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ON THE MAPPING CLASS GROUP OF SPHERICAL ORBIFOLDS

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INTRODUCTION

In this manuscript techniques of 3-dimensional topology are used to (try to) prove the following results.

Proposition 1 Let $G \subset Diff^+(S^3)$ be a finite group of diffeomorphisms acting nonfreely on S^3. Then the mapping class group of the orbifold S^3/G, $MC^+(S^3/G)$, is a finite group.

(here $MC^+(S^3/G)$=homeomorphisms of the orbifold S^3/G modulo isotopies which fix the singular set , see paragraph 0) .

Corollaries 1.6 and 1.8 Exists a finite group $\Gamma \subset Diff^+(S^3/G)$ which projects on $MC^+(S^3/G)$ with kernel either trivial or Z_2 .

As an immediate consequence one has

Corollary If G is a finite group of diffeomorphisms which acts nonfreely on S^3, any diffeomorphism $f:S^3 \to S^3$ for which $fGf^{-1}=G$ is G-equivariantly isotopic to a diffeomorphism of finite order .

If $G \subset SO(4)$, a first step towards a computation of $MC^+(S^3/G)$ is given by the following

Proposition 2 Let G be a finite group and let $F=\cup Fix\ g \quad g \in G \quad g \neq identity \quad F \neq \phi$:

if S^3-F is Seifert fibered , then $MC^+(S^3/G) \cong NG/ZG.G$

if S^3-F is hyperbolic (and not Seifert fibered) then $MC^+(S^3/G) \cong NG/G$

(where NG and ZG are the normalizer and the centralizer of G in SO(4))

The first obvious question looking at the above statements is why one assumes that G acts nonfreely. One has to distinguish two cases . If we know that a group with free action is conjugate to an orthogonal group , I believe that the above results continue to remain true for it (certainly prop. 1 is true). If it is not known if the given group is conjugate to an orthogonal group , then I believe that proposition 1 is unknown .

It is conjectured that

finite subgroups of diffeomorphisms in the 3-sphere are conjugate to orthogonal groups.

The conjecture has a positive solution if one considers groups with nonfree action (this is a special case of a more general result of Thurston ,see theorem 0.3 below) . In the free action case the problem has only partial positive answers (for an account of this case see [Thomas]).

We will use Thurston's result (in it's general form) only in the proof of proposition 2. The reason for which in all the manuscript we assume that F(=U Fix g $g \in G$ $g \neq$identity) is nonempty is that in this case S^3-F is a Haken manifold .

Here I try to explain what roughly Haken manifolds look like .

Consider for a moment compact surfaces . Except in the case of the disk and of the sphere, all surfaces have 1-dimensional submanifolds which are homotopically non trivial , which I call incompressible curves . If one considers a torus , for example , and cuts it along a meridian and a longitude , one gets a quadrangle . Then one can imagine to embed this quadrangle in the plain , to give it a geometric structure (in this case a flat one) making the opposite sides of the same length (i.e. to make it a rectangle) . Finally one can reglue all back to obtain a geometric structure on the torus . This process (i.e. to cut along incompressible curves and to embed somewhere the poligon obtained in this way) can be repeated with all other surfaces different from the sphere and the disk.

In 2-dimensional topology it is easy to prove that a compact surface with no incompressible subcurves is either the sphere or the disk . But suppose for a moment that this result where not known . Suppose that the classical uniformization theorem where not known. Then it would always be possible to prove along the lines which follow a uniformization theorem for surfaces which ,first , are such that all their subsurfaces with no incompressible subcurves are balls or disks , and ,second , contain incompressible subcurves :

(i) one could prove that it is possible , cutting the surface along a finite number of incompressible curves (a "hierarcy") to obtain a finite set of disks and spheres (this is the topological part of the proof) ;

(ii) then one could give geometric structures to the (topological) disks and reglue all back, taking care that the geometries overlap well on the common sides of couples of disks (this is the geometric part of the proof) .

The topological part of the proof here is to cut along well chosen curves to obtain simplier surfaces . If one cutted along compressible surfaces one would never simplify his surface.

Haken manifolds are the 3-dimensional version of the situation described above . They contain incompressible orientable surfaces (a translation of the concept of incompressible curve in a surface) ; they are irreducible (i.e. something which rules out the presence ,for example , of fake cells) .

Haken manifolds are "uniformizable" and the philosophy is to cut them along incompressible surfaces to obtain a set of balls , to give to the balls a geometric structure and to reglue all back (geometrization theorem for Haken manifols) .

We will use heavily the fact that Haken manifolds are geometrizable (along with other properties, for example the fact that , for orientation preserving homeomorphisms , homotopy and isotopy relations coincide).

Another obvious question is what happens in the two dimensional case of our propositions . We will assume the following

Theorem Let $G \subset \mathrm{Diff} S^2$ be a finite group . then $MC^+(S^2/G)$ is finite and the epimorphism $\mathrm{Diff}(S^2/G) \to MC^+(S^2/G)$ admits a section . Moreover G is conjugate to an orthogonal group and , assuming $G \subset O(3)$, one has an isomorphism

$MC^+(S^2/G) \cong NG/\langle ZG,G \rangle$

(NG and ZG are the normalizer and the centralizer of G in O(3)) .

Finally we give sketch of the proofs .

Proposition 1 If $F = \cup Fix\ g\ \ g \in G\ \ g \neq identity$ $(S^3\text{-int } N(F))/G$ where $N(F)$ is a tubular neighbourhood of F, is a Haken atoroidal manifold (by the positive solution of the Smith conjecture and it's generalizations contained in [Smith], and by equivariant loop theorem). Two cases can occure.

$(S^3\text{-int } N(F))/G$ is Seifert fibered. In this case the singular set F' of the orbifold S^3/G is formed by circles . This implies that the stabilizers of balls in S^3 do not contain subgroups isomorphic to A_5 . Therefore , without the assumption of Thurston's claim (here stated as theorem 0.3) S^3/G is irreducible both as a manifold and as an orbifold . The Seifert structure in $(S^3\text{-int } N(F))/G$ be chosen to extend in a foliation of circles in all S^3/G (if this does not happen at the "first attempt " one uses [Heil] to show that , by the irriducibility of S^3/G and the fact that as an orbifold (and consequently as a manifold) has finite first homotopy group ,indeed it is G , S^3/G is a lens space where F' is formed by cores for a toral Heegaard splitting . Therefore also in this case one concludes that the original foliation on $(S^3\text{-int } N(F))/G$ could be chosed to be extendable) . Then , by a result of Davies-Morgan (remark 1.5) , S^3/G is a Seifert fibered orbifold .

We show that in each mapping class there are fibre preserving homeomorphisms . Then we use the analysis of the mapping class group of a Seifert fibered manifold given in ch 9 of [Johannson] , to show that $MC^+(S^3/G) \cong MC(O)$, where O is the base orbifold. As O is a spherical 2-orbifold , $MC(O)$ is a finite group.

$(S^3\text{-int } N(F))/G$ is hyperbolic and not Seifert fibered .

If the volume is finite the result follows easily from Mostow rigidity theorem (every homeomorphism is homotopic to exactly one isometry) , a theorem of Waldhausen (which asserts that in Haken manifolds homotopy and isotopy relations coincide for orientation preserving homeomorphisms) and elementary hyperbolic geometry (in a hyperbolic manifold of finite volume the group of isometries is finite).

In the case of infinite volume one can double the manifold to obtain a Haken atoroidal manifold whose nonempty boundary is a union of tori , as in theorem 2' of [Zimmermann] . If the double is hyperbolic one reduces to a finite volume case which can be worked out as above. If the double is Seifert fibered, then, by classical theorems on incompressible surfaces in Seifert fibered manifolds (stated for example in ch VI of [Jaco]) it follows that the double is a surface bundle over S^1. It follows easily, considering the concrete situation in which we are involved, that S^3-int N(F) is a handlebody of genus 2. This implies that S^3/G has as a manifold a Heegaard splitting of genus 2 preserved up to isotopy by orbifold homeomorphisms $f:S^3/G \to S^3/G$. This induces a Heegaard splitting in S^3 preserved by the liftings of our f's. Then the result follows easily.

Moreover a necessary step before all the above work is to show that the restriction map gives a well defined immersion $MC^+(S^3/G) \to MC^+(S^3/G$- int N(F')). Here we sketch our strategy for the injectivity:

if $f:S^3/G \to S^3/G$ preserves M'=S^3/G- int N(F') and f|M' is isotopic to the identity, then $f|\partial N$, N=N(F'), is isotopic to the identity. After an easy isotopy, assume $f|\partial N$=identity. Then we define an isotopy in N constant in ∂N after which f|N=identity. Then, considered the isotopy in M' which makes f|M'=identity, we extend it in N (assuming without discussion that the isotopy in M' is standard in some sense in $\partial M'$).

