is used the other (upper) sign. The lagrangian (2.63) can be rewritten in the following way:

$$L = \int d^2r \left\{ -\frac{1}{4}F_{ij}F_{ij} + \frac{2}{m}\phi^\dagger D\bar{D}\phi - \frac{1}{2e}\vec{\nabla} \wedge \vec{J} - \frac{e}{2m}B|\phi|^2 - g(v^2 - |\phi|^2)^2 \right\} \tag{2.73}$$

We stress that this is the *same* lagrangian as equation (2.65). Again we can compute the equations of motion:

$$\frac{\delta L}{\delta \phi^\dagger} = 0 \quad \Rightarrow \quad \frac{2}{m}D\bar{D}\phi - \frac{e}{2m}B\phi + 2g(v^2 - |\phi|^2)\phi = 0 \tag{2.74}$$

$$\frac{\delta L}{\delta A_i} = 0 \quad \Rightarrow \quad \partial_j F_{ji} = -J_i \tag{2.75}$$

These equations are solved by:

$$\begin{cases} \bar{D}\phi = 0 \\ B = \dfrac{2\pi}{ke}(v^2 - |\phi|^2) \end{cases} \tag{2.76}$$

provided we take $g = \frac{\pi}{2mk}$ and $m = \frac{ke^2}{4\pi}$.
These are the equations that are obtained from (2.69) using parity transformation (2.43). Their solution have exactly the same form of the solutions of (2.69) but with positive values for n.
So we have found that in a Maxwell framework, which is parity conserving, we get both the vortex and the antivortex solutions.

2.3. Non-self-dual vortices

In the previous sections we have discussed exact self-dual vortex-like solutions, and we have discovered that vortex and antivortex of this kind cannot exist as solutions of the same theory.
In this section we give up the project of exact solutions and try to find them in a variational way. With this approach we will manage to find both vortex and antivortex solutions.
Our starting point is again the hamiltonian (2.9):

$$2mH = \int d^2r \left\{ (\partial_r f)^2 + \frac{1}{r^2}(n - erA_\theta)^2 f^2 + 2mg(v^2 - |\phi|^2)^2 \right\} \tag{2.77}$$

with:

$$B = \frac{1}{r}\frac{\partial}{\partial r}(rA_\theta) = -\frac{2\pi}{ke}(v^2 - f^2) \qquad A_\theta = -\frac{2\pi}{ke}\frac{1}{r}\int_0^r dr' r' \Big(v^2 - f^2(r')\Big) \tag{2.78}$$

and with the asymptotic conditions (necessary for the energy to be finite):

$$\lim_{r\to\infty} erA_\theta = n \qquad A_\theta(0) = 0 \qquad \lim_{r\to\infty} f = v \qquad f(0) = 0 \tag{2.79}$$

We make the following ansatz:

$$erA_\theta = n\Big[1 - \Big(1 + \frac{p^2 r^2}{2}\Big)e^{-\omega r^2}\Big] \tag{2.80}$$

p and ω being two positive real parameters to be determined minimizing the energy. Correspondently we get:

$$f^2(r) = v^2 + \frac{nk}{2\pi}(p^2\omega r^2 - p^2 + 2\omega)e^{-\omega r^2} \tag{2.81}$$

Notice that the asymptotic conditions on A_θ are verified; imposing the condition for $f(r)$ in $r=0$ we get the following relation between the parameters:

$$p^2 = 2\omega + \frac{2\pi v^2}{nk} \tag{2.82}$$

So we have:

$$f^2(r) = v^2 - \left(v^2 - \frac{nk}{2\pi}p^2\omega r^2\right)e^{-\omega r^2} \tag{2.83}$$

With this ansatz the hamiltonian (2.77) becomes:

$$2mH = \int d^2r \left\{ \frac{\omega r^2[v^2 + \frac{nk}{2\pi}(1-\omega r^2)]^2 e^{-2\omega r^2}}{v^2 - (v^2 - \frac{nk}{2\pi}p^2\omega r^2)e^{-\omega r^2}} + \right.$$

$$\left. + \frac{n^2}{r^2}\left(1+\frac{p^2r^2}{2}\right)^2 e^{-2\omega r^2}\left[v^2 - \left(v^2 - \frac{nk}{2\pi}p^2\omega r^2\right)e^{-\omega r^2}\right] + 2mg\left(v^2 - \frac{nk}{2\pi}p^2\omega r^2\right)^2 e^{-2\omega r^2} \right\} \tag{2.84}$$

The condition $f^2(r) \geq 0$ implies:

$$v^2(1 - e^{-\omega r^2}) + p^2\omega r^2 \frac{nk}{2\pi}e^{-\omega r^2} \geq 0 \tag{2.85}$$

That is

$$v^2 \geq -p^2 \frac{nk}{2\pi} \frac{\omega r^2}{e^{-\omega r^2} - 1} \tag{2.86}$$

Now since $0 \leq \frac{\omega r^2}{e^{-\omega r^2}-1} \leq 1$ this condition is always satisfied for $n > 0$ but for $n = -|n| < 0$ we have the constraint:

$$v^2 \geq p^2 \frac{|n|k}{2\pi} \tag{2.87}$$

Now we have to minimize this hamiltonian with respect to one of the two parameters p and ω. It is convenient to introduce the following adimensional quantities:

$$\xi = vr \qquad g = \frac{1}{v}f \tag{2.88}$$

and

$$a = \frac{1}{v} p \qquad b = \frac{1}{v^2}\omega \tag{2.89}$$

with the new parameters relation (2.82) reads:

$$a^2 = 2b + \frac{2\pi v^2}{nk} \tag{2.90}$$

Then the hamiltonian (2.84) can rewritten as:

$$\frac{2m}{v^2}H = \int d^2\xi \left\{ \frac{b^2\xi^2[1+\frac{nk}{2\pi}a^2(1-b\,\xi^2)]^2 e^{-2b\xi^2}}{1-(1-\frac{nk}{2\pi}a^2 b\,\xi^2)e^{-b\xi^2}} + \right.$$

$$\left. + \frac{n^2}{\xi^2}\left(1+\frac{a^2\xi^2}{2}\right)^2 e^{-2b\xi^2}\left[1-\left(1-\frac{nk}{2\pi}a^2 b\,\xi^2\right)e^{-b\xi^2}\right] + 2mg\left(1-\frac{nk}{2\pi}a^2 b\,\xi^2\right)^2 e^{-2b\xi^2} \right\}$$

$$\frac{2m}{v^2}\frac{1}{\pi}H = \int_0^\infty d\zeta\,\frac{1}{b}\left\{ \frac{b\zeta[1+\frac{nk}{2\pi}a^2(1-\zeta)]^2 e^{-2\zeta}}{1-(1-\frac{nk}{2\pi}a^2\zeta)e^{-\zeta}} + \right.$$

$$\left. + \frac{n^2 b}{\zeta}\left(1+\frac{a^2}{2b}\zeta\right)^2 e^{-2\zeta}\left[1-\left(1-\frac{nk}{2\pi}a^2\zeta\right)e^{-\zeta}\right] + 2mg\left(1-\frac{nk}{2\pi}a^2\zeta\right)^2 e^{-2\zeta} \right\} \tag{2.91}$$

here we have performed the angular integration and put $\zeta = b\,\xi^2$.
Now let us study the vortex case supposing $n > 0$.
To find the minimum of H we have computed the three integrals in (2.91) (with $|n| = 1$, $g = \frac{\pi}{2m}$ and $k = 2$) for different value of b. H is minimal for $b = 2.6$ and its value is $H_{\min} = 20.64\,(\frac{v^2}{2m})$.
Let us now suppose $n = -|n| < 0$ and study the antivortex case.
We have to take into account the constraint (2.87) which in terms of the adimensional quantities (2.89) reads:

$$a^2 < \frac{2\pi}{|n|k} \tag{2.92}$$

We have computed the three integrals in (2.91) for different values of a respecting (2.92). H is minimal for $a = 0$ and its value is $H_{\min} = 6.44\,(\frac{v^2}{2m})$. We see that this energy is very close ($6.44 \simeq 2\pi$) to the minimum found while treating the self-dual case, proving the efficiency of our variational method.
Notice the difference of the energy of the vortex and that of the antivortex. We saw in section 2.1 that the minimum is obtained only in the antivortex case; this is confirmed by the present analysis in which the energy of the vortex is more than three times that of the antivortex.
In tables 5, 6, 7 and 8 we report the plots of g and A_θ in the two cases.

Notice that the variation in the number of the particles due to these vortices is:

$$\delta N = \int d^2r\, \delta\rho = \int d^2r\, \left(f^2(r) - v^2\right) = \int d^2r \left[\left(\frac{nk}{2\pi}p^2\omega r^2 - v^2\right)e^{-\omega r^2}\right] = nk \tag{2.94}$$

So if $n > 0$ we have a "mountain" in the density *i.e.* an increased number of particles with respect to the mean value; conversely if $n < 0$ we have a "valley" *i.e.* a decreased number of particles. Notice that the total number of particles is preserved only if the total vorticity is zero as we promised at the end of section 2.1.

We further notice that these vortex and antivortex solutions are not related by a parity transformation and therefore parity is still a broken symmetry.

Chapter 3.

Effective theory

3.1. Effective theory as a limit from the torus

In the Landau-Ginzburg theory considered in the previous chapter we have taken:

$$\vec{\nabla} \wedge \vec{A} \propto \delta\rho \tag{3.1}$$

to study the fluctuations of the density from the mean value. One of the peculiarities of this approach is that the ground state corresponds to a translational invariant configuration since $\delta\rho = 0$.
This kind of approach should be compared to the standard approach to CS theory where the magnetic field is proportional to the density and not to its fluctuation:

$$\vec{\nabla} \wedge \vec{A} \propto \rho \tag{3.2}$$

In this formulation to have translational invariance one has to take care of the boundary conditions in a proper way and to take a torus [14]. In this way one shows that the ground state of the full quantum solution of the mean field theory corresponds to a constant density. Due to the non-trivial topological properties of the torus one has to take into account the topological components of the gauge potential (also called "flat connections" since $\vec{\nabla} \wedge \vec{a} = 0$) defined by:

$$a_x = \oint dx A_x \qquad a_y = \oint dy A_y \tag{3.3}$$

here the two integrals are performed along the two non-trivial loops of the torus. In this way the hamiltonian can be written as follows [14]:

$$H = \int d^2x \int d^2a \left\{ \frac{1}{2m} |\vec{D}\phi|^2 + c \left| \left(\frac{k}{4\pi} a_i + i\, \epsilon_{ij} \frac{\partial}{\partial a_j} \right) \psi \right|^2 \right\} \tag{3.4}$$

where the covariant derivative is:

$$D_i = \partial_i - \frac{i\pi}{2k} v^2 \epsilon_{ij} x_j - i \frac{a_i}{L} - ie\tilde{A}_i \tag{3.5}$$

here L is the length of a side of the torus (we are supposing, for simplicity that our torus is a square with identified edges), c is a constant which will play no rôle in what follows and $\vec{R}$ is the fluctuation part of $\vec{A}$ such that $\vec{\nabla} \wedge \vec{R} \propto \delta\rho$.
If we turn to the plane we have to take the limit $L \to \infty$ at constant density (we will call it *thermodynamical limit*), so $\frac{a_i}{L} \to 0$ if a_i is bounded (we will see below that indeed a_i is bounded) but in the hamiltonian still survives a term in a_i.
Note that now the wavefunction ψ is now a function of $\vec{a}$ beside $\vec{x}$; *i.e.* $\psi = \psi(\vec{x}, \vec{a})$ so we define the density $\rho(\vec{x})$ to be:

$$\rho(\vec{x}) = \int d^2a \, |\psi(\vec{x}, \vec{a})|^2 \tag{3.6}$$

It is convenient to introduce a complex notation:

$$A = A_M - i\tilde{A} \qquad \bar{A} = \bar{A}_M + i\bar{\tilde{A}} \tag{3.7}$$

where

$$A_M = \frac{i}{2} \frac{\pi}{ke} v^2 \bar{z} \qquad \bar{A}_M = -\frac{i}{2} \frac{\pi}{ke} v^2 z \tag{3.8}$$

are the "mean field" parts and

$$\tilde{A} = \frac{i}{2}(\tilde{A}_1 - i\tilde{A}_2) \qquad \bar{\tilde{A}} = -\frac{i}{2}(\tilde{A}_1 + i\tilde{A}_2) \tag{3.9}$$

are the "fluctuating" parts; for the flat connections we define:

$$a = \frac{1}{2\pi}(ia_1 + a_2) \qquad \bar{a} = \frac{1}{2\pi}(-ia_1 + a_2) \tag{3.10}$$

For the magnetic field we have:

$$\begin{aligned} B &= 2i(\bar{\partial} A - \partial \bar{A}) = -\frac{2\pi}{ke}\rho \\ &= 2i(\bar{\partial} A_M - \partial \bar{A}_M) + (\bar{\partial}\tilde{A} - \partial \bar{\tilde{A}}) = B_M + \tilde{B} = -\frac{2\pi}{ke} v^2 + \tilde{B} \end{aligned} \tag{3.11}$$

therefore:

$$\tilde{B} = \frac{2\pi}{ke}(v^2 - \rho). \tag{3.12}$$

Let us now suppose that $\tilde{A} = 0$ and study the mean field solution. The hamiltonian is:

$$H = \int d^2z \int d^2a \left\{ \frac{2}{m} \left| (\partial - ieA_M)\psi \right|^2 - \frac{e}{2m} B_M |\psi|^2 + c' \left| \left(\frac{\partial}{\partial a} + \frac{\pi k}{2} \bar{a} \right) \psi \right|^2 \right\} =$$

$$= \int d^2z \int d^2a \left\{ \frac{2}{m} \left| \left(\partial + \frac{\pi}{2k} v^2 \bar{z}\right) \psi \right|^2 - \frac{e}{2m} B_M |\psi|^2 + c' \left| \left(\frac{\partial}{\partial a} + \frac{\pi k}{2} \bar{a} \right) \psi \right|^2 \right\} \quad (3.13)$$

The state of minimal energy corresponds to:

$$\psi = \psi_M = e^{-\frac{\pi v^2}{2k} z\bar{z} - \frac{\pi k}{2} a\bar{a}} g(\bar{z}, \bar{a}) \quad (3.14)$$

here g is an arbitrary antiholomorphic function.
From (3.14) we can see that a is bounded.
Let us look for the constant density solutions, which are known to correspond to the ground state of the full quantum mechanical mean field problem. Choosing $g(\bar{z}, \bar{a}) = e^{\pi v \bar{z} \bar{a}} \, v$ we get $\rho_M = v^2$, in fact

$$\rho_M = \int d^2a \, |\psi_M|^2 = \int d^2a \, e^{-\pi k |a - \frac{v}{k} \bar{z}|^2} v^2 = v^2 \quad (3.15)$$

provided we normalize the measure d^2a such that

$$\int d^2 a e^{-\pi k a \bar{a}} = 1.$$

For the hamiltonian (3.13) we get:

$$H_M = -\frac{e}{2m} B_M \int d^2 z v^2 = -\frac{e}{2m} B_M N \quad (3.16)$$

this is the energy of N particles in the lower Landau level. We know that in the mean field solution actually the N particles fill exactly k levels corresponding to the energy:

$$E_M = -\frac{e}{2m} B_M N k \quad (3.17)$$

So, in order to reproduce the correct mean field energy, we must add:

$$E' = -\frac{e}{2m} B_M N(k-1) = \frac{2\pi}{m} \left(1 - \frac{1}{k}\right) \int d^2a \int d^2z |\psi|^4 \quad (3.18)$$

So our correct starting hamiltonian is:

$$H = \int d^2x \int d^2a \left\{ \frac{1}{2m} |\vec{D}\phi|^2 + c' \left| \left(\frac{\partial}{\partial a} + \frac{\pi k}{2} \bar{a} \right) \psi \right|^2 + \frac{2\pi}{m} \left(1 - \frac{1}{k}\right) |\psi|^4 \right\} \quad (3.19)$$

Notice that more in general a constant density is also obtained taking:

$$\psi_M = e^{-\frac{\pi v^2}{2k} z\bar{z} - \frac{\pi k}{2} a\bar{a} + \pi v \bar{z} \bar{a} + i(pz + \bar{p}\bar{z})} \, v \quad (3.20)$$

then the hamiltonian (3.19) becomes:

$$H_M = \int d^2z \left\{ \frac{2}{m}|ip|^2 v^2 - \frac{ke}{2m} B_M v^2 \right\} = \frac{1}{2m}(p_x^2 + p_y^2)N - \frac{ke}{2m} B_M N \qquad (3.21)$$

so we have found, beside the standard mean field energy (3.17) a kinetic energy equal to that of one particle times N. So our system moves like a condensate where all particles have the same momentum. In other words it represent a collective motion. If we compute the currents:

$$\begin{aligned} J &= \int d^2a \, \frac{e}{2mi}\left[\psi^\dagger D\psi - \psi(D\psi)^\dagger\right] \\ \bar{J} &= \int d^2a \, \frac{e}{2mi}\left[\psi^\dagger \bar{D}\psi - \psi(\bar{D}\psi)^\dagger\right] \end{aligned} \qquad (3.22)$$

we get:

$$J = \frac{ev^2}{m} p \qquad \bar{J} = \frac{ev^2}{m}\bar{p} \qquad (3.23)$$

So we have found for the current exactly the charge density times the velocity.