In corollary 1.6 we show that if M' is hyperbolic then $MC^+(S^3/G)$ is realizable (i.e. embeddable in a natural way) in $Diff^+(S^3/G)$. First we realize it in MC+(M'). If we are not in the special case of the handlebody of genus 2 this step is immediate (each mapping class contains exactly one isometry (more exactly the extension also to the boundary of an isometry defined in the interior: the group action of the isometries in the interior extends naturally also to the boundary). Then associate to each mapping class it's isometry). In the special case we used the following geometric argument: first we realized $MC^+(S^3/G)$ in $Diff^+(\partial M')$ as a group of isometries for some hyperbolic stucture (using [Kerckhoff]). Then as all these isometries are extendable in M', we used uniformization theorem by Schottky groups (which says that the hyperbolic stucture in $\partial M'$ comes from a hyperbolic stucture defined in all M'), and we concluded that one can extend the action on the boundary in a isometric action in the interior of M' (using the fact that conformal homeomorphisms on the 2-sphere are naturally identified with isometries in the 3-dimensional hyperbolic space).

Having realized $MC^+(S^3/G)$ in $Diff^+(\partial M')$, one has a finite group of diffeomorphisms Γ acting on $\partial N(F')$ with the diffeomorphisms singularly all extendable. I consider a finite cover $X \to N(F')$ where X is a union of handlebodies with empty singular set. Then I lift $\Gamma \subset Diff^+(\partial N)$ in a $\tilde{\Gamma} \subset Diff^+(\partial X)$ and I extend it's action in the interior of X, with no change of the action of G (by means of equivariant loop theorem, one considers a G-equivariant hierarcy in X. Then one extends the action on this union of disks, cuts X along these disks, reduces to the case of a union of balls, extends the action in the interior of the balls and reglues all back). The extension of $\tilde{\Gamma}$ induces an extension of Γ.

In corollary 1.8 I reduce to the case where $G \subset SO(4)$ and G preserves the "standard" Hopf fibration and it's orientation, let's denote it by Φ (I call it standard here: I don't know if this it's standard name).If I indicate $\Phi P = \{$elements of SO(4) which preserve $\Phi\}$.

I use the sequences

$$1 \to Z_2 \to \Phi P \to O(2) x SO(3) \to 1$$

$$\text{(S3)} \qquad 1 \to SO(2) x \{1\} \to O(2) x SO(3) \to O(3) \to 1$$

which show how fibre preserving elements induce transformations in S^2 (up to conjugation). The second sequence splits.One has $MC^+(S^3/G) \cong MC(S^3/G_2)$ by proposition 1 (where G_2 comes from the first sequence where G projects to a subgroup of $G_1 x G_2 \subset SO(2) x SO(3)$). Then, first I realize $MC^+(S^3/G_2)$ in a $\Gamma \subset O(3)$ and then I embed $\Gamma \subset O(2) x SO(3)$. Finally I lift Γ in $\tilde{\Gamma} \subset \Phi P$. By the commutativity of the following diagram the result follows

$$\begin{array}{ccccc} 1 \to Z_2 \to & \tilde{\Gamma} & \to & \Gamma \\ & \downarrow & & \downarrow \\ & MC^+(S^3/G) & \cong & MC(S^3/G_2) \end{array}$$

Proposition 2 Lift the group Γ of corollaries 1.6 and 1.8 in a finite $\Gamma \subset Diff^+(S^3)$. Then, by Thurston's claim I assume , after a conjugation , $\Gamma \subset SO(4)$. Moreover I assume that G is fixed by this conjugation . Then $NG \to MC^+(S^3/G)$ is surjective and on has to fin it's kernel.

For S^3- int N(F) Seifert fibered , the kernel is $\langle ZG,G \rangle$ and one reduces to the 2-dimensional case to prove it .

For S^3- int N(F) hyperbolic then NG/G acts as a group of isometries in S^3/G-intN(F') to conjugation (if the action is free this follows from uniformization theorem for Haken manifolds; for not free actions it follows by Thurston's claim) . As nontrivial isometries are homotopically nontrivial (byelementary hyperbolic geometry the centralizer of $\pi_1(S^3$- int N(F) in PSL(2,C) is trivial) the kernel is exactly G.

Preliminaries 0 Here are collected definitions and statements of theorems of 3-dimensional topology which will be used later . General references for what follows are [Jaco] , [Scott] , [Smith] , [Takeuchi] , [Thurston] , [Waldhausen] .

Orbifolds Here are presented only 2-orbifolds without reflection points and orientable 3-orbifolds .

2-orbifolds Are pairs $O'=(|O'|,\Sigma O')$,where $|O'|$ is a surface , $\Sigma O'$ is a discrete subset of points of $|O'|$, called singularities , labelled with positive integers , the indices of singularity

Orientable 3-orbifolds Are pairs $O=(|O|,\Sigma O)$,where $|O|$ is an orientable 3-manifold ,ΣO is an embedded graph in $|O|$,whose points are called the singularities of O , whse edges are labelled with positive integers , the indices of singularity . Moreover ,ΣO is locally of the form group , i.e. if $A_5 \not\subset \Gamma$, then by the positive solution of the Smith conjecture and by others results which generalize it (contained in the book [Smith]) Γ is conjugate in $Diff^+(D^3)$ to an orthogonal group . If $A_5 \subseteq \Gamma$, the same result follows from the claim of Thurston.

(b) (see[Takeuchi]) By (a) O is irreducible . Let $O' \subset O$ be an orientable incompressible suborbifold and let $\Sigma = p^{-1}(O')$. If Σ were compressible , as it's components are not spheres , it would have compressing disks . By the equivariant loop theorem of Meeks and Yau this set of disks could be chosen to be G-equivariant , i.e.for $g \in G$ either $gD \cap D = \phi$ or $gD=D$. These disks would project into compressing discal orbifolds for O' .

A homeomorphism $f:O \to O'$ (in the orbifold sense) is a homeomorphism $f:(|O|,\Sigma O) \to (|O'|,\Sigma O')$ which preserves the indices .

Two homeomorphisms $f,g:O \to O'$ are homotopic (isotopic) if exists a homotopy (isotopy) $F:(|O|,\Sigma O) \times I \to (|O'|,\Sigma O')$ of f and g as maps between pairs) .

Theorem (0.2) ([Waldhausen]) Let M be a Haken 3-manifold . If $f:M \to M$ is homotopic to the identity and is orientation preserving , then it is isotopic to the identity .

Seifert fibered orbifolds An orbifold O (3-dimensional , orientable) is Seifert fibered if it is covered by 3-dimensional suborbifolds $\{O_i\}$ (i.e. their boundaries are in general position with ΣO and $\Sigma O_i = \Sigma O \cap O_i$)

with $O_i = T(p,q)/G$ where $T(p,q)$ is the fibered solid torus obtained cutting $D^2 \times S^1$ along $D^2 \times \{1\}$ and regluing back after having rotated one end through q/p of a full turn and G is a finite fibre preserving group of diffeomorphisms (T(p,q) and G vary with O_i) .

Moreover , the foliations defined locally , define a global foliation in O .

A 2-suborbifold $O' \subset O$ is boundary parallel if $|O'|$ can be isotopically moved in a component of $\partial|O|$ remaining at any step in general position with ΣO .

A hyperbolic orbifold O has its interior homeomorphic (in orbifold sense) to $\mathbb{H}^3/\Gamma$ ($\mathbb{H}^3$ is the simply connected model of the hyperbolic geometry , Γ is a dicrete group of isometries for $\mathbb{H}^3$) and is not Seifert fibered .

In a hyperbolic orbifold O there are no essential toral suborbifolds (i.e. 2-orbifolds finitely covered by tori , which are incompressible in O and are not boundary parallel)

(Of course the above definitions remain valid if one replaces everywhere the word orbifold with the word manifold)

There are , up to isometry , only eight 3-dimensional simply connected homogeneous spaces which have locally isometric (complete) quotients of finite volume :

S^3 , R^3 , SL(2,R) , Nil , S^2xR , H^3 , Sol .

A compact manifold whose interior is is locally isometric to one of the above spaces (with a complete metric) is said to have a geometry . Except T^2xI , which is both euclidean and hyperbolic , all geometrizable compact manifolds have at most one geometry .

A manifold turns out to be Seifert fibered iff it is the quotient by a group of isometries of one of the first five geometries in the list .

Geometrization theorem for Haken manifolds Let M be a Haken manifold .Then either $O \cong Sol/\Gamma$ ($\Gamma \subset Iso(Sol)$ where Sol is the 3-dimensional non abelian solvable group which covers compact homogeneus spaces , see [Scott]) or M admits a canonical (i.e. a ,maybe empty , maximal and unique up to isotopy) family of essential 2-dimensional tori which subdivides M in pieces which are Seifert fibered or hyperbolic .

The geometrization theorem for Haken manifolds says that Haken manifolds admit a "canonical" decomposition in geometric pieces. Thurston conjectured that this might be true for all 3-manifolds.

I state finally the claim of Thurston mentioned in the introduction (all mistakes in the formulation are of my responsability : I know very indirectly what Thurston claims)

Theorem 0.3 Consider a finite group of homeomorphisms (sufficiently regular) G that acts on an irreducible manifold M and assume that exists $g \in G$ $g \neq$ identity with $\dim \mathrm{Fix} g > 0$ .Then M admits a canonical decomposition in geometric pieces which is G-invariant. Restricted to the geometric pieces the elements of G are isometries .

Proof of proposition 1

Let $F = \cup \mathrm{Fix} g$ $g \in G$ $g \neq$ identity , $F' \subset S^3/G$ it's image , $N(F')$ a regular neighbourhood of F' , $M' = S^3/G - \mathrm{int}\, N(F')$.

Lemma 1 1

(i) For each class in $MC^+(S^3/G)$ exists a representative f such that $f(M')=M'$,and the restriction map $MC^+(S^3/G) \to MC^+(M')$ $[f] \to [f|M]$ is well defined .

(ii) The restriction map is injective .

Proof (i) Consider $g: S^3/G \to S^3/G$ a homeomorphism .As $g(F')=F'$ and by continuity , exists a regular neighbourhood N' of F' with $N' \subset \mathrm{int}\, N(F') \cup \mathrm{int}\, g(N(F'))$. As $\partial M' = \partial N(F')$ and $g(\partial M')$ are peripheral in $S^3/G - \mathrm{int}\, N'$, exists a regular neighbourhood N'' of F' with $N(F') \cup g(N(F')) \subset \mathrm{int}\, N''$. Then $\partial M'$ and it's image by g are two incompressible surfaces in an I-bundle over a closed surface . By [Waldhausen] $g\partial M'$ can be moved on $\partial M'$ by a global isotopy in $N'' - \mathrm{int}\, N'$ which is constant in $\partial(N'' - \mathrm{int}\, N')$.

If $f,g: S^3/G \to S^3/G$ are isotopic and both preserve M' , their restrictions in M' are homotopic and , by [Waldhausen] , isotopic in M' .

(ii) Consider $f: S^3/G \to S^3/G$ and assume $f|M'$ isotopic to the identity .

CLAIM f , after an isotopy , can be assumed to be the identity in $N(F')$.

Proof of the CLAIM $f|\partial M'$ is isotopic to the identity in $\partial M'$. Extending the isotopy in an isotopy in all S^3/G with support in a thin nbhd of $\partial M'$, we assume $f|\partial M'$ equal to the identity . Consider $N=N(F')$ and forget the rest . We will show that exists in N an isotopy between f and the identity constant on the boundary ∂N and which preserves F' .

We discuss the case of a component of N which is a solid torus (and which we call again N) .

1. Fix a section 1-punctured disk $D \subset N$.

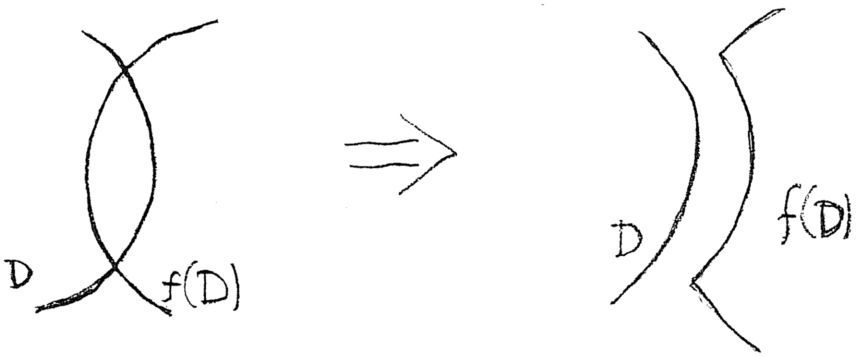
2. If $f(D)=D$ then f is isotopic to the identity in D by an isotopy constant on ∂D which preserves the puncture . Then cut N along D to obtain a couple (B,F) , where F is an unknotted segment in a ball B (unknotted becouse B retracts on F , as N was a regular nbhd of the related component of the singular set) . Now (B,F) is isomorphic to $(D^3$,z axis) , f is equal to the identity on the boundary of B and , arranging that $f(0)=0$ (here we identify conjugate situations) , one can apply the Alexander trick $f(x,t)=t\, f(x/t)$ to obtain an isotopy between f and the identity in (B,F) constant in ∂B . Finally glue back to obtain the wanted isotopy in N .

3. If $f(D)\neq D$ I procede as follows . First I assume the following "principle" :

PRINCIPLE. If two surfaces intersect , one can always move slightly them in order to make them mutually transverse , respecting whichever reasonable conditions on has for this move .

Therefore assume $f(D)$ and D to be transverse . If $f(\text{ int } D) \cap \text{int } D=\phi$, then $D \cup f(D)$ is a sphere and bounds a ball B in N . Then $(B,B\cap F')$ is isomorphic to $(D^3$,z axis) .(i.e. $B\cap F'$ is unknotted in B . This follows from the fact that , given $k=k_1\# k_2$, if k is unknotted , then k_1 and k_2 are unknotted .Apply this to our case , imagining N immersed in S^3 as an unknotted torus . Consider the fibration in D^3 (i.e. B) defined by the z-direction and consider a z-parallel isotopy which sends the hemisphere $f(D)$ onto D . Extend in an isotopy defined in all N with support in a nbhd of B with the associated time dependent field tangent to F' in F' .

If $f(\text{ int}D) \cap \text{int } D \neq\phi$, then it is a finite union of circles . Consider the inner circles . These are boundaries of disks in $f(\text{int } D)$ and $\text{int } D$. By an isotopy rel F' it is possible (with the above procedure) to erase these inner intersections (see the picture). After a finite number of moves , one reduces to the case $f(\text{int } D) \cap \text{int } D = \phi$.



We now discuss the case when N is a handlebody of genus greater than 1 . Consider a hierarcy of 1-punctured disks for N . Fix one disk D . Then , working exactly as before, make f=identity in D . Then cut N along D and do the same for another disk and so on . One reduces ,up to isomorphism , to the case $f:(D^3,\Sigma) \to (D^3,\Sigma)$, where Σ is formed by the three coordinate axis and f=identity in ∂D^3 . Applying the Alexander trick , one obtains an isotopy between f and the identity constant in the boundary and $\text{rel}\Sigma$ in D^3 . Then , reglue all back .

<u>Lemma 1.2</u>

(i) S^3/G is an irreducible non Haken orbifold

(ii) $|S^3/G|$ is an irreducible non Haken manifold with finite first homotopy group .

(iii) M' is an atoroidal Haken manifold .

<u>Remark</u> (i) and the general case of (ii) depend on the theorem 0.3 , i.e. on Thurston's claim. For propositon 1 only (iii) and a special case of (ii) , which don't depend on the mentioned theorem of Thurston , are really important .

<u>Proof</u> (i) As S^3 is irreducible and non Haken (i) follows from thm 0.1

(ii) $|S^3/G|$ is irreducible :

Given a sphere $S \subset |S^3/G|$, put it in general position with F' , to obtain an orbifold $O' \subset S^3/G$. If O' is a spherical orbifold , by (i) (i.e. by irreducibility of S^3/G) it bounds a ballic orbifold and therefore S bounds a ball . O' cannot be a flat or a hyperbolic sphere with three punctures, for otherwise it would be an incompressible suborbifold of S^3/G_(indeed in this case all closed loops on O' are already compressible in O') and by (i) we know that S^3/G it is non-Haken . Assume by induction that if a punctured sphere has strictly less than n punctures then it bounds a sphere . Suppose that O' has n punctures : considered a compressing disk for O' , make surgery along this diskand consider the two resulting suborbifolds O" and O''' .These are spheres with strictly less than n punctures and therefore both bound balls , say B" and B''' , in $|S^3/G|$. These balls are either disjoint or one of the two contains the other . In any case it is easy to find a ball B' with $\partial B'=|O'|$.

$|S^3/G|$ has finite first homotopy group . Indeed one has an epimorphism $\pi_1{}^{orb}(S^3/G)(\equiv G)\to\pi_1(|S^3/G)$.

(This last epimorphism can be explained as follows.