Now let us introduce the fluctuations taking $\tilde{A} \neq 0$:

$$H = \int d^2z \int d^2a \left\{ \frac{2}{m}\left|(\partial + \frac{\pi}{2k}v^2\bar{z} - e\tilde{A})\psi\right|^2 - \frac{e}{2m}B|\psi|^2 + \frac{2\pi}{m}\left(1 - \frac{1}{k}\right)|\psi|^4 + \right.$$

$$\left. +c'\left|\left(\frac{\partial}{\partial a} + \frac{\pi k}{2}\bar{a}\right)\psi\right|^2 \right\} \qquad (3.24)$$

Then if we take:

$$\psi = e^{-\frac{\pi v^2}{2k} z\bar{z} - \frac{\pi k}{2} a\bar{a} + \pi v \bar{z} \bar{a}} \, \phi(z, \bar{z}) \qquad (3.25)$$

for the density we get:

$$\rho = \int d^2a \, |\psi|^2 = \int d^2a \, e^{-\pi k |\bar{a} - \frac{v}{k} z|^2} \, |\phi(z,\bar{z})|^2 = |\phi(z,\bar{z})|^2 \qquad (3.26)$$

and for the hamiltonian:

$$H = \int d^2z \left\{ \frac{2}{m}|(\partial - e\tilde{A})\phi|^2 - \frac{2}{2m}B\rho + \frac{\pi}{m}\left(1 - \frac{1}{k}\right)\rho^2 \right\} =$$

$$= \int d^2z \left\{ \frac{2}{m}|(\partial - e\tilde{A})\phi|^2 + \frac{\pi}{m}\rho^2 \right\} \qquad (3.27)$$

So we have found an hamiltonian which in the "mean field" case gives the correct value of the energy and in the "fluctuating" case recovers the covariant derivatives built with the fluctuation of the gauge potential like it was in the theory developed in the previous chapter.

3.2. Small deformations approach

Our starting point will be the following lagrangian density (which is the same as (2.1) after solving for A_μ^{CS})[1]:

$$\mathcal{L} = i\phi^\dagger \partial_0 \phi - \frac{1}{2m}\left|(\vec{\nabla} - ie\vec{\tilde{A}})\phi\right|^2 - \frac{\pi}{m}\left(1 - \frac{1}{k}\right)|\phi|^4 = i\phi^\dagger \partial_0 \phi - \mathcal{H} \tag{3.28}$$

here $\tilde{A}$ is the "fluctuating" gauge potential defined by:

$$\vec{\nabla} \wedge \vec{\tilde{A}} = -\frac{2\pi}{ke}(v^2 - \rho) = \frac{2\pi}{ke}\delta\rho \tag{3.29}$$

Furthermore we choose to impose the constraint:

$$N = \int d^2x\, v^2 = \int d^2x\, |\phi|^2 \tag{3.30}$$

which is simply the statement of conservation of the number of particles.
The first analysis we want to perform on this lagrangian is based on a small deformation approach. Let us take the following parameterization:

$$\phi = v\, e^{i\theta + \frac{\eta}{v}} \simeq v\, e^{i\theta}\left(1 + \frac{\eta}{v}\right) \tag{3.31}$$

here θ is the phase and $\eta(\vec{x}, t)$ a small parameter. Then we have for the density:

$$\rho = v^2\, e^{\frac{2\eta}{v}} \simeq v^2 + 2v\eta \tag{3.32}$$

and for the fluctuation:

$$\delta\rho = \rho - v^2 \simeq 2v\eta \tag{3.33}$$

[1] A similar lagrangian has already been introduced by other authors to describe FQHE [15]. Also is ref. [16] a lagrangian like (3.28) is used for anyon superconductivity, but the discussion there is quite different from ours.

Now from equation (3.30) we can see that the integral over all the plane of $\eta(\vec{x},t)$ is zero, then we can write it as the divergence of some quantity $\vec{u}(\vec{x},t)$:

$$\eta(x) = \vec{\nabla}\cdot\vec{u}(\vec{x},t) \tag{3.34}$$

furthermore we have the freedom to take $\vec{u}(\vec{x},t)$ irrotational, that is

$$\vec{\nabla}\wedge\vec{u} = \partial_x u_y - \partial_y u_x = 0 \tag{3.35}$$

Then we have from (3.29) and (3.33):

$$\vec{\nabla}\wedge\vec{\tilde{A}} = \frac{4\pi v}{ke}\vec{\nabla}\cdot\vec{u} \tag{3.36}$$

this equation can be solved by:

$$\tilde{A}_x = -\frac{4\pi v}{ke}u_y \qquad \tilde{A}_y = \frac{4\pi v}{ke}u_x \tag{3.37}$$

Notice that $\vec{\nabla}\cdot\vec{\tilde{A}}=0$ as it should.
With the new parameterization the lagrangian (3.28) becomes:

$$\mathcal{L} = 2v\theta\partial_0(\vec{\nabla}\cdot\vec{u}) - \frac{1}{2m}v^2(\vec{\nabla}\theta)^2 - \frac{1}{2m}\left(\frac{4\pi v^2}{k}\right)^2\vec{u}^2 - \frac{1}{2m}(\triangle\vec{u})^2 - \frac{8\pi v^2}{m}\left(1-\frac{1}{k}\right)(\vec{\nabla}\cdot\vec{u})^2 \tag{3.38}$$

Performing the variation with respect to θ we get:

$$2v\partial_0(\vec{\nabla}\cdot\vec{u}) + \frac{v^2}{m}\vec{\nabla}\cdot\vec{\nabla}\theta = 0 \qquad \Rightarrow \qquad \frac{v^2}{m}\vec{\nabla}\theta + 2v\dot{\vec{u}} = 0 \tag{3.39}$$

which can be regarded as the continuity equation:

$$\partial_0\rho + \vec{\nabla}\cdot(\rho\vec{v}) = 0 \tag{3.40}$$

provided we identify the velocity $\vec{v}=\frac{\vec{\nabla}\theta}{m}$.
Inserting back this equation in the lagrangian we get:

$$\mathcal{L} = 2m\dot{\vec{u}}^2 - \frac{1}{2m}\left(\frac{4\pi v^2}{k}\right)^2\vec{u}^2 - \frac{1}{2m}(\triangle\vec{u})^2 - \frac{8\pi v^2}{m}\left(1-\frac{1}{k}\right)(\vec{\nabla}\cdot\vec{u})^2 \tag{3.41}$$

so we have managed to write the lagrangian as a kinetic part minus a potential part. Taking $\vec{u}=\vec{u}_0 e^{i(Et+\vec{p}\cdot\vec{x})}$ we find the spectrum of the energy [1]:

$$2mE^2 = \frac{1}{2m}\Big(\frac{4\pi v^2}{k}\Big)^2 + \frac{1}{2m}(p_x^2+p_y^2)^2 + \frac{8\pi v^2}{m}\Big(1-\frac{1}{k}\Big)(p_x^2+p_y^2) \tag{3.42}$$

the minimum of the energy, for $p=0$, is not zero, so we have a gap:

$$E(0) = \frac{2\pi}{mk}v^2 \equiv \mathcal{E} \tag{3.43}$$

which is exactly the energy of the gap between Landau levels in the mean field theory. We note here that the spectrum of the small deformations is less or equal than the spectrum of the vortex (topological) excitations studied in the previous chapter for, as we noted in the previous chapter (equation (2.22)), the energy of the vortex excitations is always greater or equal than $\mathcal{E}$.
Also currents can be computed:

$$\begin{aligned} J_x &= \frac{e}{2mi}\Big[\phi^\dagger D_x\phi - \phi(D_x\phi)^\dagger\Big] = \frac{ev^2}{m}\Big(\partial_x\theta + \frac{4\pi}{k}vu_y\Big) \\ J_y &= \frac{e}{2mi}\Big[\phi^\dagger D_y\phi - \phi(D_y\phi)^\dagger\Big] = \frac{ev^2}{m}\Big(\partial_y\theta - \frac{4\pi}{k}vu_x\Big) \end{aligned} \tag{3.44}$$

which, in a compact form, can be rewritten as:

$$J_i = -2ev(\dot{u}_i - \mathcal{E}\epsilon_{ij}u_j) \tag{3.45}$$

This current exhibit the chiral property of our system.
To see this let us parameterize $\vec{u}(\vec{x},t)$ as follows:

$$u_x(\vec{x},t) = u_{0x}\cos(Et+\varphi_x) \qquad u_y(\vec{y},t) = u_{0y}\cos(Et+\varphi_y) \tag{3.46}$$

then if $E=\mathcal{E}$ the currents (3.44) can be rewritten:

$$J_x = J_0\cos(\mathcal{E}t+\varphi_0) \qquad J_y = J_0\sin(\mathcal{E}t+\varphi_0) \tag{3.47}$$

---

[1] Here we are neglecting a contribution to the energy coming from an electrostatic interaction between fluctuations, which will play an essential rôle in the next chapter. The piece to be added to the hamiltonian density is

$$\frac{e^2}{8\pi}\delta\rho(\vec{x})\int d^2x'\frac{1}{|\vec{x}-\vec{x}'|}\delta\rho(\vec{x}');$$

its contribution to the r.h.s. of equation (3.42) is $+\frac{e^2}{4}|\vec{p}|$.

where J_0 and φ_0 are constants depending on $\vec{u}_0$.
Notice that this is a circularly polarized current with defined direction of rotation. It can be easily seen that, since such direction depends on k, it reverses performing a parity transformation. This chiral property emerges only if the energy is equal to the gap $\mathcal{E}$ that is for $p \to 0$.

3.3. Correlation functions

We want to compute:

$$\langle 0|\delta\rho(\vec{x})\delta\rho(\vec{y})|0\rangle = \lim_{t\to 0^+}\langle 0|T(\delta\rho(\vec{x},t)\delta\rho(\vec{y},0)|0\rangle \tag{3.48}$$

As a consequence of (3.33) and (3.34) we have

$$\langle 0|\delta\rho(\vec{x})\delta\rho(\vec{y})|0\rangle = 4v^2\langle 0|\vec{\partial}\cdot\vec{u}(\vec{x})\vec{\partial}\cdot\vec{u}(\vec{y})|0\rangle \tag{3.49}$$

so we start computing the correlation functions for the $\vec{u}$'s.
Let us consider the following lagrangian density:

$$\mathcal{L}_J = \mathcal{L} + \vec{J}(\vec{x},t)\cdot\vec{u}(\vec{x},t) \tag{3.50}$$

then if we build the partition function:

$$Z[\vec{J}] = \int \mathcal{D}\vec{u}\, e^{i\int d^3x\,\mathcal{L}_J} \tag{3.51}$$

we have:

$$\langle 0|T(\vec{u}(\vec{x},t)\vec{u}(\vec{y},t')|0\rangle = -i\frac{\delta}{\delta\vec{J}(\vec{x},t)}\frac{\delta}{\delta\vec{J}(\vec{y},t')}Z[\vec{J}]\bigg|_{\vec{J}=0}. \tag{3.52}$$

We can write:

$$Z[\vec{J}] = e^{-\frac{i}{2}\int d^3x d^3y\,\vec{J}(\vec{x},t)G(\vec{x}-\vec{y},t-t')\vec{J}(\vec{y},t')} \tag{3.53}$$

here $G(\vec{x}-\vec{y},t-t')$ is the Green's function defined by:

$$\left[-\partial_0^2 + \frac{1}{4m^2}\left(\triangle^2 - 16\pi v^2\left(1-\frac{1}{k}\right)\triangle + \left(\frac{4\pi v^2}{k}\right)^2\right)\right]G(\vec{x},t) = \frac{i}{4m}\delta^{(2)}(\vec{x})\delta(t) \tag{3.54}$$

By Fourier analysis we get:

$$G(\vec{x},t) = \frac{i}{4m}\frac{1}{(2\pi)^3}\int d\omega d^2p\, e^{i(\omega t+\vec{p}\cdot\vec{x})}\frac{1}{\omega^2 + \frac{(p^2+\mu_1^2)(p^2+\mu_2^2)}{4m^2} + i\epsilon} \tag{3.55}$$

where $\mu_1^2 < \mu_2^2$ are such that $(p^2+\mu_1^2)(p^2+\mu_2^2) = p^4 + 16\pi v^2\left(1-\frac{1}{k}\right)p^2 + 4m^2\mathcal{E}^2$. Then

$$\langle 0|T(\vec{u}(\vec{x},t)\vec{u}(\vec{y},t')|0\rangle = -iG(\vec{x}-\vec{y},t) \tag{3.56}$$

Taking the limit for $t \to 0$ and the divergencies we get to:

$$\langle 0|\delta\rho(\vec{x})\delta\rho(\vec{y})|0\rangle = \frac{v^2}{(2\pi)^2}\int d^2p\,\frac{p^2}{\sqrt{(p^2+\mu_1^2)(p^2+\mu_2^2)}}\,e^{i\vec{p}\cdot(\vec{x}-\vec{y})} \tag{3.57}$$

which can be rewritten as:

$$\langle 0|\delta\rho(\vec{x})\delta\rho(\vec{y})|0\rangle =$$

$$= v^2\delta^{(2)}(\vec{x}-\vec{y}) - \frac{v^2}{(2\pi)^2}\int d^2p\,\frac{\sqrt{(p^2+\mu_1^2)(p^2+\mu_2^2)}-p^2}{\sqrt{(p^2+\mu_1^2)(p^2+\mu_2^2)}}\,e^{i\vec{p}\cdot(\vec{x}-\vec{y})} \tag{3.58}$$