Every orbifold O of any dimension has a first fundamental group of homotopy defined roughly as follows :

1. O is simply connected , i.e. $\pi_1{}^{orb}(O)=1$, if there are no proper regular branching coverings $O'\to O$.

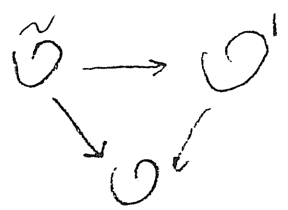
2. for every orbifold O exists a regular branching covering $O\to O$ where O is simply connected and which is unique (in the same sense one has unicity and iniversality for coverings in elementary topology).. Then $\pi_1{}^{orb}(O)$ is the group of deck transformations for this covering .

If |O| is a manifold , on considers the orbifold $(O',\Sigma O')$ defined as follows :

(a) $p:|O'|\to|O|$ is the universal covering of |O| .

(b) $\Sigma O'=p^{-1}(\Sigma O)$

Then remains induced a regular covering $p:O'\to O$ and one has the following commutative diagram of coverings :



(see [Smith] lemma 1.1 ch.X and [Thurston] ch.13))

(iii) M' is irreducible becouse any embedded sphere bounds a D^3/Γ in S^3/G which has no singular points , for it's boundary is a nonsingular sphere (if D^3/Γ had singular points , also it's boundary would have punctures . Indeed , if $\gamma\in\Gamma$ has nonempty fixed set , if it is contained in intD^3 , this set is a union of circles . But , assuming that γ has prime order p , this possibility is ruled out by the following inequality.(with coefficients in $\mathbb{Z}_p$)

$0<\text{rank } H_1(\text{Fix}\gamma)\leq \text{rank } H_1(D^3) + \text{rank } H_2(D^3)=0$).

As $\partial M\neq\phi$ M' is Haken ([Hempel] lemma 6.7)

The only nontrivial part of (iii) is to show that M' is atoroidal . Let $T\subset M'$ be an incompressible torus . By the proof of thm 0.1 (ii) , there is a compressing (punctured) disk for T in S^3/G . Making surgery along this disk one obtains a 2-punctured sphere in S^3/G. As the stabilizersof the components of it's counterimages in S^3 are cyclic (and therefore they are not isomorphic to A_5) , then this 2-punctured sphere bounds a ball whose nonempty singular set is an (unknotted) segment . This segment is contained in a circular component of F' and therefore T is boundary parallel in M' .

By the geometrization theorem :
Corollary 1.3 M' is either seifert fibered or a hyperbolic manifold

Lemma 1.4 $MC^+(S^3/G)$ is a finite group .

Proof M' is either Seifert fibered or hyperbolic . We distinguish two cases :

(1) M' is Seifert fibered .

CLAIM: M' admits a Seifert fibration Φ' which extends to a Seifert fibration Φ in $|S^3/G|$ with $F'\subset\Phi$.

Remark 1.5 By prop. 7.11 ch.X in [Smith] this implies that there is a Seifert fibration in the orbifold S^3/G which induces Φ' in M' .

Proof of the claim If a given Seifert fibration Φ' on M' does not extend to all $|S^3/G|$, then by [Heil] $|S^3/G|=L_1\#\ldots\#L_m\#(S^1xS^2)^{2g+n-q-1}$ where :

is the connected sum operation ,

L_i are nontrivial lens spaces and m is not less the number of singular fibres of M' ,

n is the number of components of F' ,

n-q>0 is the number of components of F' for which a meridian of a tubular neighbourhood is homologous to regular fibres and g is the genus of the orbit 2-orbifold

$|S^3/G|$ is irreducible (as the singular set is a union of circles , the stabilizers of G-invariant balls in S^3 are not A_5's , therefore the stated irreducibilty of $|S^3/G|$ follows as a special case of lemma1.2 (ii) , see the remark after the lemma) . This implies g=0 and n-q=1 . Therefore $|S^3/G|$ is a lens space . If k' is the only component of F' where the fibration Φ' of M' does not extend , then $|S^3/G|=(|S^3/G|-$ int N(k') $)\cup$N(k') is a Heegaard splitting . This lifts to a Heegaard splitting by tori in S^3 and therefore the Seifert orbifold S^3/G- int N(k') is finitely covered by a solid torus without singular points . Then the orbit orbifold of S^3/G- int N(k') has to be a punctured disk , i.e. S^3/G- int N(k') has at most one singular fibre . But then F' is formed by one or two connected components which are cores of a toral Heegaard splitting of $|S^3/G|$ and therefore the claim is proved .

By remark 1.5 S^3/G admits a Seifert structure and fibres on a spherical 2-orbifold O . O is either a 2 or 3 punctured sphere or a 1 punctured projective space .

If O is a 1 punctured projective space , there is a Seifert fibration in S^3/G whose orbit orbifold is a 3 punctured sphere of type $S^2(2,2,n)$. Indeed $M\cong K\tilde{\times}I$ (the twisted I-bundle over the Klein bottle) . $K\tilde{\times}I$ admits two Seifert fibrations (up to isotopy) which come from the only two fibrations in the double TxI which are invariant for the deck transformation . For one of these structures $K\tilde{\times}I$ is an S^1-bundle over the Moebius band . For the other $K\tilde{\times}I$ fibres over a disk with two singular fibres of index 2 (singular fibres are two disjoint 1-sided curves on the zero section $K\subset K\tilde{\times}I$, up to isotopy) . Both Seifert fibrations extend to all $|S^3/G|$ for , if fibers where meridians for N(F') , then one would have a manifold with infinite fundamental group (indeed , lifting the Dehn surgery in T^2xI , one would obtain S^2xS^1) which is not possible by lemma 1.2 .

We will refer now to [Johannson] chapter 9 .

Let $H^+(S^3/G)$= fiber preserving homeomorphisms modulo fiber preserving isotopies . One has a natural well defined immersion $H^+(S^3/G) \to MC^+(S^3/G)$. To see this it will suffice to show that if $f:S^3/G \to S^3/G$ is fibre preserving and isotopic to the identity in S^3/G , the isotopy can be chosen to be fibre preserving .

This can be shown as follows .

1. If $f:O \to O$ is a fibre preserving homeomorphism isotopic to the identity , where O is a Seifert orbifold with $|O|$ a solid torus and ΣO is either empty or a core circle , we show below that the isotopy can be chosen fibre preserving .

We have a fibration $O \to D(n)$ and

$$\begin{array}{ccc} O & \xrightarrow{f} & O \\ \downarrow & & \downarrow \\ D(n) & \xrightarrow[\bar{f}]{} & D(n) \end{array}$$

$\bar{f}$ is isotopic to the identity (here $n=r.p$ where r is the index of singularity of ΣO and p comes from the isomorphism as Seifert fibered manifolds , $T(p,q) \cong |O|$). Consider an isotopy $F: D(n)xI \to D(n)$. Lift it in a Z_p-equivariant isotopy $\tilde{F}: D(r)xI \to D(r)$ between a lift of f and the identity , for the covering $D(r) \to D(n)$. Embed properly $D(r)$ in O as an incompressible suborbifold . Consider the diagram

$$\begin{array}{ccc} \tilde{F} : D(r)xI & \to & D(r) \\ \downarrow & & \downarrow \\ \tilde{\tilde{F}} : OxI & \to & O \end{array}$$

$\tilde{\tilde{F}}$ is an isotopy defined as follows :

cut O along $D(r)$. In such a way one obtains a I-fibered ball D^2xI . One has an isotopy $\tilde{F}_1:D^2x\{0,1\}xI \to D^2x\{0,1\}$. The fact that F was equivariant with respect to the holonomy $D(r) \to D(r)$ implies that $\tilde{F}_1$ can be extended linearly in an isotopy $\tilde{F}_1 : D^2xIxI \to D^2xI$. Gluing back one obtains a well defined $\tilde{\tilde{F}} : OxI \to O$.

So far we have changed f ,by means of a fibre preserving isotopy which preserves the singular set , in a map which induces the identity in the base orbifold .

Now consider again the disk $D(r) \subset O$. We would like to change f ,by a fibre preserving isotopy , in such a way that $f(D(r))=D(r)$ .Assume that this had been already done . Then cut O along $D(r)$ to obtain a $D^2 x I$. As $\bar{f}: D^2(n) \to D^2(n)$ is the identity , after one more isotopy ,i.e.a rotation around the core curve of the solid torus O (which is the singular fibre) , one can cut O along $D(r)$ and reduce to a $f: D^2 x I \to D^2 x I$ where f preserves the I-foliation and is the identity in the upper and lower disks . Then , using the linear structure of $D^2 x I$, define a fibre preserving isotopy between f and the identity .Then glue back .