The integral in (3.58) can be rewritten in the following way:

$$-\frac{2v^2}{(2\pi)^2}\int_{\mu_1^2}^{\mu_2^2} dt\,\frac{t}{\sqrt{(t-\mu_1^2)(\mu_2^2-t)}}K_0\left(\sqrt{t}\,|\vec{x}-\vec{y}|\right) \tag{3.59}$$

this is a negative defined function, which we will call $F(\vec{x}-\vec{y})$, which is zero for $|\vec{x}-\vec{y}| \to \infty$ and diverges logarithmically for $|\vec{x}-\vec{y}| \to 0$.
This solution can be compared with the multiparticle density solution.
If we take for the density the usual N-particle expression:

$$\rho(\vec{x}) = \sum_{n=1}^{N}\delta^{(2)}(\vec{x}-\vec{x}_n) \tag{3.60}$$

then we have:

$$\rho(\vec{x})\rho(\vec{y}) = \sum_{n=1}^{N}\sum_{m=1}^{N}\delta^{(2)}(\vec{x}-\vec{x}_n)\delta^{(2)}(\vec{y}-\vec{x}_m) =$$

$$= \sum_{n=1}^{N}\delta^{(2)}(\vec{x}-\vec{x}_n)\delta^{(2)}(\vec{y}-\vec{x}_n) + \sum_{i\neq j}\delta^{(2)}(\vec{x}-\vec{x}_i)\delta^{(2)}(\vec{y}-\vec{x}_j) =$$

$$= \sum_{n=1}^{N}\delta^{(2)}(\vec{x}-\vec{y})\delta^{(2)}(\vec{y}-\vec{x}_n) + \sum_{i\neq j}\delta^{(2)}(\vec{x}-\vec{x}_i)\delta^{(2)}(\vec{y}-\vec{x}_j) =$$

$$= \rho(\vec{x})\delta^{(2)}(\vec{x}-\vec{y}) + :\rho(\vec{x})\rho(\vec{y}): \tag{3.61}$$

where we have introduced the "normal order" notation

$$\sum_{i\neq j}\delta^{(2)}(\vec{x}-\vec{x}_i)\delta^{(2)}(\vec{y}-\vec{x}_j) =: \rho(\vec{x})\rho(\vec{y}): \tag{3.62}$$

Then

$$\langle 0|\rho(\vec{x})\rho(\vec{y})|0\rangle = \langle 0|\rho(\vec{x})|0\rangle\delta^{(2)}(\vec{x}-\vec{y}) + \langle 0| :\rho(\vec{x})\rho(\vec{y}): |0\rangle \tag{3.63}$$

But since $\rho(\vec{x})=v^2+\delta\rho$ and $\langle 0|\delta\rho(\vec{x})|0\rangle=0$ we have $\langle 0|\rho(\vec{x})|0\rangle=v^2$ so

$$\langle 0|\rho(\vec{x})\rho(\vec{y})|0\rangle = v^4 + \langle 0|\delta\rho(\vec{x})\delta\rho(\vec{y})|0\rangle = v^2\delta(\vec{x}-\vec{y}) + \langle 0| :\rho(\vec{x})\rho(\vec{y}): |0\rangle \tag{3.64}$$

But recalling that we have found:

$$\langle 0|\delta\rho(\vec{x})\delta\rho(\vec{y})|0\rangle = v^2\delta^{(2)}(\vec{x}-\vec{y}) - v^2F(\vec{x}-\vec{y}) \tag{3.65}$$

so we conclude

$$\langle 0| :\rho(\vec{x})\rho(\vec{y}): |0\rangle = v^4 - v^2F(\vec{x}-\vec{y}) \tag{3.66}$$

3.3.1. Compressibility

The compressibility χ of a fluid can be defined in the following way. If $F(\vec{p},\omega)$ is the Fourier transform of $\langle 0|\delta\rho(\vec{x},t)\delta\rho(\vec{y},t')|0\rangle$ then χ is proportional to

$$\lim_{\vec{p}\to 0} F(p,0) \qquad (3,67)$$

in our case we have:

$$F(\vec{p},\omega) = \frac{1}{4m}\frac{p^2}{\omega^2+\frac{(p^2+\mu_1^2)(p^2+\mu_2^2)}{4m^2}} \tag{3.68}$$

Therefore we immediately get $\chi=0$.
It is important to note that if the spectrum of the energy had not a gap, *i.e.* if $\mathcal{E}=0$, we would have had $\mu_1^2=0$ and therefore $\chi\neq 0$. So we can conclude that there is a relation between compressibility and gaplessness in the energy spectrum. We will turn again on this point later.

3.4. Small perturbations

In this section we point our attention to how our system reacts if we apply a small perturbation of density from outside, *i.e.* we fix a non-zero value of $\delta\rho$ at some point and see how the deformation propagates. We find two qualitatively different behaviours according to whether the fluid is incompressible or not.
We consider the static case. Let us define

$$H_R = H + \int d^2x\, \vec{\nabla}\cdot\vec{u}(\vec{x})R(\vec{x}) = H - \int d^2x\, \vec{u}(\vec{x})\cdot\vec{\partial}R(\vec{x}) \tag{3.69}$$

here $R(\vec{x})$ is an external current coupled to $\vec{\partial}\cdot\vec{u} = \frac{1}{2v}\delta\rho$ and H is the hamiltonian corresponding to the lagrangian density in equation (3.41):

$$H = \int d^2x \left\{ \frac{1}{2m}(\Delta\vec{u})^2 + \frac{8\pi v^2}{m}\Big(1-\frac{1}{k}\Big)(\vec{\nabla}\cdot\vec{u})^2 + \frac{1}{2m}\Big(\frac{4\pi v^2}{k}\Big)^2\vec{u}^2 \right\} \tag{3.70}$$

We consider the "static" Green's function defined by:

$$\left[\frac{1}{m}\left(\Delta^2 - 16\pi v^2\Big(1-\frac{1}{k}\Big)\Delta + \Big(\frac{4\pi v^2}{k}\Big)^2\right)\right]G(\vec{x}) = i\delta^{(2)}(\vec{x})\delta(t) \tag{3.54}$$

By Fourier analysis we find

$$G(\vec{x}-\vec{y}) = \frac{m}{(2\pi)^2}\int d^2p\, \frac{e^{i\vec{p}\cdot(\vec{x}-\vec{y})}}{(p^2+\mu_1^2)(p^2+\mu_2^2)} =$$

$$= \frac{m}{\mu_2^2-\mu_1^2}\frac{1}{(2\pi)^2}\int d^2p\, e^{i\vec{p}\cdot(\vec{x}-\vec{y})}\left\{\frac{1}{p^2+\mu_1^2} - \frac{1}{p^2+\mu_2^2}\right\} =$$

$$= \frac{m}{\mu_2^2-\mu_1^2}\frac{1}{(2\pi)}\Big[K_0\big(\mu_1|\vec{x}-\vec{y}|\big) - K_0\big(\mu_2|\vec{x}-\vec{y}|\big)\Big] \tag{3.71}$$

The asymptotic behaviours of $G(\vec{x})$ are:

$$\begin{aligned} \lim_{x\to 0} G(x) &= \frac{m}{\mu_2^2-\mu_1^2}\frac{1}{(2\pi)}\log\frac{\mu_2}{\mu_1} > 0 \\ \lim_{x\to\infty} G(x) &= \frac{m}{\mu_2^2-\mu_1^2}\frac{1}{(2\pi)}\sqrt{\frac{\pi}{2\mu_1|x|}}e^{-\mu_1 x} \to 0 \end{aligned} \tag{3.72}$$

The qualitative plot of $G(x)$ is therefore

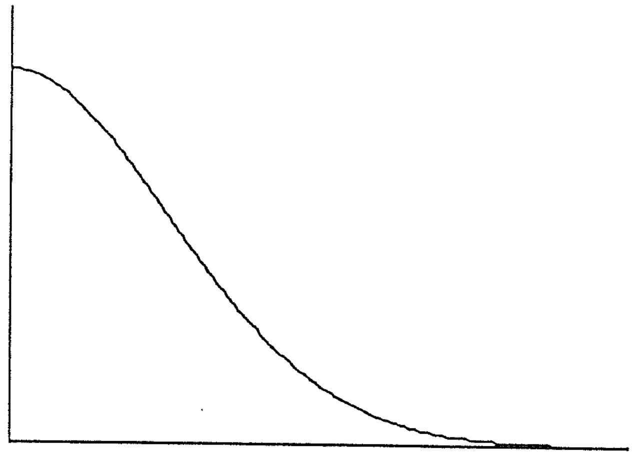

Fig. 2

Now using the fact that

$$\vec{u} = \int d^2y\, G(\vec{x}-\vec{y})\vec{\nabla}R(\vec{y}) = \vec{\partial}\int d^2y\, G(\vec{x}-\vec{y})R(\vec{y}) \tag{3.73}$$

$\delta\rho(\vec{x})$ can be written as:

$$\delta\rho(\vec{x}) = \frac{m}{\mu_2^2-\mu_1^2}\frac{v}{\pi}\int d^2y\left[\mu_1^2 K_0(\mu_1|\vec{x}-\vec{y}|) - \mu_2^2 K_0(\mu_2|\vec{x}-\vec{y}|)\right]R(\vec{y}) \tag{3.74}$$

Now we choose $R(\vec{y}) = R_0\,\delta^{(2)}(\vec{y})$ which corresponds to fixing the value of $\delta\rho$ at some point, so we get:

$$\delta\rho(\vec{x}) = \frac{mv}{\mu_2^2-\mu_1^2}\frac{R_0}{\pi}\left[\mu_1^2 K_0(\mu_1|\vec{x}|) - \mu_2^2 K_0(\mu_2|\vec{x}|)\right] \tag{3.75}$$

For $|\vec{x}|\to 0$ this behaves as:

$$\delta\rho(\vec{x}) \to \frac{mv}{\mu_2^2-\mu_1^2}\frac{R_0}{\pi}\left[-\mu_1^2\log\frac{\mu_1|\vec{x}|}{2} + \mu_2^2\log\frac{\mu_2|\vec{x}|}{2}\right] \tag{3.76}$$

which is ill-defined for $|\vec{x}|\to 0$ but, notice, it is negative. In order to get a well behaviour at the origin we relax the condition on $R(\vec{y})$ and introduce a small real parameter α such that:

$$R_\alpha(\vec{x}) = \frac{R_0}{\alpha\pi}e^{-\frac{|\vec{x}|}{\alpha}} \equiv R_0\, g_\alpha(\vec{x}) \tag{3.77}$$

notice that for $\alpha\to 0$ we recover the delta function. Then defining

$$\delta\rho_\alpha(\vec{x}) = \int d^2y\, g_\alpha(\vec{x}-\vec{y})\delta\rho(\vec{y}) \tag{3.78}$$

we get

$$\delta\rho_\alpha(\vec{x}) = \frac{mv}{\mu_2^2-\mu_1^2}\frac{R_0}{\pi}\int d^2y\, d^2z\, g_\alpha(\vec{x}-\vec{y})\Big[\mu_1^2 K_0(\mu_1|\vec{y}-\vec{z}|) - \mu_2^2 K_0(\mu_2|\vec{y}-\vec{z}|)\Big] g_\alpha(\vec{z}) \tag{3.79}$$

The actual $\delta\rho(\vec{x})$ is recovered in the limit $\alpha \to 0$.
If α is small enough the integral in (3.79) is well defined for $|\vec{x}| \to 0$ and it yields a *negative* number which we will call $-\xi_\alpha$:

$$\lim_{|\vec{x}|\to 0} \delta\rho_\alpha(\vec{x}) = -\frac{mv}{\mu_2^2-\mu_1^2}\frac{R_0}{\pi}\xi_\alpha \tag{3.80}$$

We can fix the value $\rho_\alpha(0)$ of $\delta\rho_\alpha(\vec{x})$ at some point, say $\vec{x}=0$, and determine consequently the value of R_0:

$$R_0 = -\frac{\mu_2^2-\mu_1^2}{mv}\frac{\pi}{\xi_\alpha}\rho_\alpha(0) \tag{3.81}$$

So equation (3.79) becomes:

$$\delta\rho_\alpha(\vec{x}) = \frac{\rho_\alpha(0)}{\xi_\alpha}\int d^2y\, d^2z\, g_\alpha(\vec{x}-\vec{y})\Big[\mu_1^2 K_0(\mu_1|\vec{y}-\vec{z}|) - \mu_2^2 K_0(\mu_2|\vec{y}-\vec{z}|)\Big] g_\alpha(\vec{z}) \tag{3.82}$$

For $|\vec{x}| \to \infty$ $K_0(\mu_1|\vec{x}|)$ dominates on $K_0(\mu_2|\vec{x}|)$ because $\mu_1^2 < \mu_2^2$ so we have:

$$\lim_{|\vec{x}|\to\infty} \delta\rho_\alpha(\vec{x}) = -\frac{\rho_\alpha(0)}{\xi_\alpha}\int d^2y\, d^2z\, g_\alpha(\vec{x}-\vec{y})\mu_1^2 K_0(\mu_1|\vec{y}-\vec{z}|) g_\alpha(\vec{z}) \to 0 \tag{3.83}$$

but notice that this is a negative quantity so the qualitative plot of $\delta\rho_\alpha(\vec{x})$ is:

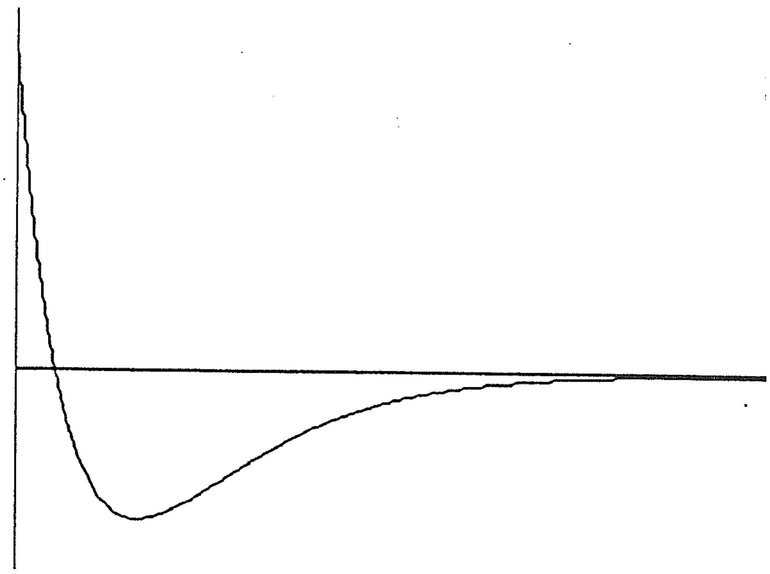

Fig. 3

such that $\int d^2x\, \delta\rho_\alpha(\vec{x}) = 0$ as it should for an incompressible fluid.
As it was noted in the previous section if the spectrum were gapless, *i.e.* if $\mu_1^2 = 0$, we had a compressible fluid. Indeed if this is the case our $\delta\rho_\alpha(\vec{x})$ becomes:

$$\delta\rho_\alpha(\vec{x}) = \frac{\rho_\alpha(0)}{\xi_\alpha} \int d^2y\, d^2z\, g_\alpha(\vec{x}-\vec{y})\mu_2^2 K_0(\mu_2|\vec{y}-\vec{z}|)g_\alpha(\vec{z}) \tag{3.84}$$

which evidently is always positive and we get the qualitative plot:

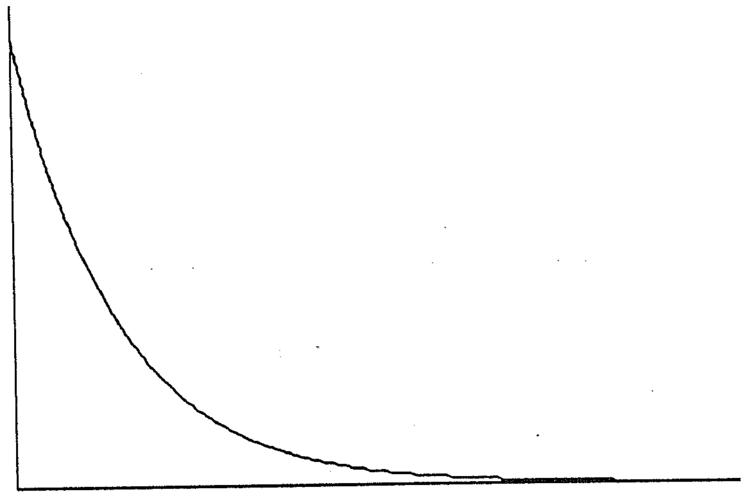

Fig. 4

Therefore also this analysis of the small density perturbations confirms the relation between incompressibility and the existence of a gap in the energy spectrum.

3.4.1. Currents

We compute here the currents corresponding to the small perturbations of density just studied. Since we are considering a static problem the currents (3.45) become:

$$J_i = 2ev\mathcal{E}\epsilon_{ij}u_j \tag{3.85}$$

let us, for example, compute J_θ :

$$J_\theta = -2v\mathcal{E}u_r = -\frac{\rho_\alpha(0)}{\xi_\alpha}\mathcal{E} \int d^2r' \left[\mu_1 K_1(\mu_1|r-r'|) - \mu_2 K_1(\mu_2|r-r'|)\right] g_\alpha(r') \tag{3.86}$$

The total current is

$$I_\theta = \int_0^\infty dr\, J_\theta(r) = \frac{\mathcal{E}}{\xi_0}\rho_0(0)\,\log\frac{\mu_2}{\mu_1} \tag{3.86}$$

here we have, safely, performed the limit $\alpha \to 0$.
We have found a total current of negative sign. It can be checked that performing a parity transformation we get exactly the same current with the opposite sign as we expect from a chiral fluid.

3.5. Polarization effect

In this section we test our two dimensional chiral system by scattering an electromagnetic wave perpendicularly incident [17]. We begin the analysis studying how the electromagnetic potential couples with our system on the plane. The small deformation lagrangian, equation (3.38), is modified adding the incoming electromagnetic potential $\vec{A}$:

$$\mathcal{L} = 2v\theta\vec{\nabla}\cdot\dot{\vec{u}} - \frac{1}{2m}v^2\left(\vec{\nabla}\theta - \frac{4\pi v^2}{k}\tilde{\vec{u}} + e\vec{A}\right)^2 - \frac{1}{2m}(\triangle\vec{u})^2 - \frac{8\pi v^2}{m}\left(1-\frac{1}{k}\right)(\vec{\nabla}\cdot\vec{u})^2 \tag{3.87}$$

here $\tilde{\vec{u}}$ is the dual of $\vec{u}$ in the sense that $\tilde{u}_i = \epsilon_{ij}u_j$.
Performing the variation with respect to θ we find:

$$\frac{\delta\mathcal{L}}{\delta\theta} = 2v\vec{\nabla}\cdot\dot{\vec{u}} + \frac{v^2}{m}\vec{\nabla}\cdot\left(\vec{\nabla}\theta - \frac{4\pi v^2}{k}\tilde{\vec{u}} + e\vec{A}\right) = 0 \tag{3.88}$$

from (3.35) we have $\vec{\nabla}\cdot\dot{\vec{u}} = 0$ and if we decompose $\vec{A} = \vec{A}_L + \vec{A}_\perp$ we have $\vec{\nabla}\cdot\vec{A}_\perp = 0$ so:

$$\vec{\nabla}\theta = -\frac{2m}{v}\tilde{\vec{u}} - e\vec{A}_L \tag{3.89}$$

Then the lagrangian (3.87) can be rewritten as:

$$\mathcal{L} = -2v\dot{\vec{u}}\cdot\vec{\nabla}\theta - \frac{1}{2m}v^2\left(\vec{\nabla}\theta + e\vec{A}_L\right)^2 - \frac{1}{2m}v^2\left(e\vec{A}_\perp - \frac{4\pi v^2}{k}\tilde{\vec{u}}\right)^2 - \frac{1}{2m}(\triangle\vec{u})^2 - \frac{8\pi v^2}{m}\left(1-\frac{1}{k}\right)(\vec{\nabla}\cdot\vec{u})^2 \tag{3.90}$$

Now we introduce the following parameterization:

$$\vec{A} = \vec{\nabla}\varphi + \tilde{\vec{\nabla}}\psi \qquad A_i = \partial_i\varphi + \epsilon_{ij}\partial_j\psi \tag{3.91}$$

So equation (3.90) becomes:

$$\mathcal{L}=2m\dot{\vec{u}}^2+2ve\dot{\vec{u}}\cdot\vec{\nabla}\varphi-2m\mathcal{E}^2\vec{u}^2+2v\mathcal{E}\vec{u}\cdot\vec{\nabla}\psi-\frac{e^2v^2}{2m}(\vec{\nabla}\psi)^2-\frac{1}{2m}(\triangle\vec{u})^2-$$

$$-\frac{8\pi v^2}{m}\Big(1-\frac{1}{k}\Big)(\vec{\nabla}\cdot\vec{u})^2 \tag{3.92}$$

If we take $\vec{u}=\vec{u}_0e^{i(\omega t+\vec{p}\cdot\vec{x})}$ we find:

$$\mathcal{L}=2m\Big(\omega^2-\mathcal{E}^2-g(p)\Big)\vec{u}^2+2v\vec{u}\cdot(i\omega\vec{\nabla}\varphi+\mathcal{E}\vec{\nabla}\psi)-\frac{e^2v^2}{2m}(\vec{\nabla}\psi)^2 \tag{3.93}$$

here $g(p)$ stands for the p-dependent terms in (3.42).
Now let $A_i(p)$ be the Fourier components of $\vec{A}$ on the plane. Then we can identify:

$$\vec{A}=\vec{A}_L+\vec{A}_\perp=\frac{\vec{p}\cdot\vec{A}}{p^2}\,\vec{p}+\frac{\epsilon_{ij}A_ip_j}{p^2}\,\tilde{\vec{p}} \tag{3.94}$$

or, in terms of φ and ψ:

$$\vec{\nabla}\varphi=\frac{\vec{p}\cdot\vec{A}}{p^2}\,\vec{p}\qquad \vec{\nabla}\psi=\frac{\epsilon_{ij}A_ip_j}{p^2}\,\vec{p} \tag{3.95}$$

Now equation (3.93) can be rewritten as:

$$\mathcal{L}=2m\Big(\omega^2-\mathcal{E}^2-g(p)\Big)\left(\vec{u}+\frac{ev}{2m}\frac{i\omega\vec{\nabla}\varphi+\mathcal{E}\vec{\nabla}\psi}{\omega^2-\mathcal{E}^2-g(p)}\right)\left(\vec{u}+\frac{ev}{2m}\frac{i\omega\vec{\nabla}\varphi+\mathcal{E}\vec{\nabla}\psi}{\omega^2-\mathcal{E}^2-g(p)}\right)^*-$$

$$-\frac{e^2v^2}{2m}\frac{|i\omega\vec{\nabla}\varphi+\mathcal{E}\vec{\nabla}\psi|^2}{\omega^2-\mathcal{E}^2-g(p)}-\frac{e^2v^2}{2m}(\vec{\nabla}\psi)^2 \tag{3.96}$$

We can eliminate the first term performing the functional integration over $\vec{u}$; we remain with something which we write as:

$$\mathcal{L}=-\frac{1}{2}A^\dagger\hat{\mathcal{L}}A \tag{3.97}$$

here we have introduced the complex notation:

$$A=\begin{pmatrix}A_x\\A_y\end{pmatrix} \tag{3.98}$$

and

$$\hat{\mathcal{L}}=\frac{e^2v^2}{m}\frac{1}{\omega^2-\mathcal{E}^2-g(p)}\frac{1}{p^2}\times$$

$$\times \begin{pmatrix} |ip_x\omega + p_y\mathcal{E}|^2 & (ip_x\omega + p_y\mathcal{E})^*(ip_y\omega - p_x\mathcal{E}) \\ (ip_y\omega - p_x\mathcal{E})^*(ip_x\omega + p_y\mathcal{E}) & |ip_y\omega - p_x\mathcal{E}|^2 \end{pmatrix} +$$
$$+\frac{e^2v^2}{mp^2}\begin{pmatrix} p_y^2 & -p_xp_y \\ -p_xp_y & p_x^2 \end{pmatrix} \tag{3.99}$$

Equation (3.99) is the contribution to the total lagrangian coming from the interaction between the incoming wave and the planar chiral fluid. Now we suppose $\vec{A}$ propagating in the z direction and the plane situated at $z = 0$; if we write $A = A(z)e^{i(\omega t + \vec{p}\cdot\vec{x})}$, with $A(z)$ discussed below, the total lagrangian is:

$$\mathcal{L}(A) = \frac{1}{2}\omega^2 a^2 - \frac{1}{2}(p^2A^2 + (\partial_z A)^2) - \frac{1}{2}A^\dagger\,\hat{\mathcal{L}}\,A\,\delta(z) \tag{3.100}$$

from equation (3.100) we extract the equation of motion for A:

$$\partial_z^2 A + (\omega^2 - p^2)A - \hat{\mathcal{L}}\,A\,\delta(z) = 0 \tag{3.101}$$

For $A(z)$ we have:

$$A(z) = \begin{cases} \alpha_- e^{ikz} + \beta_- e^{-ikz} & \text{if } z < 0 \\ \alpha_+ e^{ikz} & \text{if } z > 0 \end{cases} \tag{3.102}$$

We have to impose continuity at $z = 0$ for $A(z)$ and its derivative:

$$\begin{aligned} A(0^+) = A(0^-) &\Rightarrow \alpha_+ = \alpha_- + \beta_- \\ \partial A(0^+) = \partial A(0^-) + \hat{\mathcal{L}}A(0) &\Rightarrow ik(\alpha_- - \beta_-) = (ik - \hat{\mathcal{L}})\alpha_+ \end{aligned} \tag{3.103}$$

So:

$$\begin{aligned} \alpha_+ &= 2ik\,(2ik - \hat{\mathcal{L}})^{-1}\alpha_- \\ \beta_- &= \alpha_+ - \alpha_- = \hat{\mathcal{L}}\,(2ik - \hat{\mathcal{L}})^{-1}\alpha_- \end{aligned} \tag{3.104}$$

From equation (3.104) we can read the coefficients of reflection and transmission:

$$R = \frac{\beta_-}{\alpha_-} = \frac{\frac{\hat{\mathcal{L}}}{2ik}}{1 - \frac{\hat{\mathcal{L}}}{2ik}} \qquad T = \frac{\alpha_+}{\alpha_-} = \frac{1}{1 - \frac{\hat{\mathcal{L}}}{2ik}} \tag{3.105}$$

Notice that $R = T - 1$ and $|T|^2 + |R|^2 = 1$ as it should.
In particular we are interested in the coefficient of transmission in the limit when $\omega \to \mathcal{E}$. In this case only the first part of equation (3.99) is important. If we compute the inverse of the matrix $1 - \frac{1}{2ik}\hat{\mathcal{L}}$ we get, taking for simplicity $\vec{p} = (0, p)$:

$$T = \frac{1}{1 - 2i(q+s) - 2qs}\begin{pmatrix} 1 - iq & -q \\ q & 1 - i(q+2s) \end{pmatrix} \tag{3.106}$$

here we have set $q = -\frac{1}{2k}\frac{e^2v^2}{m}\frac{\mathcal{E}^2}{\omega^2-\mathcal{E}^2}^{2}$ and $s = -\frac{1}{2}\frac{1}{2k}\frac{e^2v^2}{m}$.
The limit $\omega \to \mathcal{E}$ correspond to $q \to \infty$ that is:

$$T = \frac{1}{2}\frac{i}{i+s}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} \tag{3.107}$$

Now let us consider the following parameterization:

$$\begin{aligned} A_x &= Re\big(\cos\theta e^{-i\omega t}\big) = \cos\theta\cos\omega t \\ A_y &= Re\big(\sin\theta e^{i\varphi}e^{-i\omega t}\big) = \sin\theta\cos(\varphi - \omega t) \end{aligned} \tag{3.108}$$

we can easily see that the circular polarization is for $\theta = \frac{\pi}{4}$ and $\varphi = \frac{\pi}{2}$, so that:

$$A_x = \frac{\sqrt{2}}{2}\cos\omega t \qquad A_y = \frac{\sqrt{2}}{2}\sin\omega t \tag{3.109}$$

in our complex formalism this corresponds to:

$$A = \begin{pmatrix} \cos\theta \\ \sin\theta e^{i\varphi} \end{pmatrix} e^{-i\omega t} = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 \\ i \end{pmatrix} e^{-i\omega t} \tag{3.110}$$

It is easy to check that T project a state in a circularly polarized one *i.e.* if the circularly polarized state is

$$|c\rangle = \begin{pmatrix} 1 \\ i \end{pmatrix} \tag{3.111}$$

then

$$T = \frac{1}{2}\frac{i}{i+s}|c\rangle\langle c| \tag{3.112}$$

That is at the resonance $\omega \to \mathcal{E}$ our chiral planar system behaves like a perfect polarizer.
Computing the reflection coefficient we find:

$$R = T - 1 = \frac{1}{2}\frac{s}{i+s}|c\rangle\langle c| - \frac{1}{2}|\bar{c}\rangle\langle\bar{c}| \tag{3.113}$$

here $|\bar{c}\rangle$ is the state polarized in the opposite direction. The reflection is only partially polarized.
Finally note that if $\omega \to \infty$ we get:

$$\begin{aligned} \hat{\mathcal{L}} &= \frac{e^2v^2}{m}\frac{1}{p^2}\left[\begin{pmatrix} p_x^2 & p_xp_y \\ p_xp_y & p_y^2 \end{pmatrix} + \begin{pmatrix} p_y^2 & -p_xp_y \\ -p_xp_y & p_x^2 \end{pmatrix}\right] = \\ &= \frac{e^2v^2}{m}\frac{1}{p^2}\begin{pmatrix} p^2 & 0 \\ 0 & p^2 \end{pmatrix} = \frac{e^2v^2}{m}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned} \tag{3.114}$$

so we have lost any polarizing effect.
If conversely we take $\omega \to 0$ and $p \to 0$ we get $\hat{\mathcal{L}} = 0$, so there is no more coupling between the electromagnetic wave and the planar system.