If $f(D(r)) \neq D(r)$, assume first that $f(D(r)) \cap D(r) = \phi$. It is possible to define a global fibre preserving isotopy in O which projects into the identity in $D(n)$ and sends $f(D(r))$in $D(r)$, as follows :

$f(D(r))$ and $D(r)$ are sections for O . Now find a third section disjoint from them and cut O along it , to obtain a $D^2 x I$ .Here $D(r)$ and $f(D(r))$ are two graphs . Then it is trivial to define a fibre preserving isotopy in $D^2 x I$ which is constant in the upper and lower disks sending $f(D(r))$ into $D(r)$.

If $f(D(r)) \cap D(r) \neq \phi$ we assume that they intersect transversely . Then one reduces to the case $f(D(r)) \cap D(r) = \phi$ using the arguments contained in lemma 1.1 (ii) (taking care of the fact that the moves now have to be fibre preserving and $f(D(r)) \cap D(r)$ contains also segments , but these are not real problems) .

among other things we have showed that isotopies indisks of the base orbifold "lift".

2. If $f: S^3/G \to S^3/G$ is isotopic to the identity , the first step allows us to assume that it is equal to the identity in a fibered nbhd of each singular fibre

(Each singular fibre is preserved by f . This is of course true for the components of $\Sigma(S^3/G)$ .Moreover f does not interchange singular fibres of M' for otherwise the induced map $\bar{f}$ in the base orbifold would be a homotopically trivial map which interchanges two punctures . This last thing never occures for then the lifting to the universal covering $\tilde{\bar{f}}: \tilde{O} \to \tilde{O}$

would interchange fixed points of different orbits,i.e conjugating nonconjugate subgroups of $\pi_1^{orb}(O')$ (here O' is a good orbifold) .

If $D(n) \subset O$, $p : S^3/G \to O$, is the projection of the "candidate" fibered nbhd N , choose a grater disk D' such that $f(D(n)) \subset D'$. Then for a moment forget S^3/G and consider only $N' = p^{-1}(D')$. It is possible to change f in such a way that f(N)=N (lift an isotopy of D'). After this forget N' and consider only N . Here isotop f to the identity as in the first step . Finally extend these isotopies in all S^3/G in a standard way).

At this point consider $M = S^3/G - N$ where N is the union of fibered nbhd's of the singular fibres (M is an S^1-bundle) . Assume f to preserve M and to bethe identity in N .Then $f:M \to M$ is homotopic and , by [Waldhausen] ,isotopic to the identity in M. I assume (without further discussion here) that the isotopy is , of the following form in the boundary $\partial M = \partial N$:

$$\begin{array}{c} \partial N \times I \to \partial N \\ \downarrow \quad \downarrow \\ \Phi : \partial T(p,q) \times I \to \partial T(p,q) \end{array}$$

$\Phi((x,y),t) = ((\exp t\, 2\pi i \alpha)\, x, (\exp t\, 2\pi i \beta)\, y)$ α and β are rationals (here assume that $\Sigma N \to \{0\} \times S^1$ (here N is one component of N) .

CLAIM : $H^+(S^3/G) \cong MC^+(S^3/G)$

<u>Proof</u>

1. Assume O to be a 3-punctured sphere . Then we will show that all $f:M' \to M'$ preserve the fibration , up to isotopy.

The group defined in $\pi_1(M')$ by the fibre is characteristic. Moreover , for each component $T \subset \partial M'$, $\pi_1(T)$ embeds in $\pi_1(M')$ (see the presentation of $\pi_1(M')$). These two observations imply that $f: \partial M' \to \partial M'$ preserves the fibration induced in $\partial M'$ up to isotopy (this a well known fact for 2-tori) . This implies f can be changed , by an isotopy with support in an very thin nbhd of ∂N in order to make $f|\partial N$ fibre preserving .

Now we consider the following general situation :

Let M be a Seifert fibered manifold with nonempty boundary and let $f:M\to M$ be a homeomorphism whose restriction on the boundary preserves the induced fibration. Then $f:M\to M$ preserves the fibration on M up to isotopy .

SKETCH of proof

(i) $M=T(p,q)$. Let $\varphi:M\to M$ fibre preserving which equals f on the boundary and preserves the core(I don't discuss the existence of this φ) . Then consider $\varphi^{-1}f$ i.e. reduce to the case where $f:M\to M$ is the identity on the boundary . Now we content ourselves to sketch the argument .

By procedures analogues to the ones discussed earlier , fix a trasversal incompressible disk D and define an isotopy after which $f(D)=D$. Then, a further isotopy to make f=identity in D . Finnally reduce to the case of I-bundles on the disk and ,then , reglue back to obtain the wanted isotopy which turns out to be constant on ∂M .

(ii) In the general case we proceed as follows:

1. we assume without discussion that it is possible to change f in order that , considered $f|\partial M$, the induced map $f:\partial O\to\partial O$ is of finite order .

2. Then we assume the existence of a finite family of mutually disjoint vertical tori and annuli which subdivide M in a union of solid tori and which has the following property:

after a modification of f , the above family of surfaces is f-invariant .

Make the restriction of f in this family of verical annuli and tori fibre preserving .Then solve the problem for couples formed by solid tori V and $f(V)$ (working as in the above case (i)).defining isotopies which are constant on the boundaries . Finally glue all toghether .

This solves the case when the base orbifold O is a 3-punctured sphere .

2. If O is a 2-punctured sphere then $|S^3/G|$ is a lens space . The two or one components of F' are cores of a toral Heegaard splitting and with no harm one can assume that our $f:S^3/G\to S^3/G$ preserves the splitting . Then one has the following situation (we take from [Bonahon]):

f is isotopic relF' to an element of the group generated by (here we fix a parametrization $V_1 \cong V_2 \cong D^2 x S^1$ of the two solid tori for which our initial Seifert fibration reduces to a standard one) :

(i) the identity

(ii) the involution $\tau V_i \to V_i$ $\tau(u,v)=(v,u)$

(iii) the involution σ which interchanges the tori $\sigma(u,v)=(u,v)$ (σ exists iff $|S^3/G|=L(p,q)$ with $q^2=1$ [p] and F' has two components with the same index).

Indeed if f preserves each torus then homologically on the middle torus it is either the identity or it's inverse , i.e. it is isotopic in T either to the identity or to τ . Make it actually equal to one of them in T in the standard way . Then arguments described before allow us to conclude . If f interchanges the V's , simply compose it with σ to reduce to the previous case.

It is trivial that identity, σ, τ , $\sigma\tau$ are not mutually isotopic relF' and that $\sigma\tau=\tau\sigma$. Therefore we are done.

We have proved the following thing (among others) :

If O is a 2-punctured sphere then

either $H^+(S^3/G)=Z_2$

or $H^+(S^3/G)=Z_2 \oplus Z_2$ (exactly when $q^2=1$ [p]) .

This concludes the proof of the claim .

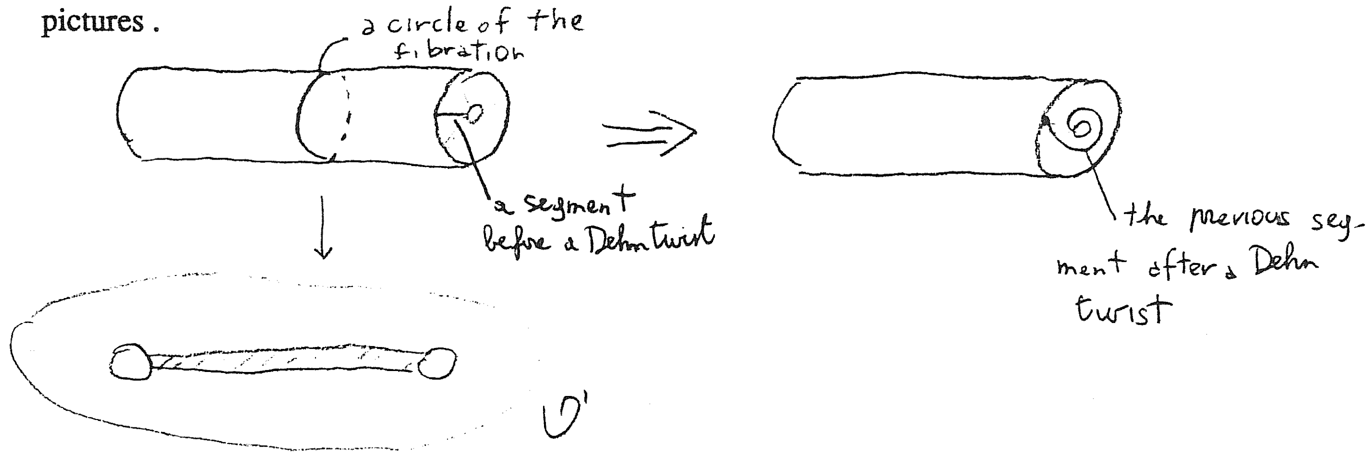
Now one has an obvious map $H^+(S^3/G) \to MC(O)$

CLAIM : the above map is an isomorphism .