Chapter 4.

Introducing an external magnetic field

4.1. Meissner effect

In this section we test our chiral charged superfluid with an external magnetic field, and study its Meissner effect.
Up to now we have been considering a two dimensional system; now we are going to study an effect which is essentially three-dimensional. Therefore we suppose to have a multilayered bulk of many two-dimensional thin films separated by a spacing d; we further suppose that at the edge of the bulk there is a uniform, constant magnetic field orthogonal to the layers' plane [1] as in figure.

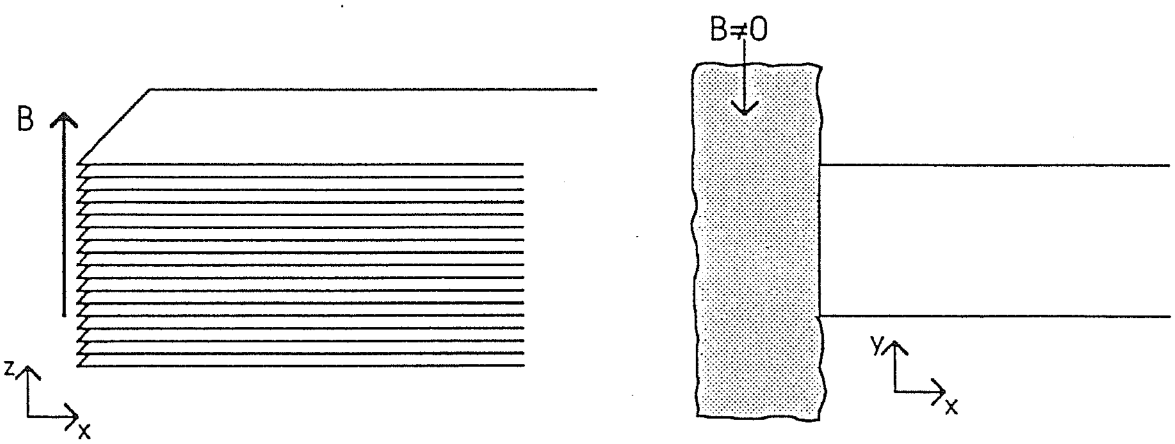

Fig. 5

[1] A similar analysis can be done for the case of the magnetic field with direction parallel to the layers' plane [17]. In this case the results are very different, in that the magnetic penetration length increases (according to a fractional power law) with the dimensions of the system.

In this three dimensional system we must take into account also an electrostatic contribution, which we have neglected in the previous chapter (see footnote at page 29) and which will play an essential rôle here. This contribution is essentially due to an electric field $\vec{E}$ which comes from the fluctuations of the charged matter and obeys the Maxwell equation:

$$\vec{\nabla} \cdot \vec{E} = e\,\delta\rho^{(3)} \tag{4.1}$$

here $\delta\rho^{(3)} = \frac{\delta\rho}{d}$, is the three dimensional density.
In other words if there is a fluctuation $\delta\rho$ of matter the system will not be in electrostatic equilibrium anymore, for there will be some zones where there is lack of charged matter and others where there is abundance: $\vec{E}$ is the electric field resulting from this non-equilibrium situation.
So let us consider the following three-dimensional Hamiltonian density [1]:

$$\mathcal{H} = \frac{1}{2md}|\vec{D}\phi|^2 + \frac{1}{2}\left(\frac{\vec{B}^2}{e^2} + \vec{E}^2\right) + \frac{g}{d}(\rho - v^2)^2 \tag{4.2}$$

here

$$\vec{B} = e\vec{\nabla} \wedge \vec{A}^{em} \qquad \vec{E} = -\vec{\nabla} A_0^{em} \qquad \vec{D} = \vec{\nabla} - i\vec{A}^{CS} - ie\vec{A}^{em} \tag{4.3}$$

Notice that the following equations hold:

$$\begin{cases} \vec{\nabla} \cdot \vec{E} = \frac{e}{d}\delta\rho \\ \vec{\nabla} \wedge \vec{E} = 0 \end{cases} \tag{4.4}$$

$$\begin{cases} \vec{\nabla} \cdot \vec{A}^{CS} = 0 \\ \vec{\nabla} \wedge \vec{A}^{CS} = \frac{2\pi}{k}\delta\rho \end{cases} \tag{4.5}$$

here $\vec{E}$ is a three-dimensional vector whereas $\vec{A}^{CS}$ is two-dimensional. If we suppose that the matter distribution is constant in the z direction, which means that we have exactly the same matter distribution in every layer, then we have no electric field orthogonal to the x-y layers' plane. With this assumption equations (4.4) and (4.5) tell that $\vec{E}$ and $\vec{A}^{CS}$ are dual two-dimensional vectors, *i.e.* :

$$E_i = \frac{ke}{2\pi d}\epsilon_{ij} A_j^{CS} \tag{4.6}$$

[1] In this chapter we have redefined $e\vec{A}^{CS} \rightarrow \vec{A}^{CS}$ so that the electric charge e appears only coupled to $\vec{A}^{em}$.

So we can write the hamiltonian as follows:

$$\frac{H}{L_z} = \int dxdy \left\{ \frac{1}{2md}|\vec{D}\phi|^2 + \frac{1}{2}\left[\left(\vec{\nabla} \wedge \vec{A}^{em}\right)^2 + \frac{k^2 e^2}{4\pi^2 d^2}(\vec{A}^{CS})^2 \right] + \frac{g}{d}(\rho - v^2)^2 \right\} \quad (4.7)$$

here we have performed the integration in z.
At this point we make one further assumption: we suppose that the external magnetic field is constant on the edge of the bulk, so we can take $\vec{A}^{em}$ in the y direction:

$$\vec{A}^{em} = (0, A^{em}, 0) \quad (4.8)$$

furthermore we still have the freedom to take also $\vec{A}^{CS}$ in the y direction:

$$\vec{A}^{CS} = (0, A^{CS}, 0) \quad (4.9)$$

In the gauge:

$$\vec{\nabla} \cdot \vec{A}^{em} = \vec{\nabla} \cdot \vec{A}^{CS} = 0 \quad (4.10)$$

A^{em} and A^{CS} depend only on x. With these assumptions all quantities in (4.7) depend only on x, so we can perform the integration in y and get:

$$\frac{H}{L_y L_z} = \int_0^\infty dx \left\{ \frac{1}{2md}\left[|\partial_x \phi|^2 + |eA^{em} + A^{CS}|^2 \rho \right] + \frac{1}{2}(\partial_x A^{em})^2 + \right.$$

$$\left. + \frac{k^2 e^2}{8\pi^2 d^2}(A^{CS})^2 + \frac{g}{d}\left(\rho - v^2\right)^2 \right\} \quad (4.11)$$

L_y being the length of the edge of the bulk.

Now, before going into computational details, we want to spend some time analyzing the reason why we expect a Meissner effect. To this end we will make some simplificatory assumptions. The first is to suppose that some external device is keeping constant the magnetic flux $\Phi_0 = L_y l_x B$, where L_y is the length of the edge of the sample and l_x is the penetration length of the magnetic field, so that

$$Bl_x = \frac{\Phi_0}{L_y} = \varphi_0 \quad (4.12)$$

is given to the system from outside.
The second assumption we make is to suppose B constant for $0 < x < l_x$ and zero for $x > l_x$ whereas actually it is exponentially decreasing.
The third assumption we make is to take a constant value for the density $\rho = v^2$.

So we have:

$$B = \frac{\varphi_0}{l_x} \qquad A^{em}(x) = \frac{x}{l_x}\varphi_0 \tag{4.13}$$

To test the meaningfulness of what we are doing let us see what happens for the well known case of the standard superconductor, *i.e.* with $\vec{A}^{CS}=0$; the hamiltonian (4.11) becomes:

$$\frac{H}{L_y L_z} = \int_0^{l_x} dx \left\{ \frac{1}{2}\vec{B}^2 + \frac{1}{2md}e^2(A^{em})^2 v^2 \right\} = \frac{1}{2}\frac{\varphi_0^2}{l_x} + \frac{1}{6md}e^2 v^2 \varphi_0^2 l_x \tag{4.14}$$

Minimizing H with respect to l_x we find:

$$\frac{\partial H}{\partial l_x} = 0 \qquad \Rightarrow \qquad l_x = \sqrt{\frac{3md}{e^2 v^2}} \tag{4.15}$$

which is to be compared with the standard value of the penetration length for type II superconductors:

$$l_x = \sqrt{\frac{m}{e^2 \rho^{(3)}}} \tag{4.16}$$

Notice that in what we have done a fundamental rôle is played by the term quadratic in A^{em}, the "mass term" for the electromagnetic field.
In our case, with $\vec{A}^{CS} \neq 0$, this effect could be ruined by the possible cancellation $\vec{A}^{CS} = -e\vec{A}^{em}$ but, notice, we have also the electrostatic interaction term quadratic in $\vec{A}^{CS}$ which now plays the dominant rôle. We have:

$$\frac{H}{L_y L_z} = \int_0^{l_x} dx \left\{ \frac{1}{2}\vec{B}^2 + \frac{1}{2}\vec{E}^2 + \frac{1}{2md}(eA^{em} + A^{CS})v^2 \right\} = \frac{1}{2}\frac{\varphi_0^2}{l_x} + \frac{k^2 e^4}{24\pi^2 d^2}\varphi_0^2 l_x \tag{4.17}$$

Minimizing with respect to l_x we get:

$$\frac{\partial H}{\partial l_x} = 0 \qquad \Rightarrow \qquad l_x = \frac{2\pi d\sqrt{3}}{ke^2 v^2} \tag{4.18}$$

we see that we get a *finite* penetration length, so we expect to have Meissner effect.

Now we turn back to hamiltonian (4.11) and analyze it in the framework of the small deformation approach introduced in the previous chapter.
So we have, recalling the basic definitions:

$$\phi = v e^{\frac{\eta}{v}} = v + \eta \qquad \partial_x \phi(x) = \partial_x \eta(x) = \partial_x^2 u(x)$$

$$\rho = v^2 + 2v\eta \qquad \delta\rho = 2v\eta$$
$$A^{CS} = \frac{4\pi v}{k}u \qquad g = \frac{\pi}{m}\left(1 - \frac{1}{k}\right)$$

Notice that in this one-dimensional case ψ is real, so there is no phase. The hamiltonian becomes:

$$\frac{H}{L_y L_z} = \frac{1}{d}\int_0^\infty dx \left\{ \frac{1}{2m}\left[(\partial_x^2 u)^2 + \left(eA^{em} + \frac{4\pi v}{k}u\right)^2\right] + \right.$$
$$\left. + \frac{d}{2}(\partial_x A^{em})^2 + \frac{2v^2 e^2}{d}u^2 + \frac{4\pi v^2}{m}\left(1 - \frac{1}{k}\right)(\partial_x u)^2 \right\} \tag{4.19}$$

Where we have made use of the fact that $\int dx\, \eta(x) = 0$.
We now make the following ansatz:

$$u(x) = u_0 e^{-\lambda x} \qquad A^{em}(x) = A_0 e^{-\lambda x} \tag{4.20}$$

here u_0 and A_0 are the values at the edge of the bulk. If we minimize H with respect to λ we find, as we will see below, different leading values of λ for $u(x)$ and $A^{em}(x)$ from which we find the penetration length of the magnetic field and the characteristic scale for the spatial variations of the order parameter ψ, which we may call coherence length. The ratio of these two lengths is what characterizes the different types of superconductivity: for superconductors of type I this ratio is a small number, that is the coherence length is much bigger that the penetration length; for the superconductors of type II the situation is reversed.
With the ansatz (4.20) the hamiltonian becomes:

$$\frac{H}{L_y L_z} = \frac{1}{d}\int_0^\infty dx \left\{ \left[\frac{\lambda^4}{2m} - \frac{4\pi v^2}{m}\left(1 - \frac{1}{k}\right)\lambda^2 + \frac{8\pi^2 v^4}{mk^2} + \frac{2e^2 v^2}{d}\right]u^2(x) + \right.$$
$$\left. + \frac{4\pi v^3 e}{mk}u(x)A^{em}(x) + \left(\frac{e^2 v^2}{2m} - \frac{d}{2}\lambda^2\right)(A^{em}(x))^2 \right\} \tag{4.21}$$

What is in curly brackets can be rewritten in a matricial form as follows:

$$\begin{pmatrix} u & A^{em} \end{pmatrix} \begin{pmatrix} \frac{\lambda^4}{2m} - \frac{4\pi v^2}{m}(1 - \frac{1}{k})\lambda^2 + \frac{8\pi^2 v^4}{mk^2} + \frac{2e^2 v^2}{d} & \frac{2\pi v^3 e}{mk} \\ \frac{2\pi v^3 e}{mk} & \frac{e^2 v^2}{2m} - \frac{d}{2}\lambda^2 \end{pmatrix} \begin{pmatrix} u \\ A^{em} \end{pmatrix} \tag{4.22}$$

so H is minimal when the determinant of this matrix is zero, that is when:

$$\frac{d}{4m}\lambda^6 - \left(\frac{2\pi v^2 d}{m}\left(1 - \frac{1}{k}\right) + \frac{e^2 v^2}{4m^2}\right)\lambda^4 + \left(\frac{8\pi v^4 d}{2mk^2} + e^2 v^2 + \frac{2\pi e^2 v^4}{m^2}\left(1 - \frac{1}{k}\right)\right)\lambda^2 -$$

$$-\frac{e^4 v^4}{md} = 0 \tag{4.23}$$

This is an equation of the third order in λ^2. If we take the following typical values for the parameters:

$$d = 1\text{Å} \qquad e^2 = \frac{4\pi}{137} \qquad v^2 = 4 \cdot 10^{-3}\text{Å}^{-2} \qquad m = 250\text{Å}^{-1} \qquad k = 2 \tag{4.24}$$

we get the solutions:

$$\lambda_1^2 = 1.46 \cdot 10^{-6} \qquad \lambda_{2,3}^2 = 0.025 \pm i0.61 \equiv \mu \pm i\varrho \tag{4.25}$$

from here we get:

$$\lambda_1 = 1.21 \cdot 10^{-3} \qquad \lambda_{2,3} = 0.56 \pm i0.54 \equiv \alpha \pm i\beta \tag{4.26}$$

We will see in a moment that the leading value of λ for B is λ_1 which is a rather small number; before going on it is instructive to rewrite equation (4.23) in an approximate form taking only the leading terms:

$$\lambda^2\left[\frac{d}{4m}\lambda^4 - \left(\frac{2\pi v^2 d}{m}\left(1 - \frac{1}{k}\right) + \frac{e^2 v^2}{4m^2}\right)\lambda^2 + e^2 v^2\right] = \frac{e^4 v^4}{md} \tag{4.27}$$

which for small λ^2 has the solution:

$$\lambda^2 = \frac{e^4 v^4}{md}\frac{1}{e^2 v^2} = \frac{e^2 v^2}{md} \tag{4.28}$$

which, compared with equation (4.16), is exactly the expression for the inverse of the square of the standard penetration length, and, as we will see in the following, corresponds numerically to λ_1

The general solution for $u(x)$ is the linear combination:

$$u(x) = u_1 e^{-\lambda_1 x} + u_2 e^{-\lambda_2 x} + u_3 e^{-\lambda_3 x} \tag{4.29}$$

On this general solution we can impose the reality condition $u^*(x) = u(x)$ and the constraint for vanishing at the edge of the bulk.

So we get:

$$u_1^* = u_1 \qquad u_2^* = u_3 \qquad u_1 = -2\,\mathrm{Re}\,(u_2) \tag{4.30}$$

Parameterizing $u_2 = \frac{1}{2}(-u_1 + iw)$ we get:

$$u(x) = u_1 e^{-\lambda_1 x} - e^{-\alpha x}(u_1 \cos\beta x - w \sin\beta x) \tag{4.31}$$

For $\delta\rho$ we have:

$$\delta\rho = 2v\partial u = 2v\left[-\lambda_1 u_1 e^{-\lambda_1 x} + e^{-\alpha x}\Big((\alpha u_1 + \beta w)\cos\beta x + (\beta u_1 - \alpha w)\sin\beta x\Big)\right] \tag{4.32}$$

Since $\delta\rho(0) = -v^2$ we can determine the value of w:

$$w = -\frac{v}{2\beta} - \frac{u_1}{\beta}(\alpha - \lambda_1) \tag{4.33}$$

Then let us turn to $A^{em}(x)$. The following relation holds between $A^{em}(x)$ and $u(x)$:

$$A^{em}(x) = \frac{4\pi v^3 e}{mkd}\frac{1}{\lambda^2 - \frac{e^2v^2}{md}}u(x) \tag{4.34}$$

So we get:

$$A^{em}(x) = \frac{4\pi v^3 e}{mkd}\left\{\frac{u_1}{\lambda_1^2 - \frac{e^2v^2}{md}}e^{-\lambda_1 x} - \right.$$
$$-\frac{1}{\frac{e^4v^4}{m^2d^2} - 2\mu\frac{e^2v^2}{md} + \mu^2 + \varrho^2}e^{-\alpha x}\left[\left(u_1\left(\mu - \frac{e^2v^2}{md}\right) - w\varrho\right)\cos\beta x + \right.$$
$$\left.\left. + \left(u_1\varrho + w\left(\mu - \frac{e^2v^2}{md}\right)\right)\sin\beta x\right]\right\} \tag{4.35}$$

From this expression we can immediately obtain $B = \partial_x A^{em}$:

$$B = \frac{4\pi v^3 e}{mkd}\left\{\frac{-\lambda_1 u_1}{\lambda_1^2 - \frac{e^2v^2}{md}}e^{-\lambda_1 x} + \right.$$
$$+\frac{\alpha}{\frac{e^4v^4}{m^2d^2} - 2\mu\frac{e^2v^2}{md} + \mu^2 + \varrho^2}e^{-\alpha x}\left[\left(u_1\left(\mu - \frac{e^2v^2}{md}\right) - w\varrho\right)(\alpha\cos\beta x - \beta\sin\beta x) + \right.$$
$$\left.\left. + \left(u_1\varrho + w\left(\mu - \frac{e^2v^2}{md}\right)\right)(\alpha\sin\beta x + \beta\cos\beta x)\right]\right\} \tag{4.36}$$

Imposing $B(0) = B_0$ we get a value for u_1, the only parameter still undetermined. If we substitute the correct values of the parameters we immediately see that that dominant term is the first, *i.e.* the penetration length is $\frac{1}{\lambda_1} = 826.45$Å. We can compare this value to the numerical value of the standard penetration length (4.16) which is 825.57Å. This almost perfect correspondence is in agreement with the approximate analysis of equation (4.28).

For $\delta\rho$ the dominant is the second, *i.e.* the coherence length is $\frac{1}{\alpha} = 1.78$Å.

Notice that the penetration length is about 400 times the coherence length, so our system behaves like a type II superconductor.
In tables 9 and 10 we report the plots of $\delta\rho(x)$ and $B(x)$.

4.2. Vortices

In this section we study, with a variational method, the vortices in presence of an external electromagnetic field.
These vortices have an origin completely different from the vortices studied in chapter 2. Those were originated by fluctuations of the CS magnetic field, *i.e.* of the matter density from the mean value. These are the standard well known vortex configurations of superconductors of type II between the two critical temperatures. They are small regions of the specimen of normal behaviour, all surrounded by a superconducting region, where the external magnetic field penetrates completely and uniformly. There are superconducting currents flowing around the vortices. Furthermore there is a penetration of the magnetic field from the vortex region to the surrounding superconducting region (Meissner effect).
As we saw in the previous section the hamiltonian is:

$$\frac{H}{L_z} = \int d^2r \left\{ \frac{1}{2md}|\vec{D}\phi|^2 + \frac{1}{2}\left(\frac{\vec{B}^2}{e^2} + \vec{E}^2\right) + \frac{g}{d}(\rho - v^2)^2 \right\} \tag{4.37}$$

We make the following ansatz:

$$\begin{aligned}
&\phi = f(r)e^{in\theta} \\
&erA_\theta^{em} = n\left(1 - e^{-\lambda r^2}\right) \qquad A_r^{em} = 0 \\
&rA_\theta^{CS} = \frac{2\pi}{k}\int_0^r dr'r'\left(f^2(r') - v^2\right) = \frac{\pi}{k}v^2r^2e^{-\omega r^2} \qquad A_r^{CS} = 0
\end{aligned} \tag{4.38}$$

from the last one we easily get:

$$f^2(r) = v^2 - v^2(1 - \omega r^2)e^{-\omega r^2} \tag{4.39}$$

Notice that our ansatz is such that we have quantization of the flux of the external magnetic field, differently from the vortices studied in the first chapter when it was the flux of the CS magnetic field to be quantized. Notice also that since the vorticity is no longer connected with the CS field now, differently from the vortices

studied in chapter 2, it is possible to have an isolated vortex or antivortex since $\int_0^\infty drr(v^2 - f^2(r)) = 0$ as can be checked from the last of equations (4.38). Lastly notice that the ansatz for erA_θ^{em} is exactly the same we made for erA_θ in chapter 2 with $p=0$ (see equation (2.80)).
Substituting in (4.37) we get:

$$\frac{H}{L_z} = \int d^2r \left\{ \frac{1}{2md} \left[v^2 \frac{\omega^2(2-\omega r^2)^2 r^2 e^{-2\omega r^2}}{1-(1-\omega r^2)e^{-\omega r^2}} + \right.\right.$$

$$\left. + \frac{v^2}{r^2}\left(ne^{-\lambda r^2} - \frac{\pi}{k} v^2 r^2 e^{-\omega r^2}\right)^2 \left(1 - (1-\omega r^2)e^{-\omega r^2}\right)\right] +$$

$$\left. + \frac{2}{e^2} n^2 \lambda^2 e^{-\lambda r^2} + \frac{e^2 v^4}{8d^2} r^2 e^{-2\omega r^2} + \frac{g}{d} v^4 (1-\omega r^2)^2 e^{-2\omega r^2} \right\} \tag{4.40}$$

If we introduce the adimensional quantities:

$$\xi = \sqrt{\omega}\, r \qquad a = \frac{\lambda}{\omega} \qquad b = \frac{v^2}{\omega} \tag{4.41}$$

(4.40) becomes:

$$\frac{m}{\pi v^2 L_z} H = \frac{1}{d}\int_0^\infty d\zeta \left\{ \frac{1}{2}\left[\zeta \frac{(2-\zeta)^2 e^{-2\zeta}}{1-(1-\zeta)e^{-\zeta}} + \frac{1}{\zeta}\left(ne^{-a\zeta} - \frac{\pi}{k} b\zeta e^{-\zeta}\right)^2 \left(1-(1-\zeta)e^{-\zeta}\right)\right] + \right.$$

$$\left. + \frac{2mdn^2}{e^2}\frac{a^2}{b} e^{-2a\zeta} + \frac{m}{8d}\frac{e^2}{v^2} b^2 \zeta e^{-2\zeta} + mgb(1-\zeta)^2 e^{-2\zeta} \right\} \tag{4.41}$$

here we have performed the angular integration and put $\xi^2 = \zeta$. The first term yield a positive number (~ 0.8) and we will not care about it for our minimization. All other terms yield:

$$\frac{md}{\pi v^2 L_z} H = \frac{1}{2}\left[\frac{23}{108}\frac{\pi^2}{k^2} b^2 - \frac{2n\pi}{k}\frac{5+2a}{(a+1)(a+2)^2} b + \frac{n^2}{1+2a} + \log\left(1 + \frac{1}{2a}\right)\right] +$$

$$+ \frac{dmn^2}{e^2}\frac{a}{b} + \frac{me^2}{32 v^2 d} b^2 + \frac{gm}{4} b \tag{4.42}$$

We have found the values of a and b that minimize this expression using numerical methods. With the usual (4.24) values of the parameters we get:
for the case of the vortex ($n=1$)

$$a = 7.26 \cdot 10^{-6} \qquad b = 0.040 \tag{4.44}$$

which correspond to:

$$\omega = 0.101\text{Å}^{-2} \qquad \lambda = 7.336 \cdot 10^{-7}\text{Å}^{-2} \tag{4.45}$$

for the case of the antivortex ($n = -1$)

$$a = 6.27 \cdot 10^{-6} \qquad b = 0.034 \tag{4.46}$$

which correspond to:

$$\omega = 0.117\text{Å}^{-2} \qquad \lambda = 7.337 \cdot 10^{-7}\text{Å}^{-2} \tag{4.47}$$

From these results we get the values of the dimensions of the vortex and of the fluctuations of the magnetic field:

$$\frac{1}{\sqrt{\omega}} = 3.147\text{Å} \qquad \frac{1}{\sqrt{\lambda}} = 1167.54\text{Å} \tag{4.48}$$

for the vortex, and

$$\frac{1}{\sqrt{\omega}} = 2.924\text{Å} \qquad \frac{1}{\sqrt{\lambda}} = 1167.46\text{Å} \tag{4.49}$$

for the antivortex. Notice that for the vortex and for the antivortex we have got different results as it should have been expected since our system is chiral.
We see that again the behaviour is that typical for a type II superconductor.

4.2.1. Currents

We have said in section 4.2 that around the vortices circulate a superconducting current. Here we evaluate such current. From the definition of current, equation (2.13), it is straightforward to check that:

$$\begin{aligned} J_r &= 0 \\ J_\theta &= \frac{e}{m}\left(v^2 - v^2\left(1 - \omega r^2\right)e^{-\omega r^2}\right)\left(\frac{n}{r}e^{-\lambda r^2} - \frac{\pi}{k}v^2 r e^{-\omega r^2}\right) \end{aligned} \tag{4.50}$$

From (4.50) we can compute the total current flow:

$$I = \int_0^\infty dr\, J_\theta = \frac{ev^2}{m}\left[\frac{n}{2}\frac{\omega}{\omega + \lambda} - \frac{3}{8}\frac{\pi v^2}{k}\frac{1}{\omega} + \frac{n}{2}\log\left(\frac{\omega}{\lambda} - 1\right)\right] \tag{4.51}$$

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Tables caption

1: Plots of the adimensional quantities $G(\xi)$ and $a(\xi)$, solutions of equation (2.23) for $|n|=1$.

2: Plots of the adimensional quantities $G(\xi)$ and $a(\xi)$, solutions of equation (2.23) for $|n|=2$.

3: Comparison between the theoretical prediction of the radii of the self-dual antivortices for different values of n as from equation (2.41) (continuous line) and the same number measured on the numerical solutions of equation (2.23) in correspondence to $|\psi|=\frac{v}{2}$ (dots).

4: Plot of the current density $J_\theta(\xi)$, see equation (2.54), $(n=-1)$.

5: Plot of $g^2(\xi)$, see equation (2.88), for the antivortex case $(n=-1)$.

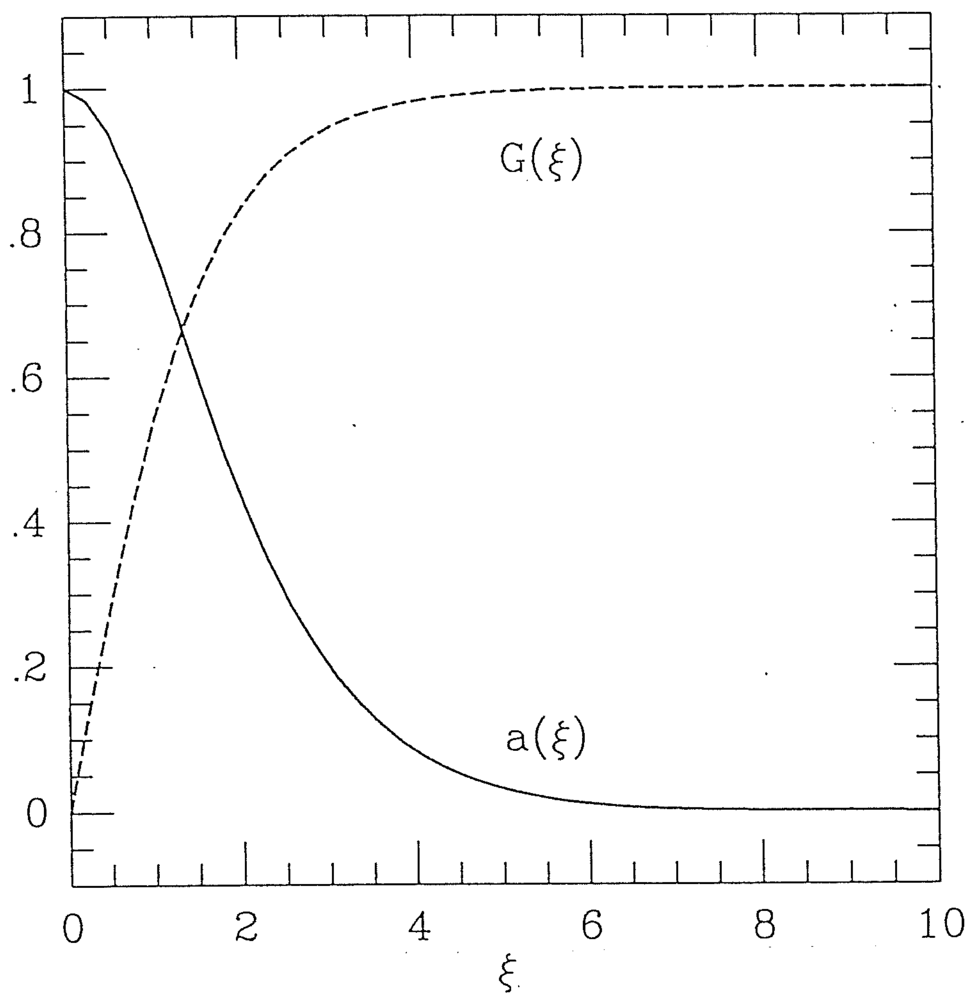
6: Plot of $erA_\theta(\xi)$, see equation (2.88), for the antivortex case $(n=-1)$.

7: Plot of $g^2(\xi)$, see equation (2.88), for the vortex case $(n=1)$.

8: Plot of $erA_\theta(\xi)$, see equation (2.88), for the vortex case $(n=1)$.

9: Plot of $\delta\rho(x)$ as from equation (4.32).

10: Plot of $B(x)$ as from equation (4.36).