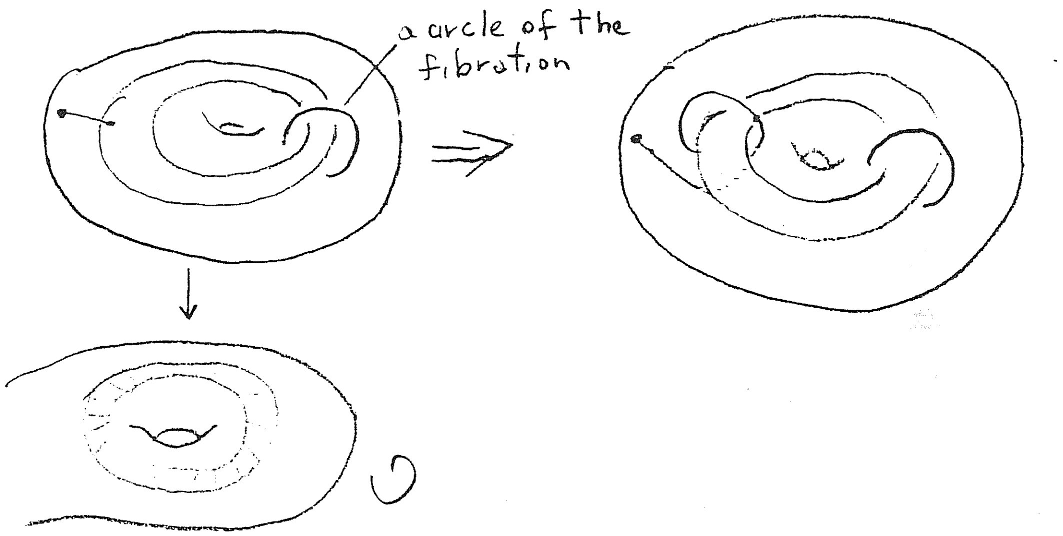
Proof of the claim .

Injectivity .Let $f:S^3/G \to S^3/G$ be fibre preserving and suppose that projects to a map $f:O \to O$ which is isotopic to the identity. After a fibre preserving isotopy it is possible to assume that f preserves M' . In [Johannson] is proved that $f|M'$ is isotopic to a composition of maps Which are Dehn twists around a finite family o vertical annuli and tori which cover a family of mutually disjoint essential curves which form a system of cuts for O' (i.e. it is a minimal family of curves such that , cutting O' along them , one obtains a disk) ,see the pictures .

(a)



(b)



In our case there are no essential simply closed curves in O' (i.e. homotopically nontrivial and not peripheral) and nontrivial Dehn twists around annuli do not extend to all S^3/G . Therefore $f|M'$ is isotopic to the identity . We have showed already that this is enough to conclude that f is isotopic to the identity in all S^3/G .

<u>Surjectivity</u> . Given $f:O\rightarrow O$, chosen a disk for each puncture and one more disk of regular points , it is possible to assume , after an isotopy , that f preserves the disks . Call O" the complement of the disks and M" its preimage in S^3/G . Then M" is a trivial bundle over O". Fix a structure $M''=O''\times S^1$. The first define f in the zero section , and then extend it in all

M". This map $f:M''\rightarrow M''$ extends also in the tori of $|S^3|-M''$ (it extends at least in a nonfibre preseving way becouse , as it preserves the zero section and is fibre preserving , the meridians of the solid tori are invariant . To make fibre preserving in the wanted way also in the tori , define in each of them an isotopy as in the above claim) .

Therefore $MC^+(S^3/G)\cong MC(O)$ .As $MC(O)$ (it is isomorphic to a spherical group acting on the 2-sphere) is a finite group , we are done .

(2) M' is hyperbolic

If volume $M'<\infty$, as a consequence of Mostow's rigidity theorem and of theorem 7 of [Waldhausen] , any class of $MC^+(M')$ is represented by one and only one isometry (this is a consequence of Mostow's rigidity theorem ; the unicity is an elementry fact of hyperbolic geometry : as $\pi_1(M')$ has trivial centralizer in PSL(2,C) , except in the case it is an abelian group , which certanly is not our case , two isometries cannot be homotopic) and the epimorphism $Diff^+(M')\rightarrow MC^+(M')$ admits a section (to each class associate it's isometry), what is meant by saying that $MC^+(M')$ is realized in $Diff^+(M')$. As the group of isometries of a hyperbolic manifold of finite volume is finite (by elementary hyperbolic geometry) $MC^+(M')$ is finite .

If volume$M'=\infty$, i.e. if $\partial M'$is not a union of tori , one cannot apply directely Mostow's rigidity . One can work out this difficulty repeating the argument in the proof of thm 2' in [Zimmermann] . Here is the construction :

∂M consists of pieces of the following type :

(i) tori , which come from the S^1 components of F'

(ii) spheres with three holes , coming from points in F' where three branches meet

(iii) annuli , which connect pieces of type (ii) .

Let $\partial_0 M'$ consist of all pieces of type (ii) and $\partial_1 M'$ of all pieces of type (i) and (iii) and $\partial M'=\partial_0 M' \cup \partial_1 M'$.

CLAIM (a) M' is irreducible and atoroidal .

(b) Every embedded disk $(D,\partial D)$ in $(M',\partial_0 M')$ is isotopic rel ∂D into $\partial_1 M'$.

(c) Every essential embedded annulus $(A,\partial A)$ in $(M',\partial_0 M')$ is isotopic in $(M',\partial_0 M')$ into $\partial_1 M'$.

(a) follows by lemma 1.2 .

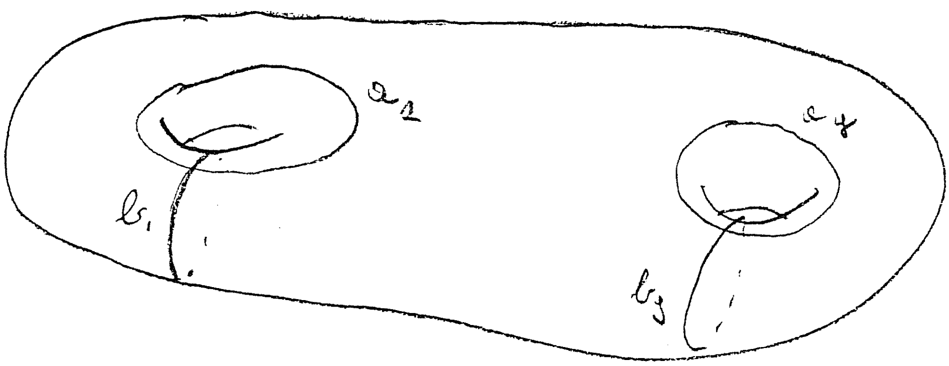
(b) If $(D,\partial D) \subset (M',\partial_0 M')$ is a properly embedded disk and if it not $\partial_0 M'$-parallel (i.e. if D does not compresses , by means of a homotopy , on $\partial_0 M'$) then ∂D is boundary parallel in $\partial_0 M'$ (as $\partial_0 M'$ is a union of 3-holed spheres) . But then one can build easily a 1-punctured sphere , call it O , in S^3/G . Now , O is a bad 2-suborbifold embedded in a good 3-orbifold . This is not possible .

(c) If $(A,\partial A) \subset (M',\partial_0 M')$ is an incompressible annulus , it's two boundary curves are parallel in $\partial_0 M'$ to two (different) holes . But then , gluing along the boundaries two punctured disks , one obtains from A a 2-punctured sphere in S^3/G . It must be good (i.e the two indices are equal) and , by lemma 1.2 (ii) (and here there are no A_5's) , it bounds a ball whose nonempty singular set is a segment . The ball is a regular nbhd for this segment and A is $\partial_1 M'$ parallel .

Let D_0M' be the double of M' along ∂_0M'. Then D_0M' is a 3-manifold whose boundary consists of a nonempty union of tori and (a), (b), (c) imply that it is irreducible and atoroidal. Therefore D_0M' is either Seifert fibered or hyperbolic.

<u>First case: D_0M' is Seifert fibered</u>.