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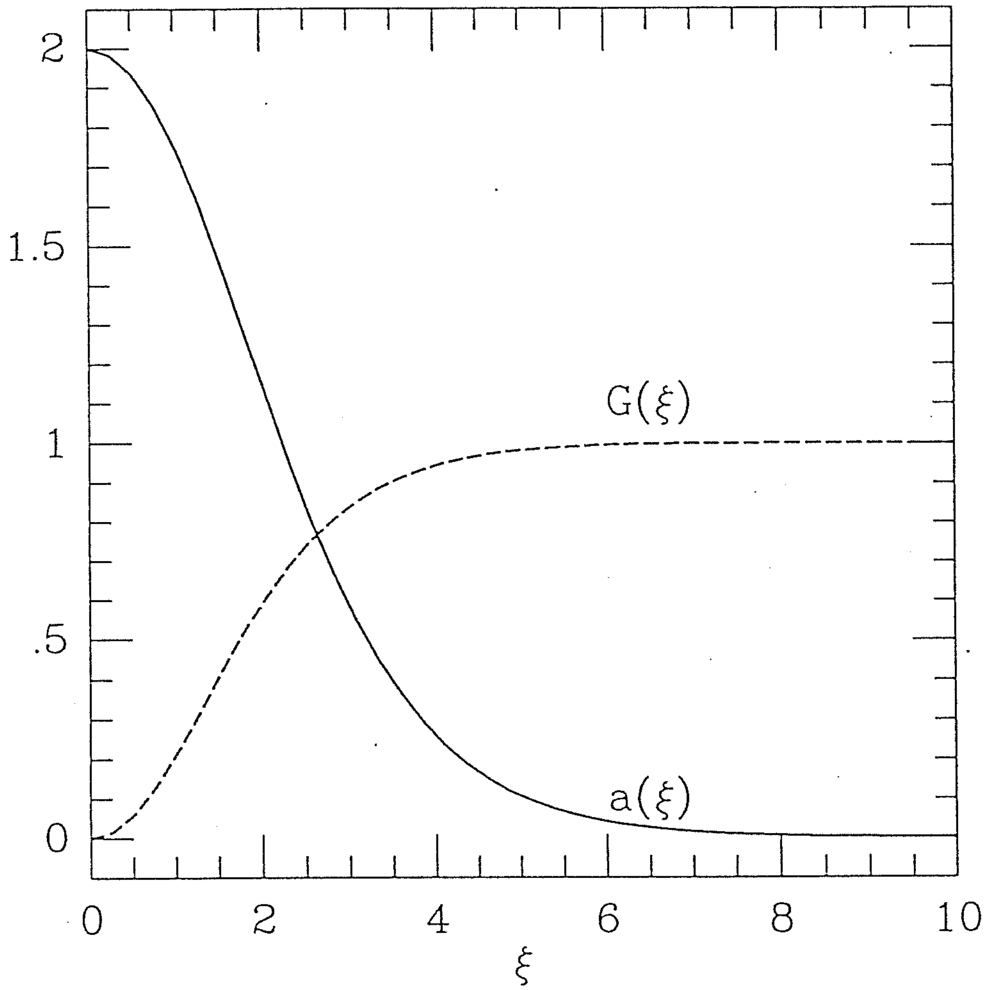
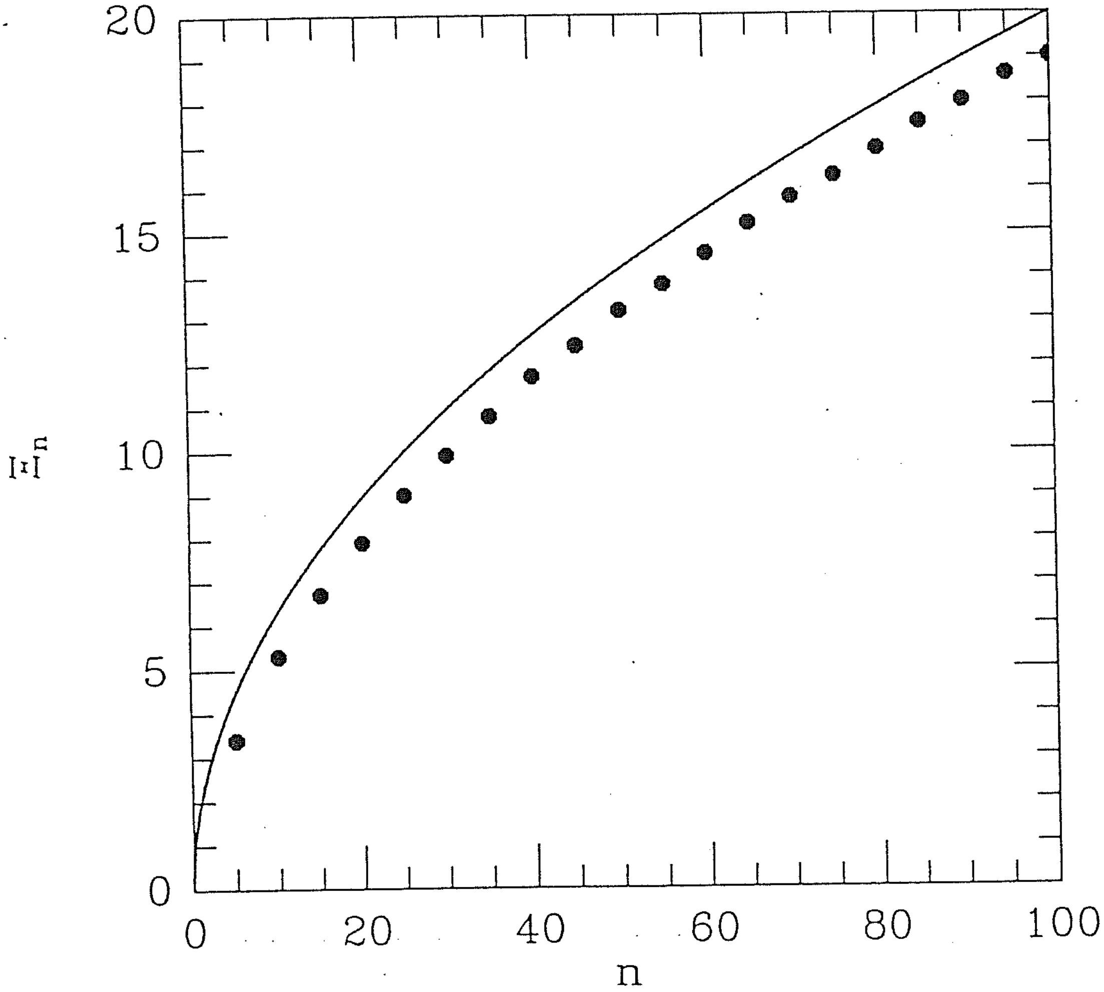


Table 2



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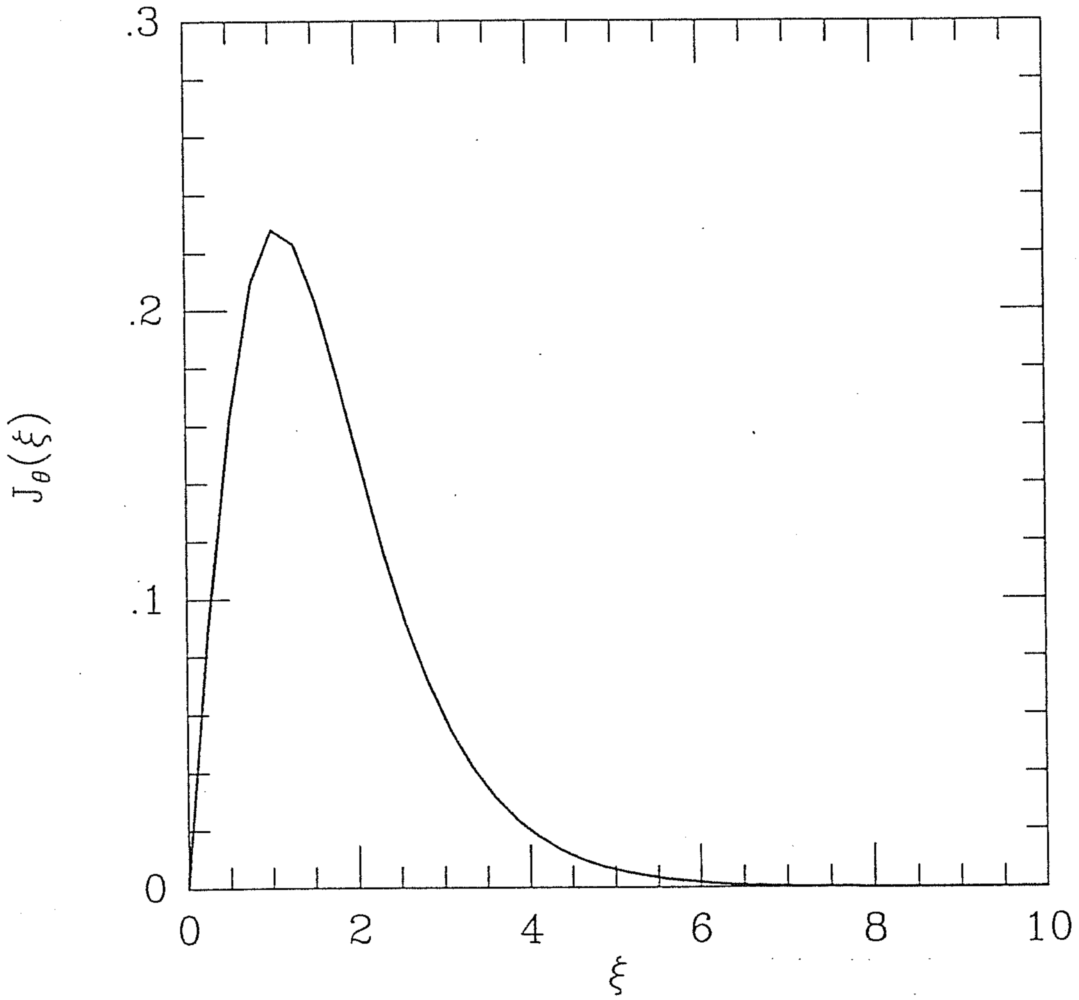

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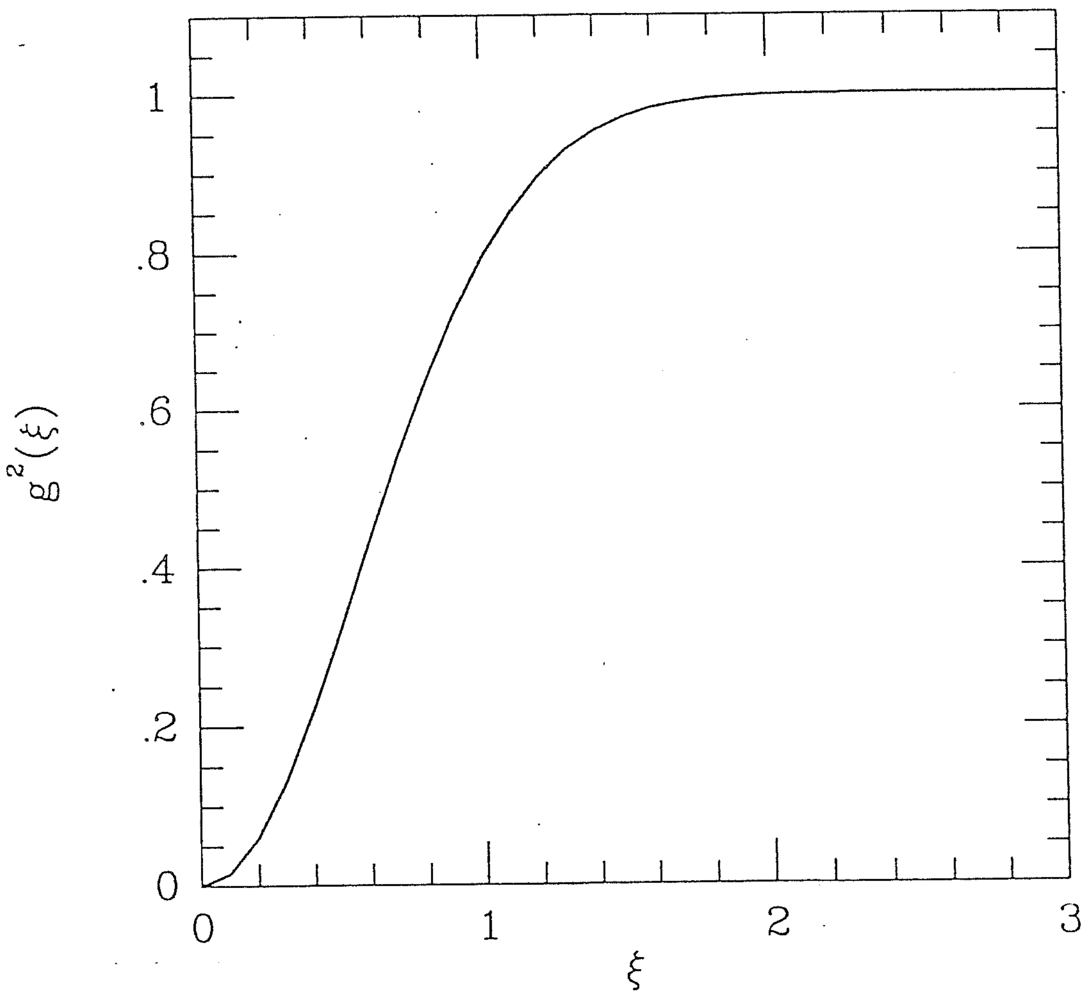
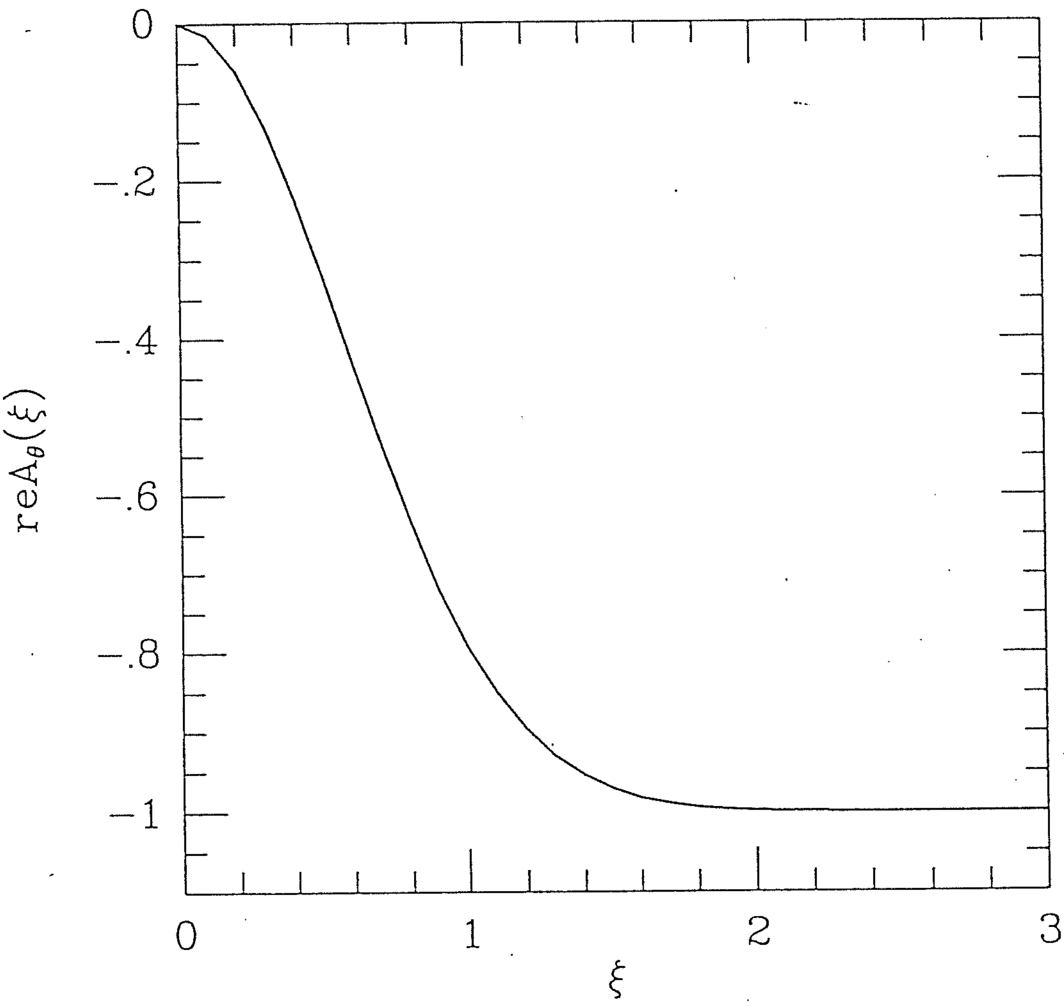


Table 5



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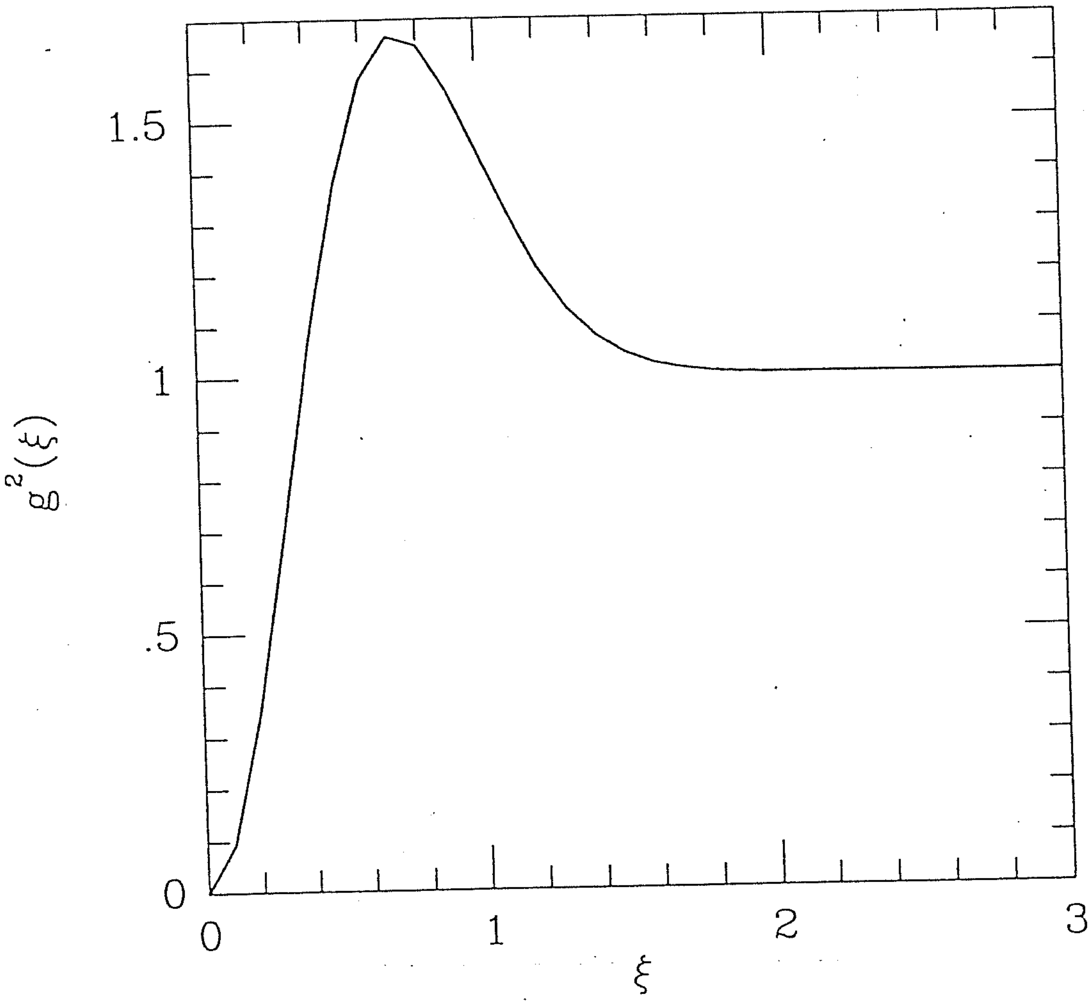

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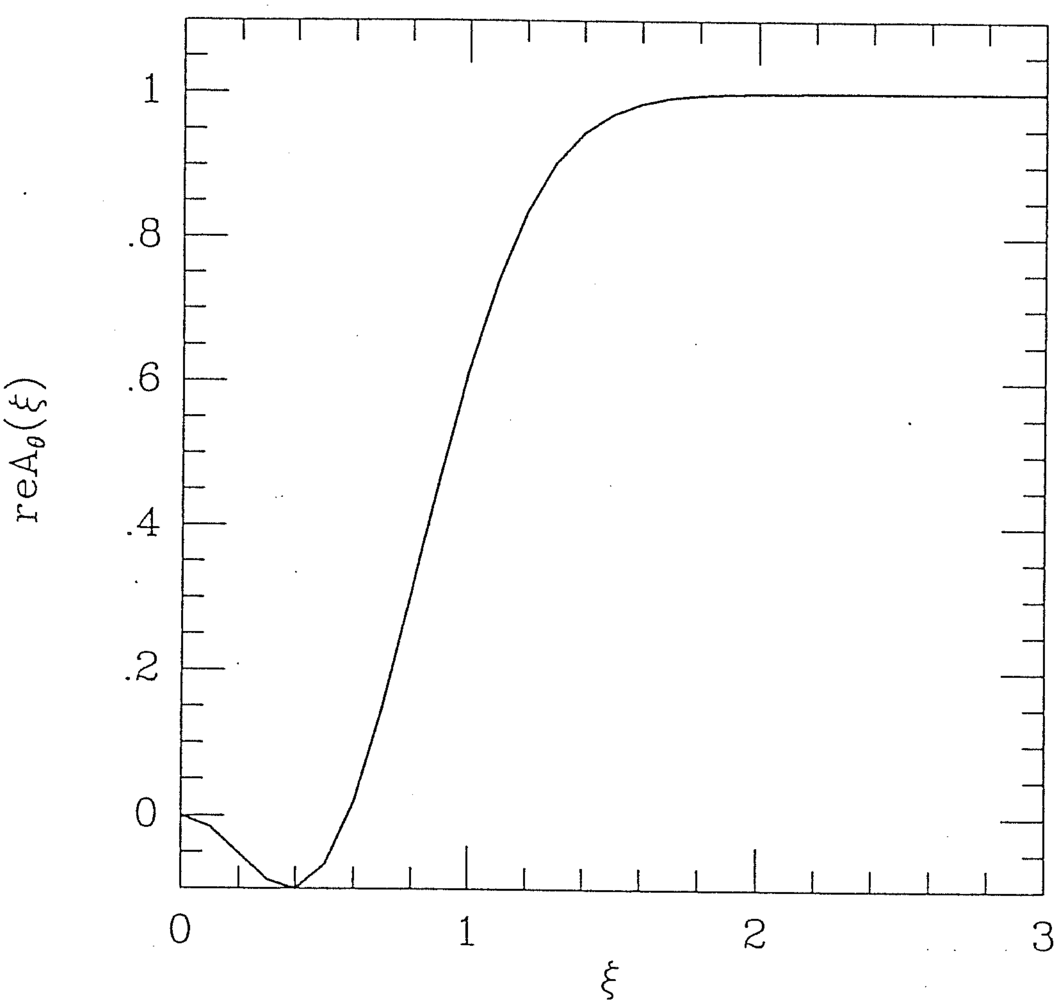

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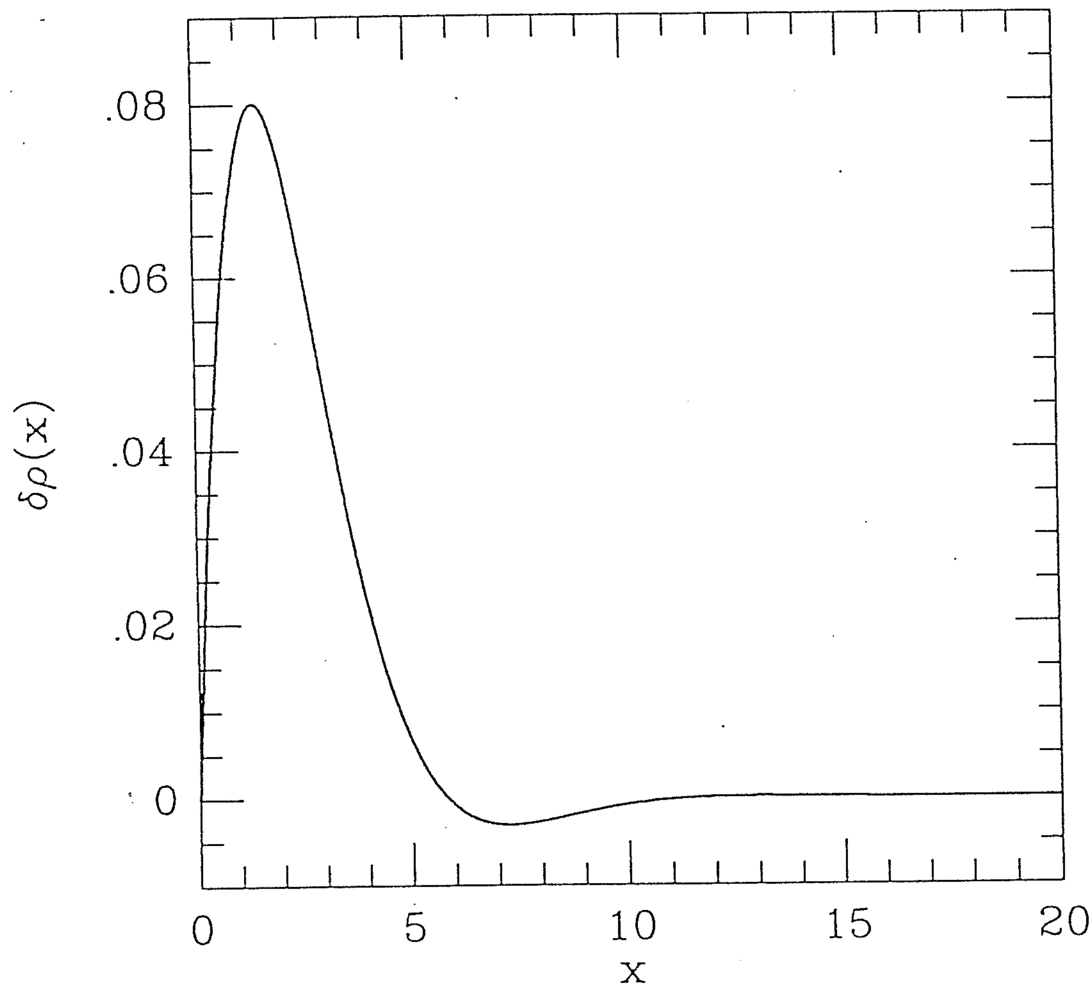

Table 9

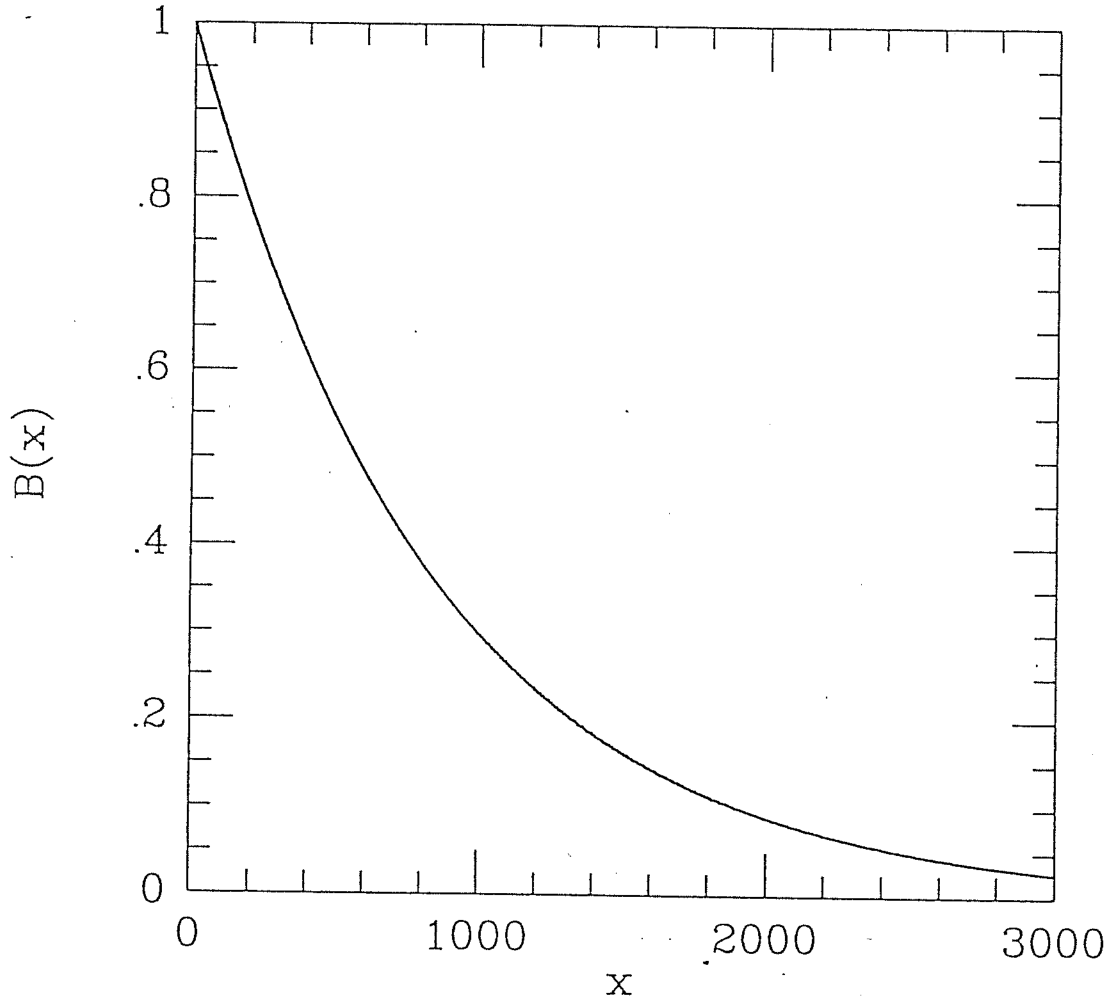

Table 10