Then M' is homeomorphic to $Sx[0,1]$, where S is a sphere with three holes (see [Jaco] ch. VI for what follows. Incompressible surfaces in Seifert fibered manifolds are either vertical or horizontal. ∂_0M' is horizontal .Each component of ∂_0M' does not separate D_0M'.This implies that D_0M' is a surface bundle over S^1 (the components of ∂_0M' are fibres). As ∂_0M' separates D_0M' in two components, ∂_0M' has exactly two components) .Then $|S^3/G|=N(F')\cup M'$ is a Heegaard splitting of genus 2, preserved (up to isotopy) by all $f:S^3/G\rightarrow S^3/G$. This decompositon lifts to a Heegaard splitting $S^3=N(F)\cup M$ and all liftings f of f preserve each of the handlebodies (assuming that f does). As all Heegaard splittings of S^3 of a given genus are isotopically equivalent (and therefore in what follows one considers the "better" Heegaard splitting that one can imagine), and as for a handlebody H one has an immersion $MC^+(H)\rightarrow Out\ \pi_1(\partial H)$,$MC^+(S^3,F)$ immerses in a group of permutations S_{4g} (imagine to have the crosscuts a's and b's in the picture).



For each class of $MC^+(S^3/G)$ choose a representative which fixes $\partial M'$ .If the following diagrams commute

$$1 \to \pi_1(\partial M,x) \to \pi_1(\partial M',x') \to G \to 1$$

$$\alpha\downarrow \qquad \downarrow f_* \qquad \downarrow\beta$$

$$1 \to \pi_1(\partial M,y) \to \pi_1(\partial M,y') \to G \to 1$$

then f_* is uniquely determined by α and β , for the centralizer of surface groups are trivial . Therefore $MC^+(S^3/G)$ is finite .

<u>Second case : D_0M' is hyperbolic .</u>

For each class in $MC^+(M')$ there are representatives which preserve the decomposition $\partial M'=\partial_0 M' \cup \partial_1 M'$ (here I assume this without further discussion) .

By very hard arguments contained in the proof of the geometrization theorem for Haken manifolds , it is possible to assume $\partial_0 M' \subset D_0 M'$ to be totally geodetic (see chV [Smith]). As this is a special situation , ther is an alternative way to see it ([Zimmermann]. In D_0M' , $\partial_0 M'=\mathrm{Fix}\tau$ with τ the (topological for the moment) reflection which interchanges the two copies of M' which form D_0M' .

For any $f:(M',\partial_0 M') \to (M',\partial_0 M')$, D_0f and τ commute . By [Tollefson] τ is conjugate to an isometry in D_0M' and conjugation is by means of a homeomorphism isotopic to the identity . Then , changing the metric tensor , it is possible to assume that τ is an isometry (of course $\mathrm{Fix}\,\tau$ becomes geodetic) . By Mostow's rigidity theorem and by theorem 7 of [Waldhausen] , D_0f is isotopic to an isometry h . As D_0f does , h commutes with τ , and therefore h preserves $\partial_0 M'$ and M' . We will see now that the isometry can be chosen to preserve $\partial_0 M'$.

Assume the above not true . Consider $F:\partial_0 M'xI \to D_0 M'$ the restriction in $\partial_0 M'$ of one isotopy between $D_0 f$ and h . Observe that the restrictions of $D_0 f$ and h in $\partial_0 M'$ are isotopic in $\partial_0 M'$ by an isotopy defined in all $D_0 M'$ (Indeed , the fact that $D_0 f$ and $h:D_0 M' \to D_0 M'$ are homotopic means that two of their liftings $\tilde{f}$ and $\tilde{h}$ define by conjugation the same homomorphism in the group of the deck transformations , for a fixed locally isometric covering $H^3 \to D_0 M'$. $\partial_0 M'$ lifts in a union of mutually disjoint planes which is preseved by $\tilde{f}$ and $\tilde{h}$. Fixed one plain H , then $\tilde{f},\tilde{h}:H \to \tilde{f}(H)=\tilde{h}(H)=H'$ are equivariant respect the stabilizers Γ and Γ' of H and H' and induce the same homomorphism between Γ and Γ' . Therefore $f,h:\Sigma \to \Sigma'$ are homotopic and ,therefore ,isotopic (here the Σ's are projections of the H's). Then it is possible to assume $f=h$ in $\partial_0 M'$ (after an isotopy with support in a thin nbhd of $\partial_0 M'$).

As we have done before , we assume that

$F:D_0 M'xI \to D_0 M'$ can be chosen to be ,when restricted to the boundary , of the form

$F:S^1 x S^1 x I \to S^1 x S^1$ $\quad F(x,y,t)=((\exp t\, 2\pi i\, \alpha)\, x,(\exp t\, 2\pi i\, \beta)\, y)$ with α and β rationals. Assume also that the boundary of $\partial_0 M'$ is formed by geodesics for this parametrization of $\partial D_0 M'$.

Therefore a component of the boundary of $\partial_0 M'$ is either

(i) fixed by the isotopy F , or

(ii) spans the related component of $\partial D_0 M'$.

If case (i) happens for all the boundary components of $\partial_0 M'$ then one could assume that the isotopy F preserves $\partial_0 M'$, i.e.one can find another isotopy which in $\partial D_0 M'$ is equal to F and makes what we expect . Indeed $F:\partial_0 M'xI \to D_0 M'$ is isotopic rel $\partial(\partial_0 M'xI)$ to a $F':\partial_0 M'xI \to D_0 M'$ with $\mathrm{Image}F' \subset \partial_0 M'$. This turns out to be a restriction of a

$D_0M'xI \rightarrow D_0M'$ which is equal to the original isotopy F in $\partial D_0M'$ (the argument is described in detail in [Waldhausen] pg 81-82)

If case (ii) happens for a component $C \subset \partial\Sigma$ ($\Sigma \subset \partial_0 M'$ a component), then consider the map (obtained when one glues the top and the bottom of ΣxI)

$F:\Sigma xS^1 \rightarrow D_0M'$, and the induced map

$F_*: \pi_1(\Sigma xS^1) \rightarrow \pi_1(D_0M')$.

Now $\pi_1(\Sigma xS^1) = < a , b , c , f \mid f$ commutes with a , b , c and $a.b.c=1 >$ .The image of f should commute with the images of a b c , i.e. with three parabolic elements with fixed points contained in three different orbits . Then the image of f should be the identity , which is not true from the incompressibility of the components of $\partial D_0M'$ and the fact that $\{x\}xS^1$ embeds in a nontrivial loop of $\partial D_0M'$, for x a point of a boundary component which meets Σ). therefore assuming case (ii) we obtain a contradiction .

Therefore, given $f:(M',\partial M') \rightarrow (M',\partial M')$, the induced map $D_0f:D_0M' \rightarrow D_0M'$ is isotopic to an isometry which preserves M' by an isometry which preserves M' . .

This gives a well defined immersion $MC^+(M') \rightarrow Isom(D_0M')$. As the last group is finite, $MC^+(S^3/G)$ is finite .

<u>Corollary 1.6</u> If M' is hyperbolic , there is a section for the epimorphism $Diff^+(S^3/G) \rightarrow MC^+(S^3/G)$.

<u>Proof</u> We prove first that exists a section $MC^+(S^3/G) \rightarrow Diff^+(M')$.

For M' hyperbolic but not a handlebody of genus 2 this has been proved in lemma 1.4 (we have showed that ,immediately or after some work , one is in condition to use Mostow's rigidity).

For M' a handlebody of genus 2 one has a well defined homomorphism $MC^+(S^3/G) \to MC^+(\partial M')$ which is an injection (these things follow easily from the fact that $MC^+(S^3/G) \to MC^+(M')$ is well defined and injective).By [Kerckhoff] there is a group of conformal transformations Γ for a conformal structure in $\partial M'$ which realizes $MC^+(S^3/G) \subset MC^+(\partial M')$. Fixing our attention to $\partial M'$, we see that the elements of Γ are isotopic to restrictions of homeomorphisms $S^3/G \to S^3/G$. Therefore the single elements of Γ are extendable to all M'. Moreover, the conformal structures on $\partial M'$ can be thought always as being induced by hyperbolic structures in M' (uniformization by Schottky groups). These two things , along with the fact that one can identify in a natural way conformal homeomorphisms of the 2-sphere with hyperbolic isometries , imply that Γ extends to a group of isometries for some hyperbolic structure in the interior of M .

The proof is completed by means of the following :

Lemma 1.7 .Let $O=(|O|,\Sigma O)$ be a good 3-orbifold whose components are finitely covered by handlebodies , with $|O|$ a union of handlebodies . Assume moreover that $|O|$ retracts in ΣO Then ,if $\Gamma \subset \mathrm{Diff}(\partial O)$ is a finite group of diffeomorphisms all extendable in all O , the action of Γ extends in all O .

Sketch of proof .Consider $X \to O$ a covering , with X a union of handlebodies . Lift Γ in a Γ. Then Γ acts on ∂X and it's single elements are extendable in X. Exists a hyperbolic structure on ∂X for which Γ acts as a group of isometries. Consider a decomposition in pants of ∂X, where the boundary curves of the pants are geodesics on ∂X which bound compressing disks in X. Then Meeks and Yau prove that there is in X a G-invariant family of disks which bound tthese geodesics. Extend the action of Γ (i.e. of the remaining elements) in the union of these disks as follows :

the action $G \times \cup D_\alpha \rightarrow D_\alpha$ is conjugate to an isometric action for a fixed flat structure . Extend first Γ on the centers of the disks and then , using the linear structure of the flat structure , extend linearly) .

Then cut X along the disks , to obtain a union of balls . The action of G in the union of these balls is equivalent to an isometric action for a fixed flat structure . Then extend the action of Γ first to the centers of the balls and then , using the linear structure , to the full balls . After this process glue all back. Then Γ induces the wanted action of Γ in O .

Corollary 1.8 . If M' is Seifert fibered , $Diff^+(S^3/G)$ contains a finite subgroup which projects into $MC^+(S^3/G)$ with kernel ,say K , either trivial or Z_2 .

(For O ,the base orbifold , a 2-punctured sphere this has been discussed in the proof of lemma 1.4 ,i.e.

either $H^+(S^3/G)=Z_2$

or $H^+(S^3/G)=Z_2 \oplus Z_2$ (exactly when $q^2=1$ [p])).

We need some preliminaries on the relation between Seifer fibrations in S^3 and fibre preserving actions of subgroups of SO(4) (this material is taken by [Scott]) .

One has a sequence

(S1) $1 \rightarrow Z_2 \rightarrow S^3 x S^3 \rightarrow SO(4) \rightarrow 1$

where (x,y) goes to the transformation $A(x,y)z=xzy^{-1}$ (here "A(x,y)" is not a conventional notation ; here S^3 is thought as the moltiplicative group of the unitary quaternions).

This induces a sequence

$$1 \to Z_2 \to S^3 \to SO(3) \to 1$$

for A(x,x) preserves the 2-spheres in S^3 formed by points with a given distance from $1 \in S^3$.

A sequence

(S2) $1 \to Z_2 \to SO(4) \to SO(3) x SO(3) \to 1$

remains induced .

For any 1-dimensional compact group $S \subset SO(4)$ remains induced a Seifert fibration (the leaves are the orbits) which is a Hopf fibration (i.e. the S^1-bundle over S^2 with Euler characteristic equal to 1) . Any two fibrations of this type are conjugate in O(4) . Therefore we will content to consider the standard one

(S^3,Φ) $\Phi=\{ L_\lambda \mid \lambda \in C \cup \{\infty\} \}$

$L_\lambda = \{ (x,y) \in C^2 \mid |x|^2+|y|^2=1 \quad x/y=\lambda \}$

From the multiplication rule for the quaternions

$$(z_1,z_2)\,(w_1,w_2) = (z_1w_1 - z_2w_2, z_2w_1 + z_1w_2)$$

one observes that Φ is preserved always by right multiplication

($\lambda \to (\lambda w_1 - w_2)/(\lambda w_2 + w_1)$) and by left multiplication by (w_1,w_2) iff either $w_1=o$ (then $\lambda \to -1/\lambda$) or $w_2=0$ (then $\lambda \to \lambda$) .

This implies easily that the set of fibre preserving elements $\Phi P \subset SO(4)$ is exactly the counterimage in (S2) of O(2)xSO(3) .

From $\Phi P \to P\Gamma L(2,C) \subset Diff(S^2)$ (where $P\Gamma L(2,C)$ is the group of conformal transformations on S^2, the base orbifold) , is induced a sequence

$1 \to SO(2) x \{1\} \to O(2) x SO(3) \to P\Gamma L(2,C)$

where $O(2) x \{1\}$ goes to the transformation $\lambda \to -1/\bar{\lambda}$.

The image in $P\Gamma L(2,C)$ is a conjugate of $O(3) \subset P\Gamma L(2,C)$ and , composing by an inner authomorphism of $P\Gamma L(2,C)$, we can assume to have

(S3) $1 \to SO(2) x \{1\} \to O(2) x SO(3) \to O(3) \to 1$

where $\{1\} x SO(3) \to O(3)$ is the standard immersion and $O(2) x \{1\}$ goes to some orientation reversing orthogonal involution .

<u>Proof of the corollary</u> . We know that G preserves in S^3 a Seifert fibration . A classical result of Vogt asserts that a subgroup of $Diff^+(S^3)$,non necessarily finite , which preserves a Seifert fibration in S^3 is conjugate to an orthogonal group . Therefore we assume that $G \subset SO(4)$. We have seen , in the discussion of lemma 1.4 , that the Seifert fibration is S^3/G can be chosed in such a way that S^3/G fibres on a punctured sphere . As G is a group of orientation preserving diffeomorphisms , this implies that exists in S^3/G an orientable Seifert fibration. This implies that there is an S^1-action on S^3/G which preserves the fibres (this can be seen as follows :

let $O"=O-\cup D$ where the D's are disks , one for each puncture plus one more disk. Then it's counterimage is a trivial bundle where is easy to define an S^1-action . Then one extends this action in the interiors of remaining solid tori (here i don't discuss this))

Then G is in the normalizer (in this case in the centralizer) of a group isomorphic to S^1 contained in $Diff^+(S^3/G)$ .By the result of Vogt , the group $<G , S^1>$ is conjugate to an orthogonal group . Now I make the following assumption :

<u>Assumption 1.9</u> . If G , G' are finite orthogonal groups conjugate in $Diff^+(S^3)$ then they are conjugate in SO(4) .

Note .This should follow from a classification (or a list) of finite subgroups of SO(4) which should be contained in [Seifert-Threllfal] , a paper which I have not read .

Therefore , we conclude that , after one more conjugation if necessary , $G \subset ZS^1$ (i.e. in the centralizer) in SO(4) .But this S^1 determines a Hopf fibration in the sense considerd above , aand that G preseves the orientation of this fibration . From the above discussion the projection of G in the sequence (S2) is contained in a $G_1 x G_2 \subset SO(2)xSO(3)$. Then $G \subset Diff^+(S^3)$ projects to $G_2 \subset Diff(S^2)$ and for the base orbifold one has $O \cong S^2/G_2$. Now , MC(O) is realized by a finite subgroup of Diff(O) , which , on it's turn , lifts to a finite subgroup of O(3) . But the sequence (S3) splits . Therefore one obtains a finite subgroup of O(2)xSO(3). Consider finally the counterimage ,call it Γ , of the last group in SO(4). Then $\Gamma/G \rightarrow MC^+(S^3/G)$ will be an epimorphism with kernel equal to Z_2 .

2- Proof of proposition 2

Proposition 2 Let G be a finite group and let F=U Fix g $g \in G$ $g \neq$identity :
if S^3-F is Seifert fibered , then $MC^+(S^3/G) \cong NG/ZG.G$
if S^3-F is hyperbolic (and not Seifert fibered) then $MC^+(S^3/G) \cong NG/G$

(where NG and ZG are the normalizer and the centralizer of G in SO(4))

Proof .(i)

(i) If F is nonempty then we distinguish two cases :

S^3-F is Seifert fibered .

By the 2-dimensional case of our problem $MC(O) \cong NG_2/<ZG_2,G_2>$. But , as NG_2 and ZG_2 lift to the normalizer and ,respectively , to the centralizer of G in SO(4) , and from the diagram

$$\begin{array}{ccc} NG & \to & NG_2 \\ \downarrow & & \downarrow \\ MC^+(S^3/G) & \cong & MC(O) \end{array}$$

the result follows .

S^3-F is hyperbolic and not Seifert fibered.

Then NG/G is a finite group (otherwise NG would contain an S^1 and there would be a free S^1-action on S^3-F , which therefore would be Seifert fibered). NG/G acts effectively on the manifold with boundary $M'=S^3/G$- int N(F'). If it's action is free then , as the quotient manifold is hyperbolic (for it is a Haken manifold, therefore a manifold which admits a decomposition in geometric pieces ,covered by a hyperbolic manifold, and thus is hyperbolic) , NG/G acts as a group of isometries for a complete Riemannian metric of constant negative curvature in M' . If the action is nonfree the same follows assuming Thurston's claim. As different isometries cannot be homotopic ($\pi_1(M')$ has trivial center in PSL(2,C) , unless it is a group of roto-translations respect to a line or a parabolic group, which is not our case. If two different isometries commuted , then the center would be nontrivial) the above implies that $NG/G \to MC^+(S^3/G)$ is injective .

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